ADER-DG Implementation

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1 ADERDG Implementation

1.1 Notation

- Scalar quantities with physical meaning (coordinates, pressures, ...) are written in italic letters. This also includes the components of vectors and tensors with physical meaning; see the next bullet point.
- Vectors and tensors with physical meaning (points, velocities, forces, stresses,...) are printed in bold and using upright letters, e.g., $\mathbf{A}$, $\mathbf{f}$.
- Linear algebra vectors and matrices are written in upright letters and are underlined, e.g., $\underline{\mathrm{A}}$, $\underline{\mathrm{f}}$. Elements belonging to linear algebra vectors and matrices are also written in an upright font but are not underlined, e.g. A_{ij}, f_i.

1.2 Computational mesh and trace operators

Let $\mathcal{T}_h$ be a partition of the domain $\Omega \subset \mathbb{R}^d$ consisting of quadrilateral/hexahedral elements. We refer to the disjoint open sets $K \in \mathcal{T}_h$ as mesh cells. We denote by h_K the diameter of K. We store the local quantities h_K in the vector $\underline{\mathrm{h}} = \{h_K\}_{K \in \mathcal{T}_h}$, and set $h_{\max} = \max_{K \in \mathcal{T}_h} h_K$. Finally, $\mathbf{n}$ denotes the outward normal unit vector to the cell boundary ∂K. We will denote the number of cells by N_K.

An interior face of $\mathcal{T}_h$ is the $d-1$ dimensional intersection $\partial K^+ \cap \partial K^-$, where K^+ and K^- are two adjacent elements of $\mathcal{T}_h$. Similarly, a boundary face of $\mathcal{T}_h$ is the $d-1$ dimensional intersection $\partial K \cap \partial \Omega$ which consists of entire faces of ∂K. We denote by Γ_h the union of all interior faces of $\mathcal{T}_h$.

Here and in the following, we refer generically to a "face" even in the case $d = 2$.

1.3 Mappings

In this section, we introduce mappings between the mesh cells $K \in \mathcal{T}_h$ and a reference cell. Mappings allow us to treat integrals over mesh cells with non-uniform extent in an uniform manner. To ease the presentation, we only consider the two-dimensional case here.

Let us define the *reference cell* $\hat{K} = [0,1]^2$ and let us denote by $\hat{\mathbf{x}} = (\hat{x}, \hat{y})^{\mathrm{T}}$ a point belonging to $\hat{K}$. Let $K \in \mathcal{T}_h$ denote a nondegenerate quadrilateral cell with center $\mathbf{P}_0$ and cell size $(\Delta x,\, \Delta y)^T$.

We can express K according to

$$K = \boldsymbol{\mathcal{F}}_K(\hat{K}),$$

where we define the mapping $\boldsymbol{\mathcal{F}}_K : \hat{K} \to K$ as

$$\mathbf{x} = \boldsymbol{\mathcal{F}}_K(\hat{\mathbf{x}}) = \begin{pmatrix} \mathcal{F}_{K,x}(\hat{\mathbf{x}}) \\ \mathcal{F}_{K,y}(\hat{\mathbf{x}}) \end{pmatrix} = \mathbf{P}_0 + \begin{pmatrix} \Delta x & 0 \\ 0 & \Delta y \end{pmatrix} \begin{pmatrix} \hat{x} - 0.5 \\ \hat{y} - 0.5 \end{pmatrix} \tag{1.1}$$

We will further denote the inverse mapping by $\boldsymbol{\mathcal{F}}_K^{-1} \colon\ K \to \hat{K}$.

The considered quadrilateral cells have a constant Jacobian matrix, i.e.,

$$\mathrm{D}\boldsymbol{\mathcal{F}}_K(\hat{\mathbf{x}}) = \begin{pmatrix} \frac{\partial \mathcal{F}_{K,x}}{\partial \hat{x}} & \frac{\partial \mathcal{F}_{K,x}}{\partial \hat{y}} \\ \frac{\partial \mathcal{F}_{K,y}}{\partial \hat{x}} & \frac{\partial \mathcal{F}_{K,y}}{\partial \hat{y}} \end{pmatrix} (\hat{\mathbf{x}}) = \begin{pmatrix} \Delta x & 0 \\ 0 & \Delta y \end{pmatrix},$$

and thus also a constant Jacobian determinant:

$$\begin{aligned} J_K(\hat{\mathbf{x}}) &= \det\left(\mathrm{D}\boldsymbol{\mathcal{F}}_K\right)(\hat{\mathbf{x}}) \\ &= \Delta x\, \Delta y. \end{aligned} \tag{1.2}$$

Let $\hat{f}$ be a sufficiently regular function on the reference cell $\hat{K}$. Further let f be sufficiently regular on K and such that

$$f(\mathbf{x}) = (\hat{f} \circ \boldsymbol{\mathcal{F}}_K^{-1})(\mathbf{x}) = \hat{f}(\hat{\mathbf{x}}), \qquad \hat{\mathbf{x}} \in \hat{K},\, \mathbf{x} = \boldsymbol{\mathcal{F}}_K(\hat{\mathbf{x}}) \in K. \tag{1.3}$$

Then, it holds that

$$\nabla f(\mathbf{x}) = \left(\mathrm{D}\boldsymbol{\mathcal{F}}_K^{-\mathrm{T}} \cdot \hat{\nabla} \hat{f}\right)(\hat{\mathbf{x}}) = \begin{pmatrix} \frac{1}{\Delta x} & 0 \\ 0 & \frac{1}{\Delta y} \end{pmatrix} \hat{\nabla} \hat{f}(\hat{\mathbf{x}}) \tag{1.4}$$

where

$$\hat{\nabla} = \left(\frac{\partial}{\partial \hat{x}}, \frac{\partial}{\partial \hat{y}}\right)^{\mathrm{T}}$$

denotes the gradient with respect to the reference coordinates. This follows from (1.3.1) and using the chain rule. A similar identity can be derived for vector-valued functions.

We can also express integrals over the volume of cell K with respect to the reference coordinates and the invertible mapping $\boldsymbol{\mathcal{F}}_K$. Let $\hat{f}$ and f be defined as in (1.3.1), we have

$$\begin{aligned} \int_K f(\mathbf{x})\,\mathrm{d}\mathbf{x} &= \int_{\hat{K}} \hat{f}(\hat{\mathbf{x}})\,|J_K(\hat{\mathbf{x}})|\,\mathrm{d}\hat{\mathbf{x}} = \int_0^1 \int_0^1 \hat{f}(\hat{x},\hat{y})\,|J_K(\hat{x},\hat{y})|\,\mathrm{d}\hat{x}\,\mathrm{d}\hat{y} \\ &\overset{(1.2)}{=} J_K \int_0^1 \int_0^1 \hat{f}(\hat{x},\hat{y})\ \mathrm{d}\hat{x}\,\mathrm{d}\hat{y} \\ &= \Delta x\,\Delta y \int_0^1 \int_0^1 \hat{f}(\hat{x},\hat{y})\ \mathrm{d}\hat{x}\,\mathrm{d}\hat{y}. \end{aligned}$$

Let us indicate the faces of the reference cell using a tuple (ξ, f), $\xi \in \{0, 1, \ldots, d-1\}$, $f \in \{0, 1\}$, where the first index corresponds to the fixed coordinate direction:

$$\hat{e}^{1,0} = \{\hat{\mathbf{x}} \in \hat{K} : \hat{x} = 0\}, \qquad \hat{e}^{1,1} = \{\hat{\mathbf{x}} \in \hat{K} : \hat{y} = 0\},$$
$$\hat{e}^{2,0} = \{\hat{\mathbf{x}} \in \hat{K} : \hat{x} = 1\}, \qquad \hat{e}^{2,1} = \{\hat{\mathbf{x}} \in \hat{K} : \hat{y} = 1\},$$

Then, we can define the four faces belonging to the boundary of K according to:

$$e^{1,0} = \boldsymbol{\mathcal{F}}_K(\hat{e}^{1,0}), \qquad e^{1,1} = \boldsymbol{\mathcal{F}}_K(\hat{e}^{1,1}), \qquad e^{2,0} = \boldsymbol{\mathcal{F}}_K(\hat{e}^{2,0}), \qquad e^{2,1} = \boldsymbol{\mathcal{F}}_K(\hat{e}^{2,1}).$$

Now we can express integrals over faces belonging to the boundary of K with respect to the reference coordinates and the mapping $\boldsymbol{\mathcal{F}}_K$. Let $\hat{f}$ and f be defined as in (1.3.1), we have

$$\int_{e^{1,0}} f(\mathbf{x})\,\mathrm{d}s(\mathbf{x}) = \int_0^1 \hat{f}(0,\hat{y}) \left| \frac{\partial \boldsymbol{\mathcal{F}}_K(0,\hat{y})}{\partial\,\hat{y}} \right| \mathrm{d}\hat{y}$$
$$\stackrel{(1.1)}{=} \Delta y \int_0^1 \hat{f}(0,\hat{y})\ \mathrm{d}\hat{y}, \tag{1.5}$$
$$\int_{e^{1,1}} f(\mathbf{x})\,\mathrm{d}s(\mathbf{x}) = \int_0^1 \hat{f}(1,\hat{y}) \left| \frac{\partial \boldsymbol{\mathcal{F}}_K(1,\hat{y})}{\partial\,\hat{y}} \right| \mathrm{d}\hat{y}$$
$$= \Delta y \int_0^1 \hat{f}(1,\hat{y})\ \mathrm{d}\hat{y}, \tag{1.6}$$
$$\int_{e^{2,0}} f(\mathbf{x})\,\mathrm{d}s(\mathbf{x}) = \int_0^1 \hat{f}(\hat{x},0) \left| \frac{\partial \boldsymbol{\mathcal{F}}_K(\hat{x},0)}{\partial\,\hat{x}} \right| \mathrm{d}\hat{x}$$
$$\stackrel{(1.1)}{=} \Delta x \int_0^1 \hat{f}(\hat{x},0)\ \mathrm{d}\hat{x}, \tag{1.7}$$
$$\int_{e^{2,1}} f(\mathbf{x})\,\mathrm{d}s(\mathbf{x}) = \int_0^1 \hat{f}(\hat{x},1) \left| \frac{\partial \boldsymbol{\mathcal{F}}_K(\hat{x},1)}{\partial\,\hat{x}} \right| \mathrm{d}\hat{x}$$
$$= \Delta x \int_0^1 \hat{f}(\hat{x},1)\ \mathrm{d}\hat{x}, \tag{1.8}$$

1.3.1 Mappings in arbitrary dimensions

All concepts considered in the previous section naturally extend to higher dimensions. Let us consider in this sections a reference cell $\hat{K} \in [0,1]^d$.

For the Jacobian determinant, we obtain in this case:

$$J_K = \prod_{\xi=1}^{d} \Delta x_\xi. \tag{1.9}$$

For a volume integral, we obtain:

$$\begin{aligned}\int_K f(\mathbf{x})\,\mathrm{d}\mathbf{x} &= J_K \int_{\hat{K}} f(\hat{\mathbf{x}})\,\mathrm{d}\mathbf{x} \\ &= \prod_{\xi=1}^{d} \Delta x_\xi \int_{\hat{K}} \hat{f}(\hat{\mathbf{x}})\,\mathrm{d}\hat{\mathbf{x}} \end{aligned} \tag{1.10}$$

And for an integral over a face $e^{\xi,f}$, $\xi \in \{1,2,\dots,d\}$, $f \in \{0,1\}$, we obtain:

$$\begin{aligned}\int_{e^{\xi,f}} f(\mathbf{x})\,\mathrm{d}s(\mathbf{x}) &= \prod_{\zeta=1,\zeta\neq\xi}^{d} \Delta x_\zeta \int_{\hat{e}^{\xi,f}} \hat{f}(\hat{\mathbf{x}})\,\mathrm{d}\hat{s}(\hat{\mathbf{x}}) \\ &= \frac{J_K}{\Delta x_\xi} \int_{\hat{e}^{\xi,f}} \hat{f}(\hat{\mathbf{x}})\,\mathrm{d}\hat{s}(\hat{\mathbf{x}}) \end{aligned} \tag{1.11}$$

Let $\hat{f}$ be a sufficiently regular function on the reference cell $\hat{K}$, and let f be sufficiently regular on K and such that

$$f(\mathbf{x}) = (\hat{f} \circ \boldsymbol{\mathcal{F}}_K^{-1})(\mathbf{x}) = \hat{f}(\hat{\mathbf{x}}), \qquad \hat{\mathbf{x}} \in \hat{K},\ \mathbf{x} = \boldsymbol{\mathcal{F}}_K(\hat{\mathbf{x}}) \in K.$$

Then, partial derivatives of f can be expressed with respect to the reference coordinates according to:

$$\frac{\partial f(\mathbf{x})}{\partial x_\xi} = \frac{1}{\Delta x_\xi} \frac{\partial \hat{f}(\hat{\mathbf{x}})}{\partial \hat{x}_\xi}, \tag{1.12}$$

where $\xi \in \{1,2,\dots,d\}$.

