Agenda

1. CI / Hyp Test
   Duality

2. p-values

3. GLRT

4. Problems

5. Your Questions.

Bootstrap

Normal
Percentile

$\Big\}$ Assumptions about normality

Pivotal $\Big\}$ Assumption about pivotness to be exact.

A pivot

A function of the estimator and the parameter whose distribution doesn't depend on the parameter

$$X_i \overset{ind}{\sim} N(\mu, 1)$$

$$\bar{X} \sim N(\mu, \tfrac{1}{n})$$

$$\bar{X} - \mu \sim N(0, \tfrac{1}{n})$$

Tobe exact pivotal interval
requires

$\hat{\theta} - \theta$ to be a pivot.

$\hat{\theta} - \theta \sim F$, F has no dependence on θ.

Test / CI duality

Theorem B Ch 9

Suppose r.v. $\underset{\sim}{X}$

parameter space Θ

Suppose that

$C(\underset{\sim}{X})$ is a $(1-\alpha)100\%$ CI

for θ_0.

So
$$P\left(\theta_0 \in C(\underset{\sim}{X}) \mid \theta=\theta_0\right) = 1-\alpha$$

Then acceptance region

of level α test

$H_0: \theta = \theta_0$ $\quad A(\theta_0) = \{\underline{x} \mid \theta_0 \in C(\underline{x})\}$

Pf.

$$P(\underline{X} \in A(\theta_0) \mid \theta = \theta_0)$$

$$= P(\theta_0 \in C(\underline{X}) \mid \theta = \theta_0) = 1 - \alpha$$

$$= 1 - P(\underline{X} \notin A(\theta_0) \mid \theta = \theta_0)$$

$$= 1 - P_{H_0}(\text{rejecting}) = 1 - \alpha$$

$$P_{H_0}(\text{rejecting}) = \alpha$$

P-values

So far

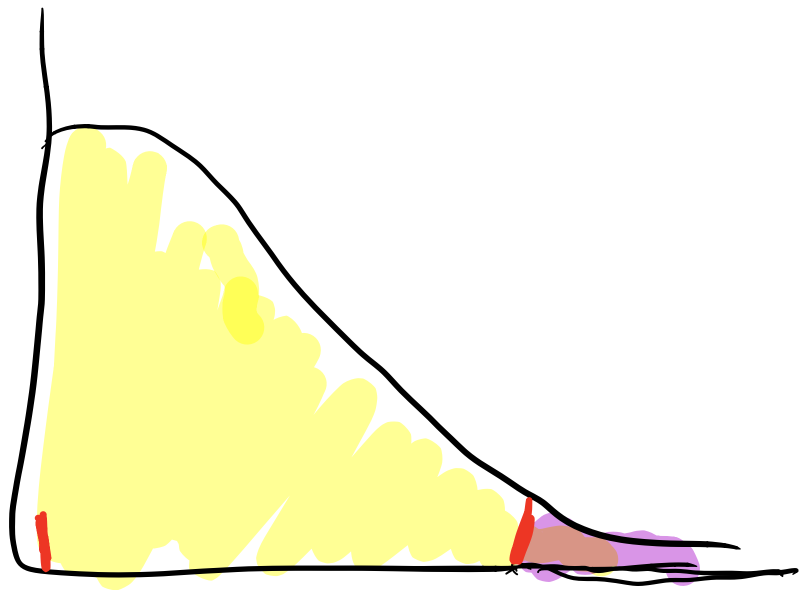
1. Pick α
2. Constructing rejection criterion

Def. p-value (in cases where rejection criteria is $T > t$, $T < t$)

is the probability under the null of seeing an as extreme or more extreme result we reject when p-value is small.

H_0

$T(X) > t$



$P(\text{something worse} \mid \text{null})$

Generalized LRT

NP-Lemma and LRT (Simple-Simple)

we were testing

$$H_0 : \theta = \theta_0$$

$$H_1 : \theta = \theta_1$$

$$LR(\underline{X}) = \Lambda = \frac{L(\theta_0)}{L(\theta_1)}$$

we reject when $LR(\underline{X})$ small

For the null

$\theta \in \Theta$

$$H_0 : \theta \in \omega_0$$

$$H_1 : \theta \in \omega_1$$

$$\Omega = \omega_0 \cup \omega_1$$

$$\Lambda = \frac{\max_{\theta \in \omega_0} L(\theta)}{\max_{\theta \in \Omega} L(\theta)}$$

Ex.

