

# Quantum Computation, Categorical Semantics and Linear Logic

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## Abstract

We develop a type theory and provide a denotational semantics for a simple fragment of the quantum lambda calculus, a formal language for quantum computation based on linear logic. In our semantics, terms inhabit certain Hilbert bundles, and computations are interpreted as the appropriate inner product preserving maps between Hilbert bundles. These bundles and maps form a symmetric monoidal closed category, as expected for a calculus based on linear logic.<sup>1</sup>

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Keywords: Quantum Computing, Lambda Calculus, Linear Logic, Denotational Semantics

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<sup>1</sup>**Note added Feb '07:** The construction of the category in section 5 contains a flaw that invalidates the assertion that Hilbert bundles provide a monoidal closed category. An alternative model is under active research. To the knowledge of the authors, the syntax, type system, and equational proof system remain correct, as does the presentation of the semantics in section 7, provided a suitable category can be constructed.

# 1 Introduction

The quantum lambda calculus developed by the author in [1] may be regarded both as a programming language and as a formal algebraic system for reasoning about quantum algorithms. It provides a model of quantum computation that combines the universality of the quantum Turing machine [2, 3] and the compositionality of the quantum circuit models [4]. The calculus turned out to be closely related to the linear lambda calculi used in the study of linear logic [5, 6, 7, 8]. In [1], we set up a computational model, or operational semantics, and an equational proof system for this calculus, and argued that it was equivalent to the quantum Turing machine, and therefore universal for quantum computation.

In the present article, we report progress in developing a type theory and a denotational semantics for a fragment of the quantum lambda calculus.

In our semantics, terms inhabit certain Hilbert bundles, and computations are interpreted as the appropriate inner product preserving maps between Hilbert bundles. These bundles and maps form a symmetric monoidal closed category, as expected for a calculus based on linear logic [9, 10].

For simplicity, in this paper we restrict our attention to a purely linear fragment of the full quantum calculus. This fragment is not universal for quantum computation, but is at least as expressive as the quantum circuit model. Future work should address the full calculus.

## 2 A simple quantum calculus

We consider only a simple linear, multiplicative fragment  $\lambda_q^\otimes$  of the quantum lambda calculus. First, we define a grammar for types in figure 1.

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$A, B ::=$	<i>types:</i>
Qbit	<i>qubit</i>
$(A \multimap B)$	<i>function</i>
$(A \otimes B)$	<i>tensor product</i>

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**Figure 1:** Types of the quantum calculus  $\lambda_q^\otimes$

The grammar for raw terms, given in figure 2, is based on a syntax for linear logic developed by Wadler in [6]. We have added constants representing qubit states and unitary maps, and we have provided primitives for addition and scalar multiplication of terms, to enable us to express quantum superpositions.

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$p, q ::=$	<i>patterns:</i>
$x$	<i>variable</i>
$(p \otimes q)$	<i>product</i>
$t, u ::=$	<i>terms:</i>
$x \mid y \mid \dots$	<i>variable</i>
$(\lambda p : A. t)$	<i>function abstraction</i>
$(t \ u)$	<i>function application</i>
$(t \otimes u)$	<i>product</i>
$\mathbf{0} \mid \mathbf{1}$	<i>qubit</i>
$c_U$	<i>constant representing <math>U</math>, for every <math>U \in U(2^n), n = 1, 2, \dots</math></i>
$(t + u)$	<i>superposition</i>
$\alpha \cdot t$	<i>scalar product, for <math>\alpha \in \mathbf{C}</math></i>

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**Figure 2: Syntax of the quantum calculus  $\lambda_q^\otimes$**

In the following, we will omit brackets wherever convenient, with the convention that products, sums and applications associate to the left. For example,  $(f \ u \ v \ w) \equiv (((f \ u) \ v) \ w)$ .

We will implicitly identify terms that differ only in the renaming of bound variables (called  $\alpha$ -equivalence). In addition, we identify terms according to the equivalence relation generated by the equations of figure 3.

Notice that modulo this equivalence, the set of terms becomes a complex vector space. Indeed, the postulates of a vector space are satisfied by construction. The symbol  $\otimes$  can now be regarded as a bilinear tensor product, while the application bracket is bilinear and lambda abstraction linear over the vector space operations.

In the following, the notation  $t$  will be abused to denote the equivalence class to which  $t$  belongs under  $\alpha$ -renaming and the linear equivalences of

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$t_1 + t_2 \sim t_2 + t_1$	$(comm_+)$
$(t_1 + t_2) + t_3 \sim t_1 + (t_2 + t_3)$	$(assoc_+)$
$\alpha \cdot (t_1 + t_2) \sim \alpha \cdot t_1 + \alpha \cdot t_2$	$(dist)$
$\alpha \cdot (\beta \cdot t) \sim \alpha\beta \cdot t$	$(assoc\cdot)$
$t_1 + 0 \cdot t_2 \sim t_1$	$(zero)$
$1 \cdot t \sim t$	$(one)$
$(\alpha \cdot t_1 + \beta \cdot t_2) \otimes u \sim \alpha \cdot t_1 \otimes u + \beta \cdot t_2 \otimes u$	$(lin_{\otimes}^1)$
$u \otimes (\alpha \cdot t_1 + \beta \cdot t_2) \sim \alpha \cdot u \otimes t_1 + \beta \cdot u \otimes t_2$	$(lin_{\otimes}^2)$
$((\alpha \cdot t_1 + \beta \cdot t_2) u) \sim \alpha \cdot (t_1 u) + \beta \cdot (t_2 u)$	$(lin_{app}^1)$
$(u (\alpha \cdot t_1 + \beta \cdot t_2)) \sim \alpha \cdot (u t_1) + \beta \cdot (u t_2)$	$(lin_{app}^2)$
$\lambda p. (\alpha \cdot t_1 + \beta \cdot t_2) \sim \alpha \cdot (\lambda p. t_1) + \beta \cdot (\lambda p. t_2)$	$(lin_{\lambda})$

---

**Figure 3: Linear equivalences on terms**

figure 3. As a consequence, we can freely perform linear manipulations on terms, such as commuting sums, distributing tensor products or application brackets over sums, and so forth.

