

# L7.1 MDP Planning

Zonghua Gu 2022

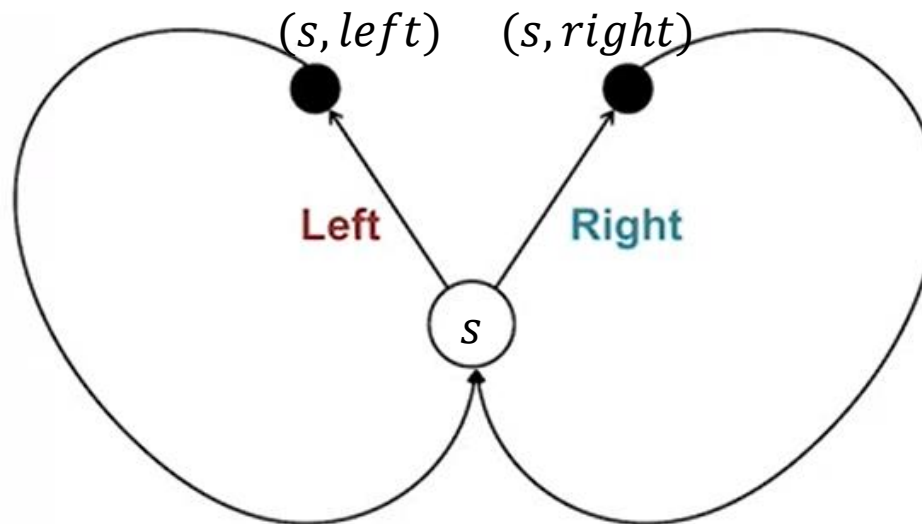


# Markov Decision Process (MDP)

- An MDP consists of:
  - Set of states  $S$
  - Start state  $s_0$
  - Set of actions  $A$
  - Transitions and rewards  $p(r, s' | s, a)$  (w. discount  $\gamma$ )
- Policy maps from states to actions:
  - Deterministic policy  $a = \pi(s)$  defines a deterministic action  $a$  for state  $s$ .
  - Stochastic policy  $\pi(a | s)$  defines a probability distribution over possible actions  $a$  for state  $s$ .
- Markov means that next state only depends on current state
  - $P(S_{t+1} = s' | S_t = s_t, A_t = a_t, S_{t-1} = s_{t-1}, A_{t-1} = a_{t-1}, \dots, S_0 = s_0, A_0 = a_0)$
  - $= P(S_{t+1} = s' | S_t = s_t, A_t = a_t)$
  - Given the present state, the future and the past are independent
  - e.g., for driving task, current vehicle position  $x$  as the state does not satisfy the Markov property, since the next state depends on not only  $x$ , but also velocity  $\dot{x}$ , acceleration  $\ddot{x}$ , (assuming acceleration  $\ddot{x}$  stays constant within each step) If we redefine the state as vector  $[x, \dot{x}, \ddot{x}]^T$ , then it satisfies the Markov property.
  - Or, current snapshot of front camera view can be used as the state (e.g., NVIDIA's PilotNet), but some works use past  $N$  video frames as the state to capture more dynamics (e.g., Waymo's ChauffeurNet).

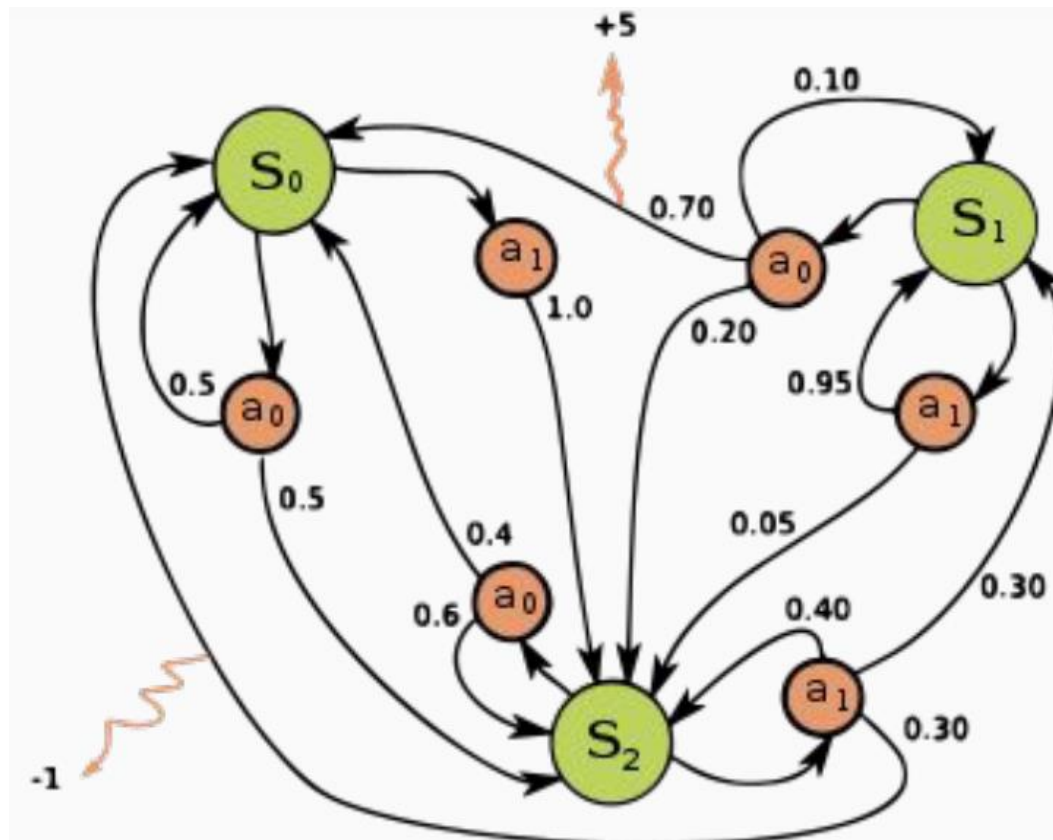
# MDP Quiz

- For this MDP with a single state  $s$  and two possible actions  $left$  and  $right$ . Are these valid policies?
  - 1)  $\pi(left|s) = \pi(right|s) = 0.5$  (goes left or right with equal probability. uniform random policy)
  - 2)  $\pi(left|s) = 1.0, \pi(right|s) = 0$  (always goes left)
  - 3) Alternating left and right, i.e., if previous action is left, then current action must be right, next action must be left, and so on.
  - ANS: 3) is not a valid policy, since it depends on the history of actions. To be a valid policy, the action must depend on the current state only.
- We can redefine the MDP' extended state to include the last action as part of it, then 3) is a valid policy for the new MDP.



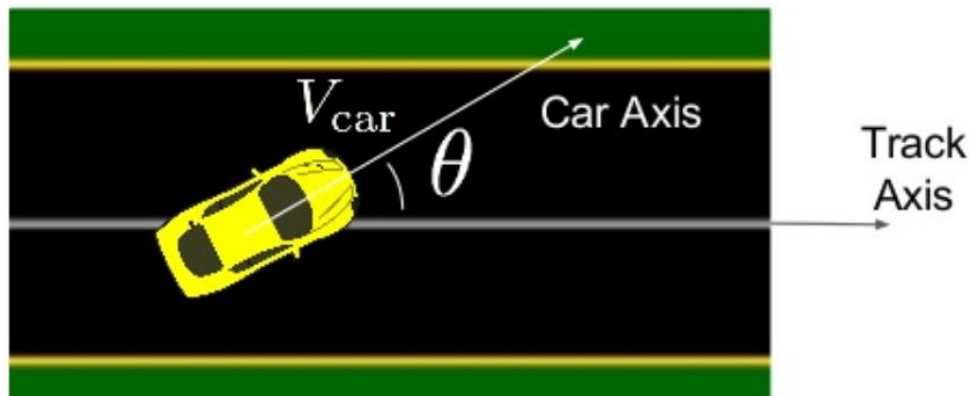
# An Example MDP

- Green nodes denote 3 states  $s_0, s_1, s_2$ ; Red nodes denote 2 possible actions  $a_0, a_1$  in each state. Each red node can also be denoted as  $(s, a)$ .
- Agent taking action  $a$  in state  $s$  may get different reward  $r$  and next state  $s'$ , denoted as state transition  $(s, a, r, s')$ , due to environment uncertainty (all rewards are 0 expect +5 and -1 show in fig).



# RL Reward Function

- For the vehicle in left fig:
  - state: Pose of ego-car ( $x, y, \theta$ ) and environment map; action: Steering wheel/brake/acceleration
- Possible reward function:  $R_t = w_1 V_{car} \cos \theta - w_2 |cte|$ 
  - Weight sum to maximum longitudinal velocity (first term), and minimize cross-track error (distance to lane center)
  - This is an example of dense reward (e.g., at every time step), as opposed to sparse reward (e.g., only at the end of each episode)
- Compare with twiddle() :
  - twiddle() can be viewed as an RL algorithm (policy gradient), that learns PID parameters with sparse reward (cost function is average cross-track error (cte), computed at the end of each simulation episode, as sum of squares of ctes for N timesteps divided by N. But it is very crude:
  - It does not use the numeric value of cte, only its relative size (if  $err < best\_err$ );
  - Cost function does not include heading angle  $\theta$ ;
  - if the track is very long and irregular, then we can make the reward denser, to adjust PID parameters every K timesteps instead of at the end of each episode.



```
if err < best_err:
    best_err = err
    dp[i] *= 1.1
else:
    p[i] -= 2 * dp[i]
    robot = make_robot()
    x_trajectory, y_trajectory, err = run(robot, p)

    if err < best_err:
        best_err = err
        dp[i] *= 1.1
    else:
        p[i] += dp[i]
        dp[i] *= 0.9
```

twiddle()

# Amazon DeepRacer

- Amazon Web Services (AWS) launched DeepRacer in 2018 for training AD algorithms with RL
  - <https://aws.amazon.com/deepracer/>
- You can train RL algorithm in the simulator on AWS cloud, but it costs money after some free time.
- They hold competitions, both online and in real-world. 1/10<sup>th</sup> scale race car costs USD \$349.



# Params for Writing Reward Function

all\_wheels\_on\_track

x

y

distance\_from\_center

is\_left\_of\_center

is\_reversed

heading

progress

steps

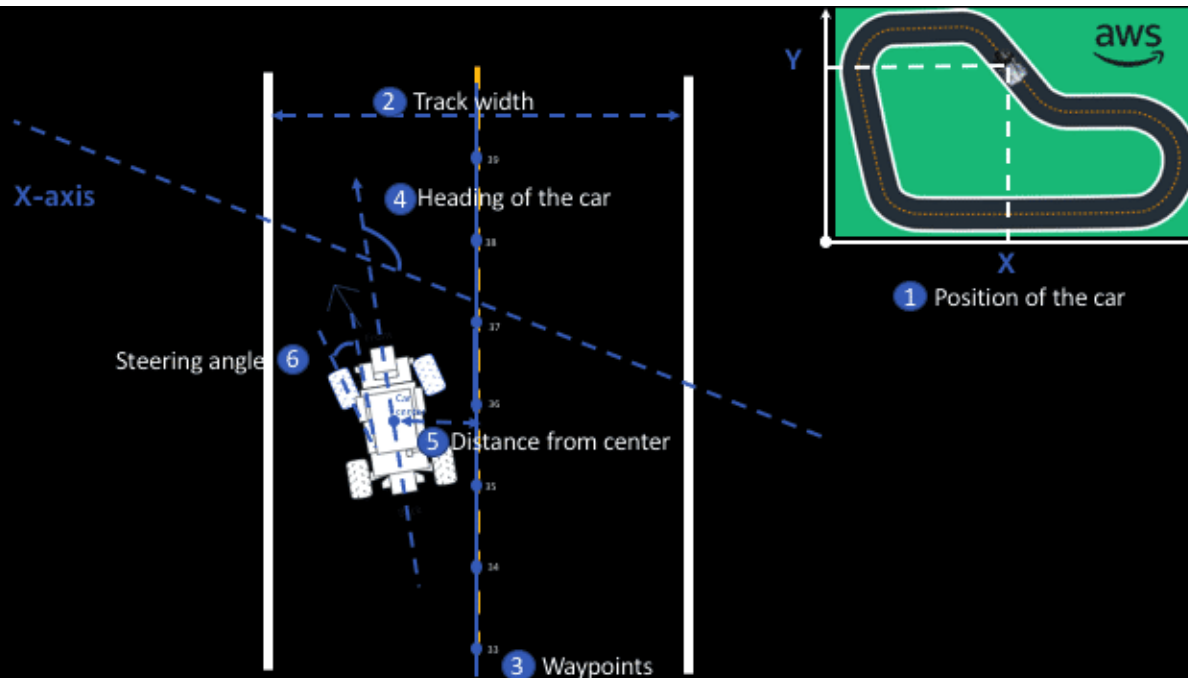
speed

steering\_angle

track\_width

waypoints

closest\_waypoints



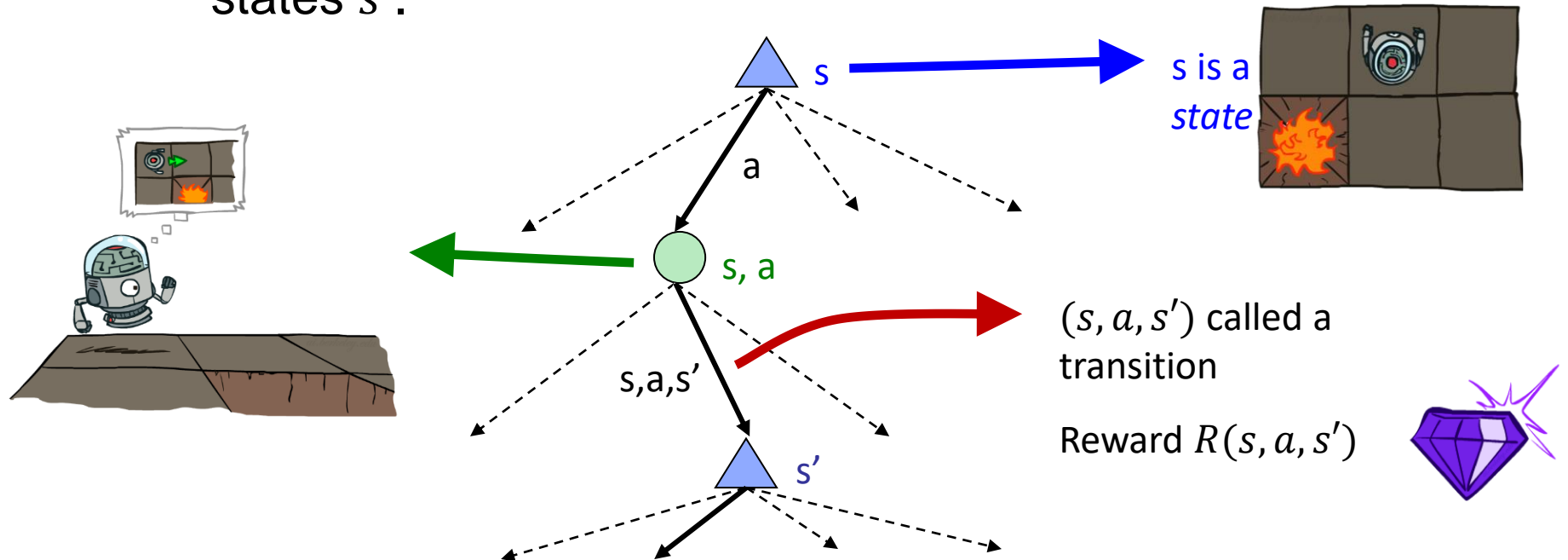
# Example Reward Function

- ```
def reward_function(params):  
    """Example of penalize steering, which helps mitigate zig-zag behaviors"""  
    # Read input parameters  
    distance_from_center = params['distance_from_center']  
    track_width = params['track_width']  
    steering = abs(params['steering_angle']) # Only need the absolute steering angle  
  
    # Calculate 3 markers that are at varying distances away from the center line  
    marker_1 = 0.1 * track_width  
    marker_2 = 0.25 * track_width  
    marker_3 = 0.5 * track_width  
  
    # Give higher reward if the agent is closer to center line and vice versa  
    if distance_from_center <= marker_1:  
        reward = 1  
    elif distance_from_center <= marker_2:  
        reward = 0.5  
    elif distance_from_center <= marker_3:  
        reward = 0.1  
    else:  
        reward = 1e-3 # likely crashed/ close to off track  
  
    # Steering penalty threshold, change the number based on your action space setting  
    ABS_STEERING_THRESHOLD = 15  
  
    # Penalize reward if the agent is steering too much  
    if steering > ABS_STEERING_THRESHOLD:  
        reward *= 0.8  
  
    return float(reward)
```
- A more realistic and complex reward function: <https://www.middleware-solutions.fr/2019/08/14/an-introduction-to-aws-deepracer>



# MDP Search Tree

- Each MDP state  $s$  projects a search tree starting from it.
- In general, both policy and environment may be stochastic
  - Policy  $\pi(a|s)$ : probability distribution over possible actions  $a$  from state  $s$ .
  - Environment  $p(r, s'|s, a)$ : if agent takes action  $a$  in state  $s$ , env gives probability distribution over reward  $r$  and next states  $s'$ .



