Numerical calculation of manifolds of periodic orbits of the restricted three body problem

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Introduction

This is a brief dissertation on how to calculate the tube manifolds of the planar and vertical **Lyapunov orbits** and **Halo orbits** in the **restricted circular three body problem**. As such, it will mainly focus on the practical procedures needed to numerically compute these trajectories. We give a brief recap on the necessary theory, without proving it, so that the reader can have the mathematical foundation and tools to understand the methodology.

It is by no means a complete review on how to calculate all these orbits, in fact we do not consider many analytic techniques of computation, since they require a quite advanced level of mathematical skills. Instead we will use the physical understanding of the problem and its properties to elaborate a methodology that is heavily based on numerical computations. It is somewhat less elegant, but it should be enough for a large variety of applications.

Even though it is a basic introduction to the topic, many prerequisites are taken for granted. In particular, the reader should be familiar with linear algebra, systems of linear and nonlinear ODEs, Hamiltonian mechanics and numerical methods of integration and root finding.

The reader should also be familiar with a programming language oriented towards scientific computations. Python, Matlab, C++, etc. are some of the most used.

This work is aimed towards STEM undergraduate students that are interested in this topic.

Part I

Theory

Chapter 1

The three-body problem

1.1 Assumptions for the three body problem

We are all familiar with the two body problem, especially in orbital mechanics. Two bodies of masses m_1 and m_2 are attracted by one another by a force

$$\mathbf{F} = -G\frac{m_1 m_2}{|r|^3}\mathbf{r} \tag{1.1}$$

and follow elliptical orbits around their center of mass.

This is a pretty good approximation of what happens between the Earth and the Moon, for example, because the distance from the Sun and the other planets is so large that their influence is pretty negligible.

If we were to account for all the bodies in the solar system we would have the *n-body problem*, which is a hard problem to solve.

So we will limit ourselves to three bodies, which is not as complicated as the *n-body problem*, but still offers interesting dynamics to study.

Since the orbits of planets and moon are ellipses with a very low eccentricity, we can approximate them to circles. Moreover we are really interested in the dynamics of a small object (a spacecraft, a comet) moving in the gravitational field of two massive bodies (planets, moons), therefore the third body does not really affect the gravitational field of the whole system.

Therefore we can simplify again our problem by studying the **circular restricted three body problem**(CR3BP from now on).

Circular: Because the massive bodies follow elliptical orbits that are very close to circumferences

Restricted: Because we are interested in the trajectories of a spacecraft,which has a mass so small compared to the other two bodies that it does not influence the gravitational field of the system

So, if we were to study the Earth-Moon-spacecraft example we would have that the Earth and the Moon follow circular trajectories (given by the 2-body problem) around their center of mass, and the spacecraft is moving without interfering in the gravitational potential given by the 2 massive bodies.

1.2 The two-body problem

We will first study the two-body problem, and then we will use it as a foundation to build the restricted three-body problem. We do this because the main bodies are bound to a plane by the dynamics of the gravitational force (a central force), therefore we can **neglect the third coordinate** (Z) and simplify our calculations.

This is just to study the motion of the two main bodies, when we will treat the spacecraft we will use all three dimensions.

How do to the massive bodies move? We will see how the two bodies interact with each other and how we can reduce this problem to a simpler one.

The first body is of mass m_1, whilst the second is of mass m_2 (usually $m_1 > m_2$).

Their position in a **inertial** frame of reference is indicated by $\mathbf{R}_1$ and $\mathbf{R}_2$ respectively. By following Newton's law of gravitation we have

$$\mathbf{F}_1 = -G\frac{m_1 m_2}{||\mathbf{R}_2 - \mathbf{R}_1||^3}(\mathbf{R}_2 - \mathbf{R}_1) \tag{1.2}$$

$$\mathbf{F}_2 = G\frac{m_1 m_2}{||\mathbf{R}_2 - \mathbf{R}_1||^3}(\mathbf{R}_2 - \mathbf{R}_1) \tag{1.3}$$

While the center of mass of the system is:

$$\mathbf{B} = \frac{m_1\mathbf{R}_1 + m_2\mathbf{R}_2}{m_1 + m_2} = \mathbf{0} \tag{1.4}$$

We say that $\mathbf{B} = \mathbf{0}$ without loss of generality because the system is isolated.

We can simplify this problem by introducing:

$$\tilde{\mathbf{R}} = \mathbf{R}_2 - \mathbf{R}_1 \tag{1.5}$$

$$M = m_1 + m_2 \tag{1.6}$$

$$\mu = \frac{m_2}{M} \tag{1.7}$$

From Newton's second law, each body moves according to:

$$\ddot{\mathbf{R}}_1 = -G\frac{m_2}{||\tilde{\mathbf{R}}||^3}\tilde{\mathbf{R}} \tag{1.8}$$

$$\ddot{\mathbf{R}}_2 = G\frac{m_1}{||\tilde{\mathbf{R}}||^3}\tilde{\mathbf{R}} \tag{1.9}$$

Therefore, since $\ddot{\tilde{\mathbf{R}}} = \ddot{\mathbf{R}}_2 - \ddot{\mathbf{R}}_1$, we have:

$$\ddot{\tilde{\mathbf{R}}} = \frac{GM}{||\tilde{\mathbf{R}}||^3}\tilde{\mathbf{R}} \tag{1.10}$$

Which is the equation of motion of a Kepler problem with mass $M = m_1 + m_2$. Solving it allows us to find back $\mathbf{R}_1$ and $\mathbf{R}_2$ by substituting $\tilde{\mathbf{R}}$ in 1.5 and 1.4.

Since we are dealing with a Kepler problem,which is a central force problem, we can also express $\tilde{\mathbf{R}}$ in polar coordinates as:

$$\tilde{\mathbf{R}} = R(\cos(\theta(t)), \quad \sin(\theta(t)))$$

From equation 1.10 we know we are dealing with a central force potential:

$$V(R) = -\frac{GM}{R}$$

Using the Hamiltonian in polar coordinates:[1]:

$$H = \frac{p_R^2}{2} + \frac{p_\theta^2}{2R^2} - \frac{GM}{R} \tag{1.11}$$

Where p_R is the radial momentum, p_θ is the angular momentum and $R = \|\mathbf{R}\|$. We want a circular motion, which is found by solving

$$\dot{p}_R = -\frac{\partial H}{\partial R} = -\frac{p_\theta^2}{R^3} + \frac{GM}{R^2} = 0 \tag{1.12}$$

giving the radius of the circular orbit

$$R^* = \frac{p_\theta^2}{GM} \tag{1.13}$$

We know that on the circular orbit:

$$p_\theta = \dot{\theta} R^*$$

and by substituting p_θ we have

$$\frac{GM}{R^{*3}} = \dot{\theta}^2$$

and it is clear that $\dot{\theta}$ is constant, since the left side is also constant.

By calling $\dot{\theta} = \omega$ and integrating we have:

$$\theta(t) = \omega t + C \tag{1.14}$$

but we can neglect the integration constant.

Therefore the solution is:

$$\tilde{\mathbf{R}}(t) = R^* \left(\cos(\omega t), \sin(\omega t)\right) \tag{1.15}$$

the expressions of $\mathbf{R}_1$ and $\mathbf{R}_2$ follow immediately from 1.4, because:

$$\mathbf{B} = \frac{m_1}{M}\mathbf{R}_1 + \frac{m_2}{M}\mathbf{R}_2 =$$

$$(1-\mu)\mathbf{R}_1 + \mu\mathbf{R}_2 = \mathbf{0} \tag{1.16}$$

Therefore

$$\mathbf{R}_1(t) = -\mu R^* \left(\cos(\omega t), \sin(\omega t)\right) \tag{1.17}$$

$$\mathbf{R}_2(t) = (1-\mu) R^* \left(\cos(\omega t), \sin(\omega t)\right) \tag{1.18}$$

We can normalize the equations by choosing the physical units of measure in such a way that

$$R^* = 1$$

---

[1] Clearly, we skip some steps here. See later for a definition of the Hamiltonian

$$\omega = \sqrt{\frac{GM}{R^{*3}}} = 1 \tag{1.19}$$
$$M = 1$$

which also has to give

$$G = 1$$

Now we have the normalized solution in a **inertial** frame of reference:

$$\mathbf{R}_1(t) = -\mu(\cos t, \sin t)$$

$$\mathbf{R}_2(t) = (1-\mu)(\cos t, \sin t)$$

In a three dimensional frame of reference **aligned with orbital plane** they are:

$$\mathbf{R}_1(t) = \left(-\mu\cos t, -\mu\sin t, 0\right) \tag{1.20}$$

$$\mathbf{R}_2(t) = \left((1-\mu)\cos t, (1-\mu)\sin t, 0\right) \tag{1.21}$$

1.3 Inertial and rotating frames

Now we have two massive bodies that move by following 1.20 and 1.21 and generate a gravitational field. What happens if we put a body of negligible mass (a comet, or a spacecraft) in this field?

We will see that in a **inertial** frame of reference its motion is quite complicated, but in a **rotating** it will become mush simpler, while giving us come interesting insights on the dynamics.

Let us call the position of the particle in a three dimensional **inertial** frame of reference:

$$\mathbf{R} = (X, Y, Z)$$

and the gravitational potential generated by the two primaries:

$$V = -\frac{1-\mu}{d_1} - \frac{\mu}{d_2} \tag{1.22}$$

where

$$d_1 = \|(X, Y, Z) - \mathbf{R}_1(t)\|$$

$$d_2 = \|(X, Y, Z) - \mathbf{R}_2(t)\|$$

are the distances of **R** from the two primaries.
Therefore the particle is subject to the Lagrangian:

$$L(X, Y, Z\dot{X}, \dot{Y}, \dot{Z}, t) = T - V = \tfrac{1}{2}\|\dot{X}, \dot{Y}, \dot{Z}\|^2 + \frac{1-\mu}{d_1} + \frac{\mu}{d_2} \tag{1.23}$$

Notice that the potential **V** is time dependent, since the position of the bodies generating gravity is changing in time following 1.20 and 1.21. This is not desirable.

But by analyzing 1.20 and 1.21 we notice that the two primaries are always opposite to each other, are always lined up with the origin and are orbiting O with the same period $\dfrac{2\pi}{w}$.

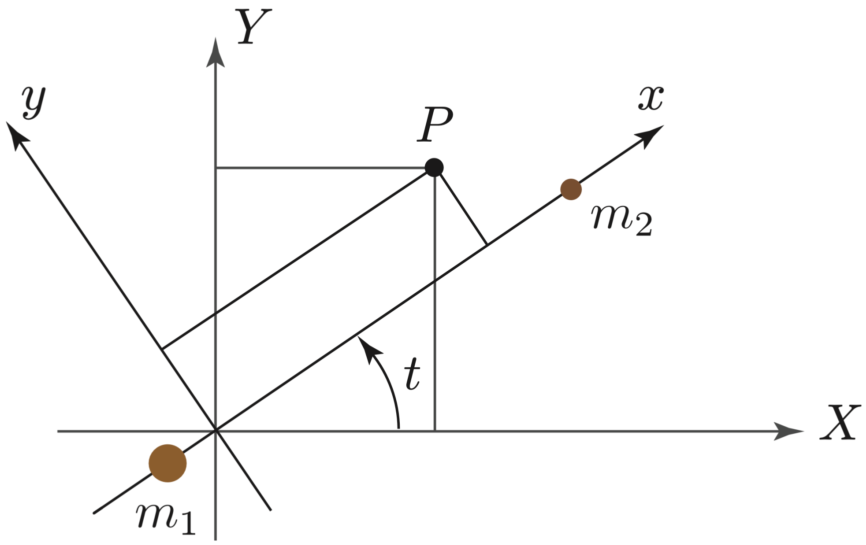

Figure 1.1: Rotating frame

Therefore we can introduce an axis passing through m_1 and m_2 that is rotating with the bodies around O with angular velocity $\omega = 1$. We call this axis $\hat{x}$, $\hat{z}$ the axis parallel to the vector of angular momentum of the main bodies and $\hat{y}$ the remaining axis defined by the right-hand rule, with the center of the new frame in the center of mass of the system. Now we have a **rotating** frame of reference $(\hat{x}, \hat{y}, \hat{z})$.

We can use a rotation matrix to go from the **inertial** frame to the **rotating** one:

$$\mathbf{N} = \begin{bmatrix} \cos t & \sin t & 0 \\ -\sin t & \cos t & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1.24}$$

if we call the position of the two primaries in the **rotating** frame of reference $\mathbf{r}_1$ and $\mathbf{r}_2$, we have:

$$\mathbf{r}_1 = \mathbf{N} \cdot \mathbf{R}_1 = -\mu(1,0,0) \tag{1.25}$$

$$\mathbf{r}_2 = \mathbf{N} \cdot \mathbf{R}_2 = (1-\mu)(1,0,0) \tag{1.26}$$

If we call the position of the particle in the **rotating** frame of reference:

$$\mathbf{r} = (x, y, z)$$

We have that the distances becomes:

$$d_1 = \sqrt{(x+\mu)^2 + y^2 + z^2} \qquad d_2 = \sqrt{(x+\mu-1)^2 + y^2 + z^2}$$

which means that the potential $V(x, y, z)$ is no longer time dependent!

But we also want to translate the kinetic energy T to the rotating coordinates. Notice that, since $\mathbf{N}$ is a rotation matrix, it is orthogonal:

$$\mathbf{N}^{-1} = \mathbf{N}^T$$

therefore, to go from **rotating** coordinates to **inertial**:

$$(X, Y, Z) = \mathbf{N}^T(x, y, z)$$

$$(\dot{X}, \dot{Y}, \dot{Z}) = \dot{\mathbf{N}}(x, y, z) + \mathbf{N}(\dot{x}, \dot{y}, \dot{z})$$

Now we only need the norm of these vectors:

$$\|(\dot{X}, \dot{Y}, \dot{Z})\|^2 = \|\dot{\mathbf{N}}(x, y, z)\|^2 + \|\mathbf{N}(\dot{x}, \dot{y}, \dot{z})\|^2 + 2\dot{\mathbf{N}}(x, y, z) * \mathbf{N}(\dot{x}, \dot{y}, \dot{z})$$

By doing some calculations:

$$\|(\dot{X}, \dot{Y})\|^2 = (x^2 + y^2) + (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + 2(\dot{y}x - \dot{x}y)$$

Therefore the Lagrangian in the **rotating** frame of reference is:

$$L(\dot{x}, \dot{y}, \dot{z}, x, y, z) = \frac{1}{2}(x^2 + y^2) + \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + (\dot{y}x - \dot{x}y) + \frac{1-\mu}{d_1} + \frac{\mu}{d_2} \tag{1.27}$$

$$d_1 = \sqrt{(x+\mu)^2 + y^2 + z^2} \qquad d_2 = \sqrt{(x+\mu-1)^2 + y^2 + z^2} \tag{1.28}$$

From now on we will use only this **rotating** frame of reference, since it is time-independent and tells us the position of the spacecraft relatively to the position of the primaries, which is what we are interested in.

1.4 Hamiltonian and Hamilton equations

Now that we have the Lagrangian we can use the Legendre transformation to get the Hamiltonian.
Let the momentum be:

$$p^i = \frac{\partial L}{\partial \dot{q}^i} \tag{1.29}$$

where the superscript indicates the i-th component.
The Hamiltonian is defined by:

$$H(q, p) = \sum_{i=1}^{n} p^i \dot{q}^i - L(q, \dot{q}^i)$$

where $\dot{q}^i$ is a function of p^i, the inverse of 1.29.

In our case the **Hamiltonian** of the CR3BP in the **rotating** frame of reference is:

$$\boxed{H(\mathbf{r}) = \tfrac{1}{2}\left(p_x^2 + p_y^2 + p_z^2\right) + p_x y - p_y x - \frac{1-\mu}{\sqrt{(x+\mu)^2 + y^2 + z^2}} - \frac{\mu}{\sqrt{(x+\mu-1)^2 + y^2 + z^2}} \quad (1.30)}$$

and the **Hamilton equations** are:

$$\dot{x} = \frac{\partial H}{\partial p_x} = p_x + y \tag{1.31}$$

$$\dot{y} = \frac{\partial H}{\partial p_y} = p_y - x \tag{1.32}$$

$$\dot{z} = \frac{\partial H}{\partial p_z} = p_z \tag{1.33}$$

$$\dot{p}_x = -\frac{\partial H}{\partial x} = p_y - (1-\mu)\frac{x+\mu}{\left((x+\mu)^2+y^2+z^2\right)^{3/2}} - \mu\frac{x+\mu-1}{\left((x+\mu-1)^2+y^2+z^2\right)^{3/2}} \tag{1.34}$$

$$\dot{p}_y = -\frac{\partial H}{\partial y} = -p_x - (1-\mu)\frac{y}{\left((x+\mu)^2+y^2+z^2\right)^{3/2}} - \mu\frac{y}{\left((x+\mu-1)^2+y^2+z^2\right)^{3/2}} \tag{1.35}$$

$$\dot{p}_z = -\frac{\partial H}{\partial z} = -(1-\mu)\frac{z}{\left((x+\mu)^2+y^2+z^2\right)^{3/2}} - \mu\frac{z}{\left((x+\mu-1)^2+y^2+z^2\right)^{3/2}} \tag{1.36}$$

These are the equations that we will use to determine the spacecraft trajectory.

In more rigorous term equations 1.31 to 1.36 can be rewritten as[2]:

$$\mathbf{r} \in \mathbb{R}^6 \qquad \dot{\mathbf{r}} = \mathbf{X}(\mathbf{r}) \tag{1.37}$$

with:

$$\mathbf{X}(\mathbf{r}) = \mathbf{J} \cdot \nabla H(\mathbf{r})$$

where **J** is the symplectic matrix and $\mathbf{r} = \left(x, y, z, p_x, p_y, p_z\right)$

$$\mathbf{J} = \begin{bmatrix} 0 & I_2 \\ -I_2 & 0 \end{bmatrix} \tag{1.38}$$

We call 1.37 an *Hamiltonian vector field* and its solution an *Hamiltonian flow*, written as:

$$\mathbf{r}(t) = \phi(t, \mathbf{r})$$

Unfortunately there is no analytic solution, so we will have to solve 1.37 numerically (more on that in *Part II* and *Part III*).

Since this is an *autonomous* and *positional* system, the Hamiltonian is also the mechanical energy:

$$H = T + V = E$$

---

[2] Before **r** was the position of the particle in the position space $\mathbf{r} = (x, y, z)$.
From now on we will use **r** to indicate the *state vector* of the particle

Chapter 2

Equilibrium points and linearization

Once one has a dynamical system it is interesting to study its equilibrium points (if they exist) and the stability of such points.

In this chapter we will give a brief theoretical explanation of such topic, limiting ourselves to only the necessary theorems, without proving them.

We will first see how to find the equilibrium points, then how to calculate the Jacobian matrix in these points and what it tells us about (linear) stability.

We will explain the theory through a working example, the **planar circular restricted 3 body problem**, which is what we have introduced in the previous chapter, but limited to the plane. Therefore for this chapter the formulas and examples will be referred to a **two-dimensional problem**.

We do this because the notation is far less cumbersome and the theoretical explanation does not lose generalization. Everything we say in this chapter can be easily "scaled up" to a spatial problem.

In any case in *Part III* we will calculate the trajectories in three dimensions, and we will give there all the necessary formulas and remarks .

2.1 Equilibrium points

We have a dynamical system:

$$\dot{\mathbf{r}} = \mathbf{X}(\mathbf{r}) \tag{2.1}$$

An equilibrium point is a point in the phase space where the particle does not change its state with time. Let $\mathbf{c}$ be the *equilibrium point*:

$$\mathbf{r}(t) = \mathbf{c} \qquad \forall t$$

We can find this point by searching for the state vector $\mathbf{c}$ that satisfies:

$$\mathbf{X}(\mathbf{c}) = \mathbf{0}$$

which is:

$$\dot{x} = p_x + y = 0 \tag{2.2}$$

$$\dot{y} = p_y - x = 0 \tag{2.3}$$

$$\dot{p}_x = p_y - (1-\mu)\frac{x+\mu}{\left((x+\mu)^2+y^2\right)^{3/2}} - \mu\frac{x+\mu-1}{\left((x+\mu-1)^2+y^2\right)^{3/2}} = 0 \tag{2.4}$$

$$\dot{p}_y = -p_x - (1-\mu)\frac{y}{\left((x+\mu)^2+y^2\right)^{3/2}} - \mu\frac{y}{\left((x+\mu-1)^2+y^2\right)^{3/2}} = 0 \tag{2.5}$$

Notice that the solution it has to be:

$$\boxed{\mathbf{c} = (x_c, y_c, -y_c, x_c)}$$

Substituting, equations 2.4 and 2.5 become :

$$x - (1-\mu)\frac{x+\mu}{\left((x+\mu)^2+y^2\right)^{3/2}} - \mu\frac{x+\mu-1}{\left((x+\mu-1)^2+y^2\right)^{3/2}} = 0$$

$$y - (1-\mu)\frac{y}{\left((x+\mu)^2+y^2\right)^{3/2}} - \mu\frac{y}{\left((x+\mu-1)^2+y^2\right)^{3/2}} = 0$$

solving these equations gives the coordinates of the equilibrium points in the **rotating** frame. These are the famous Lagrange points, and will be denoted by L_1, L_2, L_3, L_4, L_5.

2.1.1 Collinear equilibrium points

If $y = 0$ we are aligned with the primaries, and 2.4 becomes:

$$x - (1-\mu)\frac{x+\mu}{|x+\mu|^3} - \mu\frac{x-1+\mu}{|x-1+\mu|^3} = 0 \tag{2.6}$$

which once solved[1], gives the following equilibrium state space points:

$$L_1 = (x_{L_1}, 0, 0, x_{L_1}) \qquad L_2 = (x_{L_2}, 0, 0, x_{L_2}) \qquad L_3 = (x_{L_3}, 0, 0, x_{L_3})$$

with

$$x_{L_3} < -\mu < x_{L_1} < 1-\mu < x_{L_2}$$

In the following chapters we will only consider collinear equilibrium points, because their instability gives origin to much more interesting dynamics. More on that later.

2.1.2 Equilateral equilibrium points

If $y \neq 0$ equation 2.5 becomes:

$$1 - \frac{1-\mu}{\left((x+\mu)^2+y^2\right)^{3/2}} - \frac{\mu}{\left((x+\mu-1)^2+y^2\right)^{3/2}} = 0 \tag{2.7}$$

which has solutions:

$$L_4 = \left(\frac{1}{2}-\mu, \frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}, \frac{1}{2}-\mu\right) \qquad L_5 = \left(\frac{1}{2}-\mu, -\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}, \frac{1}{2}-\mu\right)$$

---

[1] The numerical solution will be treated in part *II*

2.2 Linearization

Now that we have an equilibrium point **c** we can use it as the center of a Taylor expansion of first order.
This is useful because it will give us:

- A local (close to **c**) linear approximation of the original non-linear system
- Information about the stability of the equilibrium point **c**

The state vector can be expressed as:

$$\mathbf{r} = \mathbf{c} + \delta\mathbf{r} \tag{2.8}$$

where $\delta\mathbf{r} = (\delta x, \delta y, \delta p_x, \delta p_y)$ is a small perturbation (or a small distance between **r** and **c**).

$$\begin{aligned} x &= x_c + \delta x \\ y &= y_c + \delta y \\ p_x &= p_{x,c} + \delta p_x \\ p_y &= p_{y,c} + \delta p_y \end{aligned}$$

Notice that **c** does not depend on time,therefore:

$$\dot{\mathbf{r}} = \delta\dot{\mathbf{r}} \tag{2.9}$$

The Taylor expansion of $\mathbf{X}(\mathbf{r})$ around **c** is:

$$\dot{\mathbf{r}} = \mathbf{X}(\mathbf{r}) = \mathbf{X}(\mathbf{c}) + \nabla\mathbf{X}(\mathbf{c}) \cdot (\mathbf{r} - \mathbf{c}) + \ldots$$

and, because of 2.8 and 2.9 and the definition of equilibrium point, we get:

$$\delta\dot{\mathbf{r}} = \mathbf{0} + \nabla\mathbf{X}(\mathbf{c}) \cdot \delta\mathbf{r} + \ldots$$

Considering only the linear dynamics:

$$\delta\dot{\mathbf{r}} = \nabla\mathbf{X}(\mathbf{c}) \cdot \delta\mathbf{r} \tag{2.10}$$

where $\nabla\mathbf{X}$ is the **Jacobian** of **X** calculated in **c**.