Not all terms constructible from the above grammar will represent physically meaningful computations. In order to encode the latter, we shall restrict attention to well-typed terms. The rules for constructing well-typed terms, based on the syntax provided in [6], are given in figure 4.

The typing rules allow us to construct only physically realizable computational states. In particular, the *Qbit-I* rule is used to form normalized superpositions of only equal-length sequences of qubits. The *Cut* rule then allows us to form superpositions of more complex terms containing interspersed qubits in a way that guarantees that all terms in a superposition at most differ in positions containing 0's and 1's. We will call such superpositions *congruent* [1]. These are the terms that can be obtained in computations where reductions proceed in parallel in all the branches of a quantum superposition, as required by quantum mechanics [1].

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$$\begin{array}{c}
\overline{x : A \vdash x : A} \quad (Id) \\
\\
\frac{\Gamma, p : A, q : B, \Delta \vdash t : C}{\Gamma, q : B, p : A, \Delta \vdash t : C} \quad (Exch) \\
\\
\frac{\Gamma \vdash \sum_i \alpha_i \cdot t_1^i \otimes \dots \otimes t_n^i : A_1 \otimes \dots \otimes A_n \quad x_1 : A_1, \dots, x_n : A_n, \Delta \vdash u : B}{\Gamma, \Delta \vdash \sum_i \alpha_i \cdot u[t_1^i/x_1, \dots, t_n^i/x_n] : B} \quad (Cut) \\
\\
\frac{\Gamma, p : A \vdash t : B}{\Gamma \vdash (\lambda p : A. t) : A \multimap B} \quad (\multimap-I) \\
\\
\frac{\Gamma \vdash t : A \quad y : B, \Delta \vdash u : C}{\Gamma, f : A \multimap B, \Delta \vdash u[(f t)/y] : C} \quad (\multimap-E) \\
\\
\frac{\Gamma \vdash t_1 : A \quad \Delta \vdash t_2 : B}{\Gamma, \Delta \vdash t_1 \otimes t_2 : A \otimes B} \quad (\otimes-I) \\
\\
\frac{\Gamma, p : A, q : B \vdash t : C}{\Gamma, p \otimes q : A \otimes B \vdash t : C} \quad (\otimes-E) \\
\\
\frac{\sum |\alpha_{b_1 \dots b_n}|^2 = 1}{\vdash \alpha_{0 \dots 0} \cdot \mathbf{0} \otimes \dots \otimes \mathbf{0} + \dots + \alpha_{1 \dots 1} \cdot \mathbf{1} \otimes \dots \otimes \mathbf{1} : \text{Qbit}^n} \quad (\text{Qbit-}I) \\
\\
\frac{U \in U(2^n)}{\vdash c_U : \text{Qbit}^n \multimap \text{Qbit}^n} \quad (c_U-I)
\end{array}$$


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**Figure 4: Well-typed terms of the quantum calculus  $\lambda_q^\otimes$**

Notice that the set of well-typed terms is not a vector space any longer, since the typing rules of figure 4 do not allow unrestricted superpositions of well-typed terms. For example, the term

$$\frac{1}{\sqrt{2}}(\mathbf{0} + \mathbf{0} \otimes \mathbf{0})$$

is not derivable from the above rules and is therefore not well-typed. Neither is the non-normalized term  $\mathbf{0} + \mathbf{1}$ . Although we have lost the vector space structure, when we develop the denotational semantics of the calculus in

section 4, we shall see that well-typed terms fit naturally within a Hilbert bundle category.

Next we postulate a set of equations that we intend to be true in our calculus. Suppressing type annotations for brevity, these are given in figure 5.

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$$\begin{array}{ll}
t = t & (refl) \\
\\
\frac{t_1 = t_2}{t_2 = t_1} & (sym) \\
\\
\frac{t_1 = t_2 \quad t_2 = t_3}{t_1 = t_3} & (trans) \\
\\
\frac{t_1 = t_2 \quad u_1 = u_2}{\alpha \cdot t_1 + \beta \cdot u_1 = \alpha \cdot t_2 + \beta \cdot u_2} & (lin) \\
\\
\frac{t_1 = t_2 \quad u_1 = u_2}{(t_1 \ u_1) = (t_2 \ u_2)} & (app) \\
\\
(\lambda x_1 \otimes \cdots \otimes x_n. u) \sum_i \alpha_i \cdot t_1 \otimes \cdots \otimes t_n = \sum_i \alpha_i \cdot u [t_1/x_1, \dots, t_n/x_n] & (\beta) \\
\\
\frac{x \text{ not free in } t}{\lambda x. (t \ x) = t} & (\eta) \\
\\
\frac{U \in U(2^n) \quad b_1, \dots, b_n \in \{\mathbf{0}, \mathbf{1}\}}{c_U \ b_1 \otimes \cdots \otimes b_n = \sum_{b'_1, \dots, b'_n \in \{0,1\}} U_{b_1 \dots b_n}^{b'_1 \dots b'_n} \cdot b'_1 \otimes \cdots \otimes b'_n} & (U)
\end{array}$$


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**Figure 5: Equational proof system for the quantum calculus  $\lambda_q^\otimes$**