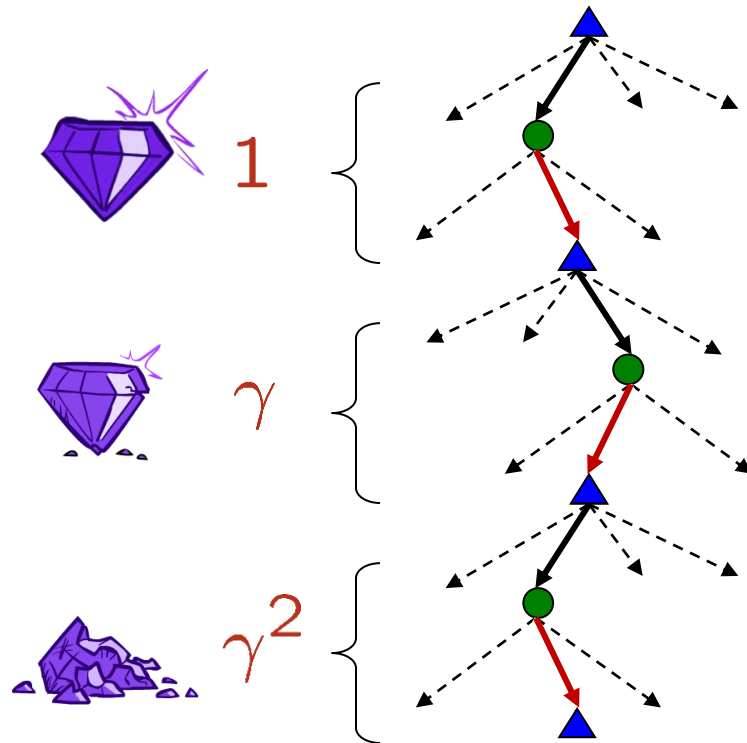
# Preventing Infinite Rewards

- Problem: What if the game lasts forever? Do we get infinite rewards? No. Possible solutions:
- **Finite horizon:** (limit search tree depth)
  - Terminate episodes after a fixed  $T$  timesteps
- **Discount factor:**  $0 < \gamma \leq 1$ 
  - Think of it as a  $1 - \gamma$  chance of ending the episode at every step. Effective horizon (expected episode length):  $\sum_{t=0}^{\infty} \gamma^t = \frac{1}{1-\gamma}$
  - Smaller  $\gamma$  leads to shorter horizon, and preference of short-term to long-term rewards, and vice versa
- (Can have both finite horizon and discount factor)
- **Absorbing state:** guarantee that for every policy, a terminal state will eventually be reached

# Discount Factor Example

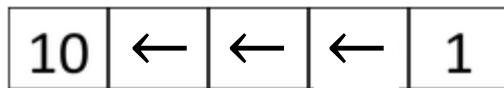
- Each time we descend a level in the search tree, we multiply in the discount once

- Example:  $\gamma = 0.5$ 
  - $U([1,2,3]) = 1 * 1 + 0.5 * 2 + 0.25 * 3 < U([3,2,1])$



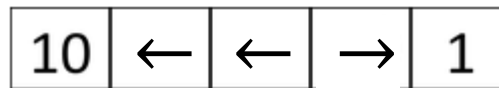
# Discounting Example

- Given:
  - Actions: East, West, and Exit (only available in exit states a, e)
  - Transitions: deterministic
- For  $\gamma = 1$ , optimal policy in each state is always moving West
  - From state d, reward of going West is  $\gamma^3 \cdot 10 = 10$ , larger than reward of going East  $\gamma \cdot 1 = 1$
- For  $\gamma = 0.1$ , optimal policy in each state is shown below
  - From state d, reward from going West is  $\gamma^3 \cdot 10 = 0.01$ , less than reward from going East  $\gamma \cdot 1 = 0.1$ .
- For which  $\gamma$  are West and East equally good when in state d?
  - $\gamma^3 \cdot 10 = \gamma \cdot 1 \Rightarrow \gamma = \frac{1}{\sqrt{10}} \approx .32$



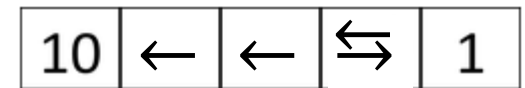
a      b      c      d      e

$$\gamma = 1$$



a      b      c      d      e

$$\gamma = 0.1$$

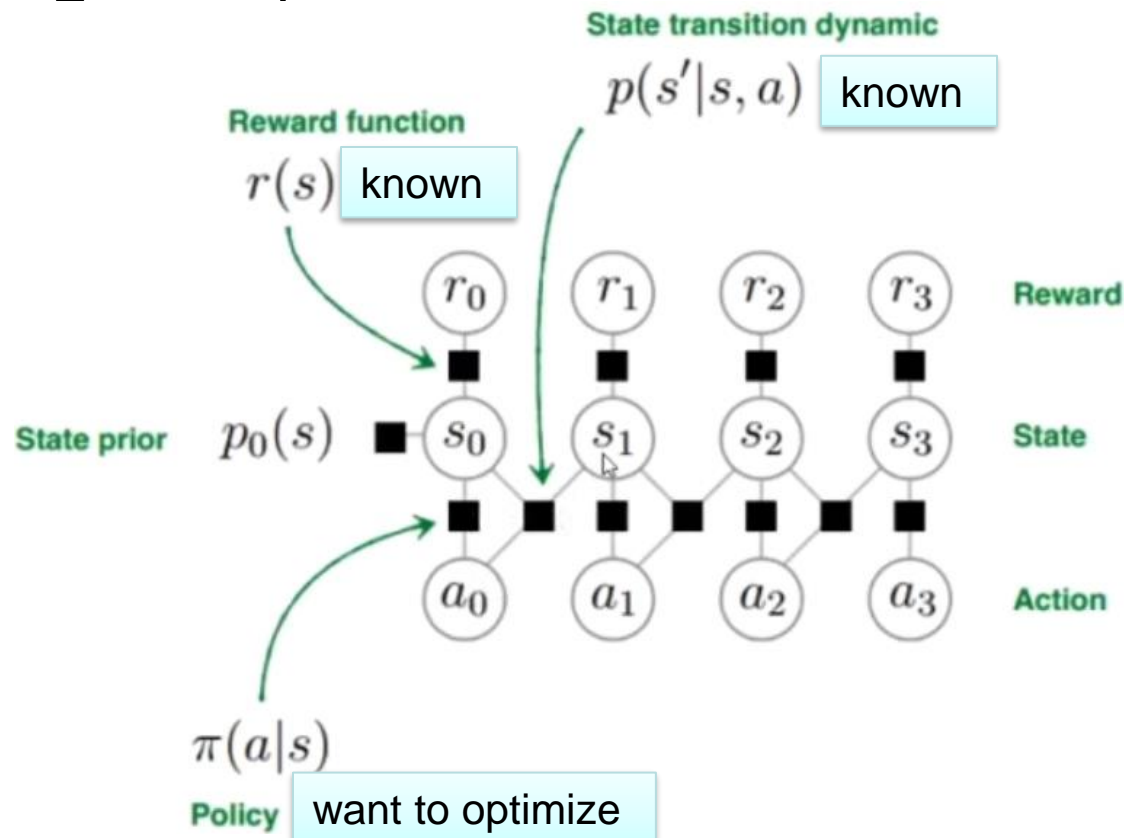


a      b      c      d      e

$$\gamma = \frac{1}{\sqrt{10}}$$

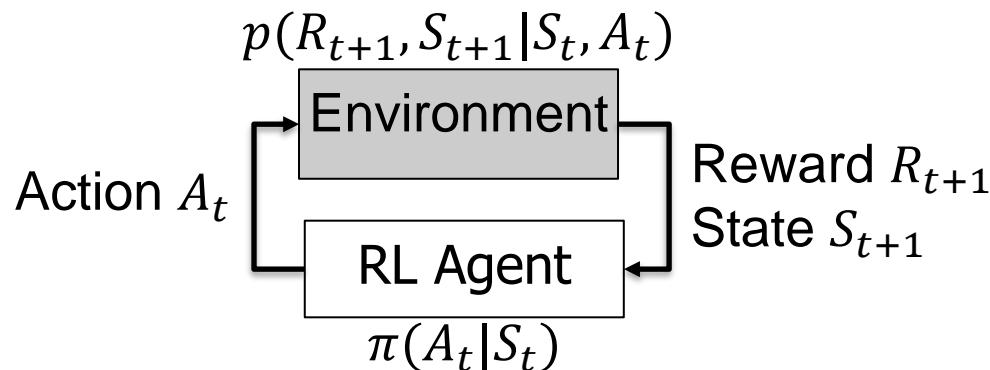
# Known MDP

- In this lecture, we assume known MDP, and use dynamic programming to solve Bellman Equations and find the optimal policy (no learning here).



# Formal Definition of MDP

- **Return** (cumulative discounted reward) at time  $t$ :  $G_t \doteq R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \dots + \gamma^{T-t-1} R_T = \sum_{k=0}^{T-t-1} \gamma^k R_{t+k+1}$ 
  - At each step  $t \in [0, T-1]$ , agent takes an action  $A_t$  in state  $S_t$ ; at step  $t+1$ , agent receives a reward  $R_{t+1}$  and transitions into the next state  $S_{t+1}$  with the trace  $(S_t, A_t, R_{t+1}, S_{t+1})$
  - We assume episodic tasks, and this specific episode has length of  $T$  steps. ( $T = \infty$  for continuous tasks)
- **State Value Function**: expected return under policy  $\pi$ :  $v_\pi(s) \doteq \mathbb{E}_\pi[G_t | S_t = s]$
- **Action Value Function**: expected return from taking action  $a$ , then follow policy  $\pi$ :  $q_\pi(s, a) \doteq \mathbb{E}_\pi[G_t | S_t = s, A_t = a]$
- The RL problem: find the optimal policy  $\pi(a|s)$  that maximizes the expected return from each state (the state VF)



# Bellman Expectation Equations

- Bellman Expectation Equation for State Value Function:
- $v_{\pi}(s) = \sum_a \pi(a|s) \sum_{r,s'} p(r, s'|s, a) [r + \gamma v_{\pi}(s')]$ 
  - Expected value starting from state  $s$  and following policy  $\pi$ .
- Bellman Expectation Equation for Action Value Function
- $q_{\pi}(s, a) = \sum_{r,s'} p(r, s'|s, a) [r + \gamma \sum_{a'} \pi(a'|s') q_{\pi}(s', a')]$ 
  - Expected value starting from state  $s$ , taking action  $a$ , and thereafter following policy  $\pi$ .

Important

# Bellman Optimality Equations

- Bellman Optimality Equation for State Value Function:
- $v_*(s) = \max_a \sum_{r,s'} p(r, s'|s, a) [r + \gamma v_*(s')]$ 
  - Max value starting from state  $s$  and following the greedy policy  $\pi(s) = \operatorname{argmax}_a q_*(s, a)$
- Bellman Optimality Equation for Action Value Function
- $q_*(s, a) = \sum_{r,s'} p(r, s'|s, a) \left[ r + \gamma \max_{a'} q_*(s', a') \right]$ 
  - Max value starting from state  $s$ , taking action  $a$ , and thereafter following the greedy policy  $\pi(s) = \operatorname{argmax}_a q_*(s, a)$

$$V(s) = \max_a (R(s, a) + \gamma \sum_{s'} T(s, a, s') V(s'))$$

$$V(s) = \max_{a_1} (R(s_1, a_1) + \gamma \sum_{s_2} T(s_1, a_1, s_2) \max_{a_2} (R(s_2, a_2) + \gamma \sum_{s_3} T(s_2, a_2, s_3) \dots))$$

$$Q(s, a) = R(s, a) + \gamma \sum_{s'} T(s, a, s') \max_{a'} Q(s', a')$$

- Notations in left fig:
- $\sum_{s'} T(s, a, s') [\dots] = \sum_{r,s'} p(r, s'|s, a) [\dots]$
- $R(s, a) = \sum_{r,s'} p(r, s'|s, a) r$

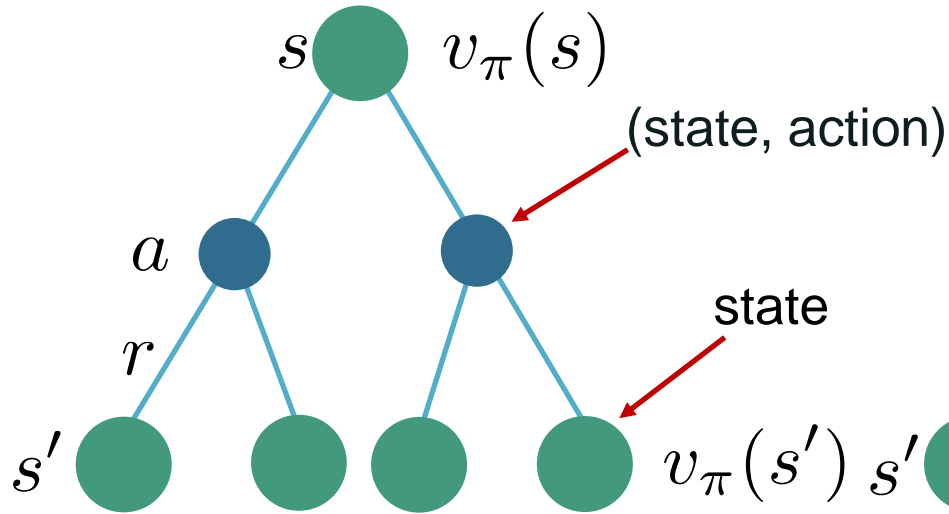


# Bellman Equations written with Expectation Symbols

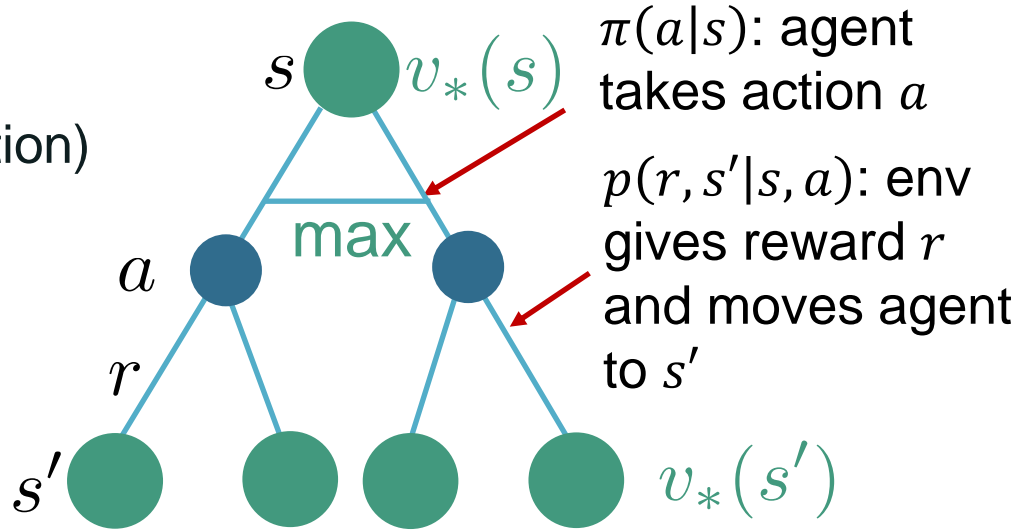
- Bellman Exp Equations:
- $v_{\pi}(s) = \mathbb{E}_a \mathbb{E}_{r, s'} [r + \gamma v_{\pi}(s')]$
- $q_{\pi}(s, a) = \mathbb{E}_{r, s'} [r + \gamma \mathbb{E}_a q_{\pi}(s, a)]$
- Bellman Opt Equations:
- $v_*(s) = \max_a \mathbb{E}_{r, s'} [r + \gamma v_*(s')]$
- $q_*(s, a) = \mathbb{E}_{r, s'} \left[ r + \gamma \max_a q_*(s, a) \right]$
- Detailed derivations:
  - $v_{\pi}(s) = \sum_a \pi(a|s) q_{\pi}(s, a) = \mathbb{E}_{a \sim \pi(a|s)} q_{\pi}(s, a) = \mathbb{E}_{a \sim \pi(a|s)} \mathbb{E}_{r, s' \sim p(r, s'|s, a)} [r + \gamma v_{\pi}(s')]$
  - $q_{\pi}(s, a) = \sum_{r, s'} p(r, s'|s, a) [r + \gamma v_{\pi}(s')] = \mathbb{E}_{r, s' \sim p(r, s'|s, a)} [r + \gamma v_{\pi}(s')] = \mathbb{E}_{r, s' \sim p(r, s'|s, a)} [r + \gamma \mathbb{E}_{a \sim \pi(a|s)} q_{\pi}(s, a)]$
  - $v_*(s) = \max_a q_*(s, a) = \max_a \mathbb{E}_{r, s' \sim p(r, s'|s, a)} [r + \gamma v_*(s')]$
  - $q_*(s, a) = \sum_{r, s'} p(r, s'|s, a) [r + \gamma v_*(s')] = \mathbb{E}_{r, s' \sim p(r, s'|s, a)} [r + \gamma v_*(s')]$

Important

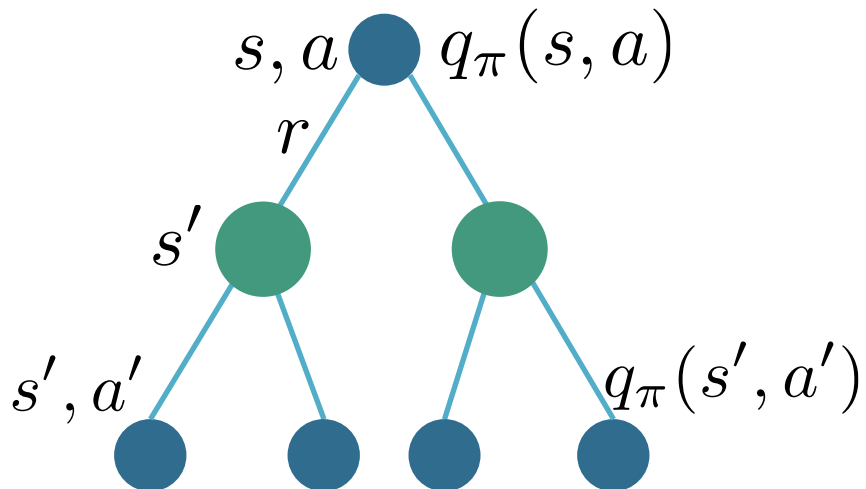
# Backup Diagrams



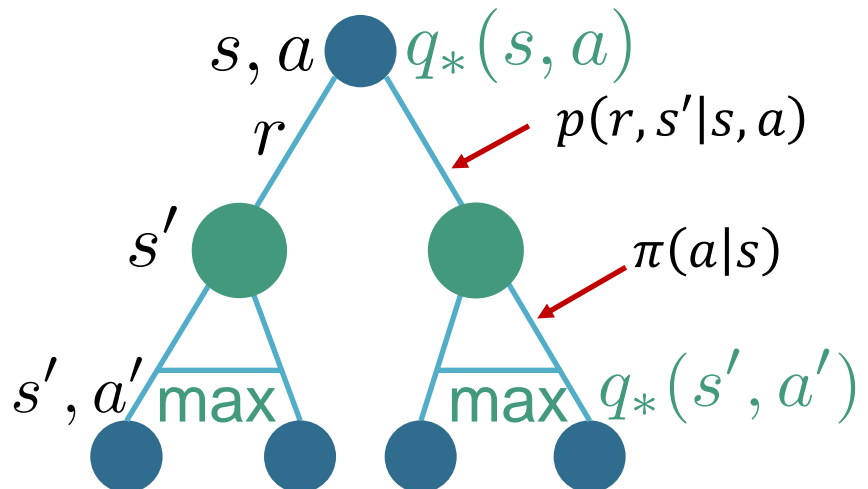
Bellman Exp Eqn for  $v_\pi(s)$



Bellman Opt Eqn for  $v_*(s)$



Bellman Exp Eqn for  $q_\pi(s, a)$



Bellman Opt Eqn for  $q_*(s, a)$

## $v(s)$ vs. $q(s, a)$

- State-action Value Function  $q(s, a)$  contains more information than State value function  $v(s)$ . Given  $q_*(s, a)$ , optimal policy  $\pi_*(s) = \underset{a}{\operatorname{argmax}} q_*(s, a)$ .
- **Can always** go from  $q_\pi(s, a)$  to  $v_\pi(s)$ , or from  $q_*(s, a)$  to  $v_*(s)$ :
  - $v_\pi(s) = \sum_a \pi(a|s) q_\pi(s, a)$ ;  $v_*(s) = \max_a q_*(s, a)$
- With known MDP ( $p(r, s'|s, a)$ , i.e., model-based): **can** go from  $v_\pi(s)$  to  $q_\pi(s, a)$ , or from  $v_*(s)$  to  $q_*(s, a)$ :
  - $q_\pi(s, a) = \sum_{r, s'} p(r, s'|s, a) [r + \gamma v_\pi(s')]$
  - $q_*(s, a) = \sum_{r, s'} p(r, s'|s, a) [r + \gamma v_*(s')]$
- With unknown MDP (unknown  $p(r, s'|s, a)$ , i.e., model-free) : **cannot** go from  $v_\pi(s)$  to  $q_\pi(s, a)$ , or from  $v_*(s)$  to  $q_*(s, a)$

# Simplified Bellman Equations for Deterministic Env

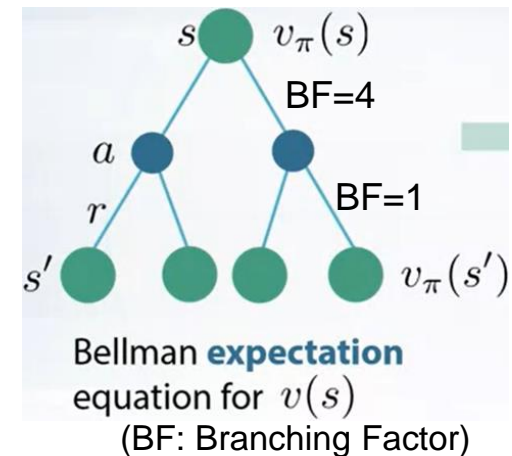
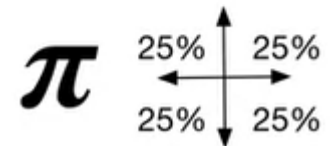
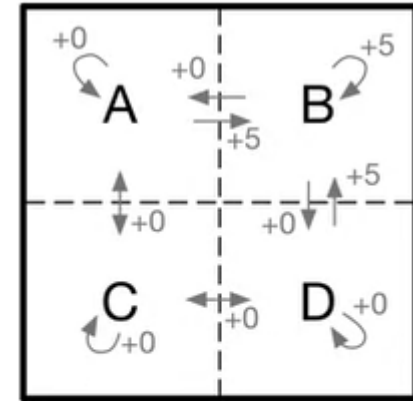
- Bellman Equations:
  - $v_{\pi}(s) = \sum_a \pi(a|s) q_{\pi}(s, a)$
  - $q_{\pi}(s, a) = \sum_{r, s'} p(r, s'|s, a) [r + \gamma v_{\pi}(s')]$
  - $v_*(s) = \max_a q_*(s, a)$
  - $q_*(s, a) = \sum_{r, s'} p(r, s'|s, a) [r + \gamma v_*(s')]$
- For Deterministic Env: there is only one possible  $(r, s')$  for a given  $(s, a)$  (we use  $R_s^a$  to emphasize that reward  $r$  is specific to this  $(s, a)$ ):
  - $q_{\pi}(s, a) = R_s^a + \gamma v_{\pi}(s')$
  - $q_*(s, a) = R_s^a + \gamma v_*(s')$

