Machine Learning and Optimal Transport for shape parametrisation

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1a) Generative Models

First line of research consist in studying **generative models** for shape optimization of complex geometries with a large number of parameters; the objective is to learn the shape of the geometry using a minor number of parameters, for example for modeling naval hulls, and creating new artificial geometries similar to real data, as creating new real geometries can be costly.

1b) Optimal transport

In the second line **semidiscrete optimal transport** is used to obtain an optimal transport map to find some intermediate geometries with some regularity constraints. An application is design optimization, in which the quantity of material for similar models must be the same.

2) Optimal transport map

Given Ω a Borel set and two measures μ and ν on Ω such that $\mu(\Omega) = \nu(\Omega)$, c a convex function $T : \Omega \to \Omega$ such that

$$\begin{cases} \nu(X) = \mu(T^{-1}(X)) & \text{for any Borel (i.e. measurable) subset } X \text{ of } \Omega \\ \int_\Omega c(x, T(x)) d\mu & \text{is minimal} \end{cases}$$

is called the **optimal transport** map from μ to ν. We are interested in μ continuos and ν discrete. It this case we talk of Semi Discrete Optimal Transport.

We can approximate the optimal transport map with the following algorithm:

Algorithm 1: Semi Discrete Optimal Transport

Input: Two tetrahedral meshes M and M', and k the desired number of vertices in the result

Output: A tetrahedral mesh G with k vertices and a pair of points p_i^0 and p_i^1 attached to each vertex. Transport is parameterized by time $t \in [0,1]$ with $p_i(t) = (1-t)p_i^0 + tp_i^1$

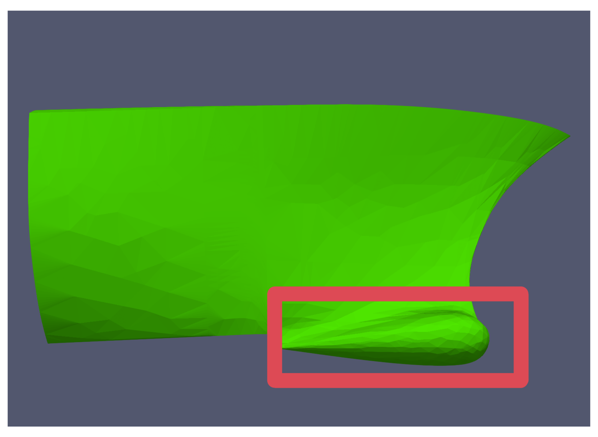
1. Sample M' with a set Y of k points
2. Compute the weight vector W that realizes the optimal transport between M and Y
3. Construct $E = Del(Y)$ where Del it the Delaunay Triangulation.
4. For each $i \in [1 \dots k]$, $(p_i)^0 \leftarrow \text{centroid}\left(\text{Pow}_W(y_i) \cap M\right)$, $\quad (p_i)^1 \leftarrow y_i$
5. G will be the mesh defined by the topology of E with the pair of points $(p_i)^0$,$(p_i)^1$.

It can be proven that the linear map does not clash particles. This algorithm is implemented in the library **Geogram** created by Bruno Levy.
We modified the map to be

$$\phi_{MM'}(t) = Vol(M)^{\frac{1}{3}} \frac{tM' + (1-t)M}{Vol(tM' + (1-t)M)^{\frac{1}{3}}}$$

in order to preserve volume in intermediate times.

Let's now show a ship hull.