1.3.2 Space-time mappings

We can express the tuple $(K,[t^K, t^K+\Delta t])$ according to

$$(K,[t^K, t^K+\Delta t]) = \boldsymbol{\mathcal{F}}_{K,t}(\hat{K},[0,1]),$$

where we have introduced the space-time mapping $\boldsymbol{\mathcal{F}}_{K,t}\colon\ \hat{K}\times[0,1] \to K \times [t^K, t^K+\Delta t]$ with:

$$(\mathbf{x},t) = \boldsymbol{\mathcal{F}}_{K,t}(\hat{\mathbf{x}}) = (\mathbf{P}_0, t^K) + \begin{pmatrix} \Delta x & 0 & 0 \\ 0 & \Delta y & 0 \\ 0 & 0 & \Delta t \end{pmatrix} \begin{pmatrix} \hat{x}-0.5 \\ \hat{y}-0.5 \\ \hat{t} \end{pmatrix}. \tag{1.13}$$

1.4 Basis functions

Let be f be a sufficiently regular univariate function such that it can be approximated by a polynomial of order N, i.e., the leading approximation error term is of order $\mathcal{O}(N+1)$.

Given a set of support point and function value pairs $\{(x_i, f(x_i))\}_{0 \leq i \leq N}$, the corresponding prolongation polynomial in the Lagrange form can be constructed according to:

$$f_N(x) = \sum_{i=0}^{N} f(x_i)\,\varphi_i(x), \tag{1.14}$$

where the Lagrange basis polynomials are defined as

$$L_i(x) = \left(\prod_{\substack{0 \leq l \leq N \\ l \neq i}} \frac{x - x_l}{x_i - x_l} \right), \qquad i = 0, \dots, N,$$

Since we exclude the $(x - x_i)$ term in the product, the basis functions have the property

$$L_i(x_j) = \delta_{ij} = \begin{cases} 1 & i = j, \\ 0 & \text{else.} \end{cases} \qquad i, j = 0, \dots, N, \tag{1.15}$$

Let $I \in \mathbb{R}$ be a real interval. With respect to the Gauss-Legendre quadrature of degree $2\,(N+1) - 1$, we define a discrete scalar product $\langle \cdot, \cdot \rangle_{L^2(I)}$ according to:

$$\langle f, g \rangle_{L^2(I)} = \sum_{n=0}^{N} w_n\, f(x_n)\, g(x_n),$$

where x_n denotes a quadrature node and w_n denotes a quadrature weight, $n = 0, 1, \dots, N$.

The next result is very important for the definition of the DG basis functions:

Lemma 1.1: *Let us denote by $\{\varphi_i\}_{i=0,1,\dots,N}$ a Lagrange basis utilising basis polynomials located at the nodes of a Gauss-Legendre quadrature of degree $2\,(N+1)-1$. Let the nodes lie in the interval I. Then, $\{\varphi_i\}_{i=0,1,\dots,N}$ is a orthogonal basis with respect to the $(\cdot,\cdot)_{L^2(I)}$ scalar product. Furthermore, it is a orthogonal basis with respect to all Gauss-Legendre quadratures with at least a degree of $2\,(N+1) - 1$ that are used to evaluate $(\cdot,\cdot)_{L^2(I)}$.*

Proof: We only need to proof that the basis functions are orthogonal with respect to the Gauss-Legendre quadrature of degree $2\,(N+1) - 1$. The rest follows from the fact that the Gauss-Legendre quadrature of degree $2\,(N+1) - 1$ is exact for all polynomials of degree $2\,(N+1) - 1 > 2\,N$, where $2\,N$ is the maximum total degree of the product of two basis polynomials.

Select two basis functions φ_i and φ_j,$i, j = 0, 1, \dots, N$. Since both functions are Lagrange polynomials, we have

$$\varphi_i(x_j) = \delta_{ij} = \begin{cases} 1 & i = j, \\ 0 & \text{else,} \end{cases} \tag{1.16}$$

where x_j denotes a support point, and δ_{ij} denotes the Kronecker delta. An analogous condition holds for φ_j.

We want to evaluate the scalar product $(\cdot, \cdot)_{L^2(I)}$ by using the Gauss-Legendre quadrature of degree $2\,(N+1)-1$. We obtain:

$$(\varphi_i,\, \varphi_j)_{L^2(I)} = \langle \varphi_i,\, \varphi_j \rangle_{L^2(I)} = \sum_{n=0}^{N} w_n\, \varphi_i(x_n)\, \varphi_j(x_n) = \sum_{n=0}^{N} w_n\, \delta_{ij}\, \delta_{nj} = w_i\, \delta_{ij}. \qquad \square$$

1.4.1 Definition of the DG basis functions

Let us introduce the space of polynomials of order at most N with support in $[0,\, 1]$:

$$Q_N([0,\, 1]) = \operatorname{span}\left\{\hat{x}^n \text{ such that } 0 \le n \le N \text{ and } \hat{x} \in [0,\, 1]\right\}.$$

As a basis of the space $Q_N([0, 1])$, we use a set of $N+1$ Lagrange basis polynomials. The support points of the basis polynomials are chosen such that they coincide with the nodes of a Gauss-Legendre quadrature of degree $2\,(N+1)-1$ which uses the interval $[0,\, 1]$ as domain of integration. We denote the scalar-valued univariate reference basis functions by $\hat{\varphi}_i(\hat{x})$, $i = 0, 1, \ldots, N$.

Let us define scalar-valued multivariate polynomials on the reference cell $\hat{K} = [0, 1]^d$, and on each mesh cell $K \in \mathcal{T}_h$ according to:

$$\hat{\phi}_n(\hat{\mathbf{x}}) = \prod_{\xi=1}^{d} \hat{\varphi}_{n_\xi}(\hat{x}_\xi), \tag{1.17}$$

$$\phi_n^K(\mathbf{x}) = \begin{cases} (\hat{\phi}_n \circ \boldsymbol{\mathcal{F}}_K^{-1})(\mathbf{x}) & \mathbf{x} \in K, \\ 0 & \text{else}, \end{cases} \tag{1.18}$$

for $n_\xi = 0, 1, \ldots, N$, $n_\xi \in \{1, \ldots, d\}$, and a linearised index $n = 0, 1, \ldots, (N+1)^d - 1$ that is constructed according to:

$$n = \sum_{\xi=1}^{d} N_\xi\, n_\xi,$$

using the one-dimensional basis indices $n_\xi = 0, 1, \ldots, N$ and some strides $N_\xi \in \{0, N{+}1\}$, $\xi \in \{1, 2, \ldots, d\}$ that define an unique order of the degrees of freedom.

Notice that (1.18) implies that

$$\mathbf{x} = \boldsymbol{\mathcal{F}}_K \hat{\mathbf{x}} \qquad \Leftrightarrow \qquad \phi_n^K(\mathbf{x}) = \hat{\phi}_n(\hat{\mathbf{x}}),$$

for $\mathbf{x} \in K$, $\hat{\mathbf{x}} \in \hat{K}$, and $n = 0, 1, \ldots, (N+1)^d - 1$.

The corresponding d-dimensional support points (d-dimensional Gauss-Legendre quadrature nodes) on the reference cell are constructed as tensor products of the one-dimensional support points:

$$\hat{\mathbf{x}}_n = (\hat{x}_{n,1}, \ldots, \hat{x}_{n,d})^{\mathrm{T}},$$

with $n = 0, 1, \ldots, (N+1)^d - 1$.

One can show that the polynomials $\{\hat{\phi}_n(\hat{\mathbf{x}})\}_{n=0,1,\ldots,(N+1)^d-1}$ form a basis of the space

$$Q_N(\hat{K}) = \operatorname{span}\left\{\Pi_{i=1}^d \hat{x}_i^{n_i} \text{ such that } 0 \le n_i \le N,\ i = 1, \ldots, d, \text{ and } \hat{\mathbf{x}} \in \hat{K}\right\}$$

The d-dimensional support points for the multivariate basis functions on the mesh cells $K \in \mathcal{T}_h$ are then constructed by means of the mapping $\boldsymbol{\mathcal{F}}_K$:

$$(x_{n,1}, \ldots, x_{n,d})^{\mathrm{T}} = \boldsymbol{\mathcal{F}}_K \hat{\mathbf{x}}_n,$$

with $n = 0, 1 \ldots, (N+1)^d - 1$ denoting the global index.

Since we only consider affine reference cell to mesh cell mappings, we can further show that the polynomials $\{\phi_n^K(\mathbf{x})\}_{n=0,1,\ldots,(N+1)^d-1}$ form a basis of the space

$$Q_N(K) = \operatorname{span}\left\{\Pi_{i=1}^d x_i^{n_i} \text{ such that } 0 \le n_i \le N,\ i = 1, \ldots, d, \text{ and } \mathbf{x} \in K\right\}.$$

Let us introduce another index $v = 0, 1, \ldots, N_{\mathrm{var}} - 1$ that numbers the variables. We construct the basis $\{\phi_n^{K;v}(\mathbf{x})\}_{v=0,1,\ldots,N_{\mathrm{var}}-1;\, n=0,1,\ldots,(N+1)^d-1}$ of the space $Q_N(K)^{N_{\mathrm{var}}}$ according to:

$$\begin{aligned}
&\{\phi_n^{K;v}(\mathbf{x})\}_{v=0,1,\ldots,N_{\mathrm{var}}-1;\, n=0,1,\ldots,(N+1)^d-1} = \\
&\qquad \{(1,\,0,\,\ldots,0)^{\mathrm{T}}_{N_{\mathrm{var}}}, (0,\,1,\,\ldots,0)^{\mathrm{T}}_{N_{\mathrm{var}}}, \ldots, (0,\,0,\,\ldots,1)^{\mathrm{T}}_{N_{\mathrm{var}}}\} \otimes \\
&\qquad \{\phi_n^K(\mathbf{x})\}_{n=0,1,\ldots,(N+1)^d-1}
\end{aligned}$$

We further construct a basis $\{\phi_n^{K;v;e}(\mathbf{x})\}_{e=1,2,\ldots,d;\, v=0,1,\ldots,N_{\mathrm{var}}-1;\, n=0,1,\ldots,(N+1)^d-1}$ of the space $Q_N(K)^{N_{\mathrm{var}} \times d}$ according to:

$$\begin{aligned}
&\{\phi_n^{K;v;e}(\mathbf{x})\}_{e=1,2,\ldots,d;\, v=0,1,\ldots,N_{\mathrm{var}}-1;\, n=0,1,\ldots,(N+1)^d-1} = \\
&\qquad \{(1,\,0,\,\ldots,0)^{\mathrm{T}}_{N_{\mathrm{var}}}, (0,\,1,\,\ldots,0)^{\mathrm{T}}_{N_{\mathrm{var}}}, \ldots, (0,\,0,\,\ldots,1)^{\mathrm{T}}_{N_{\mathrm{var}}}\} \otimes \\
&\qquad \{(1,\,0,\,\ldots,0)^{\mathrm{T}}_{d}, (0,\,1,\,\ldots,0)^{\mathrm{T}}_{d}, \ldots, (0,\,0,\,\ldots,1)^{\mathrm{T}}_{d}\} \otimes \\
&\qquad \{\phi_n^K(\mathbf{x})\}_{n=0,1,\ldots,(N+1)^d-1}
\end{aligned}$$

Lastly, let us introduce the space-time basis polynomials

$$\begin{aligned}
\hat{\theta}_{l;n}(\hat{\mathbf{x}}, \hat{t}) &= \hat{\varphi}_l(\hat{t})\, \hat{\phi}_{l;n}(\hat{\mathbf{x}}, \hat{t}) \\
\theta_{l;n}^K(\mathbf{x}, t) &= \begin{cases} (\hat{\theta}_{l;n} \circ \boldsymbol{\mathcal{F}}_{K,t}^{-1})(\mathbf{x}, t) & (\mathbf{x}, t) \in K \times [t^K, t^K + \Delta t] \\ 0 & \text{else,} \end{cases}
\end{aligned}$$

with $l = 0, 1, \ldots, N$, and $n = 0, 1, \ldots, (N+1)^d - 1$. Here, t^K denotes a time stamp associated with the cell K.

The corresponding $(d+1)$-dimensional support points ($(d+1)$-dimensional Gauss-Legendre quadrature nodes) on the reference cell are constructed as tensor products of the one-dimensional support points:

$$(\hat{\mathbf{x}}_n, \hat{t}_l) = (\hat{x}_{n,1}, \ldots, \hat{x}_{n,d}, \hat{t}_l)^{\mathrm{T}},$$

with $n \in \{0, 1, \ldots, (N+1)^d - 1\}$ denoting the index of the spatial basis and with $l \in \{0, 1, \ldots, N\}$ denoting the index of the temporal basis.

We can now construct a basis $\{\theta^{K;v}_{l;n}(\mathbf{x})\}_{v=0,1,\ldots,N_{\text{var}}-1;\, l=0,1,\ldots,N;\, n=0,1,\ldots,(N+1)^d-1}$ of the space $Q_N(K \times [t^K, t^K + \Delta t])^{N_{\text{var}}}$ according to:

$$\begin{aligned}\{\theta^{K;v}_{l;n}(\mathbf{x})\}&_{v=0,1,\ldots,N_{\text{var}}-1;\, l=0,1,\ldots,N;\, n=0,1,\ldots,(N+1)^d-1} = \\ &\{(1,\, 0,\, \ldots, 0)^{\mathrm{T}}_{N_{\text{var}}}, (0,\, 1,\, \ldots, 0)^{\mathrm{T}}_{N_{\text{var}}}, \ldots, (0,\, 0,\, \ldots, 1)^{\mathrm{T}}_{N_{\text{var}}}\} \\ &\otimes \{\theta^{K}_{l;n}(\mathbf{x})\}_{l=0,1,\ldots,N;\, n=0,1,\ldots,(N+1)^d-1}.\end{aligned}$$

1.4.2 Properties of the basis functions

Let us number the d-dimensional Gauss-Legendre quadrature weights on the reference cell with the linearised index $n = 0, 1, \ldots, (N+1)^d - 1$ we used to number the basis functions and the corresponding d-dimensional spatial support points, i.e.,

$$w_n = \prod_{\xi=1} w_{n_\xi}, \tag{1.19}$$

with $n_\xi = 0, 1, \ldots, N$, $n_\xi \in \{1, \ldots, d\}$.