# Policy Evaluation

- The prediction problem: predict Value Function for given policy  $\pi$  by solving Bellman Exp. Equation for State Value Function
  - $v_{\pi}(s) = \sum_a \pi(a|s) \sum_{r,s'} p(r, s'|s, a) [r + \gamma v_{\pi}(s')] = \mathbb{E}_{\pi}[R_{t+1} + \gamma v_{\pi}(S_{t+1}) | S_t = s]$
- Can also be written as:
  - $v_{\pi}(s) = \sum_a \pi(a|s) q_{\pi}(s, a)$
  - $q_{\pi}(s, a) = \sum_{r,s'} p(r, s'|s, a) [r + \gamma v_{\pi}(s')]$  denotes the State-Action Value Function for taking action  $a$  in state  $s$ , then follow policy  $\pi$  afterwards
- A set of linear equations that can be solved analytically for small system
  - # unknowns = # equations = # states

# Grid World1: Policy Evaluation

- Non-episodic MDP w. **deterministic env**: Agent in state  $s \in \{A, B, C, D\}$  taking action  $a \in \{l, r, u, d\}$  always moves to the next state in the movement direction, unless it is blocked by the walls. Discount factor  $\gamma = 0.7$ .
- Random policy**: Agent in state  $s \in \{A, B, C, D\}$  takes a random action  $a \in \{l, r, u, d\}$  with equal probability of 0.25 each.
- Bellman Exp. Equation for det env:  $v_\pi(s) = \sum_a \pi(a|s) q_\pi(s, a)$ ;  $q_\pi(s, a) = R_s^a + \gamma v_\pi(s')$
- $v_\pi(A) = 0.25(q_\pi(A, l) + q_\pi(A, r) + q_\pi(A, u) + q_\pi(A, d)) = 0.5 \cdot 0.7v_\pi(A) + 0.25 \cdot (5 + 0.7v_\pi(B)) + 0.25 \cdot 0.7v_\pi(C)$ 
  - $q_\pi(A, l) = q_\pi(A, u) = 0 + 0.7v_\pi(A)$
  - $q_\pi(A, r) = 5 + 0.7v_\pi(B)$
  - $q_\pi(A, d) = 0 + 0.7v_\pi(C)$
- $v_\pi(B) = 0.25(q_\pi(B, l) + q_\pi(B, r) + q_\pi(B, u) + q_\pi(B, d)) = 0.25 \cdot 0.7v_\pi(A) + 0.5 \cdot (5 + 0.7v_\pi(B)) + 0.25 \cdot 0.7v_\pi(D)$ 
  - $q_\pi(B, l) = 0 + 0.7v_\pi(A)$
  - $q_\pi(B, r) = q_\pi(A, u) = 5 + 0.7v_\pi(B)$
  - $q_\pi(B, d) = 0 + 0.7v_\pi(D)$
- $v_\pi(C) = 0.25(q_\pi(C, l) + q_\pi(C, r) + q_\pi(C, u) + q_\pi(C, d)) = 0.25 \cdot 0.7v_\pi(A) + 0.5 \cdot 0.7v_\pi(C) + 0.25 \cdot 0.7v_\pi(D)$ 
  - $q_\pi(C, l) = q_\pi(C, d) = 0 + 0.7v_\pi(C)$
  - $q_\pi(C, r) = 0 + 0.7v_\pi(D)$
  - $q_\pi(C, u) = 0 + 0.7v_\pi(A)$
- $v_\pi(D) = 0.25(q_\pi(D, l) + q_\pi(D, r) + q_\pi(D, u) + q_\pi(D, d)) = 0.25 \cdot (5 + 0.7v_\pi(B)) + 0.25 \cdot 0.7v_\pi(C) + 0.5 \cdot 0.7v_\pi(D)$ 
  - $q_\pi(D, l) = 0 + 0.7v_\pi(C)$
  - $q_\pi(D, r) = q_\pi(D, d) = 0 + 0.7v_\pi(D)$
  - $q_\pi(D, u) = 5 + 0.7v_\pi(B)$
- Solution**:  $v_\pi(A) = 4.2, v_\pi(B) = 6.1, v_\pi(C) = 2.2, v_\pi(D) = 4.2$ .  $q_\pi(s, a)$  can also be obtained.



# Iterative Policy Evaluation

- If # states are too large, it may be too expensive to analytically solve the set of linear Bellman Exp. Equations. We can use Iterative Policy Evaluation and solve the recursive Bellman equations iteratively:
  - $v(s) = \text{Bellman}(v(s')) \Rightarrow v_k(s) \leftarrow \text{Bellman}(v_{k-1}(s'))$
- In-place updates:
  - For faster convergence, update  $v_k(s)$  using the value of  $v_k(s')$  that has been updated in the same iteration, instead of  $v_{k-1}(s')$  from the previous iteration.
  - $v_k(s) \leftarrow \text{Bellman}(v_k(s'))$

## Iterative Policy Evaluation, for estimating $V \approx v_\pi$

Input  $\pi$ , the policy to be evaluated

Algorithm parameter: a small threshold  $\theta > 0$  determining accuracy of estimation

Initialize  $V(s)$ , for all  $s \in \mathcal{S}^+$ , arbitrarily except that  $V(\text{terminal}) = 0$

Loop:

$\Delta \leftarrow 0$

Loop for each  $s \in \mathcal{S}$ :

$v \leftarrow V(s)$

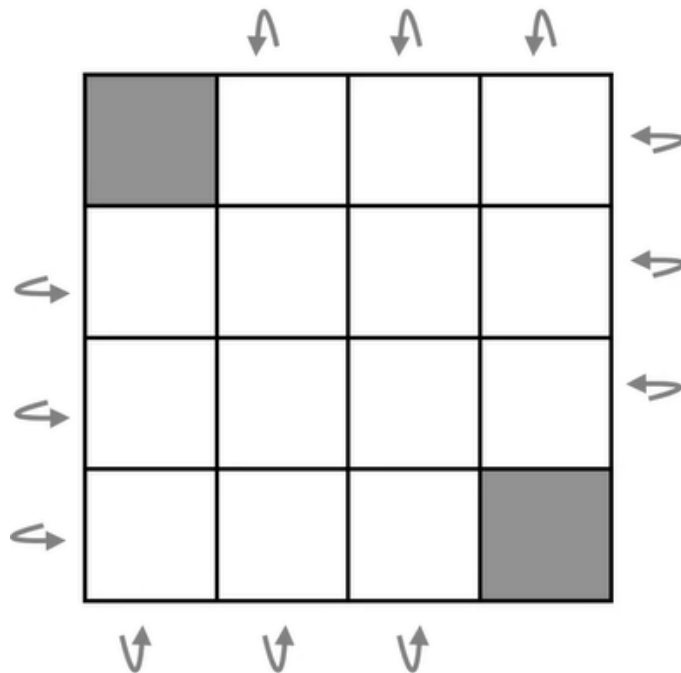
$V(s) \leftarrow \sum_a \pi(a|s) \sum_{s',r} p(s',r|s,a) [r + \gamma V(s')]$

$\Delta \leftarrow \max(\Delta, |v - V(s)|)$

until  $\Delta < \theta$

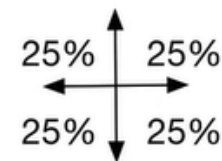
# Iterative Policy Evaluation Example

- An episodic MDP with terminal states with  $v(s) = 0$  located in the top left and bottom right corners. Reward  $R = -1$  for every transition (agent is punished for delays before reaching the terminal state and ending the episode). Discount factor  $\gamma = 1$ . Four possible actions in each state: up, down, left, and right. Environment is deterministic. If the action would move the agent off the grid, it instead leaves the agent in the same state.
- **Random policy:** Agent in any state  $s$  takes a random action  $a \in \{l, r, u, d\}$  with equal probability of 0.25 each.



$$R = -1$$

$$\gamma = 1$$



**Policy**



# Iterative Policy Evaluation Example

- For fixed random policy:
- Bellman Exp. Equation for Det Env:  $v_{\pi}(s) = \sum_a \pi(a|s)q_{\pi}(s, a)$ ;  $q_{\pi}(s, a) = R_s^a + \gamma v_{\pi}(s')$
- 1<sup>st</sup> sweep:
  - for all states  $v_{\pi}(s) = .25((-1 + 0) + (-1 + 0) + (-1 + 0) + (-1 + 0)) = -1$
- 2<sup>nd</sup> sweep:
  - For states marked -2:  $v_{\pi}(s) = .25((-1 - 1) + (-1 - 1) + (-1 - 1) + (-1 - 1)) = -2$
  - For states marked -1.7:  $v_{\pi}(s) = .25((-1 + 0) + (-1 - 1) + (-1 - 1) + (-1 - 1)) = -1.75$
- 3<sup>rd</sup> sweep:
  - For states marked -2.4:  $v_{\pi}(s) = .25((-1 + 0) + (-1 - 2) + (-1 - 1.75) + (-1 - 2)) = -2.43$
  - For states marked -2.9:  $v_{\pi}(s) = .25((-1 - 1.7) + (-1 - 2) + (-1 - 1.75) + (-1 - 2)) = -2.85$
  - For states marked -3:  $v_{\pi}(s) = .25((-1 - 2) + (-1 - 2) + (-1 - 2) + (-1 - 2)) = -3$
- Influence of terminal states gradually spreads through the entire grid.

|   |   |   |   |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 |

Initial

|    |    |    |    |
|----|----|----|----|
| 0  | -1 | -1 | -1 |
| -1 | -1 | -1 | -1 |
| -1 | -1 | -1 | -1 |
| -1 | -1 | -1 | 0  |

After 1<sup>st</sup> sweep

|      |      |      |      |
|------|------|------|------|
| 0    | -1.7 | -2   | -2   |
| -1.7 | -2   | -2   | -2   |
| -2   | -2   | -2   | -1.7 |
| -2   | -2   | -1.7 | 0    |

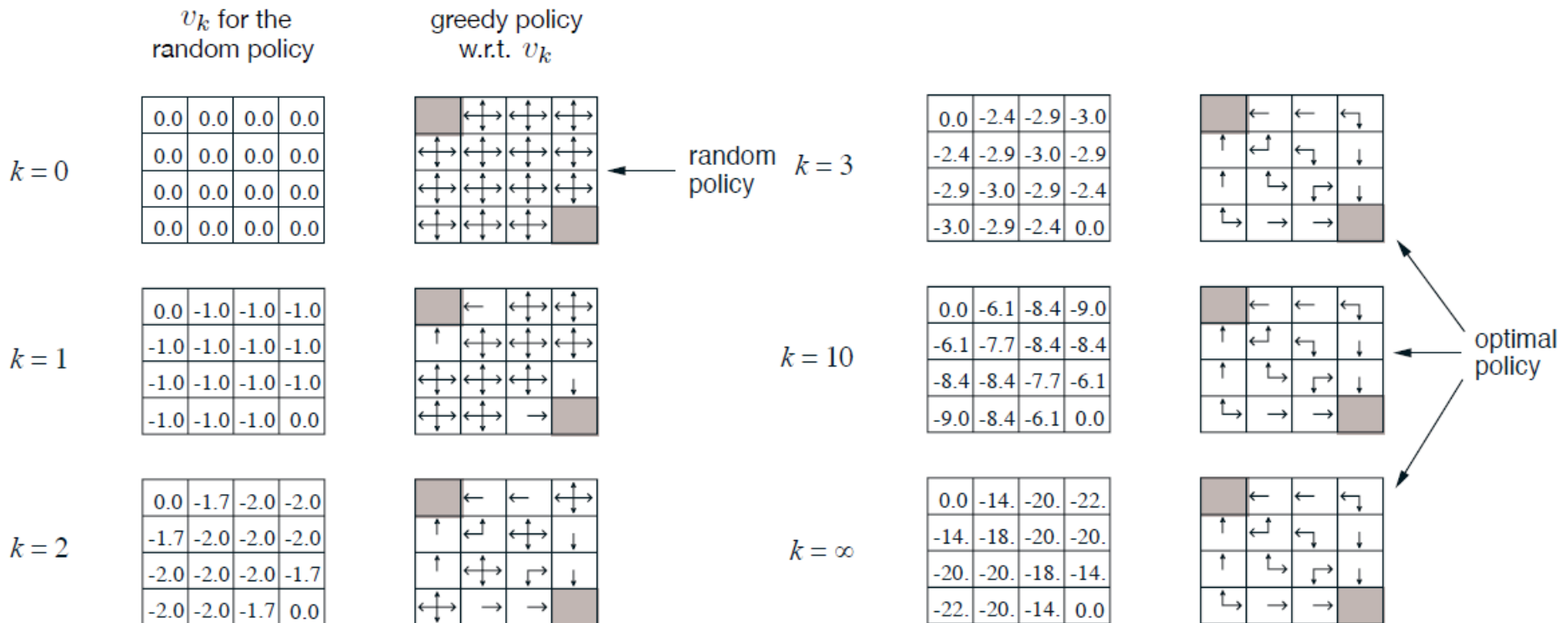
After 2<sup>nd</sup> sweep

|      |      |      |      |
|------|------|------|------|
| 0    | -2.4 | -2.9 | -3   |
| -2.4 | -2.9 | -3   | -2.9 |
| -2.9 | -3   | -2.9 | -2.4 |
| -3   | -2.9 | -2.4 | 0    |

After 3<sup>rd</sup> sweep

# Iterative Policy Evaluation Results

- Figure 4.1: Convergence of iterative policy evaluation on a small gridworld with the random policy (all actions equally likely). The left column is the sequence of approximations of the state-value function. The right column is the sequence of greedy policies corresponding to the value function  $v_\pi(s)$  estimates.
- Note that we are not updating the policy (always the random policy) across iterations. If you follow the greedy action at the current step, then follow the random policy in the future, then it is better than following the random policy from the current step.



# Policy Iteration

Policy Iteration (using iterative policy evaluation) for estimating  $\pi \approx \pi_*$

## 1. Initialization

$V(s) \in \mathbb{R}$  and  $\pi(s) \in \mathcal{A}(s)$  arbitrarily for all  $s \in \mathcal{S}$

## 2. Policy Evaluation

Loop:

$\Delta \leftarrow 0$

Loop for each  $s \in \mathcal{S}$ :

$v \leftarrow V(s)$

$V(s) \leftarrow \sum_{s',r} p(s',r|s,\pi(s)) [r + \gamma V(s')]$

$\Delta \leftarrow \max(\Delta, |v - V(s)|)$

until  $\Delta < \theta$  (a small positive number determining the accuracy of estimation)

Repeat until policy converges:

Policy Evaluation: Estimate state value function  $v_\pi$  for some fixed policy  $\pi$  with Iterative Policy Evaluation (or solving linear equations).

Policy Improvement: generate new policy based on the newly estimated  $v_\pi$ :  $\pi = \text{greedy}(v_\pi)$ .

## 3. Policy Improvement

*policy-stable*  $\leftarrow$  true

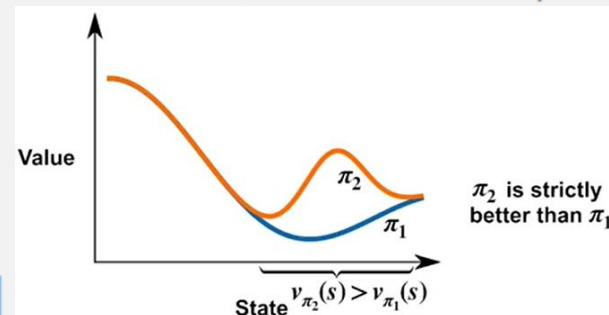
For each  $s \in \mathcal{S}$ :

*old-action*  $\leftarrow \pi(s)$

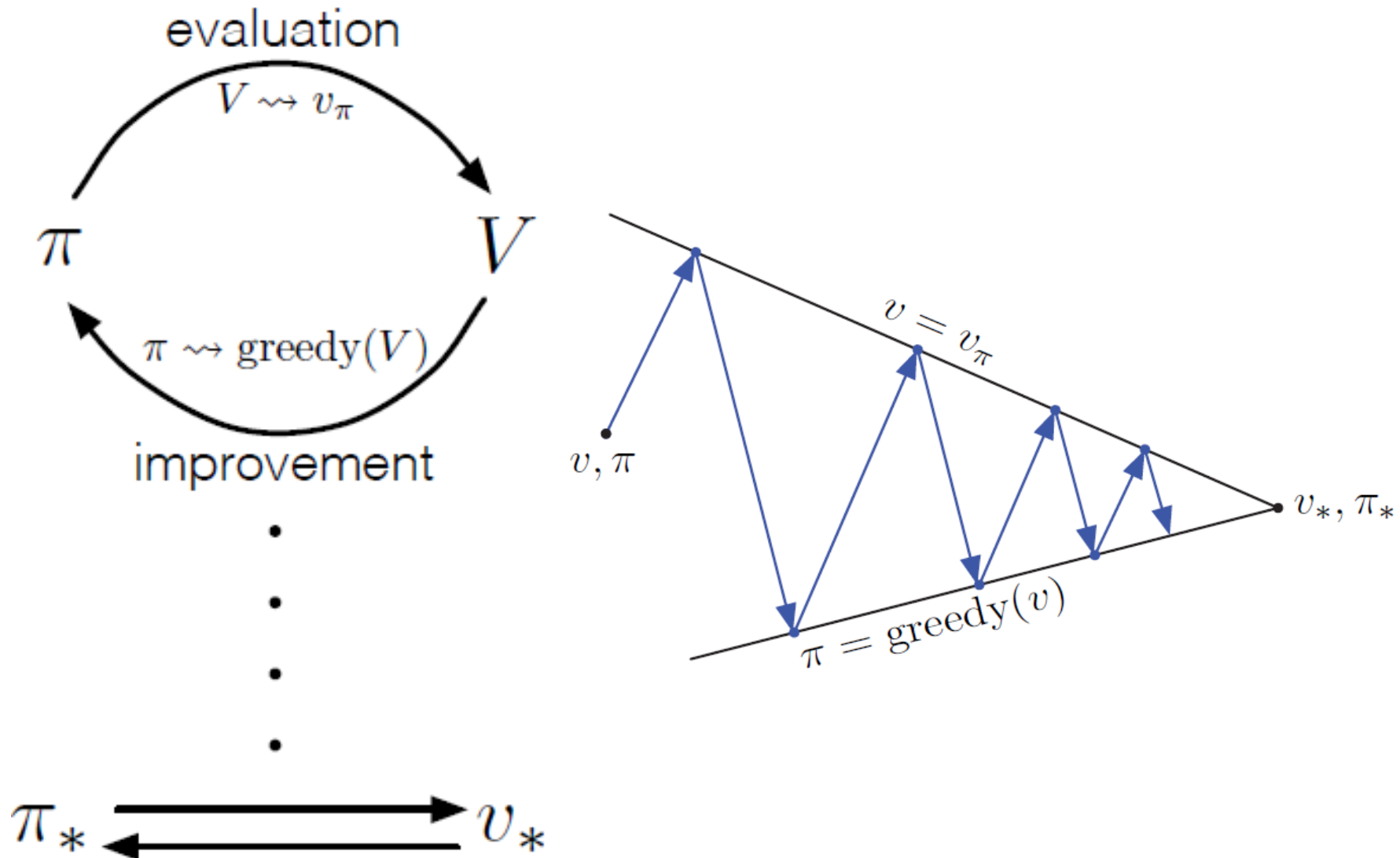
$\pi(s) \leftarrow \arg\max_a \sum_{s',r} p(s',r|s,a) [r + \gamma V(s')]$

