Agenda

1. CI / Hypothesis Test Duality

2. P-values

3. GLRT

4. Problems

5. Your Questions

Theorem B

Suppose r.v. $\underline{X}$

parameter space Θ

Suppose that $C(\underline{X})$

is a $(1-\alpha)\cdot 100\%$ CI

for θ. So for every

$\theta_0 \in \Theta$

$$P\left(\theta_0 \in C(\underline{X}) \mid \theta = \theta_0\right)$$

$$= 1 - \alpha$$

Then acceptance region at level α of test $H_0: \theta = \theta_0$

$$A(\theta_0) = \{ \underline{X} \mid \theta_0 \in C(\underline{X}) \}$$

Pf.

$$P(\underline{X} \in A(\theta_0) \mid \theta = \theta_0)$$

$$= P(\theta_0 \in C(\underline{X}) \mid \theta = \theta_0)$$

$$= 1 - \alpha$$

$$= 1 - P(\underline{X} \notin A(\theta_0) \mid \theta = \theta_0)$$

$$= 1 - P_{H_0}(\text{rejection})$$

$\Rightarrow P_{H_0}(\text{rejection}) = \alpha$

p-values

So far

1. Pick α
2. Find rejection criterion.

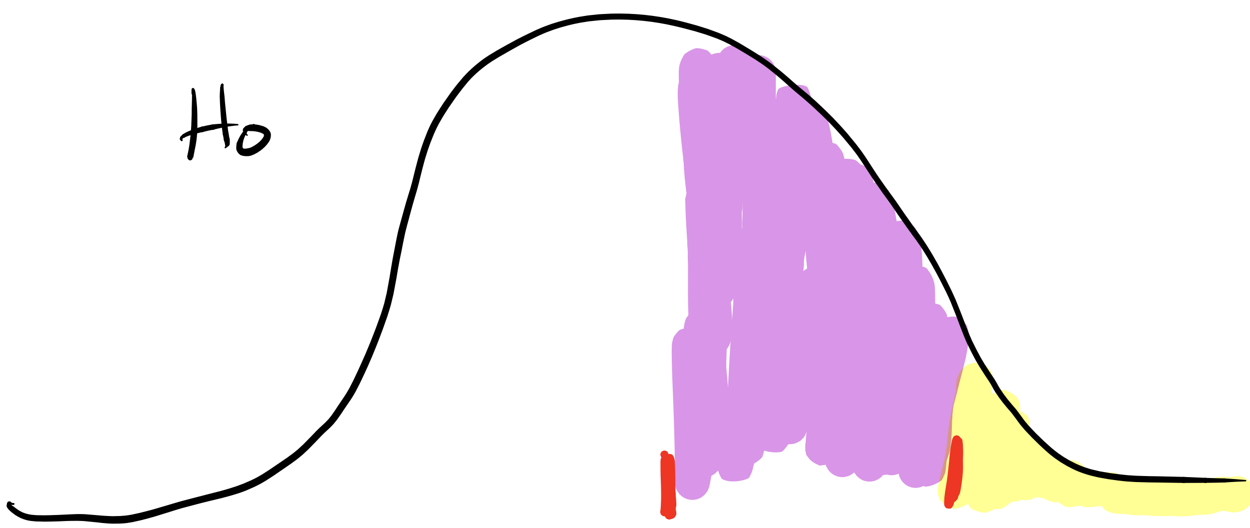
Def.

Smallest significance level such that we reject the null hypothesis given a specific test statistic.

If we have a rejection criteria of the form

$T > t$ or $T < t$

then p-value is probability of seeing an as extreme or more extreme result under the null.



suppose reject $T(x) > t$

3. Generalized LRT

Simple - Simple LRT

we were testing

$$H_0: \theta = \theta_0$$
$$H_1: \theta = \theta_1$$

$$LR(\underset{\sim}{X}) = \Lambda = \frac{L(\theta_0)}{L(\theta_1)}$$

we rejected when

$LR(\underset{\sim}{x})$ small.

H_0 , ω_0 is all the values of θ possible under H_0.

ω_1 all the values of θ possible under H_1

$$\Omega = \omega_0 \cup \omega_1$$

$$\Lambda = \frac{\max_{\theta \in \omega_0} L(\theta)}{\max_{\theta \in \Omega} L(\theta)}$$

special case $\Theta \in \mathbb{R}$

$H_0: \theta = \theta_0 \qquad \omega_0 = \{\theta_0\}$

$H_1: \theta \neq \theta_0 \qquad \Omega = \mathbb{R}$

$$\Lambda = \frac{L(\theta_0)}{\max\limits_{\theta \in \Theta} L(\theta)} \leftarrow \text{MLE.}$$

In general

$$-2\log(\Lambda) \xrightarrow{d} \chi^2_{df}$$

$$df = \dim(\Omega) - \dim(\omega_0)$$

$H_0: p_1 = p_2 = p_3 = p \quad p_i \in \mathbb{R}$

H_1: atleast one is not equal to the others.

