Geometry Morphing

June 19, 2022

The objective

The objective of this internship is to morph some tetrahedral meshes preserving some properties (for now, the volume). We want to achieve this in two ways:

- Using (semi-discrete) Optimal Transport
- Using Variational Autoencoders

Some notation

Let A a tetrahedral mesh, composed by N tetrahedrals, then A_i will denote the volume matrix of the i-th tetrahedra:

$$\begin{vmatrix} a_1 - d_1 & b_1 - d_1 & c_1 - d_1 \\ a_2 - d_2 & b_2 - d_2 & c_2 - d_2 \\ a_3 - d_3 & b_3 - d_3 & c_3 - d_3 \end{vmatrix}$$

where a, b, c, d are the points of the i-th tetrahedral. It is called volume matrix because the volume of the tetrahedra is $\frac{|\det(A_i)|}{6}$, by swapping a, b, c, d we can assume that $det(A_i) > 0$. The volume of the mesh so can be calculated with $\frac{\sum_{i=1}^{N} det(A_i)}{6}$

Optimal Transport map I

Given Ω a Borel set and two measures μ and ν on Ω such that $\mu(\Omega) = \nu(\Omega)$, c a convex function. $T : \Omega \rightarrow \Omega$ such that

$$\begin{cases} \nu(X) = \mu(T^{-1}(X)) \text{for any Borel (i.e. measurable) subset } X \text{ of } \Omega \\ \int_\Omega c(x, T(x)) d\mu \text{ is minimal} \end{cases}$$

is called the optimal transport map from μ to ν. We are interested in μ continuos and ν discrete.

Optimal Transport map II

The Voronoi diagram $\text{Vor}\,(P)$ is the partition of $\mathbb{R}^d$ into the subsets $\text{Vor}\,(p_i)$ defined by :
$\text{Vor}\,(p_i) := \left\{x \| \|x - p_i\|^2 < \| x - p_j\|^2 \quad \forall j \neq i\right\}$ the power diagram $\text{Pow}_W(P)$ is the partition of $\mathbb{R}^d$ into the subsets $\text{Pow}_W\,(p_i)$ defined by:
$\text{Pow}_W\,(p_i) := \left\{x \mid \|x - p_i\|^2 - w_i < \|x - p_j\|^2 - w_j \quad \forall j \neq i\right\}$
the map T_W defined by $\forall i, \forall p \in \text{Pow}_W\,(p_i)\,, T_W(p) = p_i$ is called the assignment defined by the power diagram $\text{Pow}_W(P)$ **and is an optimal transport map**.

Knowing this we can approximate the optimal transport map in the following algorithm:

Data: Two tetrahedral meshes M and M', and k the desired number of vertices in the result

Result: A tetrahedral mesh G with k vertices and a pair of points p_i^0 and p_i^1 attached to each vertex. Transport is parameterized by time $t \in [0, 1]$ with $p_i(t) = (1 - t)p_i^0 + tp_i^1$.

(1) Sample M' with a set Y of k points

(2) Compute the weight vector W that realizes the optimal transport between M and Y

(3) Compute $E = \mathrm{Del}(Y) \mid M'$ and $F = \mathrm{Pow}_W(Y) \mid M$

$\mathrm{Tets}(\mathrm{G}) \leftarrow E \cap F$ whereDel is the Delaunay triangulation.

(4) Foreach $i \in [1 \dots k], (p_i)^0 \leftarrow \mathrm{centroid}\left(\mathrm{Pow}_W\left(y_i\right) \cap M\right)$, $(p_i)^1 \leftarrow y_i$

Problems

There are two main problems with this approach:

- Step 3 may modify substantially the two meshes, so they may end up having different volumes (this howewer can be resolved by using more tetrahedras and rescaling at the end)
- Let A and B the two meshes obtained with p_i^0 and p_i^1, then the map $f(t) = t * A + (1 - t) * B$ will preserve volume with only very restrictive assumptions.

Theorem

The volume is preserved for every $t \in [0, 1]$ *iff*
$\sum_{i=1}^{M} det(B_i)(Tr(B_i^{-1}A_i) - 3) = 0$ *and*
$\sum_{i=1}^{M} det(A_i)(Tr(A_i^{-1}B_i) - 3) = 0$

A weaker version also holds

Theorem

$|Vol(t * A + (1 - t) * B) - Vol(B)| \leq \frac{4}{27} |\sum_{i=1}^{M} det(B_i)(Tr(B_i^{-1}A_i) - det(A_i)Tr(A_i^{-1}B_i))| + \frac{1}{2} |\sum_{i=1}^{M} det(A_i)(Tr(A_i^{-1}B_i) - 3)|$