If *old-action*  $\neq \pi(s)$ , then *policy-stable*  $\leftarrow$  false

If *policy-stable*, then stop and return  $V \approx v_*$  and  $\pi \approx \pi_*$ ; else go to 2

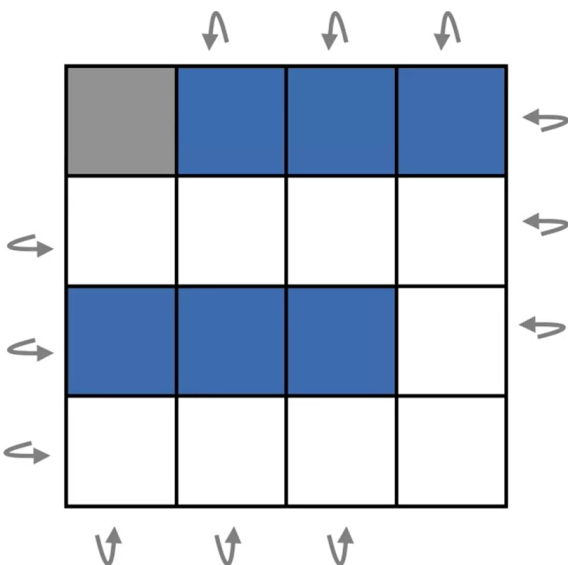


# Policy Iteration Illustration



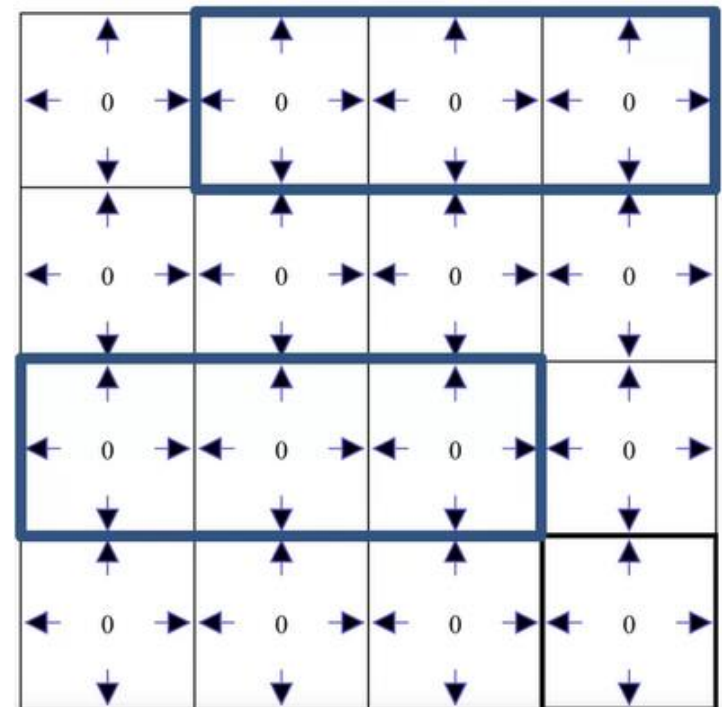
# Policy Iteration Example

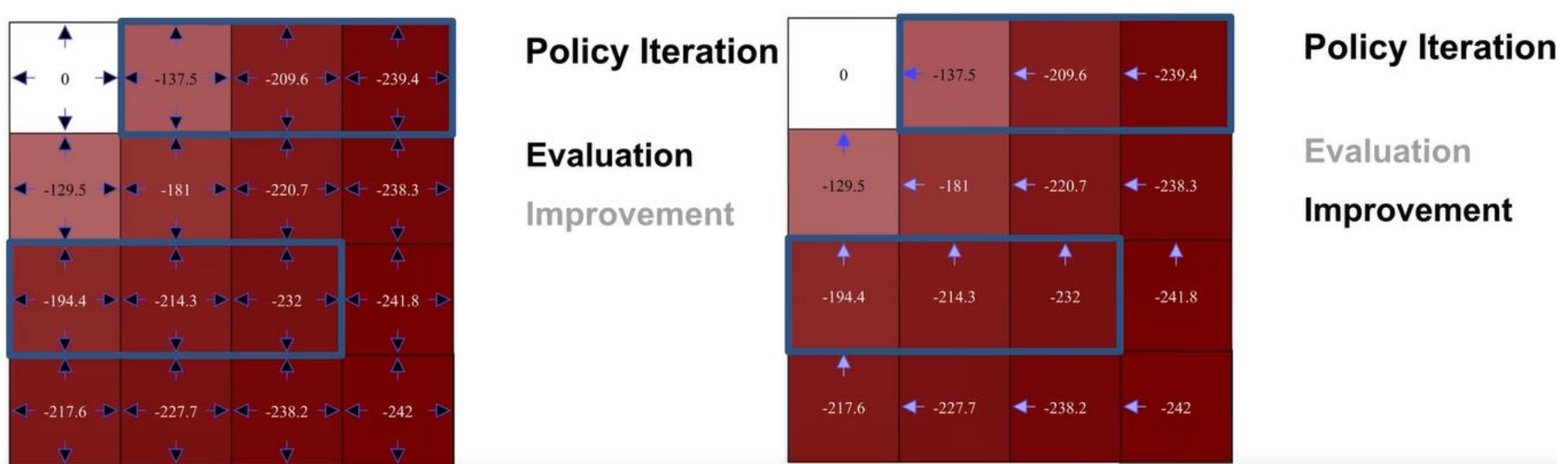
- An episodic MDP with terminal states with  $v(s) = 0$  located in the top left corner. Blue states are bad states with large negative reward.
- Start with uniform random policy.
- Initialize  $v(s) = 0$  for all  $s$ .



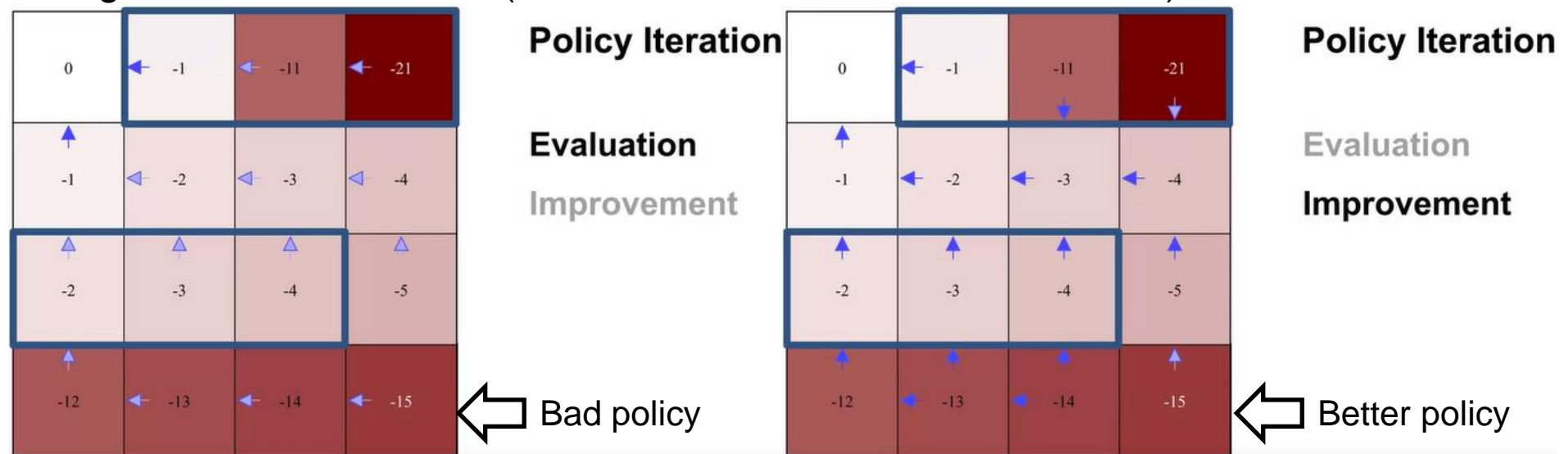
$$R = \begin{cases} -10 & \text{Blue states} \\ -1 & \text{Other states} \end{cases}$$

$$\gamma = 1$$

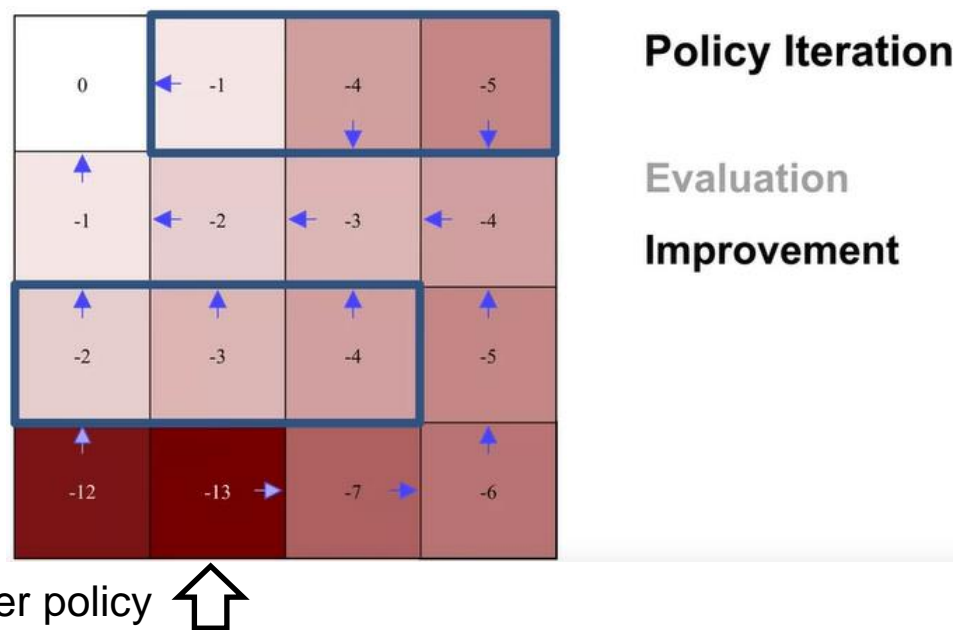
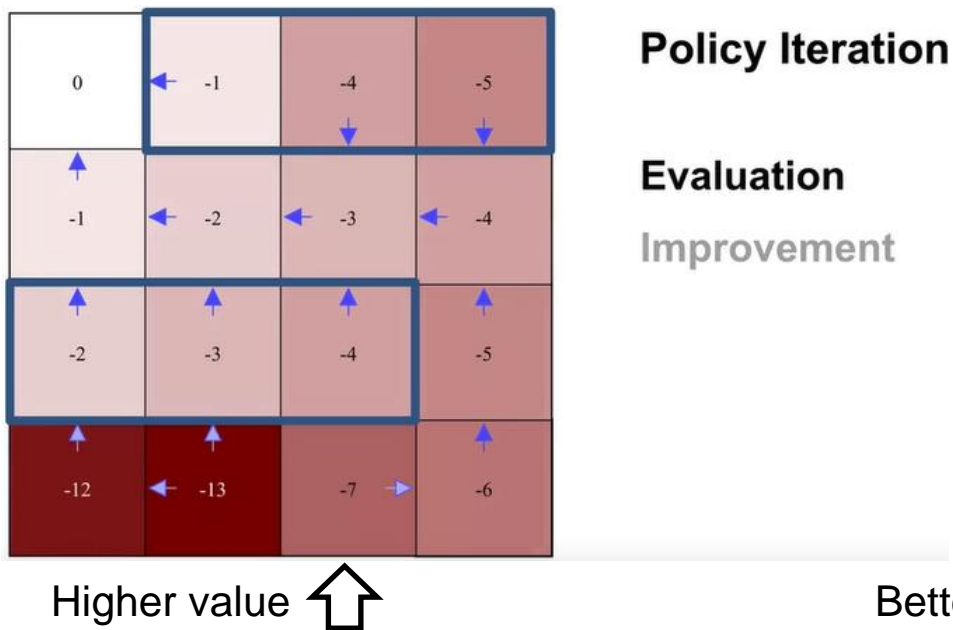
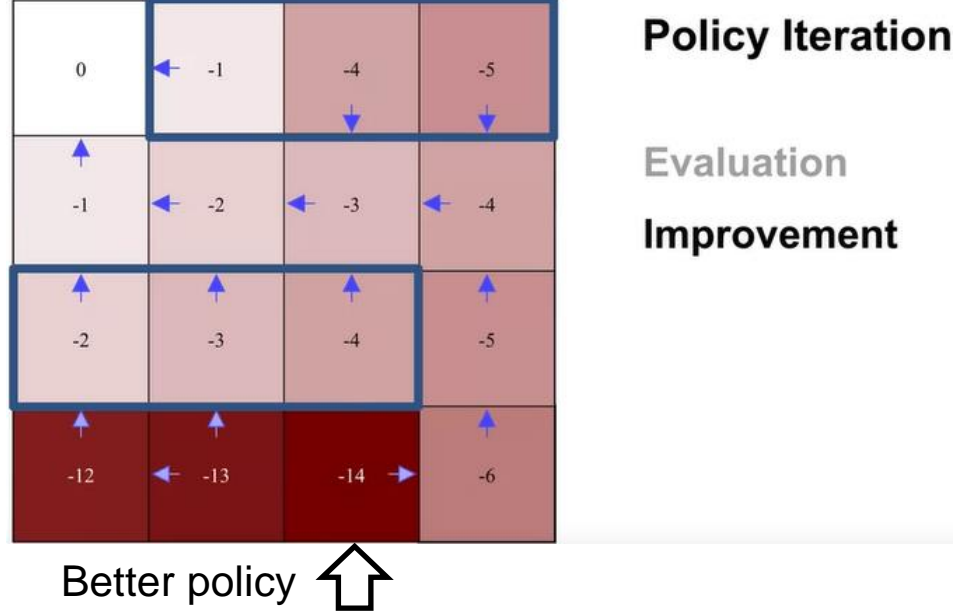
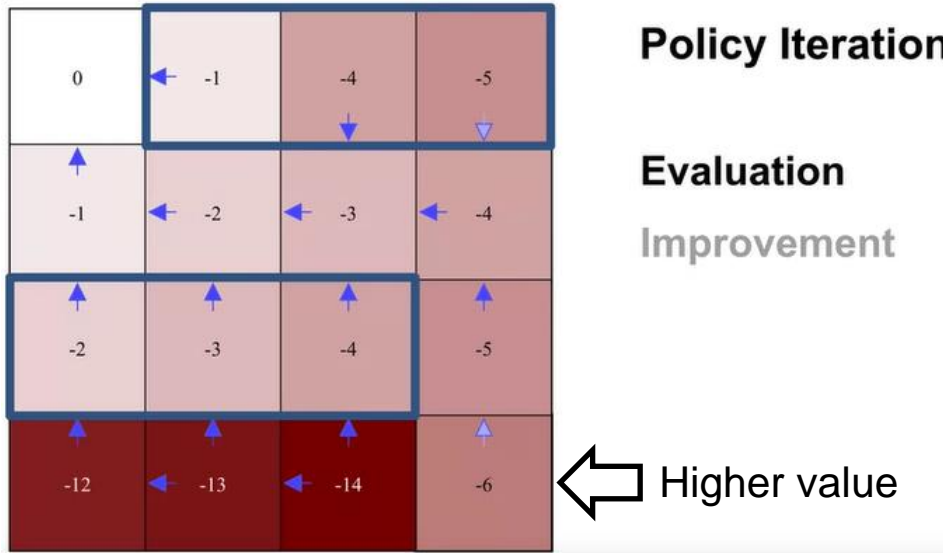


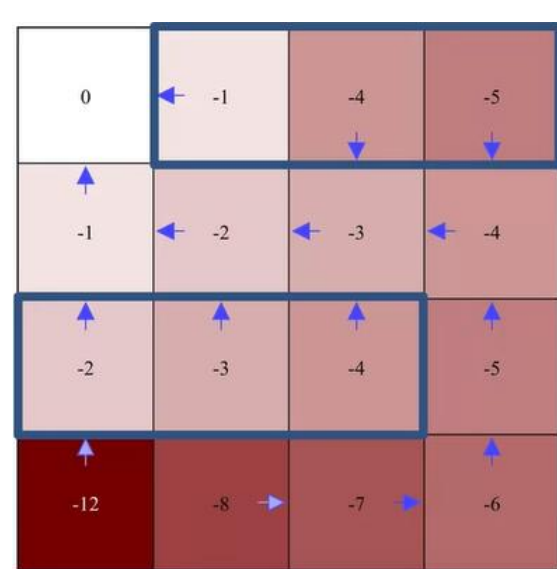


After 1<sup>st</sup> sweep of PE+PI. Most state's policies are not good since they go towards blue states (darker colors denote low-value states).



After 2<sup>nd</sup> sweep of PE+PI. Policies are much better, but a few states' policies still go towards blue states.

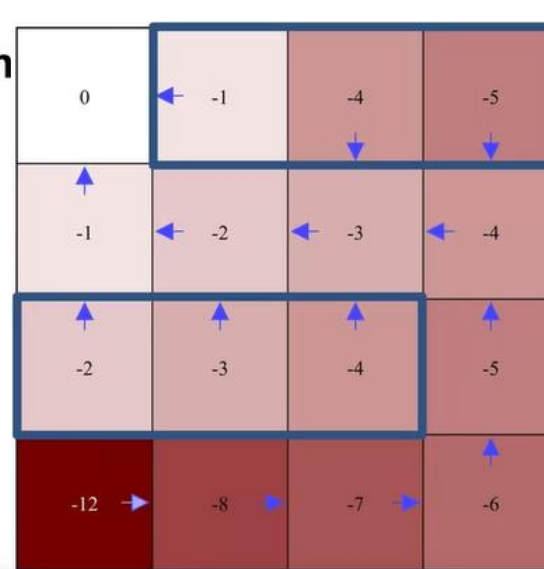




↑ Higher value

Policy Iteration

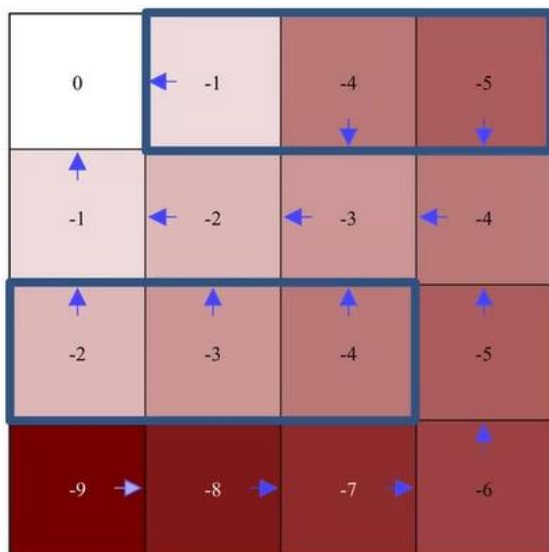
Evaluation  
Improvement



Better policy ↑

Policy Iteration

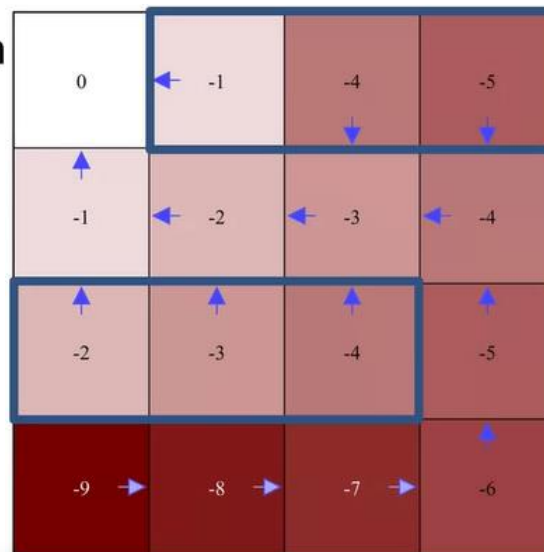
Evaluation  
Improvement



Value functions converged  
(May not converge in general. Policy often  
stable long before value functions converge.)

