Agenda

1) Hypothesis Testing motivation

2) Type 1 and 2 errors

3) NP - Lemma

4) Problem.

Intro to Hypothesis Testing

We get some estimate

$\hat{\theta}$, how can I test

hypotheses with this estimate?

we've already talked about

CI's

Another option is

Hypothesis Testing

So we do Null Hypothesis Significance Testing.

So we assume a null hypothesis and look to reject null.

Simple Hypotheses:
Completely specifies the distribution when we assume hypothesis.

Suppose our model
$X \sim bin(10, p)$.

Simple hypothesis $p = 0.5$.

Composite hypotheses: not simple.

in previous example

$$p \in (0.5, 1)$$

In general H_0: null hypothesis

H_1: alternative.

<u>Def.</u> The set of values of some statistic T for which H_0 is rejected is called a rejection region.

The set of values for which H_0 is not rejected the acceptance region.

We never say accept the null. We say fail to reject the null.

How do we pick these regions?

Let's first see what can go wrong

In framework H_0 vs H_1

we can make two mistakes.

1. We reject H_0 when it is actually true.

(Type 1 error)

2. We fail to reject H_0 when it is actually false

(Type 2 error)

which is worse?

Court room analogy

Rejecting the null is conviction.

Failing to reject is not convicting.

So essentially in this analogy

Type 1 error

Sending an innocent person to jail

Type 2 error

Letting a guilty person go free

Notation.

$P_{H_0}(\text{event})$

"Probability given H_0 is true of the event"

$P(\text{Type 1 error})$

$= P_{H_0}(\text{rejecting } H_0) = \alpha$

This is called the significance level.

Under composite null.

$$\alpha = \max_{H \in H_0} P_H(\text{type 1 error})$$

$P_{H_1}(\text{fail to reject } H_0) = \beta$

we call $1-\beta$ the power

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we want $\alpha, \beta = 0$, this is impossible.

1) Fix apriori α to small value.

2) Max $(1-\beta)$ power while holding constrained α.

This is Q3 from 9.11

Suppose $X \sim \text{bin}(100, p)$

Consider a test that rejects $H_0: p = .5$ vs. $H_1: p \neq .5$ for $|X - 50| > 10$.

Use normal approx to find α, graph power function.

$$P_{H_0}(\text{rejecting } H_0)$$

$$= P_{H_0}(|X-50| > 10)$$

$$= 1 - P_{H_0}(|X-50| \leq 10)$$

$$= 1 - P_{H_0}(-10 \leq X-50 \leq 10)$$

$$= 1 - P_{H_0}(40 \leq X \leq 60)$$

$$E(X) = np$$

$$\text{var}(X) = np(1-p)$$

$$X \dot{\sim} N(np, np(1-p))$$

$$= 1 - P_{H_0}\left(\frac{40 - np}{\sqrt{np(1-p)}} \le Z \le \frac{60 - np}{\sqrt{np(1-p)}}\right)$$

$$Z = \frac{X - np}{\sqrt{np(1-p)}} \sim N(0,1)$$

$$= 1 - P\left(\frac{40-50}{5} \le Z \le \frac{60-50}{5}\right)$$

$$= 1 - P(-2 \le z \le 2)$$

$$= 1 - (\Phi(2) - \Phi(-2))$$

standard normal CDF

≈ 0.0455

b) we can get power

$$1-\beta = 1 - P_{H_1}(\text{fail to reject } H_0)$$

$$= P_{H_1}(\text{reject } H_0)$$

$$= 1 - P_{H_1}\left(\frac{40-np}{\sqrt{np(1-p)}} \le Z \le \frac{60-np}{\sqrt{np(1-p)}}\right)$$

$$= 1 - P_{H_1}\left(\frac{40-100p}{10\sqrt{p(1-p)}} \le Z \le \frac{60-100p}{10\sqrt{p(1-p)}}\right)$$

All I'm told is that $p \neq 0.5$

$$= 1 - \left(\Phi\left(\frac{60 - 100p}{10\sqrt{p(1-p)}} \right) - \Phi\left(\frac{40 - 100p}{10\sqrt{p(1-p)}} \right) \right)$$

at $p = 0.5$

$$P_{H_0}(\text{reject}) = \alpha$$

How do we actually construct a test

Is there a way to find a most powerful α level test easily?

Neyman-Pearson Lemma is a way for simple-simple tests.

Likelihood Ratio test.

Ex. of a simple-simple test.

$H_0: p = 0.5 \quad H_1: p = 0.7$ (binom)

Ex of simple - composite

$H_0: p = 0.5 \quad H_1: p \neq 0.5$

Interlude

$$P_{H_1}(\text{fail to reject } H_0) = \beta$$

we call $1-\beta$ the power

$$1 - P_{H_1}(\text{fail to reject } H_0)$$

$$= P_{H_1}(\text{reject } H_0)$$

Likelihood Ratio Test

Suppose simple-simple hypotheses

We can get a likelihood under both the alternative and the null

Denote $f_{H_0}(\underline{x})$ or $f_0(\underline{x})$ as the likelihood under the null. (similar notation alternative)

$$LR(\underline{x}) = \frac{f_{H_0}(\underline{x})}{f_{H_1}(\underline{x})}$$

if H_1 is more "likely"

$$f_{H_1}(x) > f_{H_0}(x)$$

equiv.

$$\frac{f_{H_0}(x)}{f_{H_1}(x)} \text{ is small}$$

NP-Lemma

Suppose we have a test that rejects H_0 when

$$\frac{f_{H_0}(x)}{f_{H_1}(x)} < c(\alpha)$$ has

significance level α.