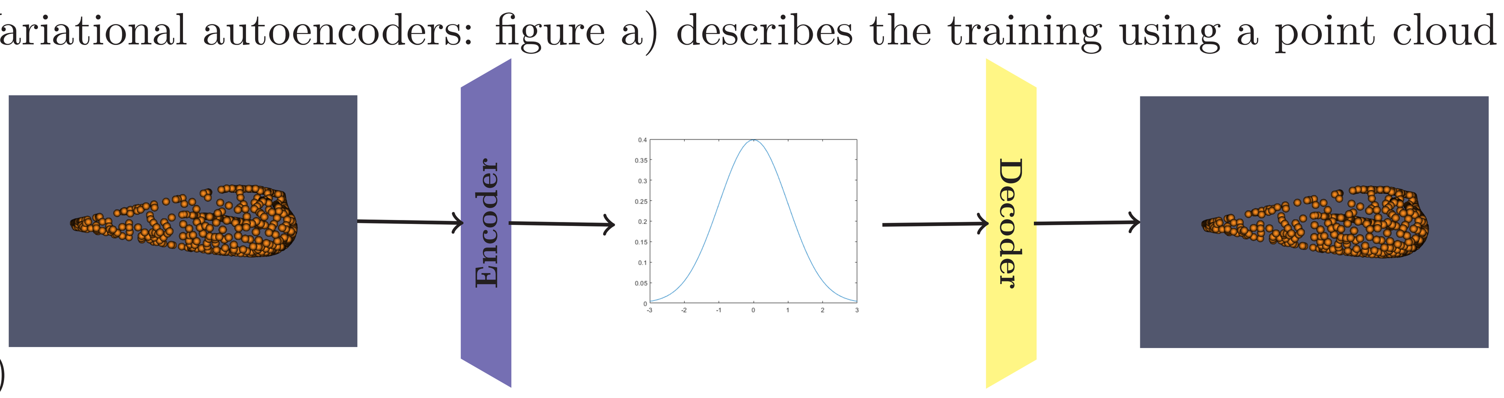


Evidenced in the red square there is the bulb, to which we applied optimal transport with one of it's deformation, as the following figure show: the first and the last figure are the bulb and its deformation respectively, in the middle there is the intermediate mesh at time $t = \frac{1}{2}$.

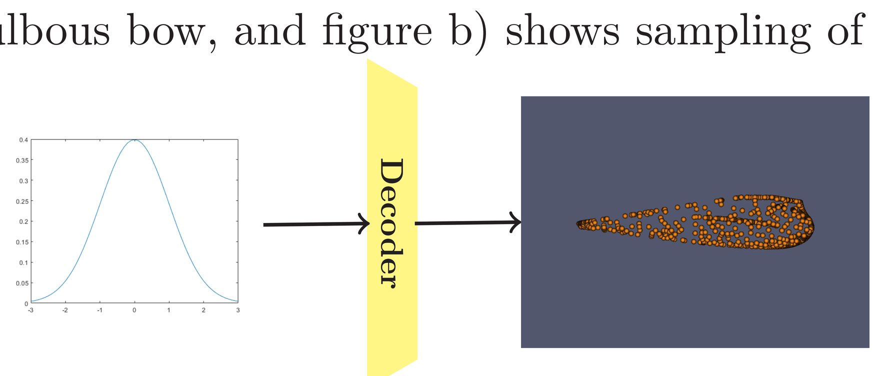
3) Generative Models

Two main model classes:

- Variational autoencoders: figure a) describes the training using a point cloud mesh of Bulbous bow, and figure b) shows sampling of a deformed Bulbous.

