Introduction to Probability

Techniques for Counting

- Combinations
- Permutations
- Permutations with constraints

Permuations of subsets

The number of permutations of subsets of k elements selected from a set of n different elements is

$$P(n,r) = \frac{n!}{(n-k)!}$$

Combinations of subsets

The number of combinations that can be selected from n items is

$$\binom{n}{k} = \frac{n!}{k! \times (n-k)!}$$

Counting Problems

- Sampling without replacement.
- Factorials
- Permutations
- Combinations

Sampling without replacement

- Sampling is said to be "without replacement" when a unit is selected at random from the population and it is not returned to the main lot.
- The first unit is selected out of a population of size N and the second unit is selected out of the remaining population of $N-1$ units and so on.
- For example, if you draw one card out of a deck of 52, there are only 51 cards left to draw from if you are selecting a second card.

Sampling without replacement

A lot of 100 semiconductor chips contains 20 that are defective. Two chips are selected at random, without replacement from the lot.

- What is the probability that the first one is defective?
  (Answer : 20/100 , i.e 0.20)
- What is the probability that the second one is defective given that the first one was defective?
  (Answer: 19/99)
- What is the probability that the second one is defective given that the first one was not defective?
  (Answer: 20/99)

Sampling With Replacement

Sampling is called “with replacement” when a unit selected at random from the population is returned to the population and then a second element is selected at random. Whenever a unit is selected, the population contains all the same units.

- What is the probability of guessing a PIN number for an ATM card at the first attempt.
- Importantly a digit can be used twice, or more, in PIN codes.
- For example 1337 is a valid pin number, where 3 appears twice.
- We have a one-in-ten chance of picking the first digit correctly, a one-in-ten chance of the guessing the second, and so on.
- All of these events are independent, so the probability of guess the correct PIN is $0.1 \times 0.1 \times 0.1 \times 0.1 = 0.0001$

Factorials Numbers

A factorial is a positive whole number, based on a number n , and which is written as "$n!$". The factorial $n!$ is defined as follows:

$$n! = n \times (n-1) \times (n-2) \times \ldots \times 2 \times 1$$

Remark $n! = n \times (n-1)!$

Example:

- $3! = 3 \times 2 \times 1 = 6$
- $4! = 4 \times 3! = 4 \times 3 \times 2 \times 1 = 24$

Remark $0! = 1$ not 0.

Permutations and Combinations

Often we are concerned with computing the number of ways of selecting and arranging groups of items.

- A ***combination*** describes the selection of items from a larger group of items.
- A ***permutation*** is a combination that is arranged in a particular way.

- Suppose we have items A,B,C and D to choose two items from.
- AB is one possible selection, BD is another. AB and BD are both combinations.
- More importantly, AB is one combination, for which there are two distinct permutations: AB and BA.

Combinations

Combinations: The number of ways of selecting k objects from n unique objects is:

$$^{n}C_{k} = \frac{n!}{k! \times (n-k)!}$$

In some texts, the notation for finding the number of possible combination is written

$$^{n}C_{k} = \binom{n}{k}$$

Example of Combinations

How many ways are there of selecting two items from possible 5?

$$^5C_2 \left(\text{also } \binom{5}{2} \right) = \frac{5!}{2! \times 3!} = \frac{5 \times 4 \times 3!}{2 \times 1 \times 3!} = 10$$

Discuss how combinations can be used to compute the number of rugby matches for each group in the Rugby World Cup.

The Permutation Formula

The number of different permutations of r items from n unique items is written as ${}^{n}P_{k}$

$$ {}^{n}P_{k} = \frac{n!}{(n-k)!} $$

Permutations

Example: How many ways are there of arranging 3 different jobs, between 5 workers, where each worker can only do one job?

$$^5P_3 = \frac{5!}{(5-3)!} = \frac{5!}{2!} = 60$$

Example of Combinations

A committee of 4 must be chosen from 3 females and 4 males.

- In how many ways can the committee be chosen.
- In how many cans 2 males and 2 females be chosen.
- Compute the probability of a committee of 2 males and 2 females are chosen.
- Compute the probability of at least two females.