The general Jacobian for $\mathbf{X}(\mathbf{r})$ is:

$$\nabla\mathbf{X}(\mathbf{r}) = \begin{bmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ \frac{\partial \dot{p}_x}{\partial x} & \frac{\partial \dot{p}_x}{\partial y} & 0 & 1 \\ \frac{\partial \dot{p}_y}{\partial x} & \frac{\partial \dot{p}_y}{\partial y} & -1 & 0 \end{bmatrix} \tag{2.11}$$

where:

$$\frac{\partial \dot{p}_x}{\partial x} = -(1-\mu)\left(\frac{1}{d_1^3} - 3\frac{(x+\mu)^2}{d_1^5}\right) - \mu\left(\frac{1}{d_2^3} - 3\frac{(x+\mu-1)^2}{d_2^5}\right)$$

$$\frac{\partial \dot{p}_y}{\partial y} = -(1-\mu)\left(\frac{1}{d_1^3} - 3\frac{y^2}{d_1^5}\right) - \mu\left(\frac{1}{d_2^3} - 3\frac{y^2}{d_2^5}\right)$$

$$\frac{\partial \dot{p}_x}{\partial y} = \frac{\partial \dot{p}_y}{\partial x} = 3(1-\mu)\frac{(x+\mu)y}{d_1^5} + 3\mu\frac{(x+\mu-1)y}{d_2^5}$$

Once $\nabla\mathbf{X}$ is evaluated in $\mathbf{c}$ it gives a 4×4 constant coefficients matrix, which makes 2.10 a **linear system of differential equations with constant coefficients**.

This is a very easy system to solve, and it gives the *linearized dynamics* near the equilibrium point, which, as we will see later, can give very interesting insights even on the non-linear dynamics close to $\mathbf{c}$.

2.3 Linear stability of equilibrium points

To know the linear stability of an equilibrium point $\mathbf{c}$ we need the eigenvalues λ_i and eigenvectors $\mathbf{v}_i$ of $\nabla\mathbf{X}(\mathbf{c})$,

$$\nabla\mathbf{X}(\mathbf{c}) \cdot \mathbf{v}_i = \lambda_i \mathbf{v}_i$$

since we are considering only *collinear equilibrium points* we obtain (we do not prove it) that:

$$\lambda_1 = \lambda \qquad \lambda_2 = -\lambda \qquad \lambda_3 = i\omega \qquad \lambda_4 = -i\omega$$

with $\lambda, \omega \in \mathbb{R}$.

Knowing the eigenvalues we can categorize the equilibrium:

Equilibrium points categorization Let $\mathbf{c}$ be an equilibrium point and $\nabla\mathbf{X}(\mathbf{c})$ the $n \times n$ Jacobian calculated in $\mathbf{c}$. The equilibrium is:

- *Elliptic* if $Re(\lambda_n) = 0 \quad \forall n$
- *Hyperbolic* if $Re(\lambda_n) \neq 0 \quad \forall n$
- *Partially hyperbolic* if there are $Re(\lambda) = 0$ and $Re(\lambda) \neq 0$

In our case the equilibrium is *partially hyperbolic*.

The next step is to define the diagonal matrix of eigenvalues:

$$\mathbf{\Lambda} = \begin{bmatrix} \lambda & 0 & 0 & \\ 0 & -\lambda & 0 & 0 \\ 0 & 0 & i\nu & 0 \\ 0 & 0 & 0 & -i\nu \end{bmatrix} \tag{2.12}$$

and the matrix of the respective eigenvectors:

$$\boldsymbol{P} = [\mathbf{v}_1, \quad \mathbf{v}_2, \quad \mathbf{v}_3, \quad \mathbf{v}_4] \tag{2.13}$$

By inspection we find that the eigenvectors are complex and linearly independent, thus they form a basis (again, we do not prove it), therefore if $\mathbf{u}$ is a vector[2] in this eigenbasis we have:

$$\mathbf{u} = (u_1, u_2, u_3, u_4)$$

with $u_1, u_2, u_3, u_4 \in \mathbb{C}$.

---

[2] Mind the notation! u_i is the i-th component of $\mathbf{u}$, while $\mathbf{u}_i$ is the vector where only the i-th component is non-zero.
$\mathbf{u}_1 = (u_1, 0, 0, 0)$
$\mathbf{v}_1 = \mathbf{P}\mathbf{u}_1$

To obtain again the vector in the original basis:

$$\delta\mathbf{r} = \mathbf{P}\cdot\mathbf{u} \tag{2.14}$$

and also:

$$\delta\dot{\mathbf{r}} = \mathbf{P}\cdot\dot{\mathbf{u}} \tag{2.15}$$

since $\mathbf{P}$ is constant. If we substitute this in 2.10 we get:

$$\mathbf{P}\cdot\dot{\mathbf{u}} = \nabla\mathbf{X}(\mathbf{c})\cdot\mathbf{P}\cdot\mathbf{u}$$

$$\dot{\mathbf{u}} = \mathbf{P}^{-1}\cdot\nabla\mathbf{X}(\mathbf{c})\cdot\mathbf{P}\cdot\mathbf{u}$$

but because $\nabla\mathbf{X}(\mathbf{c})$ is diagonalizable:

$$\dot{\mathbf{u}} = \Lambda\cdot\mathbf{u} \tag{2.16}$$

and the problem is now trivial, since:

$$\dot{u}_i = \lambda_i u_i$$

which means:

$$u_i(t) = u_i(0)e^{\lambda_i t}$$

by using 2.14 we can easily find:

$$\delta\mathbf{r}(t) = \mathbf{P}\cdot\mathbf{u}(t)$$

However this is a problem. Since both $\mathbf{P}$ and $\mathbf{u}$ have complex components the result $\delta\mathbf{r}$ is generally complex too. This is not good for our purposes, because a complex state vector would not make any physical sense. Therefore we need to impose some restrictions on $\mathbf{u}$ to get a $\delta\mathbf{r}\in\mathbb{R}$.

By inspecting $\mathbf{P}$ one sees that the first two columns are real, therefore u_1 and u_2 need only to be real.

The third and fourth column are complex conjugates:

$$P_{i3} = \begin{pmatrix} a \\ -ib \\ ic \\ d \end{pmatrix} \qquad\qquad P_{i4} = \begin{pmatrix} a \\ ib \\ -ic \\ d \end{pmatrix}$$

If $u_1 = u_2 = 0$ we obtain

$$\delta\mathbf{r} = u_3\begin{pmatrix} a \\ -ib \\ ic \\ d \end{pmatrix} + u_4\begin{pmatrix} a \\ ib \\ -ic \\ d \end{pmatrix}$$

Suppose that

$$u_3 = \alpha + i\beta$$

$$u_4 = \eta + i\xi$$

we get

$$\begin{aligned}
\delta x &= a(\alpha + i\beta + \eta + i\xi) \\
\delta y &= ib(-\alpha - i\beta + \eta + i\xi) \\
\delta p_x &= ic(\alpha + i\beta - \eta - i\xi) \\
\delta p_y &= d(\alpha + i\beta + \eta + i\xi)
\end{aligned}$$

and we want the result to be real, which means that:

$$\beta + \xi = 0$$
$$\alpha - \eta = 0$$

therefore

$$\eta = \alpha \qquad\qquad \xi = -\beta$$

which means that u_4 is the complex conjugate of u_3:

$$\boxed{u_4 = \alpha - i\beta = u_3^*} \tag{2.17}$$

Therefore to obtain real results we need to choose $u_1, u_2 \in \mathbb{R}$ and $u_4 = u_3^* \in \mathbb{C}$.

Finally, the solution in the eigenbasis is:

$$u_1(t) = u_1(0)e^{\lambda t}$$
$$u_2(t) = u_2(0)e^{-\lambda t}$$
$$u_3(t) = u_3(0)e^{i\omega t}$$
$$u_4(t) = u_4(0)e^{-i\omega t}$$

where:

$$u_1(0), u_2(0) \in \mathbb{R} \qquad u_4(0) = u_3(0)^* \in \mathbb{C}$$

2.3.1 Unstable, stable and center spaces

Now we will give a definition of the **unstable**, **stable**, and **center** spaces based on the value of the eigenvalues.

Let $\phi(t, \delta\mathbf{r})$ be the flow of the linearized system, with initial condition $\delta\mathbf{r}$:

- A **stable** space is a set of points $\delta\mathbf{r} \in \mathbf{E}^S$ such that:

$$\lim_{t\to\infty} \phi(t, \delta\mathbf{r}) = \mathbf{c}$$

- An **unstable** space is a set of points $\delta\mathbf{r} \in \mathbf{E}^U$ such that:

$$\lim_{t\to-\infty} \phi(t, \delta\mathbf{r}) = \mathbf{c}$$

- A **center** space is a set of points $\delta\mathbf{r} \in \mathbf{E}^C$ such that:

$$\|\phi(t, \delta\mathbf{r})\| < \epsilon \qquad \epsilon > 0 \quad \forall t > 0$$

From 2.14 we know that:

$$\mathbf{v}_1 = \mathbf{P} \cdot \mathbf{u}_1$$
$$\mathbf{v}_2 = \mathbf{P} \cdot \mathbf{u}_2$$
$$\mathbf{v}_3 = \mathbf{P} \cdot \mathbf{u}_3$$
$$\mathbf{v}_4 = \mathbf{v}_3^* = \mathbf{P} \cdot \mathbf{u}_3^*$$

We call the space spanned by $\mathbf{v}_1$ the **unstable** space, the space spanned by $\mathbf{v}_2$ the **stable** space, the one spanned by $\mathbf{v}_3, \mathbf{v}_4$ the **center** space.

$$\mathbb{E}^U = \langle \mathbf{v}_1 \rangle$$

$$\mathbb{E}^S = \langle \mathbf{v}_2 \rangle$$

$$\mathbb{E}^C = \langle \mathbf{v}_3, \mathbf{v}_4 \rangle$$

Thus we associate the *positive* eigenvalue to the **unstable** space, the *negative* eigenvalue to the **stable** space and the *imaginary* eigenvalues to the **center** space.

These spaces have the following properties:

1. We can decompose $\mathbb{R}^4$ into the three spaces

   $$\mathbb{E}^S \oplus \mathbb{E}^U \oplus \mathbb{E}^C = \mathbb{R}^4$$

   which means that the composition of these spaces gives a complete description of the linearized phase space.

2. $\mathbb{E}^U$, $\mathbb{E}^S$, $\mathbb{E}^C$ are invariant, which means that any starting point on these spaces will give a point on these positions at time t:

   $$\begin{aligned} &\forall \delta\mathbf{r} \in \mathbb{E}^S \qquad \phi(t, \delta\mathbf{r}) \in \mathbb{E}^S \quad t > 0 \\ &\forall \delta\mathbf{r} \in \mathbb{E}^U \qquad \phi(t, \delta\mathbf{r}) \in \mathbb{E}^U \quad t > 0 \\ &\forall \delta\mathbf{r} \in \mathbb{E}^C \qquad \phi(t, \delta\mathbf{r}) \in \mathbb{E}^C \quad t > 0 \end{aligned}$$

Notice how, for our system, E^S and E^U are lines while E^C is a 2D surface with at its core complex exponentials

$$\mathbf{v}_3 = \mathbf{P} * \left(0, 0, u_3(0)e^{i\nu t}, 0\right)$$

$$\mathbf{v}_4 = \mathbf{P} * \left(0, 0, 0, u_4(0)e^{-i\nu t}\right)$$

that have a characteristic period of

$$T = \frac{2\pi}{\nu} \tag{2.18}$$

In fact a starting condition on E^C describes an orbit of period T. We can rearrange the coordinates u_3, u_4 to see this better. Let $u_3(0)$ and $u_4(0)$ be:

$$u_3(0) = \rho e^{i\phi} \qquad u_4(0) = \rho e^{-\phi}$$

$$\alpha = \frac{u_3 + u_4}{2} = Re(u_3) = \rho \cos(\nu t + \phi) \tag{2.19}$$

$$\beta = \frac{u_3 - u_4}{2i} = Im(u_3) = \rho \sin(\nu t + \phi) \tag{2.20}$$

We have an harmonic oscillator on the complex plane.

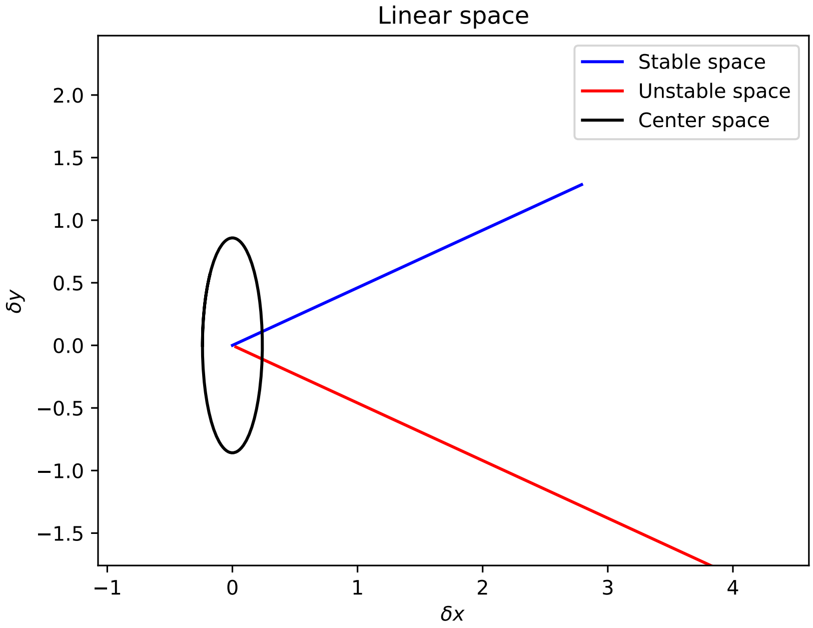

Figure 2.1: Projection on the position space $(\delta x, \delta y)$ of the center space and parts of the stable and unstable spaces

Chapter 3

Manifold theory

Here we briefly introduce the local manifolds theorems without proving them. These theorems are the necessary foundation that allows us to move from the linearized space (which is easy to compute) to the nonlinear space.

In the nonlinear space the stable, unstable and center spaces that we just saw are generalized to the stable, unstable and center manifolds. We can imagine a manifold as a sort of higher dimensional surface in the phase space that has a certain property.

We will give a very simple definition of manifold, hoping to not anger any mathematician reading these book. Then we will state the local stable and center manifold theorems.

3.1 Definitions

What is a manifold? In simple terms a n-dimensional manifold is a topological space where each point and its neighborhood are homeomorphic to $\mathbb{R}^n$.

In other words it means that a n-dimensional manifold is n-dimensional surface that can be parametrized by n coordinates.

Example The circular cylinder in polar coordinates is a 2-dimensional manifold in a 3-dimensional space:

$$\begin{aligned} x &= R\cos(\theta) \\ y &= R\sin(\theta) \\ z &= z \end{aligned}$$

So the 3-dimensional space $(R, \theta, z) \in \mathbb{R}^3$ completely describes the manifold. In other words, any point on (R, θ, z) maps to a point on the manifold.

The rigorous mathematical definition of manifold is quite complex and there graduate courses only on this topic,but for our purposes this intuitive definition of manifold should suffice.

3.1.1 Stable, unstable and center manifolds

Here we give the definitions of the manifolds that interest us:

- **Stable manifold** Let $W^S \in \mathbb{R}^n$
$$W^S = \{\mathbf{r} \in D \setminus \mathbf{c} : \lim_{t\to+\infty} \phi(t,\mathbf{r}) = \mathbf{c}\}$$
which is the set of initial conditions **r** in the phase space that end up in the equilibrium
- **Unstable manifold** Let $W^U \in \mathbb{R}^n$
$$W^U = \{\mathbf{r} \in D \setminus \mathbf{c} : \lim_{t\to-\infty} \phi(t,\mathbf{r}) = \mathbf{c}\}$$
which is the set of initial conditions in the phase space that go away from the equilibrium
- **Center manifold** Let $W^C \in \mathbb{R}^n$
$$W^C = \{\mathbf{r} \in D \setminus \mathbf{c} : \|\phi(t,\mathbf{r}) - \mathbf{c}\| < \epsilon\} \qquad \epsilon > 0, t \in \tau$$
Where τ is a neighborhood of $t = 0$.
This is the set of initial conditions that orbit the equilibrium point for a certain amount of time.

So if the particle is on any point of the *stable* manifold it will naturally approach the equilibrium, without external action.

If it is on the *unstable* manifold it will naturally depart from the proximity of the equilibrium.

If it is on the *center* manifold it will orbit the equilibrium for a certain time before departing along the unstable manifold.

3.2 Theorems

The unstable manifold can be treated as a stable manifold if the time flow is reversed, so here we will introduce only the local *stable* manifold theorem.

Local stable manifold theorem Consider a dynamical system $\dot{\mathbf{r}} = \mathbf{X}(\mathbf{r})$ and suppose it has an *hyperbolic* equilibrium point **c**.

Then:

- there exists a local smooth stable manifold W^S_{loc} tangent to the stable space E^S in the equilibrium point **c**
- This manifold is invariant, which means that $\forall \mathbf{r} \in W^S_{loc}$
$$\phi(t,\mathbf{r}) \in W^S_{loc} \quad \forall t > 0$$
- Moreover, the local stable manifold is part of the stable manifold:
$$W^S_{loc} \subset W^S$$
so that:
$$W^S = \left\{\phi\left(t, W^S_{loc}\right), \quad t < 0\right\}$$

What does this mean? Because the linear space and the manifold are tangent in the equilibrium point, if a particle is on the stable space and sufficiently close to the equilibrium point it will also be very close to the nonlinear stable manifold. We can consider the linear stable space as an approximation of the local stable manifold when close to the equilibrium.

Because the local stable manifold theorem states that the manifold is invariant, a starting condition on the local manifold *has to* generate a trajectory on the "complete" manifold.

Therefore, if we take an initial point on the linear space *very close* to the equilibrium, and we integrate from that point in the nonlinear space, we obtain a trajectory that *approximates* a trajectory on the manifold.

This property is **very important** for us, as we will see later.

Center manifold theorem Consider a dynamical system $\dot{\mathbf{r}} = \mathbf{X}(\mathbf{r})$ and suppose it has a partially hyperbolic equilibrium point $\mathbf{c}$. Then:

- There exists a local smooth center manifold W^C_{loc} tangent to the center space E^C in the equilibrium point $\mathbf{c}$.
- This manifold is also locally invariant, which means that

  $$\phi(t, W^C_{loc}) \in W^C_{loc}$$

  only for a $|t| < \varepsilon$ with $\varepsilon \in \mathbb{R}, \varepsilon > 0$.

Therefore each collinear equilibrium point will have a *stable*, *unstable* and *center* manifolds *tangent* in the equilibrium to the respective *spaces*[1] .

We are greatly interested in the center manifold, because a particle on it will perform a *periodic orbit* around the equilibrium point, which has many uses in mission design. This periodic orbit is also called the **Lyapunov orbit**.

3.3 Heteroclinic points

Here we introduce the heteroclinic points, which will be useful to us to calculate transfer trajectories (heteroclinic connections) between equilibrium points. Basically they allow movement from one equilibrium to the other without external action.

Definition If a dynamical system $\dot{\mathbf{r}} = \mathbf{X}(\mathbf{r})$ has more than one hyperbolic (or partially hyperbolic) equilibrium points $\mathbf{c}_1, \mathbf{c}_2$, the transverse intersection of the stable manifold of one point W^S_1 with the unstable manifold of another point W^U_2 is called a *heteroclinic point*.

$$\mathbf{p}_h \in W^S_1 \cap W^U_2$$

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[1] for a more exhaustive explanation see: Bressan, Alberto. (2007). Tutorial on the center manifold theorem. Lecture Notes in Mathematics,Springer

Example We know that L_1 and L_2 are both partially hyperbolic and both have stable and unstable manifolds. Therefore it is possible to find a crossing of their manifolds, a heteroclinic point, and establish a heteroclinic connection between L_1 and L_2.

A particle on this trajectory would start on the $W^U_{L_2}$ of L_2, arrive to the heteroclinic point $\mathbf{p}_h$ and take the $W^S_{L_1}$ to approach L_1 for $t \to \infty$.

Chapter 4

Poincaré section

Here we introduce the theory of Poincaré sections. These are particular restrictions of the phase space that are quite useful when studying high dimensional systems.

We will first introduce the Poincaré section and Poincaré map, and then see how they simply the study of high dimensional phase spaces, with some examples for our problem.

4.1 Poincaré section and Poincaré map

We introduce a surface Σ,usually of dimension $n-1$, in the phase space that has the following properties

- For any starting condition on the surface, there exists a time in which we return to the surface
$$\forall \mathbf{r} \in \Sigma \quad \exists t(\mathbf{r}) > 0 : \phi(t(\mathbf{r}), \mathbf{r}) \in \Sigma$$
- The flow $\phi(t(\mathbf{r}), \mathbf{r})$ crosses the surface Π in the same direction as the starting point $\mathbf{r}$

Depending on the dimensions of Σ, we reduce the degrees of freedom of the system by one or more. This is more clear if we take as an example the CR3BP:

Example Let the horizontal plane be the Poincaré section

$$\Sigma = \left\{\mathbf{r} \in \mathbb{R}^6 : \quad z = 0\right\}$$

Now the state vector will always be:

$$\mathbf{r} = \left(x, y, 0, p_x, p_y, p_z\right)$$

Therefore we could also define the points on Σ as:

$$\mathbf{s} = \left(x, y, p_x, p_y, p_z\right) \in \mathbb{R}^5 \qquad \mathbf{s} \in \Sigma$$

Notice that there is not a unique definition of a Poincaré section,each problem has its most convenient choice.

Since by definition, the flow starting on any point of the surface has to return to the surface, we can define the *Poincaré map* $\psi(\mathbf{s})$:

$$\begin{gathered}\Pi \to \Pi \\ \mathbf{s} \mapsto \mathbf{s}' = \psi(\mathbf{s}) \qquad (4.1)\end{gathered}$$

with $\mathbf{s} \in \Sigma$, where **s**' is the return point on Σ of the trajectory.

The successive iterations of a Poincaré map can be expressed as:

$$\begin{gathered}\mathbf{s}'' = \psi(\mathbf{s}') = \psi \circ \psi(\mathbf{s}) \\ \vdots \\ \mathbf{s}^{(k)} = \psi^k(\mathbf{s}) \qquad (4.2)\end{gathered}$$

for $k \in \mathbb{Z}$.

We have transformed the *continuous* flow into a *discrete* map. Instead of knowing the continuous state of the particle, we only know it in certain instances, but we have reduced the dimension of the problem by 1 or more.

It is then natural to define an analogous of the equilibrium, called the **fixed point** of the Poincaré map, as:

$$\mathbf{p} = \psi^k(\mathbf{p}) \qquad (4.3)$$

which means that every iteration starting on **p** returns on **p**.

Remark
Assume that a particle is traveling along a periodic orbit. This translates to the Poincaré section as a fixed point, because at each iteration the particle will intersect the surface in the same point, as per definition of periodic orbit. In our case a periodic orbit around a Lagrange point (Lyapunov orbit) becomes a fixed point for Poincaré the map.

4.2 Linearization

We have just seen how to define a fixed point for a Poincaré map. Then, just as we did for the flow, we can do a Taylor expansion around the fixed point and study its linear stability:

$$\begin{gathered}\mathbf{s} = \mathbf{p} + \delta\mathbf{s} \\ \psi(\mathbf{s}) = \mathbf{p} + \nabla\psi(\mathbf{p}) \cdot (\mathbf{s} - \mathbf{p}) + \dots\end{gathered}$$

The linear dynamics are defined by:

$$\delta\mathbf{s}' = \mathbf{M} \cdot \delta\mathbf{s} \qquad (4.4)$$

where $\mathbf{M} = \nabla\psi(\mathbf{p})$ is the Jacobian matrix calculated in **p**.

Now the linearized succession is given by:

$$\delta\mathbf{s}^{(k)} = \mathbf{M}^k \cdot \delta\mathbf{s} \qquad (4.5)$$

Remark Notice how the equations for the flow $\dot{\mathbf{r}} = \mathbf{X}(\mathbf{r})$ express the *rate of change* of the state vector as a function of the current state vector, while 4.1 or 4.5 express the *next* point as a function of the current one.

4.2.1 Stability

Now that we have the Jacobian in the fixed point we can calculate the eigenvalues η and eigenvectors $\boldsymbol{\zeta}$:

$$\mathbf{M} \cdot \boldsymbol{\zeta} = \eta \boldsymbol{\zeta}$$

and, from 4.5:

$$\mathbf{M}^k \cdot \boldsymbol{\zeta} = \eta^k \boldsymbol{\zeta}$$

As for the flow, the eigenvectors span the *stable*, *unstable* and *center* spaces if:

- **Stable space** : $\langle \boldsymbol{\zeta}_i \rangle = \mathbb{E}^S$ if $|Re(\eta_i)| < 1$
- **Unstable space** : $\langle \boldsymbol{\zeta}_i \rangle = \mathbb{E}^U$ if $|Re(\eta_i)| > 1$
- **Center space** : $\langle \boldsymbol{\zeta}_i, \boldsymbol{\zeta}_j \rangle = \mathbb{E}^C$ if $|Re(\eta_i)| = |Re(\eta_j)| = 0$ and $Im(\eta_i) \neq 0$, $Im(\eta_j) \neq 0$. Also $\eta_j = \eta_i^*$ and $\boldsymbol{\zeta}_j = \boldsymbol{\zeta}_i^*$

4.2.2 Manifolds

The local manifolds theorems are still true for the Poincaré map, therefore we have that the stable, unstable and center manifolds are tangent to their respective spaces in the fixed point.

