# Machine Learning and Optimal Transport for shape parametrisation

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## Introduction: 2 lines of research

First line consist in studying **generative models** for shape optimization of complex geometries with a large number of parameters; the objective is to learn the shape of the geometry using a minor number of parameters, for example for modeling naval hulls, and creating new artificial geometries similar to real data, as creating new real geometries can be costly.

In the second line **semidiscrete optimal transport** is used to obtain an optimal transport map to find some intermediate geometries with some regularity constraints. An application is design optimization, in which the quantity of material for similar models must be the same. The extension of semidiscrete optimal transport to three meshes will also be discussed.

## 2 - Optimal Transport

Given $\Omega$ a Borel set and two measures $\mu$ and $\nu$ on $\Omega$ such that $\mu(\Omega) = \nu(\Omega)$, $c$ a convex function $T : \Omega \to \Omega$ such that

$$\begin{cases} \nu(X) = \mu(T^{-1}(X)) & \text{for any Borel (i.e. measurable) subset } X \text{ of } \Omega \\ \int_\Omega c(x, T(x))d\mu & \text{is minimal} \end{cases}$$

is called the **optimal transport** map from $\mu$ to $\nu$. We are interested in $\mu$ continuos and $\nu$ discrete. It this case we talk of Semi Discrete Optimal Transport.

We can approximate the optimal transport map with the following algorithm:

Data: Two tetrahedral meshes $M$ and $M'$, and $k$ the desired number of vertices in the result

Result: A tetrahedral mesh $G$ with $k$ vertices and a pair of points $p_i^0$ and $p_i^1$ attached to each vertex. Transport is parameterized by time $t \in [0,1]$ with $p_i(t) = (1-t)p_i^0 + tp_i^1$.

(1) Sample $M'$ with a set $Y$ of $k$ points

(2) Compute the weight vector $W$ that realizes the optimal transport between $M$ and $Y$

(3) Construct $E = Del(Y)$ where $Del$ it the Delaunay Triangulation.

(4) For each $i \in [1 \dots k], (p_i)^0 \leftarrow \text{centroid}\left(\text{Pow}_W(y_i) \cap M\right), \quad (p_i)^1 \leftarrow y_i$

G will be the mesh defined by the topology of $E$ with the pair of points $(p_i)^0$,$(p_i)^1$. It can be proven that the linear map does not clash particles. This algorithm is implemented in the library **Geogram** created by Bruno Levy.

We modified the map to be

$$\phi_{MM'}(t) = Vol(M)^{\frac{1}{3}} \frac{tM' + (1-t)M}{Vol(tM' + (1-t)M)^{\frac{1}{3}}}$$

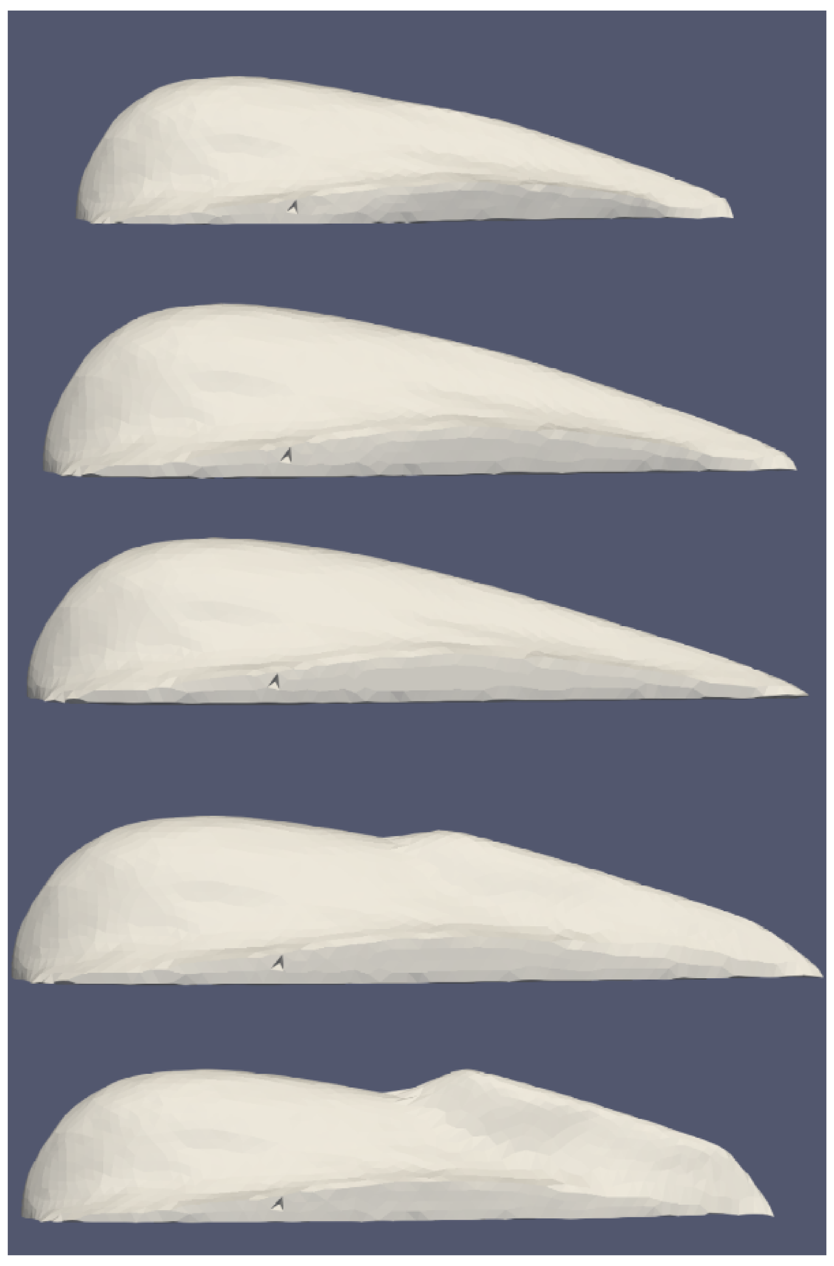
in order to preserve volume in intermediate times.