$\Theta = \mathbb{R}$

$\omega_0 = \{\theta_0\}$

$H_0: \theta = \theta_0$

$H_1: \theta \neq \theta_0$

$\Omega = \mathbb{R} = \Theta$

$$\Lambda = \frac{L(\theta_0)}{\max_{\theta \in \Theta} L(\theta)} \leftarrow \text{MLE}$$

In general   <u>Wilk's Theorem</u>

$$-2\log(\Lambda) \xrightarrow{d} \chi^2_{df}$$

$$df = \dim(\Omega) - \dim(\omega_0)$$

$$\dim(\mathbb{R}^n) = n$$

$H_0: p_1 = p_2 = p_3$ $p_i \in \mathbb{R}$

H_1: At least one not equal to others.

$$\omega_0 = \mathbb{R}$$

$\omega_1 = \mathbb{R}^3$

$\Omega = \mathbb{R}^3$

$\dim(\Omega) = 3 \quad \dim(\omega_0) = 1$

Q41 Ch 4

$X_i \overset{ind}{\sim} \text{bin}(n_i, p_i)$

for $i = 1, \dots, m$

Derive a LRT for

$H_0: p_1 = p_2 = \dots = p_m$

H_1: They're not all equal

$\omega_0 = (0,1)$

$\omega_1 = (0,1)^m$

$$\Lambda(X) = \frac{\max_{p_1 = p_2 = \dots = p_m = p} L_n(\underline{p})}{\max L_n(\underline{p})}$$

$$L_n(\underline{p}) = \prod_{i=1}^{m} \binom{n_i}{x_i} p_i^{x_i} (1-p_i)^{n_i - x_i}$$

$$\overset{H_0}{=} \left(\prod_{i=1}^{m} \binom{n_i}{x_i} \right) \prod_{i=1}^{n} p^{x_i} (1-p)^{n_i - x_i}$$

$$\overset{H_0}{=} \left(\prod_{i=1}^{m} \binom{n_i}{x_i} \right) p^{\sum x_i} (1-p)^{\sum (n_i - x_i)}$$

$$l_n(\underline{p}) = C$$

$$+ \sum x_i \log p$$

$$+ \sum (n_i - x_i) \log (1-p)$$

$$\frac{\partial \ln(p)}{\partial p} = \frac{\sum x_i}{p} - \frac{\sum (n_i - x_i)}{1-p}$$

$$= 0$$

$$\hat{p} = \frac{\sum_{i=1}^{m} x_i}{\sum_{i=1}^{m} n_i}$$

(numerator maximizer argument)

$$L(p) = \prod_{i=1}^{m} \binom{n_i}{x_i} p_i^{x_i} (1-p_i)^{n_i - x_i}$$

$$\hat{p}_i = \frac{x_i}{n_i} \text{ for } i=1,\dots,m$$

$$\Lambda = \frac{\prod_{i=1}^{m} \cancel{\binom{n_i}{x_i}} \cdot \hat{p}^{x_i} (1-\hat{p})^{n_i - x_i}}{\prod_{i=1}^{m} \cancel{\binom{n_i}{x_i}} \hat{p}_i^{x_i} (1-\hat{p}_i)^{n_i - x_i}}$$

$$= \frac{\hat{p}^{\sum_{i=1}^{m} x_i} (1-\hat{p})^{\sum_{i=1}^{m} (n_i - x_i)}}{\prod_{i=1}^{m} \hat{p}_i^{x_i} (1-\hat{p}_i)^{n_i - x_i}}$$

$$L(\underline{p}) = \prod_{i=1}^{m} \binom{n_i}{x_i} p_i^{x_i} (1-p_i)^{n_i - x_i}$$

equiv to max

$$\prod_{i=1}^{m} p_i^{x_i} (1-p_i)^{n_i - x_i}$$

equiv to max

$$p_i^{x_i} (1-p_i)^{n_i - x_i} \quad \forall i$$

$$\Rightarrow \quad \hat{p}_i = \frac{x_i}{n_i}$$

$$-2\log(\Lambda) \xrightarrow{d} \chi^2_{df}$$

$$df = \dim(\Omega) - \dim(\omega_0)$$

$$\Omega = (0,1)^n \quad = n-1$$

$$\omega_0 = (0,1)$$

$$P_{H_0}\left(-2\log\Lambda \geq c(\alpha)\right) = \alpha$$

$$1 - P_{H_0}\left(-2\log\Lambda \leq c(\alpha)\right) = \alpha$$

$$= 1 - F(c(\alpha)) = \alpha$$

F is cdf of χ^2_{n-1}

<u>p-value</u>

Suppose $-2\log(\Lambda) = t_0$

$$P_{H_0}(-2\log(\Lambda) > t_0)$$

$$= 1 - P_{H_0}(-2\log\Lambda \leq t_0)$$

$$= 1 - F(t)$$

F is cdf of χ^2_{n-1}

Q 42 Ch 9 a, b

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280 Nylon bars
tested in 5 places
for breaks