Since they are totally analogous, we do not state them nor prove them.

Let us take the unstable manifold as an example. Being on this manifold means that at every iteration we get points on the P. section that are further and further away from the fixed point. How does this translate to the phase space?
The fixed point is the Lyapunov orbit in the phase space, and a point on the unstable manifold translates to the phase space as a trajectory that at each passage through the P. section get further away from the ideal periodic orbit, until it leaves it completely.
These are the so called **tube manifolds**.

Chapter 5

Reduction to center manifold

In the previous chapters we have laid down the basic theory for equilibrium points, stability and manifolds, where we use the linear spaces as starting conditions to approximate trajectories on the manifolds.

However, sometimes this method may be impractical. In this chapter we briefly explain the theory necessary to do this. We will give some basic notions of canonical transformations, Lie generating functions, normal form Hamiltonians and reductions to center manifolds.

5.1 Canonical transformations

In the Hamiltonian formalism we can always express an Hamiltonian H given in coordinates (q, p) as an other Hamiltonian $\tilde{H}$ in coordinates (Q, P). To do this we need a so called *canonical transformation* $\mathbf{w}$, which a continuous, differentiable function that maps the new coordinates to the old ones[1].

$$(q, p) = \mathbf{w}(Q, P) \tag{5.1}$$

And the new Hamiltonian is given by substitution[2] :

$$\tilde{H}(Q, P) = H(q, p) = H(\mathbf{w}(Q, P)) \tag{5.2}$$

We do this because by using the right transformation we can obtain a new Hamiltonian that can be substantially easier to study or give greater insight into the dynamics of the problem.

But not any function $\mathbf{w}(Q, P)$ can be a canonical transformation. We do not prove it here, but a generic function **is a canonical transformation** if and only if it is **symplectic**.

Given the Jacobian of $\mathbf{w} = \big(q(Q, P), p(Q, P)\big)$:

$$\nabla \mathbf{w} = \begin{bmatrix} \dfrac{\partial q}{\partial Q} & \dfrac{\partial q}{\partial P} \\ \dfrac{\partial p}{\partial Q} & \dfrac{\partial p}{\partial Q} \end{bmatrix} \tag{5.3}$$

---

[1] Why it is useful to express the old coordinates as a function of the old ones and not vice versa will be clear in later chapters

[2] Notice that it is not sufficient to simply substitute the old coordinates with the new ones. The new $\tilde{H}$ could have a form quite different from the original H

the function $\mathbf{w}$ is symplectic if and only if:

$$\nabla\mathbf{w}\cdot\mathbf{J}\cdot\nabla\mathbf{w}^T = \nabla\mathbf{w}^T\cdot\mathbf{J}\cdot\nabla\mathbf{w} = \mathbf{J} \tag{5.4}$$

where $\mathbf{J}$ is the $2n\times 2n$ *simplectic matrix*:

$$\mathbf{J} = \begin{bmatrix} 0 & \mathbb{I}_n \\ -\mathbb{I}_n & 0 \end{bmatrix} \tag{5.5}$$

where $2n$ is the dimension of the phase space.

Notice that 5.4 is equivalent to checking the conservation of the Poisson brackets.

Example: The harmonic oscillator has Hamiltonian:

$$H\left(q,p\right) = \frac{\omega}{2}(p^2+q^2)$$

but by using the canonical transformations:

$$p = \sqrt{2I}\cos(\phi) \qquad q = \sqrt{2I}\sin(\phi)$$

we get the simplified:

$$\tilde{H}\left(\phi, I\right) = \omega I$$

This new Hamiltonian may not have much physical sense in the real world, but it substantially easier to solve (I is an integral of motion).

5.1.1 Generating functions

The main problem with the canonical transformations is that the the function $\mathbf{w}$ is not always trivial, and, if we were to build one from scratch, it may not always be so easy to ensure that the Poisson brackets are preserved.

Luckily, there exists a method, called *method of the generating function*, that allows us to always obtain a canonical transformation.

Suppose that we have an unknown generic function of the new coordinates (Q,P) called $\chi\,(Q,P)$.

If we treat this function as an Hamiltonian we can calculate its flow $\phi(t,\mathbf{r})$. This would require to integrate the Hamilton equations of χ, which may not always be that easy or desirable.

$$\dot{\mathbf{r}} = \mathbf{X}(\mathbf{r}) = \begin{bmatrix} \frac{\partial\chi}{\partial P} \\ -\frac{\partial\chi}{\partial Q} \end{bmatrix} \tag{5.6}$$

However we can note that for a time sufficiently small, the flow can be expressed as a Taylor series around the initial condition:

$$\mathbf{r}(t) = \mathbf{r}(0) + \left.\frac{d\mathbf{r}}{dt}\right|_{t=0} t + \frac{1}{2}\left.\frac{d^2\mathbf{r}}{dt^2}\right|_{t=0} t^2 + \dots \tag{5.7}$$

using 5.6 we get:

$$\mathbf{r}(t) = \mathbf{r}_0 + \mathbf{X}|_{t=0}\,t + \tfrac{1}{2}\nabla\mathbf{X}\cdot\mathbf{X}|_{t=0}\,t^2 + \dots \tag{5.8}$$

What is interesting is that the flow of an Hamiltonian is always simplectic (we do not prove it), therefore the map:

$$\mathbf{r}_0 \longmapsto \mathbf{r}(t) = \phi(t, \mathbf{r}_0)$$

is a canonical transformation that maps the new coordinates $\mathbf{r}_0$ to the old coordinates $\mathbf{r}(t)$.

Notice also that any function $\mathbf{a}(\mathbf{r}(t))$ calculated on the Hamiltonian flow of χ can be similarly expanded through a Taylor series:

$$\mathbf{a}(\mathbf{r}(t)) = \mathbf{a}(\mathbf{r}_0) + \left.\frac{d\mathbf{a}}{dt}\right|_{t=0} t + \frac{1}{2} \left.\frac{d^2\mathbf{a}}{dt^2}\right|_{t=0} t^2 + \dots \tag{5.9}$$

Pay attention to :

$$\frac{d\mathbf{a}}{dt} = \nabla\mathbf{a} \cdot \dot{\mathbf{r}} = \frac{\partial\chi}{\partial P}\frac{\partial a_Q}{\partial Q} - \frac{\partial\chi}{\partial Q}\frac{\partial a_P}{\partial P}$$

This is exactly the *Lie derivative* of $\mathbf{a}$ evaluated on the flow of Hamiltonian χ ($\mathscr{L}_\chi$).

$$\mathscr{L}_\chi \cdot = \frac{\partial\chi}{\partial P}\frac{\partial\cdot}{\partial Q} - \frac{\partial\chi}{\partial Q}\frac{\partial\cdot}{\partial P} \tag{5.10}$$

Therefore the time derivative of any function calculated on the flow of the Hamiltonian χ is:

$$\frac{d\mathbf{a}}{dt} = \mathscr{L}_\chi \mathbf{a} \tag{5.11}$$

It is easy to see that:

$$\frac{d^n a}{dt^n} = \mathscr{L}_\chi^n \mathbf{a} = \mathscr{L}_\chi \circ \mathscr{L}_\chi \circ \dots \circ \mathscr{L}_\chi \mathbf{a} \tag{5.12}$$

With this we can redefine both 5.8, 5.9 as

$$\mathbf{r}(t) = \exp(t\mathscr{L}_\chi)\mathbf{r}_0 \tag{5.13}$$

$$\mathbf{a}(\mathbf{r}(t)) = \exp(t\mathscr{L}_\chi)\mathbf{a}(\mathbf{r}_0) \tag{5.14}$$

If we call $\mathbf{r}_0 \equiv \mathbf{R}$ the new coordinates and $\mathbf{r}(1) \equiv \mathbf{r}$ the old coordinates, we get a canonical transformation:

$$\mathbf{r} = \exp(\mathscr{L}_\chi)\mathbf{R} \tag{5.15}$$

and any function calculated on the Hamiltonian flow is:

$$\mathbf{a}(\mathbf{r}) = \exp(\mathscr{L}_\chi)\mathbf{a}(\mathbf{R}) \tag{5.16}$$

The new Hamiltonian is simply:

$$\tilde{H}(\mathbf{R}) = H(\mathbf{r}) = \exp(\mathscr{L}_\chi) H(\mathbf{R}) \tag{5.17}$$

Now we know how to build a transformation that is always symplectic. What is interesting is that by tweaking χ we can obtain a canonical transformation with the desired properties.

5.2 Normal form

Sometimes there can be terms in the Hamiltonian that are undesirable. Here we explain a method that generates a canonical transformation which erases those terms.

We organize our Hamiltonian to account for the unwanted terms:

$$H(\mathbf{r}) = N + \xi \tag{5.18}$$

where N is the *normal form* and ξ is the *remainder.*

The normal form is a representation of the Hamiltonian that is easy to study or of interest, while the remainder are the undesirable terms that we want to eliminate through the canonical transformation.

Book-keeping parameter It is useful to categorize the normal form and the remainder through a so called *book-keeping parameter* λ^m. The parameter is $\lambda = 1$, therefore does not effect the Hamiltonian in any way, but the exponent m gives us information on the effect of the terms on the dynamics.

We give the normal form λ^0 and the remainder a $\lambda^1, \lambda^2, \ldots$ depending on how much we expect the terms to affect the dynamics (terms with a higher exponent are less relevant).

Therefore we could write our Hamiltonian as:

$$H = \lambda^0 N_0 + \lambda\xi_1 + \lambda^2\xi_2 + \ldots$$

5.2.1 Finding the generating function

Now we will discuss a way to find a generating function χ using the normal form.

It is an iterative process. We indicate with $H^{(k)}$ the k-th Hamiltonian calculated. $H^{(k)}$ means that the unwanted terms up to order λ^k (included) have been eliminated.

Similarly $N_0^{(k)}$ means that the normal form contains terms up to order $O(\lambda^k)$, while $\xi_j^{(k)}$ are the unwanted terms of order j that still exist at step k.

Suppose that χ is a function like:

$$\chi_1 = \lambda g_1(\mathbf{R})$$

Notice that:

$$H^{(1)}\left(\mathbf{R}^{(1)}\right) = \exp\left(\mathscr{L}_{\chi_1}\right) H^{(0)}\left(\mathbf{R}^{(1)}\right) \tag{5.19}$$

$$\begin{aligned}\exp(\mathscr{L}_{\chi_1})H^{(0)} =& N_0^{(0)} + \{N_0^{(0)}, \chi_1\} + \frac{1}{2}\{\{N_0^{(0)}, \chi_1\}, \chi_1\} + \\ & \lambda\xi_1^{(0)} + \lambda\{\xi_1^{(0)}, \chi_1\} + \lambda\frac{1}{2}\{\{\xi_1^{(0)}, \chi_1\}, \chi_1\} + \ldots \\ & \lambda^2\xi_2^{(0)} + \lambda^2\{\xi_2^{(0)}, \chi_1\} + \lambda^2\frac{1}{2}\{\{\xi_2^{(0)}, \chi_1\}, \chi_1\} + \ldots\end{aligned}$$

Notice that because $\chi = \lambda g$ and the linearity of the Lie derivative, both χ and $\{N, \chi\}$ are of order $O(\lambda)$.

We want to find a generating function χ that satisfies the *homological equation*:

$$\{N_0^{(0)}, \chi_1\} + \lambda\xi_1^{(0)} = 0 \tag{5.20}$$

If this is true then the unwanted terms of order $O(\lambda)$ will be eliminated after the transformation.

What is the best function χ that we could try? There is no single answer as the most appropriate generating function will depend on the problem at hand and on the normal form we want to achieve.

However, after we have found the most suitable one, we should get:

$$H^{(1)} = N^{(1)} + \lambda^2\xi_2^{(1)} + \lambda^3\xi_3^{(1)} + \dots \tag{5.21}$$

Which means that the "unwanted" terms have been moved to terms of higher order of λ, meaning that they are less relevant.

For example, now we have removed all the terms of order $O(\lambda^1)$, leaving only terms of order $O(\lambda^2)$ or higher. Repeating these steps another time will remove the terms of order $O(\lambda^2)$, leaving only terms of order $O(\lambda^3)$ or higher.

This means that by iterating n times we will get a new Hamiltonian[3] of order $O(\lambda^n)$, with unwanted terms of order $O(\lambda^{n+1})$.

Example Suppose that we have the pendulum:

$$h = \frac{p^2}{2} - \omega^2\cos\theta$$

and we want to study its motion around the phase point $(0, \tilde{p})$, with $\tilde{p} >> 1$.

We can use the translation

$$P = p - \tilde{p}$$

which gives the canonical transformation (the reader can easily check this affirmation):

$$p = P + \tilde{p} \qquad \theta = Q$$

Substituting we get the new Hamiltonian:

$$H(Q,P) = h\big(\theta(Q,P), p(Q,P)\big) = \frac{P^2}{2} + \tilde{p}P + \frac{\tilde{p}^2}{2} - \omega^2\cos Q$$

we call $\tilde{p} \equiv \alpha$ for simplicity of notation and disregard $\frac{\tilde{p}^2}{2}$ since it is a constant and does not effect the dynamics:

$$H(Q,P) = \frac{P^2}{2} + \alpha P - \omega^2\cos Q$$

We want to get a normal form where there are no angles Q, only momenta. This way the momenta will become integrals of motion. Organizing the equation through book-keeping:

$$H = N_0 + \lambda\xi_1$$

---

[3] It should be noted that each time that we apply this transformation the set of initial conditions on the phase space that lead to convergence of the series shrinks. The proof of this is beyond the scope of this text, but it is something to keep in mind.

$$N_0 = \frac{P^2}{2} + \alpha P$$
$$\xi_1 = -\omega^2 \cos Q$$

For the generic generating function χ we get the transformation:

$$\tilde{H} = \exp(\mathscr{L}_\chi) H = N_0 + \lambda\xi_1 + \{N_0, \chi_1\} + \lambda\{\xi_1, \chi_1\} + \frac{1}{2}\{\{N_0, \chi_1\}, \chi_1\} + \lambda\frac{1}{2}\{\{\xi_1, \chi_1\}, \chi_1\} + \dots$$

Now we focus on solving the *homological equation* 5.20:

$$\{N_0, \chi_1\} + \lambda\xi_1 = 0$$

Notice:

$$\{N, \chi\} = \frac{\partial\chi}{\partial P}\frac{\partial N}{\partial Q} - \frac{\partial\chi}{\partial Q}\frac{\partial N}{\partial P} = -\frac{\partial\chi}{\partial Q}(P + \alpha)$$

Therefore:

$$\{N, \chi\} + \lambda\xi = -\frac{\partial\chi}{\partial Q}(P + \alpha) - \lambda\omega^2 \cos Q = 0$$

Since the term we want to erase is a trigonometric function of Q, it might be a good idea to use a χ such as:

$$\chi = \lambda\beta(P) \sin Q$$
$$\frac{\partial\chi}{\partial Q} = \lambda\beta(P) \cos Q$$

Substituting we get:

$$-\lambda\beta(P) \cos Q(P + \alpha) - \lambda\omega^2 \cos Q = 0$$

Therefore to nullify the equation we have that parameter β has to be:

$$\beta(P) = -\frac{\omega^2}{P + \alpha}$$

The generating function is:

$$\chi_1 = -\lambda\frac{\omega^2}{P + \alpha} \sin Q$$

Therefore the new Hamiltonian is:

$$\tilde{H} = N_0 + \lambda\{\xi_1, \chi_1\} + \dots$$

which are:

$$\tilde{H} = \frac{P^2}{2} + \alpha P + \lambda^2\frac{\omega^4 \sin^2 Q}{(P + \alpha)^2} + O(\lambda^3)$$

Notice that since α is supposed to be quite big the third term is small, and the higher order of the book-keeping parameter reflects that.

Approximating:

$$\tilde{H} \approx N_0 = \frac{P^2}{2} + \alpha P$$

$$\dot{Q} = P + \alpha$$
$$\dot{P} = 0$$

Part II

2D calculations

Chapter 6

Numerical integration and linearization

In *Part II* we will work through the calculations needed to find the tube manifolds of the collinear Lagrange points in the **planar** CR3BP. We start from the planar problem because it is substantially easier and will be a good way to familiarize ourselves with the methodology, before moving on to the **spatial** problem.

We will develop a numerical method that uses as little math as possible, so that even the reader not familiar with the complex mathematics behind manifold dynamics will be able to calculate these trajectories.

To better explain this process we will use an example: the dynamics of a spacecraft in the Earth - Moon system, mainly around L_1.

We have multiple objectives:

1. Find a family of periodic orbit
2. Find the tube manifolds of one of these orbits
3. Find a heteroclinic connection between two Lagrange points

Since we are dealing with numerical methods we will also have to:

- Build a numerical integrator of the Hamilton equations and verify that it is reliable
- Find the collinear Lagrange points
- Calculate the Jacobian and linearized space
- Calculate a first approximation of the periodic orbit derived from the linear space
- Build a root finding method
- Use the first approximation as a starting point for the root finding method
- Find the periodic orbit

Once found the periodic orbit we will move to the discrete dynamics of the Poincaré map, and:

- Find the fixed point of the periodic orbit
- Calculate the Jacobian of the map in that point
- Use its eigenvectors to find the stable and unstable tube manifolds

Then we will calculate the tube manifolds of both L_1 and L_2 and find a heteroclinic connection between them.

6.1 Integration

The first thing to do is to numerically simulate the dynamics of the system.

We start by choosing a mass parameter μ of interest:

System	μ
Sun - Jupiter	9.537×10^{-4}
Sun - Earth	3.036×10^{-6}
Earth - Moon	0.01215
Jupiter - Ganymede	7.804×10^{-5}

Since our example is in the Earth - Moon system we take $\mu = 0.01215$.

Then we have to numerically solve the Hamilton equations of the planar problem:

$$H(\mathbf{r}) = \tfrac{1}{2}\left(p_x^2 + p_y^2\right) + p_x y - p_y x - \frac{1-\mu}{\sqrt{(x+\mu)^2+y^2}} - \frac{\mu}{\sqrt{(x+\mu-1)^2+y^2}} \tag{6.1}$$

$$\dot{x} = \frac{\partial H}{\partial p_x} = p_x + y \tag{6.2}$$

$$\dot{y} = \frac{\partial H}{\partial p_y} = p_y - x \tag{6.3}$$

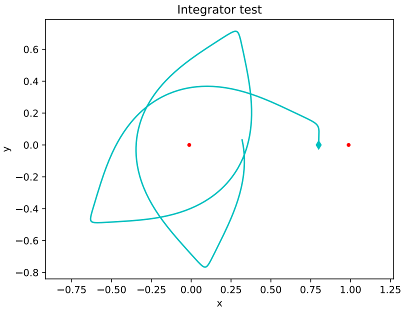
$$\dot{p}_x = -\frac{\partial H}{\partial x} = p_y - (1-\mu)\frac{x+\mu}{\left((x+\mu)^2+y^2\right)^{3/2}} - \mu\frac{x+\mu-1}{\left((x+\mu-1)^2+y^2\right)^{3/2}} \tag{6.4}$$

$$\dot{p}_y = -\frac{\partial H}{\partial y} = -p_x - (1-\mu)\frac{y}{\left((x+\mu)^2+y^2\right)^{3/2}} - \mu\frac{y}{\left((x+\mu-1)^2+y^2\right)^{3/2}} \tag{6.5}$$

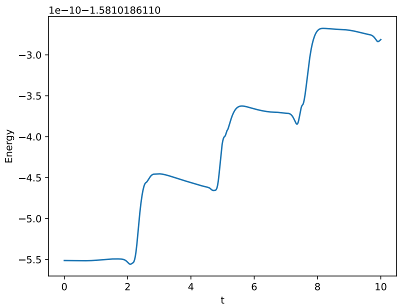
this can be done by using any differential equations integrator. All major scientific programming suites should provide one. It is important to achieve a good accuracy, since the system we are dealing with is quite sensible and even small errors can dramatically change the trajectory of the particle.

A simple and reliable way to test the quality of the integration is to check if the energy is conserved. Since we are dealing with conservative potentials the energy of a particular trajectory should stay constant. Therefore we can calculate the energy at each point of our trajectory and check that it does not change.

There will be some unavoidable numerical errors, but if the energy is conserved up to the 8-10 decimal digits it should be good enough for our purposes.



(a) Starting condition $\mathbf{r}_0 = (0.8, 0, 0, 1)$ and energy $E = -1.581018611$



(b) Error on the conservation of energy

Example For our example we chose the system Earth - Moon, therefore a mass parameter $\mu = 0.01215$. Then we test our integrator with a starting condition of $\mathbf{r}_0 = (0.8, 0, 0, 1)$. The energy of the orbit is $E = -1.581018611$

6.2 Collinear equilibrium points

We have already seen in chapter **??** that the collinear points can be found by solving the nonlinear system:

$$\begin{cases} x - (1-\mu)\dfrac{x+\mu}{|x+\mu|^3} - \mu\dfrac{x-1+\mu}{|x-1+\mu|^3} = 0 \\ y = 0 \end{cases} \tag{6.6}$$

The easiest way to find the solutions is to use a numerical nonlinear root finder. Again, any scientific programming suite should have one, so we do not explain how to actually find these roots.

Since usually these solvers need an initial guess to converge, we provide guesses that can be found in literature:

$$L_1 : 1 - \mu - \sqrt[3]{\frac{\mu}{3}} \qquad L_2 : 1 - \mu + \sqrt[3]{\frac{\mu}{3}} \qquad L_3 : -1 + \frac{7}{12}\mu$$

Where $\sqrt[3]{\frac{\mu}{3}}$ is the so called *Hill radius*, an approximation of the radius of the sphere of gravitational influence of a body.

Once we have obtained the roots of the system, the equilibrium points in the phase space are:

$$\mathbf{c}_1 = (x_{L_1}, 0, 0, x_{L_1}) \qquad \mathbf{c}_2 = (x_{L_2}, 0, 0, x_{L_2}) \qquad \mathbf{c}_3 = (x_{L_3}, 0, 0, x_{L_3})$$

Example The roots of 6.6 are:

$$x_{L_1} = 0.836918007 \qquad x_{L_2} = 1.155679913 \qquad x_{L_3} = -1.005062402$$

6.3 Linearization

We want to find a starting condition that gives a good first approximation of the periodic orbit.

Now that we have the equilibrium points we linearize the system around **c**, obtaining:

$$\delta\dot{\mathbf{r}} = \nabla\mathbf{X}(\mathbf{c}) \cdot \delta\mathbf{r}$$

From chapter 2 we know that the Jacobian is 2.11. The reader can easily see that for the planar problem the Jacobian calculated in any collinear Lagrange point is:

$$\nabla\mathbf{X}(\mathbf{c}) = \begin{bmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ \frac{\partial \dot{p}_x}{\partial x} & 0 & 0 & 1 \\ 0 & \frac{\partial \dot{p}_y}{\partial y} & -1 & 0 \end{bmatrix}$$

because $y = 0$.

We are interested in studying the eigenvalues λ and eigenvectors ρ:

$$\begin{array}{llll} \lambda_1 = \lambda & \lambda_2 = -\lambda & \lambda_3 = i\omega & \lambda_4 = -i\omega \\ \rho_1 & \rho_2 & \rho_3 & \rho_4 \end{array}$$

and the transformation matrix **P**:

$$\mathbf{\Lambda} \cdot \mathbf{P} = \nabla\mathbf{X}(\mathbf{c}) \cdot \mathbf{P}$$

$$\delta\mathbf{r} = \mathbf{P} \cdot \mathbf{u}$$

With that we can write the diagonalized equations:

$$\dot{\mathbf{u}} = \mathbf{\Lambda} \cdot \mathbf{u}$$

which are:

$$\begin{aligned} \dot{u}_1 &= \lambda u_1 \\ \dot{u}_2 &= -\lambda u_2 \\ \dot{u}_3 &= i\omega u_3 \\ \dot{u}_4 &= -i\omega u_4 \end{aligned}$$

with $u_1, u_2, u_3, u_4 \in \mathbb{C}$ and $u_4 = u_3^*$, as we have already seen in chapter 2.

6.3.1 First approximation of periodic orbit

We know from chapter 3 that the center space is spanned by the eigenvectors ρ_3, ρ_4 and that any orbit in this space is a periodic orbit called a **Lyapunov orbit**.