Let $M$, $M'$, $M''$ three meshes. Let us suppose we calculate the OTM from $M$ to $M'$ and from $M''$ selecting $M'$ to be sampled in the same way in both transports, so the resulting mesh will be the same (because in the algorithm does not depend on the other). So with the same assumption as before we can define a new transport map

$$\phi_{MM'M^{(2)}}(t) = \begin{cases} Vol(M')^{\frac{1}{3}} \frac{2tM' + (1-2t)M}{Vol(2tM' + (1-2t)M)^{\frac{1}{3}}} & 0 \le t \le \frac{1}{2} \\ Vol(M')^{\frac{1}{3}} \frac{(2t-1)M'' + (2-2t)M'}{Vol((2t-1)M'' + (2-2t)M')^{\frac{1}{3}}} & \frac{1}{2} \le t \le 1 \end{cases}$$

that goes from $M$ to $M'$ and then from $M'$ to $M''$.

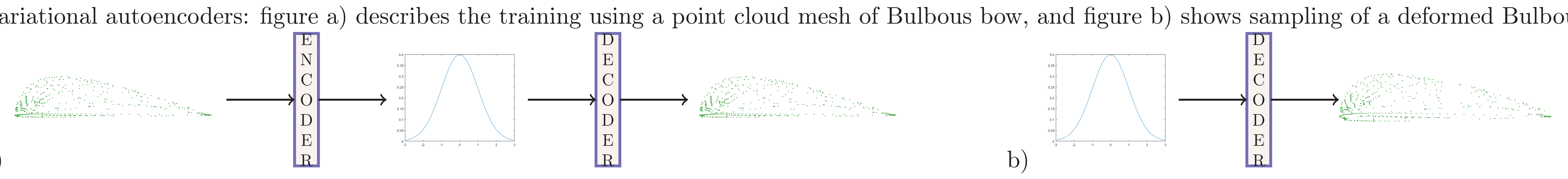
Here is showed an example of three mesh OTM of a Bulbous bow, the three meshes are images in odd position, while in even position we have the intermediate meshes:



## 4 - Generative Models

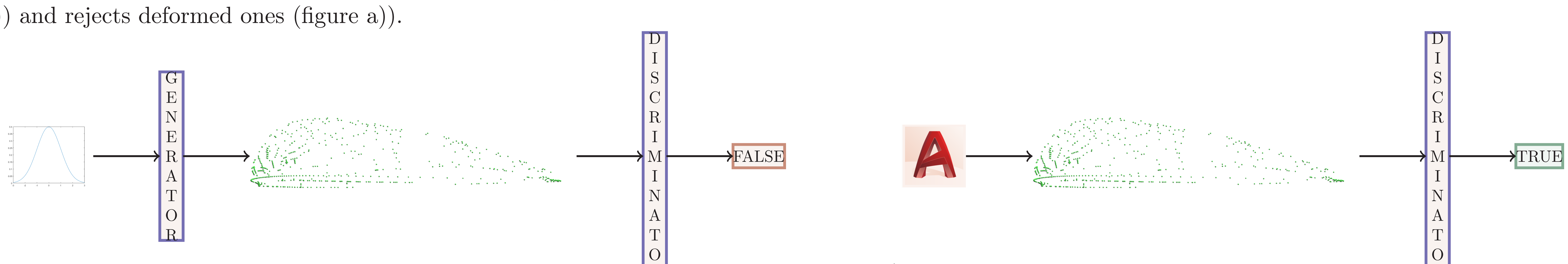
Two main model classes:

- Variational autoencoders: figure a) describes the training using a point cloud mesh of Bulbous bow, and figure b) shows sampling of a deformed Bulbous




a) b)

- Generative adversarial network:it is characterized by a generator that samples point cloud mesh of deformed Bulbous and by a discriminator that accepts real Bulbous (figure b)) and rejects deformed ones (figure a)).




a) b)

## 5 - Preliminary results and future work

Summary of our results:

- We are able to do volume preserving continuos deformation between three regular meshes using semidiscrete optimal trasport. Our next step is to generalize it for more than three meshes, which is nontrivial (as $M$ and $M'$ are nonsampled), and also to test less regular meshes.
- We are able to sample parallelepipeds centered in 0 with volume 1 using Variational Autoencoders. However, the space $Z$ is too much sparse even for these simple meshes, so we started studying Generative Adversarial Networks.

## Bibliography and Software References


- Geogram, `http://alice.loria.fr/software/geogram/doc/html/index.html`
- A numerical algorithm for $L_2$ semi-discrete optimal transport in 3D, Bruno Levy, arXiv, 2014
- Variational Autoencoders for Deforming 3D Mesh Models, Qingyang Tan, arXiv, 2018