# A Hermitian distance operator to represent directed natural processes

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## Introduction

### Rationales

Natural processes that operate in time are generally directed as they often cannot be reversed. For instances, erosion cannot reverse to generate mountains decomposing leaves can't be reversed to life leaves, bush fires can't be reverted back to grasslands, and so on. Also, natural processes that operate in space may do so in a directed manner. For examples, seeds may disperse more readily down dominant winds, river fish may prefer locations some distance downstream of tributaries carrying drifting preys on which they can feed, parasite may more readily transfer from one area to another than the other way around because of spatiotemporal patterns in migrations. It is not particularly hard to imaging all sorts of situation where time, space, or space-time interactions would occur in directed fashions to influence natural variability. Have a way to describe that variability and its directed nature would thus be a valuable asset to ecology, evolutionary science, and other disciplines that are concerned with the dispersal for organisms or genes in space and time.

### Fourier transforms

Fourier transforms is, perhaps, the most common approach to time series analysis. It consists in representing the time series as a set of basis functions that are the sum of a cosine and a sine as follows:

$$\psi_k(x) = \sum_{j=0}^{N-1} x_j e^{-\frac{i2\pi}{N}kj}$$

where $k = 0, 1, 2, ..., N-1$ is the order of the transform and $j$ is the index of each of the $N$ discrete observations $x_j$ in the time series ($e$ and $\pi$ are the usual constants and $i$, the imaginary unit solving the equation $i^2 + 1 = 0$). Because the exponential of an imaginary number corresponds to a sum of a cosine and a sine ($e^{ia} = \cos(a) + i\sin(a)$), $\psi_k(x)$ can also be represented as follows:

$$\psi_k(x) = \sum_{j=0}^{N-1} x_j \left[\cos\left(\frac{2\pi}{N}kj\right) - i\sin\left(\frac{2\pi}{N}kj\right)\right]$$

and thus each components of the transform correspond to the combination of a cosine whose amplitude corresponds to the real part of $\psi_k(x)$, which we will hereafter refer to as $\mathrm{Re}(\psi_k(x))$, and a sine whose amplitude corresponds to the imaginary part of $\psi_k(x)$, and which we will hereafter refer to as $\mathrm{Im}(\psi_k(x))$. The inverse of that transform is obtained by the following:

$$x_j(\psi) = \frac{1}{N} \sum_{k=0}^{N-1} \psi_k e^{\frac{i2\pi}{N}kj}$$

which corresponds to superposing each an every cosine and sine combinations as follows:

$$x_j(\psi) = \frac{1}{N} \sum_{k=0}^{N-1} \psi_k \left[ \cos\left(\frac{2\pi}{N} kj\right) + i \sin\left(\frac{2\pi}{N} kj\right) \right]$$

Special considerations arise as $x_j$ are all real values. When $k = 0$, the cosine is a constant 1 and the sine a constant 0 irrespective of $j$, and thus $\psi_0$ is the sum of all $x_j$. Moreover, coefficients with $k > 0$ are conjugate symmetric (i.e., $\psi_l = \mathrm{Conj}(\psi_{N-l})$ for $l = 1, 2, ..., N-1$), which is expected as pairs of end to end coefficients have to cancel out one another's imaginary part to yield real numbers.