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Tarea 2 Cosmología

P1) $$ds^2 = a^2(\tau)\left[-(1+2\psi(\vec{x},t))d\tau^2 + (1-2\phi(\vec{x},t))\delta_{ij}dx^i dx^j\right]$$

Podemos expresar la métrica como:

$$g_{\mu\nu} = \underbrace{\bar{g}_{\mu\nu}}_{\text{término a background}} + \underbrace{\delta g_{\mu\nu}}_{\text{término perturbado}} \Rightarrow \bar{g}_{00} = -a^2 \;;\; \bar{g}_{0i} = \bar{g}_{i0} = 0 \;;\; \bar{g}_{ij} = a^2\delta_{ij}$$

$$\delta g_{00} = -2a^2\psi \;;\; \delta g_{0i} = \delta g_{i0} = 0 \;;\; \delta g_{ij} = -2\phi\,\delta_{ij}a^2$$

Esto nos permite separar los símbolos de Christoffel:

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2}g^{\alpha\rho}\left[g_{\rho\mu,\nu} + g_{\rho\nu,\mu} - g_{\mu\nu,\rho}\right] = \bar{\Gamma}^\alpha_{\mu\nu} + \delta\Gamma^\alpha_{\mu\nu}$$

Background:

$$\bar{\Gamma}^0_{\mu\nu} = \frac{1}{2}\bar{g}^{0\rho}\left[\bar{g}_{\rho\mu,\nu} + \bar{g}_{\rho\nu,\mu} - \bar{g}_{\mu\nu,\rho}\right] \overset{\bar{g}^{0i}=0 \Rightarrow \rho=0}{=} \frac{1}{2}\bar{g}^{00}\left[\bar{g}_{0\mu,\nu} + \bar{g}_{0\nu,\mu} - \bar{g}_{\mu\nu,0}\right]$$

$\bar{g}_{\mu\nu}$ es diagonal $\Rightarrow \bar{g}^{\mu\nu} = \frac{1}{\bar{g}_{\mu\nu}}$

$$\bar{\Gamma}^0_{00} = \frac{1}{2}\bar{g}^{00}\left[\bar{g}_{00,0} + \bar{g}_{00,0} - \bar{g}_{00,0}\right] = \frac{1}{2}\bar{g}^{00}\cdot\bar{g}_{00,0}$$

$$= \frac{1}{2}\cdot -\frac{1}{a^2}\cdot\frac{\partial}{\partial\tau}(-a^2) = \frac{1}{\cancel{2}}\frac{1}{a^2}\cancel{2}a\dot{a}$$

$$= \frac{\dot{a}}{a} = \mathcal{H}$$

$$\bar{\Gamma}^0_{i0} = \frac{1}{2}\bar{g}^{00}\left[\bar{g}_{0i,0} + \bar{g}_{00,i} - \bar{g}_{i0,0}\right] = \bar{\Gamma}^0_{0i} \Rightarrow \bar{\Gamma}^0_{i0} = \bar{g}_{00,i} = \partial_i(a^2(\tau)) = 0$$

$\bar{g}_{0i} = \bar{g}_{i0} = 0$

$$\bar{\Gamma}^0_{ij} = \frac{1}{2}\bar{g}^{00}\left[\cancel{\bar{g}_{0i,j}} + \cancel{\bar{g}_{0j,i}} - \bar{g}_{ij,0}\right] = -\frac{1}{2}\bar{g}^{00}\bar{g}_{ij,0} = -\frac{1}{2}\cdot -\frac{1}{a^2}\cdot\partial_\tau(a^2\delta_{ij})$$

$g_{0i} = 0$

$$= \frac{1}{\cancel{2}}\frac{1}{a^2}\cancel{2}a\dot{a}\,\delta_{ij} = \frac{\dot{a}}{a}\delta_{ij} = \mathcal{H}\delta_{ij}$$

$$\bar{\Gamma}^i_{\mu\nu} = \frac{1}{2}\bar{g}^{i\rho}\left[\bar{g}_{\rho\mu,\nu} + \bar{g}_{\rho\nu,\mu} - \bar{g}_{\mu\nu,\rho}\right] \overset{\bar{g}^{i\rho}=0 \text{ si } i\neq\rho}{=} \frac{1}{2}\bar{g}^{ii}\left[\bar{g}_{i\mu,\nu} + \bar{g}_{i\nu,\mu} - \bar{g}_{\mu\nu,i}\right]$$

$$\bar{\Gamma}^i_{00} = \frac{1}{2}\bar{g}^{ii}\left[\underbrace{\bar{g}_{i0,0}}_{0} + \underbrace{\bar{g}_{i0,0}}_{0} - \bar{g}_{00,i}\right] = \frac{1}{2}\bar{g}^{ii}\partial_i(a^2) = 0$$

$$\bar{\Gamma}^i_{j0} = \frac{1}{2}\bar{g}^{ii}\left[\bar{g}_{ij,0} + \underbrace{\bar{g}_{i0,j}}_{0} - \underbrace{\bar{g}_{j0,i}}_{0}\right] = \frac{1}{2}\cdot\frac{1}{a^2}\cdot\partial_\tau(a^2\delta_{ij}) = \frac{1}{2}\frac{1}{a^2}2a\dot{a}\,\delta_{ij} = \mathcal{H}\delta_{ij}$$

$$\bar{\Gamma}^i_{0j} = \frac{1}{2}\bar{g}^{ii}\left[\underbrace{\bar{g}_{i0,j}}_{0} + \bar{g}_{ij,0} - \underbrace{\bar{g}_{0j,i}}_{0}\right] = \bar{\Gamma}^i_{j0} = \mathcal{H}\delta_{ij}$$

$$\bar{\Gamma}^i_{jk} = \frac{1}{2}\bar{g}^{ii}\left[\underset{0}{\bar{g}_{ij,k}} + \underset{0}{\bar{g}_{ik,j}} - \underset{0}{\bar{g}_{jk,i}}\right] = 0$$

Perturbados: (Consideramos perturbaciones a primer orden, por lo que no hay términos $\delta g^{\alpha\rho}(\delta g_{\mu\rho,\nu}+\dots)$)

$$\delta\Gamma^\alpha_{\mu\nu} = \frac{1}{2}\bar{g}^{\alpha\rho}\left[\delta g_{\rho\mu,\nu} + \delta g_{\rho\nu,\mu} - \delta g_{\mu\nu,\rho}\right] + \frac{1}{2}\delta g^{\alpha\rho}\left[\bar{g}_{\rho\mu,\nu} + \bar{g}_{\rho\nu,\mu} - \bar{g}_{\mu\nu,\rho}\right]$$

