Image Deblurring

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Mathematical background

Relevant topics relevant to image deblurring include:

- Convolution: An operation that expresses how a function g(x) can shape another function f(x) reflected on the y axis to create third function h(x) = (f*g)
- Fast Fourier transformation: FFT convolution uses the principle that multiplication in the frequency domain corresponds to convolution in the time domain. The input signal is transformed into the frequency domain using the DFT(Direct Fourier transformation), multiplied by the frequency response of the filter, and then transformed back into the time domain using the Inverse DFT
- Deconvolution: the inverse process of convolution, f*g = h, find f

Introduction to Image Deblurring

- Given a blurred image and a linear model for the blurring, reconstruct the original image
- Consider the linear system of equations:

  $Kf = g,$

  Where K is a real n x n matrix called the blurring matrix, f is a vector to represent the image and g is a vector to represent the blurred image

Why do Images get blurred?

Blur caused by camera:

- Defocus - Caused by having a scene with multiple depth layers and only the layer on a focal plane will focus on the camera sensor which leads to the other layers being out of focus
- Motion - Caused by moving the camera during image taking

Blur caused intentionally:

- Rectangular/Box - Caused by having each pixel have a value equal to the average value of its neighbouring pixels in the original image
- Gaussian - Caused by a Gaussian function which takes the base form: $f(x) = \exp(-x^2)$

Introduction to Image Deblurring

- It seems so simple, to just solve for f if you are given K and g
- However, naively solving it will not work in a majority of cases because of noise
- Our new equation that now factors for noise looks like:

$$Kf + \eta = g$$

- Most of the time, noise is not known so it is very difficult to work with it.

Types of image noise

- Shot noise - Caused by variation in the number of photons sensed at a given exposure level
- Periodic noise - Caused by electrical interference during image capturing
- Salt-and-pepper noise - Caused by analog-to-digital converter errors, bit error in transmission, etc.

Image with salt and pepper noise (Dark pixels in bright regions and bright pixels in dark regions)

Applications of Image Deblurring

- Interpret CAT Scans
- Astronomical images
- Blur removal from images (such as vacation photos)
- Noise removal from 3d renders

Convolution of a Matrix

-convolution of a matrix is basically just taking a weighted averages of the values of the surrounding cells

Syntax : F * G

$$y[m,n] = x[m,n] * h[m,n] = \sum_{j=-\infty}^{\infty} \sum_{i=-\infty}^{\infty} x[i,j] \cdot h[m-i, n-j]$$

Image Convolution

-Image convolution is the process by which the value of a pixel in an image is evaluated by the weighted average of its surrounding neighboring pixels

Building the blur matrix

- The blurring Matrix can be composed by any distributive function, or point separated function, in our example, we went with a Gaussian distributed matrix.
- A gaussian distributed matrix spreads relevance to the center of the matrix, meaning that values away from the center become smaller.
- We build the blur matrix by taking the PSF and applying it to the toeplitz matrix

$$\begin{bmatrix}
h_1 & 0 & \cdots & 0 & 0 \\
h_2 & h_1 & & \vdots & \vdots \\
h_3 & h_2 & \cdots & 0 & 0 \\
\vdots & h_3 & \cdots & h_1 & 0 \\
h_{m-1} & \vdots & \ddots & h_2 & h_1 \\
h_m & h_{m-1} & & \vdots & h_2 \\
0 & h_m & \ddots & h_{m-2} & \vdots \\
0 & 0 & \cdots & h_{m-1} & h_{m-2} \\
\vdots & \vdots & & h_m & h_{m-1} \\
0 & 0 & 0 & \cdots & h_m
\end{bmatrix}$$

Singular Value Decomposition (SVD)

Singular value Decomposition is a method for factoring a matrix

Done by separating a matrix n x m A in to components $U\Sigma V$ such that

$A = U\Sigma V^*$

Where U and V are unitary matrices

To solve SVD we need to take the eigenvectors of AA^* and A^*A and use them as the columns for U and V and Σ is diagonal matrix defined by the eigenvalues of the AA^* and A^*A

Kronecker Products

- A blurring matrix of a n by n will contain n^4 many elements making
- Kronecker products have a property where if a K can be decomposed into A and B such that kron(A,B) = K then

K vec(f) = vec(A f B^T)