$X_i \overset{ind}{\sim} bin(5, p_i)$

$i = 1, \ldots, 280$

Null that all bars have equal chance of breaks (p_i's equal)

$H_0: p_1 = p_2 = \dots = p_{280} = p$

H_1: Not H_0.

a) Already proved

$$\hat{p}_{MLE} = \frac{\sum_{i=1}^{280} X_i}{\sum_{i=1}^{280} n_i} = \frac{\sum_{i=1}^{280} X_i}{280 \cdot 5}$$

$$= \frac{\text{Total \# breaks}}{\text{Total \# trials}}$$

$$= \frac{199}{5 \cdot 280} = 0.142$$

From book

Breaks/Bar	Frequency
0	157
1	69
2	35
3	17
4	1
5	1

b) Test statistic

For Pearson's chi-squared test

$$X^2 = \sum_{k=0}^{S} \frac{(O_k - E_k)^2}{O_k}$$

$$\Theta_i(\hat{p}) = \binom{S}{i} \hat{p}^i (1-p)^{S-i}$$

Get this value

for $i = 0, \ldots, S$

i	O_i	E_i
0	157	$280 \cdot \theta_0(\hat{p})$
1	69	$280 \theta_1(\hat{p})$
2	35	$\vdots$
3	17	
4	1	
5	1	$280 \theta_5(\hat{p})$

E_i = Total # obs · probability of being in bin.

$$= 280 \cdot \theta_i(\hat{p})$$

$$X^2 = \sum_{k=0}^{S} \frac{(O_k - E_k)^2}{E_k}$$

For Pearson's Chisquared

$$X^2 \xrightarrow{d} \chi^2_{k - \dim(p) - 1}$$

"X" "chi"

k is number of bins

$\dim(p)$ # of terms estimated

$$X^2 \xrightarrow{d} \chi^2_{6-1-1=4}$$

$$\chi^2 = \sum_{k=0}^{5} \frac{(O_k - E_k)^2}{E_k} = 44.265$$

p-value

$$P_{H_0}(\chi^2 > 44.265)$$

$$= 1 - P_{H_0}(\chi^2 \leq 44.265)$$

$$= 1 - F(44.265)$$

F is cdf of χ^2_4

p-value

$= 5.652 \cdot 10^{-9}$

So we reject.

$$P(a(\underline{x}) \le \theta \le b(\underline{x})) = 1-\alpha$$

$$X_i \sim \exp\left(\frac{1}{\tau}\right)$$

$$E(X_i) = \tau$$

$$\hat{\tau} = \bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

a sum of exponentials is Gamma.

$$\bar{X} \sim \text{Gamma}\left(n, \frac{n}{\tau}\right)$$

equiv.

$$\tau \cdot \bar{X} \sim \text{Gamma}(n, n)$$

this is a pivot

$$P\left(g_{\frac{\alpha}{2}, (n,n)} \leq \tau \cdot \bar{X} \leq g_{1-\frac{\alpha}{2} (n,n)}\right)$$

$$= 1 - \alpha$$

$g_{\alpha,(n,n)}$ is the α quartile of $G(n,n)$ distribution

$$P\left(\frac{g_{\frac{\alpha}{2},(n,n)}}{\overline{X}} \leq T \leq \frac{g_{1-\frac{\alpha}{2},(n,n)}}{\overline{X}}\right)$$

$$= 1-\alpha$$

MLE CI is from pivot

$$\frac{\sqrt{n}\left(\hat{\theta}_{MLE} - \theta\right)}{I(\hat{\theta})^{-1}} \dot{\sim} N(0,1)$$

this is a pivot

0. Important Probability Theory

- Distributions
- Expected Value
- Variance

- MGF (ren)
- CLT
- LLN

1. Estimation

a) MLE

i) Computing the MLE

ii) Asymptotic Dist of MLE

iii) CI's with MLE

b) Delta method

c) MoM

d) General Estimation properties

i) Consistency

$\hat{\theta} \xrightarrow{P} \theta$

ii) Sufficiency (use Factorization Thm)

iii) Efficiency (Cramer-Rao Lower Bound)

iv) Unbiasedness

$$E(\hat{\theta}) = \theta$$

(MLE has ii), i)
and iii), iv) asymptotically)

e) Computational methods

i) Bootstrap
Non-Param
Param

Good for variability of estimator

f) Data Vis

i) Histograms

ii) Box-Plots

iii) Q-Q Plots

(use for tails, skew)

g) Sufficiency and Rao-Blackwell

Note you will probably not have to compute Rao-Blackwell.

h) Testing

i) Type 1 and 2 errors

ii) Significance Level

iii) Power

iv) NP-Lemma

v) GLRT

vi) Duality of CI and testing

vii) p-values

viii) Chi-square test