Además:

$$g^{\mu\nu}g_{\mu\nu} = \mathbb{1} \ / \delta(\,)$$

$$\Rightarrow \delta(g^{\mu\nu}g_{\mu\nu}) = \delta(\mathbb{1}) \Rightarrow \delta g^{\mu\nu}g_{\mu\nu} + g^{\mu\nu}\delta g_{\mu\nu} = 0 \Rightarrow \delta g^{\mu\nu}g_{\mu\nu} = -g^{\mu\nu}\delta g_{\mu\nu} \ / g^{\mu\nu}$$

$$\Rightarrow \delta g^{\mu\nu} = -g^{\mu\alpha}\delta g_{\alpha\beta}g^{\beta\nu} \xRightarrow[\text{de perturbaciones}]{\text{A primer orden}} \delta g^{\mu\nu} = -\bar{g}^{\mu\alpha}\delta g_{\alpha\beta}\bar{g}^{\beta\nu}$$

$$\Rightarrow \delta g^{00} = -\bar{g}^{00}\delta g_{00}\bar{g}^{00}$$

$$= -(-a^{-2})(-2a^2\psi)(-a^{-2}) = \frac{2\psi}{a^2}$$

$$\delta g^{ij} = -\bar{g}^{i\alpha}\delta g_{\alpha\beta}\bar{g}^{j\beta} = -\bar{g}^{ii}\bar{g}^{jj}\delta^{i\alpha}\delta^{j\beta}\delta g_{\alpha\beta} = -(a^{-2}\delta^{i\alpha})(a^{-2}\delta^{j\beta})(-2\phi a^2\delta_{\alpha\beta})$$

$\bar{g}^{ij}=0$

$$= 2\phi a^{-2}\delta^{i\alpha}\delta^j_\alpha = 2\phi a^{-2}\delta^{ij}$$

$$\delta g^{i0} = \delta g^{0i} = -\bar{g}^{i\alpha}\delta g_{\alpha\beta}\bar{g}^{0\beta} = -\left[\bar{g}^{i0}\underset{0}{\delta g_{00}}\bar{g}^{00} + \bar{g}^{ik}\underset{0}{\delta g_{k0}}\bar{g}^{00} + \bar{g}^{i0}\underset{0}{\delta g_{0k}}\bar{g}^{0k}\right] = 0$$

$$\Rightarrow \delta\Gamma^0_{\mu\nu} = \frac{1}{2}\bar{g}^{0\rho}\left[\delta g_{\rho\mu,\nu} + \delta g_{\rho\nu,\mu} - \delta g_{\mu\nu,\rho}\right] + \frac{1}{2}\delta g^{0\rho}\left[\bar{g}_{\rho\mu,\nu} + \bar{g}_{\rho\nu,\mu} - \bar{g}_{\mu\nu,\rho}\right]$$

$\bar{g}^{0\rho}=0$ si $0\neq\rho$

$$= \frac{1}{2}\bar{g}^{00}\left[\delta g_{0\mu,\nu} + \delta g_{0\nu,\mu} - \delta g_{\mu\nu,0}\right] + \frac{1}{2}\delta g^{00}\left[\bar{g}_{0\mu,\nu} + \bar{g}_{0\nu,\mu} - \bar{g}_{\mu\nu,0}\right]$$

$$\Rightarrow \delta\Gamma^0_{00} = \frac{1}{2}\bar{g}^{00}\left[\delta g_{00,0} + \delta g_{00,0} - \delta g_{00,0}\right] + \frac{1}{2}\delta g^{00}\left[\bar{g}_{00,0} + \bar{g}_{00,0} - \bar{g}_{00,0}\right]$$

$$= \frac{1}{2}\bar{g}^{00}\delta g_{00,0} + \frac{1}{2}\delta g^{00}\bar{g}_{00,0}$$

$$= \frac{1}{2}\cdot -a^{-2}\cdot\partial_t(-a^2\cdot 2\psi) + \frac{1}{2}\cdot\frac{2\psi}{a^2}\cdot\partial_t(-a^2)$$

$$= -\frac{1}{2}a^{-2}\cdot 2\cdot\left[2a\dot{a}\psi - a^2\dot{\psi}\right] + \frac{1}{a^2}\psi\cdot -2a\dot{a}$$

$$= \frac{2\dot{a}}{a}\psi + \dot{\psi} - \frac{2\dot{a}}{a}\psi = \dot{\psi} \qquad \Rightarrow \delta\Gamma^0_{00} = \dot{\psi}$$

$$\delta\Gamma^0_{i0} = \frac{1}{2}\delta g^{00}\left[\underset{0}{\cancel{\bar g_{0i,0}}} + \underset{\partial_i(-a^2)=0}{\cancel{\bar g_{00,i}}} - \underset{0}{\cancel{\bar g_{i0,0}}}\right] + \frac{1}{2}\bar g^{00}\left[\underset{0}{\cancel{\delta g_{0i,0}}} + \delta g_{00,i} - \underset{0}{\cancel{\delta g_{i0,0}}}\right]$$

$$= \frac{1}{2}\bar g^{00}\delta g_{00,i} = \frac{1}{2}\cdot -\frac{1}{a^2}\cdot\partial_i\left[-2a^2\Psi\right] = \frac{-1}{2a^2}\cdot -2a^2\partial_i\Psi = \partial_i\Psi$$

$$\delta\Gamma^0_{i0} = \delta\Gamma^0_{0i}$$

→ propiedad de los símbolos

$$\delta\Gamma^0_{ij} = \frac{1}{2}\bar g^{00}\left[\underset{0}{\cancel{\delta g_{0i,j}}} + \underset{0}{\cancel{\delta g_{0j,i}}} - \delta g_{ij,0}\right] + \frac{1}{2}\delta g^{00}\left[\underset{0}{\cancel{\bar g_{0i,j}}} + \underset{0}{\cancel{\bar g_{0j,i}}} - \bar g_{ij,0}\right]$$

$$= \frac{1}{2}\bar g^{00}\delta g_{ij,0} - \frac{1}{2}\delta g^{00}\bar g_{ij,0} = -\frac{1}{2}\;\frac{-1}{a^2}\cdot\partial_t(-2\phi\delta_{ij}a^2) - \frac{1}{2}\frac{2\Psi}{a^2}\partial_t(+a^2\delta_{ij})$$

$$= \frac{-\delta_{ij}}{\cancel{2}a^{\cancel{2}}}\left[a^{\cancel{2}}\cdot\cancel{2}\dot\phi + \cancel{2}2a\dot a\phi\cdot 2\right] - \frac{1}{\cancel{2}}\frac{\cancel{2}\Psi}{a^2}\cdot 2a\dot a\,\delta_{ij}$$

$$= -\delta_{ij}\dot\phi - 2\frac{\dot a}{a}\phi\delta_{ij} - \frac{2\dot a}{a}\Psi\delta_{ij} = -\delta_{ij}\left[\dot\phi + 2\mathcal{H}(\phi+\Psi)\right]$$