$\dim(\omega_0) = 1$

$\dim(\omega_1) = 3$

$\dim(\Omega) = 3$

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$X_i \overset{ind}{\sim} bin(n_i, p_i)$

for $i = 1, \ldots, m$

Derive a likelihood ratio test for

$H_0: p_1 = p_2 = \ldots = p_m$

H_1: They're not all equal

$$\omega_0 = p \in (0,1)$$

$$\omega_1 = (p_1, \dots, p_m) \in (0,1)^m$$

$$\Lambda(x) = \frac{\max_{p_1 = \dots = p_m = p} L_n(\underline{p})}{\max L_n(\underline{p})}$$

$$L_n(p) = \prod_{i=1}^{m} \binom{n_i}{x_i} p_i^{x_i} (1-p_i)^{n_i - x_i}$$

$$\overset{H_0}{=} \left(\prod_{i=1}^{m} \binom{n_i}{x_i} \right) \prod_{i=1}^{n} p^{x_i} (1-p)^{n_i - x_i}$$

$$\ell_n(p) \overset{H_0}{=} C + \log\left(\prod_{i=1}^{m} p^{x_i}\right) + \log\left(\prod_{i=1}^{m} (1-p)^{n_i - x_i}\right)$$

$$= C + \sum_{i=1}^{m} x_i \log(p) + \sum_{i=1}^{m} (n_i - x_i) \log((1-p)$$

$$\frac{\partial \ell_n(p)}{\partial p} = \frac{\sum_{i=1}^{m} x_i}{p} - \frac{\sum_{i=1}^{m} (n_i - x_i)}{1-p}$$

$$= 0$$

$$\hat{p} = \frac{\sum_{i=1}^{m} x_i}{\sum_{i=1}^{m} n_i}$$

$$L(p) = \prod_{i=1}^{m} \binom{n_i}{x_i} p_i^{x_i} (1-p_i)^{n_i - x_i}$$

$$\hat{p}_i = \frac{x_i}{n_i} \text{ for } i = 1, \ldots, m$$

$$\Lambda = \frac{\prod_{i=1}^{m} \cancel{\binom{n_i}{x_i}} \cdot \hat{p}^{x_i} \cdot (1-\hat{p})^{n_i - x_i}}{\prod_{i=1}^{m} \cancel{\binom{n_i}{x_i}} \hat{p}_i^{x_i} (1-\hat{p}_i)^{n_i - x_i}}$$

$$\Lambda = \frac{\hat{p}^{\sum_{i=1}^{m} x_i} (1-\hat{p})^{\sum^{m} (n_i - x_i)}}{\prod_{i=1}^{m} \hat{p}_i^{x_i} (1-\hat{p}_i)^{n_i - x_i}}$$

$$-2 \log(\Lambda) \xrightarrow{d} \chi^2_{df}$$

$$df = m-1$$

$$\dim(\omega_0) = 1$$

$$\dim(\Omega) = m$$

To get Rejection region.