These rules represent an equality relation on well-typed terms only. In other words, we impose the constraint that the total term on either side of each  $=$  sign be well-typed. If we were to ignore this constraint, we would be able to prove certain well-typed terms equal to terms that are not well-typed. The constraint avoids equalities such as

$$\frac{1}{\sqrt{2}}((\lambda x : \text{Qbit. } x) \ \mathbf{0} + (\lambda x : \text{Qbit. } x) \ \mathbf{1}) = \frac{1}{\sqrt{2}}(\mathbf{0} + (\lambda x : \text{Qbit. } x) \ \mathbf{1}),$$

which is invalid because the left hand side is well-typed (as follows from using the *Qbit-I* and *Cut* rules) while the right hand side is not, because terms in the superposition are not congruent. On the other hand,

$$\frac{1}{\sqrt{2}}((\lambda x : \text{Qbit}. x) \mathbf{0} + (\lambda x : \text{Qbit}. x) \mathbf{1}) = \frac{1}{\sqrt{2}}(\mathbf{0} + \mathbf{1})$$

is a valid equality, since substitution has been performed in parallel in both branches of the superposition.

In order to be able to satisfy this constraint, note that the rule (*lin*) has to be formulated so as to allow parallel substitution in all branches of a superposition, giving a result which is a normalized sum of congruent terms. Also, note that rule ( $\beta$ ) cannot be formulated recursively in terms of binary sums only. Indeed, as in the (*Cut*) rule of figure 4, substitution has to occur in all branches of a superposition in parallel.

Note that the rules (*app*) and ( $\beta$ ) imply the following substitution rules under  $\otimes$  and lambda abstraction:

$$\frac{t_1 = t_2 \quad u_1 = u_2}{t_1 \otimes u_1 = t_2 \otimes u_2} \qquad \frac{t_1 = t_2}{\lambda p. t_1 = \lambda p. t_2}$$

Notice also that the rule ( $\beta$ ) implies, for example, the equation

$$(\lambda p \otimes q. t) u \otimes v = \lambda p. ((\lambda q. t) v) u$$

relating uncurried and curried versions of a function.

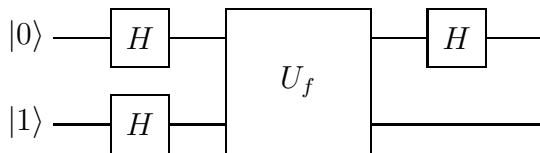
### 3 An example

Deutsch's algorithm [2, 11] can be expressed as follows:

```
deutsch  $\equiv$ 
   $\lambda U_f : (\text{Qbit} \otimes \text{Qbit} \multimap \text{Qbit} \otimes \text{Qbit}).$ 
    let  $x \otimes y : \text{Qbit} \otimes \text{Qbit} = U_f (H \mathbf{0}) \otimes (H \mathbf{1})$  in
       $(H x) \otimes y$ 
```

where we have used the common longhand

$$(\text{let } p : A = t \text{ in } u) \equiv (\lambda p : A. u) t$$



**Figure 6: Deutsch's algorithm**

and where  $H : \text{Qbit} \rightarrow \text{Qbit}$  is the Hadamard gate [11].

Here the argument  $U_f$  is assumed to be a unitary map that takes  $x \otimes y$  to  $x \otimes (y + f(x) \bmod 2)$ , where  $f$  is some unknown function of one bit. For example, if  $f$  is the identity function, we could take  $U_f$  to be the *cnot* gate.

It is an easy exercise to check that **deutsch** is well-typed according to the rules of figure 4. The equational proof rules of figure 5 allow us to simplify the expression  $(\text{deutsch } \textit{cnot})$  just as we normally would in a pen and paper calculation, to obtain

$$\text{deutsch } \textit{cnot} = \mathbf{1} \otimes \frac{1}{\sqrt{2}} \left( \mathbf{0} + (-1) \cdot \mathbf{1} \right),$$

where the result  $1 = f(0) + f(1) \bmod 2$  can be read off from the first qubit, indicating that the function is balanced, as required.

## 4 Categorical semantics

We now address the main question of this paper, namely that of providing a denotational semantics for the fragment of the quantum lambda calculus considered here. An operational semantics for an untyped version of a full lambda calculus for universal quantum computation, including a description of how computations in the calculus can be physically realized on a quantum Turing machine, was provided in the earlier paper [1].

The fragment  $\lambda_q^\otimes$  considered here is not universal. For example, it cannot express recursion. However, it is already quite powerful. Indeed, any fixed-size acyclic quantum circuit can be straightforwardly encoded in  $\lambda_q^\otimes$ . In addition, the calculus can express higher-order functions, representing operations on quantum circuits, higher order operations on these operations, and so forth.



The model for our interpretation will be a category. Types in the calculus will be interpreted as objects in the category, and judgments  $\Gamma \vdash t : A$  will be interpreted as morphisms between objects [12, 13, 14, 15]. In the rest of this section we will give some motivation for the particular category that we will choose, which will then be constructed in more detail in the following sections.

To begin our search for the appropriate category, consider first the types  $\text{Qbit}^n$ , intended to represent state vectors consisting of a finite number of qubits. To model these, it is clear that our category should have objects corresponding to  $2^n$ -dimensional Hilbert spaces. As a first attempt, we may therefore try to model our calculus in a category of Hilbert spaces. States would correspond, up to a phase, with unit vectors in a Hilbert space. The morphisms, corresponding to computations, would be unitary maps between Hilbert spaces.

