Thm: $(\Omega, \sigma(\mathcal{A}), m)$을 시그마유한측도공간 ($\sigma$-finite measure space) 이라고하자. 그리고 $\mathcal{A}$를 아래의 조건을 만족하는 π-system 이라고하자.

① $\exists A_1, A_2, \dots \in \mathcal{A}$ such that $\bigcup_{i=1}^{\infty} A_i = \Omega$.

② $\forall i \in \mathbb{N}: m(A_i) < \infty$.

그렇다면 메저 $m: \sigma(\mathcal{A}) \to [0, \infty]$의 값은 $m: \mathcal{A} \to [0, \infty]$의 값에 의하여 유일하게 결정된다.

머리속상상

	m_1	m_2
$\mathcal{A}$	''	''
$\sigma(\mathcal{A}) - \mathcal{A}$	○	✓

$\tilde{\mathcal{D}} = \{B \in \sigma(\mathcal{A}) : m_1(B) \neq m_2(B)\}$.

$\mathcal{D} = \{B \in \sigma(\mathcal{A}) : m_1(B) = m_2(B)\}$.

* $\mathcal{D}_A = \{B \in \sigma(\mathcal{A}) : m_1(A \cap B) = m_2(A \cap B)\}$.

WTS: $\mathcal{D} = \sigma(\mathcal{A}) \Leftrightarrow \forall B \in \sigma(\mathcal{A}): m_1(B) = m_2(B) \quad \cdots ⊛$

그런데 ⊛를 보이기위해서 $\forall A \in \mathcal{A}: \mathcal{D}_A = \sigma(\mathcal{A})$ 임을 보여도 충분함.

$\because$ Fix $B \in \sigma(\mathcal{A})$.

Choose $A_1, A_2, \dots \in \mathcal{A}$ st. $\bigcup_{i=1}^{\infty} A_i = \Omega$. From $A_1, A_2, \dots$ construct $\Omega_i \uparrow \Omega$. as following:

$\Omega_1 = A_1$.

$\Omega_2 = A_1 \cup A_2 = A_1 \uplus (A_2 - \Omega_1)$

$$\Omega_3 = A_1 \cup A_2 \cup A_3 = A_1 \uplus (A_2 - \Omega_1) \uplus (A_3 - \Omega_2)$$

$\vdots$

$$\Omega_n = A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_n = \biguplus_{i=1}^{n} (A_i - \Omega_{i-1}), \quad \Omega_0 = \phi.$$

$$m_1(B) = m_1(B \cap \Omega) = m_1\Big(B \cap \bigcup_{n=1}^{\infty} \Omega_n\Big)$$

$$= m_1\Big(\bigcup_{n=1}^{\infty} B \cap \Omega_n\Big) = m_1\Big(\lim_{n\to\infty} B \cap \Omega_n\Big) = \lim_{n\to\infty} m_1(B \cap \Omega_n)$$

$$\underset{?}{=} \lim_{n\to\infty} m_2(B \cap \Omega_n) = m_2(B)$$

$$?: \; m_1(B \cap \Omega_n) = m_1\Big(B \cap \biguplus_{i=1}^{n} (A_i - \Omega_{i-1})\Big)$$

$$= m_1\Big(B \cap \biguplus_{i=1}^{\infty} (A_i \cap \Omega_{i-1}^c)\Big) = m_1\Big(\biguplus_{i=1}^{\infty} (B \cap A_i \cap \Omega_{i-1}^c)\Big)$$

$$= \sum_{i=1}^{\infty} m_1(B \cap A_i \cap \Omega_{i-1}^c) = \sum_{i=1}^{\infty} m_1(B \cap \Omega_{i-1}^c \cap A_i)$$

$$= \sum_{i=1}^{\infty} m_2(B \cap \Omega_{i-1}^c \cap A_i)$$

Note: $B \cap \Omega_{i-1}^c \in \sigma(\mathcal{A})$ & $A_i \in \mathcal{A}$. 」

그런데 「…」의 논리가 임의의 $B \in \sigma(\mathcal{A})$에 대하여 성립하므로,

$$\forall A \in \mathcal{A} : \mathcal{D}_A = \sigma(\mathcal{A}) \Rightarrow \forall B \in \sigma(\mathcal{A}) : m_1(B) = m_2(B)$$

$\boxed{\text{WTS}}$: $\forall A \in \mathcal{A} : \mathcal{D}_A = \sigma(\mathcal{A})$.