This rejects for large

$$-2\log(\Lambda)$$

$$P_{H_0}\left(-2\log(\Lambda) > c(\alpha)\right) = \alpha$$

$$1 - P_{H_0}\left(-2\log\Lambda \le c(\alpha)\right) = \alpha$$

$$1 - F(c(\alpha)) = \alpha$$

F is cdf of χ^2_{n-1}

p-value

suppose our instance has

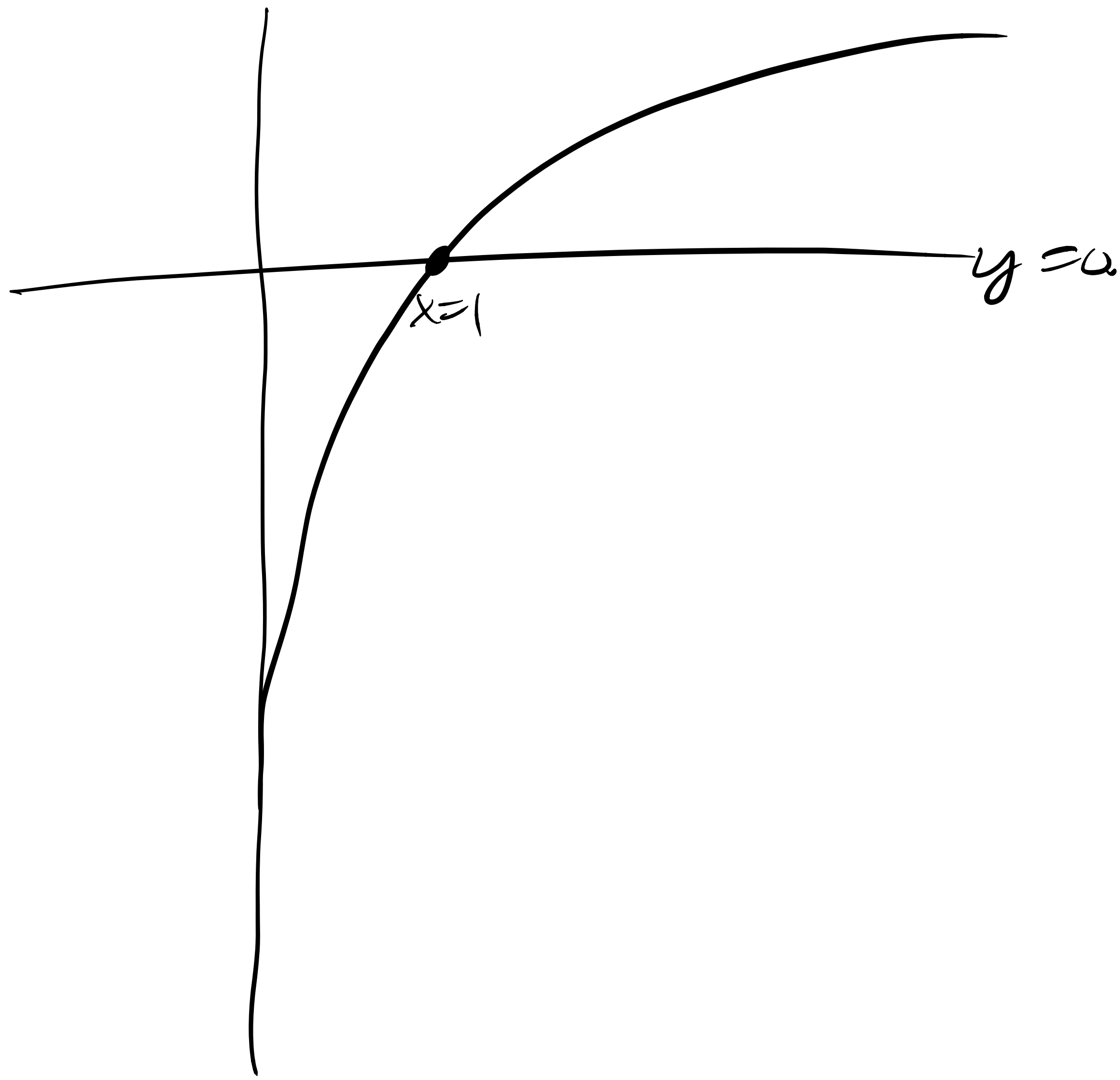
$$-2\log(\Lambda) = t$$

p-value

$$P_{H_0}(-2\log(\Lambda) \geq t)$$

$$= 1 - P_{H_0}(-2\log(\Lambda) \leq t)$$

$$= 1 - F(t)$$



$x=1$
$y=a$

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280 bars tested in 5 places for breaks

So $X_i \sim bin(5, p_i)$

$i=1,\ldots,280$

Null is that all bars have equal chance of break p_i's equal

$H_0: p_1 = p_2 = \ldots = p_{280} = p$

H_1: Not H_0

Breaks/Bar	Frequency
0	157
1	69
2	35
3	17
4	1
5	1

a) MLE under null same as numerator of 41

$$\hat{p} = \frac{\sum_{i=1}^{280} x_i}{\sum_{i=1}^{280} n_i}$$

$$= \frac{\text{Total \# breaks}}{\text{Total \# trials}}$$

$$= \frac{199}{5 \cdot 280} = 0.142$$

$$X^2 = \sum_{k=0}^{5} \frac{(O_k - E_k)^2}{E_k}$$

E_k ?