However, in our calculus, computations (functions) are also first-order objects, and it is easily seen that a Hilbert space category is not appropriate for modeling function types such as  $\text{Qbit} \multimap \text{Qbit}$  and the towers of higher-order operations built upon them. For example, there is no sensible way of representing functions in  $\text{Qbit} \multimap \text{Qbit}$  as unit vectors in an appropriate Hilbert space. To see this, compare the operations that are available in  $\text{Qbit}$  with those that are available in  $\text{Qbit} \multimap \text{Qbit}$ . Up to normalization,  $\text{Qbit}$  is closed under addition. However, this is not true for elements of  $\text{Qbit} \multimap \text{Qbit}$ , if we interpret these as  $2 \times 2$  unitary matrices. Indeed, a nonzero sum of two unitary matrices cannot in general be normalized to give another unitary matrix unless one is a multiple of the other. Instead of a Hilbert space, we therefore need a mathematical structure where the applicability of addition is restricted, in this case to collinear matrices.

In fact, the complex one-dimensional vector spaces consisting of scalar multiples of unitary matrices naturally fit together to give a vector bundle over base space  $U(2)$ . We shall soon see that each fiber can be regarded as a one-dimensional Hilbert space. We will therefore be led to interpret  $\text{Qbit} \multimap \text{Qbit}$  as a vector bundle whose fibers are Hilbert spaces, or a *Hilbert bundle*.

Note that a Hilbert space such as  $\text{Qbit}^n$  can be considered as a special case of a Hilbert bundle, where the base space is a single point.

Further motivation for the bundle abstraction is given by considering the function

$$\text{apply} \equiv \lambda u \otimes x : (\text{Qbit} \multimap \text{Qbit}) \otimes \text{Qbit}. (u \ x)$$

with domain  $(\text{Qbit} \multimap \text{Qbit}) \otimes \text{Qbit}$  and range  $\text{Qbit}$ . The vector space dimension of the domain is larger than that of the range, meaning that in a Hilbert space category, no inner-product-preserving map would exist to give a semantics to **apply**. However, interpreting  $\text{Qbit} \multimap \text{Qbit}$ , as above, as a Hilbert bundle with fiber dimension 1, with any reasonable definition of the tensor product the total fiber dimension of the domain would be 2, just like that of the range, and we shall see that a fiberwise unitary map will exist that can be used to give a meaning to **apply**.

As another example, consider the function space  $\text{Qbit} \multimap \text{Qbit} \otimes \text{Qbit}$ , which contains, for example, the functions  $x \mapsto x \otimes 0$  and  $x \mapsto x \otimes 1$ . These are both isometries (they conserve the Hilbert space inner product) which can be added and normalized to give another isometry, generating one fiber of a Hilbert bundle with fiber dimension 2. Another fiber of the same bundle is generated by  $x \mapsto 0 \otimes x$  and  $x \mapsto 1 \otimes x$ . It is clear that functions from different fibers cannot sensibly be added to give an isometry. And indeed, the resulting term would not be well-typed.

## 5 Hilbert bundles

We have given some motivation that just as qubits inhabit Hilbert spaces, higher order types will inhabit appropriate Hilbert bundles. Just as operations on qubits may be represented as unitary transformations between Hilbert spaces, higher-order operations will be represented by the appropriate class of bundle morphisms. In this section we make these intuitions more precise. After defining the concept of a Hilbert bundle, we introduce a product  $\mathcal{A} \boxtimes \mathcal{B}$  and a quotient  $\mathcal{A} \multimap \mathcal{B}$  of Hilbert bundles for later use in our semantics. These satisfy the very pleasing property that if  $m$  is the rank (fiber dimension) of  $\mathcal{A}$  and  $n$  the rank of  $\mathcal{B}$ , then  $\mathcal{A} \boxtimes \mathcal{B}$  has rank  $mn$  and  $\mathcal{A} \multimap \mathcal{B}$  has rank  $\lfloor n/m \rfloor$ .

**Definition 5.1.** A *Hilbert bundle* is a vector bundle in which the fibers are isomorphic to a fixed Hilbert space  $\mathcal{H}$ .

More precisely, a Hilbert bundle consists of a total space  $E$ , a base space  $B$ , a typical fiber  $\mathcal{H}$ , and a projection  $\pi : E \rightarrow B$  such that for each point  $b \in B$ , there exists an open neighborhood  $U$  and a homeomorphism  $h : \pi^{-1}(U) \rightarrow U \times \mathcal{H}$ . Moreover, if  $h'$  is another such homeomorphism whose domain intersects with that of  $h$ , then  $h' \cdot h^{-1}$  is a fiberwise Hilbert space isomorphism.  $\square$

**Definition 5.2.** The *rank* of a Hilbert bundle is the dimension of its fibers.  $\square$

We will mainly deal with certain subbundles of complex *Grassmann bundles*, which are built out of subspaces of a complex vector space as follows [16]:

**Definition 5.3.** Given a complex vector space  $V$ , the *Grassmann bundle*  $\Gamma(k, V)$  is the vector bundle with base the *Grassmann manifold*

$$B \equiv G(k, V) \equiv \{W \mid W \text{ a } k\text{-dimensional subspace of } V\},$$

with total space

$$E \equiv \{(W, x) \mid x \in W\},$$

and projection  $\pi : E \rightarrow B : (W, x) = W$ .  $\square$

Notice that a Grassmann bundle comes with a canonical *Gauss map* [16] to the original vector space in which the fibers are embedded, given by

$$g : E \rightarrow V : (W, x) = x.$$

Generalising this situation, we define:

**Definition 5.4.** An *embedded Hilbert bundle with carrier space*  $V$ , where  $V$  is a complex vector space, is a Hilbert bundle  $\pi : E \rightarrow B$  together with a *Gauss map*  $g : E \rightarrow V$  taking fibers to subspaces of constant dimension. Restricted to each fiber, we require that  $g$  be linear and one-to-one. We also require that if  $v$  and  $w$  are both in the intersection of the images of two fibers, the inner product of their preimages in the two fibers coincide.  $\square$

If  $V$  is a Hilbert space, the Grassmann bundles associated with  $V$  can be regarded as embedded Hilbert bundles. So can any proper subbundles of a Grassmann bundle.