In this section, we summarise the properties of most of the basis polynomials that appeared in the previous section. We exclude the basis polynomials of the volume flux and space-time volume flux ansatz spaces in the summary. In the following, let $l, l' \in \{0, 1, \ldots, N\}$, $n, n' \in \{0, 1, \ldots, (N+1)^d - 1\}$, $K, K' \in \mathcal{T}_h$ and $v, v' \in \{0, 1, \ldots, N_{\text{var}}\}$. The basis functions have the following properties:

- Lagrange basis property (reference cell):

$$\hat{\varphi}_l(\hat{x}_{l'}) = \delta_{ll'}, \tag{1.20}$$

$$\hat{\phi}_n(\hat{\mathbf{x}}_{n'}) = \prod_{\xi=1}^{d} \hat{\varphi}_{n_\xi}(\hat{x}_{m,\xi}) = \prod_{\xi=1}^{d} \delta_{n_\xi n'_\xi} = \delta_{nn'}, \tag{1.21}$$

$$\hat{\theta}_{l;n}(\hat{\mathbf{x}}_{n'}, \hat{t}_{l'}) = \hat{\varphi}_l(\hat{t}_{l'})\, \hat{\phi}_{n'}(\hat{\mathbf{x}}_{n'}) = \delta_{ll'}\, \delta_{nn'}. \tag{1.22}$$

- Sampling property (reference cell):

$$\langle \hat{f},\, \hat{\varphi}_{l'} \rangle_{L^2([0,1])} = w_{l'}\, \hat{f}(\hat{x}_{l'}), \tag{1.23}$$

$$\begin{aligned}\langle \hat{f},\, \hat{\phi}_{n'} \rangle_{L^2(\hat{K})} &= \prod_{\xi=1}^{d} \langle \hat{f},\, \hat{\varphi}_{n'_\xi} \rangle_{L^2([0,1])} \\ &= w_n\, \hat{f}(\hat{\mathbf{x}}_n),\end{aligned} \tag{1.24}$$

$$\begin{aligned}\langle \hat{f},\, \hat{\theta}_{l';n'} \rangle_{L^2(\hat{K} \times [0,1])} &= \langle \hat{f},\, \hat{\varphi}_{l'} \rangle_{L^2([0,1])}\, \langle \hat{f},\, \hat{\phi}_{n'} \rangle_{L^2(\hat{K})} \\ &= w_l\, w_n\, \hat{f}(\hat{\mathbf{x}}_n, t_l),\end{aligned} \tag{1.25}$$

- Sampling property (mesh cell):

$$\begin{aligned}
\langle f,\, \varphi_{l'} \rangle_{L^2([t^K, t^K+\Delta t])} &\overset{(1.10)}{=} \Delta t\, \langle \hat{f},\, \hat{\varphi}_{l'} \rangle_{L^2([0,1])} && (1.26)\\
&= \Delta t\, w_{l'}\, f(t_{l'}), && (1.27)\\
\langle f,\, \phi_{n'} \rangle_{L^2(K)} &\overset{(1.10)}{=} J_K\, \langle \hat{f},\, \hat{\phi}_{n'} \rangle_{L^2(\hat{K})}\\
&= J_K\, w_{n'}\, f(\mathbf{x}_{n'}), && (1.28)\\
\langle f,\, \theta_{l';n'} \rangle_{L^2(K\times[t^K, t^K+\Delta t])} &= \langle f,\, \varphi_l \rangle_{L^2([t^K, t^K+\Delta t])}\, \langle f,\, \phi_n \rangle_{L^2(K)}\\
&= \Delta t\, J_K\, w_{l'}\, w_{n'}\, f(\mathbf{x}_{n'}, t_{l'}), && (1.29)
\end{aligned}$$

- Discrete orthogonality (reference cell)

$$\begin{aligned}
\langle \hat{\varphi}_l,\, \hat{\varphi}_{l'} \rangle_{L^2([0,1])} &= w_l\, \delta_{ll'}, && (1.30)\\
\langle \hat{\phi}_n,\, \hat{\phi}_{n'} \rangle_{L^2(\hat{K})} &= \prod_{\xi=1}^{d} \langle \hat{\varphi}_{n_\xi},\, \hat{\varphi}_{n'_\xi} \rangle_{L^2([0,1])} = \prod_{\xi=1}^{d} w_{n_\xi} \delta_{n_\xi n'_\xi}\\
&= w_n\, \delta_{nn'}, && (1.31)\\
\langle \hat{\theta}_{l;n},\, \hat{\theta}_{l';n'} \rangle_{L^2(\hat{K}\times[0,1])} &= \langle \hat{\varphi}_l,\, \hat{\varphi}_{l'} \rangle_{L^2([0,1])}\, \langle \hat{\phi}_n,\, \hat{\phi}_{n'} \rangle_{L^2(\hat{K})}\\
&= w_l\, w_n\, \delta_{ll'}\, \delta_{nn'}. && (1.32)
\end{aligned}$$

- Discrete orthogonality (mesh cell)

$$\begin{aligned}
\langle \varphi_l^K,\, \varphi_{l'}^K \rangle_{L^2([t^K, t^K+\Delta t])} &= \Delta t\, w_l\, \delta_{ll'}, && (1.33)\\
\langle \phi_n^K,\, \phi_{n'}^K \rangle_{L^2(K)} &= J_K\, \langle \hat{\phi}_n,\, \hat{\phi}_{n'} \rangle_{L^2(\hat{K})}\\
&= J_K\, w_n\, \delta_{nn'}, && (1.34)\\
\langle \theta_{l;n}^{K;v},\, \theta_{l';n'}^{K;v} \rangle_{L^2(K\times[t^K, t^K+\Delta t])} &= \Delta t\, \langle \hat{\varphi}_l,\, \hat{\varphi}_{l'} \rangle_{L^2([0,\,1])}\, J_K\, \langle \hat{\phi}_n,\, \hat{\phi}_{n'} \rangle_{L^2(\hat{K})}\\
&= \Delta t\, J_K\, w_l\, w_n\, \delta_{ll'}\, \delta_{nn'}. && (1.35)
\end{aligned}$$

- Compact support:

$$\begin{aligned}
\langle \phi_n^K,\, \phi_{n'}^{K'} \rangle_{L^2(K)} &= \langle \phi_n^K,\, \phi_{n'}^K \rangle_{L^2(K)}\, \delta_{KK'}\\
&= J_K\, w_n\, \delta_{KK'}\, \delta_{nn'}, && (1.36)\\
\langle \theta_{l;n}^K,\, \theta_{l';n'}^{K'} \rangle_{L^2(K\times[t^K, t^K+\Delta t])} &= \Delta t\, \langle \hat{\varphi}_l,\, \hat{\varphi}_{l'} \rangle_{L^2([0,\,1])}\, J_K\, \langle \hat{\phi}_n,\, \hat{\phi}_{n'} \rangle_{L^2(\hat{K})} \delta_{KK'}\\
&= \Delta t\, J_K\, w_l\, w_n\, \delta_{KK'}\, \delta_{ll'}\, \delta_{nn'}. && (1.37)
\end{aligned}$$

- Orthogonal variables:

$$
\begin{aligned}
\langle \phi_n^{K;v},\, \phi_{n'}^{K';v'} \rangle_{L^2(K)} &= \delta_{vv'}\, \langle \phi_n^{K},\, \phi_{n'}^{K} \rangle_{L^2(K)} \\
&= J_K\, w_n\, \delta_{KK'}\, \delta_{vv'}\, \delta_{nn'}, \qquad (1.38) \\
\langle \theta_{l;n}^{K;v},\, \theta_{l';n'}^{K';v'} \rangle_{L^2(K\times[t^K,t^K+\Delta t])} &= \langle \theta_{l;n}^{K},\, \theta_{l';n'}^{K'} \rangle_{L^2(K\times[t^K,t^K+\Delta t])}\, \delta_{vv'} \\
&= \Delta t\, J_K\, w_l\, w_n\, \delta_{KK'}\, \delta_{vv'}\, \delta_{ll'}\, \delta_{nn'}. \qquad (1.39)
\end{aligned}
$$

1.5 Operators

In this section, we introduce operators that appear within the derivation of the ADER-DG operators and vectors.

Below, let $l, l' \in \{0, 1, \dots, N\}$, $n, n' \in \{0, 1, \dots, (N+1)^d - 1\}$, $K, K' \in \mathcal{T}_h$ and $v, v' \in \{0, 1, \dots, N_{\text{var}}\}$.

First, let us introduce an operator $\hat{P} \in \mathbb{R}^{(N+1)\times(N+1)}$ that projects the coefficients associated with the univariate reference basis functions on the $(N+1)$ nodes of a regular partition of interval $[0, 1]$:

$$
\hat{P}_{ij} = \varphi_i\left(\frac{j}{N}\right), \qquad i, j = 0, 1, \dots, N. \qquad (1.40)
$$

Let us further introduce the operators

$$
\left[\hat{f}, \hat{g}\right]^{\tau}_{L^2(\hat{K})} = \int_{\hat{K}} \hat{f}(\hat{\mathbf{x}}, \tau)\, \hat{g}(\hat{\mathbf{x}}, \tau)\, \mathrm{d}\hat{\mathbf{x}}, \qquad (1.41)
$$

$$
[f, g]^{\tau}_{L^2(K)} = \int_{K} f(\mathbf{x}, t^K + \Delta t\, \tau)\, g(\mathbf{x}, t^K + \Delta t\, \tau)\, \mathrm{d}\mathbf{x}, \qquad (1.42)
$$

where $\tau \in \{0,\, 1\}$, $\hat{f}$ and $\hat{g}$ are square integrable on $\hat{K} \times [0,\, 1]$, and f and g are square integrable on $K \times [t^K,\, t^K + \Delta t]$.

For our basis polynomials, we obtain

$$
\begin{aligned}
\left[\hat{\theta}_{l;n},\, \hat{\phi}_{n'}\right]^{\tau}_{L^2(\hat{K})} &= \hat{\varphi}_l(\tau)\ \langle \hat{\phi}_n,\, \hat{\phi}_{n'} \rangle_{L^2(\hat{K})} \\
&= \hat{\varphi}_l(\tau)\, \langle \hat{\phi}_n,\, \hat{\phi}_{n'} \rangle_{L^2(\hat{K})} \\
&= w_n\, \hat{F}_l^{\tau}\, \delta_{nn'}, \qquad (1.43) \\
\left[\theta_{l;n}^{K;v},\, \phi_{n'}^{K;v}\right]^{\tau}_{L^2(K)} &= J_K\, \left[\hat{\theta}_{l;n},\, \hat{\phi}_{n'}\right]^{\tau}_{L^2(\hat{K})} \\
&= J_K\, w_n\, \hat{F}_l^{\tau}\, \delta_{KK'}\, \delta_{vv'}\, \delta_{nn'}, \qquad (1.44) \\
\left[\hat{\theta}_{l;n},\, \hat{\theta}_{l';n'}\right]^{\tau}_{L^2(\hat{K})} &= \hat{\varphi}_l(\tau)\, \hat{\varphi}_{l'}(\tau)\ \langle \hat{\phi}_n,\, \hat{\phi}_{n'} \rangle_{L^2(\hat{K})} \\
&= w_n\, \hat{F}_l^{\tau}\, \hat{F}_{l'}^{\tau}\, \delta_{nn'}, \qquad (1.45) \\
\left[\theta_{l;n}^{K;v},\, \theta_{l';n'}^{K;v}\right]^{\tau}_{L^2(K)} &= J_K\, \left[\hat{\theta}_{l;n},\, \hat{\theta}_{l';n'}\right]^{\tau}_{L^2(\hat{K})} \\
&= J_K\, w_n\, \hat{F}_l^{\tau}\, \hat{F}_{l'}^{\tau}\, \delta_{KK'}\, \delta_{vv'}\, \delta_{nn'}, \qquad (1.46)
\end{aligned}
$$

where we have introduced the vectors $\hat{F}^f \in \mathbb{R}^{N+1}$, $f = 0, 1$, with

$$\hat{F}_i^f = \begin{cases} \hat{\varphi}_i(0) & f = 0, \\ \hat{\varphi}_i(1) & f = 1, \end{cases} \tag{1.47}$$

$$= \begin{cases} \hat{P}_{i0} & f = 0, \\ \hat{P}_{iN} & f = 1, \end{cases} \tag{1.48}$$

where $i \in \{0, 1, \ldots, N\}$.

