Moment Generating Functions

Section 4.5 of textbook.

Def. Random variable X

$$M_X(t) = E\left(e^{tX}\right)$$

for discrete $= \displaystyle\sum_{x} e^{tx} p(x)$

for continuous $= \displaystyle\int_{\mathbb{R}} e^{tx} f(x)\, dx.$

The MGF does not necessarily exist for any particular value of t. (For ex. Cauchy)

What is a moment?

$E(X^k)$ is called the kth moment.

(adv. side note, characteristic function which always exists)

<u>Property A</u>

If $M_X(t)$ exists for t in some open interval that contains 0 then it uniquely determines the probability distribution of X.

Property B

The moment generating aspect.

$$M_X^{(r)}(0) = E(X^r)$$

(rth derivative)

"Pseudo-proof"

for cts X

$$M_X(t) = E(e^{tX})$$

$$= \int_{\mathbb{R}} e^{tx} f(x)\, dx$$

$$= \int_{\mathbb{R}} \underbrace{\left(\sum_{k=0}^{\infty} \frac{(tx)^k}{k!} \right)}_{\text{Taylor exp.}} \cdot f(x)\, dx$$

(Assuming ability to exchange int and sum)

$$= \sum_{k=0}^{\infty} \frac{t^k}{k!} \underbrace{\int_{\mathbb{R}} x^k f(x)\, dx}_{E(X^k)}$$

$$= \sum_{k=0}^{\infty} \frac{t^k}{k!} E(X^k)$$

$$= 1 + \frac{t}{1!}E(X) + \frac{t^2}{2!}E(X^2)$$

$$+ \ldots$$

$$\left.\frac{d^r}{dt^r} M_X(t)\right|_{t=0} = E(X^r)$$

Poisson Example

$X \sim \text{poisson}(\lambda)$

$$P(X=k) = \frac{\lambda^k e^{-\lambda}}{k!}$$

$$M_X(t) = E(e^{tX})$$

$$= \sum_{k=0}^{\infty} e^{tk} \cdot \frac{\lambda^k e^{-\lambda}}{k!}$$

$$= e^{-\lambda} \underbrace{\sum_{k=0}^{\infty} \frac{(\lambda e^t)^k}{k!}}_{\text{Taylor series exponential}}$$

$$= e^{-\lambda} e^{\lambda e^t} = e^{\lambda(e^t - 1)}$$

Expectation and Variance

want $E(X)$ and $Var(X)$.

Know $Var(x) = E(x^2) - (E(x))^2$

$$\frac{d}{dt} e^{\lambda(e^t-1)} = e^{\lambda(e^t-1)} \cdot \lambda e^t$$

evaluate at $t=0$

$E(x) = \lambda$

Now $E(x^2)$

$$\frac{d}{dt} e^{\lambda(e^t-1)} \cdot \lambda e^t$$

$$= \lambda e^t e^{\lambda(e^t-1)} + \lambda^2 e^t e^{\lambda(e^t-1)}$$

evaluate at $t=0$.

$= \lambda + \lambda^2$.

$\Rightarrow Var(X) = E(X^2) - (E(X))^2$

$= \lambda + \lambda^2 - \lambda^2$

$= \lambda$

<u>Property c</u>

$Y = a + bX$, X is random variable, a and b are constants.

$M_Y(t) = e^{at} M_X(bt)$.

Property D

Suppose $X \perp\!\!\!\perp Y$

define $Z = X + Y$

then $M_Z(t) = M_X(t) \cdot M_Y(t)$

Pf. $M_Z(t) = E(e^{tZ})$

$$= E\left(e^{t(X+Y)}\right)$$

$$= E\left(e^{tX} \cdot e^{tY}\right)$$

$$= E(e^{tX}) E(e^{tY})$$ (by ind.)

$$= M_X(t) \cdot M_Y(t) \qquad \square$$

Poisson example pt 2.

Suppose $X \sim \text{pois}(\lambda)$

$Y \sim \text{pois}(\mu)$ and

$X \perp\!\!\!\perp Y$

then $X+Y \sim \text{pois}(\lambda+\mu)$.

<u>Pf.</u> $M_{X+Y}(t) = M_X(t) M_Y(t)$ (Prop D)

$$= e^{\lambda(e^t-1)} e^{\mu(e^t-1)}$$

$$= e^{(\lambda+\mu)(e^t-1)}$$

So by property A

$X+Y \sim pois(\lambda+\mu)$ $\square$

Suppose $X_i \overset{ind}{\sim} pois(\lambda_i)$

then

$$\sum_{i=1}^{n} X_i \sim pois\left(\sum_{i=1}^{n} \lambda_i\right)$$

<u>Pf.</u> (induction)

base case $n=2$. (just proved)

Induction step.