따라서 ① $\forall A \in \mathcal{A} : \mathcal{D}_A \supset \sigma(\mathcal{A})$ ② $\forall A \in \mathcal{A} : \mathcal{D}_A \subset \sigma(\mathcal{A})$

임을 보이면 되다. 그런데 ②은 당연하므로, ①만 보이면 된다.

Note: IF ① $\mathcal{D}_A$ is contianing $\mathcal{A}$ ② $\mathcal{D}_A$ is λ-system

THEN $\mathcal{D}_A \supset l(\mathcal{A}) = \sigma(\mathcal{A})$.

그런데 여기에서 ① $\mathcal{D}_A \supset \mathcal{A}$ 는 당연하므로. ② 만 보이면 된다.

Fix $A \in \mathcal{A}$ 「 (i) $\Omega \in \mathcal{D}_A$ (ii) $B_1, B_2 \in \mathcal{D}_A$, $B_1 \subset B_2 \Rightarrow$ $B_2 - B_1 \in \mathcal{D}_A$ (iii) $B_1, B_2, \cdots \in \mathcal{D}_A \Rightarrow \biguplus_{i=1}^{\infty} B_i \in \mathcal{D}_A$.

Check (i) $\Omega \in \mathcal{D}_A \Leftrightarrow m_1(A \cap \Omega) = m_2(A \cap \Omega)$ ... 성립 o.

Check (ii) $B_1, B_2 \in \mathcal{D}_A \Leftrightarrow m_1(A \cap B_1) = m_2(A \cap B_1)$ &
$m_1(A \cap B_2) = m_2(A \cap B_2)$

$B_2 - B_1 \in \mathcal{D}_A \Leftrightarrow m_1(A \cap (B_2 - B_1)) = m_2(A \cap (B_2 - B_1))$

$\Leftrightarrow m_1(A \cap B_2) - m_1(A \cap B_1) = m_2(A \cap B_2) - m_2(A \cap B_1)$

$\therefore$ (ii) 도 성립함.

Check (iii) : $B_1, B_2, \cdots \in \mathcal{D}_A \Rightarrow \biguplus_{i=1}^{\infty} B_i \in \mathcal{D}_A$.

$\biguplus_{i=1}^{\infty} B_i \in \mathcal{D}_A \Leftrightarrow m_1\left(A \cap \biguplus_{i=1}^{\infty} B_i\right) = m_2\left(A \cap \biguplus_{i=1}^{\infty} B_i\right)$

그런데 $m_1\left(A \cap \biguplus_{i=1}^{\infty} B_i\right) = \sum_{i=1}^{\infty} m_1\left(A \cap \biguplus_{i=1}^{\infty} B_i\right)$

$= \sum_{i=1}^{\infty} m_2\left(A \cap \biguplus_{i=1}^{\infty} B_i\right) = m_2\left(A \cap \biguplus_{i=1}^{\infty} B_i\right)$

이므로 (iii) 도 성립」. 그런데 「 ... 」의 논의가 임의의 $A \in \mathcal{A}$. 에 대하여 동일하게 성립하므로, $\mathcal{D}_A$는 $\forall A \in \mathcal{A}$ 에 대하여 λ-system 이다. 따라서 모든 $A \in \mathcal{A}$에 대하여 $\mathcal{D}_A = \sigma(\mathcal{A})$ 이다.