We define the reference stiffness operator $\underline{\hat{\mathrm{K}}} \in \mathbb{R}^{(N+1)^2}$ according to:

$$\begin{aligned} \hat{\mathrm{K}}_{ij} &= \langle \partial_{\hat{x}}\, \hat{\varphi}_i,\, \hat{\varphi}_j \rangle_{L^2([0,1])} \\ &\overset{(1.26)}{=} w_j\, \partial_{\hat{x}}\, \hat{\varphi}_i(\hat{x}_j), \qquad i, j \in \{0, 1, \ldots, N\}. \end{aligned} \tag{1.49}$$

Let us further introduce time derivative operators on the space-time reference cell and the space-time mesh cells $K \in \mathcal{T}_h$:

$$\begin{aligned} \left\langle \partial_{\hat{t}} \hat{\theta}_{l;n},\, \hat{\theta}_{l';n'} \right\rangle_{L^2(\hat{K}\times[0,1])} &= \langle \partial_{\hat{x}}\, \hat{\varphi}_l,\, \hat{\varphi}_{l'} \rangle_{L^2([0,1])} \, \langle \hat{\phi}_n,\, \hat{\phi}_{n'} \rangle_{L^2(\hat{K})} \\ &\overset{\text{(I)}}{=} w_n\, \hat{K}_{ll'}\, \delta_{nn'}, \end{aligned} \tag{1.50}$$

$$\begin{aligned} \left\langle \partial_{\hat{t}} \theta^{K;v}_{l;n},\, \theta^{K;v}_{l';n'} \right\rangle_{L^2(K\times[t^K, t^K+\Delta t])} &= J_K \left\langle \partial_{\hat{t}} \hat{\theta}_{l;n},\, \hat{\theta}_{l';n'} \right\rangle_{L^2(\hat{K}\times[0,1])} \\ &= J_K\, w_n\, \hat{K}_{ll'}\, \delta_{KK'}\, \delta_{vv'}\, \delta_{nn'}, \end{aligned} \tag{1.51}$$

where we have used the operator (1.49) and the discrete orthogonality property of the basis functions (1.31) in step I.

Let $\xi \in \{1, 2, \ldots, d\}$. We also introduce spatial derivative operators on the spatial and space-time reference cells:

$$\begin{aligned} \left\langle \hat{\phi}_n,\, \partial_{\hat{x}_\xi} \hat{\phi}_{n'} \right\rangle_{L^2(\hat{K})} &= \langle \hat{\phi}_n,\, \partial_{\hat{x}_\xi} \hat{\phi}_{n'} \rangle_{L^2(\hat{K})}, \\ &= \langle \hat{\varphi}_{n_\xi},\, \partial_{\hat{x}} \hat{\varphi}_{n'_\xi} \rangle_{L^2[0,1])} \prod_{\zeta=1,\zeta\neq\xi}^{d} \langle \hat{\varphi}_{n_\zeta},\, \hat{\varphi}_{n'_\zeta} \rangle_{L^2([0,1])}, \\ &= \hat{K}_{n'_\xi n_\xi} \prod_{\zeta=1,\zeta\neq\xi}^{d} w_{n_\zeta}\, \delta_{n_\zeta n'_\zeta}, \end{aligned} \tag{1.52}$$

$$\begin{aligned} \left\langle \hat{\theta}_{l;n},\, \partial_{\hat{x}_\xi} \hat{\theta}_{l';n'} \right\rangle_{L^2(\hat{K}\times[0,1])} &= \langle \hat{\varphi}_l,\, \hat{\varphi}_{l'} \rangle_{L^2([0,1])} \, \langle \hat{\phi}_n,\, \partial_{\hat{x}_\xi} \hat{\phi}_{n'} \rangle_{L^2(\hat{K})}, \\ &= w_l\, \hat{K}_{n'_\xi n_\xi}\, \delta_{ll'} \prod_{\zeta=1,\zeta\neq\xi}^{d} w_{n_\zeta}\, \delta_{n_\zeta n'_\zeta}. \end{aligned} \tag{1.53}$$

1.6 The ADER-DG degrees of freedom

We express the quantities that are involved in the ADER-DG scheme in terms of the space-time and spatial basis functions that we have introduced in the previous section.

- Space-time predictor $q^K \in Q_N(K \times [t^K, t^K + \Delta t])^{N_{\mathrm{var}}}$:

$$q^K = \sum_{v=0}^{N_{\mathrm{var}}} q^{K;v} = \sum_{v=0}^{N_{\mathrm{var}}} \sum_{l=0}^{N} \sum_{n=0}^{(N+1)^d-1} \widetilde{q}_{l;n}^{K;v} \, \theta_{l;n}^{K;v}.$$

  We store the space-time predictor coefficients in the vector $\underline{\widetilde{\mathrm{q}}}^K \in \mathbb{R}^{N_{\mathrm{var}}(N+1)^{d+1}}$.

- Space-time volume flux $\widetilde{\mathbf{F}}^K \in Q_N(K \times [t^K, t^K + \Delta t])^{N_{\mathrm{var}} \times d}$:

$$\begin{aligned} \widetilde{\mathbf{F}}^K = \sum_{v=0}^{N_{\mathrm{var}}} \widetilde{\mathbf{F}}^{K;v} &= \sum_{v=0}^{N_{\mathrm{var}}} \sum_{l=0}^{N} \sum_{n=0}^{(N+1)^d-1} \widetilde{\mathbf{F}}_{l;n}^{K;v} \, \theta_{l;n}^{K;v} \\ &= \sum_{v=0}^{N_{\mathrm{var}}} \sum_{l=0}^{N} \sum_{n=0}^{(N+1)^d-1} (\widetilde{F}_{l;n,1}^{K;v}, \ldots, \widetilde{F}_{l;n,d}^{K;v}) \, \theta_{l;n}^{K;v}. \end{aligned}$$

- Predictor $q_h^K \in Q_N(K)^{N_{\mathrm{var}}}$:

$$q_h^K = \sum_{v=0}^{N_{\mathrm{var}}} q_h^{K;v} = \sum_{v=0}^{N_{\mathrm{var}}} \sum_{n=0}^{(N+1)^d-1} q_n^{K;v} \, \phi_n^{K;v}$$

  We store the predictor coefficients in the vector $\underline{\mathrm{q}}^K \in \mathbb{R}^{N_{\mathrm{var}}(N+1)^d}$.

- Solution $u_h^K \in Q_N(K)^{N_{\mathrm{var}}}$:

$$u_h^K = \sum_{v=0}^{N_{\mathrm{var}}} u_h^{K;v} = \sum_{v=0}^{N_{\mathrm{var}}} \sum_{n=0}^{(N+1)^d-1} u_n^{K;v} \, \phi_n^{K;v}$$

  We store the solution coefficients in the vector $\underline{\mathrm{u}}^K \in \mathbb{R}^{N_{\mathrm{var}}(N+1)^d}$.

- Volume flux $\mathbf{F}_h^K \in Q_N(K)^{N_{\mathrm{var}} \times d}$:

$$\begin{aligned} \mathbf{F}_h^K = \sum_{v=0}^{N_{\mathrm{var}}} \mathbf{F}^{K;v} &= \sum_{v=0}^{N_{\mathrm{var}}} \sum_{n=0}^{(N+1)^d-1} \mathbf{F}_n^{K;v} \, \phi_n^{K;v} \\ &= \sum_{v=0}^{N_{\mathrm{var}}} \sum_{n=0}^{(N+1)^d-1} (F_{n,1}^{K;v}, \ldots, F_{n,d}^{K;v}) \, \phi_n^{K;v}. \end{aligned}$$

1.7 Space-time predictor computation

The space-time predictor computation requires us to perform N_{iter} Picard iterations in order to obtain the space-time predictor coefficients $\widetilde{q}^{K;v}_{l;n}$, $l \in \{0, 1, \ldots, N\}$, $n \in \{0, 1, \ldots, (N+1)^d - 1\}$:

$$\sum_{l'=0}^{N} \sum_{n'=0}^{(N+1)^d-1} \left(\left[\theta^{K;v}_{l;n}, \theta^{K;v}_{l';n'}\right]^1_{L^2(K)} - \left\langle \partial_{\hat{t}} \theta^{K;v}_{l;n}, \theta^{K;v}_{l';n'} \right\rangle_{L^2(K\times[t^K, t^K+\Delta t])} \right) \widetilde{q}^{K;v;(r+1)}_{l';n'}$$

$$= \sum_{n'=0}^{(N+1)^d-1} \left[\theta^{K;v}_{l;n}, \phi^{K;v}_{n'}\right]^0_{L^2(K)} u^{K;v}_{n'}$$

$$- \sum_{l'=0}^{N} \sum_{n'=0}^{(N+1)^d-1} \widetilde{\mathbf{F}}^{K;v}_{l';n'}(\widetilde{q}^{K;(r)}) \left\langle \theta^{K;v}_{l';n'}, \nabla \theta^{K;v}_{l;n} \right\rangle_{L^2(K\times[t^K, t^K+\Delta t])} \tag{1.54}$$

$$+ \sum_{l'=0}^{N} \sum_{n'=0}^{(N+1)^d-1} S^{K;v}_{l';n'}(\widetilde{q}^{K;(r)}) \left\langle \theta^{K;v}_{l';n'}, \theta^{K;v}_{l;n} \right\rangle_{L^2(K\times[t^K, t^K+\Delta t])}, \tag{1.55}$$

where $r \in \{0, 1, \ldots, N_{\text{var}}\}$ denotes the current iteration. In matrix notation, we obtain:

$$\underline{\mathrm{L}}^{KK}\, \underline{\mathrm{q}}^{K;(r+1)} = \underline{\mathrm{v}}^K(\underline{\mathrm{u}}^K) - \underline{\mathrm{w}}^K(\underline{\widetilde{\mathrm{q}}}^{K;(r)}) \tag{1.56}$$

$$\Rightarrow \underline{\mathrm{q}}^{K;(r+1)} = (\underline{\mathrm{L}}^{KK})^{-1} \left(\underline{\mathrm{v}}^K(\underline{\mathrm{u}}^K) - \underline{\mathrm{w}}^K(\underline{\widetilde{\mathrm{q}}}^{K;(r)}) \right), \tag{1.57}$$

where we identify the left-hand side operator $\underline{\mathrm{L}}^{KK} \in \mathbb{R}^{N_{\text{var}}\,(N+1)^{d+1} \times N_{\text{var}}\,(N+1)^{d+1}}$, and the right-hand side vectors $\underline{\mathrm{v}}^K(\underline{\mathrm{u}}^K) \in \mathbb{R}^{N_{\text{var}}\,(N+1)^{d+1}}$, and $\underline{\mathrm{w}}^{K;(r)}(\underline{\widetilde{\mathrm{q}}}^{K;(r)}) \in \mathbb{R}^{N_{\text{var}}\,(N+1)^{d+1}}$.

1.7.1 Left-hand side operator

Let us pick two variables $v, v' \in \{0, 1, \ldots, N_{\text{var}} - 1\}$ and two cells $K, K' \in \mathcal{T}_h$. The block $\underline{\mathrm{L}}^{vv'} \in \mathbb{R}^{(N+1)^{d+1} \times (N+1)^{d+1}}$ has the elements:

$$\begin{aligned}
\mathrm{L}^{KK';vv'}_{ll';nn'} &= \left[\theta^{K;v}_{l;n}, \theta^{K;v}_{l';n'}\right]^1_{L^2(K)} - \left\langle \partial_{\hat{t}} \theta^{K;v}_{l;n}, \theta^{K;v}_{l';n'} \right\rangle_{L^2(K\times[t^K, t^K+\Delta t])} \\
&\overset{(\mathrm{I})}{=} J_K\, w_n\, (\hat{F}^1_l\, \hat{F}^1_{l'} - \hat{K}_{ll'})\, \delta_{KK'}\, \delta_{vv'}\, \delta_{nn'} \\
&= J_K\, w_n\, \hat{L}_{ll'}\, \delta_{KK'}\, \delta_{vv'}\, \delta_{nn'},
\end{aligned} \tag{1.58}$$

where we have introduced the reference cell operator

$$\hat{L}_{ll'} = \hat{F}^1_l\, \hat{F}^1_{l'} - \hat{K}_{ll'}, \tag{1.59}$$

where $l, l' \in \{0, 1, \ldots, N\}$, and $n, n' \in \{0, 1, \ldots, (N+1)^d - 1\}$. In step I of the above derivations, we used (1.44) and (1.51).

1.7.2 Constant right-hand side term

The elements of vector $\underline{\mathrm{v}}^{K;v}$, $v \in \{0,1,\dots,N_{\text{var}}\}$, are computed according to:

$$
\begin{aligned}
\mathrm{v}_{l;n}^{K;v} &= \sum_{n'=0}^{(N+1)^d-1} \left[\theta_{l;n}^{K;v},\, \phi_{n'}^{K;v}\right]_{L^2(K)}^{0} u_{n'}^{K;v} \\
&+ \sum_{l'=0}^{N} \sum_{n'=0}^{(N+1)^d-1} S_{l';n'}^{K;v}(\widetilde{q}^{K;(r)}) \left\langle \theta_{l';n'}^{K;v},\, \theta_{l;n}^{K;v} \right\rangle_{L^2(K\times[t^K,t^K+\Delta t])} \\
&\overset{\text{(I)}}{=} \sum_{n'=0}^{(N+1)^d-1} J_K\, w_n\, \hat{F}_l^0\, \delta_{KK'}\, \delta_{vv'}\, \delta_{nn'}\, u_{n'}^{K;v} \\
&+ \sum_{l'=0}^{N} \sum_{n'=0}^{(N+1)^d-1} S_{l';n'}^{K;v}\, \Delta t\, J_K\, w_l\, w_n\, \delta_{KK'}\, \delta_{vv'}\, \delta_{ll'}\, \delta_{nn'} \\
&= J_K\, w_n\, \hat{F}_l^0\, u_n^{K;v} + \Delta t\, J_K\, w_l\, w_n\, S_{l;n}^{K;v},
\end{aligned}
$$

where $l \in \{0,1,\dots,N\}$, and $n \in \{0,1,\dots,(N+1)^d-1\}$. We have used (1.39) and (1.44) in step I of the above derivations.

