Tutorial - Extraction of Hadronic Quantities

1 Mass of a pion

A pion two-point function is given by

$$C(\vec{p},t) = \sum_{\vec{x},\vec{y}} e^{-i\vec{p}\cdot(\vec{x}-\vec{y})} \left\langle \phi(\vec{x},t)\phi^\dagger(\vec{y},0) \right\rangle = \left\langle \tilde{\phi}(\vec{p},t)\tilde{\phi}^\dagger(\vec{p},0) \right\rangle \tag{1}$$

where $\phi(x) = \overline{d}(x)\gamma_5 u(x)$ is an operator that creates a meson with quantum numbers of a pion and

$$\tilde{\phi}(\vec{p},t) \equiv \sum_{\vec{x}} e^{-i\vec{p}\cdot\vec{x}} \phi(\vec{x},t) \tag{2}$$

is the Fourier transformation to project to spatial momentum $\vec{p}$.

1.1 spectral representation

Show that the two-point function has the following spectral representation

$$C(\vec{p},t) = \sum_n \frac{|\langle 0|\,\phi\,|n\rangle|^2}{2E_n} e^{-E_n t} \tag{3}$$

Hint:

- Insert a complete set of states $\mathbb{1} = \sum_n \frac{1}{2E_n} |n\rangle \langle n|$ between the operators
- use Euclidean time evolution $O(t=0) = e^{\hat{H}t} O(t) e^{-\hat{H}t}$ for the operator
- $|n\rangle$ are eigenstates of the Hamiltonian, i.e. $\hat{H}\,|n\rangle = E_n\,|n\rangle$, with E_n the energy of state $|n\rangle$

1.2 effective mass

1. Show that the energy of the ground state (the state with the lowest energy, e.g. the pion in our example) can be obtained by

$$E_1 = \lim_{t\to\infty} E_{\text{eff}}(t) \qquad \text{with} \qquad E_{\text{eff}}(t) = \ln\left(\frac{C(\vec{p},t)}{C(\vec{p},t+1)}\right) \tag{4}$$

2. Commonly, lattice calculations use periodic boundary conditions in time. Thus, there will also be a "backward" propagating meson and equation (3) is modified as

$$C(\vec{p},t) = \sum_n \frac{|\langle 0|\,\phi\,|n\rangle|^2}{2E_n} \left(e^{-E_n t} + e^{-E_n(T-t)}\right) \tag{5}$$

where T is the time extend of the lattice. Convince yourself that the energy of the ground state can be obtained from either

$$E_{\text{eff}}(t) = \text{acosh}\left(\frac{C(\vec{p},t-1) + C(\vec{p},t+1)}{2C(\vec{p},t)}\right) \tag{6}$$

or by (numerically) solving

$$\frac{C(t)}{C(t+1)} = \frac{\cosh\big(E_{\text{eff}}(t)\ (t - T/2)\big)}{\cosh\big(E_{\text{eff}}(t)\ (t+1-T/2)\big)} \tag{7}$$

for $E_{\text{eff}}(t)$ at each t.

1.3 Numerical Excercise - effective mass of pion two-point function

In this exercise you will calculate the effective mass (6) and/or (7) of pion two-point function with spatial mometum $\vec{p} = 0$. The pion two-point correlation functions provided have been measured on a lattice with time extend $T = 96$ and periodic boundary conditions in time for 100 gauge configurations.

1. Read in the two-point function data from the provided data files `data/pion_g5-g5.<CONF>.dat` where `<CONF>` runs from 0 to 99 for the 100 gauge configuration. The files contain two columns, first is the time t, second is the two-point function $C(t)$.

   Hint: In python you can read the files as follows (e.g. to read confguration 0)

   ```
   import numpy as np

   filename = "data/pion_g5-g5.0.dat"
   t_read, twopt_read = np.loadtxt(filename, unpack=True)
   ```

   Shortcut: If you want you can start by using the provided python programme `read_twopt.py`, which reads in the data for all configurations and stores them in `twopt_conf[<t>][<conf>]` .

2. Average the two-point function over the configurations, build jackknife or bootstrap samples and determine the statistical error on the two-point function. (You may reuse code that you wrote for yesterday's tutorial!)

3. With the two-point functions (and periodic boundary conditions) of the form (5), you can "fold" the two-point function around $T/2$

$$C^{\text{fold}}(t) = \frac{1}{2}\Big(C(t) + C(T-t)\Big) \tag{8}$$

4. Optional: Plot the two-point fucntion with logarithmic y-scale (e.g., in python with `matplotlib`).

5. Implement at least one of the effective masses (6) or (7).

   Hint:

   - Numpy has functions `numpy.cosh` and `numpy.arccosh`.
   - For numerically solving as needed for (7) you can use `scipy.optimize.fsolve(func, x0, args=())`, which solves a function `f(x, *args)` with arguments `args` for x with a starting guess x_0.

6. Optional: Plot the effective mass (e.g., in python with `matplotlib`).

7. Determine the mass of the pion. You can either

   - average the effective mass from the plateau-region (i.e. the region of large enough t, where higher states are negligible)

   or

   - fit a constant to the effective mass in the plateau-region (e.g., using `scipy.optimize.curve_fit` in python).

   If the lattice spacing on this ensemble is $a = 0.12$ fm, what is the pion mass in MeV?

2 Leptonic Pion Decay

The hadronic part of the matrix element for leptonic pion decay $\pi \to \ell\nu_\ell$ can be written in terms of the pion decay constant f_π

$$\mathcal{A} = \langle 0| \, \overline{u}\gamma_0\gamma_5 d \, |\pi\rangle = M_\pi f_\pi \, . \tag{9}$$

On the lattice it can be determined from the two-point correlation function

$$C^A(t) = \sum_{\vec{y},\vec{x}} \langle 0| \, A(\vec{x},t)\phi^\dagger(\vec{y},0) \, |0\rangle \tag{10}$$

where $A(x) = \overline{u}(x)\gamma_0\gamma_5 d(x)$ and $\phi^\dagger(x) = \overline{d}(x)\gamma_5 u(x)$.

2.1 spectral representation

Write down the spectral representation of the two-point function $C^A(t)$. How can you extract f_π from $C^A(t)$ and the pion two-point function (1)?

NB: The axial-vector ($\gamma_0\gamma_5$) operator has negative parity. When using periodic boundary conditions in time, the "backward" propagating contribution will have opposite sign, such that

$$C^A(t) \sim \left(e^{-mt} - e^{-m(T-t)} \right) \, . \tag{11}$$

2.2 Numerical Exercise - determine the pion decay constant

In this exercise you will determine the pion decay constant from lattice two-point functions (using the same gauge configurations as for exercise 1.3). The data for the correlation function $C^A(t)$ is in files named `pion_g5-g0g5.<CONF>.dat`.

Determine the pion decay constant (including jackknife or bootstrap error) from $C^A(t)$ and the pion two-point function $C(t)$ used in 1.3.