EECS 475, Winter 2018
Introduction to Cryptography

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Principles and Practice (Seventh Edition)
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1 Homework 01

Problem 3.1

A generalization of the Caesar cipher, known as the affine Caesar cipher, has the following form: For each plaintext letter p, substitute the ciphertext letter C:

$$C = E([a, b], p) = (ap + b) \bmod 26$$

A basic requirement of any encryption algorithm is that it be one-to-one. That is, if $p \neq q$, then $E(k, p) \neq E(k, q)$. Otherwise, decryption is impossible, because more than one plaintext character maps into the same ciphertext character. The affine Caesar cipher is not one-to-one for all values of a. For example, for $a = 2$ and $b = 3$, then $E([a, b], 0) = E([a, b], 13) = 3$.

a. Are there any limitations on the value of b? Explain why or why not. [3 points]

Solution: Addition by b represents a shift substitution. Therefore, impose $b \in \mathbb{Z}_{26}$ so each shift is unique.

b. Determine which values of a are not allowed. [3 points]

Solution: The decryption algorithm is easy to construct.

$$D([a, b], C) := a^{-1}(C - b) \bmod 26 = p$$

For $a^{-1} \bmod 26$ to exists, we require $\gcd(a, 26) = 1$. Hence, a cannot be even or 13.

Problem 3.8

A disadvantage of the general monoalphabetic cipher is that both sender and receiver must commit the permuted cipher sequence to memory. A common technnique for avoiding this is to use a keyword from which the cipher sequence can be generated. For example, using the keyword CRYPTO, write out the keyword followed by unused letters in normal order and match this against the plaintext letters:

```
plain:  a b c d e f g h i j k l m n o p q r s t u v w x y z
CIPHER: C R Y P T O A B D E F G H I J K L M N Q S U V W X Z
```

If it is felt that this process does not produce sufficient mixing, write the remaining letters on successive lines and then generate the sequence by reading down the columns:

```
C R Y P T O
A B D E F G
H I J K L M
N Q S U V W
X Z
```

This yields the sequence:

```
C A H N X R B I Q Z Y D J S P E K U T F L V O G M W
```

Such a system is used in the example in Section 3.2 (the one that begins ”it was disclosed yesterday”). Determine the keyword. [3 points]

Solution:

From the decrypt, we know the correspondence below:

```
plain:  a b c d e f g h i j k l m n o p q r s t u v w x y z
CIPHER: S A H V P B J W U ? ? X T D M Y ? E O Z I F Q ? G ?
```

The keyword must have a length of 6 so that `A` and `B` line up horizontally.

```
S P U T ? I ?                 S P U T N I K
A B ? D E F G                 A B C D E F G
H J ? M O Q ?        ⇒        H J L M O Q R
V W X Y Z                     V W X Y Z
```

With little effort, we see the keyword is `SPUTNIK`.

Problem 3.9

When the PT-109 American patrol boat, under the command of Lieutenant John F. Kennedy, was sunk by a Japanese destroyer, a message was received at an Australian wireless station in Playfair code:

```
KXJEY UREBE ZWEHE WRYTU HEYFS
KREHE GOYFI WTTTU OLKSY CAJPO
BOTEI ZONTX BYBNT GONEY CUZWR
GDSON SXBOU YWRHE BAAHY USEDQ
```

The key used was *royal new zealand navy*. Decrypt the message. Translate `TT` into tt. [3points]

Solution:

Begin by creating the 5 × 5 Playfair matrix from the known keyword.

R	O	Y	A	L
N	E	W	Z	D
V	B	C	F	G
H	I/J	K	M	P
Q	S	T	U	X

Then, we can break up the ciphertext into digrams and decrypt each by applying the algorithm in reverse.

```
CIPHER: KX JE YU RE BE ZW E H  EW RY TU  H E YF S K  RE  H E GO YF IW TT TU OL  K S YC AJ
plain:  pt bo at on eo we ni/j ne lo st i/jn ac ti/j on i/jn bl ac ke tt st ra i/jt tw om

CIPHER: P O  BO TE IZ ON TX BY BN TG ON EY CU ZW RG DS ON SX BO UY WR  H E BA AH YU  S E DQ
plain:  i/jl es sw me re su co ve xc re wo ft we lv ex re qu es ta ny i/jn fo rm at i/jo nx
```

When rearranging the cipher, we can ignore extra x's to get the plaintext:

```
pt boat one owe nine lost in action in blackett strait two miles
sw meresu cove crew of twelve request any information
```

Problem 3.11

a. Using this Playfair matrix:

M	F	H	I/J	K
U	N	O	P	Q
Z	V	W	X	Y
E	L	A	R	G
D	S	T	B	C

Encrypt this message:

Must see you over Cadogan West. Coming at once.

Note: This message is from the Sherlock Holmes story, The Adventure of the Bruce-Partingon plans. [3 points]

Solution: Split the plaintext into digrams and add pad with an ’x’ to encrypt as shown below.

```
plain:  mu st se ey ou ov er ca do ga nw es tc om in ga to nc ex
CIPHER: UZ TB DL GZ PN NW LG TG TU ER VO LD BD UH FP ER HW QS RZ
```

b. Repeat part (a) using the keyword *largest*. [3 points]

Solution:

Using the keyword *largest*, we construct the Playfair matrix below and encrypt.

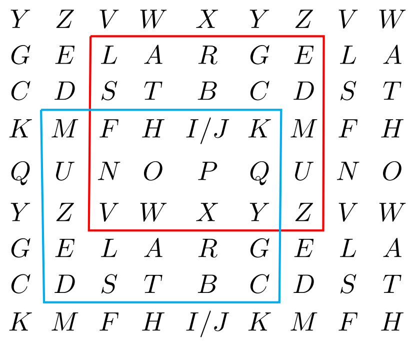
L	A	R	G	E
S	T	B	C	D
F	H	I/J	K	M
N	O	P	Q	U
V	W	X	Y	Z

```
CIPHER: UZ TB DL GZ PN NW LG TG TU ER OV DL BD UH PF ER HW QS RZ
```

c. How do you account for the results of this problem? Can you generalize your conclusion? [3 points]

Solution:

The Playfair matrix can be thought of as a torus $\mathcal{T}$ by folding and gluing the vertical edges to each other and the same with the horizontal edges. The figure to the right shows the two Playfair matrices from parts **a** and **b**, outlined in red and blue. Notice that they both generate $\mathcal{T}$. It makes sense that our ciphertexts for parts **a** and **b** were the same. If we wanted different ciphertexts, we'd need to swap rows or columns of the Playfair matrix. This wouldn't be equivalent to any shift and thus providing a new ciphertext.



Y	Z	V	W	X	Y	Z	V	W
G	E	L	A	R	G	E	L	A
C	D	S	T	B	C	D	S	T
K	M	F	H	I/J	K	M	F	H
Q	U	N	O	P	Q	U	N	O
Y	Z	V	W	X	Y	Z	V	W
G	E	L	A	R	G	E	L	A
C	D	S	T	B	C	D	S	T
K	M	F	H	I/J	K	M	F	H

Problem 3.14

a. Encrypt the message "meet me at the usual place at ten rather than eight o clock" using the Hill cipher with the key $\begin{bmatrix} 7 & 3 \\ 2 & 5 \end{bmatrix}$. Show your calculations and the result. [3 points]

Solution:

Since the Hill cipher matrix $K \in \mathbb{Z}^{2\times 2}$, we'll rewrite the plaintext into digrams and convert to numeric. We'll also pad the string with 'x' at the end to achieve an even length.

```
plain: me et me at th eu su al pl ac ea tt en ra th er th an ei gh to cl oc kx
```

This gives us the following list of vectors p_i for $1 \leq i \leq 24$:

$$(12,4),(4,19),(12,4),(0,19),(19,7),(4,20),(18,20),(0,11),(15,11),(0,2),(4,0),(19,19),$$
$$(4,13),(17,0),(19,7),(4,17),(19,7),(0,13),(4,8),(6,7),(19,14),(2,11),(14,2),(10,23).$$

To encrypt, let p_i be the i^{th} row of matrix P so that

$$\begin{aligned} PK &= \begin{bmatrix} 12 & 4 & 12 & 0 & 19 & \dots & 14 & 10 \\ 4 & 19 & 4 & 19 & 7 & \dots & 2 & 23 \end{bmatrix}^T \begin{bmatrix} 7 & 3 \\ 2 & 5 \end{bmatrix} \bmod 26 \\ &\equiv \begin{bmatrix} 14 & 14 & 14 & 12 & 17 & \dots & 24 & 12 \\ 4 & 3 & 4 & 17 & 14 & \dots & 0 & 15 \end{bmatrix}^T = C. \end{aligned}$$

We now convert each row c_i of C to alphabet characters to get the ciphertext below.

```
CIPHER: OE OD OE MR QI KY WD XW EK CM PW CZ PZ RO AN SA EB FX KJ YA MP
```

b. Show the calculations for the corresponding decryption of the ciphertext to recover the original plaintext. [3 points]

Solution:

For decryption, compute $\det(K) = 29 \equiv 3 \bmod 26$. Since $9 = 3^{-1} \bmod 26$, we have

$$K^{-1} = 9\begin{bmatrix} 5 & -3 \\ -2 & 7 \end{bmatrix} \equiv \begin{bmatrix} 19 & 25 \\ 8 & 11 \end{bmatrix} \bmod 26.$$

To obtain the plaintext, we simply take the product

$$\begin{aligned} CK^{-1} &= \begin{bmatrix} 14 & 14 & 14 & 12 & 17 & \dots & 24 & 12 \\ 4 & 3 & 4 & 17 & 14 & \dots & 0 & 15 \end{bmatrix}^T \begin{bmatrix} 19 & 25 \\ 8 & 11 \end{bmatrix} \bmod 26 \\ &\equiv \begin{bmatrix} 12 & 4 & 12 & 0 & 19 & \dots & 14 & 10 \\ 4 & 19 & 4 & 19 & 7 & \dots & 2 & 23 \end{bmatrix} = P. \end{aligned}$$

This recovers plaintext matrix P from part **a**.