1.7.3 Space-time volume flux integral

The elements of vector $\underline{\mathrm{w}}^{K;v}$, $v \in \{0,1,\dots,N_{\text{var}}\}$, are computed according to:

$$
\begin{aligned}
\mathrm{w}_{l;n}^{K;v} &= \sum_{l'=0}^{N} \sum_{n'=0}^{(N+1)^d-1} \widetilde{\mathbf{F}}_{l';n'}^{K;v} \left\langle \theta_{l';n'}^{K;v},\, \nabla\theta_{l;n}^{K;v} \right\rangle_{L^2(K\times[t^K,t^K+\Delta t])} \\
&\overset{\text{(I)}}{=} \Delta t\, J_K \sum_{l'=0}^{N} \sum_{n'=0}^{(N+1)^d-1} \widetilde{\mathbf{F}}_{l';n'}^{K;v}\, \mathrm{D}\boldsymbol{\mathcal{F}}_K^{-\mathrm{T}} \left\langle \hat{\theta}_{l';n'},\, \hat{\nabla}\hat{\theta}_{l;n} \right\rangle_{L^2(\hat{K}\times[0,1])} \\
&\overset{\text{(II)}}{=} J_K \sum_{l'=0}^{N} \sum_{n'=0}^{(N+1)^d-1} \sum_{\xi=1}^{d} \frac{\Delta t}{\Delta x_\xi} \left\langle \hat{\theta}_{l';n'},\, \partial_{\hat{x}_\xi}\hat{\theta}_{l;n} \right\rangle_{L^2(\hat{K}\times[0,1])} \widetilde{F}_{l';n',\xi}^{K;v} \\
&\overset{\text{(III)}}{=} J_K \sum_{l'=0}^{N} \sum_{n'=0}^{(N+1)^d-1} \sum_{\xi=1}^{d} \frac{\Delta t}{\Delta x_\xi}\, \hat{\mathrm{K}}_{n_\xi n'_\xi}\, \widetilde{F}_{l';n',\xi}^{K;v}\, w_l\, \delta_{ll'} \prod_{\zeta=1,\zeta\neq\xi}^{d} w_{n_\zeta}\, \delta_{n'_\zeta n_\zeta},
\end{aligned}
\tag{1.60}
$$

where $l \in \{0,1,\dots,N\}$, and $n \in \{0,1,\dots,(N+1)^d-1\}$. In step I of the above derivations, we applied a scaling argument. In step II, we expanded the d-dimensional scalar product and wrote the integral over the space-time reference cell as a discrete scalar product. In step III, we used (1.53).

1.8 Time averaging of the space-time predictor values

In the current implementation, we need to compute the time average of the space-time predictor values. This computation must be performed before the boundary extrapolation of the predictor values and before the volume integral that also relies on the boundary extrapolated predictor values.

Let us express a component of the local space-time predictor in terms of the local space-time basis

$$q^{K;v} = \sum_{l'=0}^{N} \sum_{n'=0}^{(N+1)^d-1} \widetilde{q}^{K;v}_{l';n'} \, \theta^{K;v}_{l';n'}$$

We want to compute the predictor in $[t^K, t^K + \Delta t]$ as the time average of the space-time predictor over the same interval, i.e.,

$$\begin{aligned}
q_h^{K;v} &= \sum_{n'=0}^{(N+1)^d-1} q^{K;v}_{n'} \, \phi^{K;v}_{n'} \\
&= \frac{1}{\Delta t} \int_{t^K}^{t^K+\Delta t} \sum_{l'=0}^{N} \sum_{n'=0}^{(N+1)^d-1} \widetilde{q}^{K;v}_{l';n'} \, \theta^{K;v}_{l';n'} \, \mathrm{d}t \\
&= \sum_{n'=0}^{(N+1)^d-1} \phi^{K;v}_{n'} \, \frac{1}{\Delta t} \int_{t^K}^{t^K+\Delta t} \sum_{l'=0}^{N} \widetilde{q}^{K;v}_{l';n'} \, \varphi^{K;v}_{l'} \, \mathrm{d}t \\
&= \sum_{n'=0}^{(N+1)^d-1} \phi^{K;v}_{n'} \, \frac{1}{\Delta t} \sum_{l'=0}^{N} \widetilde{q}^{K;v}_{l';n'} \, \langle 1, \, \varphi^{K;v}_{l'} \rangle_{L^2[t^K, t^K+\Delta t]} \\
&\overset{(1.26)}{=} \sum_{n'=0}^{(N+1)^d-1} \phi^{K;v}_{n'} \sum_{l'=0}^{N} w_{l'} \, \widetilde{q}^{K;v}_{l';n'}.
\end{aligned}$$

Thus,

$$q^{K;v}_{n'} = \sum_{l'=0}^{N} w_{l'} \, \widetilde{q}^{K;v}_{l';n'}, \qquad n = 0, 1, \ldots, (N+1)^d - 1. \tag{1.61}$$

A similar expression can be derived for the time average of the space-time volume flux components:

$$\mathbf{F}^{K;v}_{n'} = \sum_{l'=0}^{N} w_{l'} \, \widetilde{\mathbf{F}}^{K;v}_{l';n'}, \qquad n = 0, 1, \ldots, (N+1)^d - 1. \tag{1.62}$$

1.9 Boundary Extrapolation

The boundary extrapolation can be in some sense interpreted as the "transposed" operation to the surface integral discussed in the next section.

Let us define on the reference element $2\,d$ sets of support points

$$\{\hat{\mathbf{x}}_k\}^{\xi,f}_{k=0,1,\ldots,(N+1)^{d-1}-1} = \left\{\hat{\mathbf{x}}_n\colon\ \hat{x}_{n,\xi}=0 \text{ if } f=0 \text{ or } \hat{x}_{n,\xi}=1 \text{ if } f=1,\ n=0,1,\ldots,(N+1)^d-1\right\},$$

where $\xi = 1,2,\ldots,d$, and $f = 0,1$. We construct the corresponding support points on the mesh cells $K \in \mathcal{T}_h$ by applying a mapping:

$$\{\mathbf{x}_k\}^{\xi,f}_{k=0,1,\ldots,(N+1)^{d-1}-1} = \boldsymbol{\mathcal{F}}_K\{\hat{\mathbf{x}}_k\}^{\xi,f}_{k=0,1,\ldots,(N+1)^{d-1}-1}$$

The boundary extrapolation requires us to sum the contributions of the basis functions at the quadrature points on each face $e^{\xi,f}$.

Let us start with the extrapolation of the normal flux tensor components $(\mathbf{F}^{K;v}\,\mathbf{n}^\xi) \in \mathbb{R}^{(N+1)^d}$, $v=1,2,\ldots,N_{\text{var}}$, $\xi\in\{1,2,\ldots,d\}$, which are scalar-valued quantities.

Using the quadrature nodes $\mathbf{x}^{\xi,f}_k \in \{\hat{\mathbf{x}}_k\}^{\xi,f}_{k=0,1,\ldots,(N+1)^{d-1}-1}$,$k=0,1,\ldots,(N+1)^{d-1}-1$, located on face $e^{\xi,f}$, $\xi=1,2,\ldots,d$, $f=0,1$, we obtain for the components of the normal flux vector $\underline{\mathrm{g}}^{K;\xi,f} \in \mathbb{R}^{N_{\text{var}}\,(N+1)^{d-1}}$:

$$\begin{aligned}
\mathrm{g}^{K;\xi,f;v}_k &= \sum_{n'=0}^{(N+1)^d-1} (\mathbf{F}^{K;v}\,\mathbf{n}^\xi)_{n'}\,\phi^{K;v}_{n'}(\mathbf{x}^{\xi,f}_k)\\
&\overset{\text{(I)}}{=} \sum_{n'=0}^{(N+1)^d-1} (\mathbf{F}^{K;v}\,\mathbf{n}^\xi)_{n'}\,\hat{\phi}_{n'}(\hat{\mathbf{x}}^{\xi,f}_k)\\
&\overset{\text{(II)}}{=} \sum_{n'=0}^{(N+1)^d-1} (\mathbf{F}^{K;v}\,\mathbf{n}^\xi)_{n'}\,\hat{\varphi}_{n'_\zeta}(\hat{x}^{\xi,f}_{k,\xi}) \prod_{\zeta=1,\zeta\neq\xi}^{d} \hat{\varphi}_{n'_\zeta}(\hat{x}^{\xi,f}_{k,\zeta})\\
&\overset{\text{(III)}}{=} \sum_{n'=0}^{(N+1)^d-1} (\mathbf{F}^{K;v}\,\mathbf{n}^\xi)_{n'}\,\hat{\varphi}_{n'_\xi}(\hat{x}^{\xi,f}_{k,\xi}) \prod_{\zeta=1,\zeta\neq\xi}^{d} \delta_{n'_\zeta k_\zeta}\\
&= \sum_{n'=0}^{(N+1)^d-1} (\mathbf{F}^{K;v}\,\mathbf{n}^\xi)_{n'}\,\hat{F}^f_{n'_\xi} \prod_{\zeta=1,\zeta\neq\xi}^{d} \delta_{n'_\zeta k_\zeta},
\end{aligned}$$

where $k\in\{1,2,\ldots,(N+1)^{d-1}-1\}$, and where the vectors $\hat{F}^f \in \mathbb{R}^{N+1}$, $f=0,1$, are defined by (1.47). In step I of the above calculations, we made use of the definition of the multivariate basis functions; cf. (1.18). In step II, we split the multivariate reference basis functions into the univariate ones; cf. (1.17). In step III, we made use of the Lagrange basis property (1.20).

We obtain for the elements of the extrapolated predictor vectors $\underline{\mathrm{e}}^{K;\xi f} \in \mathbb{R}^{N_{\mathrm{var}}\,(N+1)^{d-1}}$ $\xi = 1, 2, \ldots, d,\ f = 0, 1$:

$$
\begin{aligned}
\mathrm{e}_k^{K;\xi,f;v} &= \sum_{n'=0}^{(N+1)^d-1} q_{n'}^{K;v}\, \phi_{n'}^{K;v}(\mathbf{x}_k^{\xi,f}) \\
&= \sum_{n'=0}^{(N+1)^d-1} q_{n'}^{K;v}\, \hat{F}_{n'_\xi}^{f} \prod_{\zeta=1,\zeta\neq\xi}^{d} \delta_{n'_\zeta k_\zeta},
\end{aligned}
$$

where $k \in \{1, 2, \ldots, (N+1)^{d-1} - 1\}$.

Remarks:

- Note that the predictor values used for the boundary extrapolation might be replaced by time-integrated space-time predictor values if we want to employ anarchic time stepping.

1.10 Correction

The correction phase requires us to solve the following system of equations to obtain the solutions coefficients $u_n^{K;v}$, $n \in \{0, 1, \ldots, (N+1)^d - 1\}$:

$$
\begin{aligned}
\sum_{n'=0}^{(N+1)^d-1} & \left\langle \phi_n^{K;v},\, \phi_{n'}^{K;v} \right\rangle_{L^2(K)} (u_n^{K;v} - u_n^{K;v(\mathrm{old})}) \\
& + \int_{t^K}^{t^K+\Delta t} \int_{\partial K} \phi_n^{K;v}\, \mathbf{G}(q_h^{K+}, q_h^{K-})\mathbf{n}\, \mathrm{d}\mathbf{x}\, \mathrm{d}t \\
& - \int_{t^K}^{t^K+\Delta t} \int_{K} \mathbf{F}(q_h^{K})\, \nabla\phi_n^{K;v}\, \mathrm{d}\mathbf{x}\, \mathrm{d}t = 0,
\end{aligned}
\tag{1.63}
$$

The correction step thus requires us to solve the following equation for $\underline{\mathrm{u}}^K$

$$
\underline{\mathrm{M}}^{KK}\,(\underline{\mathrm{u}}^K - \underline{\mathrm{u}}^{K;(\mathrm{old})}) = \Delta t\, \left(\underline{\Delta \mathrm{u}}^K\right)
$$

where the solution update vector is constructed according to:

$$
\underline{\Delta \mathrm{u}}^K = \underline{\mathrm{a}}^K - \underline{\mathrm{b}}^K
$$

All the operators and vectors introduced above will be discussed in detail in the next sections.

1.10.1 Mass operator

The mass operator consists of $(N_K \cdot N_{\mathrm{var}})^2$ blocks with size $(N+1)^d$, where N_K denotes the number of elements and d denotes the space dimension.

Let us pick two variables $v, v' \in \{0, 1, \dots, N_{\text{var}} - 1\}$ and two mesh cells $K, K' \in \mathcal{T}_h$. The block $\underline{\underline{\mathrm{M}}}^{KK';vv'} \in \mathbb{R}^{(N+1)^d \times (N+1)^d}$ has the elements:

$$\begin{aligned} \mathrm{M}_{nn'}^{KK';vv'} &= \langle \phi_{n'}^{K';v'}, \phi_n^{K;v} \rangle_{L^2(K)} \\ &\overset{(1.38)}{=} \delta_{KK'}\, \delta_{vv'}\, \delta_{nn'}\, J_K\, w_n. \end{aligned} \tag{1.64}$$

The mass matrix is thus completely diagonal.

1.10.2 Volume integral

The evaluation of the volume integral is very similar to the evaluation of the space-time volume flux integral (Section 1.7.3).

The compact support of the DG basis functions renders the computation of the volume flux vector $\underline{\mathrm{a}} \in \mathbb{R}^{N_K N_{\text{var}}\,(N+1)^d}$ a cell-local operation.