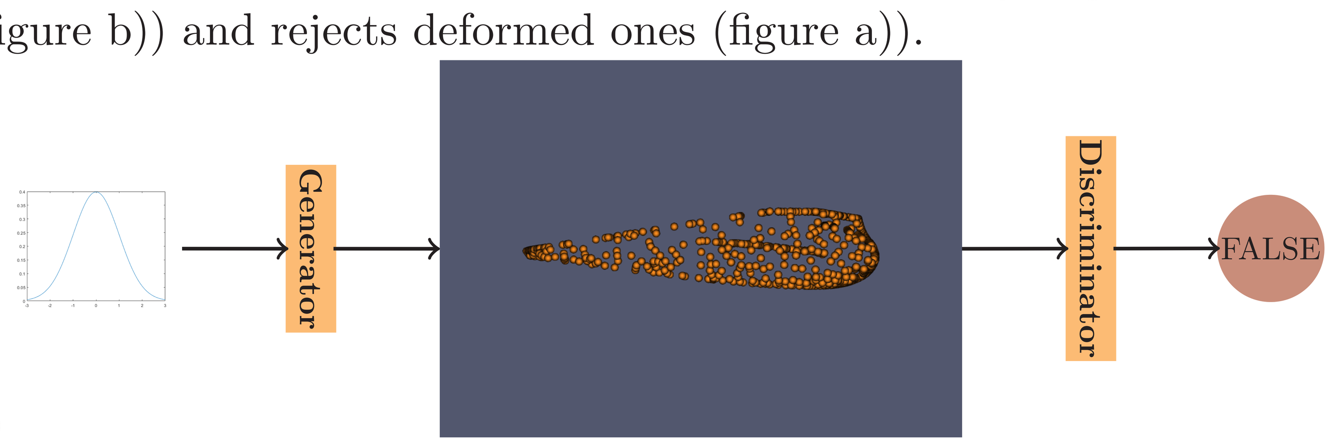


a)

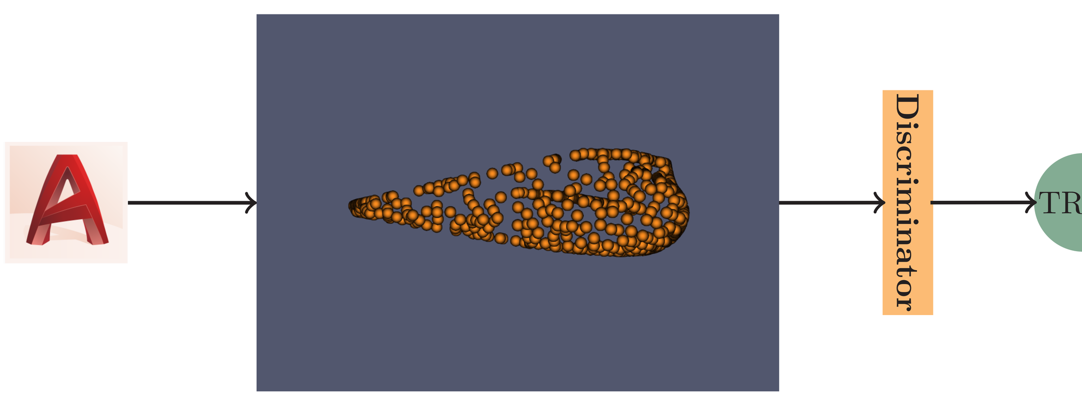


b)

- Generative adversarial network:it is characterized by a generator that samples point cloud mesh of deformed Bulbous bow and by a discriminator that accepts real Bulbous (figure b)) and rejects deformed ones (figure a)).



a)



b)

$$\mathcal{L}_{\mathrm{D}}^{\mathrm{GAN}} = -\mathbb{E}_{x\sim p_d}[\log(D(x))] - \mathbb{E}_{\hat{x}\sim p_g}[\log(1 - D(\hat{x}))]$$

$$\mathcal{L}_{\mathrm{G}}^{\mathrm{GAN}} = \mathbb{E}_{\hat{x}\sim p_g}[\log(1 - D(\hat{x}))]$$

5 - Preliminary results and future work

Summary of our results:

- We are able to do volume preserving continuos deformation between twp bulbous bows meshes using semidiscrete optimal trasport. Our next step is to generalize it for more than two meshes.
- We are able to sample bulbous bows meshes using Variational Autoencoders. However, the space Z is too much sparse, so we started studying Generative Adversarial Networks.

Bibliography and Software References

[1] A numerical algorithm for L_2 semi-discrete optimal transport in 3D, Bruno Levy, 1409.1279, 2014

[2] Variational Autoencoders for Deforming 3D Mesh Models, Qingyang Tan, arXiv:1709.04307 ,2018

[3] Geogram, `http://alice.loria.fr/software/geogram/doc/html/index.html`