Problem 3.18

b. Determine the inverse mod 26 of

$$\begin{bmatrix} 6 & 24 & 1 \\ 13 & 16 & 10 \\ 20 & 17 & 15 \end{bmatrix}. \qquad \text{[3 points]}$$

Solution:

Denote matrix A. To compute A^{-1} mod 26, we first need the determinant.

$$\det(A) = 441 \equiv 25 \bmod 26$$

We can then find the cofactors of $A^T = \begin{bmatrix} 6 & 13 & 20 \\ 24 & 16 & 17 \\ 1 & 10 & 15 \end{bmatrix}$ as shown below.

$C_{1,1} = (-1)^2(16 \cdot 15 - 10 \cdot 17)$ $\quad C_{1,2} = (-1)^3(24 \cdot 15 - 1 \cdot 17)$ $\quad C_{1,3} = (-1)^4(24 \cdot 10 - 1 \cdot 16)$
$C_{2,1} = (-1)^3(13 \cdot 15 - 10 \cdot 20)$ $\quad C_{2,2} = (-1)^4(6 \cdot 15 - 1 \cdot 20)$ $\quad C_{2,3} = (-1)^5(6 \cdot 10 - 1 \cdot 13)$
$C_{3,1} = (-1)^4(13 \cdot 17 - \cdot 16 \cdot 20)$ $\quad C_{3,2} = (-1)^5(6 \cdot 17 - 24 \cdot 20)$ $\quad C_{3,3} = (-1)^6(6 \cdot 16 - 24 \cdot 13)$

This allows us to create the adjoint matrix

$$\text{adj}(A) = \begin{bmatrix} 70 & -343 & 224 \\ 5 & 70 & -47 \\ -99 & 378 & -216 \end{bmatrix}.$$

Lastly, notice that $25 \cdot 25 \equiv 1 \bmod 26$ so we take the product

$$(\det(A))^{-1} \cdot \text{adj}(A) = 25 \cdot \text{adj}(A) \equiv \begin{bmatrix} 8 & 5 & 10 \\ 21 & 8 & 21 \\ 21 & 12 & 8 \end{bmatrix} \bmod 26 = A^{-1}.$$

Problem 3.19

Using the Vigenère cipher, encrypt the word ”explanation” using the word ”leg”. [3 points]

Solution:

Begin by repeating the keyword over the plaintext as shown below.

```
key:    legleglegle
plain:  explanation
```

We can then perform the required shifts by taking the sum mod 26.

key	11	4	6	11	4	6	11	4	6	11	4
plain	4	23	15	11	0	13	0	19	8	14	13
CIPHER	15	1	21	22	4	19	11	23	14	25	17

Encryption is completed by writing the numerical values into an alphabetic ciphertext.

```
CIPHER: PBVWETLXOZR
```

Problem A

A precursor to the ADFGVX cipher was the ADFGX cipher which used a table such as this:

	A	D	F	G	X
A	b	t	a	l	p
D	d	h	o	z	k
F	q	f	v	s	n
G	g	i/j	c	u	x
X	m	r	e	w	y

Encrypt the phrase "neither do they spin" below. Use the grid on the left below to write the two-letter substitutions row-wise. Then rearrange the columns so that the column headers are in alphabetical order in the grid on the right.

M	**E**	**R**	**I**	**T**

Write the ciphertext by reading out the grid on the right column-wise. [3 points]

Solution:

Using the provided ADFGX table, fill out the two tables below.

M	**E**	**R**	**I**	**T**
F	X	X	F	G
D	A	D	D	D
X	F	X	D	D
A	D	F	A	D
D	D	X	F	X
X	F	G	A	X
G	D	F	X	

$\Rightarrow$

E	**I**	**M**	**R**	**T**
X	F	F	X	G
A	D	D	D	D
F	D	X	X	D
D	A	A	F	D
D	F	D	X	X
F	A	X	G	X
D	X	G	F	

Thus, we have the encryption of the phrase "neither do they spin" below.

```
CIPHER: XAFDDFDFDDAFAXFDXADXGXDXFXGFGDDDXX
```

Problem B

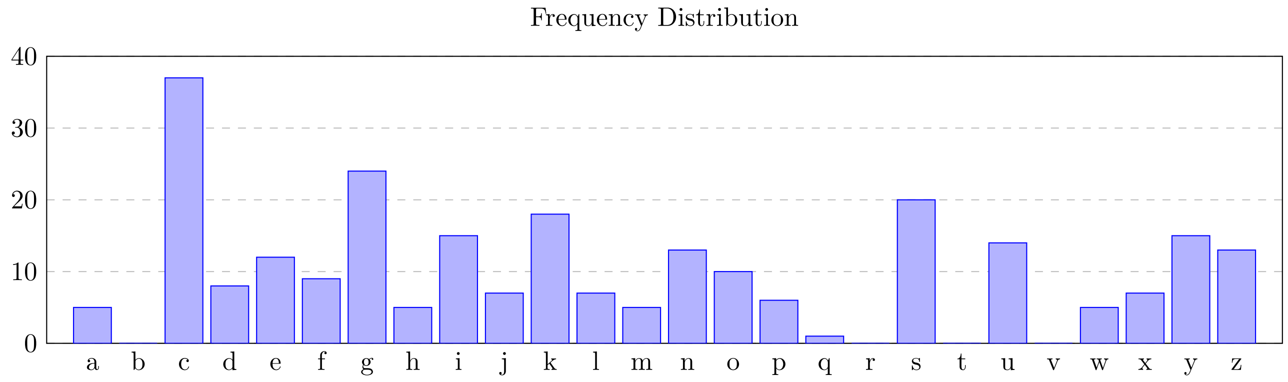
Decrypt the following permutation substitution cipher. [3 points]

```
emglosudcgdncuswysfhnsfcykdpumlwgyicoxysipjckqpkugkmgolicgincgacksnisacykzsckxecjckshy
sxcgoidpkzcnkshicgiwygkkgkgoldsilkgoiusigledspwzugfzccndgyysfuszcnxeojncgyeoweupxezgac
gnfglknsacigoiyckxcjuciuzcfzccndgyysfeuekuzcsocfzccnciaczejncshfzejzegmxcyhcjumgkucy
```

Solution:

Begin by analyzing the frequency distribution using C.

```c
while (fread(&buffer, sizeof(char), 1, in) == 1)
{
    if (isalpha(buffer) == 0) { continue; }
    freq[buffer-97]++;
}

for (int i = 0; i < ALPHABET; i++)
    { printf("%c %d\n", i+97, freq[i]); }
```



The frequency distribution is very close to that of a monoalphabetic cipher. We can begin to take guesses for certain letters based on the distribution. Notice, there is an 11-letter word repeated twice with a 5-letter prefix.

```
CIPHER: EMGLOSUDCGDNCUSWYSFHNSFCYKDPUMLWGYICOXYSIPJCKQPKUGKMGOLICGINCGACKSNI
plain:          e   e     t   te           e       e            e   e  e

CIPHER: SACYKZSCKXECJCKSHYSXCGOIDPKZCNKSHICGIWYGKKGKGOLDSILKGOIUSIGLEDSPWZUG
plain:    e  h e   e e      e      he     e                              h

CIPHER: FZCCNDGYYSF  USZCNXEOJNCGYEOWEUPXEZGACGNFGLKNSACIGOIYCKXCJUCIUZC
plain:  thee      t    he      e          h  e  t      e     e  e  e  he

CIPHER: FZCCNDGYYSF  EUEKUZCSOC  FZCCN  CIACZEJNCSHFZEJZEGMXCYHCJUMGKUCY
plain:  thee      t       he  e  thee   e  eh   e  th  h    e  e      e
```

The prefix thee suggests that F probably does not correspond to t. Consider the word wheel instead and use this to begin making some assertions and decrypt as shown below.

```
i may not be able to grow flowers but my garden produces just as many dead leave sold
over shoes pieces of rope and bushels of dead grass as any bodys and today i bought a
wheelbarrow to help in clearing it up i have always loved and respected the
wheelbarrow it is the one wheeled vehicle of which i am perfect master
```