$$\delta\Gamma^0_{ij} = \delta\Gamma^0_{ji}$$

$$\delta\Gamma^i_{\mu\nu} = \frac{1}{2}\delta g^{i\rho}\left[\bar g_{\rho\mu,\nu} + \bar g_{\rho\nu,\mu} - \bar g_{\mu\nu,\rho}\right] + \frac{1}{2}\bar g^{\rho i}\left[\delta g_{\rho\mu,\nu} + \delta g_{\rho\mu,\nu} - \delta g_{\mu\nu,\rho}\right]$$ $\delta g^{i\rho} = \bar g^{\rho i} = 0$ si $i \neq \rho$

$$= \frac{1}{2}\delta g^{ii}\left[\bar g_{i\mu,\nu} + \bar g_{i\nu,\mu} - \bar g_{\mu\nu,i}\right] + \frac{1}{2}\bar g^{ii}\left[\delta g_{i\mu,\nu} + \delta g_{i\mu,\nu} - \delta g_{\mu\nu,i}\right]$$

$$\delta\Gamma^i_{00} = \frac{1}{2}\delta g^{ii}\left[\underset{0}{\bar g_{i0,0}} + \underset{0}{\bar g_{i0,0}} - \bar g_{00,i}\right] + \frac{1}{2}\bar g^{ii}\left[\underset{0}{\cancel{\delta g_{i0,0}}} + \underset{0}{\cancel{\delta g_{i0,0}}} - \delta g_{00,i}\right]$$

$$= -\frac{1}{2}\delta g^{ii}\partial_i(-a^2) - \frac{1}{2}a^{-2}\delta^{ii}\delta g_{00,i} = -\frac{1}{2}a^{-2}\delta^{ii}\cdot\partial_i(-2a^2\Psi)$$

$$= \delta^{ii}\partial_i\Psi = \partial^i\Psi$$

$$\delta\Gamma^i_{0j} = \frac{1}{2}\delta g^{ii}\left[\underset{0}{\bar g_{i0,j}} + \bar g_{ij,0} - \underset{0}{\cancel{\bar g_{0j,i}}}\right] + \frac{1}{2}\bar g^{ii}\left[\underset{0}{\delta g_{i0,j}} + \delta g_{ij,0} - \underset{0}{\cancel{\delta g_{0j,i}}}\right]$$

$$= \frac{1}{2}\delta g^{ii}\partial_t(a^2\delta_{ij}) + \frac{1}{2}\bar g^{ii}\partial_t(-2\phi\delta_{ij}a^2)$$

$$= \cancel{\frac{1}{2}2\phi a^{-2}\delta^{ii}\left[2a\dot a\delta_{ij}\right]} + \frac{1}{2}\frac{\delta^{ii}}{a^2}\left[-2\dot\phi\delta_{ij}a^2 \cancel{- 2\cdot\phi\delta_{ij}2a\dot a}\right]$$

$$= -\delta^{ii}\delta_{ij}\dot\phi = -\delta^i_j\dot\phi$$

$$\delta\Gamma^i_{0j} = \delta\Gamma^i_{j0}$$

$$\delta\Gamma^i_{jk} = \frac{1}{2}\delta g^{ii}[\bar{g}_{ij,k} + \bar{g}_{ik,j} - \bar{g}_{jk,i}] + \frac{1}{2}\bar{g}^{ii}[\delta g_{ij,k} + \delta g_{ik,j} - \delta g_{jk,i}]$$

$$= \frac{1}{2} 2\phi a^{-2}\delta^{ii}[\underbrace{\partial_k(a^2\delta_{ij})}_{\to 0} + \underbrace{\partial_j(a^2\delta_{ik})}_{\to 0} - \underbrace{\partial_i(a^2\delta_{jk})}_{\to 0}]$$

$$+ \frac{1}{2}a^{-2}\delta^{ii}[\partial_k(-2\phi\delta_{ij}a^2) + \partial_j(-2\phi\delta_{ik}a^2) - \partial_i(-2\phi\delta_{jk}a^2)] =$$

$$= \delta^{ii}[\delta_{jk}\partial_i(\phi) - \delta_{ij}\partial_k\phi - \delta_{ik}\partial_j\phi]$$

$$= [\delta_{jk}\partial^i - \delta^i_j\partial_k - \delta^i_k\partial_j]\phi$$

$$\delta\Gamma^i_{jk} = \delta\Gamma^i_{kj}$$

Con esto, tenemos todos los símbolos de Christoffel del Conformal Newtonian Gauge:

$$\Gamma^0_{00} = \mathcal{H} + \dot{\psi} \ ; \ \Gamma^0_{0i} = \Gamma^0_{i0} = \partial_i\psi \ ; \ \Gamma^0_{ij} = \delta_{ij}[\mathcal{H} - 2\mathcal{H}(\psi+\phi) - \dot{\phi}] \ ; \ \Gamma^i_{00} = \partial^i\psi$$

$$\Gamma^i_{j0} = \Gamma^i_{0j} = \delta^i_j(\mathcal{H} - \dot{\phi}) \ ; \ \Gamma^i_{jk} = (\delta_{jk}\partial^i - \delta^i_j\partial_k - \delta^i_k\partial_j)\phi$$

Teniendo los símbolos, podemos calcular el tensor de Ricci:

$$R_{\mu\nu} = \partial_\alpha\Gamma^\alpha_{\mu\nu} - \partial_\nu\Gamma^\alpha_{\mu\alpha} + \Gamma^\alpha_{\rho\alpha}\Gamma^\rho_{\mu\nu} - \Gamma^\alpha_{\rho\nu}\Gamma^\rho_{\mu\alpha}$$

$$R_{00} = \underbrace{\partial_\alpha\Gamma^\alpha_{00}} - \underbrace{\partial_0\Gamma^\alpha_{0\alpha}} + \underbrace{\Gamma^\alpha_{\rho\alpha}\Gamma^\rho_{00}} - \underbrace{\Gamma^\alpha_{\rho 0}\Gamma^\rho_{0\alpha}}$$

$$= [\partial_0\Gamma^0_{00} + \partial_i\Gamma^i_{00}] - [\partial_0\Gamma^0_{00} + \partial_0\Gamma^i_{0i}] + [\Gamma^0_{\rho 0}\Gamma^\rho_{00} + \Gamma^i_{\rho i}\Gamma^\rho_{00}] - [\Gamma^0_{\rho 0}\Gamma^\rho_{00} + \Gamma^i_{\rho 0}\Gamma^\rho_{0i}]$$

$$= \partial_i\Gamma^i_{00} - \partial_0\Gamma^i_{0i} + [\Gamma^i_{0i}\Gamma^0_{00} + \Gamma^i_{ji}\Gamma^j_{00}] - [\Gamma^i_{00}\Gamma^0_{0i} + \Gamma^i_{j0}\Gamma^j_{0i}]$$

$$= \partial_i(\partial^i\psi) - \partial_0(\delta^i_i(\mathcal{H}-\dot{\phi})) + [(\delta^i_i(\mathcal{H}-\dot{\phi}))(\mathcal{H}+\dot{\psi}) + (\delta_{ji}\partial^i - \delta^i_j\partial_i - \delta^i_i\partial_j)\phi \cdot \partial^j\psi]$$

$$- [\partial^i\psi\partial_i\psi + \delta^i_j(\mathcal{H}-\dot{\phi})\delta^j_i(\mathcal{H}-\dot{\phi})] \ ; \ * \partial_i\partial^i = \nabla^2 \ ; \ \delta^i_i = 3$$

$$= \nabla^2\psi - 3\partial_0(\mathcal{H}-\dot{\phi}) + [3(\mathcal{H}^2 + \mathcal{H}\dot{\psi} - \dot{\phi}\mathcal{H} - \dot{\phi}\dot{\psi}) - 5\partial_i\phi\,\partial^i\psi] - [\partial^i\psi\partial_i\psi + 3(\mathcal{H}^2 - 2\mathcal{H}\dot{\phi} + \dot{\phi}^2)]$$