Assume holds for k.

$$\Rightarrow \sum_{i=1}^{k} X_i \sim pois\left(\sum_{i=1}^{k} \lambda_i\right)$$

Let $Y = \sum_{i=1}^{k} X_i$

$\mu = \sum_{i=1}^{k} \lambda_i$

Now look at

$Y + X_{k+1} \sim \text{pois}(\mu + \lambda_{k+1})$

$\Longleftrightarrow$

$\sum_{i=1}^{k+1} X_i \sim \text{pois}\left(\sum_{i=1}^{k+1} \lambda_i\right)$

$\square$

Logic and Generic Setup of Maximum Likelihood Estimation (MLE)

Suppose that we observe some data set.

$$X_1, \dots, X_n \overset{iid}{\sim} P_\theta(x)$$

θ unknown.

The joint density of the data is

$$p_\theta(x_1, \dots, x_n) = \prod_{i=1}^{n} p_\theta(x_i)$$

The likelihood is

$$L_n(\theta; \underline{x}) = \prod_{i=1}^{n} p_\theta(x_i)$$

The log-likelihood is

$$\ell_n(\theta; x) = \log \prod_{i=1}^{n} p_\theta(x_i)$$

$$= \sum_{i=1}^{n} \log p_\theta(x_i)$$

Poisson example pt 3.

$X_1, \dots, X_n$, $X_i \overset{iid}{\sim} \text{pois}(\lambda)$.

λ unknown

$$P_\lambda(X_i = x_i) = \frac{\lambda^{x_i} e^{-\lambda}}{x_i!}$$

$$L_n(\lambda; \underline{x}) = P_\lambda(X_1 = x_1, \dots, X_n = x_n)$$

$$= \prod_{i=1}^{n} P_\lambda(X_i = x_i)$$

$$\Rightarrow \ell_n(\lambda; \underline{x}) = \log \prod_{i=1}^{n} P_\lambda(X_i = x_i)$$

$$= \sum_{i=1}^{n} \log P_\lambda(X_i = x_i)$$

$$= \sum_{i=1}^{n} \log \left(\frac{\lambda^{x_i} e^{-\lambda}}{x_i!} \right)$$

$$= \sum_{i=1}^{n} \left(x_i \log \lambda - \lambda - \log(x_i!) \right)$$

$$= \log \lambda \sum_{i=1}^{n} x_i - n\lambda - \sum_{i=1}^{n} \log(x_i!)$$

$$\frac{\partial \ell_n}{\partial \lambda} = \frac{\sum_{i=1}^{n} x_i}{\lambda} - n = 0.$$

$$\hat{\lambda} = \frac{1}{n} \sum_{i=1}^{n} x_i$$

(can show that it is max).

Section 4.7 Pb 54

Let X, Y and Z be uncorrelated r.v. with variances

$\sigma^2_X, \sigma^2_Y, \sigma^2_Z$ resp.

Let $U = Z + X$

$V = Z + Y$

Find $\mathrm{Cov}(U, V)$ and

ρ_{UV}

$\mathrm{corr}(U, V)$

$$\operatorname{cov}(U, V)$$

$$= \operatorname{cov}(Z+X, Z+Y)$$

$$= \operatorname{cov}(Z, Z+Y) + \operatorname{cov}(X, Z+Y)$$

$$= \operatorname{cov}(Z, Z) + \operatorname{cov}(Z, Y) + \operatorname{cov}(X, Z) + \operatorname{cov}(X, Y)$$

Def.

$$\operatorname{Cov}(X, Y) = E\big((X - E(X))(Y - E(Y))\big)$$

if $X = Y$

$$\Rightarrow \operatorname{cov}(X, X) = E\big((X - E(X))^2\big)$$

$$= \sigma_z^2 + 0$$ (by uncorrelated)

Quick fact.

Independence $\Rightarrow \rho = 0$ and $cov = 0$.

But this does not work the other way around.

$$\rho_{UV} = \frac{cov(U,V)}{\sigma_U \sigma_V}$$

$$\sigma_U^2 = var(Z+X)$$
$$= var(Z) + var(X) + 2\cancelto{0}{cov(Z,X)}$$

$\Rightarrow \sigma_v = \sqrt{\sigma_z^2 + \sigma_x^2}$.

Case σ_v^2 is similar.

Sec 4.7 QSS

Let $T = \sum_{k=1}^{n} k \cdot X_k$

$X_k \overset{iid}{\sim} [\mu, \sigma^2]$

"with mean μ and variance σ^2"

ET, $\operatorname{var} T$

$$ET = \sum_{k=1}^{n} k E X_k = \mu \sum_{k=1}^{n} k$$

$$= \frac{n(n+1)}{2} \mu$$

$$\operatorname{Var} T = \sum_{k=1}^{n} k^2 \operatorname{Var}(X_k)$$

$$= \sigma^2 \sum_{k=1}^{n} k^2$$

$$= \sigma^2 \frac{n(n-1)(2n+1)}{6}.$$

Section 5.4 Q4

N is poisson . $E(N)=100$.

