# Exponential Distribution

The Exponential Distribution may be used to answer the following questions:

- How much time will elapse before an earthquake occurs in a given region?
- How long do we need to wait before a customer enters our shop?
- How long will it take before a call center receives the next phone call?
- How long will a piece of machinery work without breaking down?

# Exponential Distribution

- All these questions concern the time we need to wait before a given event occurs. If this waiting time is unknown, it is often appropriate to think of it as a random variable having an exponential distribution.
- Roughly speaking, the time $X$ we need to wait before an event occurs has an exponential distribution if the probability that the event occurs during a certain time interval is proportional to the length of that time interval.

# Probability density function

The probability density function (PDF) of an exponential distribution is

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

The parameter $\lambda$ is called ***rate*** parameter.

# Cumulative density function

The cumulative distribution function (CDF) of an exponential distribution is

$$P(X \leq x) = F(x) = \begin{cases} 1 - e^{-\lambda x}, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

The complemeent of the CDF (i.e. $P(X \geq x)$ is

$$P(X \geq x) = \begin{cases} e^{-\lambda x}, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

# Expected Value and Variance

The expected value of an exponential random variable $X$ is:

$$E[X] = \frac{1}{\lambda}$$

The variance of an exponential random variable $X$ is:

$$V[X] = \frac{1}{\lambda^2}$$

# Exponential Distribution: Example

Assume that the length of a phone call in minutes is an exponential random variable $X$ with parameter $\lambda = 1/10$. If someone arrives at a phone booth just before you arrive, find the probability that you will have to wait

**(a)** less than 5 minutes,

**(b)** between 5 and 10 minutes.

# Exponential Distribution: Example

**(a)** $P(X \leq 5) = 0.39346934$

**(b)** $P(5 \leq X \leq 10)$
$= P(X \leq 10) - P(X \leq 5)$
= 0.6321- 0.3934
= 0.2386
= 23.86 %

**(c)** Alternative approach to (b)
$P(5 \leq X \leq 10)$
$= P(X \geq 5) - P(X \geq 10)$
$= e^{-0.5} - e^{-1}$ =0.6065 - 0.3678
= 0.2386 = 23.86 %

# Exponential Distribution

- The Exponential Rate
- Related to the Poisson mean (m)
- If we expect 12 occurrences per hour - what is the rate?
- We would expected to wait 5 minutes between occurrences.

**Exponential Distribution**

Jobs are sent to a printer at an average of 5 jobs per hour.

1. (1 Mark) What is the expected time between jobs?
2. (1 Mark) What is the probability that the next job is sent within 6 minutes after the previous job?

## Exponential Distribution

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- How much time will elapse before an earthquake occurs in a given region?
- How long do we need to wait before a customer enters a shop?
- How long will it take before a call center receives the next phone call?
- How long will a piece of machinery work without breaking down?

# Exponential Distribution

- All these questions concern the time we need to wait before a given event occurs.
- If this waiting time is unknown, it is often appropriate to think of it as a random variable having an **exponential distribution**.
- The time $X$ we need to wait before an event occurs has an exponential distribution if the probability that the event occurs during a certain time interval is proportional to the length of that time interval.

# Probability density function

The probability density function (PDF) of an exponential distribution is

$$f(x;\lambda) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

The parameter $\lambda$ is called ***rate*** parameter. It is the inverse of the expected duration ($\mu$).

(If the expected duration is 5 (e.g. five minutes) then the rate parameter value is 0.2.)

## Exponential Distribution: Cumulative density function

The cumulative distribution function (CDF) of an exponential distribution is

$$F(x;\lambda) = \begin{cases} 1 - e^{-\lambda x}, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

The CDF can be written as the probability of the lifetime being less than some value $x$.

$$P(X \leq x) = 1 - e^{-\lambda x}$$

# Exponential Distribution: Expected Value and Variance

The expected value of an exponential random variable $X$ is:

$$E[X] = \frac{1}{\lambda}$$

The variance of an exponential random variable $X$ is:

$$V[X] = \frac{1}{\lambda^2}$$

## Exponential Distribution: Example

Assume that the length of a phone call in minutes is an exponential random variable $X$ with parameter $\lambda = 1/10$.

If someone arrives at a phone booth just before you arrive, find the probability that you will have to wait

**(a)** less than 5 minutes,

**(b)** greater than 5 minutes,

**(c)** between 5 and 10 minutes.

Also compute the expected value and variance.

# Exponential Distribution: Example

## Part a

Compute $P(X \leq 5)$ with $\lambda = 1/10$

$$P(X \leq x) = 1 - e^{-\lambda x}$$

# Exponential Distribution: Example

## Part a

Compute $P(X \leq 5)$ with $\lambda = 1/10$

- $P(X \leq x) = 1 - e^{-\lambda x}$
- $P(X \leq 5) = 1 - e^{-5/10}$
- $P(X \leq 5) = 1 - e^{-0.5}$
- $P(X \leq 5) = 1 - 0.6065$
- $P(X \leq 5) = 0.3934$

## Exponential Distribution: Example

### Part b

Compute $P(X \geq 10)$ with $\lambda = 1/10$

$$P(X \leq x) = 1 - e^{-\lambda x}$$

Complement rule

$$P(X \geq x) = 1 - P(X \leq x) = e^{-\lambda x}$$

# Exponential Distribution: Example

## Part b

Compute $P(X \geq 10)$ with $\lambda = 1/10$

$$P(X \geq x) = e^{-\lambda x}$$

- $P(X \geq x) = e^{-\lambda x}$
- $P(X \geq 10) = e^{-10/10}$
- $P(X \geq 10) = e^{-1}$
- $P(X \geq 10) = 0.3678$

# Exponential Distribution: Example

## Part c

Compute $P(5 \leq X \leq 10)$ with $\lambda = 1/10$

- Probability of being inside this interval is the complement of being outside the interval.
- The probability of being outside the interval is the composite event of being too low for the interval (i.e. $P(X \leq 5)$) and being too high for the interval (i.e. $P(X \leq 10)$).

$$P(5 \leq X \leq 10) = 1 - [P(X \leq 5) + P(X \geq 10)]$$

# Exponential Distribution: Example

## Part c

Compute $P(5 \leq X \leq 10)$ with $\lambda = 1/10$

$$P(5 \leq X \leq 10) = 1 - [P(X \leq 5) + P(X \geq 10)]$$

- **Too Low** $P(X \leq 5) = 0.3934$
- **Too High** $P(X \geq 10) = 0.3678$
- **Outside** $P(X \leq 5) + P(X \geq 10) = 0.7612$
- **Inside** $P(5 \leq X \leq 10) = 1 - 0.7612 = 0.2388$

# Exponential Distribution

## Expected Value and Variance

The expected value of an exponential random variable $X$ is:

$$E[X] = \frac{1}{\lambda} = \frac{1}{1/10} = 10$$

The variance of an exponential random variable $X$ is:

$$V[X] = \frac{1}{\lambda^2} = 100$$

# Exponential Distribution

# Exponential Distribution: Relationship to Poisson Mean

- The Exponential Rate parameter ($\lambda$) is related to the Poisson mean (m)
- If we expect 12 occurrences per hour - what is the rate of occurrences?
- We would expected to wait 1/12 of an hour (i.e. 5 minutes) between occurrences.
- Be mindful to keep your time units consistent, if working with both Poisson and Exponential.
- If working in minutes, our rate parameter values is $\lambda = 0.20$ (i.e. 1/5).
- (This could be the basis of an exam question).