Problem C

Decrypt the following Vigenère ciphertext using the Index of Coincidence method. [3 points]

```
hlfiusvsaqfpfpwaryxewdudwbrvxvrthapcjlhrlalbkwfeecmlpfvuyimxqpiovczogidthjgdhrlifyxkwhuigkull
qqqhltvyzckbseelxrpikavbrxmysirovgipszwxpiwyauszeuiqhglxesombmhuepeyplvxoethjysytwfydbxndutim
vhtzjzzazwiigktqzqpsfpeijyuklfrgnpfeckohuevmwnoiihvogamphbdseiymsogvasyfzvthckoueifegzoqbvrmh
yysqembrvxvnecpirnmihwttwtmaooirmyoqbzylszwqembrvxvnqcklalsxpkthswzhnmewtfvxohsbrlmycwfrchjkt
htifrhhyxpmxvrorbrlthdcdghxvjlllnsiekckfhbzoyifijeuklsfpxgnlfigkuegiapljgvlphwlpnpcquagyuayop
oadhjrpneihvtkivfcomgnsmvtyajwfnlivnywfxzrthtwppbvhzeyvtxxbtwoekeypfvahrpimsrckbcghxwycybboad
fiysiaqqnlecpyizswagiitafbavntlskqnembvavgtfhqqhuizlytgbbctemxtdyxigfohrfdczibghbwndgvaiosmmy
fnbncepbwyzfxvroaepbtneiduiesxzjeqqnlyptflfavpamsysllleguifwjwzrxcahbwxhiolwdubiywsqiyrthxmpme
qdghxvjtmkwhuigkxflmzwfigknyneqgvfmljjvrbyaepmylfjwggaeprphfvhuebvipaomsfofiytgbwfbtaiwnbbzwf
hoiwjhbifyymljdujmtreemsrmqwknrwwysylksnnpmysgbbvrrxrthcpgchrbrxffxzqvtrskebbuoahtxyzypjsytxh
wzoklplwaewgypigvnwmfycptsfbrgtcuizsrflgtxgbzqrsnvwzoklgvtpmysbbzghryvnrbqibqlxjyebbaheexxxeu
hmmbupeypltifqimwjinomardhaseitvwftaiglnqmflwaiwpneihaoupjxiimwfwtwmpxygknvxwfyxzwcyewfdmlbmn
rsplnnbxnsjhhywdjomjvonwbplbwigoywnrbqwtyaghqzihihghxgwzqaacswtxjcaxhsesmljcy
```

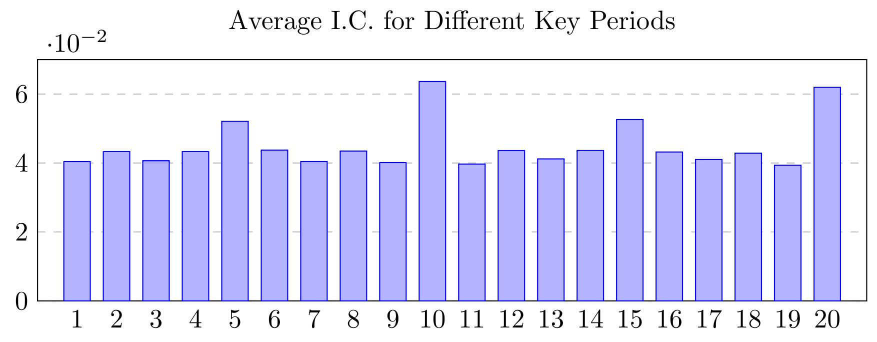
Solution:

We begin by testing keyword periods and analyzing their corresponding Index of Coincidences.

```c
double* getAverageIOCs(char *in_cipher)
{
    double *avgs = malloc(TRIALS*sizeof(double));
    int freq[ALPHA_SIZE] = {0};
    double ioc = 0;

    for (int k = 1; k <= TRIALS; k++)
    {
        for (int gap = 0; gap < k; gap++)
        {
            setFrequency(in_cipher, gap, k, CIPHER_SIZE, freq);
            ioc = getIOC(freq);
            avgs[k-1] += ioc;
        }
        avgs[k-1] /= k;
    }

    return avgs;
}

void setFrequency(char *data, int start, int jump, int end, int *freq)
{
    memset(freq, 0, ALPHA_SIZE*sizeof(int));   // set all vals to 0
    for (int i = start; i < end; i += jump)
        { freq[data[i]-97]++; }
}

double getIOC(int *freq)
{
    double out = 0;   // Index of Coincidence to return
    int length = 0;

    for (int i = 0; i < ALPHA_SIZE; i++) { length += freq[i]; }
    for (int i = 0; i < ALPHA_SIZE; i++)
    {
        out += 1.0*freq[i]*(freq[i] - 1)/(length*(length - 1));
    }

    return out;
}
```

The maximum of `avgs` yields 10 but it's clear from the results below that 20 is also a possible period.

Period	**Avg I.C.**
1	0.0403894
2	0.0433027
⋮	⋮
7	0.0404037
8	0.0434730
9	0.0400959
10	0.0636256
⋮	⋮
20	0.0619492



We choose a period of 10 and try the 26 monoalphabetic ciphers to every 10th letter of the ciphertext to obtain the Chi-Square statistics. To do so, we need the most current frequency distribution according to Peter Norvig in 2012 (`http://norvig.com/mayzner.html`).

```
for (int k = 0; k < key_size; k++)
{
    for (char ltr = 'a'; ltr <= 'z'; ltr++)
    {
        setCaesarShift(cipher, shift, k, key_size, ltr);
        setFrequency(shift, 0, 1, shift_sz, freq);
        chi_vals[ltr-97] = getChiSq(eng_dist, shift_sz, freq);
    }

    keyword[k] = getChiMin(chi_vals);
}
```

```
void setCaesarShift(char *data, char *shifted, int start, int key, char c)
{
    for (int i = start, j = 0; i < CIPHER_SIZE; i += key, j++)
    {
        if (data[i] >= c)
            { shifted[j] = data[i] - c + 'a'; }
        else
            { shifted[j] = data[i] - c + 26 + 'a'; }
    }
}

double getChiSq(double *edist, int size, int *cfreq)
{
    double expect = 0;
    double out    = 0;

    for (int i = 0; i < ALPHA_SIZE; i++)
    {
        expect = size * edist[i];
        out += pow(cfreq[i] - expect, 2) / expect;
    }

    return out;
}
```

Notice that setFrequency was defined in the previous page. The values of shift and chi_vals are shown below.

Key	Caesar Shift	Chi-Sq
a	hfwrlmqdfuyryzzsyynjtjfwmstgynwrznkwlkxljkj...sgwjmfdnwbnqwjj	1139.65
b	gevqklpcetxqxyyrxxmisievlrsfxmvqymjvkjwkiji...rfvilecmvampvii	3627.85
c	fdupjkobdswpwxxqwwlhrhdukqrewlupxliujivjhih...qeuhkdbluzlouhh	876.493
d	ectoijnacrvovwwpvvkgqgctjpqdvktowkhtihuighg...pdtgjcaktykntgg	4924.34
e	dbsnhimzbqunuvvouujfpfbsiopcujsnvjgshgthfgf...ocsfibzjsxjmsff	826.637
f	carmghlyaptmtuunttieoearhnobtirmuifrgfsgefe...nbrehayirwilree	109.124
⋮	⋮	⋮
z	igxsmnregvzszaatzzokukgxntuhzoxsaolxmlymklk...thxkngeoxcorxkk	4619.63

This suggest that the letter f was used to encode every 10th plaintext letter. Hence, it is likely the first letter of the keyword. The full Chi-Square analysis yields the keyword fluxionate.

```c
char* getDecrypt(char *data, char *kword, int size)
{
    char *ptext = malloc(CIPHER_SIZE*sizeof(char));
    int j = 0;

    for (int i = 0; i < CIPHER_SIZE; i++)
    {
        j = i % size;
        if (data[i] >= kword[j])
        { ptext[i] = data[i] - kword[j] + 'a'; }
        else
        { ptext[i] = data[i] - kword[j] + ALPHA_SIZE + 'a'; }
    }

    return ptext;
}
```

The full decrypt is shown below with spacing and punctuation applied.

```
Call me Ishmael. Some years ago never mind how long precisely having little or no
money in my purse, and nothing particular to interest me on shore, I thought I
would sail about a little and see the watery part of the world. It is a way I have
of driving off the spleen and regulating the circulation. Whenever I find myself
growing grim about the mouth; whenever it is a damp, drizzly November in my soul;
whenever I find myself involuntarily pausing before coffin warehouses, and bringing
up the rear of every funeral I meet; and especially whenever my hypos get such an
upper hand of me, that it requires a strong moral principle to prevent me from
deliberately stepping into the street, and methodically knocking peoples hats off
then, I account it high time to get to sea as soon as I can. This is my substitute
for pistol and ball. With a philosophical flourish Cato throws himself upon his
sword; I quietly take to the ship. There is nothing surprising in this. If they
but knew it, almost all men in their degree, some time or other, cherish very
nearly the same feelings towards the ocean with me. There now is your insular city
of the Manhattoes, belted round by wharves as Indian isles by coral reefs commerce
surrounds it with her surf. Right and left, the streets take you waterward. Its
extreme downtown is the battery, where that noble mole is washed by waves, and
cooled by breezes, which a few hours previous were out of sight of land. Look at
the crowds of water gazers there.
```

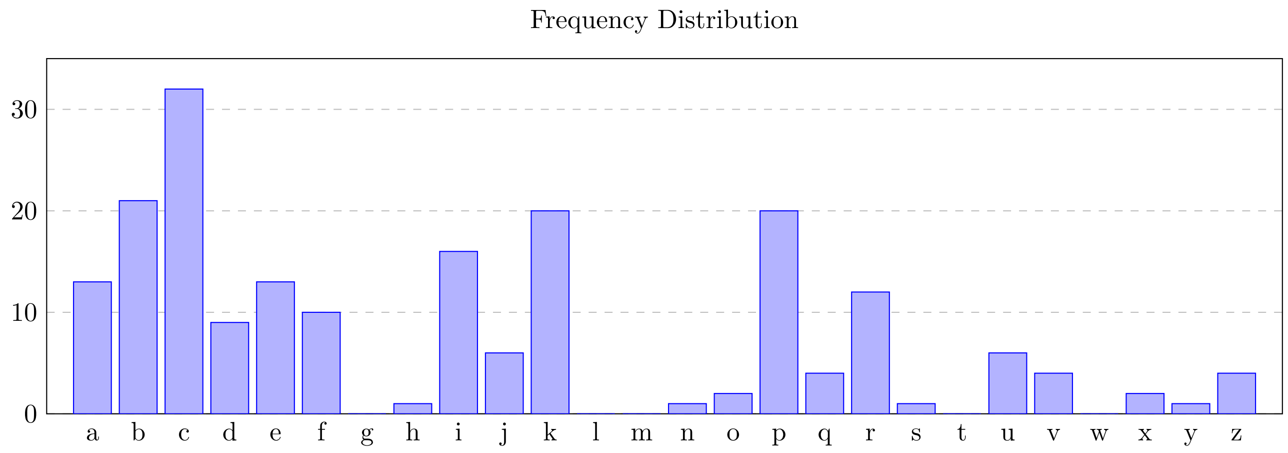
Problem D

Decrypt the following affine cipher. [3 points]

```
kqerejebcppcjcrkieacuzbkrvpkrbcibqcarbjcvfcupkriofkpacuzqepbkrxpei
ieabdkpbcpfcdccafieabdkpbcpfeqpkazbkrhaibkapcciburccdkdccjcidfuixp
afferbiczdfkabicbbenefcupjcvkabpcydccdpkbcocperkivkscpicbrkijpkabi
```

Solution:

Using scripts from previous problems, it's simple to generate our frequency distribution.