Los términos subrayados son perturbaciones de orden 2, por lo que no los consideramos (≈ 0)

$$\Rightarrow R_{00} = \nabla^2\psi - 3(\dot{\mathcal{H}} - \ddot{\phi}) + 3(\mathcal{H}^2 + \mathcal{H}\dot{\psi} - \dot{\phi}\mathcal{H}) - 3(\mathcal{H}^2 - 2\mathcal{H}\dot{\phi})$$

$$R_{00} = \nabla^2\psi - 3\dot{\mathcal{H}} + 3\ddot{\phi} + 3\mathcal{H}(\dot{\psi} + \dot{\phi})$$

$$R_{i0} = \partial_\alpha \Gamma^\alpha_{i0} - \partial_0 \Gamma^\alpha_{i\alpha} + \Gamma^\alpha_{\rho\alpha}\Gamma^\rho_{i0} - \Gamma^\alpha_{\rho 0}\Gamma^\rho_{i\alpha}$$

$$= [\partial_0 \Gamma^0_{i0} + \partial_i \Gamma^i_{i0}] - [\partial_0 \Gamma^0_{i0} + \partial_0 \Gamma^i_{ii}] + [\Gamma^0_{\rho 0}\Gamma^\rho_{i0} + \Gamma^i_{\rho i}\Gamma^\rho_{i0}] - [\Gamma^0_{\rho 0}\Gamma^\rho_{i0} + \Gamma^i_{\rho 0}\Gamma^\rho_{ii}]$$

$$= \partial_i \Gamma^i_{i0} - \partial_0 \Gamma^i_{ii} + [\Gamma^i_{0i}\Gamma^0_{i0} + \Gamma^i_{ii}\Gamma^i_{i0}] - [\Gamma^i_{00}\Gamma^0_{ii} + \Gamma^i_{i0}\Gamma^i_{ii}]$$

$$= \partial_i(\Gamma^i_{i0}) - \partial_0(\Gamma^i_{ii}) + (\Gamma^i_{0i}\Gamma^0_{i0}) - (\Gamma^i_{00}\Gamma^0_{ii})$$

$$= \partial_i(\delta^i_i(\mathcal{H} - \dot{\phi})) - \partial_\tau((\delta_{ii}\partial^i - \delta^i_i\partial_i - \delta^i_i\partial_i)\phi) + [\delta^i_i(\mathcal{H} - \dot{\phi})\partial_i\Psi]$$

$$- [(\partial^i\Psi)\delta_{ii}(\mathcal{H} - 2\mathcal{H}(\Psi + \phi) - \dot{\phi})]$$

$$= \partial_i(3(\mathcal{H} - \dot{\phi})) - \partial_\tau(-5\partial_i\phi) + 3(\mathcal{H} - \dot{\phi})\partial_i\Psi - \partial_i\Psi(\mathcal{H} - 2\mathcal{H}(\Psi + \phi) - \dot{\phi})$$

$$= \underbrace{3\partial_i\mathcal{H}}_{\substack{\mathcal{H}=\mathcal{H}(\tau) \\ \Rightarrow \partial_i\mathcal{H}=0}} - 3\partial_i\dot{\phi} + 5\partial_i(\underbrace{\partial_\tau\phi}_{\dot{\phi}}) + 3(\mathcal{H} - \dot{\phi})\partial_i\Psi - \partial_i\Psi(\mathcal{H} - 2\mathcal{H}(\Psi + \phi) - \dot{\phi})$$

: TÉRMINOS DE SEGUNDO ORDEN

$$\Rightarrow R_{i0} = 2\partial_i(\dot{\phi} + \mathcal{H}\Psi)$$

Por construcción, el tensor de Ricci es simétrico $\Rightarrow R_{i0} = R_{0i}$

$$R_{ij} = \partial_\alpha \Gamma^\alpha_{ij} - \partial_i \Gamma^\alpha_{j\alpha} + \Gamma^\alpha_{\alpha\rho}\Gamma^\rho_{ij} - \Gamma^\alpha_{i\rho}\Gamma^\rho_{\alpha j}$$

$$= [\partial_0 \Gamma^0_{ij} + \partial_k \Gamma^k_{ij}] - [\partial_i \Gamma^0_{j0} + \partial_i \Gamma^k_{jk}] + [\Gamma^0_{0\rho}\Gamma^\rho_{ij} + \Gamma^k_{k\rho}\Gamma^\rho_{ij}] - [\Gamma^0_{i\rho}\Gamma^\rho_{0j} + \Gamma^k_{i\rho}\Gamma^\rho_{kj}]$$

$$= [\partial_\tau(\delta_{ij}[\mathcal{H} - 2\mathcal{H}(\Psi + \phi) - \dot{\phi}]) + \partial_k([\delta_{ij}\partial^k - \delta^k_i\partial_j - \delta^k_j\partial_i]\phi)]$$

$$- [\partial_i(\partial_j\Psi) + \partial_i((\delta_{jk}\partial^k - \delta^k_k\partial_j - \delta^k_j\partial_k)\phi)]$$

$$+ [\Gamma^0_{00}\Gamma^0_{ij} + \Gamma^0_{0l}\Gamma^l_{ij} + \Gamma^k_{k0}\Gamma^0_{ij} + \Gamma^k_{kl}\Gamma^l_{ij}]$$

$$- [\Gamma^0_{i0}\Gamma^0_{0j} + \Gamma^0_{il}\Gamma^l_{0j} + \Gamma^k_{i0}\Gamma^0_{kj} + \Gamma^k_{il}\Gamma^l_{kj}]$$

$$= [\delta_{ij}[\dot{\mathcal{H}} - 2\dot{\mathcal{H}}(\Psi + \phi) - 2\mathcal{H}(\dot{\Psi} + \dot{\phi}) - \ddot{\phi}] + \delta_{ij}\underbrace{\partial_k\partial^k}_{\nabla^2}\phi - \partial_i\partial_j\phi - \partial_j\partial_i\phi]$$

$$- [\partial_i\partial_j\Psi + \partial_i\partial_j\phi - 3\partial_i\partial_j\phi - \partial_i\partial_j\phi]$$