The cell-local volume flux vector $\underline{\mathrm{a}}^{K;v}$ is computed according to:

$$\begin{aligned} \mathrm{a}_n^{K;v} &= \sum_{n'=0}^{(N+1)^d-1} \mathbf{F}_{n'}^{K;v} \left\langle \phi_{n'}^{K;v},\, \nabla\phi_n^{K;v} \right\rangle_{L^2(K)} \\ &\overset{\text{(I)}}{=} J_K \sum_{n'=0}^{(N+1)^d-1} \mathbf{F}_{n'}^{K;v}\, \mathrm{D}\boldsymbol{\mathcal{F}}_K^{-\mathrm{T}} \left\langle \hat{\phi}_{n'},\, \hat{\nabla}\hat{\phi}_n \right\rangle_{L^2(\hat{K})} \\ &\overset{\text{(II)}}{=} J_K \sum_{\xi=1}^{d} \frac{1}{\Delta x_\xi} \sum_{n'=0}^{(N+1)^d-1} \langle \hat{\phi}_{n'},\, \partial_{\hat{x}_\xi}\, \hat{\phi}_n \rangle_{L^2(\hat{K})}\, F_{n',\xi}^{K;v} \\ &\overset{\text{(III)}}{=} J_K \sum_{\xi=1}^{d} \frac{1}{\Delta x_\xi} \sum_{n'=0}^{(N+1)^d-1} \hat{\mathrm{K}}_{n_\xi n'_\xi}\, F_{n',\xi}^{K;v} \prod_{\zeta=1,\zeta\neq\xi}^{d} w_{n'_\zeta}\, \delta_{n'_\zeta\, n_\zeta}, \end{aligned}$$

In step I of the above derivations, we used a scaling argument. In step II, we expanded the d-dimensional scalar product and wrote the integral over the reference element as a discrete scalar product. In step III, we used (1.52).

1.10.3 Surface integral

Let us introduce positively signed normal vectors $\mathbf{n}^\xi \in \mathbb{R}^d$ that have their only non-zero at position ξ. Let us further introduce outward directed normal vectors $\mathbf{n}^{\xi f} \in \mathbb{R}^d$ that also have their only non-zero at position ξ. Their sign is however negative if $f = 0$ and positive if $f = 1$.

The face fluctuations must be unique for two cells that share an interior face. Here, they are defined with respect to one of the directions $\mathbf{n}^\xi \in \mathbb{R}^d$. From this definition of the fluctuations, it follows that the face fluctuations with respect to the outward directed normal vectors can be computed according to:

$$(\mathbf{G}^{K;v}\, \mathbf{n}^{\xi,f}) = \begin{cases} -(\mathbf{G}^{K;v}\, \mathbf{n}^\xi) & f = 0, \\ +(\mathbf{G}^{K;v}\, \mathbf{n}^\xi) & f = 1. \end{cases}, \tag{1.65}$$

where $(\mathbf{G}^{K;v}\,\mathbf{n}^{\xi}) \in \mathbb{R}^{(N+1)^{d-1}}$ denote the face fluctuations, and $\mathbf{n}^{\xi,f}$ denote the outward directed normal vectors, $\xi \in \{1,2,\ldots,d\}$, $f \in \{0,1\}$.

Evaluating the values of the mesh cell basis functions on a face of the mesh, leads to the evaluation of the reference basis functions at either $\hat{x}_\xi = 0$ or $\hat{x}_\xi = 1$, $\xi \in \{1,2,\ldots,d\}$.

The result of the projection of the boundary fluctuation values on the degrees of freedom is thus computed according to:

$$\hat{F}^f \otimes (\mathbf{G}^{K;v}\,\mathbf{n}^{\xi f}) \in \mathbb{R}^{(N+1)^d}, \tag{1.66}$$

where $\hat{F}^f \in \mathbb{R}^{N+1}$ is defined by (1.47).

Let us now procede to derive the cell-local surface integral contributions to the fluctuations vector $\underline{\mathrm{b}} \in \mathbb{R}^{N_K N_{\mathrm{var}}\,(N+1)^d}$. The vector has the entries

$$\begin{aligned}
\mathrm{b}_n^{K;v} &= \sum_{\xi=1}^{d}\sum_{f=0}^{1}\int_{e^{\xi,f}} \phi_n^{K;v}\,\hat{F}^f \otimes (\mathbf{G}^{K;v}\,\mathbf{n}^{\xi f})\,\mathrm{d}s \\
&= \sum_{\xi=1}^{d}\sum_{f=0}^{1}\int_{e^{\xi,f}} \phi_n^{K;v} \sum_{n'=0}^{(N+1)^d-1} (\hat{F}^f \otimes (\mathbf{G}^{K;v}\,\mathbf{n}^{\xi f}))_{n'}\,\phi_{n'}^{K;v}\,\mathrm{d}s \\
&= \sum_{\xi=1}^{d}\sum_{f=0}^{1}\sum_{i=0}^{(N+1)^{d-1}-1} |e^{\xi,f}| \prod_{\zeta=1,\zeta\neq\xi}^{d} w_{i_\zeta} \sum_{n'=0}^{(N+1)^d-1} \hat{\phi}_n(\hat{\mathbf{x}}_i)\,(\hat{F}^f \otimes (\mathbf{G}^{K;v}\,\mathbf{n}^{\xi,f}))_{n'}\,\hat{\phi}_{n'}(\hat{\mathbf{x}}_i) \\
&= \sum_{\xi=1}^{d}\sum_{f=0}^{1}\sum_{i=0}^{(N+1)^{d-1}-1} \prod_{\zeta=1,\zeta\neq\xi}^{d} \Delta x_\zeta\,w_{i_\zeta} \sum_{n'=0}^{(N+1)^d-1} \delta_{ni}\,(\hat{F}^f \otimes (\mathbf{G}^{K;v}\,\mathbf{n}^{\xi,f}))_{n'}\,\delta_{n'i} \\
&= \sum_{\xi=1}^{d}\sum_{f=0}^{1}\prod_{\zeta=1,\zeta\neq\xi}^{d} \Delta x_\zeta\,w_{n_\zeta}\,(\hat{F}^f \otimes (\mathbf{G}^{K;v}\,\mathbf{n}^{\xi f}))_n,
\end{aligned}$$

for $n = 1,2,\ldots,(N+1)^d - 1$.

1.10.4 Source terms

Assume that the source term $\mathbf{S} \in \mathbb{R}^{N_{\mathrm{var}}}$ is sufficiently regular such that it can be approximated by the DG basis in each cell $K \in \mathcal{T}_h$. Let S^v, $v = 0,1,\ldots,N_{\mathrm{var}}-1$ denote a component of the source term and let us further define $S^{K;v} = S^v|_K$ and $S_n^{K;v} = S^{K;v}(\mathbf{x}_n)$.

We can express the source term components according to:

$$
\begin{aligned}
S^{K;v} &\approx \sum_{n=0}^{(N+1)^d-1} \frac{\left\langle S^{K;v},\, \phi_n^{K;v} \right\rangle_{L^2(K)}}{\left\langle \phi_n^{K;v},\, \phi_n^{K;v} \right\rangle_{L^2(K)}} \phi_n^{K;v} \\
&\overset{\text{(I)}}{=} \sum_{n=0}^{(N+1)^d-1} \frac{S_n^{K;v}\, w_n}{w_n} \phi_n^{K;v} \\
&= \sum_{n=0}^{(N+1)^d-1} S_n^{K;v}\, \phi_n^{K;v}.
\end{aligned}
$$

where we have used the sampling property (1.28), and the discrete orthogonality property (1.34) of the basis polynomials in step I.

We thus obtain for the elements of the source term contribution vector $\underline{\mathrm{s}} \in \mathbb{R}^{N_K\, N_{\mathrm{var}}\, (N+1)^d}$:

$$
\begin{aligned}
\mathrm{s}_n^{K;v} = \int_K S^{K;v}\, \phi_n^{K;v}\, \mathrm{d}\mathbf{x} &= \langle\, S^{K;v},\, \phi_n^{K;v}\, \rangle_{L^2(K)} \\
&= \sum_{n'=1}^{(N+1)^d-1} S_{n'}^{K;v}\, \langle \phi_{n'}^{K;v},\, \phi_n^{K;v}\, \rangle_{L^2(K)} \\
&\overset{(1.34)}{=} \sum_{n'=1}^{(N+1)^d-1} \delta_{nn'}\, J_K\, w_{n'}\, S_{n'}^{K;v} \\
&= w_n\, J_K\, S_n^{K;v},
\end{aligned}
$$

where $K \in \mathcal{T}_h$, $v = 0, 1 \ldots, N_{\mathrm{var}}$, and $n = 0, 1, \ldots, (N+1)^d - 1$.

1.11 Solution update

Let us define the solution update vector as

$$\underline{\Delta \mathrm{u}}^K = \underline{\mathrm{a}}^K - \underline{\mathrm{b}}^K$$

The correction step requires us to solve the following equation for $\underline{\mathrm{u}}^K$

$$\underline{\mathrm{M}}^{KK}\,(\underline{\mathrm{u}}^K - \underline{\mathrm{u}}^{K;(\mathrm{old})}) = \Delta t\, \left(\underline{\Delta \mathrm{u}}^K + \underline{\mathrm{s}}^K\right)$$

Fixing the variable $v' \in 0, 1, \ldots, N_{\mathrm{var}}$ and the degree of freedom $n' \in 0, 1, \ldots, (N+1)^d - 1$, and expanding the matrix-vector product leads to (cf. (1.64)):

$$
\begin{aligned}
\sum_{v=0}^{N_{\mathrm{var}}} \sum_{n=0}^{(N+1)^d-1} \delta_{vv'}\, \delta_{nn'}\, J_K\, w_n\, (\mathrm{u}_{n'}^{K;v'} - \mathrm{u}_{n'}^{K;v';(\mathrm{old})}) &= \Delta t\, \left(\Delta \mathrm{u}_{n'}^{K;v'} + \mathrm{s}_{n'}^{K;v'}\right) \\
\Rightarrow J_K\, w_n\, (\mathrm{u}_n^{K;v} - \mathrm{u}_n^{K;v;(\mathrm{old})}) &= \Delta t\, \left(\Delta \mathrm{u}_n^{K;v} + \mathrm{s}_n^{K;v}\right)
\end{aligned}
$$

From rearranging the equation, we finally obtain

$$\mathrm{u}_n^{K;v} = \mathrm{u}_n^{K;v;(\mathrm{old})} + \frac{\Delta t}{w_n}\left(\frac{1}{J_K}\Delta \mathrm{u}_n^{K;v} + \frac{1}{J_K}\mathrm{s}_n^{K;v}\right),$$

where $v \in 0, 1, \ldots, N_{\mathrm{var}}$ and $n \in 0, 1, \ldots, (N+1)^d - 1$. Notice the division by J_K on the right-hand side.

1.12 Projecting the ADER-DG degrees of freedom on a regular mesh ("Sub output matrix")

The DG degrees of freedom can be projected on a regular partition of a cell $K \in \mathcal{T}_h$ using a d-dimensional tensor product of the 1-d regular mesh projector defined by (1.40).

Let $u_h^{(\mathrm{reg})} \in Q_N((N+1)^d)$ denote the solution u_h projected on a regular partition of K.

We compute the coefficients of $u_h^{(\mathrm{reg})}$ according to:

$$u_{h;n}^{(\mathrm{reg})} \overset{(1.18)}{=} \sum_{n'=0}^{(N+1)^d-1} u_{h;n'} \prod_{\xi=1}^{d} \hat{P}_{n'_\xi, n_\xi}, \qquad n = 0, 1, \ldots, (N+1)^d - 1, \tag{1.67}$$

1.13 Remarks on the current ExaHyPE implementation

1. If we store the 1-d regular mesh projector $\hat{P}$, we do not need to store the operators $\hat{F}^f$, $f = 0, 1$ (`FLCoeff`,`FRCoeff`,`FCoeff`). We further need not to store the large sub output operators (`subOutputMatrix`) which are d-dimensional tensor products of $\hat{P}$.

2. There is no need to allocate extra memory for the time boundary values, i.e., $\hat{F}^0$ (`F0`) since these values are already stored in `FLCoeff`, and `FCoeff`.

3. `FLCoeff` and `FRCoeff` contain the same information as `FCoeff`.

Conclusion:

- The operators `FLCoeff`$\in \mathbb{R}^{N+1}$,`FRCoeff` $\in \mathbb{R}^{N+1}$,`FCoeff`$\in \mathbb{R}^{2\,(N+1)}$, and `subOutputMatrix`$\in$ $\mathbb{R}^{(N+1)^d \times (N+1)^d \times \cdots}$ can all be realised by storing a 1-d regular mesh projector $\hat{P} \in \mathbb{R}^{(N+1)\times(N+1)}$.

2 Adaptive mesh refinement operators for maxtrixfree quadraturefree nodal DG methods

This chapter is organised as follows: In the first section, we introduce subinterval projectors for ($k = 3$)-spacetrees in one space dimension. These projectors are used to realise the prolongation operation in one space dimension. In the sections 2.2 and 2.3, we then introduce suitable subcell and subface prolongation and restriction operators for more than one space dimension.

Throughout this chapter, we consider matrixfree DG methods that utilise nodal (Lagrange) basis functions that are located at the nodes of a Gauss-Legendre quadrature.

2.1 Prolongation and restriction operators in one space dimension

In this section, we introduce the basic prolongation and restriction operators for one space dimension.