$$\Theta_i(\hat{p}) = \binom{5}{i} (\hat{p})^i (1-\hat{p})^{5-i}$$

i	O_i	E_i $(280 \cdot \theta_i(\hat{p}))$
0	157	$280 \cdot \theta_0(\hat{p})$
1	69	$\vdots$
2	35	$\vdots$
3	17	
4	1	
5	1	$280\theta_5(\hat{p})$

$$X^2 = \sum_{k=0}^{5} \frac{(O_k - E_k)^2}{E_k}$$

"X"

$$X^2 \xrightarrow{d} \chi^2_{k - \dim(p) - 1}$$

"chi"

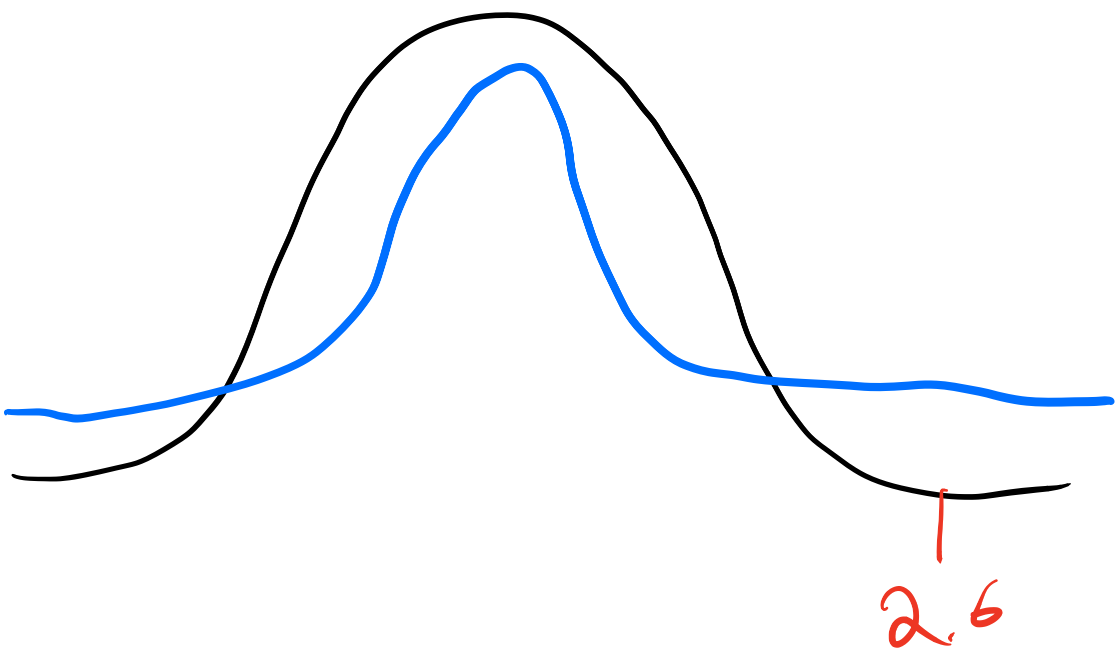
$$6-1-1=4$$

$$Q=\sum_{i=1}^{k} Z_i^2$$

$$Z_i \overset{ind}{\sim} N(0,1)$$

$$Q \sim \chi^2_k$$


Sample Quantiles
Theoretical quantiles


2.6

$$X_i \sim \exp\left(\frac{1}{\tau}\right)$$

$$E(X_i) = \tau$$

$$\hat{\tau} = \bar{X} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

$$\hat{\tau} \sim \text{Gamma}\left(n, \frac{1}{\tau}\right)$$

$$P\left(a(\underline{x}_1) \leq \tau \leq b(\underline{x})\right) = 1 - \alpha$$

A pivot is a function of the data and the parameter whose distribution is known completely.

if $\hat{\tau} \sim \text{Gamma}\left(n, \frac{n}{\tau}\right)$

$$\tau \cdot \hat{\tau} \sim \text{Gamma}(n, n)$$

$$P\left(g_{\alpha/2} \leq \tau \cdot \hat{\tau} \leq g_{1-\frac{\alpha}{2}}\right)$$

$$= 1 - \alpha$$

$$P\left(\frac{q_{\alpha/2}}{\hat{\tau}} \leq \tau \leq \frac{q_{1-\frac{\alpha}{2}}}{\hat{\tau}}\right)$$

$$= 1 - \alpha$$

$$q_{\alpha/2}$$

$$\sqrt{n}\left(\frac{\hat{\theta}_{MLE} - \theta}{I(\hat{\theta})^{-1}}\right) \sim N(0, 1)$$

$$\hat{\theta}_{MLE} \pm \frac{z_{\alpha/2}}{\sqrt{n}} I(\hat{\theta})^{-1}$$