Also from chapter 3 we know that the center space and the center manifold are tangent in the equilibrium point. So if we take a sufficiently small σ in :

$$\mathbf{u}_c = \sigma(0, 0, 1, 1)$$

we obtain a:

$$\delta\mathbf{r}_c = \mathbf{P} \cdot \mathbf{u}_c$$

that is very close (as function of σ) to this tangency point.

From this we can obtain an initial condition in the phase space:

$$\mathbf{r}_0 = \mathbf{c} + \delta\mathbf{r}_c$$

It is reasonable to expect that integrating from this initial condition should give a trajectory that is very close to the desired orbit.

This is true only for very small σ and small integration times. If σ is too large we obtain an initial condition that is too far away from the tangency point and therefore the resulting trajectory can not follow closely the manifold.

If the time of integration is too large we leave the center manifold due to its innate *local* invariance. Sadly, this is an unavoidable physical property of these orbits, and without corrections they will eventually decay.

Example The eigenvalues are:

$$\lambda_1 = 2.932048 \qquad \lambda_2 = -2.932048 \qquad \lambda_3 = 2.334381i \qquad \lambda_4 = -2.334381i$$

therefore the characteristic period is:

$$T = \frac{2\pi}{\nu} = 2.691584$$

We want a starting condition that is in the so called shooting position[1] , that is on the x - axis with $p_x = 0$. This can achieved if:

$$\mathbf{u}_c = \sigma(0,0,1,1) \qquad \sigma > 0$$

Taking a small $\sigma = 1 \times 10^{-4}$ we obtain a starting condition that is on the shooting position and is very close to the equilibrium:

$$\mathbf{r}_0 = \mathbf{c} + \mathbf{P} \cdot \mathbf{u}_c = (0.836894065205, \quad 0, \quad 0, \quad 0.837094514353)$$

This initial condition gives the trajectory:

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[1] Why we want this condition will be clear in the next chapter

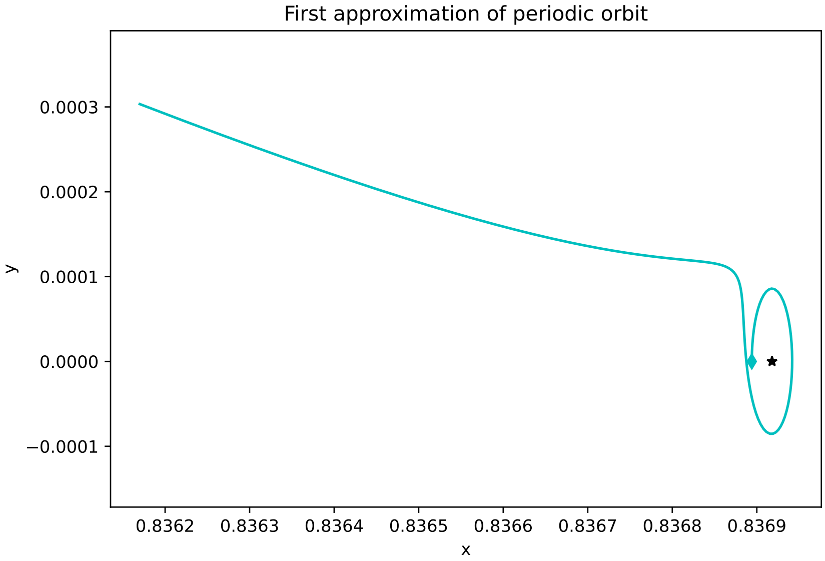

Figure 6.2: Projection in the position space of the orbit obtained with initial condition and time of integration $1.6T$. The black star is L_1

Notice how the does not close and the particle departs along the unstable manifold. This is due to the fact that the starting condition we have found is only an approximation of the correct starting condition on the center manifold. The nonlinear terms in the Hamilton equations become relevant as soon as we leave the neighborhood of L_1.

We will develop a method in the next chapter that accounts for these terms and finds a much better approximation of the Lyapunov orbit.

Counterexample What if the vector in the eigenbasis is:

$$\mathbf{u} = \sigma(0, 0, 1+i, 1-i)$$

The third and fourth components are complex conjugates, therefore we are on the center space, and σ is the same of the previous example, therefore we are also close to the tangent point. The problem is that with this $\mathbf{u}$ we will not obtain an initial condition that is on the x - axis with $p_x = 0$.

$$\mathbf{r}_0 = \mathbf{c} + \mathbf{P} \cdot \mathbf{u} = (0.836894, \quad 8.58682 \times 10^{-5}, \quad -2.99782 \times 10^{-5}, \quad 0.837095)$$

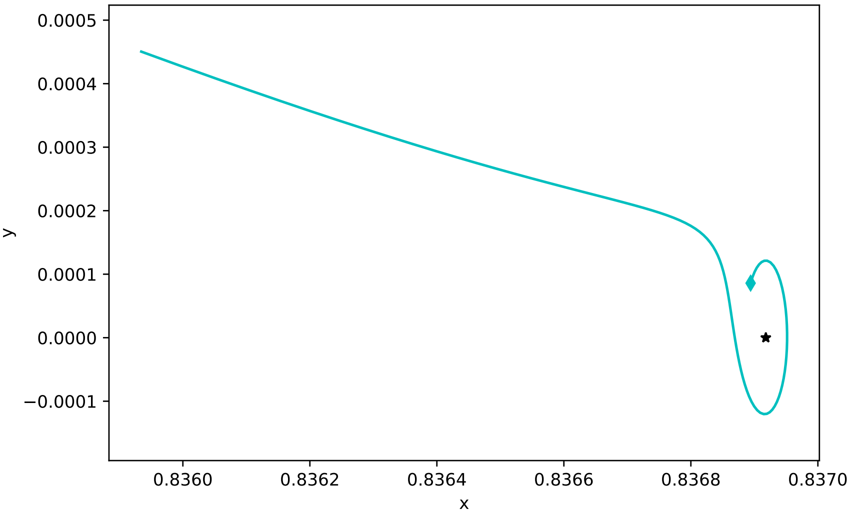

Figure 6.3: Trajectory obtained with "wrong" starting condition and integration time $1.4T$. Notice how we do not start on the x -axis

Chapter 7

Periodic orbit

Now that we have a starting condition provided by the linear approximation we have a good foundation on which to begin the research of the Lyapunov orbit.

We will first define a Poincaré section, which will allow us to handle the periodic orbit as a fixed point.

Then we will implement an algorithm that finds this fixed point and thus the periodic orbit.

7.1 Surface of intersection

In order to find the periodic orbit through numerical methods we need to treat it as a fixed point of a Poincaré map. Therefore we introduce the *surface of intersection* Γ[1] :

In the flow $\mathbf{X}(\mathbf{r}) \in \mathbb{R}^4$ we define the surface Γ as:

$$\Gamma = \left\{\mathbf{r} \in \mathbb{R}^4 : y = 0, \quad \dot{y} > 0\right\} \tag{7.1}$$

Which is the set of points on the x-axis where the y velocity is positive.

Practically we find these points by creating an event for our integrator that stops the integration when $y = 0$ and $\dot{y} = p_y - x > 0$.

Now we have a function that associates:

$$\mathbf{r}_0 = (x_0, y_0, p_{x,0}, p_{y,0}) \mapsto \mathbf{r}' = (x', 0, p'_x, p'_y)$$

If we start from $y_0 = 0$ we obtain a recursive *map* $\gamma(\mathbf{g})$ that associates:

$$\Gamma \to \Gamma$$

$$\mathbf{g} = (x, p_x, p_y) \mapsto \mathbf{g}' = (x', p'_x, p'_y) = \gamma(x, p_x, p_y)$$

This map takes a "photo" of the state vector every time that the spacecraft crosses the x-axis. Notice how we went from a 4-dimensional continuous flow to a 3-dimensional discrete map.

---

[1] The reader may notice that this is in fact a Poincaré surface. We do not call it this way because in later chapters we will define another Poincaré surface and it could lead to confusion. Just keep in mind that everything we said for Poincaré surfaces and map also applies here.

Now we would like to find a fixed point for this map,

$$\tilde{\mathbf{g}} = \gamma(\tilde{\mathbf{g}})$$

since it translates to the phase space as a periodic orbit. However there are many combinations of (x, p_x, p_y) (in fact they are infinite) that give a fixed point. Therefore we want to restrict our set of admissible solutions.

We will call a *position driven algorithm* the algorithm that, for a chosen coordinate in the vector $\mathbf{g}$ finds the remaining coordinates so that $\tilde{\mathbf{g}}$ is a fixed point.

We will call a *energy driven algorithm* the algorithm that, for a chosen energy, finds the fixed point $\tilde{\mathbf{g}}$.

Therefore, depending on which method we choose, we will get a periodic orbit that either passes through a certain point in space, or a orbit that has a certain energy.

Now we will explain the *position driven algorithm.* In *Part III* we will use the *energy driven algorithm*, since it suits better the needs of that section.

7.2 Position driven algorithm

We know that the Lyapunov orbit passing through x_0 translates to the Poincaré section as the fixed point[2] :

$$\tilde{\mathbf{g}} = (x_0, 0, \tilde{p}_y)$$

Why is $p_x = 0$? Because the Lyapunov orbit is perpendicular to the x-axis at the crossing[3] , therefore

$$\dot{x} = p_x + y = 0$$

when $y = 0$, so $p_x = 0$.

Thus we already know 2 of the 3 components of the fixed point. We only need to find $\tilde{p}_y$. We can imagine to put the spacecraft in x_0 with $p_x = 0$ and ask ourselves which is the p_y that will get us back to x_0 with $p_x = 0$.

This is the "shooting position" as we are effectively "shooting" for the right p_y.

We can find this p_y as the root of a nonlinear equation. Let γ_x be the x-component of $\gamma = \left[\gamma_x, \gamma_{p_x}, \gamma_{p_y}\right]$:

$$x' = \gamma_x(x_0, 0, p_y) = f(p_y; x_0)$$

We treat the Poincaré map as a function that takes in p_y as a variable and x_0 as a parameter, and gives the x', on the Poincaré section where we land.

$$x' = f(p_y; x_0) \tag{7.2}$$

---

[2] A word on notation: in this chapter we will use $\tilde{p}_y$ as the momentum in y that actually achieves the Lyapunov orbit.

[3] We do not prove it

We then define a new function:

$$F(p_y; x_0) = x' - x_0 = f(p_y; x_0) - x_0 \tag{7.3}$$

that tells us how far away we landed from the starting position as a function of p_y. Ideally, on a Lyapunov orbit, we should get 0 each time.

The $\tilde{p}_y$ we are searching for is the root of $F(p_y; x_0)$.
An easy way to find this root is to use Newton's method.

7.2.1 Newton's method

This method is quite famous and widely used, so we will not explain in detail how it works. We will only give the iterating formula:

$$p_y^{[n+1]} = p_y^{[n]} - \frac{F(p_y^{[n]}; x_0)}{\frac{dF}{dp_y}(p_y^{[n]}; x_0)} \tag{7.4}$$

We know that for $n \to \infty$ the method converges to a solution, given a suitable initial condition $p_y^{[0]}$ (in fact, in the right conditions, the method is very fast and may even converge in 5 - 10 iterations).

For us, the parameter x_0 and the initial condition $p_y^{[0]}$ will be the ones we have derived from the linear approximation:

$$\mathbf{r}_c = \mathbf{c} + \mathbf{P} \cdot \mathbf{u}_c$$

We cannot analytically calculate the derivative of F, therefore we are forced to derive it numerically.
Here we will use a central finite difference of first order, which is the following:

$$\frac{dF}{dp_y}\left(p_y^{[n]}; x_0\right) \approx \frac{F(p_y^{[n]} + h; x_0) - F(p_y^{[n]} - h; x_0)}{2h} + \mathcal{O}(h^2) \tag{7.5}$$

with a suitable small h.

Pay attention on how small h is, since the error on the derivation is approximately:

$$\epsilon \sim \frac{e_{int}}{h} + h^2$$

with e_{int} the error committed on the integration. Therefore the smallest error is of the order of:

$$h \approx \sqrt[3]{e_{int}}$$

Be careful! This value of h is just an estimate, you will need to find the one that gives the fastest convergence for your specific problem.

Example Taking the values that we have derived from the linear approximation:

$$x_0 = 0.836894065205$$

$$p_y^{[0]} = 0.837094514353$$

We obtain the value:

$$\tilde{p}_y = 0.837094549818$$

after 5 iterations.

Using now the initial condition $\mathbf{r}_0 = (x_0, 0, 0, \tilde{p}_y) = (0.836894065205, \quad 0, \quad 0, \quad 0.837094549818)$ we get the actual **planar Lyapunov orbit**.

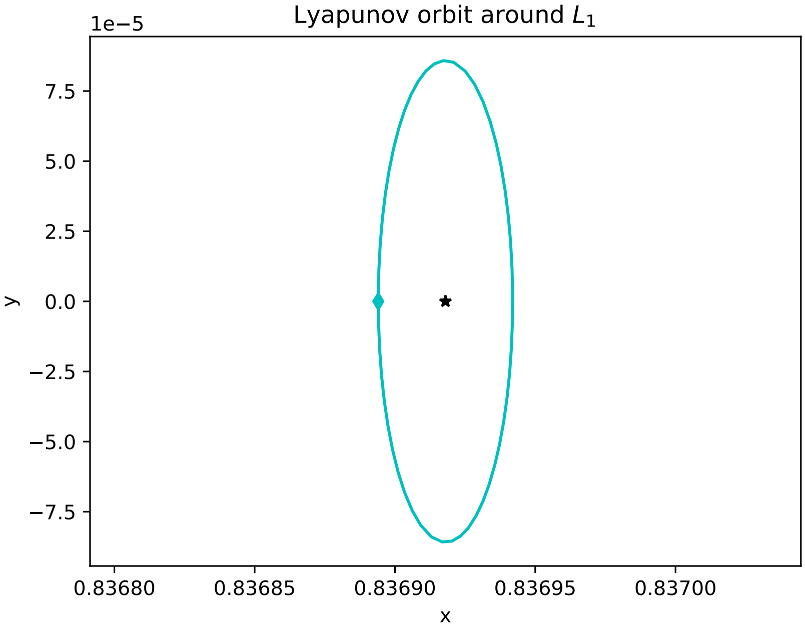

Figure 7.1: Lyapunov orbit

Notice how $\tilde{p}_y$ and $p_y^{[0]}$ differ only after the seventh decimal digit, and yet such a small difference can mean achieving the desired periodic orbit or not.

Remark Due to the local invariance of the central manifold, if we integrate for a longer time period the particle will do about 2-3 orbits before departing along the unstable manifold.

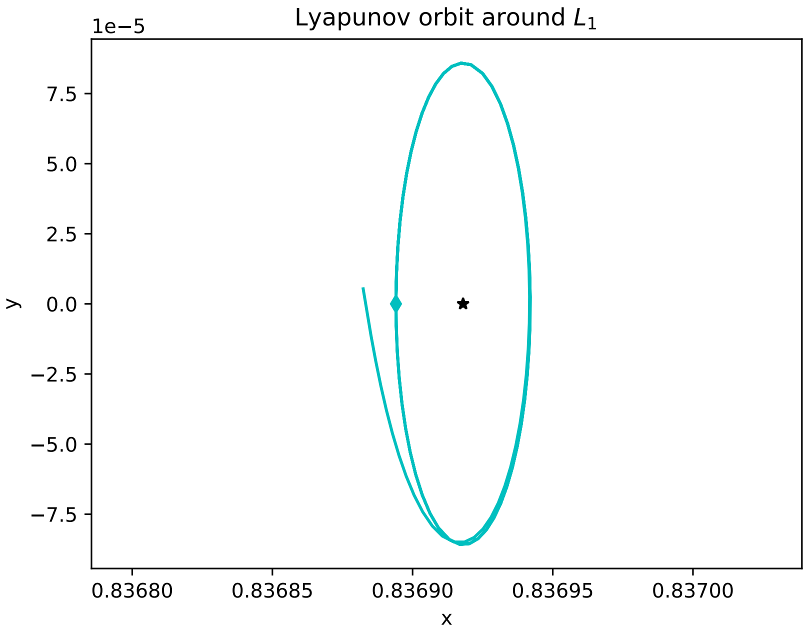

Figure 7.2: Trajectory with same starting conditions after a period of integration $3T$

7.2.2 Linear extrapolation

In the previous section we have found our first Lyapunov orbit, but since we need to stay close to the Lagrange point to use the linear approximation it is a very small orbit, with barely any practical use. We want to find orbits of any desired dimension.

The orbit is periodic, therefore its nominal dimension is dependent on the initial condition. The problem here is that if we choose a starting condition too far from L_1, that is, a large σ, we are no more negligibly close to the tangency point (L_1), which means that we do not have anymore an initial guess on the center manifold for the root finding algorithm.

If we try to use Newton's method from such a point it will probably not even converge.

So how can we find a large initial condition for Newton's method that leads to convergence?

We can use previously found small initial conditions to linearly extrapolate a larger initial guess, in the following way:

1. Pick two small σ_{-1} and σ_0 for which you can find the Lyapunov orbits
2. Obtain the two initial guesses $(x_{-1}, 0, 0, p_{y,-1})$, $(x_0, 0, 0, p_{y,0})$ and $\Delta x = x_0 - x_{-1}$
3. Obtain the two correct momenta that produce periodic orbits $\tilde{p}_{y,-1}, \tilde{p}_{y,0}$

4. Calculate the coefficients (m, q) of the line passing through the pairs $(x_{-1}, \tilde{p}_{y,-1})$ and $(x_0, \tilde{p}_{y,0})$

$$m = \frac{p_{y_0} - p_{y_{-1}}}{x_0 - x_{-1}} \qquad q = p_{y_0} - m x_0$$

5. With $x_1 = x_0 + \Delta x$ extrapolate the guess:

$$p_{y,1} = m\Delta x + p_{y_0}$$

6. Find the momentum that produces the periodic orbit $\tilde{p}_{y,1}$
7. Repeat steps 4 to 6 until you reach the desired orbit

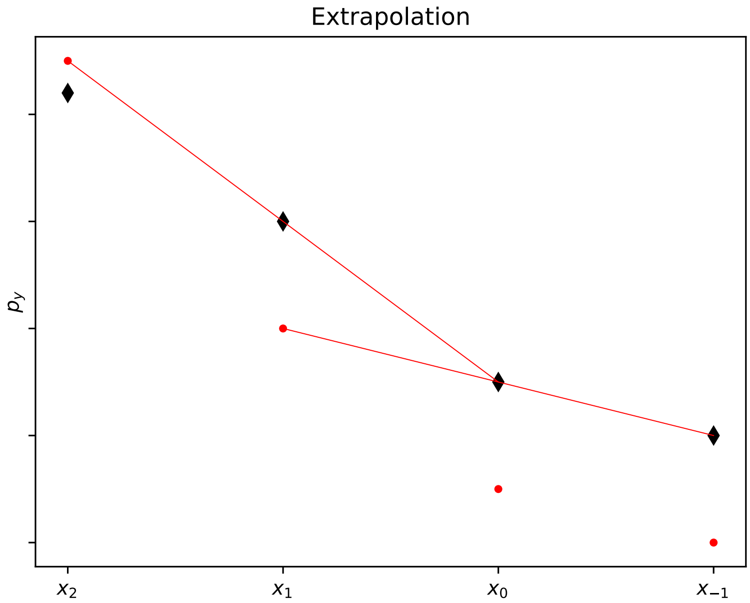

Figure 7.3: Here is an example of the process, for a few iterations. The red dots are the guesses and the black diamonds the correct momenta obtained via Newton's method

Example In our example we pick

$$\sigma_{-1} = 1 \times 10^{-4} \qquad \sigma_0 = 1 \times 10^{-3}$$

and we want to propagate the Lyapunov family until $x = x_{L_1} - \frac{r_{hill}}{5}$.

The initial positions and derived guesses are:

$$x_{-1} = 0.8368940652045109 \qquad x_0 = 0.8366785861892301$$
$$p_{y,-1} = 0.8370945143528499 \qquad p_{y,0} = 0.838683077672619$$

Here are a few iterations:

n	x_n	$\tilde{p}_{y,n}$	Energy
-1	0.8368940652045109	0.8370945498181693	-1.594167841903306
0	0.8366785861892301	0.8386866338248641	-1.594166166073831
1	0.8364631071739492	0.8402845123390982	-1.594161723852034
2	0.8362476281586684	0.841888230778558	-1.594154482180873
3	0.8360321491433875	0.843497834730994	-1.594144407577778
⋮	⋮	⋮	⋮
151	0.8050382502418416	1.1243531292562474	-1.548364297791188

Therefore we have obtained the Lyapunov orbit at the target position, with starting condition:

$$\mathbf{r} = (x_{151}, 0, 0, \tilde{p}_{y,151})$$

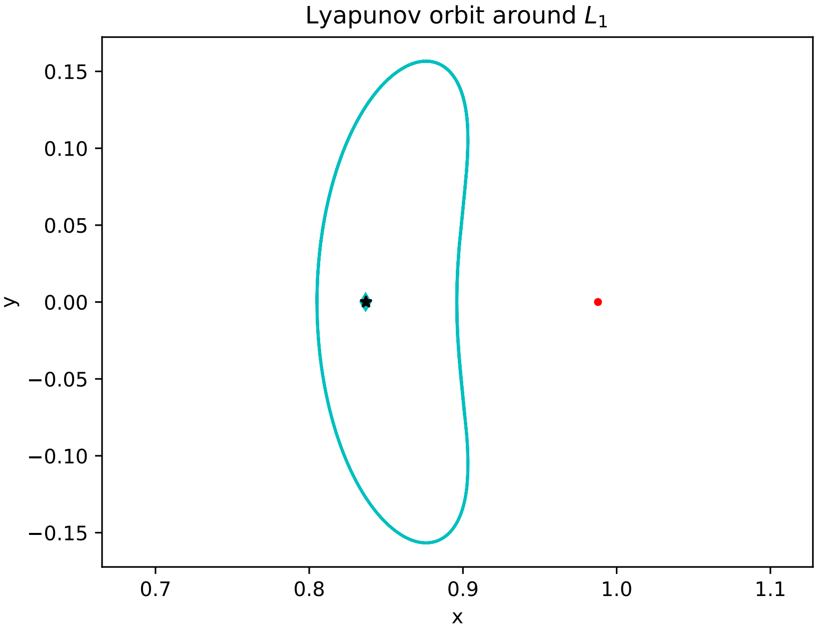

Figure 7.4: The bigger Lyapunov orbit just calculated. Notice the smaller one around L_1 that we calculated previously

Chapter 8

Manifolds

Now that we have found the periodic orbit we want to calculate its stable and unstable manifolds.

We will study the stability of the fixed point and its stable and unstable linear spaces.

Then we will generalize to the manifolds and translate to the complete phase space.

8.1 Poincaré map

In previous chapters we have defined the fixed point of a Lyapunov orbit as:

$$\tilde{\mathbf{g}} = (x_0, 0, \tilde{p}_y)$$

we can therefore also find its energy, $\tilde{E}$.

Knowing the energy of an orbit, we can define a new Poincaré surface, a restriction of Γ, called Σ, where all the points have the same energy of the Lyapunov orbit:

$$\Sigma = \left\{\mathbf{g} \in \Gamma : E(\mathbf{g}) = \tilde{E}\right\} \tag{8.1}$$

or:

$$\Sigma = \left\{\mathbf{r} \in \mathbb{R}^4 : y = 0, \quad \dot{y} > 0, \quad E(\mathbf{r}) = \tilde{E}\right\} \tag{8.2}$$

Notice that since all the points are bound to have the same energy $\tilde{E}$, we have reduced again the degrees of freedom of the system by 1. We can express a coordinate, p_y as a function of the remaining coordinates and energy.

$$\mathbf{g} = \left(x, p_x, p_y\left(x, p_x; E\right)\right)$$

The function $p_y\left(x, p_x; E\right)$ is the solution of:

$$\frac{1}{2}\left(p_x^2 + p_y^2\right) - x p_y - \frac{1-\mu}{|x+\mu|} - \frac{\mu}{|x+\mu-1|} - E = 0 \tag{8.3}$$

that gives positive:

$$\dot{y} = p_y - x > 0$$

Now we have a 2-dimensional surface Σ, with coordinates:

$$\mathbf{s} = \left(x, p_x\right) \tag{8.4}$$

On this Poincaré section we can define the Poincaré map ψ:

$$\Sigma \to \Sigma$$
$$\mathbf{s}' = \psi(\mathbf{s})$$

One way to calculate this function is:

1. From x, p_x, E calculate $p_y\left(x, p_x; E\right)$
2. From a starting condition
$$\left(x, 0, p_x, p_y\left(x, p_x; E\right)\right)$$
integrate the Hamilton equations and stop the integration when triggers the event
$$y = 0, \quad \dot{y} > 0$$
3. Extract from the vector $\mathbf{r}' = \left(x', 0, p'_x, p'_y\right)$ only the coordinates x, p_x:
$$\mathbf{s}' = \left(x', p'_x\right)$$

8.2 Fixed points and linearization

The reader can easily verify that if $\tilde{\mathbf{g}} = \left(x_0, 0, \tilde{p}_y\right)$ with energy $\tilde{E}$ is a fixed point for the map $\gamma(\mathbf{g})$, then the point:

$$\mathbf{p} = (x_0, 0)$$

is a fixed point for the map $\psi(\mathbf{s})$ and a representation of the Lyapunov orbit.