$$+ [(\mathcal{H} + \dot{\Psi})\delta_{ij}(\mathcal{H} - 2\mathcal{H}(\Psi + \phi) - \dot{\phi}) + \partial_l\Psi(\delta_{ij}\partial^l - \delta^l_i\partial_j - \delta^l_j\partial_i)\phi$$

$$+ \delta^k_k(\mathcal{H} - \dot{\phi})\delta_{ij}(\mathcal{H} - 2\mathcal{H}(\Psi + \phi) - \dot{\phi}) + (\delta_{kl}\partial^k - \delta^k_k\partial_l - \delta^k_l\partial_k)\phi \cdot (\delta_{ij}\partial^l - \delta^l_i\partial_j - \delta^l_j\partial_i)\phi]$$

$$- [\partial_i\Psi\partial_j\Psi + \delta_{il}[\mathcal{H} - 2\mathcal{H}(\Psi + \phi) - \dot{\phi}]\delta^l_j(\mathcal{H} - \dot{\phi}) + \delta^k_i(\mathcal{H} - \dot{\phi})\delta_{kj}[\mathcal{H} - 2\mathcal{H}(\Psi + \phi) - \dot{\phi}]$$

$$+ (\delta_{il}\partial^k - \delta^k_i\partial_l - \delta^k_l\partial_i)\phi \cdot (\delta_{kj}\partial^l - \delta^l_k\partial_j - \delta^l_j\partial_k)\phi]$$

: SEGUNDO ORDEN

$$\Rightarrow R_{ij} = [\delta_{ij}[\dot{\mathcal H} - 2\dot{\mathcal H}(\Psi+\phi) - 2\mathcal H(\dot\Psi+\dot\phi) - \ddot\phi + \nabla^2\phi] + \partial_i\partial_j(\phi-\Psi)]$$

$$+[\delta_{ij}[\mathcal H^2 - 2\mathcal H^2(\Psi+\phi) - \mathcal H\dot\phi + \dot\Psi\mathcal H - 2\mathcal H\dot\Psi(\Psi+\phi) - \dot\Psi\dot\phi]$$

$$+3\delta_{ij}[\mathcal H^2 - 2\mathcal H^2(\Psi+\phi) - \mathcal H\dot\phi - \dot\phi\mathcal H + 2\mathcal H\dot\phi(\Psi+\phi) + \dot\phi^2]]$$

: segundo orden

$$-[\delta_{ij}[\mathcal H^2 - 2\mathcal H^2(\Psi+\phi) - \mathcal H\dot\phi - \dot\phi\mathcal H + 2\mathcal H\dot\phi(\Psi+\phi) - \dot\phi^2]$$

$$\delta_{ij}[\mathcal H^2 - 2\mathcal H^2(\Psi+\phi) - \mathcal H\dot\phi - \dot\phi\mathcal H + 2\mathcal H\dot\phi(\Psi+\phi) + \dot\phi^2]]$$

$$\Rightarrow R_{ij} = R_{ji} = \delta_{ij}[\nabla^2\phi - \ddot\phi + \dot{\mathcal H} - \mathcal H(\dot\Psi + 5\dot\phi) + 2\mathcal H^2 - (2\dot{\mathcal H} + 4\mathcal H^2)(\Psi+\phi)] + \partial_i\partial_j(\phi-\Psi)$$

Con esto tenemos todas las componentes del tensor de Ricci, por lo que procedemos a calcular el escalar:

$$R = g^{\mu\nu}R_{\mu\nu} = g^{0\nu}R_{0\nu} + g^{i\nu}R_{i\nu} = g^{00}R_{00} + \underbrace{g^{0k}}_{0}R_{0k} + \underbrace{g^{i0}}_{0}R_{i0} + g^{ik}R_{ik}$$

la métrica es diagonal

$$\Rightarrow R = g^{00}R_{00} + g^{ik}R_{ik}$$

$$= \left[\left(-\frac{1}{a^2} + \frac{2\Psi}{a^2}\right)(\nabla^2\Psi + 3\ddot\phi - 3\dot{\mathcal H} + 3\mathcal H(\dot\Psi+\dot\phi))\right]$$

$$+\left[\delta^{ik}\left(\frac{1}{a^2} + \frac{2\phi}{a^2}\right)\delta_{ik}[\nabla^2\phi - \ddot\phi + \dot{\mathcal H} - \mathcal H(\dot\Psi+5\dot\phi) + 2\mathcal H^2 - (2\dot{\mathcal H}+4\mathcal H^2)(\Psi+\phi)] + \partial_i\partial_k(\phi-\Psi)\right]$$

$$\delta^{ik}\delta_{ik} = \delta^i_i = 3$$

$$= [-a^{-2}(\nabla^2\Psi + 3\ddot\phi - 3\dot{\mathcal H} + 3\mathcal H(\dot\Psi+\dot\phi)) + 2a^{-2}(\Psi\nabla^2\Psi + 3\Psi\ddot\phi - 3\Psi\dot{\mathcal H} + 3\mathcal H\Psi(\dot\Psi+\dot\phi))]$$

$$+[3a^{-2}[\nabla^2\phi - \ddot\phi + \dot{\mathcal H} - \mathcal H(\dot\Psi+5\dot\phi) + 2\mathcal H^2 - (2\dot{\mathcal H}+4\mathcal H^2)(\Psi+\phi)] + \delta^{ik}\partial_i\partial_k(\phi-\Psi)a^{-2}$$

$$6a^{-2}[\phi\nabla^2\phi - \phi\ddot\phi + \phi\dot{\mathcal H} - \mathcal H\phi(\dot\Psi+5\dot\phi) + 2\phi\mathcal H^2 - \phi(2\dot{\mathcal H}+4\mathcal H^2)(\Psi+\phi)]$$

$$+2a^{-2}\delta^{ik}\partial_i\partial_k(\phi-\Psi)\phi]$$

: segundo orden

$$= [-a^{-2}(\nabla^2\Psi + 3\ddot\phi - 3\dot{\mathcal H} + 3\mathcal H(\dot\Psi+\dot\phi)) - 6a^{-2}\Psi\dot{\mathcal H}]$$

$$+[a^{-2}(3\nabla^2\phi - 3\ddot\phi + 3\dot{\mathcal H} - 3\mathcal H(\dot\Psi+5\dot\phi) + 6\mathcal H^2 - 3(2\dot{\mathcal H}+4\mathcal H^2)(\Psi+\phi)) + \overbrace{\partial^k\partial_k}^{\nabla^2}(\phi-\Psi)$$

$$+6\phi\dot{\mathcal H} + 12\phi\mathcal H^2]$$

$$\Rightarrow R = 6a^{-2}(\dot{\mathcal H} + \mathcal H^2) + a^{-2}(2\nabla^2(2\phi-\Psi) - 6\ddot\phi - 6\mathcal H(\dot\Psi+3\dot\phi) - 12(\dot{\mathcal H}+\mathcal H^2)\Psi)$$

