Agenda

0. Correction
1. R setup
2. Black Scholes
3. Black Scholes δ-method
4. 8.10 Q44
5. Bootstrap motivation
6. Bootstrap coding
7. Even more Bootstrap coding.

$$\frac{\sqrt{n}\left(\hat{\theta}_{MLE}-\theta\right)}{\sqrt{I(\theta)^{-1}}} \xrightarrow{d} N(0,1)$$

$$\frac{\sqrt{n}\left(\hat{\theta}_{MLE}-\theta\right)}{\sqrt{I\left(\hat{\theta}_{MLE}\right)^{-1}}} \xrightarrow{d} N(0,1)$$

δ-method

If we have $\hat{\theta}$ s.t.

$$\sqrt{n}\,\frac{(\hat{\theta}_n - \mu)}{\sigma} \xrightarrow{d} N(0,1)$$

then

$$\sqrt{n}\,\frac{(f(\hat{\theta}_n) - f(\mu))}{|f'(\mu)|\,\sigma} \xrightarrow{d} N(0,1)$$

if $f'(\mu)$ to exist and non-zero.

So for Black-Scholes.

$$\hat{\sigma}^2 \overset{\cdot}{\sim} N\left(\sigma, \frac{v^2}{n}\right)$$

v^2 is known.

$$C(\sigma) = S_0 \Phi(d_+(\sigma)) - K \Phi(d_-(\sigma)) \exp(-rT)$$

Φ is standard normal cdf.

Let $\frac{\partial C}{\partial \sigma} C(\sigma) = C'(\sigma) \neq 0.$

then since

$$\hat{\sigma} \stackrel{\cdot}{\sim} N\left(\sigma, \frac{r^2}{n}\right)$$

$$C(\hat{\sigma}) \stackrel{\cdot}{\sim} N\left(C(\sigma), C'(\sigma)^2 \cdot \frac{r^2}{n}\right)$$

MLE and Fisher information

for $X_i \stackrel{iid}{\sim} N(\mu_0, \sigma^2)$

where μ_0 known.

write $\tau = \sigma^2$

$$l_n(\tau) = \log \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\tau}} \exp\left(\frac{-1}{2} \frac{(x_i-\mu_0)^2}{\tau}\right)$$

$$= \sum_{i=1}^{n} \log \left(\frac{1}{\sqrt{2\pi\tau}} \exp\left(-\frac{1}{2} \frac{(x_i-\mu_0)^2}{\tau} \right) \right)$$

$$= \sum_{i=1}^{n} \left(-\frac{1}{2} \log(2\pi\tau) - \frac{1}{2} \frac{(x_i-\mu_0)^2}{\tau} \right)$$

$$= c - \frac{n}{2} \log(\tau) - \frac{1}{2\tau} \sum_{i=1}^{n} (x_i-\mu_0)^2$$

$$l_n'(\tau) = \frac{-n}{2\tau} + \frac{1}{2\tau^2} \sum_{i=1}^{n} (x_i-\mu_0)^2$$

$$= 0.$$

$$\hat{\tau} = \frac{1}{n} \sum_{i=1}^{n} (x_i-\mu_0)^2$$

This is an iid sample

$$I_n(\theta) = n I_1(\theta)$$

$$l_n''(\tau) = \frac{n}{2\tau^2} - \frac{1}{\tau^3}\sum_{i=1}^{n}(x_i-\mu_0)^2$$

$$I_n(\tau) = -E\left(l_n''(\tau)\right)$$

$$= \frac{-n}{2\tau^2} + \frac{1}{\tau^3}\sum_{i=1}^{n} E\left((x_i-\mu_0)^2\right)$$

$$= \frac{-n}{2\tau^2} + \frac{1}{\tau^3} n \cdot \tau$$

$$= \frac{n}{2\tau^2}$$

asymptotic variance

$$\frac{1}{I_n(\tau)} = \frac{2\tau^2}{n}.$$

$$= \frac{2(\sigma^4)}{n}$$

THE Bootstrap

1. We have only one data set. (typically small)

All we can get is a point estimate.

2. How about we just pretend we got new datasets by resample our existing dataset.
(non-parametric).

Parametric Bootstrap

We have data drawn from P_θ, we can estimate
$\hat{\theta}(X_1, \ldots, X_n)$
However we can generate from $P_{\hat{\theta}}$

So process.

0. Estimate $\hat{\theta}(x_1, \ldots, x_n)$

1. Plug $\hat{\theta}$ into $P_{\hat{\theta}}$ and sample n, new data points

2. From 1. compute a new $\hat{\theta}^*$ based on new data points.

3. Repeat 1 and 2 a lot of times to get B estimates of $\hat{\theta}_1^*, \ldots, \hat{\theta}_B^*$

Non-Parametric

0. Compute some $\hat{\theta}(X_1, \dots, X_n)$, mean for example.

1. Sample n times from existing dataset with replacement.

2. Compute $\hat{\theta}^*$ of this new data set.

3. Repeat 1, 2 B times + get B samples from dist of "$\hat{\theta}$"

δ-method example

Suppose $E(X_i) = m$

$var(X_i) = \sigma^2 < \infty$

$$\frac{\sqrt{n}(\bar{X} - m)}{\sigma} \xrightarrow{d} N(0,1)$$

(CLT)

$$\log(\bar{X}) \xrightarrow{d} ?$$

$f(a) = \log(a)$

$f'(a) = \frac{1}{a}$

$$\frac{\sqrt{n}\left(\log(\overline{X}) - \log(m)\right)}{|f'(m)|\,\sigma}$$

$$= \frac{\sqrt{n}\left(\log(\overline{X}) - \log(m)\right)}{\left|\frac{1}{m}\right| \cdot \sigma}$$

$$\sim N(0,1)$$