We can linearize the map around the fixed point (see chapter 4):

$$\delta\mathbf{s} = \mathbf{s} - \mathbf{p}$$
$$\delta\dot{\mathbf{s}} = \mathbf{M} \cdot \delta\mathbf{s}$$

where $\mathbf{M} = \nabla\psi(\mathbf{p})$ is the Jacobian matrix calculated in the fixed point $(x_0, 0)$:

$$\mathbf{M} = \begin{bmatrix} \frac{\partial \psi_x}{\partial x} & \frac{\partial \psi_x}{\partial p_x} \\ \frac{\partial \psi_{p_x}}{\partial x} & \frac{\partial \psi_{p_x}}{\partial p_x} \end{bmatrix}$$

Since $\psi(\mathbf{s})$ is a non-analytic function we need to find the derivatives numerically, using finite differences.

The partial derivatives are:

$$\frac{\partial g_x}{\partial x} \approx \frac{g_x(x_0 + h, 0) - g_x(x_0 - h, 0)}{2h}$$
$$\frac{\partial g_x}{\partial p_x} \approx \frac{g_x(x_0, h) - g_x(x_0, -h)}{2h}$$
$$\frac{\partial g_{p_x}}{\partial x} \approx \frac{g_{p_x}(x_0 + h, 0) - g_{p_x}(x_0 - h, 0)}{2h}$$
$$\frac{\partial g_{p_x}}{\partial p_x} \approx \frac{g_{p_x}(x_0, h) - g_{p_x}(x_0, -h)}{2h}$$

again, you will have to find the h that produces the most accurate results.

8.3 Manifolds

Now that we have the linearization we can study the stability of the fixed point and the tangent spaces.

8.3.1 Eigenvalues and unstable/stable spaces

By studying the eigenvalues and eigenvectors of the Jacobian we find that **p** is a *hyperbolic* fixed point, in particular:

$$|\eta_1| > 1 \qquad |\eta_2| < 1$$

therefore we have a stable space spanned by $\boldsymbol{\zeta}_1$ and an unstable space spanned by $\boldsymbol{\zeta}_2$ for the linearized map.

$$\begin{aligned}\mathbb{E}^U &= \langle\boldsymbol{\zeta}_1\rangle\\ \mathbb{E}^S &= \langle\boldsymbol{\zeta}_2\rangle\end{aligned}$$

We can diagonalize the equations:

$$\delta\mathbf{s} = \mathbf{P}\cdot\mathbf{u}$$

where **u** is a vector in the eigenbasis and **P** is the transformation matrix.

So, to find a starting condition on the stable space we do, with a small σ:

$$\begin{aligned}\mathbf{u}_s &= \sigma(0,1)\\ \delta\mathbf{s}_s &= \mathbf{P}\cdot\mathbf{u}_s\end{aligned}$$

and to have a starting condition on the unstable space:

$$\begin{aligned}\mathbf{u}_u &= \sigma(1,0)\\ \delta\mathbf{s}_u &= \mathbf{P}\cdot\mathbf{u}_u\end{aligned}$$

Adding the fixed point in the nonlinear map gives us a points on the local manifolds.

$$\mathbf{s}_s = \mathbf{p} + \delta\mathbf{s}_s$$

$$\mathbf{s}_u = \mathbf{p} + \delta\mathbf{s}_u$$

8.3.2 Visualize the unstable manifold

We want to visualize one trajectory on the unstable[1] manifold of the Lyapunov orbit, so we do this:

1. Pick a $\mathbf{u} = (\sigma, 0)$ with σ quite small, about $\sigma \sim 10^{-6}$. Again,we need this to be close to the tangency point

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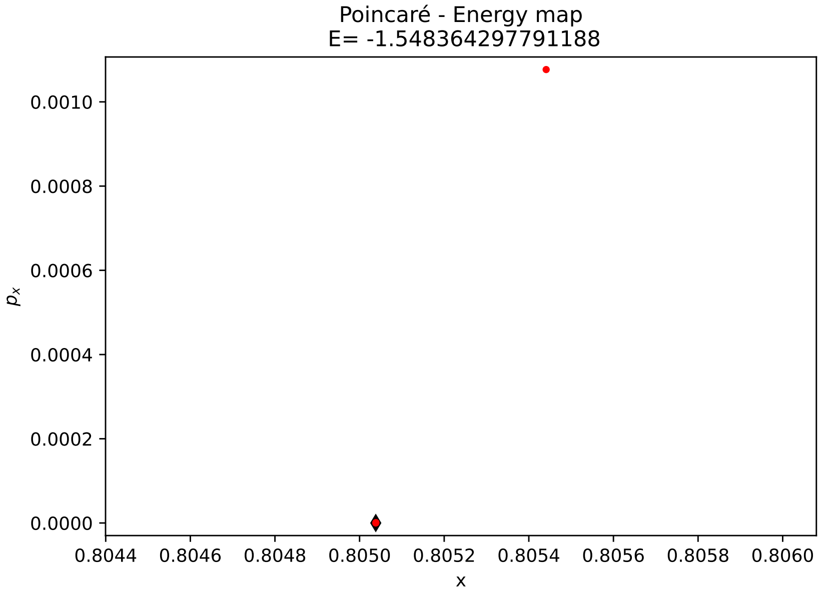
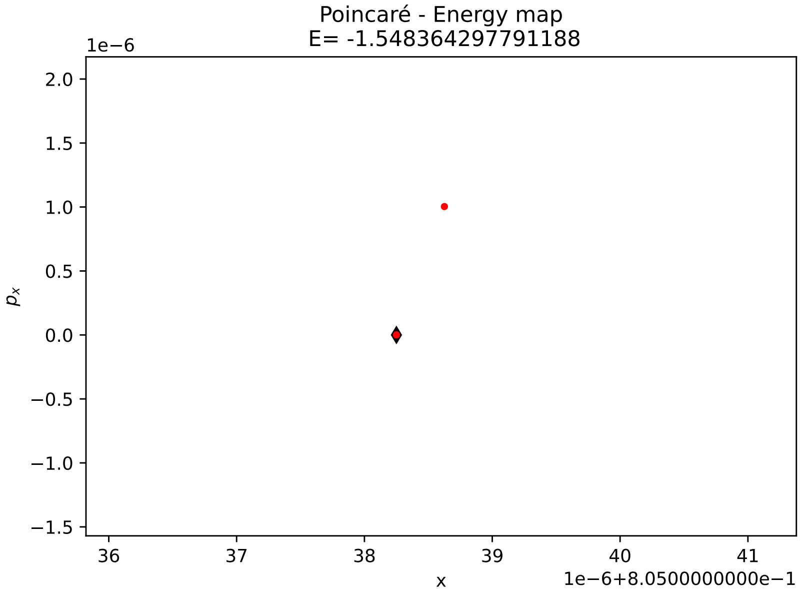
[1] The stable manifold has an analogous procedure, but with reversed time integration



(a) The black diamond is the fixed point **p**. The red dots are the iterations of the map. Notice how the second iteration, in the upper side, is already quite far from the fixed point. This is due to the large value of the unstable eigenvalue η_1



(b) This is a zoom of the previous picture, showing the starting point $\mathbf{s}_u$ and the first iteration, again as red dots. Because σ is very small the starting point is basically indistinguishable from the fixed point

Figure 8.1: A few iterations of points on the positive branch of the unstable manifold of **p**, in the $\psi(\mathbf{s})$ map

2. Calculate $\delta\mathbf{s} = \mathbf{P}\cdot\mathbf{u}$. We are on the unstable space
3. From
$$\mathbf{s} = \mathbf{p} + \delta\mathbf{s}$$
obtain $\tilde{x} = x_0 + \delta x$ and $\tilde{p}_x = 0 + \delta p_x$. We are on the local unstable manifold of the fixed point
4. Solve the quadratic equation for p_y with $E, \tilde{x}, \tilde{p}_x$ as parameters:
$$\frac{1}{2}\left(\tilde{p}_x^2 + p_y^2\right) - \tilde{x}p_y - \frac{1-\mu}{|\tilde{x}+\mu|} - \frac{\mu}{|\tilde{x}+\mu-1|} - E = 0$$
choosing p_y that satisfies:
$$\dot{y} = p_y - x > 0$$
5. Now we have a starting condition $\mathbf{r}_0 = (\tilde{x}, 0, \tilde{p}_x, \tilde{p}_y)$ in the phase space that is on the unstable local manifold of *the periodic orbit*
6. Integrate from $\mathbf{r}_0$ forward in time to obtain a trajectory on the unstable manifold

Remark Notice that the unstable and stable manifold have two branches each. These branches are given by the sign of σ.

8.3.3 Difference between manifolds

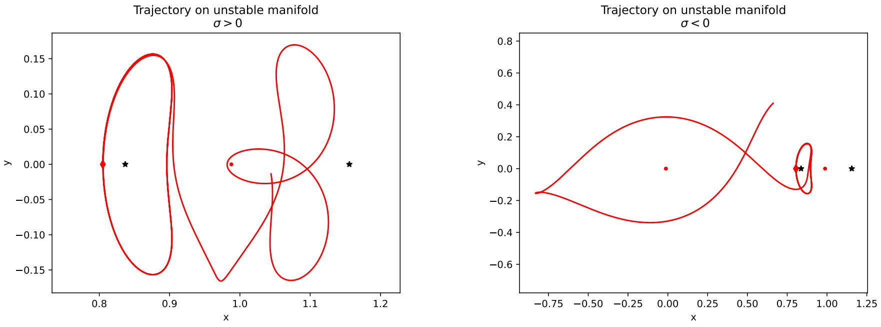
The reader may be a little confused now, since we have been talking about manifolds of equilibrium points, periodic orbits that lie on a center manifold and that themselves generate fixed points with their manifolds.

It may be useful to compare these different types of manifolds, in the hope that it will clarify any doubts the reader has.

We compare manifolds in the flow and Poincaré map by using a stable manifold as an example:

FLOW	**MAP**
Here a particle on the stable manifold is approaching the *equilibrium point* of $\dot{\mathbf{r}} = \mathbf{X}(\mathbf{r})$.	Here a particle on the stable manifold is approaching the *fixed point* of the Poincaré map $\mathbf{s}' = \psi(\mathbf{s})$, therefore in the phase space it is approaching the Lyapunov orbit.

Example In the last chapter we have found a Lyapunov orbit passing through position:
$$x_{151} = 0.8050382502418416$$
with energy:
$$E_{151} = -1.548364297791188$$
Fort this fixed point the eigenvalues of the matrix $\mathbf{M}$ are:
$$\eta_1 = 1071.41 \qquad \eta_2 = 0.000933$$



(a) Trajectory on positive branch

(b) Trajectory on negative branch

Figure 8.2: Trajectories on the two branches of the manifold, with time of integration $5T$

As an example we will calculate a trajectory on both branches of the *unstable* manifold.

We choose a $\sigma = \pm 1 \times 10^{-9}$.

Applying steps 1-6 we get the initial conditions for the two branches:

$$\mathbf{r}_{+} = (0.8050382506, \quad 0, \quad 9.365780108e - 10 \times 10^{-10}, \quad 1.1243531292657278)$$

$$\mathbf{r}_{-} = (0.8050382499, \quad 0, \quad -9.365780108e - 10 \times 10^{-10}, \quad 1.1243531292467654)$$

Visualize the tube manifolds Now that we know how to build a trajectory on the unstable manifold we want to visualize the whole tube manifold. We simply have to build many trajectories on this tube.

Build a generic :

$$\mathbf{u}_i = (0, \sigma_i)$$

with $\sigma_i = i \times 10^{-6}$ for $i = 1, 2, \dots$ and repeat steps from 1 to 6 for each one of them. Plotting all the trajectories together gives the projection on the position space of the unstable tube manifold.

Usually 30 or so starting conditions are sufficient to visualize well the manifold.

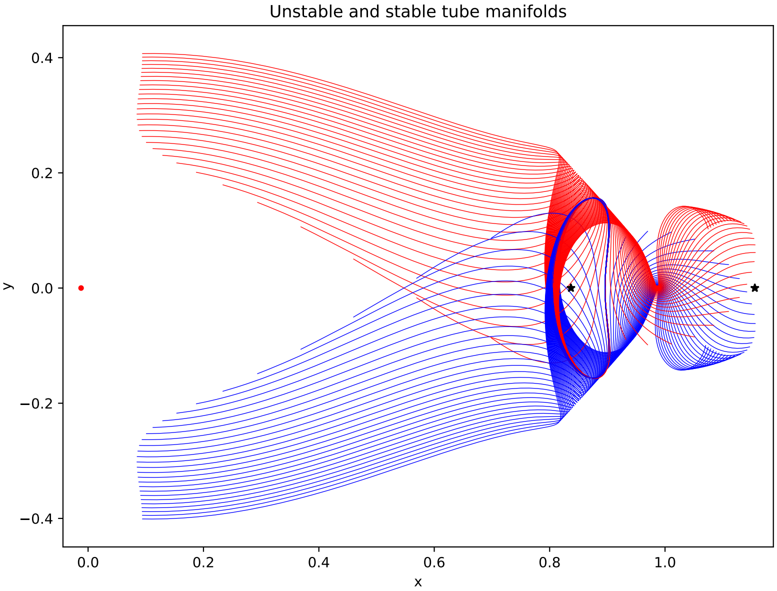


Figure 8.3: Projection of the tube manifolds on the position space. In blue is the stable manifold and in red is the unstable manifold

Chapter 9

Heteroclinic transfer

If the system has more than one hyperbolic (or partially hyperbolic) equilibrium points then it can happen that the stable manifold of one point crosses the unstable manifold of the other.
This is a heteroclinic point (or connection).

Let r_0 be a point in a generic phase space, and $\mathbf{p}_1$ and $\mathbf{p}_2$ two equilibrium points of the system:

$$\mathbf{r}_0 \in \mathbb{R}^n \qquad \mathbf{r}_0 \in W^U_{p_1} \quad and \quad \mathbf{r}_0 \in W^S_{p_2}$$

In our case this type of connection is very useful because it allows us to travel between two Lagrange points using barely any fuel.

In this chapter we will follow a trajectory going from L_1 to L_2.

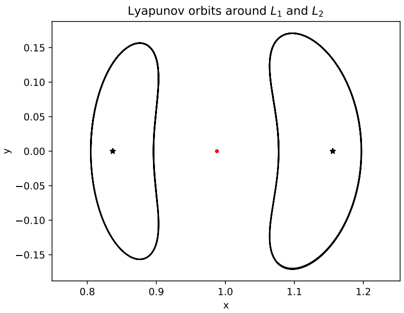
The reader should be aware that there are many other ways to compute these trajectories, some more elegant or precise. What we explain here a rather crude but simple way to calculate them. Moreover it only uses what we have developed until now in this book.

9.1 Lyapunov orbits

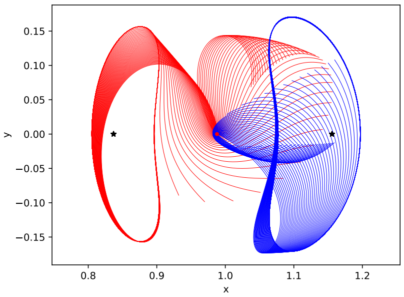
To go from L_1 to L_2 we need to calculate a periodic orbit around L_1 and then an orbit of equal energy around L_2. Mind that these orbits will probably not be used in the actual transfer, we need them as reference to find the heteroclinic intersection. The final orbits will be quite close to these, however.

Example In our example we want to go from L_1 to L_2. We find periodic orbits for the energy:

$$E = -1.5483247393843875$$



(a) The two periodic orbits for $E = -1.5483247393843875$



(b) Projection in the position space of both manifolds

9.2 Heteroclinic intersection

Now we want to find the heteroclinic intersection on the Poincaré map.

1. Prepare many starting conditions on the unstable space of the periodic orbit around L_1
2. Prepare many starting conditions on the stable space of the periodic orbit around L_2
3. Plot the first iteration of these starting conditions on the map. We obtain the manifolds of both fixed points $\mathbf{p}_{L_1}$ and $\mathbf{p}_{L_2}$
4. Find the intersection[1] $\mathbf{s}_0 = (x_0, p_{x,0})$ between the unstable manifold of $\mathbf{p}_{L_1}$ and the stable manifold of $\mathbf{p}_{L_1}$

Example For the unstable we space we prepare the following starting conditions:

$$\mathbf{u}_u = (\sigma_u, 0)$$

with $\sigma_u = k \times 10^{-5}$ where $k = [0, 400]$.

For the stable we space we prepare the following starting conditions:

$$\mathbf{u}_s = (0, \sigma_s)$$

with $\sigma_s = j \times 10^{-7}$ where $j = [-1000, 1000]$.

We plot one iteration of the map of these starting conditions. The heteroclinic intersection we found is:

$$\mathbf{s}_0 = (x_0, p_{x,0})$$

$$x_0 = 1.0649688817761498$$
$$p_{x,0} = 0.052603273137552975$$

---

[1] How to find the intersection? There are many ways, ranging from manually searching the crossing point to computer aided search algorithms. The reader can use the one more suited to his needs and skill.

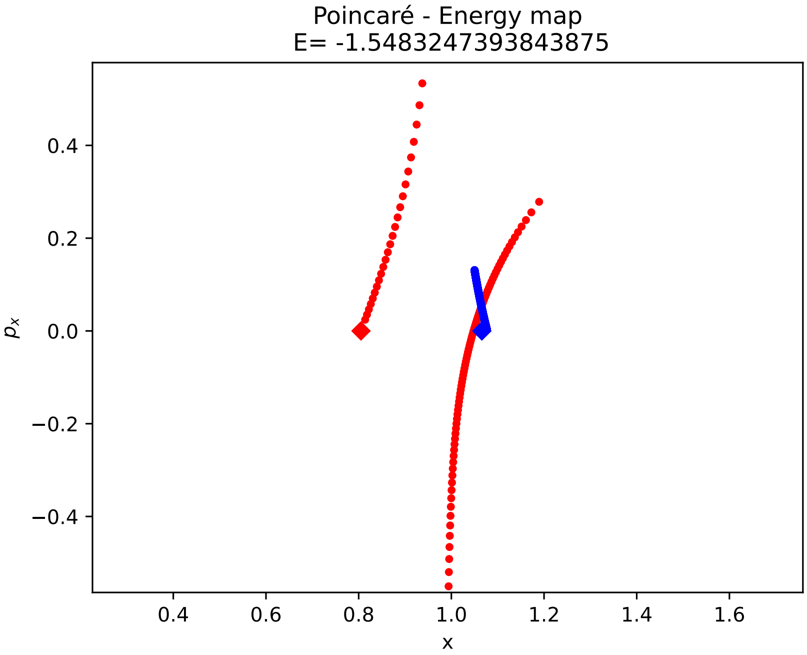


Figure 9.2: Manifolds of both fixed points in the P-E map. The red diamond is $\mathbf{p}_{L_1}$ and the blue diamond is $\mathbf{p}_{L_2}$. The intersection is: $x_0 = 1.0649688817761498$, $p_{x,0} = 0.052603273137552975$

9.3 Backward integration

Once we have found the heteroclinic intersection we need to know which exact point on the local unstable manifold of $\mathbf{p}_{L_1}$ maps to the intersection $\mathbf{s}_0$.

$$\mathbf{s}_0 = \psi(\mathbf{s}_{-1})$$

To find this point we simply do the inverse of map ψ from the intersection point:

$$\mathbf{s}_{-1} = (x_{-1}, p_{x,-1}) = \psi^{-1}(\mathbf{s}_0)$$

The inverse map ψ^{-1} is the ψ that we have developed, but with integration time reversed.

Then we search for the Lyapunov orbit passing through x_{-1}. Why?

Because this way we can put the spacecraft on a periodic orbit that passes each time through $\mathbf{s}_{-1} = (x_{-1}, 0)$. When we need to leave the orbit we can use the engines of the spacecraft to change from the momenta of the periodic orbit $\mathbf{p}_{L_1} = (0, \tilde{p}_{y,L_1})$ to the momenta of the point on the local unstable manifold $\mathbf{p}_{-1} = \big(p_{x,-1}, p_{y,-1}(E_{-1}, x_{-1}, p_{x,-1})\big)$.

Example In our case, one iteration of the inverse map from the intersection point provides:

$$x_{-1} = 0.809048555715$$

$$p_{x,-1} = -0.00869283154685$$

And the Lyapunov orbit found for x_{-1} has:

$$\tilde{p}_{y,L_1} = 1.091029333241613$$

$$E_{Lyap_1} = -1.5584125198708565$$

Notice that the energy of this new periodic orbit is different from the energy of the heteroclinic intersection (it is in fact lower). The energy needed to go from this orbit to the energy of the intersection will have to be provided by the thrusters of the spacecraft.

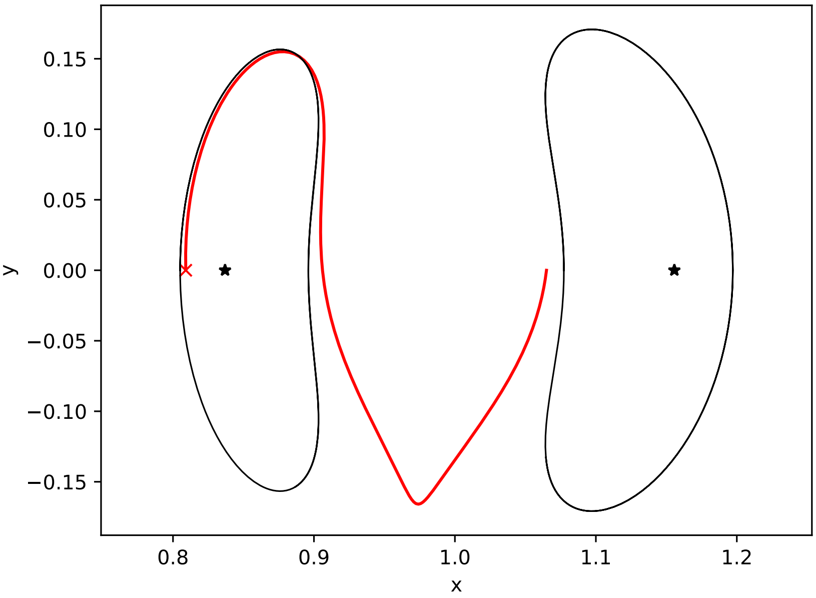


Figure 9.3: Trajectory backwards in time that from $r = (x_0, 0)$ leads to the point $(x_{-1}, 0)$ (the red cross close to L_1)

9.4 Transfer

Once we have done all these preliminary steps we can begin the actual transfer.

1. Assuming we are on the Lyapunov orbit passing through x_{-1}, when we reach the position $r = (x = x_{-1}, y = 0)$ we perform a maneuver to change the momenta of the spacecraft. We need a

$$\Delta V_1 = (p_{x,-1} - 0, p_{y,-1} - \tilde{p}_{y,L_1})$$

2. We leave the neighborhood of L_1 and reach the heteroclinic intersection

3. From the intersection we travel on the stable manifold of L_2 and we reach a neighborhood of the Lyapunov orbit of L_2

$$\mathbf{s}_1 = (x_1, p_{x,1})$$

4. If we are very precise the stable manifold should automatically lead us into the Lyapunov orbit of L_2. This may easily not be the case, due to numerical errors in our computations or the accuracy with which we have found the heteroclinic interserction.

   Therefore when we are close enough to L_2 we can perform and additional maneuver to put us on a Lyapunov orbit of L_2 passing through x_1. We simply calculate the orbit and then provide the required momenta with thrusters

$$\Delta V_2 = (0 - p_{x,1}, \tilde{p}_{y,L_2} - p_{y,1})$$

5. We are now on a periodic orbit around L_2

Example In our case the first maneuver is:

$$\Delta V_1 = (-0.008692831546845744, \quad 0.03363473958074348)$$

and when we reach the neighborhood of L_2

$$\mathbf{s}_1 = (1.0782995252401852, \quad 0.0037461341728886894)$$

a second maneuver is needed:

$$\Delta V_2 = (-0.0037461341728886894, \quad -0.0018305776818339226)$$

Therefore in our transfer we need a total ΔV of

$$\Delta V_{tot} = 0.038909382089267676$$

in normalized units.

The transfer takes a total time of

$$T = 10.401724709200904$$

again in normalized units, which translates to around 10 months for the Earth - Moon system.