Also note that any Hilbert space  $\mathcal{H}$  may trivially be considered an embedded Hilbert bundle with a single fiber and carrier space  $\mathcal{H}$ .

Homomorphisms between Hilbert bundles are defined so as to preserve all the available structure:

**Definition 5.5.** Let  $\mathcal{A} \equiv \pi_1 : E_1 \rightarrow B_1$  and  $\mathcal{B} \equiv \pi_2 : E_2 \rightarrow B_2$  be Hilbert bundles. A function  $f : E_1 \rightarrow E_2$  is a *Hilbert bundle homomorphism* if

$f$  takes fibers to fibers, is fiberwise linear, and preserves the Hilbert space inner product fiberwise.

If  $\mathcal{A}$  and  $\mathcal{B}$  are embedded Hilbert bundles with carrier spaces  $V_1$  and  $V_2$  and Gauss maps  $g_1 : E_1 \rightarrow V_1$  and  $g_2 : E_2 \rightarrow V_2$ , then  $f$  is an *embedded Hilbert bundle homomorphism* if, in addition, there exists a unique linear function  $\bar{f} : V_1 \rightarrow V_2$  such that  $g_2 \cdot f = \bar{f} \cdot g_1$ .  $\square$

We are now ready to define  $\mathcal{A} \multimap \mathcal{B}$ .

**Definition 5.6.** Let  $\mathcal{A} \equiv \pi_1 : E_1 \rightarrow B_1$  and  $\mathcal{B} \equiv \pi_2 : E_2 \rightarrow B_2$  be Hilbert bundles. Let

$$S \equiv \{c \cdot f \mid c \in \mathbf{C}, f : \mathcal{E}_1 \rightarrow \mathcal{E}_2 \text{ a Hilbert bundle homomorphism}\},$$

where scalar multiplication on bundle maps is defined via  $(c \cdot f)(x) \equiv c f(x)$ , and addition as  $(f_1 + f_2)(x) \equiv f_1(x) + f_2(x)$  as long as  $\pi_2(f_1(x)) = \pi_2(f_2(x))$  for all  $x$ . The set  $S$  is not in general closed under  $\cdot$  and  $+$ , but we may consider subsets  $W$  of  $S$  which are closed under these operations and therefore form vector spaces. We call such a vector space  $W$  *maximal* if it is not a proper subspace of another such  $W'$ . Under the condition that all maximal spaces  $W \subseteq S$  have the same dimension, we may then define the Hilbert bundle  $\mathcal{A} \multimap \mathcal{B}$  as follows:

Let the total space be

$$E \equiv \{(W, f) \mid W \subseteq S \text{ is maximal, } f \in W\},$$

with base space projection

$$\pi : E \rightarrow B : (W, f) \rightarrow W,$$

and define a Hilbert space inner product on each fiber as that obtained from the norm

$$\|cf\| \equiv |c|,$$

for  $f$  a Hilbert bundle homomorphism, via the polarization formula

$$\langle x \mid y \rangle = \frac{1}{4} (\|x + y\|^2 + i \|x + iy\|^2 - \|x - y\|^2 - i \|x - iy\|^2).$$

If  $\mathcal{A}$  and  $\mathcal{B}$  are embedded Hilbert bundles with carrier spaces  $V_1$  and  $V_2$  and corresponding Gauss maps  $g_1 : E_1 \rightarrow V_1$  and  $g_2 : E_2 \rightarrow V_2$ , we define the carrier space of  $\mathcal{A} \multimap \mathcal{B}$  to be  $V \equiv \text{Lin}(V_1, V_2)$ , and the Gauss map to be  $g : E \rightarrow V : (W, f) \rightarrow \bar{f}$ .  $\square$

Note that, by construction, the bundle homomorphisms from  $\mathcal{A}$  to  $\mathcal{B}$  are exactly the unit vectors in the fibers of the Hilbert bundle  $\mathcal{A} \multimap \mathcal{B}$ . This generalizes the interpretation of quantum states as normalized vectors in a Hilbert space and is fundamental to our description of both primitive qubit types and function types in a unified framework.

Also note that homomorphisms  $f$ , which have to preserve the Hilbert space inner product on the fibers, only exist when the dimension of the fibers of  $\mathcal{A}$  is at most that of the fibers of  $\mathcal{B}$ . In other words,  $\mathcal{A} \multimap \mathcal{B}$  is nonempty only when  $\text{rank}(\mathcal{A}) \leq \text{rank}(\mathcal{B})$ .

In fact, we have the following nice result relating the rank of  $\mathcal{A} \multimap \mathcal{B}$  to those of  $\mathcal{A}$  and  $\mathcal{B}$ .

**Theorem 5.7.** *Let  $m$  be the rank of  $\mathcal{A}$  and  $n$  the rank of  $\mathcal{B}$ . Then  $\mathcal{A} \multimap \mathcal{B}$  has rank at most  $\lfloor n/m \rfloor$ , the largest integer less than or equal to  $n/m$ .*

*Proof.* As noted,  $\mathcal{A} \multimap \mathcal{B}$  can be nonempty only if  $n \geq m$ . Assume this and let  $(W, f)$  be a normalized element of  $\mathcal{A} \multimap \mathcal{B}$ . With respect to local trivializations of  $\mathcal{A}$  and  $\mathcal{B}$ , we can represent  $f$  by a field of  $n \times m$  matrices  $F(x)$  on each neighborhood of the base manifold of  $\mathcal{A}$ , satisfying  $F(x)^\dagger F(x) = \mathbf{1}_{m \times m}$  independent of  $x$ . If  $n = km$  for some integer  $k$ , then by choosing a suitable orthonormal basis in each fiber, we can bring  $F(x)$  to block form

$$F(x) = \begin{pmatrix} U_1(x) \\ 0 \\ \vdots \end{pmatrix}, \text{ with } k \text{ blocks of } m \times m \text{ matrices.}$$