Policy Iteration

Evaluation  
Improvement



Policy stable, hence it is optimal policy)

Policy Iteration

Evaluation  
Improvement



# Another PI Example

- The greedy policy converges to the optimal policy after 5<sup>th</sup> iteration of Policy Iteration.
- Precise values of  $v(s)$  are not necessary for computing optimal policy with Policy Iteration.



# Value Iteration

- Obtain optimal Value Function by solving Bellman Opt. Equation for Optimal State Value Function iteratively (analytical solution is not possible due to the max operator)
  - $v_*(s) = \max_a \sum_{r,s'} p(r,s'|s,a) [r + \gamma v_*(s')]$
- Optimal policy  $\pi_*(s)$  is output at the end; intermediate value function  $v(s)$  may not correspond to any valid policy.

## Value Iteration, for estimating $\pi \approx \pi_*$

Algorithm parameter: a small threshold  $\theta > 0$  determining accuracy of estimation  
Initialize  $V(s)$ , for all  $s \in \mathcal{S}^+$ , arbitrarily except that  $V(\text{terminal}) = 0$

Loop:

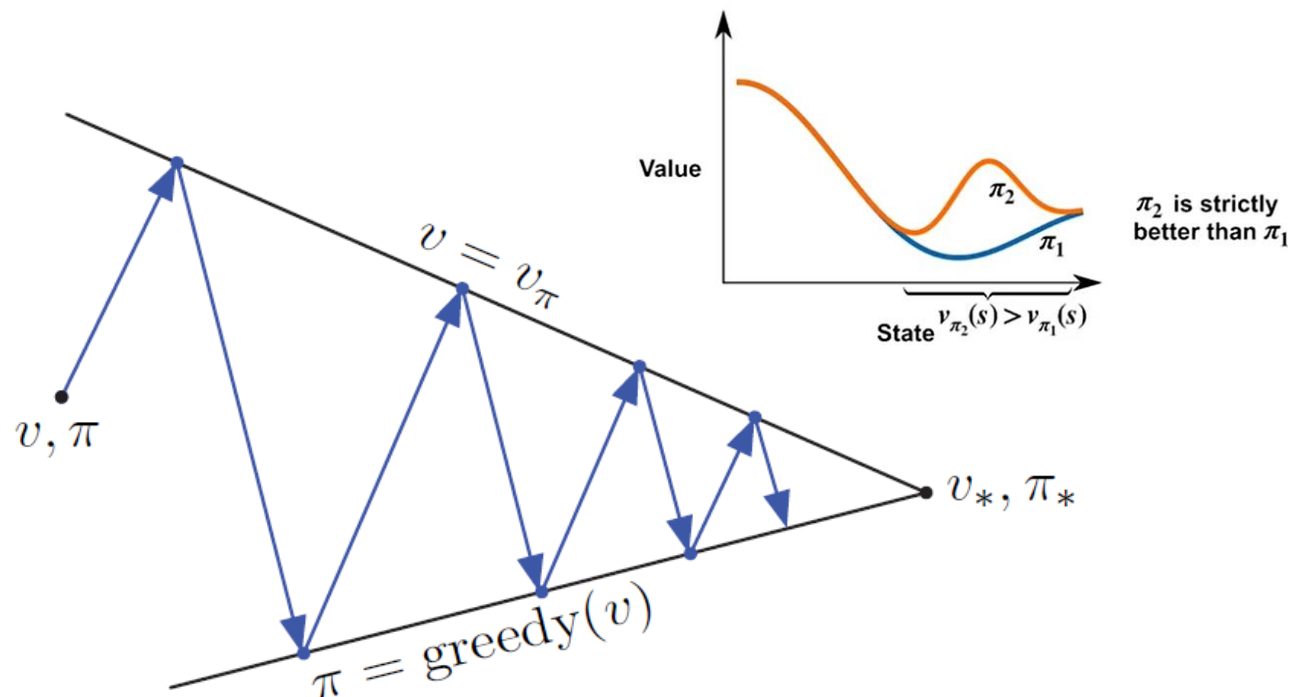
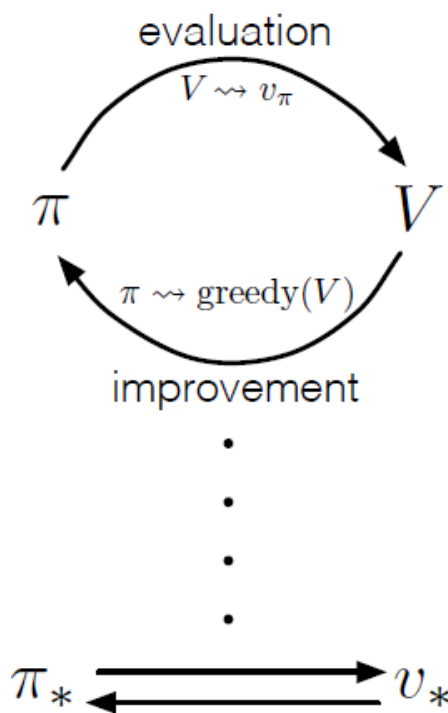
```
|  $\Delta \leftarrow 0$   
| Loop for each  $s \in \mathcal{S}$ :  
|    $v \leftarrow V(s)$   
|    $V(s) \leftarrow \max_a \sum_{s',r} p(s',r|s,a) [r + \gamma V(s')]$   
|    $\Delta \leftarrow \max(\Delta, |v - V(s)|)$   
until  $\Delta < \theta$ 
```

Output a deterministic policy,  $\pi \approx \pi_*$ , such that

$$\pi(s) = \operatorname{argmax}_a \sum_{s',r} p(s',r|s,a) [r + \gamma V(s')]$$

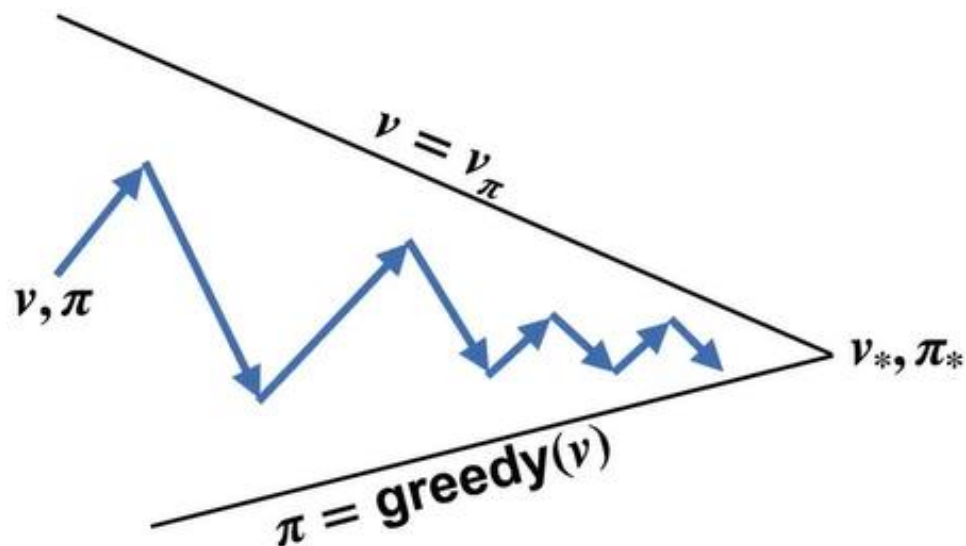
# Generalized Policy Iteration (GPI)

- GPI: 1. evaluate given policy; 2. Improve policy by acting greedily w.r.t its value function. GPI is guaranteed to converge to the optimal deterministic policy  $\pi^*$ .
- Two special cases of GPI:
- Policy Iteration (PI) (shown below):
  - 1. Policy evaluation until convergence; 2. Improve policy
- Value Iteration (VI):
  - 1. Policy evaluation with one single iteration; 2. improve policy (implicitly)



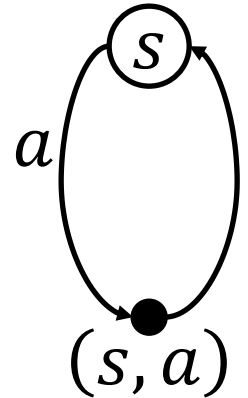
# Modified Policy Iteration

- PI is slower per sweep (cycle), with complexity  $O(|A||S|^2 + |S|^3)$ , and requires fewer sweeps.
- VI is faster per sweep, with complexity  $O(|A||S|^2)$ , and requires more sweeps.
- Modified Policy Iteration: run Policy Evaluation for # steps between 1 for VI, and that needed for convergence of  $v_\pi(s)$ , may achieve best efficiency.



# Toy Example

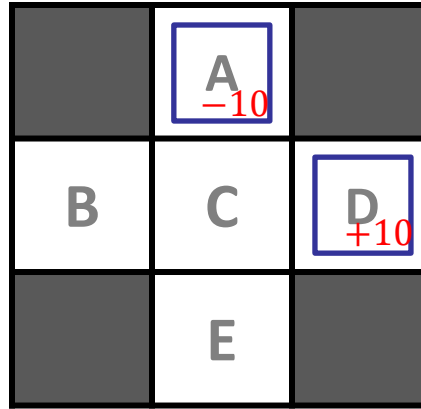
- Non-episodic MDP with a single state  $s$  and single possible action  $a$ , with reward  $r(s, a) = 1$
- Policy Iteration:
  - Policy Evaluation: initial policy  $\pi(s) = a$ ,  $v_\pi(s) = \frac{1}{1-\gamma}$   
 $\sum_a \pi(a|s)(1 + \gamma v_\pi(s)) = 1 + \gamma v_\pi(s) \Rightarrow v_\pi(s) = \frac{1}{1-\gamma}$
  - $q_\pi(s, a) = \sum_{r,s'} p(r, s'|s, a) [r + \gamma v_\pi(s')] = 1 + \frac{\gamma}{1-\gamma} = \frac{1}{1-\gamma}$
  - Policy Improvement: policy  $\pi(s) = \operatorname{argmax}_a q_\pi(s, a) = \operatorname{argmax}_a \frac{1}{1-\gamma} = a = \text{old-action}$ . So policy is now stable.
- Value Iteration:
  - $v_*(s) = \max_a (1 + \gamma v_*(s)) = 1 + \gamma v_*(s) \Rightarrow v_*(s) = \frac{1}{1-\gamma}$
  - Value function  $v_*(s)$  is now stable
  - $q_*(s, a) = \sum_{r,s'} p(r, s'|s, a) [r + \gamma v_*(s')] = \frac{1}{1-\gamma}$
  - Optimal policy  $\pi_*(s) = \operatorname{argmax}_a q_*(s, a) = \operatorname{argmax}_a \frac{1}{1-\gamma} = a$ .
- Both PI and VI converges in one step, since  $v_\pi(s) = v_*(s) = 1 + \gamma + \gamma^2 + \dots = \frac{1}{1-\gamma}$



# MiniGW Example

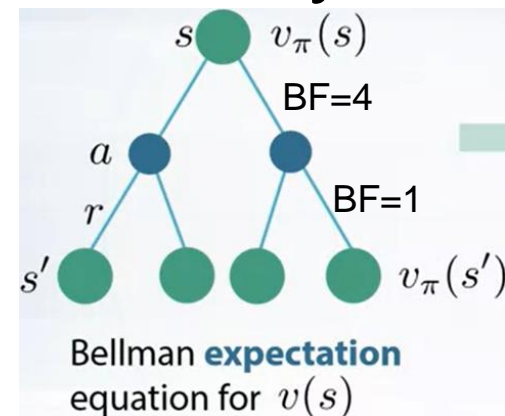
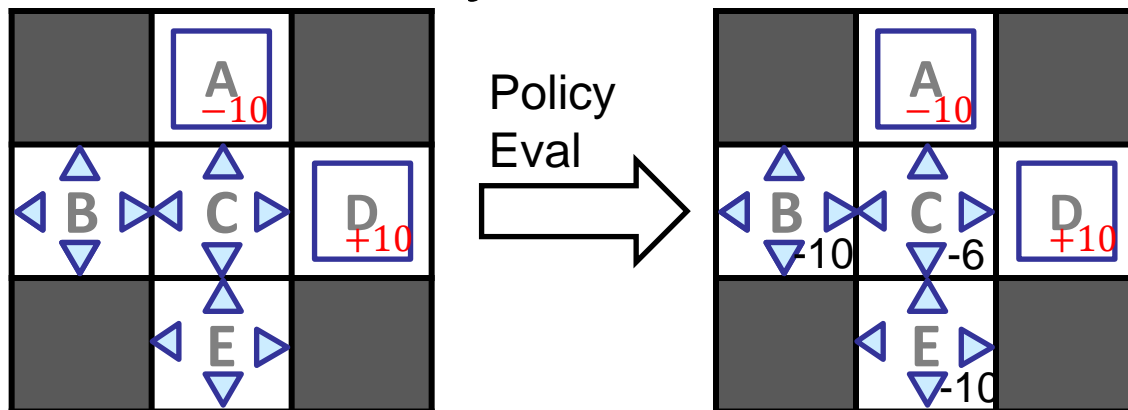
- Policy Iteration for Deterministic Environment
- Policy Iteration for Stochastic Environment
- Value Iteration for Deterministic Environment
- Value Iteration for Stochastic Environment

# MiniGW Setup



- All transitions  $(s, a, r, s')$  for  $s \in \{B, C, E\}$ ,  $a \in \{l, r, u, d\}$  have the same reward  $r = -1$ .
- $A$  and  $D$  are terminal states with fixed value function  $v(A) = -10$ ,  $v(D) = 10$ 
  - Or equivalently, you can think of an extra terminal state  $x$  with  $v(x) = 0$ , the only action in state  $A$  or  $D$  is *exit* that leads to  $x$  with  $R_A^a = -10$ ,  $R_D^a = 10$ .
- Dark squares denote obstacles which the agent cannot move into.
- Discount factor  $\gamma = 1$ .
- **Deterministic env:** Agent in state  $s \in \{B, C, E\}$  taking action  $a \in \{l, r, u, d\}$  always moves to the next state in the movement direction, unless it is blocked by an obstacle.
- **Stochastic env:** Agent in state  $s \in \{B, C, E\}$  taking action  $a \in \{l, r, u, d\}$  moves to the next state in the movement direction w. prob 0.8, or to the left or right side, each w. prob 0.1. If it is blocked by an obstacle in any direction, then it stays in the same state with prob of moving in the blocked direction.

# Iter1 Policy Evaluation of Random Policy

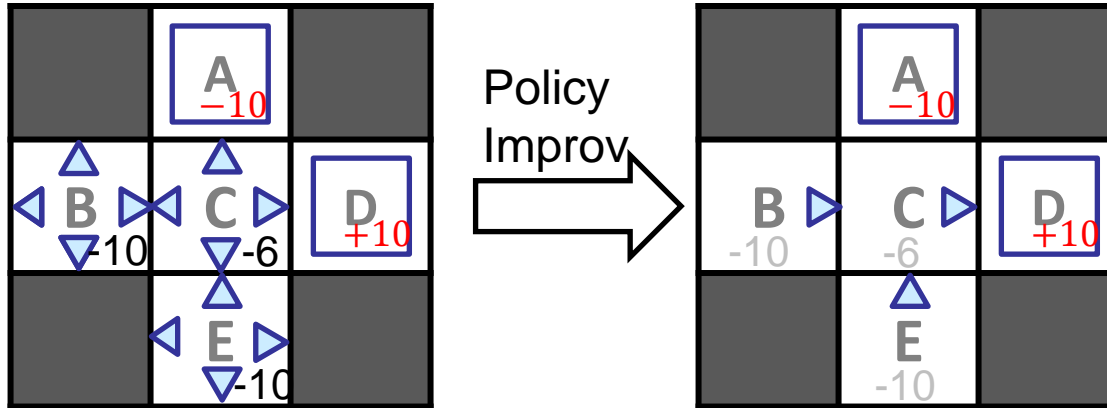


- Bellman Exp Equation for Det Env:  $v_{\pi}(s) = \sum_a \pi(a|s) q_{\pi}(s, a); q_{\pi}(s, a) = R_s^a + \gamma v_{\pi}(s')$
- Random policy:  $\pi(l|s) = \pi(r|s) = \pi(u|s) = \pi(d|s) = 0.25$
- $v_{\pi}(C) = .25(q_{\pi}(C, l) + q_{\pi}(C, r) + q_{\pi}(C, u) + q_{\pi}(C, d)) = .25(-4 + v_{\pi}(B) + v_{\pi}(E))$ 
  - $q_{\pi}(C, l) = -1 + v_{\pi}(B)$
  - $q_{\pi}(C, r) = -1 + v(D) = -1 + 10 = 9$
  - $q_{\pi}(C, u) = -1 + v(A) = -1 - 10 = -11$
  - $q_{\pi}(C, d) = -1 + v_{\pi}(E)$
- $v_{\pi}(B) = .25(q_{\pi}(B, l) + q_{\pi}(B, r) + q_{\pi}(B, u) + q_{\pi}(B, d)) = .25(-4 + 3v_{\pi}(B) + v_{\pi}(C))$ 
  - $q_{\pi}(B, l) = q_{\pi}(B, u) = q_{\pi}(B, d) = -1 + v_{\pi}(B)$
  - $q_{\pi}(B, r) = -1 + v_{\pi}(C)$
- $v_{\pi}(E) = .25(q_{\pi}(E, l) + q_{\pi}(E, r) + q_{\pi}(E, u) + q_{\pi}(E, d)) = .25(-4 + 3v_{\pi}(E) + v_{\pi}(C))$ 
  - $q_{\pi}(E, l) = q_{\pi}(E, r) = q_{\pi}(E, d) = -1 + v_{\pi}(E)$
  - $q_{\pi}(E, u) = -1 + v_{\pi}(C)$
- Analytic solution:  $v_{\pi}(C) = -6, v_{\pi}(B) = -10, v_{\pi}(E) = -10$

|       | $V_{\pi}(B)$ | $V_{\pi}(C)$ | $V_{\pi}(E)$ |
|-------|--------------|--------------|--------------|
| Iter1 | -6           | -10          | -10          |
| Iter2 |              |              |              |

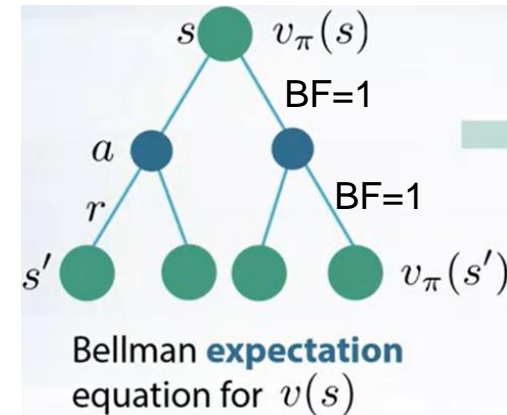
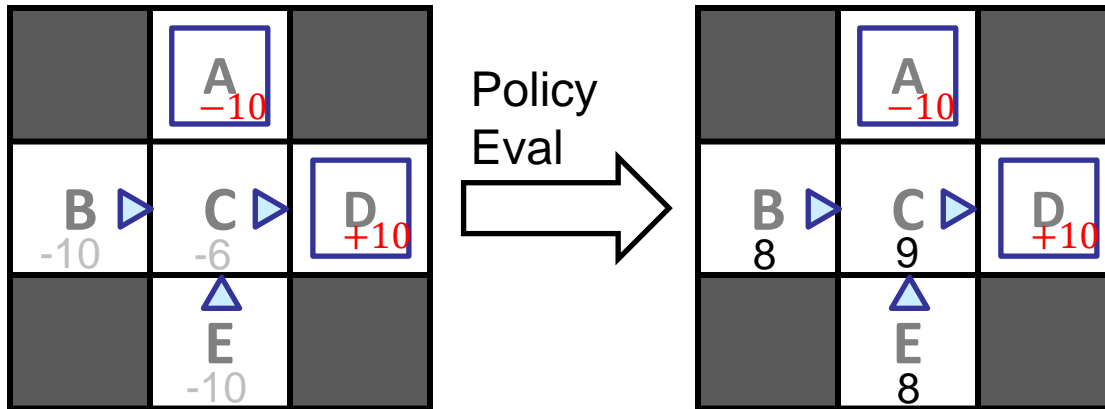