From the frequency, it's likely that `C` and `B` are ciphers for `e` and `t` respectively. Assume this is true and solve the system below.

$$\begin{bmatrix} 4 & 1 \\ 19 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \equiv \begin{bmatrix} 2 \\ 1 \end{bmatrix} \bmod 26 \qquad \Rightarrow \qquad \begin{matrix} a = 19 \\ b = 4 \end{matrix}$$

Attempt decrypting by implementing $D([19,4], C) := 11(C-4) \equiv p \bmod 26$ in the script below.

```c
while (fread(&buffer, sizeof(char), 1, in) == 1)
{
    if (isalpha(buffer) == 0) { continue; }

    dummy = (buffer - 97) - 4;
    if (dummy < 0) { dummy += 26; }
    dummy = ((dummy * 11) % 26) + 97;
    printf("%c", dummy);
}

fclose(in);
```

By introducing spacing to the plaintext generated, we see that our guess worked since this yields the Canadian anthem!

```
o canada terre de nos aieux ton front est ceint de fleurons glorieux car ton bras
sait porter lepee il sait porter la croix ton histoire est une epopee des plus
brillants exploits et ta valeur de foi trempee protegera nos foyers et nos droits
```

2 Homework 02

Problem 4.2

Consider a Feistel cipher composed of sixteen rounds with a block length of 128 bits and a key length of 128 bits. Suppose that, for a given k, the key scheduling algorithm determines values for the first eight round keys, $k_1, k_2, \ldots k_8$, and then sets

$$k_9 = k_8, k_{10} = k_7, k_{11} = k_6, \ldots, k_{16} = k_1$$

Suppose you have a ciphertext c. Explain how, with access to an encryption oracle, you can decrypt c and determine m using just a single oracle query. This shows that such a cipher is vulnerable to a chosen plaintext attack. (An encryption oracle can be thought of as a device that, when given a plaintext, returns the corresponding ciphertext. The internal details of the device are not known to you and you cannot break open the device. You can only gain information from the oracle by making queries to it and observing its responses.)

Solution: If we are given a subkey sequence

$$k_1 k_2 \ldots k_8 k_8 \ldots k_2 k_1,$$

the Feistel encryption and decryption algorithms become identical. This is due to the symmetry of the subkey sequence. Thus, with access to an encryption oracle and knowledge of a ciphertext $\mathbf{c}$, we simply query $E(\mathbf{c}) = D(\mathbf{c}) = \mathbf{m}$.

Problem 4.5

For any block cipher, the fact that it is a nonlinear function is crucial to its security. To see this, suppose that we have a linear block cipher EL that encrypts 256-bit blocks of plaintext into 256-bit blocks of ciphertext. Let $EL(k, m)$ denote the encryption of a 256-bit message m under a key k (the actual bit length of k is irrelevant). Thus,

$$EL(k, [m_1 \oplus m_2]) = EL(k, m_1) \oplus EL(k, m_2) \text{ for all 128-bit patterns } m_1, m_2.$$

Describe how, with 256 chosen ciphertexts, an adversary can decrypt any ciphertext without knowledge of the secret key k. (A "chosen ciphertext" means that an adversary has the ability to choose a ciphertext and then obtain its decryption. Here, you have 256 plaintext/ciphertext pairs to work with and you have the ability to choose the value of the ciphertexts.)

Solution: Consider the set of ciphertexts $\mathcal{C}$. Let $c_i \in \{0,1\}^{128}$ with $1 \leq i \leq 128$ be the chosen ciphertexts. Each c_i has one in the i^{th} position, zeros elsewhere and a corresponding plaintext m_i. So

$$EL(k, \mathbf{m}) = EL\left(k, \bigoplus_{i=1}^{n} m_i\right) \overset{linearity}{=\!=\!=} \bigoplus_{i=1}^{n} EL(k, m_i) = \bigoplus_{i=1}^{n} c_i = \mathbf{c}$$

where $1 \leq n \leq 128$ describes encryption. More importantly, $\bigoplus_{i=1}^{n} m_i$ corresponds to $\mathbf{c}$ and the adversary can easily compute $\mathbf{m}$. Note that $EL(k, \mathbf{0}) = \mathbf{0}$.

Problem 4.6

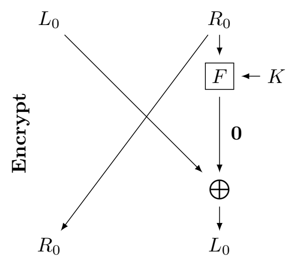
Suppose the DES F function mapped every 32-bit input R, regardless of the value of the input K, to **a** and **b**.

1. What function would DES then compute?
2. What would the decryption look like?

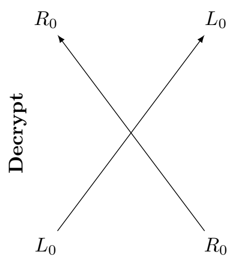
a. 32-bit string of zero

Solution:

For message $\mathbf{M}$, we have $\text{IP}(\mathbf{M}) = L_0 \| R_0$ and observe one round of DES encryption below [left].



Round	L_i	R_i
IP	L_0	R_0
1	R_0	L_0
2	L_0	R_0
16	L_0	R_0



The DES function described is $F(R) = \mathbf{0}$. It's clear from the table above [middle] that $\{L, R\}_0 = \{L, R\}_2 = \{L, R\}_{16}$. To finish encrypting, compute $\text{IP}^{-1}(R_0 \| L_0) = \mathbf{C}$.

For decryption, perform at 32-bit swap on $\text{IP}(\mathbf{C})$ which yields $L_0 \| R_0$. One round of decryption, shown above [right], is simply a swap. As with encryption, 16 rounds yields a circular result. To finish, compute $\text{IP}^{-1}\big(L_0 \| R_0\big) = \mathbf{M}$.

b. R

Solution:

The tables below, show encryption on the left and decryption on the right.

Round	L_i	R_i	Simplified
IP	L_0	R_0	
1	R_0	$L_0 \oplus R_0$	R_1
2	R_1	$R_0 \oplus R_1$	L_0
3	L_0	$R_1 \oplus L_0$	R_0
16	R_0	$L_0 \oplus R_0$	R_1

Round	L_i	Simplified	R_i
16	R_0		R_1
15	$R_0 \oplus R_1$	L_0	R_0
14	$L_0 \oplus R_0$	R_{14}	L_0
13	$R_{14} \oplus L_0$	R_0	R_{14}
12	$R_0 \oplus R_{14}$	L_0	R_0

For encryption, have $\text{IP}(\mathbf{M}) = L_0 \| R_0$ with $F(X) = X$ i.e. the identify function. Notice,

$$R_{i+1} = L_i \oplus F(R_i) = L_i \oplus R_i.$$

Then $\{L, R\}_0 = \{L, R\}_3 = \{L, R\}_{15}$. To finish encrypting, compute $\text{IP}^{-1}(R_1 \| R_0) = \mathbf{C}$.

To decrypt, certainly $F = F^{-1}$. Begin with $\text{IP}(\mathbf{C}) = R_1 \| R_0$ followed by 32-bit swap which yiels $R_0 \| R_1$. In this case,

$$L_{i-1} = R_{i-1} \oplus R_i.$$

Then $\{L, R\}_{15} = \{L, R\}_{12} = \{L, R\}_0$. To finish decrypting, compute $\text{IP}^{-1}\big(L_0 \| R_0\big) = \mathbf{M}$.

Problem 4.11

This problem provides a numerical example of encryption using a one-round version of DES. We start with the same bit pattern for the key K and the plaintext, namely:

Hexadecimal notation:	0 1 2 3 4 5 6 7 8 9 A B C D E F
Binary Notation:	0000 0001 0010 0011 0100 0101 0110 0111
	1000 1001 1010 1011 1100 1101 1110 1111

a. Derive K_1, the first-round subkey.

Solution:

The code below computes K_1 according to Figure 4.5 in the text.

```c
#define MASK_LOW_28  0xfffffff   // mask low bits
#define MASK_LOW_32 0xffffffff   // mask low bits
int main()
{
    uint64_t K0 = 0x0123456789abcdef;          // 64-S_of_Bits long

    // PART A
    // pc1 is 56-S_of_Bits long
    uint64_t PC1 = applyTransform("permuted_choice_one", K0, 64);

    uint64_t C0 = PC1 >> 28;           //  low 28-S_of_Bits of pc1
    uint64_t D0 = PC1 & MASK_LOW_28;   // high 28-S_of_Bits of pc1
    uint64_t C1 = leftShift(C0);
    uint64_t D1 = leftShift(D0);

    uint64_t concat = (C1 << 28) ^ D1;   // 56-S_of_Bits long

    // k1 is 48-S_of_Bits long
    uint64_t K1 = applyTransform("permuted_choice_two", concat, 56);
    printf("%llx\n", K1);
```

The output of our code is the 48-bit key

$$\mathbf{K_1} = \texttt{0x0b02679b49a5} = \texttt{000010 110000 001001 100111 100110 110100 100110 100101}$$

b. Derive L_0, R_0.