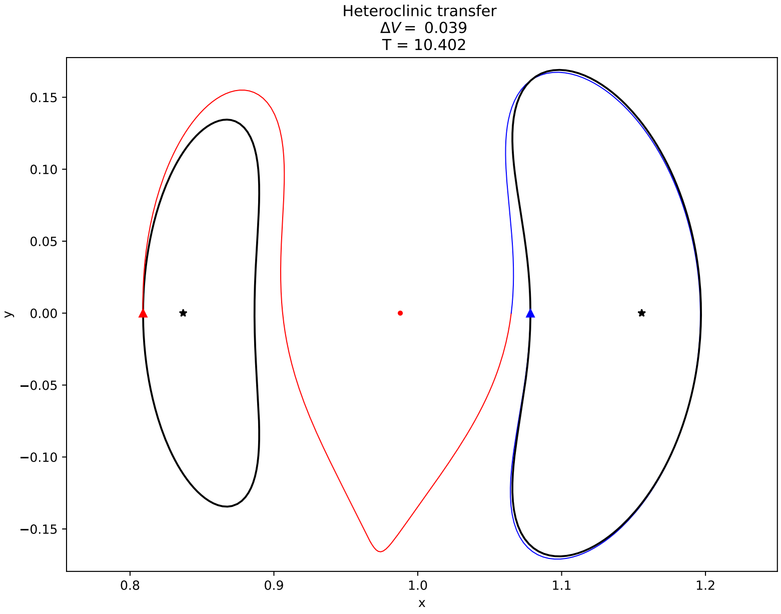


Figure 9.4: Heteroclinc transfer. The triangles are the points where a maneuver is needed.

Part III

3D calculations

Chapter 10

Numerical integration and linearization

Now we go on with the calculations for the non-planar problem. The reader will see that many of the steps are analogous to those of the planar problem.

10.1 Integration

The first step is to implement a good integrator of the Hamilton equations for the spatial problem.

The Hamiltonian and the Hamilton equations of the complete system are:

Hamiltonian:

$$H = \tfrac{1}{2}\left(p_x^2 + p_y^2 + p_z^2\right) + p_x y - p_y x - \frac{1-\mu}{\sqrt{(x+\mu)^2 + y^2 + z^2}} - \frac{\mu}{\sqrt{(x+\mu-1)^2 + y^2 + z^2}} \tag{10.1}$$

Hamilton equations: $\dot{\mathbf{r}} = \mathbf{X}(\mathbf{r})$

$$\dot{x} = \frac{\partial H}{\partial p_x} = p_x + y \tag{10.2}$$

$$\dot{y} = \frac{\partial H}{\partial p_y} = p_y - x \tag{10.3}$$

$$\dot{z} = \frac{\partial H}{\partial p_z} = p_z \tag{10.4}$$

$$\dot{p}_x = -\frac{\partial H}{\partial x} = p_y - (1-\mu)\frac{x+\mu}{\left((x+\mu)^2 + y^2 + z^2\right)^{3/2}} - \mu\frac{x+\mu-1}{\left((x+\mu-1)^2 + y^2 + z^2\right)^{3/2}} \tag{10.5}$$

$$\dot{p}_y = -\frac{\partial H}{\partial y} = -p_x - (1-\mu)\frac{y}{\left((x+\mu)^2 + y^2 + z^2\right)^{3/2}} - \mu\frac{y}{\left((x+\mu-1)^2 + y^2 + z^2\right)^{3/2}} \tag{10.6}$$

$$\dot{p}_z = -\frac{\partial H}{\partial z} = -(1-\mu)\frac{z}{\left((x+\mu)^2 + y^2 + z^2\right)^{3/2}} - \mu\frac{z}{\left((x+\mu-1)^2 + y^2 + z^2\right)^{3/2}} \tag{10.7}$$

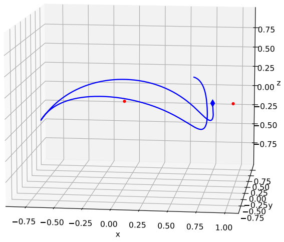
Notice how the Hamiltonian is even for the coordinates z, p_z, and the equations of motions give:

$$\dot{z} = p_z$$
$$\dot{p}_z = -c(x, y, z) \cdot z$$

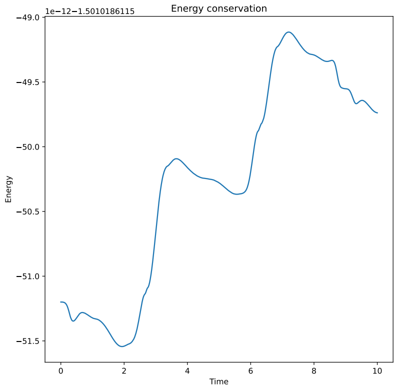
with c a positive coefficient.

This means that for the z, p_z phase space the system is basically an isolated oscillator (the coordinates x, y only affect the magnitude of the oscillation, but not its qualitative behavior). The spacecraft follows a "bouncing" motion on the horizontal plane.

Again, we can use any numerical integrator to solve these equations and check the conservation of energy.



(a) $\mu = 0.01215$, and starting conditions $(0.8, 0, 0, 0, 1, -.4)$



(b) Error on the conservation of energy

Example

10.1.1 Equilibrium points

In three dimensions the collinear equilibrium points are of the form:

$$\mathbf{c} = (x_c, 0, 0, 0, x_c, 0)$$

Notice that even for the spatial problem the equilibrium point lays on the orbital plane. This is due to the fact that the dynamics on the z-axis is symmetrical to the xy plane.

Example We take as equilibrium point the Lagrange point 1:

$$\mathbf{c}_{L_1} = (0.83691, 0, 0, 0, 0.83691, 0)$$

10.2 Linearization

Now that we have the equilibrium points we can begin to study the linearized dynamics. The general Jacobian is:

$$\nabla \mathbf{X}(\mathbf{r}) = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \frac{\partial \dot{p}_x}{\partial x} & \frac{\partial \dot{p}_x}{\partial y} & \frac{\partial \dot{p}_x}{\partial z} & 0 & 1 & 0 \\ \frac{\partial \dot{p}_y}{\partial x} & \frac{\partial \dot{p}_y}{\partial y} & \frac{\partial \dot{p}_y}{\partial z} & -1 & 0 & 0 \\ \frac{\partial \dot{p}_z}{\partial x} & \frac{\partial \dot{p}_z}{\partial y} & \frac{\partial \dot{p}_z}{\partial z} & 0 & 0 & 0 \end{bmatrix} \tag{10.8}$$

where:

$$\frac{\partial \dot{p}_x}{\partial x} = -(1-\mu)\left(\frac{1}{d_1^3} - 3\frac{(x+\mu)^2}{d_1^5}\right) - \mu\left(\frac{1}{d_2^3} - 3\frac{(x+\mu-1)^2}{d_2^5}\right) \tag{10.9}$$

$$\frac{\partial \dot{p}_y}{\partial y} = -(1-\mu)\left(\frac{1}{d_1^3} - 3\frac{y^2}{d_1^5}\right) - \mu\left(\frac{1}{d_2^3} - 3\frac{y^2}{d_2^5}\right) \tag{10.10}$$

$$\frac{\partial \dot{p}_z}{\partial z} = -(1-\mu)\left(\frac{1}{d_1^3} - 3\frac{z^2}{d_1^5}\right) - \mu\left(\frac{1}{d_2^3} - 3\frac{z^2}{d_2^5}\right) \tag{10.11}$$

$$\frac{\partial \dot{p}_x}{\partial y} = \frac{\partial \dot{p}_y}{\partial x} = 3(1-\mu)\frac{(x+\mu)y}{d_1^5} + 3\mu\frac{(x+\mu-1)y}{d_2^5} \tag{10.12}$$

$$\frac{\partial \dot{p}_x}{\partial z} = \frac{\partial \dot{p}_z}{\partial x} = 3(1-\mu)\frac{(x+\mu)z}{d_1^5} + 3\mu\frac{(x+\mu-1)z}{d_2^5} \tag{10.13}$$

$$\frac{\partial \dot{p}_y}{\partial z} = \frac{\partial \dot{p}_z}{\partial y} = 3(1-\mu)\frac{yz}{d_1^5} + 3\mu\frac{yz}{d_2^5} \tag{10.14}$$

When evaluated on a collinear point, since $y = 0$, we have:

$$\nabla \mathbf{X}(\mathbf{c}) = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \frac{\partial \dot{p}_x}{\partial x} & 0 & 0 & 0 & 1 & 0 \\ 0 & \frac{\partial \dot{p}_y}{\partial y} & 0 & -1 & 0 & 0 \\ 0 & 0 & \frac{\partial \dot{p}_z}{\partial z} & 0 & 0 & 0 \end{bmatrix} \tag{10.15}$$

The eigenvalues of $\nabla \mathbf{X}(\mathbf{c})$ are:

$$\lambda_1 = -\lambda \qquad \lambda_2 = \lambda \qquad \lambda_3 = i\omega \qquad \lambda_4 = -i\omega \qquad \lambda_5 = i\nu \qquad \lambda_6 = -i\nu$$

And the change of basis matrix is:

$$\mathbf{P} = \left[(\rho_1) \qquad (\rho_2) \qquad (\rho_3)\,(\rho_4) \qquad (\rho_5) \qquad (\rho_6)\right]$$

Their eigenvectors $\rho_1, \rho_2, \rho_3, \rho_4, \rho_5, \rho_6$ give the linearized spaces:

$$\begin{aligned} \mathbb{E}^S &= \langle \rho_1 \rangle \\ \mathbb{E}^U &= \langle \rho_2 \rangle \\ \mathbb{E}^C &= \langle \rho_3, \rho_4, \rho_5, \rho_6 \rangle \end{aligned}$$

We don't prove that to obtain real results we need $\rho_4 = \rho_3^*$ and $\rho_6 = \rho_5^*$, the reasoning is the same as for the planar problem. Again the equilibrium point is partially hyperbolic, but now the central space is **4-dimensional**.

By inspecting the matrix of the eigenvectors **P** we notice something truly interesting:

Example The eigenvalues are:

$\lambda_1 = -2.932 \qquad \lambda_2 = 2.932 \qquad \lambda_3 = 2.334i \qquad \lambda_4 = -2.334i \qquad \lambda_5 = 2.268i \qquad \lambda_6 = -2.268i$

Meaning that:

$$\lambda = 2.932 \qquad \omega = 2.334 \qquad \nu = 2.268$$

$$\mathbf{P} = \begin{bmatrix} 0.279 & 0.279 & -0.119 & -0.119 & 0 & 0 \\ 0.128 & -0.128 & -0.429j & 0.429j & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.403j & 0.403j \\ -0.946 & 0.946 & 0.149j & -0.149j & 0 & 0 \\ -0.097 & -0.097 & 0.882 & 0.882 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.915 & 0.915 \end{bmatrix} \tag{10.16}$$

Notice how only the eigenvectors related to the ν eigenvalue have a third and sixth component while all of the other components are 0. This means that if we were to multiply **P** for a vector $\mathbf{u} = (u_1, u_2, u_2, u_4, u_5, u_6)$ the coordinates $\delta z, \delta p_z$ would only be affected by u_5, u_6.

This means that (in the linearization) the dynamics on the vertical axis are *totally decoupled* from the dynamics on the plane.

In fact we can rearrange the matrix to see this more clearly:

$$\begin{pmatrix} \delta x \\ \delta y \\ \delta p_x \\ \delta p_y \\ \delta z \\ \delta p_z \end{pmatrix} = \begin{bmatrix} 0.279 & 0.279 & -0.119 & -0.119 & 0 & 0 \\ 0.128 & -0.128 & -0.429j & 0.429j & 0 & 0 \\ -0.946 & 0.946 & 0.149j & -0.149j & 0 & 0 \\ -0.097 & -0.097 & 0.882 & 0.882 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.403j & 0.403j \\ 0 & 0 & 0 & 0 & 0.915 & 0.915 \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{pmatrix}$$

The first 4×4 block-matrix is equivalent to the one that we found in the calculations for the planar problem. In fact the first four eigenvalues are exactly the ones that we found previously. This means that the linearized problem can be neatly separated in a planar part and a vertical part.

Looking at the diagonalised equations:

$$\begin{aligned}
\dot{u}_1 &= -\lambda u_1 \\
\dot{u}_2 &= \lambda u_2 \\
\dot{u}_3 &= i\omega u_3 \\
\dot{u}_4 &= -i\omega u_4 \\
\dot{u}_5 &= i\nu u_5 \\
\dot{u}_6 &= -i\nu u_6
\end{aligned}$$

are necessarily derived from a:

$$H = \lambda u_1 u_2 + i\omega u_3 u_4 + i\nu u_5 u_6$$

Notice that u_5, u_6 is an harmonic oscillator. This means that we are simply "adding" an oscillation on the vertical axis to the dynamics we have seen previously. Obviously this is the case only for the linearized dynamics, and will not translate to the nonlinear problem, but it is interesting nonetheless.

Using the Birkoff transformation we can see the oscillator more clearly:

$$u_3 = \frac{q_H - i p_H}{\sqrt{2}} \qquad u_4 = \frac{-i q_H + p_V}{\sqrt{2}} \tag{10.17}$$

$$u_5 = \frac{q_V - i p_V}{\sqrt{2}} \qquad u_6 = \frac{-i q_V + p_V}{\sqrt{2}} \tag{10.18}$$

Substituting in the Hamiltonian for the central space (when $u_1 = u_2 = 0$, which are the hyperbolic equations):

$$H(u_3, u_4, u_5, u_6) = i\omega u_3 u_4 + i\nu u_5 u_6$$

we get:

$$\tilde{H} = \frac{\omega}{2}(p_H^2 + q_H^2) + \frac{\nu}{2}(p_V^2 + q_V^2) \tag{10.19}$$

which is exactly the Hamiltonian of two decoupled harmonic oscillators.

In fact the horizontal oscillator gives the *horizontal Lyapunov orbit*, while the vertical oscillator gives the *vertical Lyapunov orbit* in the linearized space. The dynamics in the linearized central space is the *superposition of two harmonic oscillators.*

Notice however that ω and ν are:

$$\frac{\omega}{\nu} \notin \mathbb{Q} \tag{10.20}$$

therefore the superposition of the two harmonic oscillators will not generate closed (or periodic) orbits.

Since we have already discussed the horizontal Lyapunov orbit in the previous part, we will now focus on the **vertical Lyapunov orbit**.

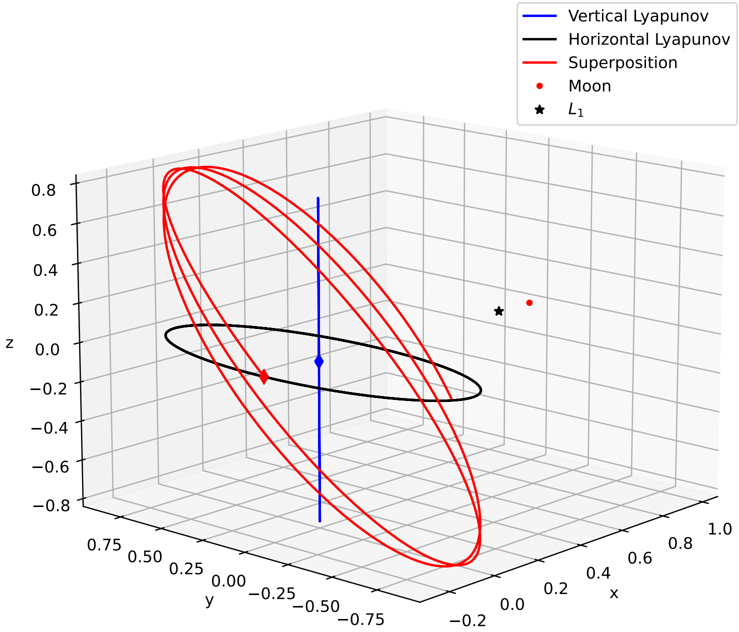


Figure 10.2: The two Lyapunov orbits and their superposition

Chapter 11

Vertical Lyapunov orbit

We have already seen how to find the planar Lyapunov orbit in the chapters related to the planar problem. Here we explain how to find the vertical Lyapunov orbit.

Again, we treat it as a fixed point of a Poincaré map and use a root finding algorithm to locate the fixed point.

11.1 Linear vertical Lyapunov orbit

Just like for the planar case we will start from the linearized space.

$$\delta\mathbf{r} = \mathbf{P} \cdot \mathbf{u}$$

We already know that the diagonalized coordinates u_5, u_6 fully control the motion on the vertical axis. Therefore we can (again) do, for a small σ:

$$\mathbf{u}_c = \sigma\,(0,0,0,0,1,1) \tag{11.1}$$

which gives us a:

$$\begin{aligned} \delta\mathbf{r}_c &= \mathbf{P} \cdot \mathbf{u}_c \\ \delta\mathbf{r} &= \left(0,0,0,0,0,p_{z,0}\right) \end{aligned}$$

Therefore we have a starting condition on the xy plane, with an initial positive vertical velocity[1] .

In the linearized dynamics we obtain an harmonic oscillator on the z-axis.

Example

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[1] The reader can check this with pen and paper.

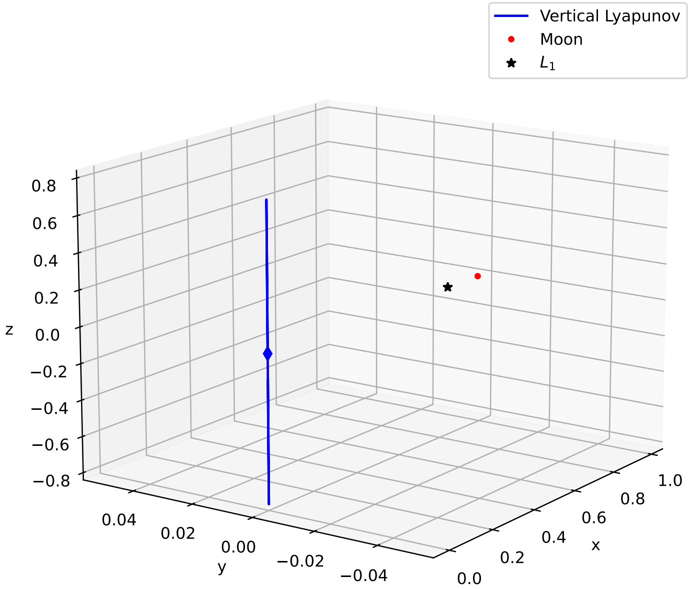


Figure 11.1: Linear vertical Lyapunov orbit

11.2 Nonlinear vertical Lyapunov orbit

Just like for the planar problem, we will treat the periodic orbit as a fixed point of a Poincaré map on a Poincaré surface and use a root finding algorithm to find this fixed point.

Here we will use a *energy driven algorithm*, which means that we will find the fixed point as a function of energy.

Let $\tilde{\mathbf{r}}$ be the initial guess obtained from the linearization and E the energy of this point:

$$\tilde{\mathbf{r}} = \mathbf{c} + \boldsymbol{\delta}\mathbf{u}_c$$

$$E = H(\tilde{\mathbf{r}})$$

Example

$$\tilde{\mathbf{r}} = \left(0.836918007, 0, 0, 0, 0.836918007, 1.830119329 \times 10^{-3}\right)$$

$$E = -1.5941661840949335$$

11.2.1 Poincaré section

We define the 4-dimensional Poincaré section Σ as the points in the phase space that lie on the horizontal plane, with positive z-velocity and with energy E:

$$\Sigma = \{\mathbf{r} \in \mathbb{R}^6 : z = 0, \dot{z} > 0, H(\mathbf{r}) = E\} \tag{11.2}$$

and therefore we can reduce the degrees of freedom to 4 by saying that the momentum in z is a function of the energy:

$$p_z = p_z(x, y, p_x, p_z; E)$$

$$p_z = \sqrt{2H - p_x^2 - p_y^2 - p_x y + p_y x + \frac{1-\mu}{\sqrt{(x+\mu)^2 + y^2 + z^2}} + \frac{\mu}{\sqrt{(x+\mu-1)^2 + y^2 + z^2}}} \tag{11.3}$$

where we have chosen the positive solution, since $p_z = \dot{z}$.

We get a map:

$$\Sigma \to \Sigma \in \mathbb{R}^4$$

$$\mathbf{s}(x, y, p_x, p_y) \mapsto \mathbf{s}' = \boldsymbol{\psi}(\mathbf{s}) \tag{11.4}$$

A periodic orbit in the phase space will be a fixed point here. We want to find the fixed point as a function of energy.

11.2.2 Root finding

We will now use a energy driven root finding algorithm, minding that the root is multidimensional. The function of which we want the roots is:

$$\mathbf{F} = \boldsymbol{\psi}(\mathbf{s}) - \mathbf{s} = 0 \tag{11.5}$$

And its Jacobian is:

$$\nabla\mathbf{F} = \nabla\boldsymbol{\psi} - \mathbf{I} \tag{11.6}$$

Where the Jacobian of $\boldsymbol{\psi}$ is calculated using finite differences:

$$\frac{\partial \psi_i}{\partial x_j} = \frac{\psi_i\left(\mathbf{s} + \epsilon\boldsymbol{\delta}_j\right) - \psi_i\left(\mathbf{s} - \epsilon\boldsymbol{\delta}_j\right)}{2\epsilon} \tag{11.7}$$

with $\boldsymbol{\delta}_j$ being a vector of Kronecker's deltas: $\boldsymbol{\delta}_j = \left(\delta_{1,j}, \delta_{2,j}, \delta_{3,j}, \delta_{4j}\right)$. As usual ϵ is a small quantity, say $\epsilon = 10^{-7}$ and should be chosen carefully to ensure the best convergence of Newton's method.

Now we check that the Poincaré map is simplectict:

$$\nabla\boldsymbol{\phi}^T \cdot \mathbf{J} \cdot \nabla\boldsymbol{\phi} = \mathbf{J}$$

Since a Poincaré map is a "selection of points" of a Hamiltonian flow, this is a good way to check that everything is correct.

Now the iteration for Newton's method is:

$$\mathbf{s}_{n+1} = \mathbf{s}_n - \nabla\mathbf{F}(\mathbf{s}_n)^{-1} \cdot \mathbf{F}(\mathbf{s}_n) \tag{11.8}$$

This allows us to find the fixed point on Σ that corresponds to a periodic orbit in the nonlinear phase space with the same energy of the linear guess:

$$\tilde{\mathbf{s}} = \left(\tilde{x}, \tilde{y}, \tilde{p_x}, \tilde{p_y}\right) \mapsto \tilde{\mathbf{r}} = \left(\tilde{x}, \tilde{y}, 0, \tilde{p_x}, \tilde{p_y}, p_z(\tilde{\mathbf{s}}; E)\right) \tag{11.9}$$

11.2.3 Propagation

Now that we know how to find the fixed point $\tilde{\mathbf{s}}$ for energy E, we can try to find orbits further away from the equilibrium point L_1.

We use again the linear propagation to find successive guesses that could be close to the next fixed point.

1. We find two close fixed points $\tilde{\mathbf{s}}_0$, $\tilde{\mathbf{s}}_1$, so that $E_1 - E_0 = dE \sim 10^{-5}$

2. We calculate the angular coefficient

$$\mathbf{m} = \frac{\tilde{\mathbf{s}}_1 - \tilde{\mathbf{s}}_0}{dE}$$

3. And the bias term

$$\mathbf{q} = \tilde{\mathbf{s}}_0 - \mathbf{m}E_0$$

4. With thew energy $E_2 = E_1 + dE$, calculate the guess

$$\boldsymbol{\xi}_2 = \mathbf{m}E_2 + \mathbf{q}$$

5. Use $\boldsymbol{\xi}_2$ as a first guess for the root finding algorithm. Obtain $\tilde{\mathbf{s}}_2$

Example A few iterations:

n	x	y	p_x	p_y	E
0	0.836918	-1.534743e-15	1.325194e-15	0.836919	-1.594166
1	0.836921	-7.383620e-15	5.068395e-15	0.836928	-1.594156
2	0.836924	-1.032484e-14	4.408001e-15	0.836937	-1.594146
3	0.836928	-1.235655e-14	8.177169e-15	0.836946	-1.594136
⋮	⋮	⋮	⋮	⋮	⋮
100	0.837295	-1.116729e-13	7.287069e-14	0.837983	-1.592996

Adding the $z = 0$ and the p_z resulting from 11.3, we get a starting condition that gives a vertical Lyapunov orbit.

$$\tilde{\mathbf{r}}_V = (0.837295, -1.116729e-13, 0, 7.287069e-14, 0.837983, 0.048419)$$

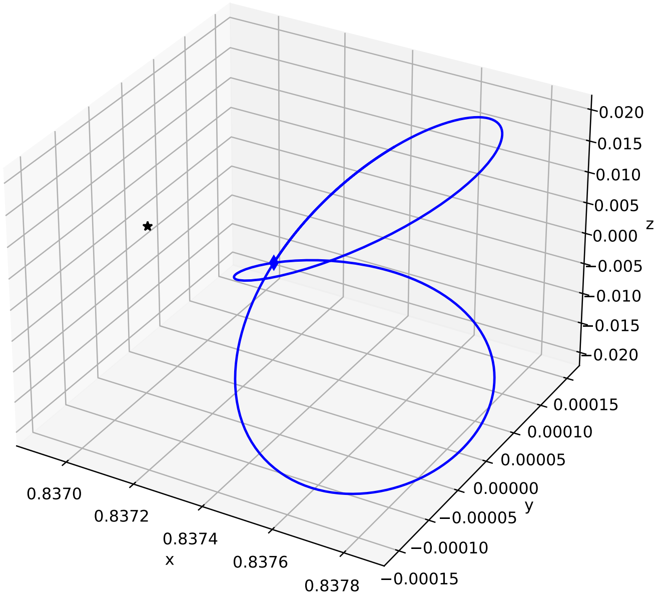


Figure 11.2: Nonlinear vertical Lyapunov orbit with starting condition $\tilde{\mathbf{r}}_V$ and time of integration one characteristic period $T = \dfrac{2\pi}{\nu}$

11.3 Manifolds

As usual to find the manifolds of a fixed point we need to analyze the linearized map. We have already seenhow to calculate the Jacobian of the Poincaré map in 11.7, now we evaluate it in the fixed point $\tilde{\mathbf{s}}$.