# Iter1 Policy Improvement



- Plug in values from PE:  $v_{\pi}(C) = -6, v_{\pi}(B) = -10, v_{\pi}(E) = -10$ , to get new policy  $\pi'$
- $\pi'(C) = \operatorname{argmax}_{a \in \{l, r, u, d\}} q_{\pi}(C, a) = r$ 
  - $q_{\pi}(C, l) = -1 + v_{\pi}(B) = -1 - 10 = -11$
  - $q_{\pi}(C, r) = 9$
  - $q_{\pi}(C, u) = -11$
  - $q_{\pi}(C, d) = -1 + v_{\pi}(E) = -1 - 10 = -11$
- $\pi'(B) = \operatorname{argmax}_{a \in \{l, r, u, d\}} q_{\pi}(B, a) = r$ 
  - $q_{\pi}(B, l) = q_{\pi}(B, u) = q_{\pi}(B, d) = -1 + v_{\pi}(B) = -1 - 10 = -11$
  - $q_{\pi}(B, r) = -1 + v_{\pi}(C) = -1 - 6 = -7$
- $\pi'(E) = \operatorname{argmax}_{a \in \{l, r, u, d\}} q_{\pi}(E, a) = u$ 
  - $q_{\pi}(E, l) = q_{\pi}(E, r) = q_{\pi}(E, d) = -1 + v_{\pi}(E) = -1 - 10 = -11$
  - $q_{\pi}(E, u) = -1 + v_{\pi}(C) = -1 - 6 = -7$

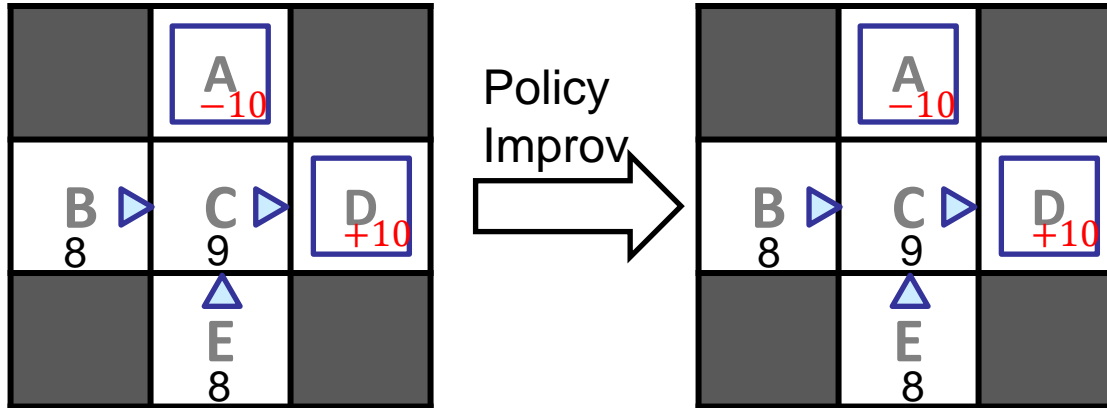
# Iter2 Policy Evaluation of Det Policy



- Bellman Exp Equation for Det Env:  $v_{\pi}(s) = \sum_a \pi(a|s) q_{\pi}(s, a); q_{\pi}(s, a) = R_s^a + \gamma v_{\pi}(s')$
- Det policy:  $\pi(r|B) = 1; \pi(r|C) = 1; \pi(u|E) = 1$
- $v_{\pi}(C) = 1.0 q_{\pi}(C, r)$ 
  - $q_{\pi}(C, r) = 9$
- $v_{\pi}(B) = 1.0 q_{\pi}(B, r)$ 
  - $q_{\pi}(B, r) = -1 + v_{\pi}(C)$
- $v_{\pi}(E) = 1.0 q_{\pi}(E, u)$ 
  - $q_{\pi}(E, u) = -1 + v_{\pi}(C)$
- Analytic solution:  $v_{\pi}(C) = 9; v_{\pi}(B) = 8; v_{\pi}(E) = 8$

|       | $V_{\pi}(B)$ | $V_{\pi}(C)$ | $V_{\pi}(E)$ |
|-------|--------------|--------------|--------------|
| Iter1 | -6           | -10          | -10          |
| Iter2 | 8            | 9            | 8            |

# Iter2 Policy Improvement

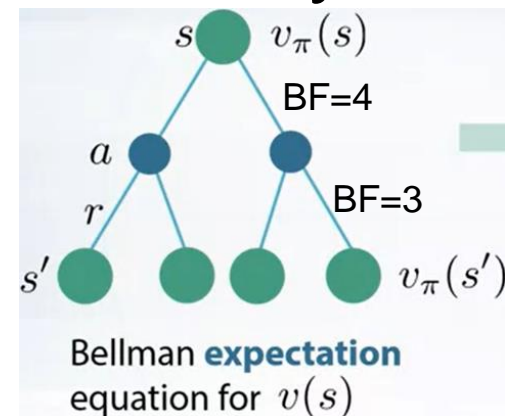
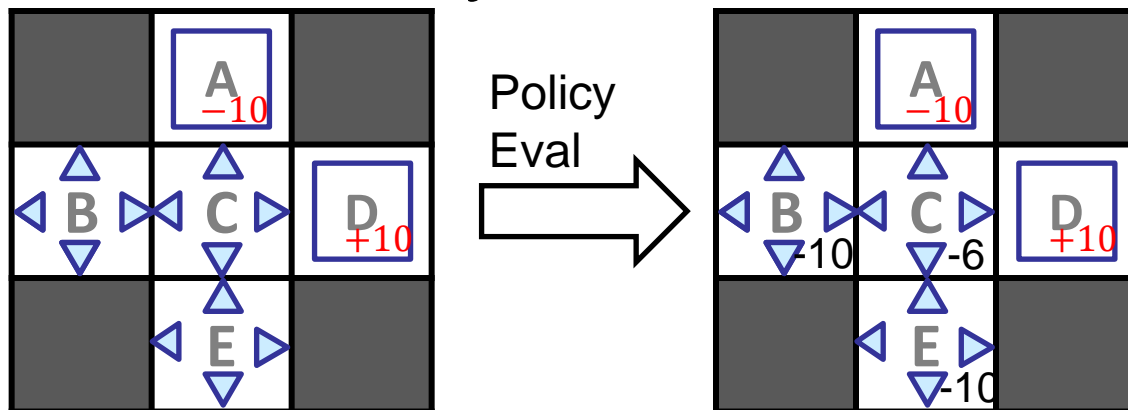


- Plug in values from PE:  $v_\pi(C) = 9, v_\pi(B) = 8, v_\pi(E) = 8$ , to get new policy  $\pi'$
- $\pi'(C) = \operatorname{argmax}_{a \in \{l, r, u, d\}} q_\pi(C, a) = r$ 
  - $q_\pi(C, l) = -1 + v_\pi(B) = -1 + 8 = 7$
  - $q_\pi(C, r) = 9$
  - $q_\pi(C, u) = -11$
  - $q_\pi(C, d) = -1 + v_\pi(E) = -1 + 8 = 7$
- $\pi'(B) = \operatorname{argmax}_{a \in \{l, r, u, d\}} q_\pi(B, a) = r$ 
  - $q_\pi(B, l) = q_\pi(B, u) = q_\pi(B, d) = -1 + v_\pi(B) = 7$
  - $q_\pi(B, r) = -1 + v_\pi(C) = -1 + 9 = 8$
- $\pi'(E) = \operatorname{argmax}_{a \in \{l, r, u, d\}} q_\pi(E, a) = u$ 
  - $q_\pi(E, l) = q_\pi(E, r) = q_\pi(E, d) = -1 + v_\pi(E) = 7$
  - $q_\pi(E, u) = -1 + v_\pi(C) = -1 + 9 = 8$
- Policy is now stable ( $\pi' = \pi$ ), so we have found the optimal policy. (We do not need to re-run Policy Evaluation, since we do not care if the value functions converge as long as the policy is stable.)

# MiniGW Example

- Policy Iteration for Deterministic Environment
- Policy Iteration for Stochastic Environment
- Value Iteration for Deterministic Environment
- Value Iteration for Stochastic Environment

# Iter1 Policy Evaluation of Random Policy

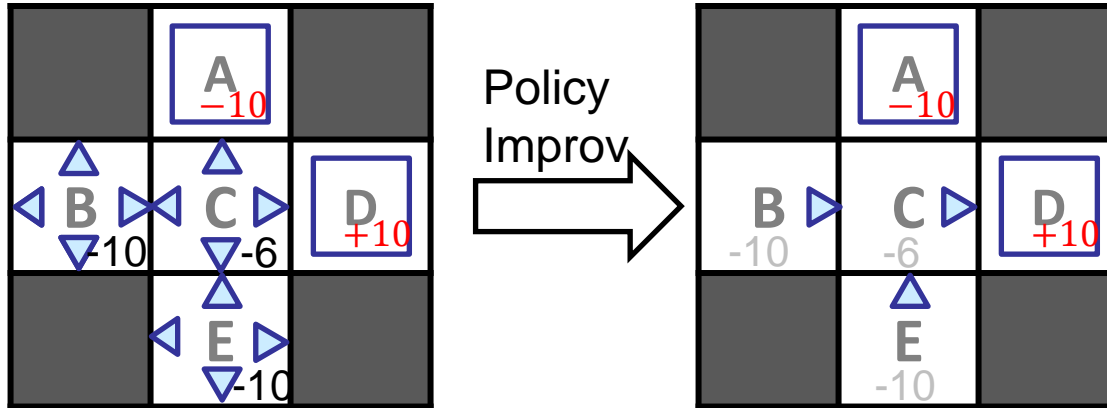


- Bellman Exp Equation:  $v_{\pi}(s) = \sum_a \pi(a|s) q_{\pi}(s, a)$ ;  $q_{\pi}(s, a) = \sum_{r, s'} p(r, s'|s, a) [r + \gamma v_{\pi}(s')]$
- Random policy:  $\pi(l|s) = \pi(r|s) = \pi(u|s) = \pi(d|s) = 0.25$
- Set of equations:

- $v_{\pi}(C) = .25(q_{\pi}(C, l) + q_{\pi}(C, r) + q_{\pi}(C, u) + q_{\pi}(C, d)) = .25(-4 + v_{\pi}(B) + v_{\pi}(E))$ 
  - $q_{\pi}(C, l) = .8[-1 + v_{\pi}(B)] + .1[-1 + v(A)] + .1[-1 + v_{\pi}(E)] = -2 + .8v_{\pi}(B) + .1v_{\pi}(E)$
  - $q_{\pi}(C, r) = .8[-1 + v(D)] + .1[-1 + v(A)] + .1[-1 + v_{\pi}(E)] = 6 + .1v_{\pi}(E)$
  - $q_{\pi}(C, u) = .8[-1 + v(A)] + .1[-1 + v_{\pi}(B)] + .1[-1 + v(D)] = -8 + .1v_{\pi}(B)$
  - $q_{\pi}(C, d) = .8[-1 + v_{\pi}(E)] + .1[-1 + v_{\pi}(B)] + .1[-1 + v(D)] = .8v_{\pi}(E) + .1v_{\pi}(B)$
- $v_{\pi}(B) = .25(q_{\pi}(B, l) + q_{\pi}(B, r) + q_{\pi}(B, u) + q_{\pi}(B, d)) = .25(-4 + 3v_{\pi}(B) + v_{\pi}(C))$ 
  - $q_{\pi}(B, l) = 1.0[-1 + v_{\pi}(B)] = -1 + v_{\pi}(B)$
  - $q_{\pi}(B, r) = .8[-1 + v_{\pi}(C)] + .2[-1 + v_{\pi}(B)] = -1 + .8v_{\pi}(C) + .2v_{\pi}(B)$
  - $q_{\pi}(B, u) = q_{\pi}(B, d) = .9[-1 + v_{\pi}(B)] + .1[-1 + v_{\pi}(C)] = -1 + .9v_{\pi}(B) + .1v_{\pi}(C)$
- $v_{\pi}(E) = 0.25(q_{\pi}(E, l) + q_{\pi}(E, r) + q_{\pi}(E, u) + q_{\pi}(E, d)) = .25(-4 + 3v_{\pi}(E) + v_{\pi}(C))$ 
  - $q_{\pi}(E, l) = q_{\pi}(B, r) = .9[-1 + v_{\pi}(E)] + .1[-1 + v_{\pi}(C)] = -1 + .9v_{\pi}(E) + .1v_{\pi}(C)$
  - $q_{\pi}(E, u) = .8[-1 + v_{\pi}(C)] + .2[-1 + v_{\pi}(E)] = -1 + .8v_{\pi}(C) + .2v_{\pi}(E)$
  - $q_{\pi}(E, d) = 1.0[-1 + v_{\pi}(E)] = -1 + v_{\pi}(E)$
- Analytic solution:  $v_{\pi}(C) = -6, v_{\pi}(B) = -10, v_{\pi}(E) = -10$

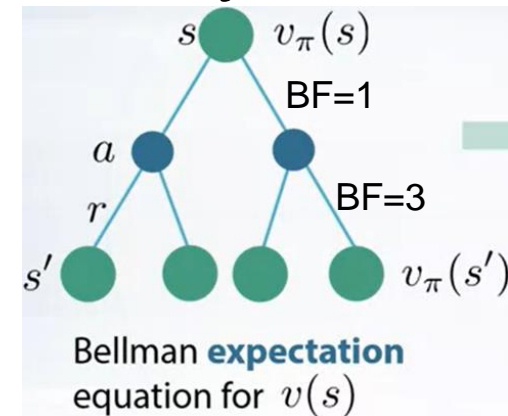
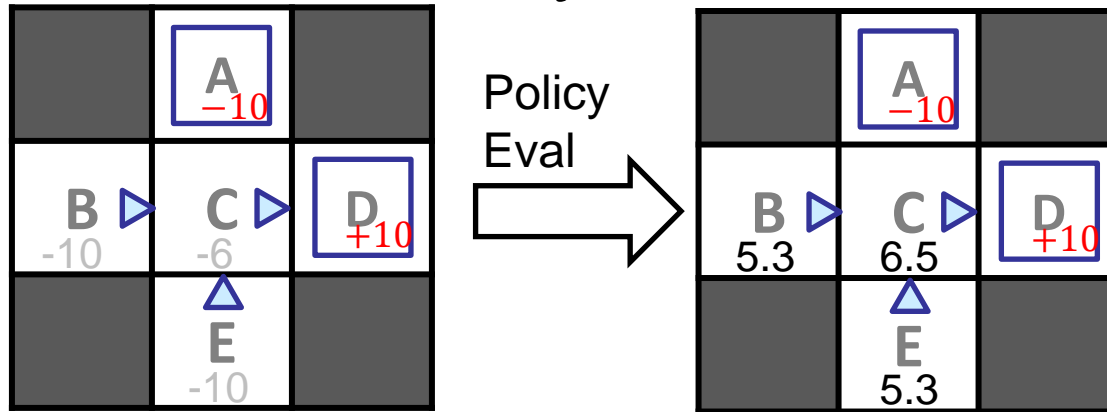
|       | $V_{\pi}(B)$ | $V_{\pi}(C)$ | $V_{\pi}(E)$ |
|-------|--------------|--------------|--------------|
| Iter1 | -6           | -10          | -10          |
| Iter2 |              |              |              |

# Iter1 Policy Improvement



- Plug in values from PE:  $v_\pi(C) = -6$ ,  $v_\pi(B) = -10$ ,  $v_\pi(E) = -10$ , to get new policy  $\pi'$
- $\pi'(C) = \operatorname{argmax}_{a \in \{l, r, u, d\}} q_\pi(C, a) = r$ 
  - $q_\pi(C, l) = -2 + .8v_\pi(B) + .1v_\pi(E) = -2 + .8 \cdot (-10) + .1 \cdot (-10) = -11$
  - $q_\pi(C, r) = 6 + .1v_\pi(E) = 6 + .1 \cdot (-10) = 5$
  - $q_\pi(C, u) = -8 + .1v_\pi(B) = -8 + .1 \cdot (-10) = -9$
  - $q_\pi(C, d) = .8v_\pi(E) + .1v_\pi(B) = .8 \cdot (-10) + 0.1 \cdot (-10) = -9$
- $\pi'(B) = \operatorname{argmax}_{a \in \{l, r, u, d\}} q_\pi(B, a) = r$ 
  - $q_\pi(B, l) = -1 + v_\pi(B) = -1 + (-10) = -11$
  - $q_\pi(B, r) = -1 + .8v_\pi(C) + .2v_\pi(B) = -1 + .8 \cdot (-6) + .2 \cdot (-10) = -7.8$
  - $q_\pi(B, u) = q_\pi(B, d) = -1 + .9v_\pi(B) + .1v_\pi(C) = -1 + .9 \cdot (-10) + .1 \cdot (-6) = -10.6$
- $\pi'(E) = \operatorname{argmax}_{a \in \{l, r, u, d\}} q_\pi(E, a) = u$ 
  - $q_\pi(E, l) = q_\pi(B, r) = -1 + .9v_\pi(E) + .1v_\pi(C) = -1 + .9 \cdot (-10) + .1 \cdot (-6) = -10.6$
  - $q_\pi(E, u) = -1 + .8v_\pi(C) + .2v_\pi(E) = -1 + .8 \cdot (-6) + .2 \cdot (-10) = -7.8$
  - $q_\pi(E, d) = -1 + 1.0v_\pi(E) = -1 + 1.0 \cdot (-10) = -11$

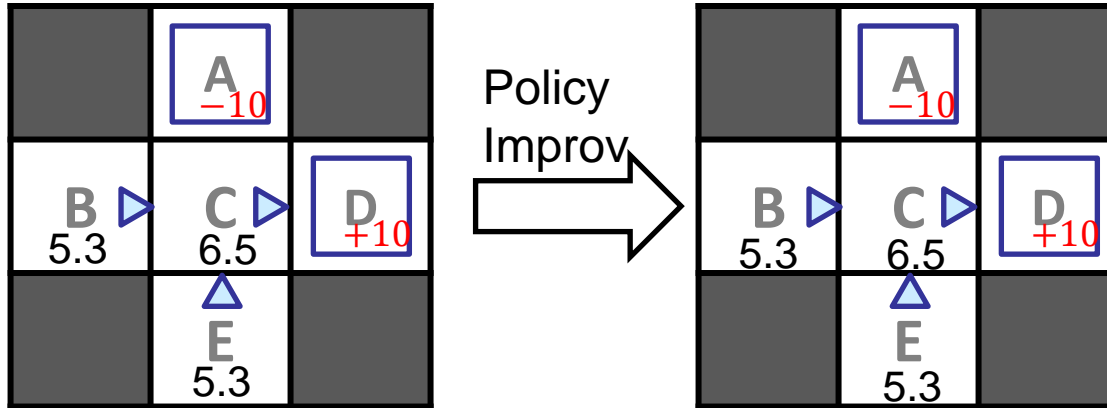
# Iter2 Policy Evaluation of Det Policy



- Bellman Exp Equation:  $v_{\pi}(s) = \sum_a \pi(a|s) q_{\pi}(s, a)$ ;  $q_{\pi}(s, a) = \sum_{r, s'} p(r, s'|s, a) [r + \gamma v_{\pi}(s')]$
- Det policy:  $\pi(r|B) = 1$ ;  $\pi(r|C) = 1$ ;  $\pi(u|C) = 1$
- Set of equations:
- $v_{\pi}(C) = 1.0 q_{\pi}(C, r)$ 
  - $q_{\pi}(C, r) = 6 + 0.1 v_{\pi}(E)$
- $v_{\pi}(B) = 1.0 q_{\pi}(B, r)$ 
  - $q_{\pi}(B, r) = -1 + .8 v_{\pi}(C) + .2 v_{\pi}(B)$
- $v_{\pi}(E) = 1.0 q_{\pi}(E, u)$ 
  - $q_{\pi}(E, u) = -1 + .8 v_{\pi}(C) + .2 v_{\pi}(E)$
- Analytic solution:  $v_{\pi}(C) \approx 6.5$ ,  $v_{\pi}(B) \approx 5.3$ ,  $v_{\pi}(E) \approx 5.3$

|       | $V_{\pi}(B)$ | $V_{\pi}(C)$ | $V_{\pi}(E)$ |
|-------|--------------|--------------|--------------|
| Iter1 | -6           | -10          | -10          |
| Iter2 | 5.3          | 6.5          | 5.3          |

