UNIVERSITY OF BERN

Linear Algebra Review

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Course: *Machine Learning*
Due date: *October 9th, 2019*

Question 1 (10 points)

$S = \{v_1, v_2, ..., v_n\}$ be an orthogonal set of non-zero vectors in R^n. Show that vectors in S are linearly independent.

Proof: consider linear combination $c_1v_1 + c_2v_2 + \ldots + c_nv_n = 0$ and show that $c_1 + c_2 + \ldots + c_n = 0$. Take dot product of equation with vector v_j.

$$0 = c_1v_1v_j + c_2v_2v_j + ... + c_iv_iv_j$$

Because S is an orthogonal set, $v_iv_j = 0$ for $i \neq j$. So all the terms but the i-th one are zero and thus we get

$$0 = c_iv_iv_i = c_i\|v_i\|^2$$

This equation implies $c_i = 0$. We conclude that $c_1 + c_2 + \ldots + c_n = 0$ and the vectors are linearly independent.

Question 2 (15 points)

Given a square matrix $A \in \mathbb{R}^{n\times n}$ and a vector $x \in \mathbb{R}^n$. Show that $x^TAx = x^T(\frac{1}{2}A + \frac{1}{2}A^T)x$.

Solution: the scalar value x^TAx is called a **quadratic form**.

$$x^T = \sum_{i=1}^{n} x_i(Ax)_i = \sum_{i=1}^{n} x_i(\sum_{j=1}^{n} A_{ij}x_j) = \sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij}x_ix_j$$

Any square matrix $A \in \mathbb{R}^{n\times n}$ can be represented as a sum of a symmetric matrix and an anti-symmetric matrix

$$A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T) = A_s + A_a$$

The quadratic form of a purely anti-symmetric matrix is

$$q = x^TA_ax = (x^TA_ax)^T = x^TA_a^Tx = -x^TA_a^Tx = -q$$

Which implies that $q = 0$.
The quadratic form of a square matrix

$$x^TAx = x^T(A_s + A_a)x = x^TA_sx + x^TA_ax$$

We know that $x^TA_ax = 0$. Finally we find that

$$x^TAx = x^TA_sx = x^T\frac{1}{2}(A + A^T)x$$

Question 3 (15 points)

Show that if $(A + B)^{-1} = A^{-1} + B^{-1}$ then $AB^{-1}A = BA^{-1}B$.

Solution: First, multiply both sides of equation with $A+B$ on the right:

$$\begin{aligned}(A+B)^{-1}(A+B) &= (A^{-1}+B^{-1})(A+B)\\ I &= I + A^{-1}B + B^{-1}A + I\end{aligned}$$

This leads to

$$A^{-1}B = -B^{-1}A - I \quad \text{and} \quad AB^{-1} = -I - BA^{-1}$$

If we multiply $A^{-1}B = -B^{-1}A - I$ with B on the left we get

$$BA^{-1}B = B(-B^{-1}A - I) = -(A+B)$$

If we multiply $AB^{-1} = -I - BA^{-1}$ with A on the right we get

$$AB^{-1}A = (-I - BA^{-1})A = -(A+B)$$

Finally we get

$$BA^{-1}B = -(A+B) = AB^{-1}A$$

Question 4 (15 points)

Use definition of trace to show that $tr(A+b) = trA + trB$, where $A, B \in \mathbb{R}^{n\times n}$.

Proof: the $(i,i)-th$ entry of $A+B$ is $a_{ii}+b_{ii}$.

$$\begin{aligned}tr(A+B) &= (a_{11}+b_{11}) + (a_{22}+b_{22}) + ... + (a_{nn}+b_{nn})\\ &= (a_{11}+a_{22}... + a_{nn}) + (b_{11}+b_{22}... + b_{nn})\\ &= trA + trB\end{aligned}$$

Question 5 (15 points)

Show that if (λ_i, x_i) are the i-th eigenvalue and i-th eigenvector of a non-singular and symmetric matrix $A \in \mathbb{R}^{n\times n}$, then $(\frac{1}{\lambda_i}, x_i)$ are the i-th eigenvalue and i-th eigenvector of A^{-1}.

Solution: multiply $Ax_i = \lambda_i x_i$ with A^{-1} on the left

$$\begin{aligned}Ax_i &= \lambda_i x_i\\ A^{-1}Ax_i &= \lambda_i A^{-1}x_i\\ x_i &= \lambda_i A^{-1}x_i\\ \frac{1}{\lambda_i}x_i &= A^{-1}x_i\end{aligned}$$

Thus $\frac{1}{\lambda_i}$ is the i-th eigenvalue and x_i the i-th eigenvector of A^{-1}

Question 6 (10 points)

Show that $rank(A) \leq min\{m,n\}$, where $A \in \mathbb{R}^{m\times n}$.

Question 7 (20 points)

In each of the following cases, state whether the real matrix A is guaranteed to be singular or not. Justify your answer in each case.

(*a*) $A \in \mathbb{R}^{(n+1)\times n}$ is a full rank matrix.

(*b*) $|A| = 0$.

(*c*) A is an orthogonal matrix.

(*d*) A has no eigenvalue equal to zero.

(*e*) A is a symmetric matrix with non-negative eigenvalues.

Solutions: Inverse of a square matrix $A \in \mathbb{R}^{n\times n}$ is denoted A^{-1} andis the unique matrix such that

$$A^{-1}A = I = AA^{-1}$$

Matrix A is singular, or non-invertible if the inverse A^{-1} of matrix does not exist.

(*a*) singular. A is not a square matrix.

(*b*) singular. $|A| = 0$ if and only if A is singular. If A is singular then it does not have full rank, and hence its columns are linearly dependent. In this case, the set S corresponds to a "flat sheet" within the n-dimensional space and hence has zero volume.

(*c*) non-singular. A square matrix $A \in \mathbb{R}^{n\times n}$ is orthogonal if all its columns are orthonormal.

$$A^T A = I = AA^T$$

We know that $|AB| = |A||B|$. Thus we have $|I| = 1 = |AA^T| = |A||A^T| = |A||A| = |A|^2$. Because we have $|A|^2 = 1$ every orthogonal matrix has a determinant either 1 or -1. If $|A| \neq 0$ matrix A is invertible, non-singular.

(*d*) non-singular. The determinant of A is the product of its eigenvalues. So if it has an eigenvalue 0 the determinant would also be 0. On the other hand if matrix has no eigenvalue equal to zero, its determinant is also non-zero and thus it is invertible.

(*e*) non-singular. Matrix which has only non-negative eigenvalues $\lambda_i \geq 0$ are positive semidefinite. Since this includes eigenvalue 0 its determinant is 0 and thus its singular. If A is positive definite, meaning that $\lambda_i > 0$ its determinant would be positive and thus non-singular.