Agenda

1. R setup
2. Black Scholes
3. Black Scholes δ-method
4. 8.10 Q4
5. Bootstrap motivation
6. Bootstrap Coding
7. Even more Bootstrap coding.

δ-method

If we have $\hat{\theta}_n$ s.t.

$$\frac{\sqrt{n}\,(\hat{\theta}_n - \mu)}{\sigma} \xrightarrow{d} N(0,1)$$

then

$$\frac{\sqrt{n}\,\left(f(\hat{\theta}_n) - f(\mu)\right)}{|f'(\mu)|\,\sigma} \xrightarrow{d} N(0,1)$$

if $f'(\mu)$ exists and non-zero at μ.

So for Black-Scholes

$$\hat{\sigma}^2 \dot{\sim} N\left(\sigma, \frac{\gamma^2}{n}\right)$$

γ^2 is known.

$$C(\sigma) = S_0 \Phi(d_+(\sigma)) - K \Phi(d_-(\sigma)) e^{-rT}$$

Φ is standard normal CDF.

Let $\frac{\partial C}{\partial \sigma} C(\sigma) = C'(\sigma)$

$\neq 0.$

since $\hat{\sigma} \cdot N\left(\sigma, \frac{\gamma^2}{n}\right)$

$$C(\hat{\sigma}) \,\dot{\sim}\, N\left(C(\sigma), C'(\sigma)^2 \frac{\gamma^2}{n}\right)$$

MLE and Fisher information

for $X_i \overset{iid}{\sim} N(\mu_0, \sigma^2)$

μ_0 known

write $\tau = \sigma^2$

$$l_n(\tau) = \log \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\tau}} \exp\left(-\frac{1}{2} \frac{(x_i - \mu_0)^2}{\tau}\right)$$

$$= \sum_{i=1}^{n} \log \left(\frac{1}{\sqrt{2\pi\tau}} \exp\left(-\frac{1}{2} \frac{(x_i - \mu_0)^2}{\tau}\right)\right)$$

$$= \sum_{i=1}^{n} -\frac{1}{2}\log(2\pi) - \frac{1}{2}\log(\tau) - \frac{1}{2\tau}(x_i - \mu_0)^2$$

$$= C - \frac{n}{2}\log(\tau) - \frac{\sum_{i=1}^{n}(x_i - \mu_0)^2}{2\tau}$$

$$l_n'(\tau) = \frac{-n}{2\tau} + \frac{1}{2\tau^2}\sum_{i=1}^{n}(x_i - \mu_0)^2$$

$$= 0.$$

$$\hat{\tau} = \frac{1}{n}\sum_{i=1}^{n}(x_i - \mu_0)^2$$

This is an iid sample

$$I_n(\tau) = n\, I_1(\tau)$$

$$\ell_n''(\tau) = \frac{n}{2\tau^2} - \frac{1}{\tau^3}\sum_{i=1}^{n}(x_i - \mu_0)^2$$

$$I_n(\tau) = -E\left(\ell_n''(\tau)\right)$$

$$= \frac{-n}{2\tau^2} + \frac{1}{\tau^3}\sum_{i=1}^{n} E\left((x_i - \mu_0)^2\right)$$

$$= \frac{-n}{2\tau^2} + \frac{1}{\tau^3}\cdot n\tau = \frac{n}{2\tau^2}$$

$\Rightarrow$ asymptotic variance is

$$1/I_n(\tau) = \frac{2\tau^2}{n} = \frac{2\sigma^4}{n}$$

$$I_1(\tau) = \frac{1}{2\tau^2}$$

$$\sqrt{n}\left(\hat{\tau} - \tau\right) \xrightarrow{d} N\left(0, 2\tau^2\right)$$

equiv.

$$\left(\hat{\tau} - \tau\right) \dot{\sim} N\left(0, \frac{2\tau^2}{n}\right)$$

The Bootstrap

1. We have only one data set (typically small)

Usually all we can get is a point estimate.

2. How about we just pretend we get new datasets by resampling our existing dataset (non-parametric)

Parametric Bootstrap.

we have data $(X_1, \ldots, X_n)$ drawn for some P_θ

we can estimate $\hat{\theta}(X_1, \ldots, X_n)$

we can't draw from P_θ

but we <u>can</u> from $P_{\hat{\theta}}$

So process.

0. Estimate $\hat{\theta}(x_1, \dots, x_n)$

1. Plug $\hat{\theta}$ into $P_{\hat{\theta}}$ and sample n data points

2. From 1 data. Compute a new $\hat{\theta}^*$ based on new data points.

3. Repeat 1, and 2 B times to get $\hat{\theta}_1^*, \dots, \hat{\theta}_B^*$.

Non-Parametric

Impose no parametric assumptions.

0. Compute some
$\hat{\theta}(X_1, \ldots, X_n)$, (mean for example)

1. Sample n times from existing dataset with replacement.

2. Compute $\hat{\theta}^*$ of this new data set.

3. Repeat 1, 2 B times to get $\hat{\theta}^*_1, \ldots, \hat{\theta}^*_B$.

A bit about pivots.

$$X_i \overset{iid}{\sim} N(\mu, 1)$$

$$\bar{X} \sim N\left(\mu, \frac{1}{n}\right)$$

$$\sqrt{n}(\bar{X} - \mu) \sim N(0, 1)$$

$$P\left(z\left(\frac{\alpha}{2}\right) \le \sqrt{n}(\bar{X} - \mu) \le z\left(1 - \frac{\alpha}{2}\right)\right) = 1 - \alpha.$$

$$\hat{\sigma}^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2$$

$$\frac{\hat{\sigma}^2}{\sigma} \sim \chi^2_{df}$$

$$\hat{\theta} - \theta$$