# Iter2 Policy Improvement



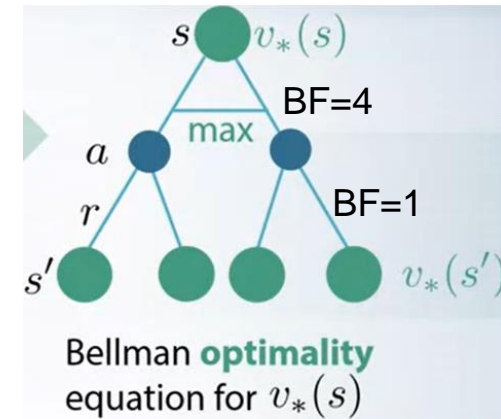
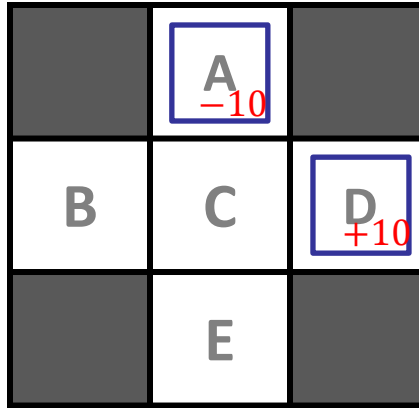
- Plugging in values from PE:  $V_\pi(C) = 6.5, V_\pi(B) = 5.3, V_\pi(E) = 5.3$ , we get new policy  $\pi'$
- $\pi'(C) = \operatorname{argmax}_{a \in \{l, r, u, d\}} q_\pi(C, a) = r$ 
  - $q_\pi(C, l) = -2 + .8v_\pi(B) + .1v_\pi(E) = -2 + .8 \cdot 5.3 + .1 \cdot 5.3 \approx 2.8$
  - $q_\pi(C, r) = 6 + .1v_\pi(E) = 6 + .1 \cdot 5.3 \approx 6.5$
  - $q_\pi(C, u) = -8 + .1v_\pi(B) = -8 + .1 \cdot 5.3 \approx -7.5$
  - $q_\pi(C, d) = .8v_\pi(E) + .1v_\pi(B) = .8 \cdot 5.3 + 0.1 \cdot 5.3 \approx 4.8$
- $\pi'(B) = \operatorname{argmax}_{a \in \{l, r, u, d\}} q_\pi(B, a) = r$ 
  - $q_\pi(B, l) = -1 + v_\pi(B) = -1 + 5.3 = 4.3$
  - $q_\pi(B, r) = -1 + .8v_\pi(C) + .2v_\pi(B) = -1 + .8 \cdot 6.5 + .2 \cdot 5.3 \approx 5.3$
  - $q_\pi(B, u) = q_\pi(B, d) = -1 + .9v_\pi(B) + .1v_\pi(C) = -1 + .9 \cdot 5.3 + .1 \cdot 6.5 \approx 4.4$
- $\pi'(E) = \operatorname{argmax}_{a \in \{l, r, u, d\}} q_\pi(E, a) = u$ 
  - $q_\pi(E, l) = q_\pi(B, r) = -1 + .9v_\pi(E) + .1V_\pi(C) = -1 + .9 \cdot 5.3 + .1 \cdot 6.5 \approx 4.4$
  - $q_\pi(E, u) = -1 + .8v_\pi(C) + .2v_\pi(E) = -1 + .8 \cdot 6.5 + .2 \cdot 5.3 \approx 5.3$
  - $q_\pi(E, d) = -1 + v_\pi(E) = -1 + 5.3 = 4.3$
- New policy is now stable ( $\pi' = \pi$ ), so we have found the optimal policy.



# MiniGW Example

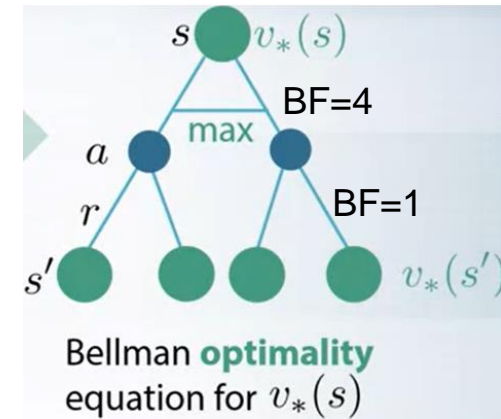
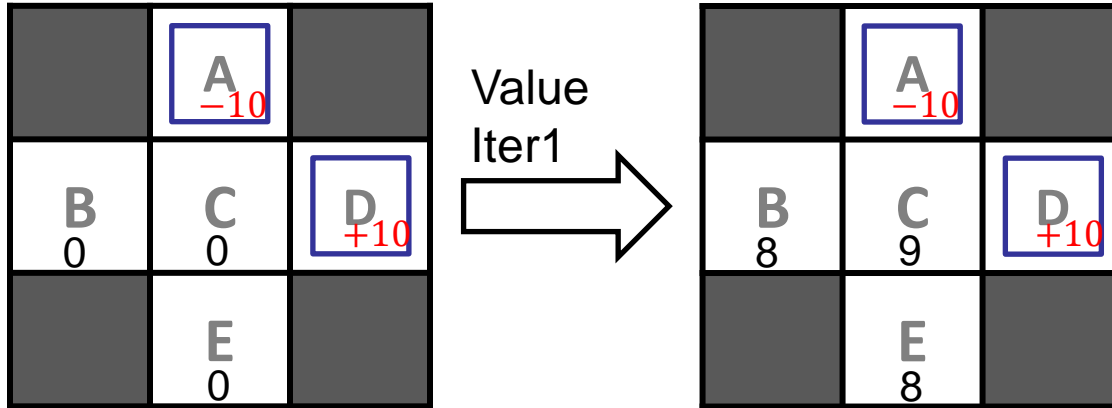
- Policy Iteration for Deterministic Environment
- Policy Iteration for Stochastic Environment
- Value Iteration for Deterministic Environment
- Value Iteration for Stochastic Environment

# Bellman Opt Equation for Det Env



- Bellman Opt Equation for Det Env:  $v_*(s) = \max_a q_*(s, a); q_*(s, a) = R_s^a + \gamma v_*(s')$
- $v_*(C) = \max_a q_*(C, a) = \max\{-1 + v_*(B), 9, -11, -1 + v_*(E)\}$ 
  - $q_*(C, l) = -1 + v_*(B)$
  - $q_*(C, r) = -1 + v(D) = 9$
  - $q_*(C, u) = -1 + v(A) = -11$
  - $q_*(C, d) = -1 + v_*(E)$
- $v_*(B) = \max_a q_*(B, a) = \max\{-1 + v_*(B), -1 + v_*(C), -1 + v_*(B), -1 + v_*(B)\}$ 
  - $q_*(B, l) = q_*(B, u) = q_*(B, d) = -1 + v_*(B)$
  - $q_*(B, r) = -1 + v_*(C)$
- $v_*(E) = \max_a q_*(E, a) = \max\{-1 + v_*(E), -1 + v_*(E), -1 + v_*(C), -1 + v_*(E)\}$ 
  - $q_*(E, l) = q_*(E, r) = q_*(E, d) = -1 + v_*(E)$
  - $q_*(E, u) = -1 + v_*(C)$
- The set of non-linear equations cannot be solved analytically due to the max operator, so we need to use Value Iteration to find  $v_*(C), v_*(B), v_*(E)$

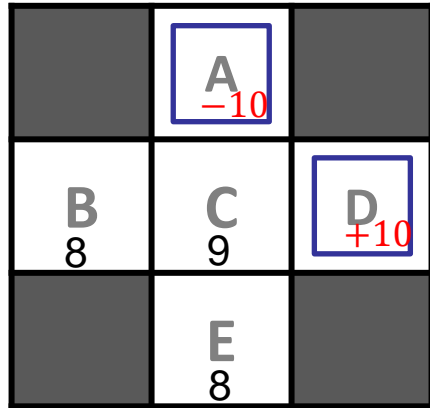
# Iter1 Value Iteration w. Det Env



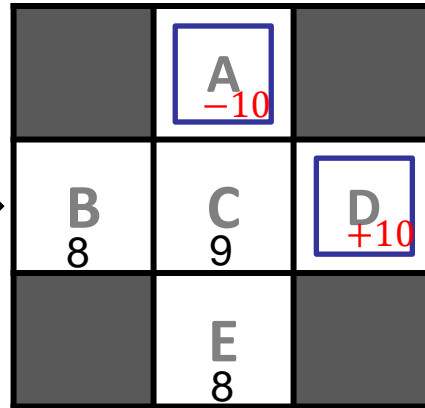
- Value Iteration:  $v_{k+1}(s) = \max_a q_{k+1}(s, a) = \max_a [r + \gamma v_k(s')]$  w. **in-place updates**.
- Initialize  $v_0(B) = v_0(C) = v_0(E) = 0$ .
- $v_1(C) = \max_a q_1(C, a) = \max(-1, 9, -11, -1) = 9$ 
  - $q_1(C, l) = -1 + v_0(B) = -1 + 0 = -1$
  - $q_1(C, r) = 9$
  - $q_1(C, u) = -11$
  - $q_1(C, d) = -1 + v_0(E) = -1 + 0 = -1$
- $v_1(B) = \max_a q_1(B, a) = \max(-1, 8, -1, -1) = 8$ 
  - $q_1(B, l) = q_1(B, u) = q_1(B, d) = -1 + v_0(B) = -1 + 0 = -1$
  - $q_1(B, r) = r + v_1(C) = -1 + 9 = 8$  ( $v_1(C)$  computed in the current iteration is used instead of  $v_0(C)$ )
- $v_1(E) = \max_a q_1(E, a) = \max(-1, -1, 8, -1) = 8$ 
  - $q_1(E, l) = q_1(E, r) = q_1(E, d) = r + v_0(E) = -1 + 0 = -1$
  - $q_1(E, u) = r + v_1(C) = -1 + 9 = 8$  ( $v_1(C)$  is used)
- After 1<sup>st</sup> iteration:  $v_1(C) = 9, v_1(B) = 8, v_1(E) = 8$

|       | $V_\pi(B)$ | $V_\pi(C)$ | $V_\pi(E)$ |
|-------|------------|------------|------------|
| Iter1 | 8          | 9          | 8          |
| Iter2 |            |            |            |

# Iter2 Value Iteration w. Det Env

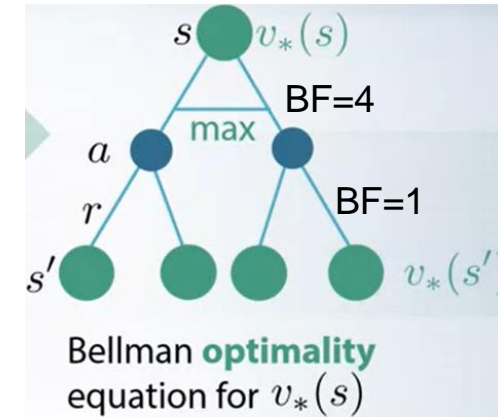
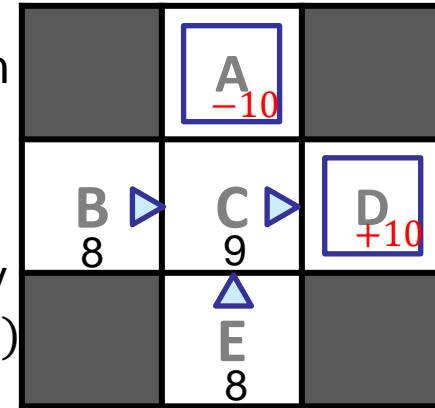


Value Iter2



Value Function converged

Optimal Policy  
 $\operatorname{argmax}_a Q_*(s, a)$



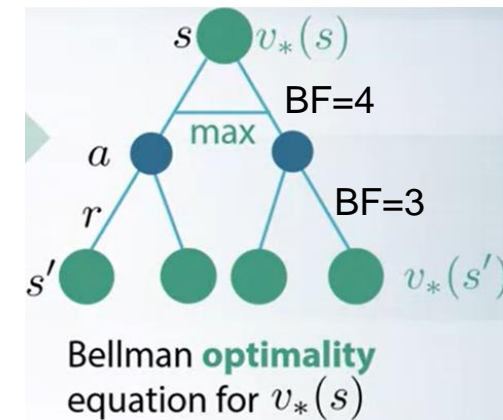
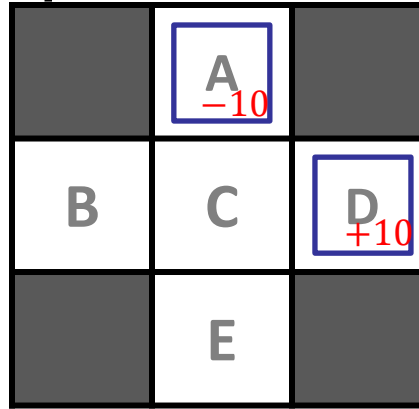
- Value Iteration:  $v_{k+1}(s) = \max_a q_{k+1}(s, a) = \max_a [r + \gamma v_k(s')]$  w. **in-place updates**.
- We have now  $v_1(B) = v_1(E) = 8, v_1(C) = 9$ .
- $v_2(C) = \max_a q_2(C, a) = \max(7, 9, -11, 7) = 9$ 
  - $q_2(C, l) = -1 + v_1(B) = -1 + 8 = 7$
  - $q_2(C, r) = 9$**
  - $q_2(C, u) = -11$
  - $q_2(C, d) = -1 + v_1(E) = -1 + 8 = 7$
- $v_2(B) = \max_a q_2(B, a) = \max(7, 8, 7, 7) = 8$ 
  - $q_2(B, l) = q_2(B, u) = q_2(B, d) = -1 + v_1(B) = -1 + 8 = 7$
  - $q_2(B, r) = -1 + v_2(C) = -1 + 9 = 8$**  ( $v_2(C)$  is used, which happens to be equal to  $v_1(C)$ )
- $v_2(E) = \max_a q_2(E, a) = \max(7, 7, 8, 7) = 8$ 
  - $q_2(E, l) = q_2(E, r) = q_2(E, d) = -1 + v_1(E) = -1 + 8 = 7$
  - $q_2(E, u) = -1 + v_2(C) = -1 + 9 = 8$**  ( $v_2(C)$  is used)
- After 2<sup>nd</sup> iteration:  $v_2(C) = 9, v_2(B) = 8, v_2(E) = 8$ . VI has converged, so  $v_*(\cdot) = v_2(\cdot)$ .
- Optimal policy:  $\pi_*(C) = \operatorname{argmax}_{a \in \{l, r, u, d\}} q_*(C, a) = r$ ;  $\pi_*(B) = \operatorname{argmax}_{a \in \{l, r, u, d\}} q_*(B, a) = r$ ;  $\pi_*(E) = \operatorname{argmax}_{a \in \{l, r, u, d\}} q_*(E, a) = u$

|       | $V_\pi(B)$ | $V_\pi(C)$ | $V_\pi(E)$ |
|-------|------------|------------|------------|
| Iter1 | 8          | 9          | 8          |
| Iter2 | 8          | 9          | 8          |

# MiniGW Example

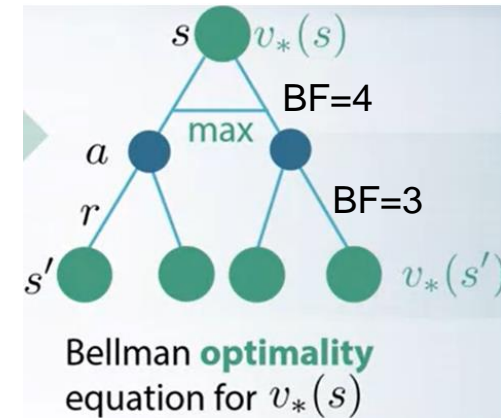
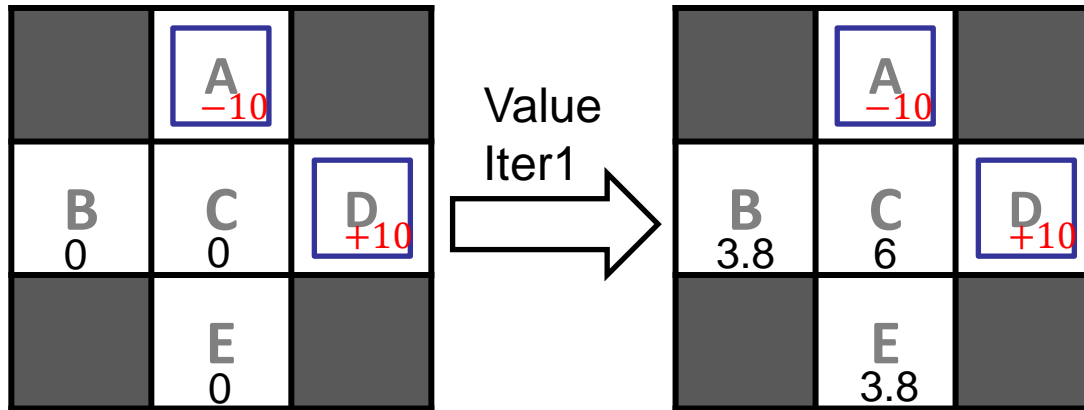
- Policy Iteration for Deterministic Environment
- Policy Iteration for Stochastic Environment
- Value Iteration for Deterministic Environment
- Value Iteration for Stochastic Environment

# Bellman Opt Equation for Stochastic Env



- Bellman Opt Equation:  $v_*(s) = \max_a q_*(s, a), q_*(s, a) = \sum_{r,s'} p(r, s'|s, a) [r + \gamma v_*(s')]$
- $v_*(C) = \max_a q_*(C, a)$ 
  - $q_*(C, l) = .8[-1 + v_*(B)] + .1[-1 + v(A)] + .1[-1 + v_*(E)] = -2 + .8v_*(B) + .1v_*(E)$
  - $q_*(C, r) = .8[-1 + v(D)] + .1[-1 + v(A)] + .1[-1 + v_*(E)] = 6 + .1v_*(E)$
  - $q_*(C, u) = .8[-1 + v(A)] + .1[-1 + v_*(B)] + .1[-1 + v(D)] = -8 + .1v_*(B)$
  - $q_*(C, d) = .8[-1 + v_*(E)] + .1[-1 + v_*(B)] + .1[-1 + v(D)] = .8v_*(E) + .1v_*(B)$
- $v_1(B) = \max_a q_*(B, a)$ 
  - $q_*(B, l) = 1.0[-1 + v_*(B)] = -1 + v_*(B)$
  - $q_*(B, r) = .8[-1 + v_*(C)] + .2[-1 + v_*(B)] = -1 + .8v_*(C) + .2v_*(B)$
  - $q_*(B, u) = q_*(B, d) = .9[-1 + v_*(B)] + .1[-1 + v_*(C)] = -1 + .9v_*(B) + .1v_*(C)$
- $v_1(E) = \max_a q_*(E, a)$ 
  - $q_*(E, l) = q_*(B, r) = .9[-1 + v_*(E)] + .1[-1 + v_*(C)] = -1 + .9v_*(E) + .1v_*(C)$
  - $q_*(E, u) = .8[-1 + v_*(C)] + .2[-1 + v_*(E)] = -1 + .8v_*(C) + .2v_*(E)$
  - $q_*(E, d) = 1.0[-1 + v_*(E)] = -1 + v_*(E)$
- The set of non-linear equations cannot be solved analytically due to the max operator, so we need to use Value Iteration to find  $v_*(C), v_*(B), v_*(E)$