Example of Combinations

Part 1

We need to choose 4 people from 7:
This can be done in

$$^{7}C_4 = \frac{7!}{4! \times 3!} = \frac{7 \times 6 \times 5 \times 4!}{4! \times 3!} = 35 \text{ ways.}$$

Part 2

With 4 men to choose from, 2 men can be selected in

$$^{4}C_2 = \frac{4!}{2! \times 2!} = \frac{4 \times 3 \times 2!}{2! \times 2!} = 6 \text{ ways.}$$

Similarly 2 women can be selected from 3 in

$$^{3}C_2 = \frac{3!}{2! \times 1!} = \frac{3 \times 2!}{2! \times 1!} = 3 \text{ ways.}$$

Example of Combinations

Part 2
Thus a committee of 2 men and 2 women can be selected in $6 \times 3 = 18$ ways.

Part 3
The probability of two men and two women on a committee is

$$\frac{\text{Number of ways of selecting 2 men and 2 women}}{\text{Number of ways of selecting 4 from 7}} = \frac{18}{35}$$

Example of Combinations

Part 4

- The probability of at least two females is the probability of 2 females or 3 females being selected.
- We can use the addition rule, noting that these are two mutually exclusive events.
- From before we know that probability of 2 females being selected is 18/35.

Example of Combinations

Part 4

- We have to compute the number of ways of selecting 1 male from 4 (4 ways) and the number of ways of selecting three females from 2 (only 1 way)
- The probability of selecting three females is therefore $\frac{4\times 1}{35} = 4/35$
- So using the addition rule

$$Pr(\text{ at least 2 females }) = Pr(\text{ 2 females }) + Pr(\text{ 3 females$$

$$Pr(\text{ at least 2 females }) = 18/35 + 4/35 = 22/35$$

Permutations with Constraints

How many different four digit numbers greater than 5000 can be formed from the digits

$$2, 4, 5, 8, 9$$

if each digit can only be used once in any given number.

Permutations with Constraints

How many of these four digit numbers are odd, given they are greater than 5000?

$$2, 4, 5, 8, 9$$

Permutations with Constraints

Introduction to Probability

Calculations using the Choose Operator

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Choose Operator

For the positive integer n and non-negative integer k (with $k \leq n$), the choose operater is calculated as follows:

$$\binom{n}{k} = \frac{n!}{k! \times (n-k)!}$$

Choose Operator

Evaluate the following: 3

1. $\binom{5}{2}$
2. $\binom{5}{0}$
3. $\binom{6}{3}$
4. $\binom{6}{6}$
5. $\binom{10}{1}$
6. $\binom{10}{9}$

Choose Operator

Part 1

$$\binom{5}{2}$$

Choose Operator

Part 2

$$\binom{5}{0}$$

Choose Operator

Part 3

$$\binom{6}{3}$$

Choose Operator

Part 4

$$\binom{6}{6}$$

Choose Operator

Part 5

$$\binom{10}{1}$$

Choose Operator

Part 6

$$\binom{10}{9}$$

Counting Sets with Venn Diagrams

- The Venn Diagram shows the number of elements in each subset of set S.
- If $P(A) = 3/10$ and $P(B) = 1/2$, find the values of x and y

Counting Sets with Venn Diagrams

- The total number of items in the data set is $x+y+5$
- There are $x+1$ items in Area A
- There are $x+y$ items in Area B
- We can say

$$P(A) = \frac{3}{10} = \frac{x+1}{x+y+5}$$

$$P(B) = \frac{1}{2} = \frac{x+y}{x+y+5}$$

Counting Sets with Venn Diagrams

Cross Multiplication

$$P(A) = \frac{3}{10} = \frac{x+1}{x+y+5}$$

Counting Sets with Venn Diagrams

Cross Multiplication

$$P(B) = \frac{1}{2} = \frac{x+y}{x+y+5}$$

Counting Sets with Venn Diagrams

Simultaneous Equations

1) $7x - 3y = 5$
2) $x + y = 5$

Counting Sets with Venn Diagrams

Simultaneous Equations

- $7x - 3y = 5$
- $x + y = 5$