Use normal approximation to find Δ such that

$$P(100-\Delta < N < 100+\Delta) \approx 0.9$$

Sol.

$E(N) = 100 \Rightarrow N \sim pois(100)$

$var(N) = 100$.

For poisson with "large" λ

we can use normal approximation

$$P(100-\Delta < N < 100+\Delta)$$

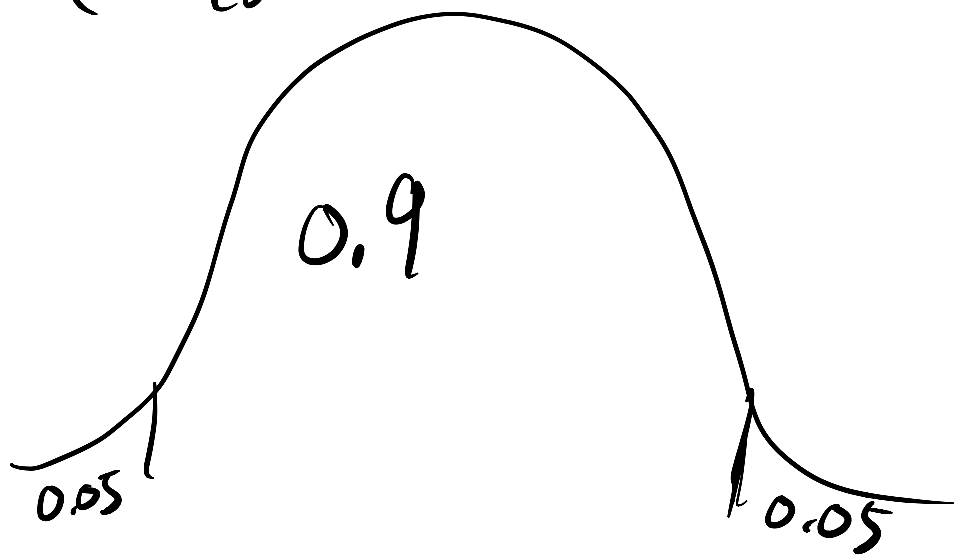
$$P\left(\frac{100-\Delta-100}{10} < \frac{N-100}{10} < \frac{100+\Delta-100}{10}\right)$$

$$= P\left(\frac{-\Delta}{10} < \frac{N-100}{10} < \frac{\Delta}{10}\right)$$

$$\frac{N-100}{10} \dot{\sim} N(0,1) \stackrel{d}{=} Z$$

" $\dot{\sim}$ " approximately distributed want

$$P\left(-\frac{\Delta}{10} < Z < \frac{\Delta}{10}\right) = 0.9$$



I want the 0.95 quantile

$$\text{qnorm}(0.95) \approx 1.64$$

$$\frac{\Delta}{10} = 1.64 \Rightarrow \Delta = 16.4$$

Section 5.4 QS.

Using MGF show that

as $n \to \infty$, $p \to 0$.

and $np \to \lambda$

$bin(n,p) \to pois(\lambda)$.

Sol

1. Find binomial MGF

Let $X \sim bin(n,p)$

$$M_X(t) = \sum_{k=0}^{n} e^{tk} \binom{n}{k} p^k (1-p)^{n-k}$$

$$= \sum_{k=0}^{n} \binom{n}{k} (e^t p)^k (1-p)^{n-k}$$

Binomial Formula

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^{k}$$

$$= \left(p e^{t} + (1-p)\right)^n$$

$$= \left(1 + (e^{t}-1)p\right)^n$$

$$\lambda = np \Rightarrow p = \frac{\lambda}{n}$$

$$= \left(1 + \frac{\lambda(e^{t}-1)}{n}\right)^n$$

$$\lim_{n \to \infty} \left(1 + \frac{\lambda(e^t - 1)}{n} \right)^n$$

$$\lim_{n \to \infty} \left(1 + \frac{x}{n} \right)^n = e^x$$

$$= e^{\lambda(e^t - 1)}$$ (By property A)

this is poisson.

Section 5.4 Q16

$X_1, \ldots, X_{20}$ are ind. r.v.
w/ density $f(x) = 2x$, $0 \le x \le 1$

Define $S = X_1 + \dots + X_{20}$

Approximate $P(S \le 10)$.