Solution:

Since $\mathbf{M} = \mathbf{K}$, have:

```c
uint64_t IP = applyTransform("initial_permutation", K0, 64);
uint32_t L0 = IP >> 32;
uint32_t R0 = IP & MASK_LOW_32;
printf("%x   %x\n", L0, R0);
```

$$\mathbf{L_0} = \texttt{0xcc00ccff} = \texttt{11001100 00000000 11001100 11111111}$$
$$\mathbf{R_0} = \texttt{0xf0aaf0aa} = \texttt{11110000 10101010 11110000 10101010}$$

c. Expand R_0 to get $E[R_0]$, where $E[\cdot]$ is the expansion function of Table S.1.

Solution: The result of `applyTransform("expansion", r_zero, 32)` is

$$\mathbf{E[R_0]} = \texttt{0x7a15557a1555} = \texttt{01111010 00010101 01010101 01111010 00010101 01010101}$$

d. Calculate $A = E[R_0] \oplus K_1$.

Solution: $\mathbf{A}$ = 0x711732e15cf0 = 01110001 00010111 00110010 11100001 01011100 11110000

e. Group the 48-bit result of (d) into sets of 6 bits and evaluate the corresponding S-box substitutions.

Solution:

Regroup as follows.

$$\begin{array}{llll} \mathbf{B_1} = \texttt{011100} & \mathbf{B_2} = \texttt{010001} & \mathbf{B_3} = \texttt{011100} & \mathbf{B_4} = \texttt{110010} \\ \mathbf{B_5} = \texttt{111000} & \mathbf{B_6} = \texttt{010101} & \mathbf{B_7} = \texttt{110011} & \mathbf{B_8} = \texttt{110000} \end{array}$$

Then apply S-box substitutions accordingly.

```
uint8_t S_of_Bi[8] = {0};    // elements are 4-bits long
uint8_t mask = 0x3f;         // mask 6 low bits
char box_name[6];

for (int j = 8; j > 0; j--)
{
    snprintf(box_name, 6, "sbox%d", j);
    S_of_Bi[j-1] = applySBox((A & mask), box_name);
    printf("%x ", S_of_Bi[j-1]);
    A = A >> 6;
}
printf("\n");
```

This gives us

$$\begin{array}{llll} \mathbf{S(B_1)} = \texttt{0000} & \mathbf{S(B_2)} = \texttt{1100} & \mathbf{S(B_3)} = \texttt{0010} & \mathbf{S(B_4)} = \texttt{0001} \\ \mathbf{S(B_5)} = \texttt{0110} & \mathbf{S(B_6)} = \texttt{1101} & \mathbf{S(B_7)} = \texttt{0101} & \mathbf{S(B_8)} = \texttt{0000}. \end{array}$$

f. Concatenate the results of (e) to get a 32-bit result, B.

Solution:

Using the code below, we get: $\mathbf{B}$ = 0x0c216d50 = 00001100001000010110110101010000.

```
uint32_t B_concat = S_of_Bi[0];

for (int j = 1; j < 8; j++)
{ B_concat = (B_concat << 4) ^ S_of_Bi[j]; }
printf("%x\n", B_concat);
```

g. Apply the permutation to get $P(B)$.

Solution: The result of `applyTransform("permutation", B, 32)` is

$$\mathbf{P(B)} = \texttt{0x921c209c} = \texttt{10010010000111000010000010011100}$$

h. Calculate $R1 = P(B) \oplus L_0$.

Solution: $\mathbf{R_1} = \mathbf{P(B)} \oplus \mathbf{L_0}$ = 0x5e1cec63 = 01011110000111001110110001100011.

i. Write down the ciphertext.

Solution:

Perform a 32-bit swap on $\mathbf{L_1} \| \mathbf{R_1}$ and apply the inverse permutation as shown below.

```c
uint64_t   swap = (R1 << 32) ^ R0;
uint64_t cipher = applyTransform("ip_inverse", swap, 64);
printf("%llx\n", cipher);
```

Thus, the ciphertext is

```
0000 0001 0110 0011 0101 0100 0111 0110 1101 1000 1010 1111 1100 1101 1010 1110
                          0x 0 1 6 3 5 4 7 6 D 8 A F C D A E
```

Solution: Finally, we made use of the three functions defined below.

```c
uint64_t applyTransform(char *name, uint64_t x, int x_size)
{
    FILE *fp = fopen(name, "r");
    int buffer = 0;
    uint8_t      nth_bit = 0;   // big endian
    uint64_t transformed = 0;

    while( fscanf(fp, "%d", &buffer) == 1 )
    {
        nth_bit = (x >> (x_size - buffer)) & 1;
        transformed = (transformed << 1) ^ nth_bit;
    }

    return transformed;
}

uint64_t leftShift(uint64_t x)
{
    uint8_t high_bit = (x >> (28 - 1)) & 1;
    uint64_t shifted = ((x << 1) & MASK_LOW_28) ^ high_bit;
    return shifted;
}

uint8_t applySBox(uint8_t partition, char *name)
{
    FILE *fp = fopen(name, "r");
    int buffer = 0;

    // 0x20 masks highest bit of 6-bit partition arg
    uint8_t row = ((partition & 0x20) >> 4) ^ (partition & 1);
    // 0x1e masks middle 4 bits
    uint8_t col = (partition & 0x1e) >> 1;
    int location = 16*row + col + 1;

    for (int i = 0; i < location; i++)
    { fscanf(fp, "%d", &buffer); }
```

Problem 4.14

a. Let X' be the bitwise complement of X. Prove that if the complement of the plaintext block is taken and the complement of an encryption key is taken, then the result of DES encryption with these values is the complement of the original ciphertext. That is,

$$\begin{aligned} \text{If} \quad Y &= E(K, X) \\ \text{Then} \quad Y' &= E(K', X') \end{aligned}$$

Solution:

To begin, notice that

$$X = L_0 \| X_0 \quad \Rightarrow \quad X' = L_0' \| R_0'.$$

On the other hand, let P be a bitwise permutation such that $P(b_i) = b_j$ where b_i, b_j are the i^{th} and j^{th} bits respectively. It follows that $P(b_i') = b_j'$. Then

$$PC1(K) = C_0 \| D_0 \quad \Rightarrow \quad PC1(K') = C_0' \| D_0'.$$

Likewise, applying a left circular shift and $PC2$ yield the expected results. See diagram below for reference.

$$\begin{tikzcd}
& X' \arrow[dl] \arrow[dr] & & & K' \arrow[d] \\
L_0' \arrow[dddrr] & & R_0' \arrow[d] \arrow[dddll] & & C_0' \| D_0' \arrow[d] \\
& & A = E(R_0') \oplus K_1' \arrow[d] & K_1' \arrow[l] & C_1' \| D_1' \arrow[l] \arrow[dd] \\
& & B \arrow[d] & & \\
L_1' & & R_1' = L_0' \oplus P(B) & & C_1' \| D_1'
\end{tikzcd}$$

Now we address the more involved portion. At the first XOR, we have $E(R_0') \oplus K_1'$. Below [left], we show that $A = E(R) \oplus K = E(R') \oplus K'$.

E	K	$E \oplus K$	E'	K'	$E' \oplus K'$
0	0	0	1	1	0
0	1	1	1	0	1
1	0	1	0	1	1
1	1	0	0	0	0

L	P	$L \oplus P$	$(L \oplus P)'$	L'	$L' \oplus P$
0	0	0	1	1	1
0	1	1	0	1	0
1	0	1	0	0	0
1	1	0	1	0	1

Applying the S-boxes then yields the same result B and for the second XOR, we see above [right] that $L' \oplus P = (L \oplus P)' = R'$.

It follows that Round 16 would yield L_{16}' and R_{16}'. Finally, since IP^{-1} is a permutation as well

$$IP^{-1}(R_{16}' \| L_{16}') = (IP^{-1}(R_{16} \| L_{16}))' = Y'$$

Thus, proving if $Y = E(K, X)$, then $Y' = E(K', X')$.

b. It has been said that a brute-force attack on DES requires searching a key space of 2^{56} keys. Does the result of part (a) change that?

Solution: In any set of 2^n bit keys, half of them are complements of the other half. This makes it computationally inexpensive to find Y' if a brute-force attack has already been performed to find Y. Hence, the true key space is actually $2^{56}/2 = 2^{55}$, which is still large.

3 Homework 03

Problem 1

Let G be a group with group operation $\circ$, and $H \subseteq G$. Recall that

$$aH = \{a \circ h : h \in H\}$$

is a *left coset* of H. Take any two elements $a, b \in G$. Show that if $aH \cup bH \neq \emptyset$, then $aH = bH$.

Solution: Assume that $aH \cap bH \neq \emptyset$. Take $h_i, h_j \in H$ such that $a \circ h_i = b \circ h_j$. Then $a \circ h_i \circ h_i^{-1} = b \circ h_j \circ h_i^{-1}$ since h_i^{-1} exists in H and clearly $a = b \circ \left(h_j \circ h_i^{-1}\right)$ by associativity of a group. Moreover, for any $h \in H$, we have $a \circ h = b \circ \left(h_j \circ h_i^{-1}\right) \circ h$. This means that $a \circ h \in bH$ which implies that $aH \subseteq bH$. By the same token, we can show that $b = a \circ \left(h_i \circ h_j^{-1}\right)$ so that for any $h \in H$, have $b \circ h = a \circ \left(h_i \circ h_j^{-1}\right) \circ h$. Hence $b \circ h \in aH$ which implies that $bH \subseteq aH$. Thus $aH = bH$.