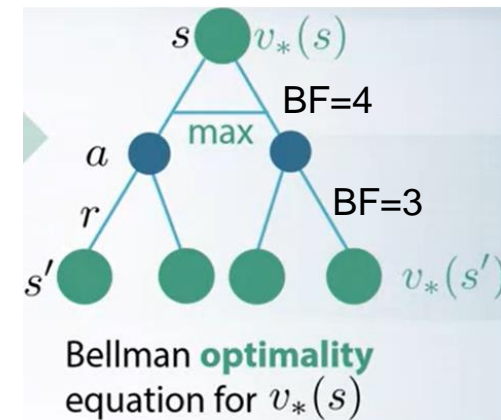
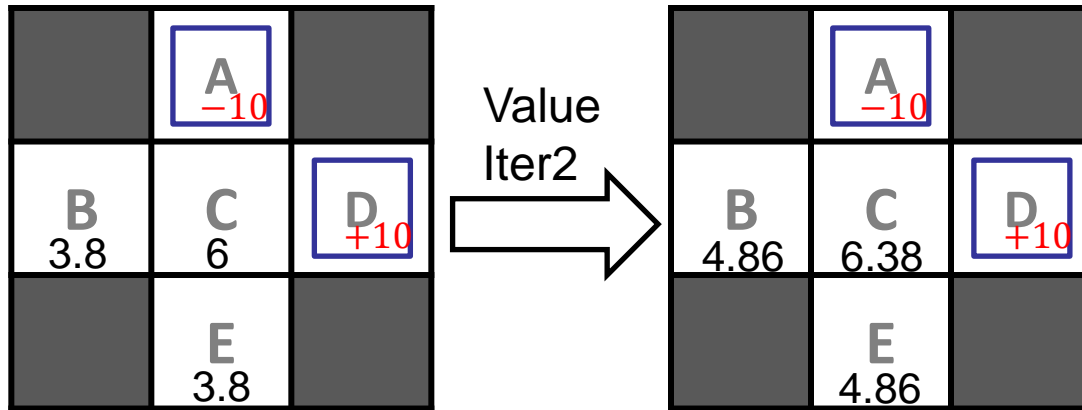
# Iter1 Value Iteration w. Sto Env



- Value Iteration:  $v_{k+1}(s) = \max_a q_{k+1}(s, a) = \max_a [r + \gamma \sum_{s'} p(s'|s, a) v_k(s')]$  w. **in-place updates**.
- Initialize  $v_0(B) = v_0(C) = v_0(E) = 0$ .
- $v_1(C) = \max_a q_1(C, a) = \max(-2, 6, -8, 0) = 6$ 
  - $q_1(C, l) = -2 + .8v_0(B) + .1v_0(E) = -2 + .8 \cdot 0 + 0.1 \cdot 0 = -2$
  - $q_1(C, r) = 6 + .1v_0(E) = 6 + 0.1 \cdot 0 = 6$
  - $q_1(C, u) = -8 + .1v_0(B) = -8 + 0.1 \cdot 0 = -8$
  - $q_1(C, d) = .8v_0(E) + .1v_0(B) = 0.8 \cdot 0 + 0.1 \cdot 0 = 0$
- $v_1(B) = \max_a q_1(B, a) = \max(-1, 3.8, -0.4, -0.4) = 3.8$ 
  - $q_1(B, l) = -1 + v_0(B) = -1 + 0 = -1$
  - $q_1(B, r) = -1 + .8v_1(C) + .2v_0(B) = -1 + .8 \cdot 6 + .2 \cdot 0 = 3.8$
  - $q_1(B, u) = q_1(B, d) = -1 + .9v_0(B) + .1v_1(C) = -1 + .9 \cdot 0 + .1 \cdot 6 = -0.4$
- $v_1(E) = \max_a q_1(E, a) = \max(-0.4, -0.4, 3.8, -1) = 3.8$ 
  - $q_1(E, l) = q_1(E, r) = -1 + .9v_0(E) + .1v_1(C) = -1 + .9 \cdot 0 + .1 \cdot 6 = -0.4$
  - $q_1(E, u) = -1 + .8v_1(C) + .2v_0(E) = -1 + .8 \cdot 6 + .2 \cdot 0 = 3.8$
  - $q_1(E, d) = -1 + v_0(E) = -1 + 1.0 \cdot 0 = -1$
- After 1<sup>st</sup> iteration:  $V_\pi(C) = 6, V_\pi(B) = 3.8, V_\pi(E) = 3.8$

|       | $V_\pi(B)$ | $V_\pi(C)$ | $V_\pi(E)$ |
|-------|------------|------------|------------|
| Iter1 | 3.80       | 6          | 3.80       |
| Iter2 |            |            |            |
| Iter3 |            |            |            |
| Iter4 |            |            |            |
| Iter5 |            |            |            |
| Iter6 |            |            |            |

# Iter2 Value Iteration w. Sto Env



- Value Iteration:  $v_{k+1}(s) = \max_a q_{k+1}(s, a) = \max_a [r + \gamma \sum_{s'} p(s'|s, a) v_k(s')]$  w. **in-place updates**.
- Now:  $v_1(B) = v_1(E) = 3.8, v_1(C) = 6$ .
- $v_2(C) = \max_a q_2(C, a) = \max(1.42, 6.38, -7.62, 3.42) = \mathbf{6.38}$ 
  - $q_2(C, l) = -2 + .8v_1(B) + .1v_1(E) = -2 + .8 \cdot 3.8 + .1 \cdot 3.8 = 1.42$
  - $q_2(C, r) = 6 + .1v_1(E) = 6 + .1 \cdot 3.8 = \mathbf{6.38}$
  - $q_2(C, u) = -8 + .1v_1(B) = -8 + .1 \cdot 3.8 = -7.62$
  - $q_2(C, d) = .8v_1(E) + .1v_1(B) = .8 \cdot 3.8 + .1 \cdot 3.8 = 3.42$
- $v_2(B) = \max_a q_2(B, a) = \max(2.8, 4.86, 3.06, 3.06) = 4.86$ 
  - $q_2(B, l) = -1 + v_1(B) = -1 + 1.0 \cdot 3.8 = 2.8$
  - $q_2(B, r) = -1 + .8v_2(C) + .2v_1(B) = -1 + .8 \cdot 6.38 + .2 \cdot 3.8 = 4.86$
  - $q_2(B, u) = q_2(B, d) = -1 + .9v_1(B) + .1v_2(C) = -1 + .9 \cdot 3.8 + .1 \cdot \mathbf{6.38} \approx 3.06$
- $v_2(E) = \max_a q_2(E, a) = \max(3.06, 3.06, 4.86, 2.8) = 4.86$ 
  - $q_2(E, l) = q_2(E, r) = -1 + .9v_1(E) + .1v_2(C) = -1 + .9 \cdot 3.8 + .1 \cdot \mathbf{6.38} \approx 3.06$
  - $q_2(E, u) = -1 + .8v_2(C) + .2v_1(E) = -1 + .8 \cdot 6.38 + .2 \cdot 3.8 \approx 4.86$
  - $q_1(E, d) = -1 + v_1(E) = -1 + 3.8 = 2.8$
- After 2<sup>nd</sup> iteration:  $V_\pi(C) = 6.38, V_\pi(B) = 4.86, V_\pi(E) = 4.86$

|       | $V_\pi(B)$ | $V_\pi(C)$ | $V_\pi(E)$ |
|-------|------------|------------|------------|
| Iter1 | 3.80       | 6          | 3.80       |
| Iter2 | 4.86       | 6.38       | 4.86       |
| Iter3 |            |            |            |
| Iter4 |            |            |            |
| Iter5 |            |            |            |
| Iter6 |            |            |            |

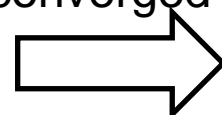


# Iter6 Value Iteration w. Sto Env

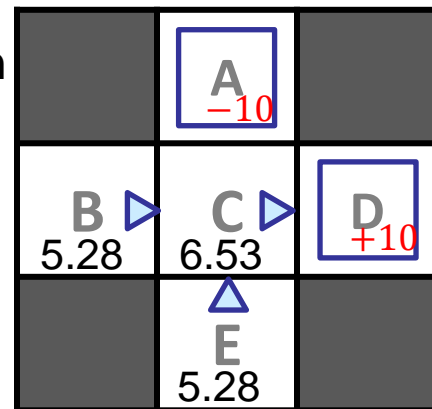
- Value functions  $v_*(s)$  converged to  $v_*(C) = 6.53$ ,  $v_*(B) = v_*(E) = 5.28$  after 6 iterations with threshold condition  $\Delta \leq \theta = 0.01$ . We can get the optimal policy:

- $\pi_*(C) = \operatorname{argmax}_{a \in (l, r, u, d)} q_*(C, a) = r$
- $\pi_*(B) = \operatorname{argmax}_{a \in (l, r, u, d)} q_*(B, a) = r;$
- $\pi_*(E) = \operatorname{argmax}_{a \in (l, r, u, d)} q_*(E, a) = u$

Value Function converged

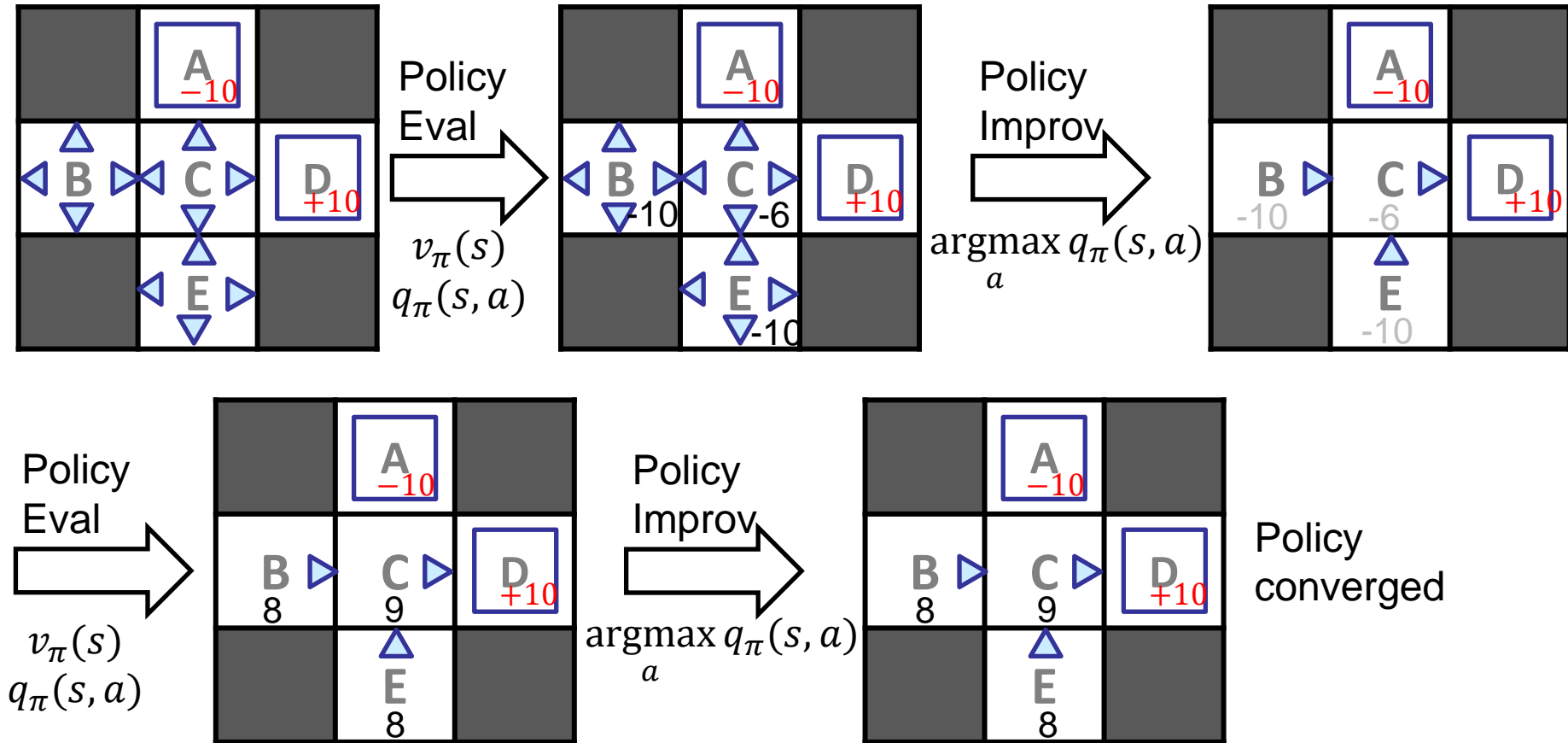


Optimal Policy  
 $\operatorname{argmax}_a Q_*(s, a)$

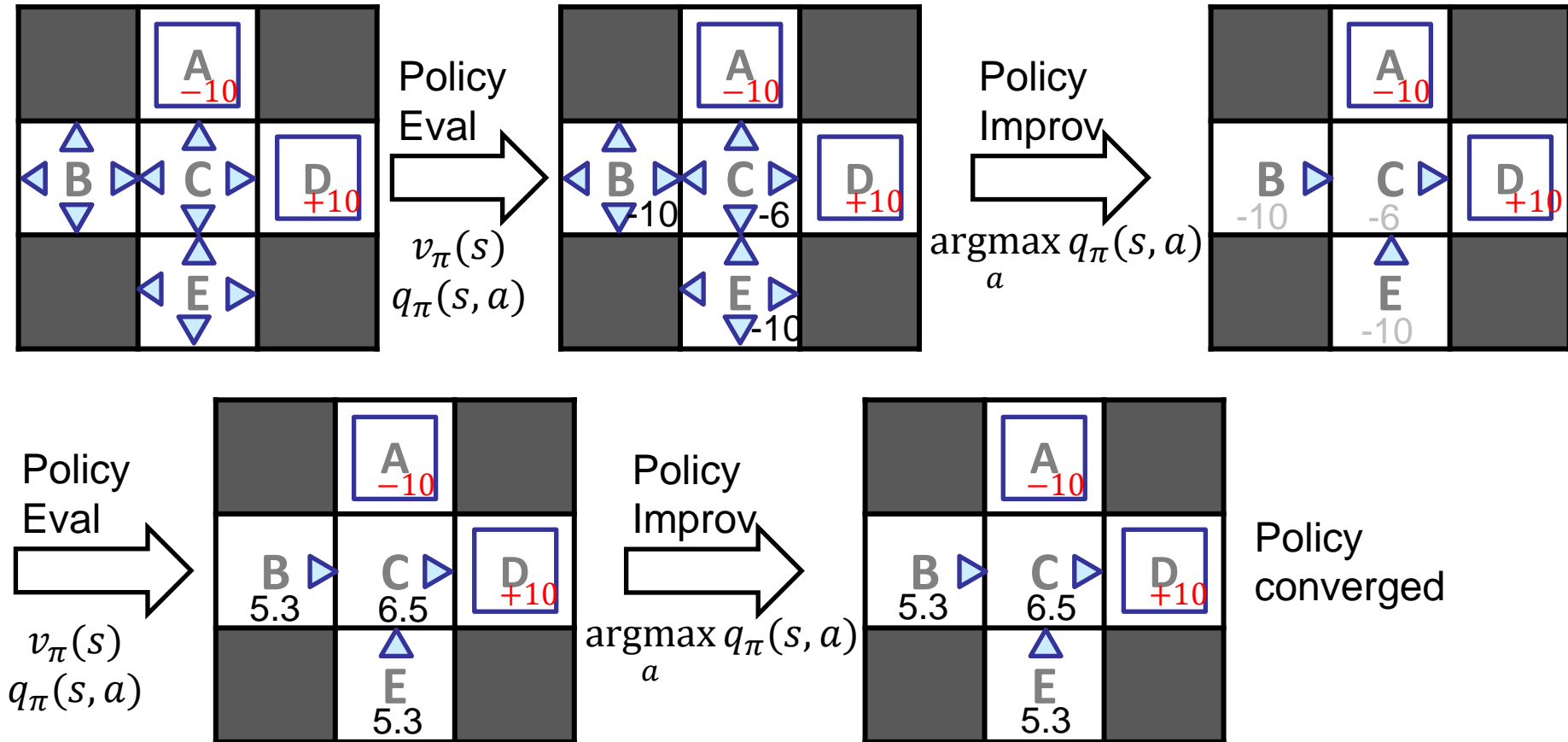


|       | $V_\pi(B)$ | $V_\pi(C)$ | $V_\pi(E)$ |
|-------|------------|------------|------------|
| Iter1 | 3.80       | 6          | 3.80       |
| Iter2 | 4.86       | 6.38       | 4.86       |
| Iter3 | 5.16       | 6.49       | 5.16       |
| Iter4 | 5.25       | 6.52       | 5.25       |
| Iter5 | 5.27       | 6.52       | 5.27       |
| Iter6 | 5.28       | 6.53       | 5.28       |

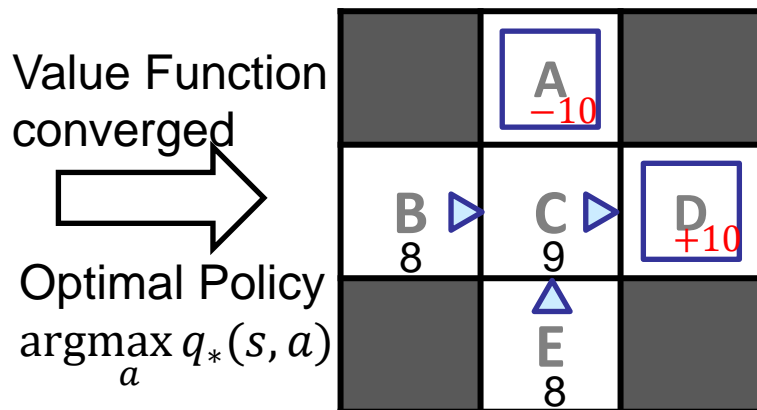
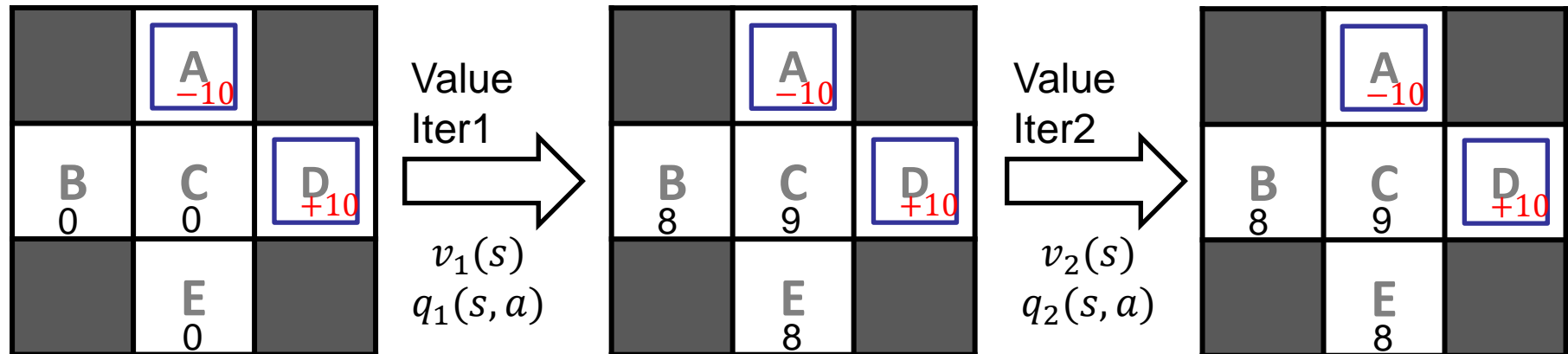
# Policy Iteration w. Deterministic Env



# Policy Iteration w. Stochastic Env



# Value Iteration w. Deterministic Env



# Value Iteration w. Stochastic Env