Sol: We need mean and variance

$$E(X) = \int_{-\infty}^{\infty} x f(x)\, dx$$

$$= \int_0^1 x\, 2x\, dx$$

$$= 2\int_0^1 x^2\, dx = 2\frac{x^3}{3}\Big|_0^1$$

$$= \frac{2}{3}$$

$$E(x^2) = \int_0^1 x^2 2x \, dx$$

$$= 2\int_0^1 x^3 \, dx.$$

$$= 2\frac{x^4}{4}\Big|_0^1 = \frac{1}{2}.$$

$$\Rightarrow \mathrm{Var}(x) = \frac{1}{2} - \frac{4}{9} = \frac{1}{18}$$

$$E(S) = E\left(\sum_{i=1}^{20} X_i\right)$$

$$= \sum_{i=1}^{20} E(X_i) = 20 \cdot \frac{2}{3}$$

$$Var(S) = \sum_{i=1}^{20} Var(X_i)$$

$$= 20 \cdot \frac{1}{18}$$

$$P(S \leq 10)$$

$$= P\left(\frac{S - 20 \cdot \frac{2}{3}}{\sqrt{20\left(\frac{1}{18}\right)}} \leq \frac{10 - 20\left(\frac{2}{3}\right)}{\sqrt{20 \frac{1}{18}}}\right)$$

$= P(Z \le c)$

$Z \sim N(0,1)$

pnorm $(c) = 0.0008$.

5.4 Q17

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independent measurements

mean μ, variance $\sigma^2 = 25$

How large n should be

so that $P(|\bar{X} - \mu| < 1) = 0.95$

Sol: $P(-1 < \bar{X} - \mu < 1)$

$$E(\bar{X}) = E\frac{1}{n}\sum X_i$$

$$= \frac{1}{n}\sum EX_i$$

$$= \frac{1}{n} \cdot \sum \mu = \frac{1}{n} \cdot n\mu$$

$$= \mu.$$

$$Var(\bar{X}) = Var\left(\frac{1}{n}\sum X_i\right)$$

$$= \frac{1}{n^2}\sum Var(X_i)$$

$$= \frac{1}{n^2}\sum \sigma^2 = \frac{1}{n^2} n \sigma^2$$

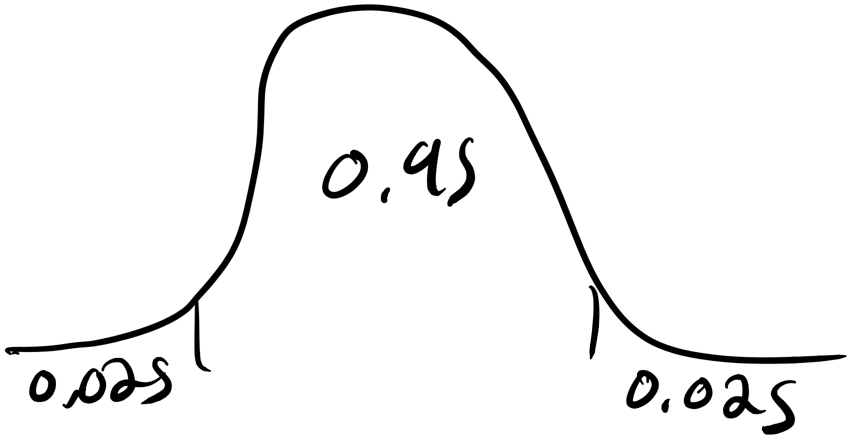
$$= \frac{\sigma^2}{n}$$

$$sd\,\bar{x} = \frac{\sigma}{\sqrt{n}} = \frac{5}{\sqrt{n}}$$

$$P\left(\frac{-1}{5/\sqrt{n}} < \frac{\bar{x}-\mu}{5/\sqrt{n}} < \frac{1}{5/\sqrt{n}}\right)$$

$$= P\left(-\frac{\sqrt{n}}{5} < z < \frac{\sqrt{n}}{5}\right)$$

$$= 0.95.$$



$$\frac{\sqrt{n}}{5} = \text{qnorm}(0.975)$$

$$= 1.96.$$

$$n = 97.$$

Sec 5.4 Q18

$$X_i \overset{ind}{\sim} [15, 10^2]$$

want

$$P\left(\sum_{i=1}^{100} X_i > 1700\right)$$

Sol.

Let $S_{100} = \sum_{i=1}^{100} X_i$

$$E\, S_{100} = 100 \cdot E X_i = 1500$$

$$Var(S_{100}) = Var\left(\sum_{i=1}^{100} X_i\right)$$

$$= \sum_{i=1}^{100} Var(X_i) = \sum_{i=1}^{100} \sigma^2$$

$$= 100 \cdot \sigma^2 = 100^2$$

$$sd(S_{100}) = 100.$$

$$P(S_n > 1700)$$

$$= P\left(\frac{S_n - 1500}{100} > \frac{1700 - 1500}{100} \right)$$

$= P(Z > 2)$

$= 1 - P(Z \leq 2)$

$= 1 - \Phi(2)$

$= 1 -$ pnorm(2) (in R)

$= 0.0228$