UNA VEZ OBTENIDOS EL TENSOR Y EL ESCALAR DE RICCI, PODEMOS CALCULAR EL TENSOR DE EINSTEIN:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \Rightarrow G^{\mu}_{\nu} = g^{\mu\alpha} G_{\alpha\nu}$$

$$\Rightarrow G^0_0 = g^{0\alpha} G_{\alpha 0} \overset{\text{MÉTRICA DIAGONAL}}{=} g^{00} G_{00}$$

$$\Rightarrow G^0_0 = g^{00}\left[R_{00} - \frac{1}{2} g_{00} R\right] = g^{00} R_{00} - \frac{1}{2} R$$

$$g^{00} R_{00} = \left[-\frac{1}{a^2} + \frac{2\Psi}{a^2}\right]\left[\nabla^2\Psi - 3\mathring{\mathcal{H}} + 3\mathring{\mathring{\phi}} + 3\mathcal{H}(\dot{\Psi} + \dot{\phi})\right]$$

$$= -a^{-2}\left[\nabla^2\Psi - 3\mathring{\mathcal{H}} + 3\mathring{\mathring{\phi}} + 3\mathcal{H}(\dot{\Psi} + \dot{\phi})\right]$$

$$+ a^{-2}\left[2\Psi\nabla^2\Psi - 6\Psi\mathring{\mathcal{H}} + 6\Psi\mathring{\mathring{\phi}} + 6\mathcal{H}\Psi(\dot{\Psi} + \dot{\phi})\right]$$: SEGUNDO ORDEN

$$\Rightarrow G^0_0 = -a^{-2}\left[\nabla^2\Psi - 3\mathring{\mathcal{H}} + 3\mathring{\mathring{\phi}} + 3\mathcal{H}(\dot{\Psi} + \dot{\phi}) + 6\mathring{\mathcal{H}}\Psi\right]$$
$$- 3a^{-2}(\mathring{\mathcal{H}} + \mathcal{H}^2) - a^{-2}\left(\nabla^2(2\phi - \Psi) - 3\mathring{\mathring{\phi}} - 3\mathcal{H}(\dot{\Psi} + 3\dot{\phi}) - 6\Psi(\mathring{\mathcal{H}} + \mathcal{H}^2)\right)$$

$$\Rightarrow G^0_0 = -\frac{3}{a^2}\mathcal{H}^2 + \frac{2}{a^2}\nabla^2\phi + \frac{6}{a^2}\mathring{\phi}\mathcal{H} + \frac{6}{a^2}\mathcal{H}^2\Psi$$

$$G^0_i = g^{0\alpha} G_{\alpha i} = g^{0\alpha}\left[R_{\alpha i} - \frac{1}{2} g_{\alpha i} R\right] = g^{0\alpha} R_{\alpha i} - \frac{1}{2}\overbrace{g^{0\alpha} g_{\alpha i}}^{\delta^0_i = 0} R$$

$$\Rightarrow G^0_i = g^{0\alpha} R_{\alpha i} \overset{g^{0\alpha} = 0 \text{ si } \alpha \neq 0}{=} g^{00} R_{0i}$$

$$= \left[-\frac{1}{a^2} + \frac{2\Psi}{a^2}\right]\left[2\partial_i(\dot{\phi} + \mathcal{H}\Psi)\right]$$

$$= -\frac{2}{a^2}\partial_i(\dot{\phi} + \mathcal{H}\Psi) + \frac{4\Psi}{a^2}\partial_i(\dot{\phi} + \mathcal{H}\Psi);$$: SEGUNDO ORDEN

$$\Rightarrow G^0_i = -\frac{2}{a^2}\partial_i(\dot{\phi} + \mathcal{H}\Psi)$$

$$G^i_j = g^{i\alpha} G_{\alpha j} \overset{g^{i\alpha} = 0 \text{ si } \alpha \neq i}{=} g^{ii} G_{ij} = g^{ii}\left[R_{ij} - \frac{1}{2} g_{ij} R\right] = g^{ii} R_{ij} - \frac{1}{2}\overbrace{g^{ii} g_{ij}}^{\delta^i_j} R$$

$$\Rightarrow G^i_j = \left[\left[\frac{1}{a^2} + \frac{2\phi}{a^2}\right] \overbrace{\delta^{ii}(\delta_{ij}}^{\delta^i_j}\left[\nabla^2\phi - \mathring{\mathring{\phi}} + \mathring{\mathcal{H}} - \mathcal{H}(\dot{\Psi} + 3\dot{\phi}) + 2\mathcal{H}^2 - (2\mathring{\mathcal{H}} + 4\mathcal{H}^2)(\Psi + \phi)\right]\right.$$
$$\left. + \partial_i\partial_j(\phi - \Psi))\right]$$
$$- \frac{1}{2}\left[\delta^i_j\left(6a^{-2}(\mathring{\mathcal{H}} + \mathcal{H}^2) + a^{-2}(2\nabla^2(2\phi - \Psi) - 6\mathring{\mathring{\phi}} - 6\mathcal{H}(\dot{\Psi} + 3\dot{\phi}) - 12(\mathcal{H}^2 + \mathring{\mathcal{H}})\Psi\right)\right]$$

$$\Rightarrow G^i_j = \Big[\delta^i_j \big[a^{-2}\big(\nabla^2\phi - \ddot{\phi} + \dot{\mathcal{H}} - \mathcal{H}(\dot{\psi} + 3\dot{\phi}) + 2\mathcal{H}^2 - (2\dot{\mathcal{H}} + 4\mathcal{H}^2)(\psi + \phi)\big)$$

$$+ \frac{2}{a^2}\big(\phi\nabla^2\phi - \phi\ddot{\phi} + \ddot{\mathcal{H}}\phi - \mathcal{H}\phi(\dot{\psi} + 3\dot{\phi}) + 2\mathcal{H}^2\phi - (2\dot{\mathcal{H}} + 4\mathcal{H}^2)(\psi + \phi)\phi\big)\big]$$

$$+ a^{-2}\overbrace{\delta^{ii}\partial_i}^{\partial^i}\partial_j(\phi - \psi) + \frac{2\phi\delta^{ii}\partial_i\partial_j(\phi - \psi)}{a^2}\Big]$$

