Bài tập 1:

a) $$\lim_{(x,y)\to(2,1)} \frac{4-xy}{x^2+3y^2}$$

- Đặt $f(x,y) = \dfrac{4-xy}{x^2+3y^2}$

- Nhận thấy $f(x,y)$ là hàm sơ cấp

$\Rightarrow f(x,y)$ xác định $\forall x,y \in \mathbb{R}^2 \setminus \{(0,0)\}$

Mà $(2,1) \neq (0,0) \Rightarrow$ tại $x=2, y=1$

$f(x,y)$ xác định $\Rightarrow \displaystyle\lim_{(x,y)\to(2,1)} \frac{4-xy}{x^2+3y^2} = \frac{4-2.1}{2^2+3.1^2}$

$= \dfrac{2}{7}$ $\blacksquare$

5) $\lim\limits_{(x,y)\to(0,0)} \dfrac{y^2 \cdot \sin^2 x}{x^4 + y^4}$

ta có $0 \le y^2 \cdot \sin^2 x \le y^2$

$\Leftrightarrow \quad 0 \le \dfrac{y^2 \cdot \sin^2 x}{y^4 + x^4} \le \dfrac{y^2}{x^4 + y^4}$

Xét $\lim\limits_{(x,y)\to(0,0)} \dfrac{y^2}{x^4 + y^4}$

ta có: $0 \le \dfrac{y^2}{x^4 + y^4} \le \dfrac{y^2}{y^4} = \dfrac{1}{y^2}$

Dễ thấy $\lim\limits_{(x,y)\to(0,0)} \dfrac{1}{y^2} = 0$

$\Rightarrow$ Áp dụng định lý giới hạn kẹp:

$\lim\limits_{(x,y)\to(0,0)} \dfrac{y^2}{x^4 + y^4} = 0$

Áp dụng định lí kẹp:

$$\lim_{(x,y)\to(0,0)} \frac{y^2 \cdot \sin^2 x}{x^4 + y^4} = \lim \frac{y^2}{x^4+y^4} = 0.$$

c) $\lim_{(x,y)\to(0,0)} \frac{2x^2 y}{x^4+y^4}$

~~Ta có: $0 \le \frac{x^2}{x^4+y^4} \le \frac{x^2}{x^4}$~~

Đặt $f(x,y) = \frac{2x^2 y}{x^4+y^4}$

Cho $(x_n, y_n) = \left(\frac{1}{n}, \frac{1}{n}\right)$

~~lim lim~~

$(x_n, y_n) \xrightarrow{n\to\infty} (0,0)$

Xét $\lim_{(x_n,y_n)\to(0,0)} \frac{2x_n^2 y_n}{x_n^4 + y_n^4}$

$= \lim_{n\to\infty} \frac{2\left(\frac{1}{n}\right)^2 \cdot \frac{1}{n}}{\left(\frac{1}{n}\right)^4 + \left(\frac{1}{n}\right)^4}$

$$= \frac{2 \cdot 0^3 \frac{1}{n^3}}{\frac{2}{n^4}} = \lim_{n \to \infty} n = +\infty = L_1$$

Chọn $(x_n', y_n') = \left(0, \frac{1}{n}\right)$

$$\lim_{n \to \infty} f(x_n, y_n) = 0$$

$$= \lim_{n \to \infty} \frac{2 \cdot 0^2 \cdot \frac{1}{n}}{0^2 + \left(\frac{1}{n}\right)^2} = 0 = L_2$$

Vì $L_1 \neq L_2 \Rightarrow$ theo định nghĩa giới hạn dãy ngôn ngữ dãy, $\nexists \lim_{(x,y) \to (0,0)} f(x,y)$

d) $$\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1}$$

$$= \lim_{(x,y)\to(0,0)} \frac{(x^2+y^2)\left(\sqrt{x^2+y^2+1}+1\right)}{(x^2+y^2)}$$

$$= \lim_{(x,y)\to(0,0)} \sqrt{x^2+y^2+1}+1 = \sqrt{0^2+0^2+1}+1 = 2$$

Bài tập 2:

a) $$f(x,y) = \frac{1+x^2+y^2}{1-x^2-y^2}$$

$\cancel{D\, f(x,y) = \mathbb{R}^2}$

ĐKXĐ: $1 - x^2 - y^2 \neq 0 \Leftrightarrow \cancel{x^2+1y^2}$ $x^2+y^2 \neq 1$

c) $f(x,y) = \begin{cases} \dfrac{xy^2}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = 0 \end{cases}$

· Xét $f(x,y) = \dfrac{xy^2}{x^2+y^2}$.