Problem 2

a. Compute $\phi(1525)$ using the formula to be given in class. (Recall that $\phi(n) = |Z_n^*|$ is Euler's totient function.)

Solution: We first factor $1525 = 5^2 \cdot 61$. Then

$$\phi(1525) = 1525(1 - 1/5)(1 - 1/61) = 1200.$$

b. Use the Extended Euclidean Algorithm to compute $27^{-1} \bmod 41$.

Solution:

To solve the congruence $27x \equiv 1 \bmod 41$, start by performing a succession of Euclidan divisions.

$$\begin{aligned} 41 &= 1 \cdot 27 + 14 \\ 27 &= 1 \cdot 14 + 13 \\ 14 &= 1 \cdot 13 + 1 \end{aligned}$$

Then we can substitute the successive remainders until we have expressed the two original numbers as a linear combination.

$$\begin{aligned} 1 &= 14 - 13 \\ &= (41 - 27) - (27 - 14) \\ &= 41 - 2 \cdot 27 + 14 \\ &= 41 - 2 \cdot 27 + (41 - 27) \\ &= 2 \cdot 41 - 3 \cdot 27 \end{aligned}$$

Thus $x = -3 \equiv 38 \bmod 41$ is the multiplicative inverse of 27 in $\mathbb{Z}_{41}$.

Problem 3

a. Recall that an isomorphism from group G to group H is a one-to-one, onto function $f : G \to H$ such that $f(a \circ b) = f(a) \circ f(b)$ for all $a, b \in G$. We say that two groups are isomorphic if there is an isomorphism between them. Show that if G is of order 4 and has an element of order 4, it is isomorphic to $\mathbb{Z}_4$.

Solution:

We wish to find a mapping $f : G \to H$ such that $(G, \circ) \simeq (\mathbb{Z}_4, +)$. Let $\text{ord}(g_1) = 4$ and assume that $g_1^k = g_k$ where we denote $g \circ g = g^2$. Since G is a group, the operation table is uniquely determined as shown below on the left.

$\circ$	g_1	g_2	g_3	g_4
g_1	g_2	g_3	g_4	g_1
g_2	g_3	g_4	g_1	g_2
g_3	g_4	g_1	g_2	g_3
g_4	g_1	g_2	g_3	g_4

$$\begin{aligned} f(g_i \circ g_j) &= f(g_1^i \circ g_1^j) = f(g_1^{i+j}) \\ &= f(g_{i+j}) = i + j \\ &= f(g_i) + f(g_j) \end{aligned}$$

Hence we define $f(g_i) = i \bmod 4$ which is one-to-one and onto. Lastly, f is a homomorphism as shown above on the right. Note that all addition is done modulo 4. Thus $(G, \circ) \simeq (\mathbb{Z}, +)$.

b. Show that if G is a group of order 4 and has no elements of order 4, it is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Solution:

This time we want $(G, \circ) \simeq (\mathbb{Z}_2 \times \mathbb{Z}_2, +)$. We can determine the table for G on the right. Here g_4 is the identity element again and we require $g_i^2 = g_4$, necessarily for all i, so all elements either have order 1 or 2.

We then define $f(g_i) = \left(\left\lfloor \frac{i}{2} \right\rfloor \bmod 2,\ i \bmod 2 \right)$. Thus by explicitly defining f such that

$$f(g_4) = (0, 0)\ ,\ f(g_1) = (0, 1)\ ,\ f(g_2) = (1, 0)\ ,\ f(g_3) = (1, 1),$$

we have shown that $(G, \circ) \simeq (\mathbb{Z}_2 \times \mathbb{Z}_2, +)$.

$\circ$	g_4	g_1	g_2	g_3
g_4	g_4	g_1	g_2	g_3
g_1	g_1	g_4	g_3	g_2
g_2	g_2	g_3	g_4	g_1
g_3	g_3	g_2	g_1	g_4

Problem 4

a. A subgroup H of group G is normal if $aH = Ha$ for all $a \in G$. Show that H is a normal subgroup of G if and only if every left coset of H is also a right coset of H.

Solution:

- ($\Rightarrow$) Assume H is a normal subgroup of G. By definition, we know that $aH = Ha$ for all $a \in G$. Since every left coset aH is the right coset Ha, we are done.
- ($\Leftarrow$) Assume that every left coset of H is also a right coset of H. Take $a \in G$ so that aH is a left coset of H. By assumption, there exists some $b \in G$ such that $aH = Hb$. Know that $(H, \circ)$ is a group so it has the identity element, call it $\mathbf{e}$. It's easy to see that $a = a \circ \mathbf{e} \in aH$ and $a = \mathbf{e} \circ a \in Ha$. However, $a \in Hb$ since $aH = Hb$. By the result of **Problem 1**, this means that $Ha = Hb$. Thus $aH = Ha$ and we conclude that H is normal subgroup of G.

b. The *index* of a subgroup H of G is the number of left cosets of H in G. Show that if H is a subgroup of index 2 in G, then H is a normal subgroup.

Solution: Know that H has index 2. This means that G is partitioned by the two cosets of H, namely H and aH for some $a \in G$. More precisely, $H \sqcup aH = G$. We also know that the number of left cosets equals the number of right cosets. Using the same logic as before, we have $H \sqcup Hb = G$ for some $b \in G$. This gives us $H \sqcup aH = H \sqcup Hb \quad \Rightarrow \quad aH = Hb$. But this is just saying that every left coset is also a right coset! Note that H is both a left and right coset of itself. Thus by the result of **a**, we conclude that H is a normal subgroup of G.

Problem 5

Recall from class that a group G is cyclic if there is an element a such every element of G is a power of a. You may use the fact that $\mathbb{Z}_p^*$ is cyclic when p is prime.

a. Show that when $p \geq 3$ is prime, there are exactly two elements $a \in \mathbb{Z}_p^*$ such that $a^2 = 1$.

Solution:

It's easy to see that two solutions to $a^2 \equiv 1 \bmod p$, where $p \geq 3$, are $a_1 = 1$ and $a_2 = p-1$. The first is trivially true while the second is easily shown below.

$$(p-1)^2 = p^2 - 2p + 1 \equiv 1 \bmod p.$$

Suppose there exists some $\tilde{a} \neq a_1 \neq a_2$ that also solves the congruence. Then

$$\begin{aligned} \tilde{a}^2 &\equiv 1 \bmod p \\ \tilde{a}^2 - 1 &\equiv 0 \bmod p \\ (\tilde{a}+1)(\tilde{a}-1) &\equiv 0 \bmod p. \end{aligned}$$

This means that $p \mid (\tilde{a}+1)(\tilde{a}-1)$ by definition. But if $p \mid (\tilde{a}+1)$, we have

$$\tilde{a} + 1 \equiv 0 \bmod p \quad \Rightarrow \quad \tilde{a} \equiv -1 \equiv p - 1 \equiv a_2 \bmod p.$$

On the other hand, if $p \mid (\tilde{a}-1)$, we have $\tilde{a} \equiv a_1 \bmod p$. This contradicts the existence of any $\tilde{a}$ from our supposition. Thus the only solutions are 1 and $p-1$.

b. Show that when $p \geq 3$ is prime,

$$(p-1)! \;\equiv\; -1 \bmod p.$$

Hint: $(p-1)!$ is the product of all of the elements in the group $\mathbb{Z}_p^*$. What are the multiplicative inverses of these elements? Which are multiplicative inverses of themselves?

Solution: Of course it's true that

$$(p-1)! \;=\; (p-1)(p-2)\cdots 3 \cdot 2 \cdot 1$$

where the right hand side of the equality consists of all the elements in $\mathbb{Z}_p^*$. Moreover, all the elements are units! So for any element $a \in \mathbb{Z}_p^*$, have $a^{-1} \in \mathbb{Z}_p^*$. It's certainly possible that $a = a^{-1}$ for some a. However, our previous result indicates that this only occurs for two numbers, namely 1 and $p-1$. If we exclude those from product, we have

$$\underbrace{(p-2)(p-3)\cdots 3 \cdot 2}_{p-3 \text{ elements}}.$$

Notice that $p-3$ is even since $p \geq 3$. Since this product is made up of units, we may rewrite

$$(p-2)(p-3)\cdots 3 \cdot 2 = \prod_{i=1}^{(p-3)/2} a_i \cdot a_i^{-1}.$$

It follows that $(p-2)! \equiv 1 \bmod p$ and thus

$$(p-1)! = (p-1)(p-2)! \equiv p - 1 \equiv -1 \bmod p.$$

4 Homework 04

Problem 1

Find the multiplicative inverse of each nonzero element in $\mathbb{Z}_7$.

Solution: The equivalences below show the multiplicative inverses.

$$\begin{aligned} 1^2 &\equiv 1 \bmod 7 \\ 2 \cdot 4 = 8 &\equiv 1 \bmod 7 \\ 3 \cdot 5 = 15 &\equiv 1 \bmod 7 \\ 6^2 = 36 &\equiv 1 \bmod 7 \end{aligned}$$

Problem 2

Demonstrate whether each of these statements is true or false for polynomials over a field.

(b) The product of polynomials of degress m and n has degree $m+n$.