Let  $(W, g)$ , with matrix representation  $G(x)$ , be in the same fiber and orthonormal to  $(W, f)$  with respect to the inner product in  $\mathcal{A} \multimap \mathcal{B}$ . By definition of  $\mathcal{A} \multimap \mathcal{B}$ , the matrix representation of  $\alpha \cdot f + \beta \cdot g$  has to satisfy

$$(\alpha F(x) + \beta G(x))^\dagger (\alpha F(x) + \beta G(x)) = \mathbf{1}_{m \times m}$$

for all  $|\alpha|^2 + |\beta|^2 = 1$ , independent of  $x$ . In other words, the columns of  $\alpha F(x) + \beta G(x)$  must remain normalized and orthogonal as we vary  $\alpha$  and  $\beta$ . It is not difficult to show that this means that  $G(x)$  can have no nonzero entries in the block occupied by  $U_1(x)$  above. We can therefore further rotate

$$\text{the basis in each fiber to make } G(x) \text{ of the form } G(x) = \begin{pmatrix} 0 \\ U_2(x) \\ 0 \\ \vdots \end{pmatrix} \text{ while}$$

keeping  $F(x)$  invariant. Since there are  $k$  vertical blocks, this can be repeated at most  $k$  times, so that the fiber to which  $(W, f)$  belongs can have dimension at most  $k = m/n$ .

In the case where  $m/n$  is not an integer, there is at most room for  $\lfloor m/n \rfloor$  square vertical blocks, and the result follows.  $\square$

All the Hilbert bundles that we will need will have ranks that are powers of two, so that  $\lfloor n/m \rfloor = n/m$  when  $n > m$ , and  $\lfloor n/m \rfloor = 0$  when  $n < m$ , in which case  $\mathcal{A} \multimap \mathcal{B}$  is the empty bundle.

Given two bundles, we can construct their *exterior tensor product* as follows:

**Definition 5.8.** Let  $\mathcal{A} \equiv \pi_1 : E_1 \rightarrow B_1$  and  $\mathcal{B} \equiv \pi_2 : E_2 \rightarrow B_2$  be Hilbert bundles. We define the *exterior tensor product bundle*  $\mathcal{A} \boxtimes \mathcal{B}$  as the Hilbert bundle with total space

$$E \equiv \bigcup_{(x,y) \in B_1 \times B_2} E_{(x,y)},$$

where the fiber  $E_{(x,y)} \equiv E_x \otimes E_y$  is the Hilbert space tensor product of the fibers

$$E_x \equiv \pi_1^{-1}(x), \quad E_y \equiv \pi_2^{-1}(y).$$

The base space is  $B \equiv B_1 \times B_2$ , with projection

$$\pi : E \rightarrow B : E_{(x,y)} \rightarrow (x, y).$$

If  $\mathcal{A}$  and  $\mathcal{B}$  are embedded Hilbert bundles with carrier spaces  $V_1$  and  $V_2$  and corresponding Gauss maps  $g_1 : E_1 \rightarrow V_1$  and  $g_2 : E_2 \rightarrow V_2$ , the carrier space of  $\mathcal{A} \boxtimes \mathcal{B}$  is the vector space tensor product  $V \equiv V_1 \otimes V_2$ , and the Gauss map is  $g \equiv g_1 \otimes g_2$ .  $\square$

In other words, in forming the exterior tensor product, we are taking tensor products of every fiber of  $\mathcal{A}$  with every fiber of  $\mathcal{B}$ . Note that this construction is different from the usual tensor product of vector bundles over a fixed base manifold, since here we are also multiplying the base spaces, which can be distinct. When  $\mathcal{A}$  and  $\mathcal{B}$  are ordinary Hilbert spaces considered as Hilbert bundles,  $\mathcal{A} \boxtimes \mathcal{B}$  reduces to the usual Hilbert space tensor product.

**Theorem 5.9.** *Let  $m$  be the rank of  $\mathcal{A}$  and  $n$  the rank of  $\mathcal{B}$ . Then  $\mathcal{A} \boxtimes \mathcal{B}$  has rank  $mn$ .*  $\square$

**Definition 5.10.** Let  $\mathbf{Unit}$  denote the complex numbers  $\mathbf{C}$  considered as an embedded Hilbert bundle with carrier space  $\mathbf{C}$ . As its name indicates, *unit* acts as a unit for the exterior tensor product in the sense that  $\mathcal{A} \boxtimes \mathbf{Unit} \simeq \mathbf{Unit} \boxtimes \mathcal{A} \simeq \mathcal{A}$ .  $\square$

We wrap up the section with a few constructions on bundle maps that will be very unseful in our semantics:

**Definition 5.11.** Let  $f : \mathcal{A} \rightarrow \mathcal{B}$  and  $g : \mathcal{C} \rightarrow \mathcal{D}$  be Hilbert bundle homomorphisms. Then we can define a Hilbert bundle homomorphism  $f \boxtimes g : \mathcal{A} \boxtimes \mathcal{C} \rightarrow \mathcal{B} \boxtimes \mathcal{D}$  fiberwise by

$$(f \boxtimes g)|_{E_x \otimes E_y} = f|_{E_x} \otimes g|_{E_y}$$

where  $E_x$  and  $E_y$  denote the fibers of  $\mathcal{A}$  and  $\mathcal{C}$  above  $x$  and  $y$  respectively.  $\square$