With this we calculate the eigenvalues and eigenvectors of the 4×4 matrix $\nabla \boldsymbol{\psi}(\tilde{\mathbf{s}})$.

$$\nabla \boldsymbol{\psi}(\tilde{\mathbf{s}}) = \mathbf{P}\boldsymbol{\Lambda}\mathbf{P}^{-1}$$

Specifically we find that it has:

$$\lambda_1 = \lambda \qquad \lambda_2 = \frac{1}{\lambda} \qquad \lambda_3 = \alpha + i\omega \qquad \lambda_4 = \alpha - i\omega$$

where $|\lambda| > 1$ and $|\alpha| < 1$.

Therefore the eigenvectors ρ_1 and ρ_2 define respectively the unstable and stable spaces.

But what about λ_3 and λ_4? Their respective eigenvectors define a sort of stable center space, since $|\alpha| < 1$, which means that in the phase space we will have small perturbations on the Lyapunov orbit that will eventually decay on the proper Lyapunov orbit.

These eigenvalues depend on the energy of the orbit, and the change of their real and imaginary parts is the cause of the bifurcation of the family of vertical Lyapunov orbits at certain energies. These bifurcations are crucial for the existence of the halo orbits, but we will come back later to this.

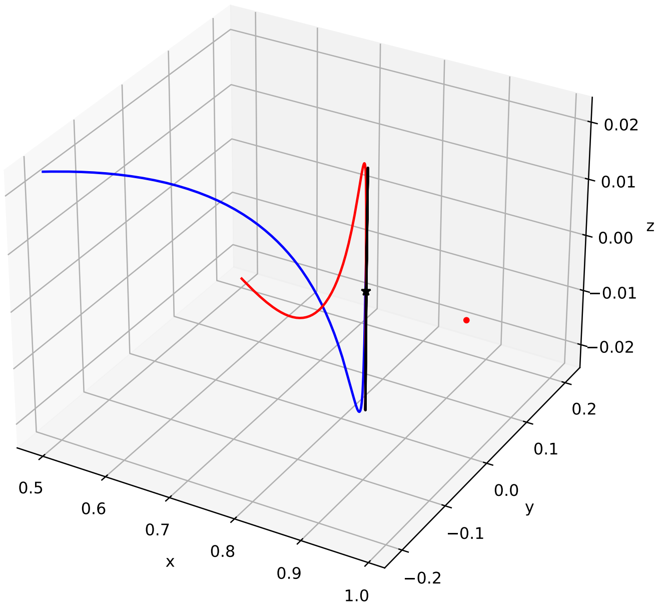


Figure 11.3: The three types of manifolds

Example

$$\lambda_1 = 3294.698 \qquad \lambda_2 = 3.03 \times 10^{-4} \qquad \lambda_3 = 0.981 + 0.194i \qquad \lambda_4 = 0.981 - 0.194i$$

11.3.1 Plot of manifolds

Since the technique is analogous to what we did for the planar problem we will only briefly list it and just give the results.

Visualize unstable manifold

1. Take many points of type $\mathbf{u} = \sigma(1, 0, 0, 0)$, with σ very small

2. Calculate $\mathbf{s} = \tilde{\mathbf{s}} + \mathbf{Pu}$
3. Take the components of $\mathbf{s}$ and build the initial condition in the phase space, with $z = 0$ and p_z from 11.3
4. Integrate and plot

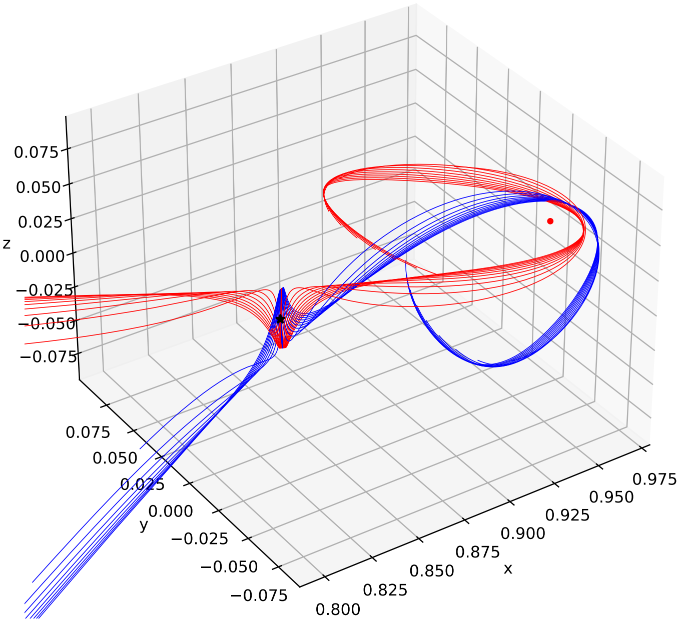


Figure 11.4: The stable (blue) and unstable (red) manifolds, with time of integration 4.5

Chapter 12

Halo orbit

12.1 Definition

Halo orbits are a type of periodic orbits centered on a collinear Lagrange point.

We have seen in section 10.2 that in the linearization the center space is made up by two harmonic oscillator. However they are not resonant, therefore a superposition of the two will never give a closed orbit.

We do not give the details, but it is interesting that in the nonlinear case the center manifold can still be divided into two nonlinear "oscillators" and that because of the nonlinear terms they are actually resonant. The superposition of these two oscillators will give a 3D closed orbit, the halo orbit.

Finding this family of orbits through numerical methods can be quite hard, since we do not have a first guess like for the vertical Lyapunov orbit. Therefore we need to reduce the phase space to the center manifold and try to find the orbits there.

12.2 Reduction to the center manifold

We know that at a collinear equilibrium point the linearized phase space is divided into *stable*, *unstable* and *center* spaces.

Thanks to the central manifold theorem[1] we also know that:

- It exists a diffeomorphism that maps the center space (close to the equilibrium) to the center manifold $\mathscr{M}$
- The manifold $\mathscr{M}$ is locally invariant for small t
- The manifold $\mathscr{M}$ is tangent to the center space in the equilibrium

Therefore it is possible to divide the phase space close to the equilibrium into its stable, unstable and center manifolds. Then we can study the dynamics on the center manifold and search for the resonant halo orbit.

We can achieve this by using a chain of canonical transformations, that will allow us to go from the original phase space to a local approximation of the center manifold.

---

[1] vedi tutorial center manifold

12.2.1 Taylor expansion

First of all we rewrite the Hamilton equations through a Taylor expansion centered around a collinear lagrange point. Basically the same steps of the linearization only that now we do not neglect all the higher order terms.

The transformation is[2] :

$$\mathbf{r} = \delta\mathbf{r} + \mathbf{c} \tag{12.1}$$

Substituting in the Hamiltonian we get:

$$H(\mathbf{r}) = H(\delta\mathbf{r}) + H(\mathbf{c}) \tag{12.2}$$

where $H(\mathbf{c})$ is the energy of the equilibrium point. In our case will be L_1. From now on we will always be in a neighborhood of the equilibrium **c**, therefore we will always refer to the energy as the total energy minus the energy of the equilibrium.

$$E = H(\delta\mathbf{r}) = H(\mathbf{r}) - H(\mathbf{c})$$

Now we can proceed with the expansions:

$$\delta\dot{\mathbf{r}} = \nabla\mathbf{X}(\mathbf{c}) \cdot \delta\mathbf{r} + \sum_{k=2}^{\infty} P_k(\delta\mathbf{r}) \tag{12.3}$$

$$H(\delta\mathbf{r}) = \frac{1}{2}\delta\mathbf{r} \cdot \nabla\nabla H(\mathbf{c}) \cdot \delta\mathbf{r} + O(\delta\mathbf{r}^3) \tag{12.4}$$

Where:

$$\sum_{k=2}^{\infty} P_k(\delta\mathbf{r})$$

are the higher order terms.

The reader can verify that

$$\nabla\mathbf{X}(\mathbf{c}) = \mathbf{J} \cdot \nabla\nabla H(\mathbf{c})$$

with **J** the symplectic matrix.

Notice that the Taylor series is not a canonical transformation, because there is no transformation at all! It is simply a rewriting of the original equations in a polynomial series.

12.2.2 Symplectic diagonalization

Like we did for the linearization in chapter 10, we can use the matrix of eigenvectors of $\nabla\mathbf{X}(\mathbf{c})$ to "diagonalize" the equations.

This will give us the coordinates[3]:

$$\mathbf{u} = (u_1, v_1, u_2, v_2, u_3, v_3) \quad \in \mathbb{C}^6 \tag{12.5}$$

which are a parametrization of the linear spaces $\mathbb{E}^U$,$\mathbb{E}^S$,$\mathbb{E}^C$ through the change of basis matrix **P**.

---

[2] A translation is always symplectic

[3] **Notation**: here we denote the components of vector **u** as u_i and v_i with $i = 1, 2, 3$. This way we have three pairs of components, with each pair referring to a eigenvalue. This is useful because it clarifies the relationship between the hyperbolic terms ($i = 1$) and the center manifold terms ($i = 2, 3$).

With the pair u_1, v_1 parametrazing the hyperbolic spaces, u_2, v_2 parametrizing the planar center space and the pair u_3, v_3 the vertical center space.

Only that now we want to be sure that the transformation is symplectic. We know that the original change of basis matrix is

$$\mathbf{P} = \begin{bmatrix} \rho_1 & \rho_2 & \rho_3 & \rho_4 & \rho_5 & \rho_6 \end{bmatrix} \tag{12.6}$$

and the diagonal matrix is:

$$\Lambda = \begin{bmatrix} \lambda & 0 & 0 & 0 & 0 & 0 \\ 0 & -\lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & i\omega & 0 & 0 & 0 \\ 0 & 0 & 0 & -i\omega & 0 & 0 \\ 0 & 0 & 0 & 0 & i\nu & 0 \\ 0 & 0 & 0 & 0 & 0 & -i\nu \end{bmatrix} \tag{12.7}$$

We modify it by creating three new coefficients

$$c_1, c_2, c_3 \in \mathbb{C}$$

so that the eigenvectors that refer to the same eigenvalue have the same coefficient. In our case:

$$\tilde{\mathbf{P}} = \begin{bmatrix} c_1\rho_1 & c_1\rho_2 & c_2\rho_3 & c_2\rho_4 & c_3\rho_5 & c_3\rho_6 \end{bmatrix} \tag{12.8}$$

Now, since the transformation is:

$$\delta\mathbf{r}(\mathbf{u}) = \tilde{\mathbf{P}} \cdot \mathbf{u} \tag{12.9}$$

we need to ensure that the matrix $\tilde{\mathbf{P}}$ is simplectic. Therefore we find the coefficients c_1, c_2, c_3 that satisfy:

$$\tilde{\mathbf{P}} \cdot \mathbf{J} \cdot \tilde{\mathbf{P}}^T = \tilde{\mathbf{P}}^T \cdot \mathbf{J} \cdot \tilde{\mathbf{P}} = \mathbf{J} \tag{12.10}$$

Now we have a canonical transformation, substituting in $H(\delta\mathbf{r}(\mathbf{u})) = H(\tilde{\mathbf{P}} \cdot \mathbf{u})$ we get :

$$\tilde{H}(\mathbf{u}) = \lambda u_1 v_1 + i\omega u_2 v_2 + i\nu u_3 v_3 + \sum_{k=3}^{\infty} P_k(u, v) \tag{12.11}$$

And the Hamilton equations:

$$\dot{u}_1 = \lambda u_1 + \frac{\partial \sum_{k=3}^{\infty} P_k(u, v)}{\partial v_1} \tag{12.12}$$

$$\dot{v}_1 = -\lambda v_1 + \dots \tag{12.13}$$

$$\dot{u}_2 = i\omega u_2 + \dots \tag{12.14}$$

$$\tag{12.15}$$

where $\sum_{k=3}^{\infty} P_k(u, v)$ is the polynomial in u, v that represents the higher order terms of the Hamiltonian.

These equations can give the impression that we have achieved to separate the phase space into its invariant manifolds. This is false.

The coordinates $\mathbf{u} = (u_1, v_1, u_2, v_2, u_3, v_3)$ are a parametrization of the *linear spaces* (or linearized manifolds), not of the proper unstable, stable and center manifolds.

Here is an example for why we are not done.

Notice that $\sum_{k=3}^{\infty} P_k(u,v)$ can have terms linear in either u_1 or v_1 for example:

$$p_k u_1 v_2^3$$

Its derivative is:

$$\frac{\partial p_k u_1 v_2^3}{\partial u_1} = p_k v_2^3$$

Therefore it is clear that if $u_1 = v_1 = 0$ the equation of motion is not zero:

$$\dot{u}_1 = \lambda u_1 + p_k v_2^3 = 0 + p_k v_2^3$$

Therefore if we were to start on the supposed center manifold $u_1 = v_1 = 0$ we would obtain equations of motion:

$$u_1(t) \neq 0$$
$$v_1(t) \neq 0$$

We need a canonical transformation that erases the linear terms of u_1, v_1. That way, with hyperbolic coordinates $u_1 = v_1 = 0$ we will always stay on the center manifold.

Example We give the numerical results of a symplectic diagonalization centered in L_1.

$$\begin{aligned}
\delta x &= 0.39335112167114683 u_1 - (0.09962535940480423 - 0.09962535940480423 i) u_2 \\
&\quad - 0.3933511216711468 v_1 + (0.09962535940480423 - 0.09962535940480423 i) v_2 \\
\delta y &= -0.18099198751253623 u_1 - (0.35730561840318537 + 0.35730561840318537 i) u_2 \\
&\quad - 0.18099198751253623 v_1 - (0.35730561840318537 + 0.35730561840318537 i) v_2 \\
\delta z &= (-0.33194747420493076 - 0.33194747420493076 i) u_3 - (0.33194747420493076 \\
&\quad + 0.33194747420493076 i) v_3 \\
\delta p_x &= 1.334316625488069 u_1 + (0.124742040825164 + 0.124742040825164 i) u_2 \\
&\quad + 1.3343166254880687 v_1 + (0.124742040825164 + 0.124742040825164 i) v_2 \\
\delta p_y &= -0.1373261968211154 u_1 + (0.7344622002388199 - 0.7344622002388199 i) u_2 \\
&\quad + 0.13732619682111538 v_1 - (0.7344622002388199 - 0.7344622002388199 i) v_2 \\
\delta p_z &= (0.7531312012504131 - 0.7531312012504131 i) u_3 - (0.7531312012504131 \\
&\quad - 0.7531312012504131 i) v_3
\end{aligned}$$

{dx → −(0.0996254 −0.0996254i)u2+(0.0996254 −0.0996254i)v2+0.393351u1−0.393351v1, dy → −(0.357306+

12.2.3 Reduction to center manifold

We have just seen that the diagonalization is not enough, we need another canonical transformation that maps the actual manifolds to their linearization.

In particular we need a canonical transformation with the following properties:

Let $\mathbf{U} = (U_1, V_1, U_2, V_2, U_3, V_3) \in \mathbb{C}^6$ be the coordinates on the manifolds and $\boldsymbol{\phi}$ be a diffeomorphism, such that:

$$\begin{aligned} \mathbb{C}^6 &\mapsto \mathbb{C}^6 \\ \mathbf{u} &= \mathbf{\Phi}(\mathbf{U}) \\ \nabla\mathbf{\Phi} \cdot \mathbf{J} \cdot \nabla\mathbf{\Phi}^T &= \mathbf{J} \\ \mathbf{\Phi}(0) &= 0 \\ \nabla\mathbf{\Phi}(0) &= \mathbf{I}_6 \end{aligned}$$

The last two conditions are called the *near identity condition.* They state that the two sets of coordinates are identical near the origin (or equilibrium). We want this because we know that the linear spaces and the manifolds are tangent in the equilibrium point, therefore in that region the two sets of coordinates should coincide.

We will use the procedure explained in chapter 5 to find this canonical transformation.

Lie series canonical transformation

First of all we define the unwanted terms. We want to eliminate the terms that are linear in u_1, v_1.

Let us define some notation: let $a_j \in \mathbb{C}$ and $\mathbf{h}_j, \mathbf{k}_j$ two triplets, representing the exponents of the coordinates u_1, u_2, u_3 and v_1, v_2, v_3

$$\xi = \sum_j a_j \mathbf{u}^{\mathbf{h}_j} \mathbf{v}^{\mathbf{k}_j} \tag{12.16}$$

meaning:

$$\xi = \sum_j a_j u_1^{h_{j1}} u_2^{h_{j2}} u_3^{h_{j3}} v_1^{k_{j1}} v_2^{k_{j2}} v_3^{k_{j3}}$$

with $\mathbf{h}, \mathbf{k} \in \mathbb{N}$ and $h_{j1} + k_{j1} = 1$, because we only want terms linear in u_1, v_1.

An example for $j = 1$ could be:

$$a_1 u_3^{h_{13}} v_1^{k_{11}} = a_1 u_3^2 v_1$$

We can also categorize the term ξ by polynomial degree, so that:

$$\xi = \xi_3 + \xi_4 + \xi_5 + \dots \tag{12.17}$$

where ξ_3 is the term of form 12.16 but of total degree 3.

The normal form is **any other term** nonlinear in u_1, v_1.

We can now rewrite the Hamiltonian $\tilde{H}$ using the book-keeping parameter λ as:

$$\tilde{H} = \lambda^2 N_2 + \lambda^3 N_3 + \dots + \lambda^3 \xi_3 + \lambda^4 \xi_4 + \dots \tag{12.18}$$

where we chose the degree of λ to match that of its associated term.

Suppose now that we have a generating function χ. We can use its Lie derivative to obtain the new Hamiltonian:

$$\begin{aligned}
\mathscr{H} &= \exp(\mathscr{L}_\chi)\tilde{H} = \tilde{H} + \mathscr{L}_\chi(\tilde{H}) + \dots \\
\mathscr{H} &= \lambda^2 N_2 + \lambda^3 N_3 + \dots + \lambda^3 \xi_3 + \lambda^4 \xi_4 + \dots + \\
&\quad \lambda^3 \mathscr{L}_\chi N_2 + \lambda^4 \mathscr{L}_\chi N_3 + \dots + \lambda^4 \mathscr{L}_\chi \xi_3 + \lambda^5 \mathscr{L}_\chi \xi_4 + \dots
\end{aligned}$$

We want to erase the unwanted terms of third order, therefore we take the homological equation:

$$\lambda^3 \left(\{N_2, \chi\} + \xi_3\right) = 0 \tag{12.19}$$

Since the unwanted terms are of the form 12.16 a good guess for the generating function is:

$$\chi = \sum_j b_j \mathbf{u}^{\mathbf{h}_j} \mathbf{v}^{\mathbf{k}_j} \tag{12.20}$$

The reader can verify that, using our notation, the term $\{N_2, \chi\}$ gives:

$$\{N_2, \chi\} = \sum_j b_j \left(\lambda(h_{j1} - k_{j1}) + i\omega(h_{j2} - k_{j2}) + i\nu(h_{j3} - k_{j3})\right) \mathbf{u}^{\mathbf{h}_j} \mathbf{v}^{\mathbf{k}_j}$$

Which means that the homological equation $\{N_2, \chi\} + \xi_3 = 0$ is:

$$\sum_j b_j \left(\lambda(h_{j1} - k_{j1}) + i\omega(h_{j2} - k_{j2}) + i\nu(h_{j3} - k_{j3})\right) \mathbf{u}^{\mathbf{h}_j} \mathbf{v}^{\mathbf{k}_j} + \sum_j a_j \mathbf{u}^{\mathbf{h}_j} \mathbf{v}^{\mathbf{k}_j} = 0 \tag{12.21}$$

Which is solved by:

$$b_j = \frac{a_j}{\lambda(h_{j1} - k_{j1}) + i\omega(h_{j2} - k_{j2}) + i\nu(h_{j3} - k_{j3})} \tag{12.22}$$

Using this generating function we obtain the canonical transformation from the coordinates on the manifolds to the ones on the linear space, and a new Hamiltonian for the dynamics on the manifolds.

$$\begin{aligned}
\mathbf{u}(\mathbf{U}) &= \mathbf{\Phi}(\mathbf{U}) = exp(\mathscr{L}_\chi)\mathbf{U} \\
\mathscr{H}(\mathbf{U}) &= \tilde{H}(\mathbf{\Phi}(\mathbf{U})) = exp(\mathscr{L}_\chi)\tilde{H}(\mathbf{U})
\end{aligned}$$

Notice that the Hamiltonian is in normal form up to the third order, meaning that we have found a polynomial approximation of the center and hyperbolic manifolds of third degree, close to the equilibrium point.