Solution: This statement is true. Take $f(x), g(x)$ to be polynomials over a field F. We may write $f(x) = ax^m + f_1(x)$ and $g(x) = bx^n + g_1(x)$ for $m, n \in \mathbb{Z}^+$ and $a, b \in \mathbb{Z}$, such that $\deg(f_1) < m$ and $\deg(g_1) < n$. If m or n equal 0, then f_1 or g_1 equal 0 respectively. Then

$$\begin{gathered} f(x) \cdot g(x) \\ (ax^m + f_1(x))(bx^n + g_1(x)) \\ abx^{m+n} \ + \ ax^m g_1(x) + bx^n f_1(x) \ + \ f_1(x) g_1(x). \end{gathered}$$

Notice that $\deg(ax^m g_1), \deg(bx^n f_1), \deg(f_1 g_1) < m+n$. Since F is a field, then $f \cdot g \in F$. Furthermore, $\deg(f \cdot g) = m + n$.

(a) The product of monic polynomials is monic.

Solution: This is merely a case of our previous proof. Let $a = b = 1$. This makes f and g monic, which consequently means that $f \cdot g$ is monic via the same steps in **(b)**.

(c) The sum of polynomials of degress m and n has degree $\max[m, n]$.

Solution: This is certainly false. Every element in a field F has an additive inverse so that $x \in F$ implies $-x \in F$. It's clear that $x + (-x) = 0 \neq \max\{\deg(x), \deg(-x)\}$.

Problem 3

Determine the multiplicative inverse of $x^3 + x + 1$ in GF(2^4) with $m(x) = x^4 + x + 1$.

Solution:

It's simple enough to use the **Extended Euclidean Algorithm** to solve this problem. First we perform successive Euclidean divisions.

$$\begin{aligned} x^4 + x + 1 &= x(x^3 + x + 1) + (x^2 + 1) \\ x^3 + x + 1 &= x(x^2 + 1) + 1 \end{aligned}$$

Then we express as a linear combination by back-solving.

$$\begin{aligned} 1 &= (x^3 + x + 1) - x(x^2 + 1) \\ &= (x^3 + x + 1) - x[(x^4 + x + 1) - x(x^3 + x + 1)] \\ &= (1 + x^2)(x^3 + x + 1) - x(x^4 + x + 1) \end{aligned}$$

Thus, we have $(x^3 + x + 1)^{-1} = x^2 + 1$ in $\mathbb{Z}_2[x]/ < x^4 + x + 1 >$.

Problem 4

Using Fermat's Theorem, find $3^{201} \bmod 11$.

Solution: We know that since $\gcd(3, 11) = 1$, then $3^{10} \equiv 1 \bmod 11$ by **Fermat's Little Theorem**. Thus, we have

$$3^{201} = (3^{10})^{20} \cdot 3 \equiv 1^{20} \cdot 3 \equiv 3 \bmod 11.$$

Problem 5

Using Fermat's Theorem to find a number x between 0 and 28 with x^{85} congruent to 6 modulo 29. (You should not need to use any brute-force searching).

Solution: Here we have the computational task of solving $x^{85} \equiv 6 \bmod 29$. Working in the field $\mathbb{Z}/29\mathbb{Z}$, call it $\mathbb{F}_{29}$, is great since $\gcd(x, 29) = 1$ for all $x \in \mathbb{F}_{29}$. Then we state an alternate form of **FLT** that says $x^p \equiv x \bmod p$. This is easy to see via multiplication. Thus we have

$$x^{85} = (x^{29})^2 \cdot x^{27} \equiv x^2 \cdot x^{27} \equiv x^{29} \equiv x \equiv 6 \bmod 29.$$

Problem 6

Use Euler's Theorem to find a number x between 0 and 28 with x^{85} congruent to 6 modulo 35. (You should not need to use any brute-force searching).

Solution:

This time we are required to solve $x^{85} \equiv 6 \bmod 35$. Begin by computing $\varphi(35)$ using **Euler's product formula**.

$$\varphi(35) = 35 \prod_{p|35} \left(1 - \frac{1}{p}\right) = 35 \left(1 - \frac{1}{5}\right) \left(1 - \frac{1}{7}\right) = 24$$

Assume that $\gcd(x, 35) = 1$. Then by **Euler's Theorem**, we have

$$(x^{24})^3 \cdot x^{13} = (x^{\varphi(35)})^3 \cdot x^{13} \equiv x^{13} \equiv 6 \bmod 35.$$

While a brute-force approach wouldn't take as long now, it's not hard to solve using the **Chinese Remainder Theorem**. We've like to solve the simultaneous congruences $x^{13} \equiv 6 \bmod 5$ and $x^{13} \equiv 6 \bmod 7$. We'll

$$x^{13} = (x^4)^3 \cdot x \equiv x \equiv 1 \bmod 5 \qquad \text{and} \qquad x^{13} = (x^6)^2 \cdot x \equiv x \equiv 6 \bmod 7.$$

Now $x = 5y + 1$ for $y \in \mathbb{Z}$ so that via substitution

$$5y + 1 \equiv 6 \bmod 7 \quad \Rightarrow \quad y \equiv 1 \bmod 7.$$

Thus, $y = 7z + 1$ for $z \in \mathbb{Z}$ implies that $x = 5(7z + 1) + 1 = 35z + 6$, which equivalently means $x \equiv 6 \bmod 35$. Indeed, $\gcd(6, 35) = 1$ and it's not hard to confirm with a computer that our result works (at least for a power as small as 85).

Problem 7

The example used by Sun-Tsu to illustrate the CRT was

$$x \equiv 2 \bmod 3\ ;\ x \equiv 3 \bmod 5\ ;\ x \equiv 2 \bmod 7$$

Solve for x using the Chinese Remainder Theorem.

Solution:

Given the three equivalence relations, we may rewrite the first as $x = 3y + 2$ for $y \in \mathbb{Z}$. Note we've simply used the definition of equivalence. Next we substitute into the second to get

$$3y + 2 \equiv 3 \bmod 5 \quad \Rightarrow \quad 3y \equiv 1 \bmod 5 \quad \Rightarrow \quad y \equiv 2 \bmod 5.$$

But then, we rewrite

$$x^{85} = (x^{29})^2 \cdot x^{27} \equiv x^2 \cdot x^{27} \equiv x^{29} \equiv x \equiv 6 \bmod 29$$

the result as $y = 5z+2$ for $z \in \mathbb{Z}$, which implies that $x = 3(5z+2)+2 = 15z+8$. Substitute into the third equivalence to get

$$15z + 8 \equiv 2 \bmod 7 \quad \Rightarrow \quad z \equiv 1 \bmod 7.$$

One last time, $z = 7w+1$ for $w \in \mathbb{Z}$. Thus, $x = 15(7w+1)+8 = 105w+23$ means that $x \equiv 23 \bmod 105$ by definition.

Problem 8

Show that every integral domain has the *cancellation property*: if $a \neq 0$ and $ax = ay$, then $x = y$.

Solution: Let R be an integral domain with $a, x, y \in R$. Assume that $a \neq 0$ and $ax = ay$. It follows that $a(x - y) = 0$. But S has no zero divisors, so either $a = 0$ or $x - y = 0$. Our assumption leads to the conclusion that $x - y = 0 \quad \Rightarrow \quad x = y$. Hence the *cancellation property* holds in every integral domain.

Problem 9

Show that every finite integral domain is a field.

Solution: Let R be an integral domain. Assume that $|R| = m \in \mathbb{Z}^+$ and consider $R/\{0\}$. Now take $a, r_i \in R/\{0\}$ and look at the products ar_i. We claim that each product is different. The product is clearly a map $f : R/\{0\} \to R/\{0\}$. If f is a bijection, we're done. Well, take $r_i, r_j \in R/\{0\}$ with $f(r_i) = f(r_j)$, then $ar_i = ar_j \quad \Rightarrow \quad r_i = r_j$ since R has the cancellation property. This shows that f is injective. However since $R/\{0\}$ is finite, f is also surjective, hence bijective. This means that there must exist some r_i such that $f(r_i) = ar_i = 1$. Notice that $1 \in R$ because it's an integral domain. Thus every nonzero element of R is a unit which means R is a field.

5 Homework 05

Problem 2.31

Show that, if n is an odd composite integer, then the Miller-Rabin test will return inconclusive for $a = 1$ and $a = (n-1)$.

Solution:

Take a composite $n \in \mathbb{O}$. We can show the result by following each step of the algorithm.

- Let $a = 1$. Clearly $\gcd(a, n) = 1$ so we proceed. Factor $n - 1 = 2^k \cdot m$ where $m \in \mathbb{O}$. Then $a_0 = a^m = 1 \bmod n$ and we are done. The test determines $a = 1$ as a prime.
- Let $a = n - 1$. Here $\gcd(a, n) = 1$ as well so we proceed. Factor as before so $n - 1 = 2^k \cdot m$. Since m is odd, write $m = 2l + 1$ for some integer l. We now compute $a_0 = (n-1)^m \bmod n$ where

$$(n-1)^m = (n-1)^{2l+1} = (n-1)^{2l}(n-1) \equiv (-1)^{2l}(-1) \equiv -1 \bmod n.$$

  Thus we have $a_0 = -1 \bmod n$ and the test determines $a = n - 1$ as a prime.

Problem 2.32

If n is composite and passes the Miller-Rabin test for the base a, then n is called a strong pseudoprime to the base a. Show that 2047 is a strong pseudoprime to the base 2.

Solution: Begin with $a = 2$ and $n = 2047$ where $\gcd(2, 2047) = 1$. Furthermore, $n - 1 = 2046 = 2^1 \cdot 1023 = 2^k \cdot m$. Since $k = 1$, we only need to compute a_0. Notice that $1023 = 99 \cdot 11$. Thus

$$a_0 = 2^{1023} = (2^{11})^{99} = (2048)^{99} \equiv 1^{99} \equiv 1 \text{ mod } 2047$$

which means 2047 passes the **MRPT** and we say that it is a strong pseudoprime to the base 2.