Let us split the interval $\hat{C} = [0, 1]$ into 3^l subintervals, where l denotes the level. We can project coefficients associated with the univariate basis functions onto each subinterval

$$\hat{C}^{0,l;i^{0;l}} = \frac{1}{3^l}[i^{0,l}, i^{0,l} + 1], \quad i^{0,l} = 0, 1, \ldots, 3^l - 1, \tag{2.1}$$

using an operator $\hat{\Pi}^{0,l;i^{0,l}} \in \mathbb{R}^{(N+1)\times(N+1)}$ with

$$\hat{\Pi}^{0,l;i^{0,l}}_{mn} = \varphi_m \left(\frac{1}{3^l}(\hat{x}_n + i^{0,l}) \right), \quad m, n = 0, 1, \ldots, N, \tag{2.2}$$

which is exactly the L^2 projector that projects the 0-th level coefficients on the l-th level coefficients located in the subinterval $\hat{C}^{0,l;i^{0;l}}$:

$$
\begin{aligned}
g_k^{l;i^{0;l}} &= \sum_{k'=0}^{N} g_{k'} \frac{\left\langle \hat{\varphi}_{k'},\, \hat{\varphi}_k^{l;i^{0;l}} \right\rangle_{L^2(\hat{C}^{0,l;i^{0;l}})}}{\left\langle \hat{\varphi}_k^{l;i^{0;l}},\, \hat{\varphi}_k^{l;i^{0;l}} \right\rangle_{L^2(\hat{C}^{0,l;i^{0;l}})}} \\
&\overset{\text{(I)}}{=} \sum_{k'=0}^{N} g_{k'} \frac{\frac{1}{3^l}\, w_k\, \hat{\varphi}_{k'}\left(\frac{1}{3^l}(\hat{x}_k + i^{0,l})\right)}{\frac{1}{3^l}\, w_k} \\
&= \sum_{k'=0}^{N} g_{k'}\, \hat{\varphi}_{k'} \left(\frac{1}{3^l}(\hat{x}_k + i^{0,l})\right) \\
&= \sum_{k'=0}^{N} g_{k'}\, \hat{\Pi}_{k'k}^{0,l;i^{0,l}},
\end{aligned} \tag{2.3}
$$

where we haved used the sampling property (1.26) and the discrete orthogonality (1.30) of the basis functions in step I.

We observe that piecewise Lagrange prolongation and L^2 projection coincide for the chosen type of basis functions. It then follows directly from the exactness of the Lagrange prolongation utilising polynomials of order N for polynomials of the same order, that the original level 0 polynomial is exactly represented by the level l subinterval polynomials in the corresponding subinterval [**?**].

2.1.1 Restriction

We define the restriction of the subinterval degrees of L^2 projection on the coarse level degrees of freedom:

$$
g_k = \sum_{i^{0,l}=0}^{3^l-1} \sum_{k'=0}^{N} \frac{\left\langle \hat{\varphi}_k, \hat{\varphi}_{k'}^{l;i^{0;l}} \right\rangle_{L^2(\hat{C})}}{\langle \hat{\varphi}_k, \hat{\varphi}_k \rangle_{L^2(\hat{C})}}\, g_{k'}^{l;i^{0,l}} \tag{2.4}
$$

$$
\overset{\text{(I)}}{=} \sum_{i^{0,l}=0}^{3^l-1} \sum_{k'=0}^{N} \frac{\left\langle \hat{\varphi}_k, \hat{\varphi}_{k'}^{l;i^{0;l}} \right\rangle_{L^2(\hat{C}^{0,l;i^{0;l}})}}{\langle \hat{\varphi}_k, \hat{\varphi}_k \rangle_{L^2(\hat{C})}}\, g_{k'}^{l;i^{0,l}} \tag{2.5}
$$

$$
\overset{\text{(II)}}{=} \sum_{i^{0,l}=0}^{3^l-1} \sum_{k'=0}^{N} \frac{\frac{1}{3^l}\, w_{k'}\, \hat{\varphi}_k \left(\frac{1}{3^l}(\hat{x}_{k'} + i^{0,l})\right)}{w_k}\, g_{k'}^{l;i^{0,l}} \tag{2.6}
$$

$$
= \sum_{i^{0,l}=0}^{3^l-1} \sum_{k'=0}^{N} \frac{1}{3^l} \left(\frac{w_{k'}}{w_k}\, \hat{\Pi}_{kk'}^{0,l;i^{0,l}} \right) g_{k'}^{l;i^{0,l}} \tag{2.7}
$$

where we have used the compact support of the basis functions in step I and the sampling property of the basis functions (1.26) in step II.

This results in an averaging of the fine grid coefficients.

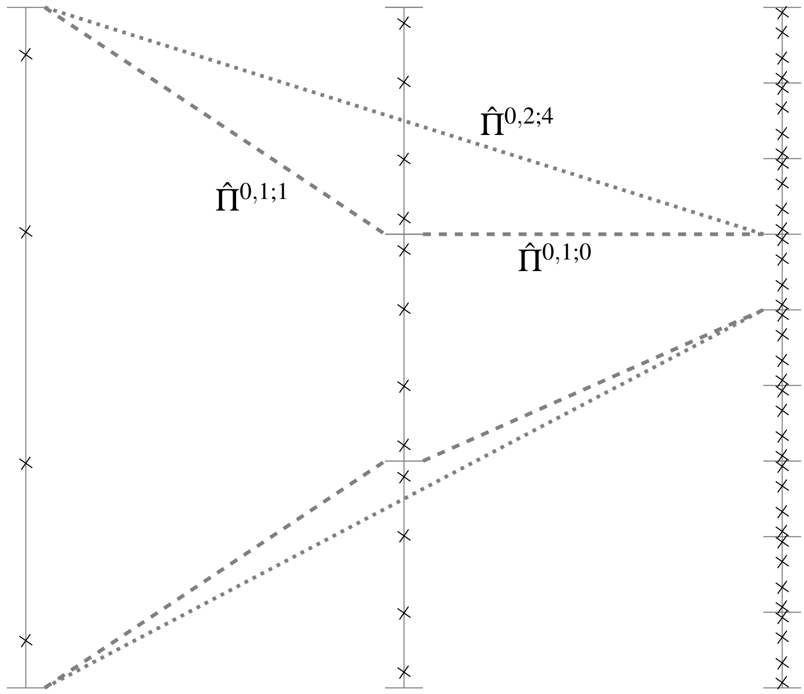

Figure 2.1: Prolongation of coefficients from level 0 down to level 2. See text for details on the prolongation operators. **TODO: Change to top-down.**

2.1.2 Recursive construction of the subinterval projectors

The projectors $\hat{\Pi}^{0,l;i_l} \in \mathbb{R}^{(N+1)\times(N+1)}$, $l > 0$, can be constructed using a cascade of the single-level projectors $\hat{\Pi}^{0,1;0}$, $\hat{\Pi}^{0,1;1}$, and $\hat{\Pi}^{0,1;2}$ (Figure 2.1). This follows directly from the exactness of the Lagrange prolongation utilising polynomials of order N for polynomials of the same order. The original polynomial is thus exactly represented by the subinterval polynomials in the corresponding subinterval.

To this end, let us express the indices $i^{0,l} = 0, 1, \ldots, 3^l - 1$ in terms of a tertiary basis:

$$i^{0,l} = \sum_{\beta=1}^{l} 3^{l-\beta}\, j_\beta, \qquad j_\beta \in \{0, 1, 2\}. \tag{2.8}$$

The support points and the corresponding Lagrange basis functions on the sub intervals can now be constructed in a recursive fashion by downscaling and shifting the parent intervals:

$$\hat{x}_k^{l;i^{0,l}} = \frac{1}{3^l}(\hat{x}_k + i^{0,l}) = \frac{1}{3}\,(\hat{x}_k^{(l-1);i^{0,l-1}} + j_l), \tag{2.9}$$

$$\hat{\varphi}_k^{l;i^{0,l}}(\hat{x}) = \left(\prod_{\substack{0\le n\le N\\ n\neq k}} \frac{\hat{x} - \hat{x}_n^{l;i^{0,l}}}{x_k^{l;i^{0,l}} - \hat{x}_n^{l;i^{0,l}}} \right), \qquad k = 0, \ldots, N, \tag{2.10}$$

where $j_{l-1} \in \{0, 1, 2\}$ denotes the sub interval index with respect to the parent interval.

The level 0 coefficients are then projected onto the $i^{0,l}$-th subinterval of level l in a recursive way similar to the way the subintervals are constructed, i.e.:

$$\hat{\Pi}^{0,l;i^{0,l}}_{m_0\, m_l} = \sum_{m_1=0,\, m_2=0,\, \ldots, m_{l-1}=0}^{N} \hat{\Pi}^{0,1;j_1}_{m_0\, m_1} \ldots \hat{\Pi}^{(l-2),(l-1);j_{l-1}}_{m_{l-2}\, m_{l-1}} \hat{\Pi}^{(l-1),l;j_l}_{m_{l-1}\, m_l} \tag{2.11}$$

$$= \sum_{m_1=0,\, m_2=0,\, \ldots, m_{l-1}=0}^{N} \hat{\Pi}^{0,1;j_1}_{m_0\, m_1} \hat{\Pi}^{0,1;j_2}_{m_1\, m_2} \ldots \hat{\Pi}^{0,1;j_{l-1}}_{m_{l-2}\, m_{l-1}} \hat{\Pi}^{0,1;j_l}_{m_{l-1}\, m_l}, \tag{2.12}$$

where $m_0, m_l = 0, 1, \ldots, N$.

2.2 Subcell prolongation and restriction operators

Subcell prolongation and restriction operators are important for the prolongation and restriction of volume data in the context of dynamic adaptive mesh refinement.

The prolongation operator can be simply constructed as tensor product of the subinterval prolongation operators.

To this end, let us split the reference cell $\hat{K} = [0, 1]^d$ into $3^{l\, d}$, $l > 0$, sub intervals

$$\hat{K}^{l;i_l} = \frac{1}{3^l}[i_{l,1},\, i_{l,1} + 1] \times \frac{1}{3^l}[i_{l,2}, i_{l,2} + 1] \times \ldots \times \frac{1}{3^l}[i_{l,d},\, i_{l,d} + 1], \tag{2.13}$$

and let us further introduce a linear index

$$i^{0,l} = \sum_{\xi=1}^{d} I_{l,\xi}\, i_{l,\xi}, \tag{2.14}$$

where the $I_{l,\xi} \in \{0, 3^l\}$, $\xi \in \{1, 2, \ldots, d\}$ denote some strides that ensure the uniqueness of the index.

We can project coefficients associated with functions that are constructed as d-dimensional tensor product of the univariate basis functions onto the subcells $\hat{K}^{l;i^{0,l}}$ by using a tensor product of the subinterval projectors $\hat{\Pi}^{0,l;i^{0;l}_\xi}$, $\xi \in \{0, 1, \ldots, d-1\}$ defined in the previous section:

$$\begin{aligned} u^{l;i^{0,l}}_n &= \sum_{k'=0}^{(N+1)^d-1} u_{n'} \frac{\left\langle \hat{\phi}_{n'}, \hat{\phi}^{l;i^{0,l}}_n \right\rangle_{L^2(\hat{K}^{l;i^{0,l}})}}{\left\langle \hat{\phi}^{l;i^{0,l}}_n \hat{\phi}^{l;i^{0,l}}_n \right\rangle_{L^2(\hat{K}^{l;i^{0,l}})}} \\ &= \sum_{k'=0}^{(N+1)^d-1} u_{n'} \prod_{\xi=1}^{d} \frac{\left\langle \hat{\varphi}_{n'_\xi}, \hat{\varphi}^{l;i^{0,l}}_{n_\xi} \right\rangle_{L^2(\hat{K}^{l;i^{0,l}})}}{\left\langle \hat{\varphi}^{l;i^{0,l}}_{n_\xi} \hat{\varphi}^{l;i^{0,l}}_{n_\xi} \right\rangle_{L^2(\hat{K}^{l;i^{0,l}})}} \\ &= \sum_{k'=0}^{(N+1)^d-1} u_{n'} \prod_{\xi=1}^{d} \hat{\Pi}^{0,l;i^{0,l}_\xi}_{n'_\xi, n_\xi}, \qquad n = 0, 1, \ldots, (N+1)^d - 1, \end{aligned} \tag{2.15}$$

where the values $u_{n'}$, $n = 0, \dots, (N+1)^d$, denote coefficients located at d-dimensional quadrature nodes on the reference element.

2.2.1 Restriction

In the context of matrixfree nodal DG, the degrees of freedom of volume data u_n, $n = 0, 1, \dots, (N+1)^d - 1$, are located at the quadrature nodes of a d-dimensional Gauss-Legendre quadrature.