**Definition 5.12.** Given a Hilbert bundle homomorphism  $f : \mathcal{C} \boxtimes \mathcal{A} \rightarrow \mathcal{B}$ , let  $x_i \otimes y_j$  be a fiberwise orthonormal basis of  $\mathcal{C} \boxtimes \mathcal{A}$  and write

$$f\left(\sum_{ij} \alpha_{ij} x_i \otimes y_j\right) = f\left(x_i \otimes \sum_{ij} \alpha_{ij} y_j\right) \equiv f_i\left(\sum_{ij} \alpha_{ij} y_j\right).$$

Given that the  $x_i$  are orthonormal, each  $f_i$  is a Hilbert bundle homomorphism  $f_i : \mathcal{A} \rightarrow \mathcal{B}$ . Let  $\tilde{f}_i$  be the corresponding element  $(W, f_i)$  of  $\mathcal{A} \multimap \mathcal{B}$ . Then the linear map  $g : \mathcal{C} \rightarrow (\mathcal{A} \multimap \mathcal{B})$  defined by  $g(x_i) = \tilde{f}_i$  is a Hilbert bundle homomorphism. We write

$$\text{curry}(f) \equiv g.$$

$\square$

The currying operation has an adjoint, defined as follows:

**Definition 5.13.** We define a Hilbert bundle homomorphism

$$\text{apply} : (\mathcal{A} \multimap \mathcal{B}) \boxtimes \mathcal{A} \rightarrow \mathcal{B}$$

by

$$\text{apply} \sum_{ij} \alpha_{ij} (W, f_i) \otimes x_j \equiv \sum_{ij} \alpha_{ij} f_i(x_j).$$

$\square$

## 6 Bundle categories

Hilbert bundles form a category with particularly nice properties, making it very suitable for providing a semantics of our linear quantum lambda calculus [9, 10].

**Theorem 6.1.** *The category consisting of (embedded) Hilbert bundles and (embedded) Hilbert bundle homomorphisms is a symmetric monoidal closed category.*

*Proof.* The monoidal operation is provided by the exterior tensor product  $\boxtimes$ , which is symmetric and associative up to natural isomorphism, and has unit **Unit**. The existence of the connective  $\multimap$  and the functors **curry** and **apply** makes the category monoidal closed.  $\square$

We now define the category that will form the arena for our semantics.

**Definition 6.2.** The category  $\mathbf{Q}^\otimes$  is a subcategory of the category of embedded Hilbert bundles. Like the latter, it has as objects embedded Hilbert bundles, and as arrows embedded Hilbert bundle homomorphisms. The objects of  $\mathbf{Q}^\otimes$  is the smallest set such that

- $\mathbf{Unit} \in \mathbf{Q}^\otimes$ .
- $\mathbf{Qbit} \in \mathbf{Q}^\otimes$ , where **Qbit** is a fixed two-dimensional Hilbert space considered as an embedded Hilbert bundle with carrier space **Qbit**.
- $\mathcal{A} \boxtimes \mathcal{B} \in \mathbf{Q}^\otimes$  for every  $\mathcal{A}, \mathcal{B} \in \mathbf{Q}^\otimes$ .
- $\mathcal{A} \multimap \mathcal{B} \in \mathbf{Q}^\otimes$  for every  $\mathcal{A}, \mathcal{B} \in \mathbf{Q}^\otimes$ .  $\square$

## 7 Semantics

We are now ready to define the semantics. The interpretation of types will be a function  $\llbracket \cdot \rrbracket$  from type denotations in our grammar to bundles in the category  $\mathbf{Q}^\otimes$ , defined inductively as follows:

$$\begin{aligned} \llbracket \mathbf{Qbit} \rrbracket &= \mathbf{Qbit} \\ \llbracket A \multimap B \rrbracket &= \llbracket A \rrbracket \multimap \llbracket B \rrbracket \\ \llbracket A \otimes B \rrbracket &= \llbracket A \rrbracket \boxtimes \llbracket B \rrbracket \end{aligned}$$



Assumptions  $\Gamma \equiv x_1 : A_1, \dots, x_n : A_n$  will also be interpreted as bundles in the category as follows:

$$\begin{aligned} \llbracket \emptyset \rrbracket &= \mathbf{Unit} \\ \llbracket \Gamma \rrbracket &= \llbracket A_1 \rrbracket \boxtimes \dots \boxtimes \llbracket A_n \rrbracket \end{aligned}$$

The interpretation  $\llbracket \Gamma \vdash t : A \rrbracket$  of a judgment is a bundle morphism  $\llbracket \Gamma \rrbracket \xrightarrow{f} \llbracket A \rrbracket$ , inductively defined according to the rules in figure 7. These rules are in one-to-one correspondence to the rules for well-typed terms and follow the exposition of Wadler in [6]. To save notation, we write  $A$  for the bundle  $\llbracket A \rrbracket$  denoted by  $A$ . We also use the notation  $\simeq$  to indicate natural isomorphisms which we do not explicitly name in the figure.

In the interpretation, the (*Cut*) rule corresponds to composition of morphisms in the category. The fact that the category is monoidal gives a meaning to products in the  $(\otimes-I)$  and  $(\otimes-E)$  rules. The symmetric monoidal structure of the category plays a role in various of the rules, providing the necessary natural isomorphisms needed in, for example, the interpretation of (*Exch*) and  $(-\circ-E)$ . The fact that the category is monoidal closed provides the **curry** and **apply** operations needed for interpreting  $(-\circ-I)$  and  $(-\circ-E)$ .

It is instructive to revisit at this point just how the necessity for a Hilbert bundle category arose. Note that in order to give a meaning to the rules  $(Qbit^n-I)$  and  $(c_U-I)$ , we needed ground objects in one-to-one correspondence to Hilbert spaces, as well as objects containing unitary maps between any pair of Hilbert spaces. The linear logic structure of the calculus further requires that the tower of higher-order objects be represented by a symmetric monoidal closed category. It seems that a minimal structure satisfying these requirements is given by the category of Hilbert bundles. Due to the linearity of our calculus, the Hilbert bundles that we obtain via our inductive construction are in fact embedded bundles with a linear carrier space.