- This helps cut down on computations

$$(\mathbf{A}_r \otimes \mathbf{A}_c)\,\mathrm{vec}(\mathbf{X}) = \mathrm{vec}(\mathbf{A}_c \mathbf{X} \mathbf{A}_r^T),$$

$$(\mathbf{A}_r \otimes \mathbf{A}_c)^T = \mathbf{A}_r^T \otimes \mathbf{A}_c^T, \qquad (\mathbf{A}_r \otimes \mathbf{A}_c)^{-1} = \mathbf{A}_r^{-1} \otimes \mathbf{A}_c^{-1},$$

$$(\mathbf{U}_r \boldsymbol{\Sigma}_r \mathbf{V}_r^T) \otimes (\mathbf{U}_c \boldsymbol{\Sigma}_c \mathbf{V}_c^T) = (\mathbf{U}_r \otimes \mathbf{U}_c)(\boldsymbol{\Sigma}_r \otimes \boldsymbol{\Sigma}_c)(\mathbf{V}_r \otimes \mathbf{V}_c)^T.$$

$$\mathbf{A} \otimes \mathbf{B} = \begin{bmatrix} a_{11}\mathbf{B} & \cdots & a_{1n}\mathbf{B} \\ \vdots & \ddots & \vdots \\ a_{m1}\mathbf{B} & \cdots & a_{mn}\mathbf{B} \end{bmatrix},$$

Separable PSF

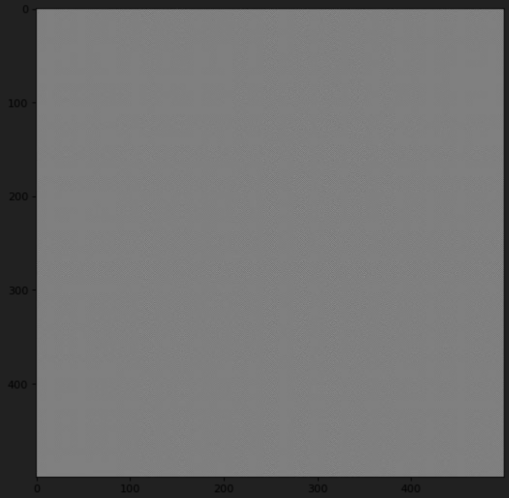
- If the PSF we use is separable then we then we can use then we can use partial SVD to compute the r and c vector such that the psf is equal to c times the transpose of r
- Once we have this we can both multiply both c and r by the square root of the s value
- Lastly calculate the toeplitz matrix respect to r and c time the sqrt s
- This will give us the blurring matrix split into its column and row components

Naive Method

Given Kf = g solve for f

Or equivalently CxB = b

Where K = kron(C,B)



Naive method

- When images are blurred there is often noise and when we try to solve the problem naively the noise is amplified often times destroying the image.
- F^(l,u) = F(l,u) + n(l,u)/H(l,u)
- Since H(l,u) should always be less than 1 n(l,u) is amplified

Tikhonov Regularization

- Regularization is a process to simplify the result of a problem often used in ill-posed problems
- Γ = ɑ I
- the controls how much noise we will let noise effect the approximation of the actual image the higher ɑ is the less noise will affect the approximation but the less the image will actually deblur

$$\min_{f}\{||g - Kf||_2^2 + \alpha^2 ||f||_2^2\}$$

$$\hat{x} = (A^\top A + \Gamma^\top \Gamma)^{-1} A^\top \mathbf{b}.$$

SVD Truncated Regularization

- Truncated SVD is the second method to regularize the image
- Truncated SVD works by taking the and truncating the matrices at a given K value
- By doing this at a given K we remove any High frequency components that will amplify the noise in the image when deblurring
- Similarly to tikhonov the more we increase K the less we actually deblur the image

$$\mathbf{x}_k = \sum_{i=1}^{k} \frac{\mathbf{u}_i^T \mathbf{b}}{\sigma_i} \mathbf{v}_i \equiv \mathbf{A}_k^\dagger \mathbf{b}$$

$$\mathbf{X}_k = (\mathbf{A}_r)_k^\dagger \mathbf{B} \left((\mathbf{A}_c)_k^\dagger \right)^T .$$

$$\mathbf{A}_k^\dagger = \begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_k \end{bmatrix} \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_k \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{u}_1^T \\ \vdots \\ \mathbf{u}_k^T \end{bmatrix}$$

Any Questions?