We obtain from the L^2 projection of the subcell degrees of freedoms:

$$u_n = \sum_{i^{0,l}=0}^{3^{l\,d}-1} \sum_{n'=0}^{(N+1)^d-1} \prod_{\xi=1}^{d} \frac{\left\langle \hat{\varphi}_{n_\xi}, \hat{\varphi}^{l;i^{0;l}_\xi}_{n'_\xi} \right\rangle_{L^2(\hat{K}^{0,l;i^{0;l}})}}{\left\langle \hat{\varphi}_{n_\xi}, \hat{\varphi}_{n_\xi} \right\rangle_{L^2(\hat{K})}} u^{l;i^{0,l}}_{n'} \tag{2.16}$$

$$= \sum_{i^{0,l}=0}^{3^{l\,d}-1} \sum_{n'=0}^{(N+1)^d-1} \prod_{\xi=1}^{d} \frac{\left\langle \hat{\varphi}_{n_\xi}, \hat{\varphi}^{l;i^{0;l}_\xi}_{n'_\xi} \right\rangle_{L^2(\hat{K}^{0,l;i^{0;l}})}}{\left\langle \hat{\varphi}_{n_\xi}, \hat{\varphi}_{n_\xi} \right\rangle_{L^2(\hat{K})}} u^{l;i^{0,l}}_{n'} \tag{2.17}$$

$$= \sum_{i^{0,l}=0}^{3^{l\,d}-1} \sum_{n'=0}^{(N+1)^d-1} \prod_{\xi=1}^{d} \frac{\frac{1}{3^l}\, w_{n'_\xi}\, \hat{\varphi}_{n_\xi}\left(\frac{1}{3^l}(\hat{x}_{n',\xi} + i^{0,l}_\xi)\right)}{w_{n_\xi}} u^{l;i^{0,l}}_{n'} \tag{2.18}$$

$$= \sum_{i^{0,l}=0}^{3^{l\,d}-1} \sum_{n'=0}^{(N+1)^d-1} \frac{1}{3^{l\,d}} \left(\prod_{\xi=1}^{d} \frac{w_{n'_\xi}}{w_{n_\xi}} \hat{\Pi}^{0,l;i^{0,l}_\xi}_{n_\xi n'_\xi} \right) u^{l;i^{0,l}}_{n'}. \tag{2.19}$$

2.3 Subface prolongation and restriction operators

Subface projectors are important for performing the numerical flux computation on the interface between two mesh cells: They are used to perform the prolongation and restriction operation that is required if the two cells that share the interface do not belong to the same refinement level.

Let us split the reference cell face $\hat{C} = [0,1]^{d-1}$ into $3^{l\,(d-1)}$ sub intervals

$$\hat{C}^{l;i_l} = \frac{1}{3^l}[i_{l,1},\, i_{l,1}+1] \times \frac{1}{3^l}[i_{l,2}, i_{l,2}+1] \times \ldots \times \frac{1}{3^l}[i_{l,d-1},\, i_{l,d-1}+1], \tag{2.20}$$

where the subinterval indices $i_{l,\zeta} = 0, 1, \dots, 3^l$, $\zeta \in \{0, 1, d-1\}$ are defined by (2.31) , and where we have introduced an unique index

$$i^{0,l} = \sum_{\zeta=1}^{d-1} I_{l,\zeta}\, i_{l,\zeta}, \tag{2.21}$$

where the $I_{l,\zeta} \in \{0, 3^l\}$, $\zeta \in \{1, 2, \dots, d-1\}$ denote some strides that ensure the uniqueness of the index.

We can again project coefficients associated with functions that are constructed as $(d-1)$-dimensional tensor product of the univariate basis functions onto the subfaces $C^{l;i^{0,l}}$ by using a tensor product of the subinterval projectors $\hat{\Pi}^{0,l;i^{0;l}_\zeta}$, $\zeta \in \{0, 1, \ldots, d-1\}$:

$$g_k^{l;i^{0,l}} = \sum_{k'=0}^{(N+1)^{d-1}-1} g_{k'} \prod_{\zeta=1}^{d-1} \hat{\Pi}^{0,l;i^{0,l}_\zeta}_{k'_\zeta,k_\zeta}, \qquad k = 0, 1, \ldots, (N+1)^{d-1} - 1, \tag{2.22}$$

where the values $g_{k'}$, $k = 0, \ldots, (N+1)^{d-1} - 1$, denote coefficients located at $(d-1)$-dimensional quadrature nodes located at the reference cell.

2.3.1 Restriction

In the context of matrixfree nodal DG, the boundary extrapolated values g_k, $k = 0, 1, \ldots, (N+1)^{d-1} - 1$ are located at the quadrature nodes of a $d-1$-dimensional Gauss-Legendre quadrature.

Since we can split the original integral over $\hat{C} = [0, 1]^{d-1}$ into $3^{l\,(d-1)}$ integrals over each subface, the inverse operation to the prolongation operation is to sum the integrals over the subfaces. The projection can thus be expressed as L^2-projection:

$$g_k \overset{\text{(I)}}{=} \sum_{i^{0,l}=0}^{3^{l\,(d-1)}-1} \sum_{k'=0}^{(N+1)^{d-1}-1} \prod_{\zeta=1}^{d-1} \frac{\left\langle \hat{\varphi}_{k_\zeta}, \hat{\varphi}^{l;i^{0;l}_\zeta}_{k'_\zeta} \right\rangle_{L^2(\hat{C}^{0,l;i^{0;l}})}}{\left\langle \hat{\varphi}_{k_\zeta}, \hat{\varphi}_{k_\zeta} \right\rangle_{L^2(\hat{C})}} g_{k'}^{l;i^{0,l}} \tag{2.23}$$

$$= \sum_{i^{0,l}=0}^{3^{l\,(d-1)}-1} \sum_{k'=0}^{(N+1)^{d-1}-1} \prod_{\zeta=1}^{d-1} \frac{\frac{1}{3^l}\, w_{k'_\zeta}\, \hat{\varphi}_{k_\zeta}\left(\frac{1}{3^l}(\hat{x}_{k',\zeta} + i^{0,l}_\zeta)\right)}{w_{k_\zeta}} g_{k'}^{l;i^{0,l}} \tag{2.24}$$

$$= \sum_{i^{0,l}=0}^{3^{l\,(d-1)}-1} \sum_{k'=0}^{(N+1)^{d-1}-1} \frac{1}{3^{l\,(d-1)}} \left(\prod_{\zeta=1}^{d-1} \frac{w_{k'_\zeta}}{w_{k_\zeta}} \hat{\Pi}^{0,l;i^{0,l}_\zeta}_{k_\zeta k'_\zeta} \right) g_{k'}^{l;i^{0,l}}. \tag{2.25}$$

2.3.2 Determining the subface index from a subcell index

For a given face (ξ, f) of a cell on level $l-1$ and a single-level subcell index $(c_1, c_2, \ldots, c_d)$, $c_\xi \in \{0, 1, 2\}$, $\xi \in \{0, 1, \ldots, d\}$ we can reconstruct the single-level subface indices $j_{l,\zeta}$, $\zeta \in \{0, 1, \ldots, d-1\}$ according to Table 2.1.

With respect to the topmost face at level $l = 0$ that contains the subface at level l, we can then construct a projector for the respective subface by recursively (bottom-up) determining the subface indices of all intermediate levels and then computing an index (top-down) according to (2.21) and (2.31)

(ξ, f)	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$	$(2,0)$	$(2,1)$	$\cdots$
c_1	0	2	$\in \{0,1,2\}$	$\in \{0,1,2\}$	$\in \{0,1,2\}$	$\in \{0,1,2\}$	$\cdots$
c_2	$\in \{0,1,2\}$	$\in \{0,1,2\}$	0	2	$\in \{0,1,2\}$	$\in \{0,1,2\}$	$\cdots$
c_3	$\in \{0,1,2\}$	$\in \{0,1,2\}$	$\in \{0,1,2\}$	$\in \{0,1,2\}$	0	2	$\cdots$
$\vdots$	$\vdots\ddots$	$\ddots$	$\ddots$	$\ddots$	$\ddots$	$\ddots$	$\ddots$

Table 2.1: Indices $(c_1, c_2, \ldots, c_d)$ of subcells that are adjacent to the parent cell face with index (ξ, f). The difference in levels of the subcells and the parent cell is 1.

2.4 Remarks on the ExaHyPE implementation

1. **Three additional DG operators:** Need to store the three order-dependent projectors $\hat{\Pi}^{0,1;j} \in \mathbb{R}^{(N+1)\times(N+1)}$ with

$$\hat{\Pi}^{0,1;j}_{mn} = \varphi_m \left(\frac{1}{3}(\hat{x}_n + j) \right), \quad m, n = 0, 1, \ldots, N, \tag{2.26}$$

   where $j \in \{0, 1, 2\}$. Remark: This can be done with a lookup table similar to the one for `Kxi`.

2. **Face data restriction** is performed over multiple levels. We want to do that in a recursive fashion; cf. (2.11). We thus need to implement a face data restriction operator such that:

$$g_k = \sum_{i^{0,l}=0}^{3^{l\,(d-1)}-1} \left(\sum_{k'=0}^{(N+1)^{d-1}-1} \frac{1}{3^{l\,(d-1)}} \prod_{\zeta=1}^{d-1} \frac{w_{k'_\zeta}}{w_{k_\zeta}} \, \hat{\Pi}^{0,l;i^{0,l}_\zeta}_{k_\zeta k'_\zeta} \; g^{l;i^{0,l}}_{k'} \right), \tag{2.27}$$

   where $k = 1, \ldots, (N+1)^{d-1} - 1$. The values g_k, $k = 0, \ldots, (N+1)^{d-1} - 1$, denote coefficients associated with $(d-1)$-dimensional quadrature nodes on the face of the coarse grid cell.

   The values $g^{l;i^{0,1}}_{k'}$, $k' = 0, \ldots, (N+1)^{d-1} - 1$, denote coefficients associated with $(d-1)$-dimensional quadrature nodes on the boundary of the fine grid cell.

   Note that in the above notation level 0 refers denotes the coarse grid level l_{coarse} and level l refers to the fine grid level l_{fine}.

   Since we loop over all subcells/subfaces in the spacetree code, only the part in the brackets has to be implemented on the solver side.

   The spacetree code supplies the restriction routine with the subinterval indices $i^{0,l}_\zeta \in \{0, 1, \ldots, 3^l - 1\}$, $\zeta \in \{1, 2, \ldots, d-1\}$. These indices are encoded in terms of a tertiary basis:

$$i^{0,l}_\zeta = \sum_{\beta=1}^{l} 3^{l-\beta} \, j_{\beta,\zeta}, \qquad j_{\beta,\zeta} \in \{0, 1, 2\}. \tag{2.28}$$

The "basis vector" of the first projection level ($\beta = 1$) is 3^{l-1}, the "basis vector" of the second projection level is 3^{l-2}, and so on. This can be used to reconstruct the sequence of subintervals $\{j_1, j_2, j_3, j_4, \ldots, j_l\}$

A straightforward decoding of the subinterval indices $i_\zeta^{0,l}$ would thus work like this:

0. Allocate extra memory to store intermediate results. We need to perform a chain of $l_{\text{fine}} - l_{\text{coarse}} - 1$ MatVecs.
1. Initialise fine grid values or temporay vector as coarse grid values.
2. The solver code loops from 1 to up to the level difference $l_{\text{fine}} - l_{\text{coarse}}$. In each iteration, it has then to determine the indices $j_{\beta,}$ (how many $3^{l-\beta}$?), restrict the fine grid data by one level and write the result to the temporary vector or the fine grid values in an alternating manner.

3. **Face data prolongation** is performed over multiple levels. We want to do that in a recursive fashion; cf. (2.11). We thus need to implement a face data prolongation operator such that:

$$g_k^{l;i^{0,l}} = \sum_{k'=0}^{(N+1)^{d-1}-1} g_{k'} \prod_{\zeta=1}^{d-1} \hat{\Pi}_{k'_\zeta,k_\zeta}^{0,l;i_\zeta^{0,l}}, \qquad k = 0, 1, \ldots, (N+1)^{d-1} - 1. \tag{2.29}$$

See above for a definition of the symbols.

4. **Volume data restriction** is performed over multiple levels. We want to do that in a recursive fashion; cf. (2.11). We thus need to implement a volume data restriction operator such that:

$$u_n = \sum_{i^{0,l}=0}^{3^{l\,d}-1} \left(\sum_{n'=0}^{(N+1)^d-1} \frac{1}{3^{l\,d}} \prod_{\zeta=1}^{d} \frac{w_{n'_\zeta}}{w_{n_\zeta}} \hat{\Pi}_{n_\zeta n'_\zeta}^{0,l;i_\zeta^{0,l}} u_{n'}^{l;i^{0,l}} \right), \tag{2.30}$$

where $n = 1, \ldots, (N+1)^d - 1$. The values u_n, $k = 0, \ldots, (N+1)^d - 1$, denote coefficients associated with d-dimensional quadrature nodes belonging to the coarse grid cell.

The values $u_{n'}^{l;i^{0,1}}$, $n' = 0, \ldots, (N+1)^d - 1$, denote coefficients associated with d-dimensional quadrature nodes belonging to the fine grid cell.

Note that in the above notation level 0 refers denotes the coarse grid level l_{coarse} and level l refers to the fine grid level l_{fine}.

Since we loop over all subcells/subfaces in the spacetree code, only the part in the brackets has to be implemented on the solver side.

The spacetree code supplies the restriction routine with the subcell indices $i_\zeta^{0,l} \in \{0, 1, \ldots, 3^l - 1\}$, $\zeta \in \{1, 2, \ldots, d\}$. These indices are encoded in terms of a tertiary basis:

$$i_\zeta^{0,l} = \sum_{\beta=1}^{l} 3^{l-\beta} j_{\beta,\zeta}, \qquad j_{\beta,\zeta} \in \{0, 1, 2\}. \tag{2.31}$$

The "basis vector" of the first projection level ($\beta = 1$) is 3^{l-1}, the "basis vector" of the second projection level is 3^{l-2}, and so on. This can be used to reconstruct the sequence of subintervals $\{j_1, j_2, j_3, j_4, \ldots, j_l\}$.

5. **Volume data prolongation** is performed over multiple levels. We want to do that in a recursive fashion; cf. (2.11). We thus need to implement a volume data prolongation operator such that:

$$u_n^{l;i^{0,l}} = \sum_{n'=0}^{(N+1)^d-1} u_{n'} \prod_{\zeta=1}^{d} \hat{\Pi}_{n'_\zeta, n_\zeta}^{0,l;i_\zeta^{0,l}}, \qquad n = 0, 1, \ldots, (N+1)^d - 1. \tag{2.32}$$

See above for a definition of the symbols.