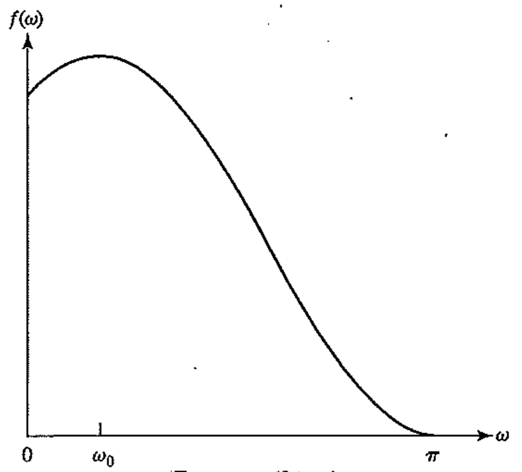

FIGURE 12.1 An example of a spectrum.

3. From (12.1.4), we have

$$\mathrm{Var}(Z_t) = \gamma_0 = \int_{-\pi}^{\pi} f(\omega)\, d\omega, \tag{12.1.10}$$

which shows that the spectrum $f(\omega)$ may be interpreted as the decomposition of the variance of a process. The term $f(\omega)\, d\omega$ is the contribution to the variance attributable to the component of the process with frequencies in the interval $(\omega, \omega + d\omega)$. A peak in the spectrum indicates an important contribution to the variance from the components at frequencies in the corresponding interval. For example, Figure 12.1 shows that the components near frequency ω_0 are of greatest importance and the high-frequency components near π are of little importance.

4. Equations (12.1.1) and (12.1.4) imply that the spectrum $f(\omega)$ and the autocovariance sequence γ_k form a Fourier transform pair, with one being uniquely determined from the other. Hence, the time domain approach and the frequency domain approach are theoretically equivalent. The reason to consider both approaches is that there are some occasions when one approach is preferable to the other for presentation or interpretation.

12.1.2 The Spectral Representation of Autocovariance Functions: The Spectral Distribution Function

Note that the spectral representation of γ_k given in (12.1.1) and (12.1.4) holds only for an absolutely summable autocovariance function. More generally, for a given autocovariance function γ_k, we can always have its spectral representation in terms of the following Fourier–Stieltjes integral:

$$\gamma_k = \int_{-\pi}^{\pi} e^{i\omega k}\, dF(\omega), \tag{12.1.11}$$

where $F(\omega)$ is known as the spectral distribution function. Equation (12.1.11) is usually referred to as the spectral representation of the autocovariance function γ_k. Like any statistical distribution function, the spectral distribution function is a nondecreasing function that can be partitioned into three components: (1) a step function consisting of a countable number of finite jumps, (2) an absolutely continuous function, and (3) a "singular" function. The third component is insignificant and is ignored in most applications. Thus, we may write the spectral distribution function as

$$F(\omega) \simeq F_s(\omega) + F_c(\omega), \tag{12.1.12}$$

where $F_s(\omega)$ is the step function and $F_c(\omega)$ is the absolutely continuous component. Equation (12.1.4) shows that for a process with absolutely summable autocovariance function, $F(\omega) = F_c(\omega)$ and $dF(\omega) = f(\omega)\,d\omega$.

To illustrate a step spectral distribution function, consider the general linear cyclical model

$$Z_t = \sum_{i=1}^{M} A_i \sin(\omega_i t + \Theta_i), \tag{12.1.13}$$

where the A_i are constants and the Θ_i are independent uniformly distributed random variables on the interval $[-\pi, \pi]$. The ω_i are distinct frequencies contained in the interval $[-\pi, \pi]$. Then

$$\begin{aligned} E(Z_t) &= \sum_{i=1}^{M} A_i E[\sin(\omega_i t + \Theta_i)] \\ &= \sum_{i=1}^{M} \frac{A_i}{2\pi} \int_{-\pi}^{\pi} \sin(\omega_i t + \Theta_i)\, d\Theta_i = 0, \end{aligned} \tag{12.1.14}$$

and

$$\begin{aligned} \gamma_k &= E(Z_t Z_{t+k}) \\ &= \sum_{i=1}^{M} A_i^2 E\,\{\sin(\omega_i t + \Theta_i)\sin[\omega_i(t + k) + \Theta_i)]\} \\ &= \sum_{i=1}^{M} A_i^2 E\left\{\frac{1}{2}[\cos \omega_i k - \cos(\omega_i(2t + k) + 2\Theta_i)]\right\} \\ &= \frac{1}{2}\sum_{i=1}^{M} A_i^2 \cos \omega_i k, \qquad k = 0, \pm 1, \pm 2, \ldots, \end{aligned} \tag{12.1.15}$$

where we note that

$$E\{\cos[\omega_i(2t + k) + 2\Theta_i]\} = \frac{1}{2\pi}\int_{-\pi}^{\pi} cos[\omega_i(2t + k) + 2\Theta_i]d\Theta_i = 0.$$