: SEGUNDO ORDEN

$$- \Big[\delta^i_j\big(3a^{-2}(\dot{\mathcal{H}} + \mathcal{H}^2) + a^{-2}(\nabla^2(2\phi - \psi) - 3\ddot{\phi} - 3\mathcal{H}(\dot{\psi} + 3\dot{\phi}) - 6(\mathcal{H}^2 + \dot{\mathcal{H}})\psi\big)\Big]$$

$$\Rightarrow G^i_j = \Big[\delta^i_j\big[a^{-2}\big(\nabla^2\phi - \ddot{\phi} + \dot{\mathcal{H}} - \mathcal{H}(\dot{\psi} + 3\dot{\phi}) + 2\mathcal{H}^2 - (2\dot{\mathcal{H}} + 4\mathcal{H}^2)(\psi + \phi)\big] + \partial^i\partial_j(\phi - \psi)$$

$$+ 2\phi(\dot{\mathcal{H}} + 2\mathcal{H}^2)a^{-2}\delta^i_j\Big]$$

$$- \Big[\delta^i_j\big[3a^{-2}(\dot{\mathcal{H}} + \mathcal{H}^2) + a^{-2}(\nabla^2(2\phi - \psi) - 3\ddot{\phi} - 3\mathcal{H}(\dot{\psi} + 3\dot{\phi}) - 6(\mathcal{H}^2 + \dot{\mathcal{H}})\psi)\big]\Big]$$

$$\Rightarrow G^i_j = \delta^i_j a^{-2}(2\dot{\mathcal{H}} - \mathcal{H}^2) + a^{-2}\delta^i_j\big(2\ddot{\phi} - \nabla^2(\phi + \psi) - 2\mathcal{H}\dot{\psi} + (4\dot{\mathcal{H}} + 2\mathcal{H}^2)\psi\big)$$

$$+ a^{-2}\partial^i\partial_j(\phi - \psi)$$

Finalmente, podemos escribir las ecuaciones de Einstein:

$$\underset{\substack{\downarrow \\ \text{Agregar una constante no altera la ecuación.}}}{G^\mu_\nu = \kappa T^\mu_\nu} \Rightarrow [\bar{G}^\mu_\nu + \delta G^\mu_\nu] = \kappa[\bar{T}^\mu_\nu + \delta T^\mu_\nu]$$

Términos de background → Términos sin potenciales

Perturbaciones (potenciales)

Por lo tanto, a background se recuperan las ecuaciones de Friedmann, luego:

$$\underset{\substack{\downarrow \\ \text{Friedmann}}}{\bar{G}^\mu_\nu = \kappa\bar{T}^\mu_\nu} \quad \wedge \quad \underset{\substack{\downarrow \\ \text{Perturbado E.Ea}}}{\delta G^\mu_\nu = \kappa\delta T^\mu_\nu}$$

$$\Rightarrow \delta G^0_0 = \kappa\delta T^0_0$$

$$\Rightarrow 2a^{-2}[\nabla^2\phi + 3\mathcal{H}(\dot{\phi} + \mathcal{H}\psi)] = \kappa\delta T^0_0$$

$$\Rightarrow \nabla^2\phi + 3\mathcal{H}(\dot{\phi} + \mathcal{H}\psi) = \frac{a^2}{2}\kappa\delta T^0_0$$

$$\delta G^0_i = -\frac{2}{a^2}\partial_i(\dot{\phi} + \mathcal{H}\psi) = \kappa\delta T^0_i \Rightarrow \partial_i(\dot{\phi} + \mathcal{H}\psi) = -\frac{a^2}{2}\kappa\delta T^0_i$$

$$\delta G^i_j = a^{-2}\big[\delta^i_j(2\ddot{\phi} - \nabla^2(\phi + \psi) - 2\mathcal{H}\dot{\psi} + (4\dot{\mathcal{H}} + 2\mathcal{H}^2)\psi) + \partial^i\partial_j(\phi - \psi)\big] = \kappa\delta T^i_j$$

$$\left.\begin{aligned} &\Rightarrow \ddot{\phi} - \mathcal{H}\dot{\psi} + (2\dot{\mathcal{H}} + \mathcal{H}^2)\psi + \frac{1}{3}\nabla^2(\phi - \psi) = \frac{a^2}{2}\kappa\delta T^i_j \\ &\quad -\Big(\partial^i\partial_j + \frac{1}{3}\delta^i_j\nabla^2\Big)(\phi - \psi) = 0 \end{aligned}\right\}$$

Se separa la ecuación entre la parte con y sin traza

P2) $T_{\mu\nu} = (\rho+P) u_\mu u_\nu + P g_{\mu\nu}$

$\Rightarrow T^\mu_\nu = g^{\mu\alpha} T_{\alpha\nu} = g^{\mu\alpha}[(\rho+P) u_\alpha u_\nu + P g_{\alpha\nu}]$

$\Rightarrow T^\mu_\nu = (\rho+P) g^{\mu\alpha} u_\alpha u_\nu + P g^{\mu\alpha} g_{\alpha\nu}$

$\Rightarrow T^\mu_\nu = (\rho+P) u^\mu u_\nu + P g^\mu_\nu$

$\nabla_\mu T^\mu_\nu = \partial_\mu T^\mu_\nu + \Gamma^\mu_{\alpha\mu} T^\alpha_\nu - \Gamma^\alpha_{\nu\mu} T^\mu_\alpha$; $\nabla_\mu T^\mu_\nu = 0$

$u^\mu = \frac{dx^\mu}{ds} = \left(\frac{1}{a(1+\Psi)}, \frac{dx^i}{a d\tau} \right) \approx \left(\frac{1}{a}(1-\Psi), \frac{v^i}{a} \right)$

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$u_\mu = g_{\mu\nu} u^\nu = \left(g_{00} \frac{1}{a}(1-\Psi), g_{ij} \frac{v^j}{a} \right) = (-a(1+\Psi), a v_i)$

$u_\mu u^\mu = -1$

Perturbando T^μ_ν :

$\delta T^\mu_\nu = \delta((\rho+P) u^\mu u_\nu) + \delta(P g^\mu_\nu)$

$= \delta(\rho+P) u^\mu u_\nu + (\rho+P)\delta(u^\mu u_\nu) + \delta P g^\mu_\nu + P \delta g^\mu_\nu$

$= (\delta\rho + \delta P) u^\mu u_\nu + (\rho+P)(\delta u^\mu u_\nu + u^\mu \delta u_\nu) + \delta P g^\mu_\nu + P\delta g^\mu_\nu$

$\Rightarrow \nabla_\mu T^\mu_\nu = \nabla_\mu (\bar{T}^\mu_\nu + \delta T^\mu_\nu)$

$\Rightarrow \delta T^\mu_{\nu;\mu} = \delta T^\mu_{\nu,\mu} - \delta\Gamma^\alpha_{\nu\beta} T^\beta_\alpha - \Gamma^\alpha_{\nu\beta} \delta T^\beta_\alpha + \delta \Gamma^\beta_{\alpha\beta} T^\alpha_\nu + \Gamma^\beta_{\alpha\beta} \delta T^\alpha_\nu = 0$

En donde sabemos que los términos a background cumplen $\nabla_\mu \bar{T}^\mu_\nu = 0$, por lo que los ignoramos

$\Rightarrow \delta T^\mu_{0;\mu} = \delta T^\mu_{0,\mu} - \delta\Gamma^\alpha_{0\beta} T^\beta_\alpha - \Gamma^\alpha_{0\beta} \delta T^\beta_\alpha + \delta\Gamma^\beta_{\alpha\beta} T^\alpha_0 + \Gamma^\beta_{\alpha\beta} \delta T^\alpha_0$