Vì $f(x,y)$ là hàm sơ cấp

$\Rightarrow f(x,y)$ liên tục trên $D = \mathbb{R}^2 \setminus \{(0,0)\}$.

* Xét tính liên tục tại O

$+ \lim\limits_{(x,y)\to(0,0)} \dfrac{xy^2}{x^2+y^2}$

Vì $0 \leq \dfrac{y^2}{x^2+y^2} \leq 1$

$\Rightarrow 0 \leq |x| \dfrac{y^2}{x^2+y^2} \leq |x|$

mà $\lim\limits_{(x,y)\to(0,0)} |x| = \lim\limits_{(x,y)\to(0,0)} 0 = 0$

$$\Rightarrow \exists \lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^2} = 0 = f(0,0)$$

$\Rightarrow f(x,y)$ liên tục tại $(0,0)$.

$\Rightarrow f(x,y)$ liên tục trên R^2.

b) $\lim\limits_{(x,y)\to(0,0)} \dfrac{y^2 \cdot \sin^2(x)}{x^4+y^4}$

Chọn $(x_n ; y_n) = \left(\frac{1}{n} ; 0\right)$

khi $n \to \infty$: $(x_n, y_n) \to (0;0)$

$\lim\limits_{n\to\infty} \dfrac{y_n^2 \cdot \sin^2(x_n)}{x_n^4+y_n^4} = \lim\limits_{n\to+\infty} \dfrac{(0)^2 \cdot \sin^2\left(\frac{1}{n}\right)}{\left(\frac{1}{n}\right)^4 + 0^4} = 0 = L_1$

Chọn $(x_n' ; y_n') = \left(\frac{1}{n} ; \frac{1}{n}\right)$

khi $n \to \infty$, $(x_n', y_n') \to (0;0)$

$\lim\limits_{n\to\infty} \dfrac{y_n^2 \cdot \sin^2(x_n)}{x_n^4+y_n^4} = \lim\limits_{n\to\infty} \dfrac{\left(\frac{1}{n}\right)^2 \cdot \sin^2\left(\frac{1}{n}\right)}{\left(\frac{1}{n}\right)^4 + \left(\frac{1}{n}\right)^4}$

$= \dfrac{1}{2} \lim\limits_{n\to\infty} \left(\dfrac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}}\right)^2 = \dfrac{1}{2} \cdot 1^2 = \dfrac{1}{2} = L_2$

$L_1 \neq L_2 \Rightarrow \nexists \lim\limits_{(x,y)\to(0,0)} \dfrac{y^2 \cdot \sin^2(x)}{x^4+y^4}$

c) $\lim\limits_{(x,y)\to(0,0)} \dfrac{e^x + e^y}{e^{xy} - 1} = f(x,y)$

$D = \{(x,y) \in \mathbb{R}^2 \mid e^{xy} - 1 \neq 0\}$

$= \{(x,y) \in \mathbb{R}^2 \mid xy \neq 0\}$

$= \{(x,y) \in \mathbb{R} \mid x \neq 0 \wedge y \neq 0\}$

mà f là hàm sơ cấp

$\Rightarrow$ f liên tục trên $D = \{(x,y) \in \mathbb{R}^2 \mid x \ldots$

d) $h(x,y) = e^{x^2 y} + \sqrt{x+y^2}$

$D = \{(x,y) \in \mathbb{R}^2 \mid x+y^2 \geq 0\}$

$= \{(x,y) \in \mathbb{R}^2 \mid x \geq 0\}$

h là hàm sơ cấp $\Rightarrow$ h liên tục trên