UNIVERSITY OF BERN

Calculus Review

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Course: *Machine Learning*
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Question 1

Consider the function $g : \mathbb{R}^n \to \mathbb{R}$ with $g(x) = x^T x$. We can readily calculate the gradient $\nabla g(x) = 2x$ by noticing that

$$\forall j = 1, ..., n \quad \frac{\partial x^T x}{\partial x_j} = \frac{\partial x_j^2}{\partial x_j} = 2x_j \to \nabla g(x) = 2x$$

Consider also the function $a : \mathbb{R}^n \to \mathbb{R}^m$ with $a(x) = Ax$ and $A \in \mathbb{R}^{m \times n}$. The Jakobian of $a(x)$ is $Da(x) = a$. Given this, answer the following questions.

(a) Consider the function $h : \mathbb{R}^n \to \mathbb{R}$ and $h(x) = x^T Q x$, where $Q \in \mathbb{R}^{n \times n}$ is a symmetric matrix. Calculate $\nabla h(x)$.

(b) Consider the function $f : \mathbb{R}^n \to \mathbb{R}$ where $f(x) = \|Ax - b\|^2$, $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. Calculate $\nabla h(x)$.

(c) Consider the function $f : \mathbb{R}^n \to \mathbb{R}$. Suppose we have a matrix $A \in \mathbb{R}^{n \times m}$ and a vector $x \in \mathbb{R}^m$. Calculate $\nabla_x f(Ax)$ as a function of $\nabla_x f(x)$.

(d) Show that

$$\frac{\partial}{\partial X} \prod_{i=1}^{n} \lambda_i = det(X) X^{-T}$$

where $X \in \mathbb{R}^{n \times n}$ and has eigenvalues $\lambda_1, ..., \lambda_i$.

Question 2 (15 points)

Assume $A \in \mathbb{R}^{m \times n}$, $X \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{m \times n}$. Show that $\nabla_X tr(AX^T B) = BA$.

Question 3 (15 points)

Solve the following equality constrained optimization problem:

$$\max_{x \in \mathbb{R}^n} x^T A x \quad \text{subject to } b^T x = 1$$

for a symmetric matrix $A \in \mathbb{S}^n$. Assume that A is invertible and $b \neq 0$.