Problem 7.4

With the ECB mode, if there is an error in a block of the transmitted ciphertext, only the corresponding plaintext block is affected. However, in the CBC mode, this error propagates. For example, an error in the transmitted C_1 (Figure 7.4) obviously corrupts P_1 and P_2.

(a) Are any blocks beyond P_2 affected?

Solution: Blocks beyond $\mathbf{P_2}$ are not affected. In the encryption stage, $\mathbf{C_1}$ is no longer required after computing $\mathbf{C_1} \oplus \mathbf{P_2}$. Once $\mathbf{C_2}$ is generated, we require an error-less transmission so that $\mathbf{P_3}$ is decrypted correctly. We emphasize that $\mathbf{C_1}$ is NOT used in the decryption of $\mathbf{P_3}$.

(b) Suppose that there is a bit error in the source version of P_1. Through how many ciphertext blocks is this error propagated? What is the effect at the receiver?

Solution: A bit error in the source version of $\mathbf{P_1}$ is propagated through all the cipher blocks. This is due to the chained dependence that all cipher blocks $\mathbf{C_i}$ have on $\mathbf{P_1}$. Since the receiver now has incorrect cipher blocks $\mathbf{C_i}$, then the decryption of $\mathbf{P_i}$ will be incorrect as well.

Problem 7.7

For the ECB, CBC, and CFB modes, the plaintext must be a sequence of one or more complete data blocks (or, for CFB mode, data segments). In other words, for these three modes, the total number of bits in the plaintext must be a positive multiple of the block (or segment) size. One common method of padding, if needed, consists of a 1 bit followed by as few zero bits, possibly none, as are necessary to complete the final block. It is considered good practice for the sender to pad every message, including messages in which the final message block is already complete. What is the motiva- tion for including a padding block when padding is not needed?

Solution: Padding every message could be a way of signaling the end of a message. More so, it can be a way of error checking. If there is tampering with the transmission channel, then the decryption will clearly have errors. The receiver will know there are clear errors if the padding structure is different from what he/she expects or has agreed upon with the sender.

Problem 7.8

If a bit error occurs in the transmission of a ciphertext character in 8-bit CFB mode, how far does the error propagate?

Solution: Suppose we have a bit error in the transmission of $\mathbf{C_j}$, call the error $\tilde{\mathbf{C}}_{\mathbf{j}}$. The receiver will compute $\tilde{\mathbf{C}}_{\mathbf{j}} \oplus \mathrm{MSB}_s(O_j) = \tilde{\mathbf{P}}_{\mathbf{j}}$. Here $\tilde{\mathbf{P}}_{\mathbf{j}}$ may not differ all that much from $\mathbf{P_j}$ since $\mathbf{C_j}$ was utilized at the end of decryption. However, there is a much bigger consequence in the $(j+1)^{\text{th}}$ step of decryption. Since $I_{j+1} = \mathrm{LSB}_{b-s}(I_j)\|\mathbf{C_j}$, it is clear that I_{j+1} depends on $\mathbf{C_j}$. Consequently, using $\tilde{\mathbf{C}}_{\mathbf{j}}$ results in $\tilde{I}_{j+1}$. Furthermore, O_{j+1} depends on I_{j+1} and $\mathbf{P_{j+1}}$ depends on O_{j+1}. Thus, we can be sure that decryption of $\mathbf{C_{j+1}}$ is significantly altered. However, this is where error propagation stops since nothing else is dependent on $\mathbf{C_j}$.

Problem 7.10

In discussing the CTR mode, it was mentioned that if any plaintext block that is encrypted using a given counter value is known, then the output of the encryption function can be determined easily from the associated ciphertext block. Show the calculation.

Solution: Know that the CTR mode for encryption is defined as $\mathbf{C_j} = \mathbf{P_j} \oplus E(K, T_j)$. We are assuming that both $\mathbf{C_j}$ and $\mathbf{P_j}$ are known. It follows that

$$\mathbf{P_j} \oplus \mathbf{C_j} = \mathbf{P_j} \oplus \mathbf{P_j} \oplus E(K, T_j) \quad \Rightarrow \quad \mathbf{P_j} \oplus \mathbf{C_j} = E(K, T_j).$$

This gives the output of the encryption function E. If we work over the decryption portion, we have

$$\mathbf{C_j} \oplus \mathbf{P_j} = \mathbf{C_j} \oplus \mathbf{C_j} \oplus E(K, T_j) \quad \Rightarrow \quad \mathbf{C_j} \oplus \mathbf{P_j} = E(K, T_j).$$

Notice that this is the exact same result as before since the E is used both in encryption and decryption.

Problem 1

Compute as much information as you can after each of the four transformations for one round of the AES encryption algorithm. The round key is `07070606050504040303020201010000`. The input block looks like this at the beginning of the round:

			7D
7C			
	D5		
		54	

Most of the bytes have been left blank. Also, write the name of the appropriate AES transformation next to each block. (Use the tables below and recall that the irreducible polynomial used to generate the field GF(2^8) in AES is $x^8 + x^4 + x^3 + x + 1$.)

```
  |  0  1  2  3  4  5  6  7  8  9  A  B  C  D  E  F
--|------------------------------------------------
0 | 63 7C 77 7B F2 6B 6F C5 30 01 67 2B FE D7 AB 76
1 | CA 82 C9 7D FA 59 47 F0 AD D4 A2 AF 9C A4 72 C0
2 | B7 FD 93 26 36 3F F7 CC 34 A5 E5 F1 71 D8 31 15
3 | 04 C7 23 C3 18 96 05 9A 07 12 80 E2 EB 27 B2 75
4 | 09 83 2C 1A 1B 6E 5A A0 52 3B D6 B3 29 E3 2F 84
5 | 53 D1 00 ED 20 FC B1 5B 6A CB BE 39 4A 4C 58 CF
6 | D0 EF AA FB 43 4D 33 85 45 F9 02 7F 50 3C 9F A8
7 | 51 A3 40 8F 92 9D 38 F5 BC B6 DA 21 10 FF F3 D2
8 | CD 0C 13 EC 5F 97 44 17 C4 A7 7E 3D 64 5D 19 73
9 | 60 81 4F DC 22 2A 90 88 46 EE B8 14 DE 5E 0B DB
A | E0 32 3A 0A 49 06 24 5C C2 D3 AC 62 91 95 E4 79
B | E7 C8 37 6D 8D D5 4E A9 6C 56 F4 EA 65 7A AE 08
C | BA 78 25 2E 1C A6 B4 C6 E8 DD 74 1F 4B BD 8B 8A
D | 70 3E B5 66 48 03 F6 0E 61 35 57 B9 86 C1 1D 9E
E | E1 F8 98 11 69 D9 8E 94 9B 1E 87 E9 CE 55 28 DF
F | 8C A1 89 0D BF E6 42 68 41 99 2D 0F B0 54 BB 16
```

Figure 1: S-box Table

$$\begin{matrix} 02 & 03 & 01 & 01 \\ 01 & 02 & 03 & 01 \\ 01 & 01 & 02 & 03 \\ 03 & 01 & 01 & 02 \end{matrix}$$

Figure 2: MixColumns Matrix

Solution:

(a) Begin by applying the S-box shown in figure 1 to the input block.

			7D
7C			
	D5		
		54	

$\Rightarrow$

			FF
10			
	03		
		20	

Substitute bytes

(b) Next we shift the rows accordingly.

			FF
10			
	03		
		20	

$\Rightarrow$

			FF
			10
			03
			20

Shift rows

(c) Now we mix the columns with the given AES MixColumns Matrix. However, since the first three columns are not known, we can only perform the transformation to the last column.

$$\begin{bmatrix} 02 & 03 & 01 & 01 \\ 01 & 02 & 03 & 01 \\ 01 & 01 & 02 & 03 \\ 03 & 01 & 01 & 02 \end{bmatrix} \cdot \begin{bmatrix} \text{FF} \\ 10 \\ 03 \\ 20 \end{bmatrix} = \begin{bmatrix} s'_{0,3} \\ s'_{1,3} \\ s'_{2,3} \\ s'_{3,3} \end{bmatrix}$$

Keep in mind that we're working modulo $f(x) = x^8 + x^4 + x^3 + x + 1$. The first multiplication step is shown below.

$$\begin{aligned} s'_{0,3} &= \text{FF} \times 02 + 10 \times 03 + 03 \times 01 + 20 \times 01 \\ &= 11111111 \times 02 + 00010000 \times 03 + 00000011 \times 01 + 00100000 \times 01 \\ &= 111101101 \end{aligned}$$

But then we work with the overflow part by XOR-ing by $f(x)$.

$$s'_{0,3} = 111101101 \oplus 100011011 = 011110101$$

Then by discarding the leading zero, we have $s'_{0,3} = 1111\ 0101 = \text{F6}$. Similarly, we have

$$\begin{bmatrix} s'_{0,3} \\ s'_{1,3} \\ s'_{2,3} \\ s'_{3,3} \end{bmatrix} = \begin{bmatrix} \text{F6} \\ \text{FA} \\ 89 \\ 49 \end{bmatrix} \quad \Longrightarrow$$

			F6
			FA
			89
			49

Mix columns

(d) The last step is to add the round key.

07	05	03	01
07	05	03	01
06	04	02	00
06	04	02	00

$\oplus$

			F6
			FA
			89
			49

$=$

			F7
			FB
			89
			49

Add round key