P3)
$$\frac{\partial f}{\partial t} + \frac{\vec{p}}{ma^2}\cdot\frac{\partial f}{\partial \vec{x}} - m\nabla\phi\cdot\frac{\partial f}{\partial \vec{p}} = 0$$

$$/\cdot \frac{m}{a^3}\int\frac{d^3p}{(2\pi)^3}$$

$$\Rightarrow \frac{m}{a^3}\int\frac{d^3p}{(2\pi)^3}\left[\frac{\partial f}{\partial t} + \frac{\vec{p}}{ma^2}\cdot\frac{\partial f}{\partial \vec{x}} - m\nabla\phi\cdot\frac{\partial f}{\partial \vec{p}}\right] = 0$$

$$\Rightarrow \int\frac{d^3p}{(2\pi)^3}\left[\frac{m}{a^3}\frac{\partial f}{\partial t} + \frac{\vec{p}}{a^5}\cdot\frac{\partial f}{\partial \vec{x}} - \frac{m^2}{a^3}\nabla\phi\cdot\frac{\partial f}{\partial \vec{p}}\right] = 0$$

$$\Leftrightarrow \frac{m}{a^3}\int\frac{d^3p}{(2\pi)^3}\frac{\partial f}{\partial t} + \int\frac{d^3p}{(2\pi)^3}\frac{\vec{p}}{a^5}\cdot\frac{\partial f}{\partial \vec{x}} - \frac{m^2}{a^3}\int\frac{d^3p}{(2\pi)^3}\nabla\phi\cdot\frac{\partial f}{\partial \vec{p}} = 0$$

$$-\int\frac{d^3p}{(2\pi)^3}\frac{\partial f}{\partial t} = \frac{\partial}{\partial t}\int\frac{d^3p}{(2\pi)^3}f$$

$$-\int\frac{d^3p}{(2\pi)^3}\frac{\vec{p}}{a^5}\frac{\partial f}{\partial \vec{x}} = \int\frac{d^3p}{(2\pi)^3}\frac{p^i}{a^5}\frac{\partial f}{\partial x^i} = \frac{1}{a^5}\frac{\partial}{\partial x^i}\int\frac{d^3p}{(2\pi)^3}p^i f$$

$$\Rightarrow \underbrace{\frac{\partial}{\partial t}\left[\frac{m}{a^3}\int\frac{d^3p}{(2\pi)^3}f\right]}_{\rho(\vec{x},t)} + \frac{1}{a}\frac{\partial}{\partial x^i}\underbrace{\left[\frac{1}{a^4}\int\frac{d^3p}{(2\pi)^3}p^i f\right]}_{\pi^i(\vec{x},t)} - \frac{m^2}{a^3}\int\frac{d^3p}{(2\pi)^3}\nabla\phi\cdot\frac{\partial f}{\partial \vec{p}} = 0$$

$$\Rightarrow \frac{\partial\rho}{\partial t} + \frac{1}{a}\frac{\partial}{\partial x^i}\pi^i - \frac{m^2}{a^3}\int\frac{d^3p}{(2\pi)^3}\nabla\phi\cdot\frac{\partial f}{\partial \vec{p}} = 0$$

$$\int\frac{d^3p}{(2\pi)^3}\nabla\phi\cdot\frac{\partial f}{\partial \vec{p}} = \nabla\phi\cdot\int\frac{d^3p}{(2\pi)^3}\frac{\partial f}{\partial \vec{p}} = \nabla\phi\cdot\underbrace{\vec{k}}_{\text{CONSTANTE}}\, f\Big|_0^\infty$$

→ PODEMOS ASUMIR QUE LA DISTRIBUCIÓN DECAE CON $p\to\infty$

$$\Rightarrow \int\frac{d^3p}{(2\pi)^3}\nabla\phi\frac{\partial f}{\partial \vec{p}} = 0$$

$$\Rightarrow \frac{\partial\rho}{\partial t} + \frac{1}{a}\partial_i\pi^i = 0$$ Ecuación de Continuidad

$$\frac{\partial f}{\partial t} + \frac{\vec{p}}{ma^2}\cdot\frac{\partial f}{\partial \vec{x}} - m\nabla\phi\cdot\frac{\partial f}{\partial \vec{p}} = 0 \quad \Big/ \; \frac{1}{a^4}\int\frac{d^3p}{(2\pi)^3}\,p^i$$

$$\Longrightarrow \frac{1}{a^4}\int\frac{d^3p}{(2\pi)^3}\,p^i\left[\frac{\partial f}{\partial t} + \frac{\vec{p}}{ma^2}\cdot\frac{\partial f}{\partial \vec{x}} - m\nabla\phi\cdot\frac{\partial f}{\partial \vec{p}}\right] = 0$$

$$\Longrightarrow \frac{1}{a^4}\int\frac{d^3p}{(2\pi)^3}\,p^i\,\frac{\partial f}{\partial t} = \frac{\partial}{\partial t}\left[\frac{1}{a^4}\int\frac{d^3p\,p^i}{(2\pi)^3}\,f\right] = \frac{\partial}{\partial t}\pi^i$$

$$\frac{1}{a^4}\int\frac{d^3p}{(2\pi)^3}\,p^i\,\frac{\vec{p}}{ma^2}\cdot\frac{\partial f}{\partial \vec{x}} = \frac{1}{a}\frac{1}{ma^5}\int\frac{d^3p}{(2\pi)^3}\,p^i p^j\cdot\frac{\partial f}{\partial x^j} = \frac{1}{a}\frac{\partial}{\partial x^j}\left[\frac{1}{ma^5}\int\frac{d^3p}{(2\pi)^3}\,p^i p^j f\right]$$

$$= \frac{1}{a}\partial_j\sigma^{ij}$$

$$\frac{1}{a^4}\int\frac{d^3p}{(2\pi)^3}\,p^i\,m\nabla\phi\cdot\frac{\partial f}{\partial \vec{p}} = \frac{\nabla\phi}{a^4}\cdot m\underbrace{\int\frac{d^3p}{(2\pi)^3}\,\overset{u}{p^i}\,\overset{dv}{\frac{\partial f}{\partial \vec{p}}}}_{\int u\,dv \,=\, uv - \int v\,du,\quad du\to 1}$$

$f \to 0$ si $p \to \infty \Rightarrow uv = 0$

$$\Longrightarrow \frac{\nabla\phi}{a}\cdot\frac{m}{a^3}\int\frac{d^3p}{(2\pi)^3}\,f = \frac{\nabla\phi}{a}\cdot\rho(x,t)$$

$\partial^i = \nabla$

$$\Longrightarrow \frac{\partial\pi^i}{\partial t} + \frac{1}{a}\partial_j\sigma^{ij} - \frac{\rho}{a}\partial^i\phi = 0$$ Conservation of momentum