Then this is the most powerful α level test for simple-simple.

Process

1) Compute Likelihood Ratio.

2) Find statistic $T(X)$ that determines when LR is small

3) Determine rejection region for T

4) Choose cutoff to ensure
α level test.

Q7 Ch 9.11

Let $X_1, \dots, X_n$ be a sample from Poisson dist.

Find LRT for

$H_0: \lambda = \lambda_0$

$H_1: \lambda = \lambda_1$, know $\lambda_1 > \lambda_0$

if $X_i \overset{ind}{\sim} pois(\lambda)$

$$P(X_i = x_i) = \frac{e^{-\lambda} \lambda^{x_i}}{x_i !}$$

$$L_n(\lambda) = \prod_{i=1}^{n} \frac{e^{-\lambda}\lambda^{x_i}}{x_i!}$$

$$= \frac{e^{-n\lambda} \cdot \lambda^{\Sigma x_i}}{\prod x_i!}$$

$$LR(x) = \frac{f_{H_0}(x)}{f_{H_1}(x)} = \frac{L_n(\lambda_0)}{L_n(\lambda_1)}$$

$$= \frac{e^{-n\lambda_0} \cdot \lambda_0^{\Sigma x_i} \Big/ \cancel{\prod x_i!}}{e^{-n\lambda}\ \lambda_1^{\Sigma x_i} \Big/ \cancel{\prod x_i!}}$$

$$= e^{-nd_0 - nd_1} \cdot \left(\frac{d_0}{d_1}\right)^{\Sigma x_i} \quad (1)$$

Let $T = \Sigma x_i$

$$= e^{-nd_0 - nd_1} \left(\frac{d_0}{d_1}\right)^{T}$$

$$d_1 > d_0 \Rightarrow \frac{d_0}{d_1} < 1$$

(2)

This $LR(X)$ is small when $T(x)$ is large

So LRT reject when

$T(x) = \sum x_i$ is large (3)

equiv.

$T(x) = \sum x_i > c(\alpha)$

(4) we want

$$P_{H_0}(\text{rejecting } H_0) = \alpha$$

$$= P_{H_0}\left(\sum_{i=1}^{n} x_i > c(\alpha)\right).$$

$$T = \sum_{i=1}^{n} x_i \sim \text{pois}(n\lambda_0)$$

under null

$$= P_{H_0}\left(T \geq c(\alpha)\right)$$

$$= 1 - P_{H_0}\left(T \leq c(\alpha)\right) = \alpha$$

$$= 1 - \sum_{t=0}^{\lfloor c(\alpha) \rfloor} \frac{(n\lambda_0)^t e^{-n\lambda_0}}{t!} = \alpha$$

<u>Power</u>

$$P_{H_1}(\text{reject } H_0)$$

$$= P_{H_1}\left(T > c(\alpha)\right)$$ now known!

$$= 1 - P_{H_1}\left(T \leq c(\alpha)\right)$$

under alt. $T \sim \text{pois}(n\lambda_1)$

$$= 1 - \sum_{t=0}^{c(\alpha)} e^{-n\lambda_1} (n\lambda_1)^t / t!$$

imagine $c(\alpha)$ is an integer

Dog Cancer problem

$H_0: p = \frac{1}{7}$ $H_1: p > \frac{1}{7}$

$$X_1, \dots, X_{54} \overset{ind}{\sim} \text{bern}(p)$$

use LRT even through not simple.

$H_0: p = p_0$

$H_1: p = p_1 > p_0$

$$L_n(p; \underline{x}) = \prod_{i=1}^{54} p^{x_i} (1-p)^{1-x_i}$$

$$= p^{\sum x_i} (1-p)^{54 - \sum x_i}$$

$$LR(\underline{x}) = \frac{f_{H_0}(\underline{x})}{f_{H_1}(\underline{x})} = \frac{p_0^{\sum x_i} (1-p_0)^{54 - \sum x_i}}{p_1^{\sum x_i} (1-p_1)^{54 - \sum x_i}}$$

$$= \left(\frac{p_0}{p_1}\right)^{\sum x_i} \left(\frac{1-p_0}{1-p_1}\right)^{54 - \sum x_i}$$

$$= \left(\frac{p_1}{p_0}\right)^{-\Sigma x_i} \left(\frac{1-p_0}{1-p_1}\right)^{54-\Sigma x_i}$$

$$= \left(\frac{p_1}{p_0}\right)^{-54} \left(\frac{p_1}{p_0}\right)^{54-\Sigma x_i} \left(\frac{1-p_0}{1-p_1}\right)^{54-\Sigma x_i}$$

$$= \left(\frac{p_1}{p_0}\right)^{-54} \left(\frac{p_1}{p_0} \frac{1-p_0}{1-p_1}\right)^{54-\Sigma x_i}$$

under null $p_1 > p_0$.

$$\frac{p_1}{p_0} > 1 \quad \frac{(1-p_0)}{1-p_1} > 1$$

$\Rightarrow$ dec function of $\sum X_i$

So we reject for small $LR(\underline{x})$

$$\Longleftrightarrow$$

reject for large $T(\underline{x}) = \sum X_i$

under null

$$P_{H_0}(T > c(\alpha)) = \alpha$$

$$= 1 - P_{H_0}(T \leq c(\alpha)) = \alpha.$$

under null

$$T \sim \text{binom}\left(59, \frac{1}{7}\right)$$

Suppose I give $\alpha = 0.05$.

$$= 1 - P_{H_0}\left(\sum X_i \leq c\right) = 0.05.$$