$$\mathscr{H} = N_2 + N_3 + \dots + \xi_4 + \xi_5 + \dots \tag{12.23}$$

$$\mathscr{H} = N_2 + N_3 + O(\lambda^4)$$

Example Here are the numerical results of $\mathbf{u}(\mathbf{U}) = \mathbf{\Phi}(\mathbf{U}) = exp(\mathscr{L}_\chi)\mathbf{U}$:

$$
\begin{aligned}
\mathrm{U1} = &-(0. + 1.84827i)\mathrm{u1u2}^2 - (0. + 1.93638i)\mathrm{u1u2v2}\\
&-(0. + 1.65946i)\mathrm{u1u3}^2 - (0. + 0.977569i)\mathrm{u1u3v3}\\
&-(0. + 1.84827i)\mathrm{u1v2}^2 - (0. + 1.65946i)\mathrm{u1v3}^2\\
&-\left((0.72582 + 2.09508i)\mathrm{u2}^3\right) - (3.01912 + 0.769632i)\mathrm{u2}^2\mathrm{v2}\\
&-(0.495003 - 0.488983i)\mathrm{u2}^2 - (0.590002 + 1.85532i)\mathrm{u2u3}^2\\
&-(2.01074 + 2.67834i)\mathrm{u2u3v3} + (0.769632 + 3.01912i)\mathrm{u2v2}^2\\
&+(0. + 1.86692i)\mathrm{u2v2} - (2.06645 - 0.628395i)\mathrm{u2v3}^2\\
&-(0.628395 - 2.06645i)\mathrm{u3}^2\mathrm{v2} - (0.434869 - 0.280995i)\mathrm{u3}^2\\
&+(2.67834 + 2.01074i)\mathrm{u3v2v3} + (0. + 1.908i)\mathrm{u3v3}\\
&+(2.09508 + 0.72582i)\mathrm{v2}^3 + (0.495003 + 0.488983i)\mathrm{v2}^2\\
&+(1.85532 + 0.590002i)\mathrm{v2v3}^2 + (0.434869 + 0.280995i)\mathrm{v3}^2 + \mathrm{u1}
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{u2} = &-(0.769632 + 3.01912i)\mathrm{u1u2}^2 + (6.03825 + 1.53926i)\mathrm{u1u2v2}\\
&+(0. + 1.86692i)\mathrm{u1u2} + (2.06645 - 0.628395i)\mathrm{u1u3}^2\\
&+(2.01074 + 2.67834i)\mathrm{u1u3v3} + (2.17746 + 6.28524i)\mathrm{u1v2}^2\\
&-(0.990006 - 0.977966i)\mathrm{u1v2} + (0.590002 + 1.85532i)\mathrm{u1v3}^2\\
&-\left((0. + 0.924133i)\mathrm{u2}^3\right) - (3.01912 + 0.769632i)\mathrm{u2}^2\mathrm{v1}\\
&-(0. + 0.968192i)\mathrm{u2}^2\mathrm{v2} - (0. + 0.811868i)\mathrm{u2u3}^2\\
&+(1.53926 + 6.03825i)\mathrm{u2v1v2} + (0. + 1.86692i)\mathrm{u2v1}\\
&-(0. + 0.924133i)\mathrm{u2v2}^2 + (0. + 0.811868i)\mathrm{u2v3}^2\\
&-(0.628395 - 2.06645i)\mathrm{u3}^2\mathrm{v1} - (0. + 0.703474i)\mathrm{u3}^2\mathrm{v2}\\
&+(2.67834 + 2.01074i)\mathrm{u3v1v3} - (0. + 1.88893i)\mathrm{u3v2v3}\\
&+(6.28524 + 2.17746i)\mathrm{v1v2}^2 + (0.990006 + 0.977966i)\mathrm{v1v2}\\
&+(1.85532 + 0.590002i)\mathrm{v1v3}^2 + (0. + 0.147101i)\mathrm{v2v3}^2\\
&-0.77444\mathrm{u1}^2\mathrm{u2} - 3.5091\mathrm{u1u2v1}\\
&-0.77444\mathrm{u2v1}^2 + \mathrm{u2}
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{U3} = &-(2.67834+2.01074i)\mathrm{u1u2u3}+(1.25679-4.1329i)\mathrm{u1u2v3}\\
&+(2.01074+2.67834i)\mathrm{u1u3v2}+(0.+1.908i)\mathrm{u1u3}\\
&+(1.18+3.71064i)\mathrm{u1v2v3}-(0.869739-0.56199i)\mathrm{u1v3}\\
&-(0.+0.944467i)\mathrm{u2}^2\mathrm{u3}-(0.+0.703474i)\mathrm{u2}^2\mathrm{v3}\\
&-(2.01074+2.67834i)\mathrm{u2u3v1}-(4.1329-1.25679i)\mathrm{u2v1v3}\\
&-(0.+1.62374i)\mathrm{u2v2v3}-\left((0.+0.829732i)\mathrm{u3}^3\right)\\
&-(0.+0.488785i)\mathrm{u3}^2\mathrm{v3}+(2.67834+2.01074i)\mathrm{u3v1v2}\\
&+(0.+1.908i)\mathrm{u3v1}+(0.+0.944467i)\mathrm{u3v2}^2\\
&-(0.+0.829732i)\mathrm{u3v3}^2+(3.71064+1.18i)\mathrm{v1v2v3}\\
&+(0.869739+0.56199i)\mathrm{v1v3}-(0.+0.147101i)\mathrm{v2}^2\mathrm{v3}\\
&-1.2841\mathrm{u1}^2\mathrm{u3}-4.08108\mathrm{u1u3v1}-1.2841\mathrm{u3v1}^2+\mathrm{u3}
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{V1} = &(2.09508+0.72582i)\mathrm{u2}^3-(0.+1.84827i)\mathrm{u2}^2\mathrm{v1}\\
&+(0.769632+3.01912i)\mathrm{u2}^2\mathrm{v2}+(1.85532+0.590002i)\mathrm{u2u3}^2\\
&+(2.67834+2.01074i)\mathrm{u2u3v3}-(0.+1.93638i)\mathrm{u2v1v2}\\
&-(3.01912+0.769632i)\mathrm{u2v2}^2-(0.+1.86692i)\mathrm{u2v2}\\
&-(0.628395-2.06645i)\mathrm{u2v3}^2-(0.+1.65946i)\mathrm{u3}^2\mathrm{v1}\\
&-(2.06645-0.628395i)\mathrm{u3}^2\mathrm{v2}-(0.434869+0.280995i)\mathrm{u3}^2\\
&-(0.+0.977569i)\mathrm{u3v1v3}-(2.01074+2.67834i)\mathrm{u3v2v3}\\
&-(0.+1.908i)\mathrm{u3v3}-(0.+1.84827i)\mathrm{v1v2}^2\\
&-(0.+1.65946i)\mathrm{v1v3}^2-(0.72582+2.09508i)\mathrm{v2}^3+(0.495003-0.488983i)\mathrm{v2}^2\\
&-(0.590002+1.85532i)\mathrm{v2v3}^2+(0.434869-0.280995i)\mathrm{v3}^2\\
&+(-0.495003-0.488983i)\mathrm{u2}^2+\mathrm{v1}
\end{aligned}
$$

$$\begin{aligned}\mathrm{V2} =&(6.28524+2.17746i)\mathrm{u1u2}^2+(1.53926+6.03825i)\mathrm{u1u2v2}\\&+(1.85532+0.590002i)\mathrm{u1u3}^2+(2.67834+2.01074i)\mathrm{u1u3v3}\\&-(3.01912+0.769632i)\mathrm{u1v2}^2-(0.+1.86692i)\mathrm{u1v2}\\&-(0.628395-2.06645i)\mathrm{u1v3}^2+(2.17746+6.28524i)\mathrm{u2}^2\mathrm{v1}\\&-(0.+0.924133i)\mathrm{u2}^2\mathrm{v2}+(0.+0.147101i)\mathrm{u2u3}^2\\&-(0.+1.88893i)\mathrm{u2u3v3}+(6.03825+1.53926i)\mathrm{u2v1v2}\\&+(0.990006-0.977966i)\mathrm{u2v1}-(0.+0.968192i)\mathrm{u2v2}^2\\&-(0.+0.703474i)\mathrm{u2v3}^2+(0.590002+1.85532i)\mathrm{u3}^2\mathrm{v1}\\&+(0.+0.811868i)\mathrm{u3}^2\mathrm{v2}+(2.01074+2.67834i)\mathrm{u3v1v3}\\&-(0.769632+3.01912i)\mathrm{v1v2}^2-(0.+1.86692i)\mathrm{v1v2}\\&+(2.06645-0.628395i)\mathrm{v1v3}^2-\left((0.+0.924133i)\mathrm{v2}^3\right)\\&-(0.+0.811868i)\mathrm{v2v3}^2-0.77444\mathrm{u1}^2\mathrm{v2}\\&+(-0.990006-0.977966i)\mathrm{u1u2}-3.5091\mathrm{u1v1v2}\\&-0.77444\mathrm{v1}^2\mathrm{v2}+\mathrm{v2}\end{aligned}$$

$$\begin{aligned}\mathrm{V3} =&(3.71064+1.18i)\mathrm{u1u2u3}+(2.67834+2.01074i)\mathrm{u1u2v3}\\&-(4.1329-1.25679i)\mathrm{u1u3v2}-(2.01074+2.67834i)\mathrm{u1v2v3}\\&-(0.+1.908i)\mathrm{u1v3}-(0.+0.147101i)\mathrm{u2}^2\mathrm{u3}+(0.+0.944467i)\mathrm{u2}^2\mathrm{v3}\\&+(1.18+3.71064i)\mathrm{u2u3v1}-(0.+1.62374i)\mathrm{u2u3v2}\\&+(2.01074+2.67834i)\mathrm{u2v1v3}\\&-(0.+0.829732i)\mathrm{u3}^2\mathrm{v3}+(1.25679-4.1329i)\mathrm{u3v1v2}\\&+(0.869739-0.56199i)\mathrm{u3v1}-(0.+0.703474i)\mathrm{u3v2}^2\\&-(0.+0.488785i)\mathrm{u3v3}^2-(2.67834+2.01074i)\mathrm{v1v2v3}\\&-(0.+1.908i)\mathrm{v1v3}-(0.+0.944467i)\mathrm{v2}^2\mathrm{v3}\\&-\left((0.+0.829732i)\mathrm{v3}^3\right)-1.2841\mathrm{u1}^2\mathrm{v3}\\&+(-0.869739-0.56199i)\mathrm{u1u3}-4.08108\mathrm{u1v1v3}\\&-1.2841\mathrm{v1}^2\mathrm{v3}+\mathrm{v3}\end{aligned}$$

12.2.4 Birkoff transformation

We can test that what we found is indeed invariant on the center manifold and also a near-identity transformation.

If we take $U_1 = V_1 = 0$ we can reduce the dynamics to the four coordinates U_2, U_3, V_2, V_3, which are a parametrization of the center manifold.

$$\mathscr{H} = i\omega U_2 V_2 + i\nu U_3 V_3 + \dots \tag{12.24}$$

However these coordinates are still complex, which is not desirable when we have to compute phase diagrams and Poincare sections. We need another canonical transformation that translates to real coordinates.

We can use the *Birkoff transformation*, that takes complex conjugate coordinates and gives real coordinates.

In our case:

$$U_2 = e^{i\phi}\frac{q_2 - i p_2}{\sqrt{2}} \qquad V_2 = e^{-i\phi}\frac{-i q_2 + p_2}{\sqrt{2}} \tag{12.25}$$

$$U_3 = e^{i\psi}\frac{q_3 - i p_3}{\sqrt{2}} \qquad V_3 = e^{-i\psi}\frac{-i q_3 + p_3}{\sqrt{2}} \tag{12.26}$$

Substituting we get:

$$\mathsf{H} = \frac{\omega}{2}\left(q_2^2 + p_2^2\right) + \frac{\nu}{2}\left(q_3^2 + p_3^2\right) + \sum_{s=3} P_s(q_2, q_3, p_2, p_3) \tag{12.27}$$

Which gets us an oscillator (the quadratic terms are in fact an harmonic oscillator). Here we can clearly see that the dynamics on the center manifold are just a superposition of the two nonlinear oscillators.

Example

$$\mathrm{u2} = \frac{\mathrm{x2} - i\mathrm{px2}}{\sqrt{2}} \qquad \mathrm{v2} = \frac{\mathrm{px2} - i\mathrm{x2}}{\sqrt{2}}$$

$$\mathrm{u3} = \frac{e^{\frac{3i\pi}{4}}(\mathrm{x3} - i\mathrm{px3})}{\sqrt{2}} \qquad \mathrm{v3} = \frac{e^{-\frac{1}{4}(3i\pi)}(\mathrm{px3} - i\mathrm{x3})}{\sqrt{2}}$$

$$\begin{aligned}
\mathsf{H} =\ & 1.45001\mathrm{px2}^4 - 5.47286\mathrm{px2}^3\mathrm{x2} + 1.10066\mathrm{px2}^3 \\
& + 1.84199\mathrm{px2}^2\mathrm{px3}^2 + 0.723251\mathrm{px2}^2\mathrm{px3x3} \\
& - 16.6296\mathrm{px2}^2\mathrm{x2}^2 + 1.34131\mathrm{px2}^2\mathrm{x2} - 0.5494\mathrm{px2}^2\mathrm{x3}^2 \\
& + 1.16719\mathrm{px2}^2 + 5.04051\mathrm{px2px3}^2\mathrm{x2} - 5.47286\mathrm{px2x2}^3 \\
& - 1.34131\mathrm{px2x2}^2 - 20.1039\mathrm{px2x2x3}^2 \\
& + 2.0038\mathrm{px2x3}^2 + 1.84199\mathrm{px3}^2\mathrm{x2}^2 + 3.76503\mathrm{px3}^2\mathrm{x3}^2 \\
& + 1.13441\mathrm{px3}^2 - 0.723251\mathrm{px3x2}^2\mathrm{x3} + 1.45001\mathrm{x2}^4 \\
& - 1.10066\mathrm{x2}^3 - 0.5494\mathrm{x2}^2\mathrm{x3}^2 + 1.16719\mathrm{x2}^2 \\
& - 2.0038\mathrm{x2x3}^2 - 4.5546\mathrm{x3}^4 + 1.13441\mathrm{x3}^2
\end{aligned}$$

12.3 Dynamics on center manifold

Equation 12.27 gives us the dynamics on the center manifold. However it is a four dimensional phase space, so it is not possible visualize it.

However, we can use a Poincaré section to reduce the dimensions. Let Ψ be the two dimensional surface:

$$\Pi = \{q_2, q_3, p_2, p_3 \in \mathbb{R}^4 : \quad q_3 = 0, \quad \dot{q}_3 > 0, \quad \mathsf{H}\left(q_2, q_3, p_2, p_3\right) = \varepsilon\} \tag{12.28}$$

These are all the points:

$$\left(q_2, p_2, 0, p_3(q_2, p_2; \varepsilon)\right) \tag{12.29}$$

Meaning that we can establish a Poincaré map:

$$\Pi \to \Pi \tag{12.30}$$

$$\left(q_2, p_2\right) \mapsto \left(q_2', p_2'\right) = \boldsymbol{\psi}\left(q_2, p_2\right) \tag{12.31}$$

As always we need to be sure that this map is symplectic, therefore we verify that:

$$\nabla\boldsymbol{\psi} \cdot \mathbf{J} \cdot \nabla\boldsymbol{\psi}^T = \mathbf{J}$$

Now we need to make a brief explanation on the nature of the halo orbits.

12.3.1 Bifurcation of Lyapunov orbit families

Due to the nature of the center manifold, for some energy levels the families of planar and vertical Lyapunov orbits bifurcate into other types of orbits.

What is a bifurcation? In simple terms, it means that at some point there are two families of periodic orbits that coexist at the same energy level, while previously only one existed.

One such family is the halo orbit, a 3D periodic orbit symmetrical to the $y = 0$ plane. These orbits correspond to the $1:1$ resonance between the planar and vertical normal frequencies.

This family of orbits arises from the bifurcation of the vertical Lyapunov orbit family. In fact there are two of such families, symmetrical to the $x = 0$ plane. These are called the north and south halo families.

Bifurcation energy Here we give the energy levels[4] of the first bifurcation for the vertical Lyapunov orbits at the Lagrange points L_1, L_2.

L_1	L_2
-1.49590	-1.48354

The halo orbits exist for energies lower than the bifurcation energy, therefore we will search them in the range $\left(E(\mathbf{c}), E_{bif}\right)$.

Example Plotting the map 12.31 for various levels of energy we get:

In each figure we have three islands. The center one revolves around the origin, which is the fixed point for the vertical Lyapunov orbit. The other two islands revolve around the fixed points of the halo orbits.

For each energy value we want to find these points.

12.3.2 Fixed points for halo orbits

To find the fixed points we will use once again a root finding algorithm. We want to find the fixed points of map $\boldsymbol{\psi}$ 12.31.

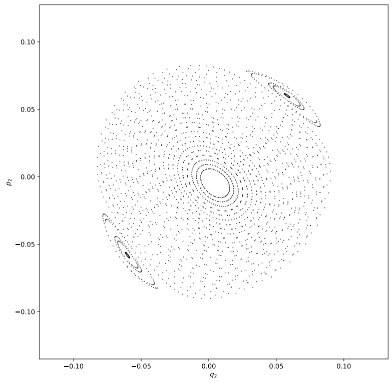
Therefore we create a function $\mathbf{F}$:

$$\mathbf{F} = \boldsymbol{\psi}(\mathbf{s}) - \mathbf{s}$$

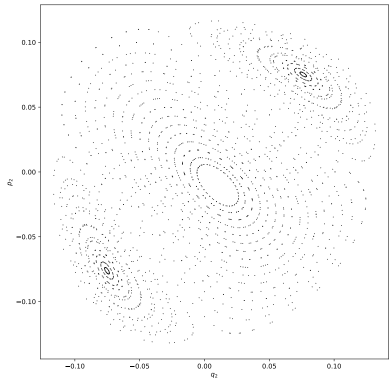
---

[4] These energy levels refer to the **complete original dynamics**. Therefore to use them in the equations we have just found we need to subtract the energy of the equilibrium point.

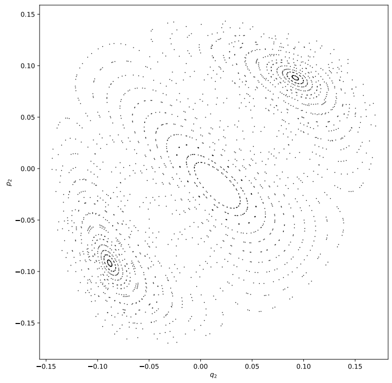
Ex: $E = E_{bif} - E_{L_1}$



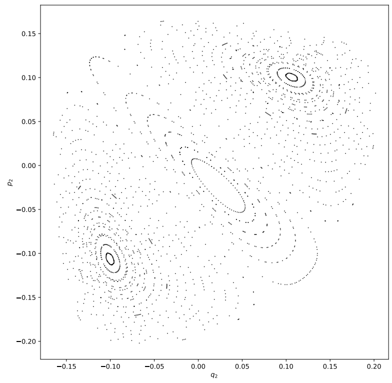
(a) E = −1.5851



(b) E = −1.5751



(c) E = −1.5651



(d) E = −1.5551

and the fixed points of $\boldsymbol{\psi}$ are the roots of $\mathbf{F}$.

We also need its jacobian:

$$\nabla\mathbf{F} = \nabla\boldsymbol{\psi}(\mathbf{s}) - \mathbf{I}_2$$

where we calculate $\nabla\boldsymbol{\psi}(\mathbf{s})$ through finite differences.

The Newton's method steps are:

$$\mathbf{s}_{n+1} = \mathbf{s}_n - \nabla\mathbf{F}(\mathbf{s}_n)^{-1}\cdot\mathbf{F}(\mathbf{s}_n)$$

As initial guess we give it a point that is close to the center of one halo islands (we can pick it by hand or through other average algorithms). After a few iterations we should get the fixed points.

Example Here we give the resulting fixed points for our example.

#	E	$\tilde{q}^{[1]}$	$\tilde{p}^{[1]}$	$\tilde{q}^{[2]}$	$\tilde{p}^{[2]}$
1	-1.5851	0.0580595	0.0601431	-0.0601431	-0.0580595
2	-1.5751	0.0761933	0.07523	-0.07523	-0.0761933
3	-1.5651	0.0919442	0.0882612	-0.0882612	-0.0919442
4	-1.5551	0.106376	0.100463	-0.100463	-0.106376

12.4 Halo orbits

Now that we have a fixed point on the Poincaré section of the center manifold, we can use the transformations to get back to the original phase space.

We have the fixed point $(\tilde{q}_2, \tilde{p}_2)$ on the Poincaré section. We need to find the corresponding $\tilde{p}_3(\tilde{q}_2, \tilde{p}_2; E)$ from the Hamiltonian H 12.27.

Now we have a point $(\tilde{q}_2, 0, \tilde{p}_2, \tilde{p}_3)$ on the real center manifold that generates a periodic orbit. We need to go back to the complex approximation of the center manifold. We use the Birkoff transformation:

$$\tilde{U}_2 = e^{i\phi}\frac{\tilde{q}_2 - i\tilde{p}_2}{\sqrt{2}} \qquad \tilde{V}_2 = e^{-i\phi}\frac{-i\tilde{q}_2 + \tilde{p}_2}{\sqrt{2}} \tag{12.32}$$

$$\tilde{U}_3 = e^{i\phi}\frac{\tilde{q}_3 - i\tilde{p}_3}{\sqrt{2}} \qquad \tilde{V}_3 = e^{-i\phi}\frac{-i\tilde{q}_3 + \tilde{p}_3}{\sqrt{2}} \tag{12.33}$$

By appending the hyperbolic coordinates $\tilde{U}_1 = \tilde{V}_1 = 0$ we obtain the starting condition on the manifolds:

$$\tilde{\mathbf{U}} = \left(0, 0, \tilde{U}_2, \tilde{V}_2, \tilde{U}_3, \tilde{V}_3\right)$$

Now we can map this point on the diagonalized pahse space:

$$\tilde{\mathbf{u}} = \exp(\mathscr{L}_\chi)\tilde{\mathbf{U}}$$

Finally we get:

$$\delta\tilde{\mathbf{r}} = \tilde{\mathbf{P}}\cdot\tilde{\mathbf{u}}$$

which translated by the equilibrium point will give the starting condition of a first approximation of a halo orbit in the original phase space:

$$\tilde{\mathbf{r}} = \mathbf{c} + \delta\tilde{\mathbf{r}}$$

Now we have a first approximation of a Halo orbit of energy E.

The root finding algorithm that we have developed previously finds any periodic orbit that passes through the xy plane, therefore we can use it to numerically identify the real halo orbit, using as initialization the guess we have just found.

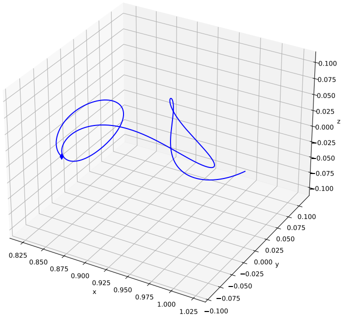
With this, we have calculated a halo orbit!

Example From the fixed points of the Poincaré map of the restricted system $\mathbf{s} = (q_2, p_2)$:

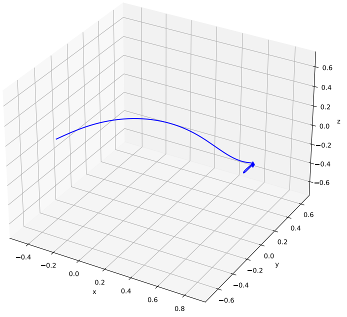
$$\tilde{\mathbf{s}}_+^{[1]} = (0.0580595, 0.0601431)$$
$$\tilde{\mathbf{s}}_-^{[1]} = (-0.0601431, -0.0580595)$$

with energy $E = -1.5851$ we obtain the initial approximations of the north and south halos:

$$\begin{aligned}\tilde{\mathbf{r}}_+ =& (8.4561288e-01, -5.9206025e-02, 3.4650667e-06,\\ & 2.1409230e-02, 8.3183966e-01, 4.8351884e-02)\\ \tilde{\mathbf{r}}_- =& (8.4561288e-01, 5.9206025e-02, -3.4650668e-06,\\ & -2.1409230e-02, 8.31839663e-01, 4.83518849e-02)\end{aligned}$$



(a) North approximation

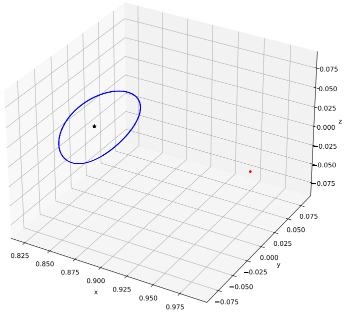


(b) South approximation

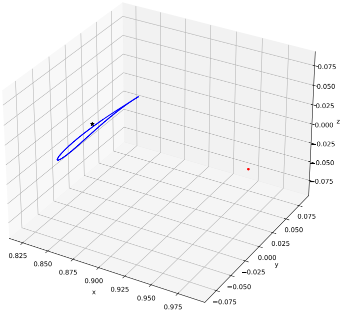
Passing these points to the root finder 11.2.2 we obtain the starting conditions:

$$\mathbf{h}_+ = (0.8458206, -0.0594533, 0, 0.0216793, 0.8306247, 0.0464978)$$
$$\mathbf{h}_- = (0.8458206, 0.0594533, 0, -0.0216793, 0.8306247, 0.0464978)$$

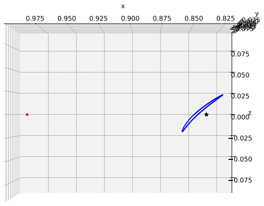
and the orbits:



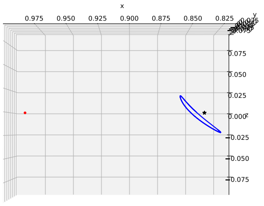
(a) North halo orbit



(b) South halo orbit



(c) Lateral view of North orbit



(d) Lateral view of South orbit

12.4.1 Manifolds

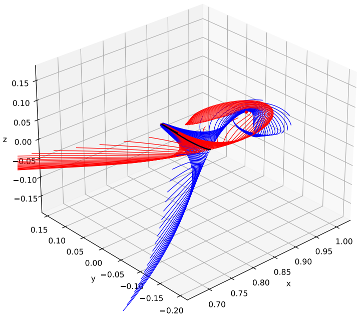
Since the procedure is the same as for the vertical Lyapunov orbit we will not explain it in details. We only give the results of our example.

Example Calculate the energy of the halo orbits $\mathbf{h}_+$ and $\mathbf{h}_-$ and verify that they are equal.

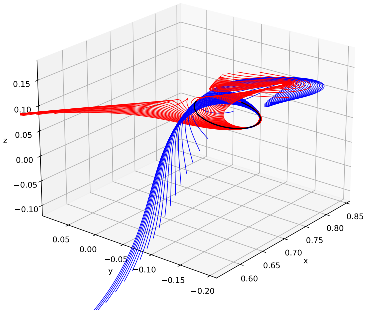
$$E_+ = E_- = -1.585100003993316$$

Therefore the coordinates x, y, p_x, p_y of $\mathbf{h}_+$ and $\mathbf{h}_-$ are a fixed point for the Poincaré map 11.4, $\mathbf{s}_+$, $\mathbf{s}_-$.

Calculate the eigenvalues and eigenvectors of both fixed points, find the stable and unstable spaces and from that plot the stable and unstable manifolds.



(a) Mamnifolds of north halo



(b) Manifolds of south halo

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