LINFO2266 - Solutions

November 15, 2021

1 Lagrangian Relaxation

1.1 Set Covering Problem

1. We introduce one Lagrangian multiplier for each constraint to obtain the Lagrangian relaxation:

$$\min \sum_{i=1}^{m} c_i x_i + \sum_{j=1}^{n} \lambda_j \left(1 - \sum_{\substack{i=1 \\ j \in S_i}}^{m} x_i \right) \tag{1}$$

with $\lambda_j \geq 0, j \in \{1, \dots, n\}$. Note how we integrated the constraints: for any *feasible* solution of the original problem, we will have

$$1 - \sum_{\substack{i=1 \\ j \in S_i}}^{m} x_i \leq 0 \tag{2}$$

which means a *negative* value will be added to the objective function and that we will thus obtain a *lower bound* on the optimal value of the problem.

For clarity, we rewrite eq. (1) as:

$$\min \sum_{i=1}^{m} \left(c_i - \sum_{j \in S_i} \lambda_j \right) x_i + \sum_{j=1}^{n} \lambda_j \tag{3}$$

For fixed values of the Lagrangian multipliers λ_j, we can obtain a lower bound by solving eq. (3). This problem is very simple to solve if we notice that:

- each x_i is multiplied by a fixed coefficient $\left(c_i - \sum_{j \in S_i} \lambda_j \right)$
- the second term $\left(\sum_{j=1}^{n} \lambda_j \right)$ is constant

In order to minimize the objective function, it is sufficient to select the sets which have a negative coefficient:

$$x_i = \begin{cases} 1 & \text{if } c_i - \sum_{j \in S_i} \lambda_j \leq 0 \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

2. By applying exactly what we described above, we get a lower bound value of 4.2.

3. We adapt the subgradient procedure covered in the lectures for this particular problem, and for multiple Lagrange multipliers in algorithm 1.

Algorithm 1 Subgradient procedure for the Set Covering Problem.

$\mathcal{L}^* \leftarrow -\infty, k \leftarrow 1, \mu_0 \leftarrow 1$
$\lambda_{0,j} \leftarrow 0, 1 \leq j \leq n$
$\mathcal{C}^* \leftarrow$ a trivial solution, or none
while $\mu_k \geq \epsilon$ **do**
    Compute cover $\mathcal{C}_k$ with weights $\left(c_i - \sum_{j \in S_i} \lambda_{k,j}\right)$ for $1 \leq i \leq m$
    $\mathcal{L}_k \leftarrow \mathit{value}\left(\mathcal{C}_k\right) + \sum_{j=1}^{n} \lambda_{k,j}$
    if $\mathcal{L}_k > \mathcal{L}^*$ **then**
        $\mathcal{L}^* \leftarrow \mathcal{L}_k$
    end if
    if $\mathcal{C}_k$ is feasible and $\mathit{value}\left(\mathcal{C}_k\right) < \mathit{value}\left(\mathcal{C}^*\right)$ **then**
        $\mathcal{C}^* \leftarrow \mathcal{C}_k$
    end if
    if $\mathcal{L}^* = \mathit{value}\left(\mathcal{C}^*\right)$ **then**
        break
    end if
    $\lambda_{k+1,j} \leftarrow \max\left(0, \lambda_{k,j} + \mu_k \left(1 - \sum_{\substack{i=1 \\ j \in S_i}}^{m} x_i\right)\right)$ for $1 \leq j \leq n$
    $\mu_{k+1} \leftarrow \frac{1}{k}$ (or another valid update rule)
    $k \leftarrow k + 1$
end while
return lb