It is straightforward to check that the semantics is sound with respect to the equational rules of figure 5.

## 8 Related work

We list some previous work on the semantics of quantum computation:

Abramsky and Coecke [17] described a realization of a categorical model of multiplicative linear logic via the quantum processes of entangling and de-

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$\overline{A \xrightarrow{id_A} A}$	$(Id)$
$\frac{\Gamma \boxtimes A \boxtimes B \boxtimes \Delta \xrightarrow{f} C}{\Gamma \boxtimes B \boxtimes A \boxtimes \Delta \simeq \Gamma \boxtimes A \boxtimes B \boxtimes \Delta \xrightarrow{f} C}$	$(Exch)$
$\frac{\Gamma \xrightarrow{f} A_1 \boxtimes \dots \boxtimes A_n \quad A_1 \boxtimes \dots \boxtimes A_n \boxtimes \Delta \xrightarrow{g} B}{\Gamma \boxtimes \Delta \xrightarrow{g \circ (f \boxtimes id_\Delta)} B}$	$(Cut)$
$\frac{\Gamma \boxtimes A \xrightarrow{f} B}{\Gamma \xrightarrow{\text{curry}(f)} (A \multimap B)}$	$(\multimap-I)$
$\frac{\Gamma \xrightarrow{f} A \quad B \boxtimes \Delta \xrightarrow{g} C}{\Gamma \boxtimes (A \multimap B) \boxtimes \Delta \xrightarrow{f \boxtimes id \boxtimes id} A \boxtimes (A \multimap B) \boxtimes \Delta \simeq (A \multimap B) \boxtimes A \boxtimes \Delta \xrightarrow{\text{apply} \boxtimes id} B \boxtimes \Delta \xrightarrow{g} B}$	$(\multimap-E)$
$\frac{\Gamma \xrightarrow{f} A \quad \Delta \xrightarrow{g} B}{\Gamma \boxtimes \Delta \xrightarrow{f \boxtimes g} A \boxtimes B}$	$(\otimes-I)$
$\frac{\Gamma \boxtimes A \boxtimes B \xrightarrow{f} C}{\Gamma \boxtimes (A \boxtimes B) \simeq \Gamma \boxtimes A \boxtimes B \xrightarrow{f} C}$	$(\otimes-E)$
$\frac{v \in \text{Qbit}^n \quad \ v\ ^2 = 1}{\text{Unit} \xrightarrow{f: \text{Unit} \rightarrow \text{Qbit}^n: c \mapsto cv} \text{Qbit}^n}$	$(\text{Qbit}^n-I)$
$\frac{\text{Unit} \boxtimes \text{Qbit}^n \simeq \text{Qbit}^n \xrightarrow{U} \text{Qbit}^n}{\text{Unit} \xrightarrow{\text{curry}(U)} (\text{Qbit}^n \multimap \text{Qbit}^n)}$	$(c_U-I)$

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**Figure 7: Semantics of the quantum calculus  $\lambda_q^\otimes$**

entangling by means of typed projectors. They discussed how these processes can be represented as terms of an affine lambda calculus.

Attempts to develop an interpretation of quantum computing using Chu space models of linear logic are described in [18] and [19].

The imperative language qGCL, developed by Sanders and Zuliani [20], is based on Dijkstra’s guarded command language. It has a formal semantics and proof system.

An alternative approach to providing an operational semantics of quantum computation based on process algebras was developed in [21].

It would be interesting to relate our efforts to the work of Selinger [22], who constructed a semantics for quantum computation based on superoperators on density matrices, and of Girard [23], who developed a linear quantum logic based on similar technology. Closely related to Selinger’s approach is the work done by Edalat [24] on the use of partial density operators to model quantum computations with a probability of nontermination. Coecke and Martin [25] introduced a domain structure for quantum information theory based on a partial order on density matrices, and in related work Kashefi [26] developed a denotational semantics for quantum computation based on a domain theory of completely positive maps over density matrices.

In contrast to the work based on density matrices, our calculus is based on the state vector formalism of quantum mechanics and does not express measurements as primitive operations within the calculus itself. However, we do not consider this a fundamental weakness, since it is known that measurements can always be deferred until the end of the computation without affecting universality [11]. Indeed, the untyped quantum lambda calculus of [1] is universal and equivalent to the quantum Turing machine (which does not have measurement primitives either). As such, our approach provides a somewhat more minimal model of quantum computation.

## 9 Conclusion

In this paper, we developed a type theory and provided a denotational semantics for a simplified version of the quantum lambda calculus first developed in [1]. In our interpretation, variables inhabit certain Hilbert bundles, and computations are interpreted as appropriate inner product preserving maps between Hilbert bundles.

There are various possible topics for future research: The interpretation considered here inductively assigns a morphism in the category  $\mathbf{Q}^\otimes$  to every judgment we can derive in our calculus. An interpretation for which the

converse is true is called *full*. We conjecture that the Hilbert bundle interpretation is indeed full, although as of this writing we have not found a general proof of this conjecture.

As opposed to the full untyped quantum calculus developed in our previous work [1], the fragment considered here is not universal for quantum computation, although it is at least as expressive as the quantum circuit model. In order to achieve universality, we would need to model typed versions of the nonlinear variables appearing in the untyped calculus (also called exponentials), include additive connectives, and add some capability for recursion. This can most likely be done by relaxing the restriction that our Hilbert bundles be embedded in linear carrier spaces.

Linear logic, which explicitly models structural operations, is very suitable for complexity analysis, a topic of great interest in Quantum Computation. It is our hope that models of quantum computation based on linear logic may contribute to our understanding of complexity in Quantum Computation.

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