L7.2.X Worked Examples

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Important

Recall: TD, Sarsa, Q Learning

- TD solves [BEV] by sampling:
  - $V(S_t) \leftarrow V(S_t) + \alpha(R_{t+1} + \gamma V(S_{t+1}) - V(S_t))$
- Sarsa and Expected Sarsa solve [BEA] by sampling:
  - $Q(S_t, A_t) \leftarrow Q(S_t, A_t) + \alpha(R_{t+1} + \gamma Q(S_{t+1}, A_{t+1}) - Q(S_t, A_t))$
  - $Q(S_t, A_t) \leftarrow Q(S_t, A_t) + \alpha(R_{t+1} + \gamma \sum_{a'} \pi(a'|S_{t+1}) Q(S_{t+1}, a') - Q(S_t, A_t))$
- Q Learning solves [BOA] by sampling:
  - $Q(S_t, A_t) \leftarrow Q(S_t, A_t) + \alpha(R_{t+1} + \gamma \max_{a'} Q(S_{t+1}, a') - Q(S_t, A_t))$

- [BEV] Bellman Expectation Equation for State Value Function:
  - $v_\pi(s) = \sum_a \pi(a|s) \sum_{r,s'} p(r, s'|s, a)\,[r + \gamma v_\pi(s')]$
- [BEA] Bellman Expectation Equation for Action Value Function
  - $q_\pi(s, a) = \sum_{r,s'} p(r, s'|s, a)\,[r + \gamma \sum_{a'} \pi(a'|s') q_\pi(s', a')]$
- [BOA] Bellman Optimality Equation for Optimal Action Value Function:
  - $q_*(s, a) = \sum_{r,s'} p(r, s'|s, a) \left[r + \gamma \max_{a'} q_*(s', a')\right]$

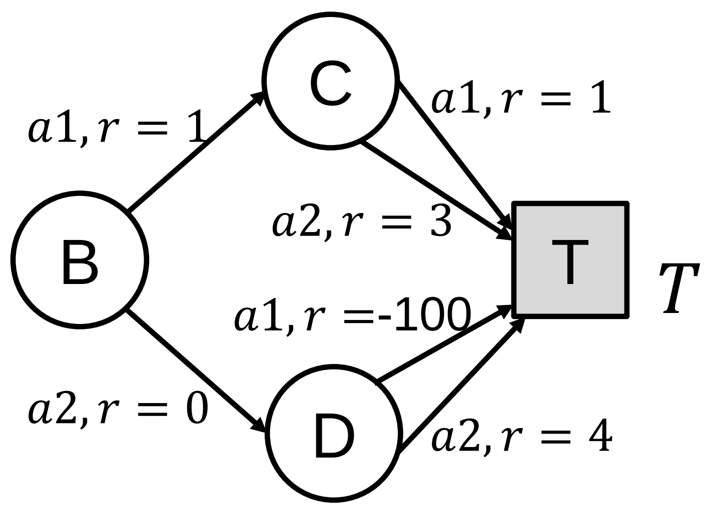
TD, Sarsa, QL w. $\alpha = 1$

- With learning rate $\alpha = 1$, each $V(S_t)$ or $Q(S_t, A_t)$ is completely overwritten by in each update
- update equations simplify to:
  – MC: $V(S_t) \leftarrow V(S_t) + \alpha\big(G_t - V(S_t)\big) = G_t$
  – TD: $V(S_t) \leftarrow V(S_t) + \alpha\big(R_{t+1} + \gamma V(S_{t+1}) - V(S_t)\big) = R_{t+1} + \gamma V(S_{t+1})$
  – Sarsa: $Q(S_t, A_t) \leftarrow Q(S_t, A_t) + \alpha\big(R_{t+1} + \gamma Q(S_{t+1}, A_{t+1}) - Q(S_t, A_t)\big) = R_{t+1} + \gamma Q(S_{t+1}, A_{t+1})$
  – QL: $Q(S_t, A_t) \leftarrow Q(S_t, A_t) + \alpha\left(R_{t+1} + \gamma \max_{a'} Q(S_{t+1}, a') - Q(S_t, A_t)\right) = R_{t+1} + \gamma \max_{a'} Q(S_{t+1}, a')$

Two-Branch Example

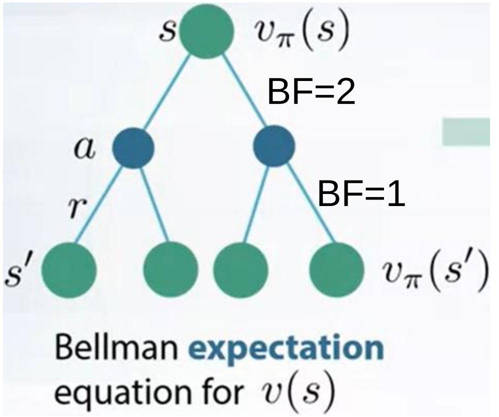
Two-Branch Example

- An episodic MDP w. deterministic env, 3 states $\{B, C, D\}$ and 2 actions $\{1,2\}$ at each state. Discount factor $\gamma = 1$, learning rate $\alpha = 1$. The initial state of each episode is B.

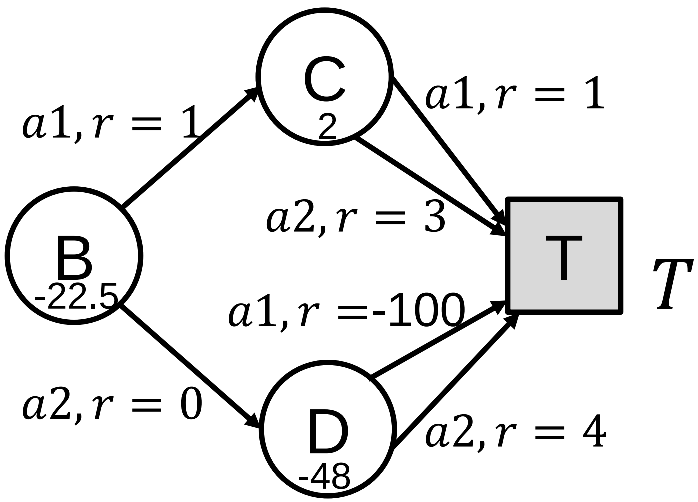


Policy Iteration

1.1 Policy Evaluation of Random Policy



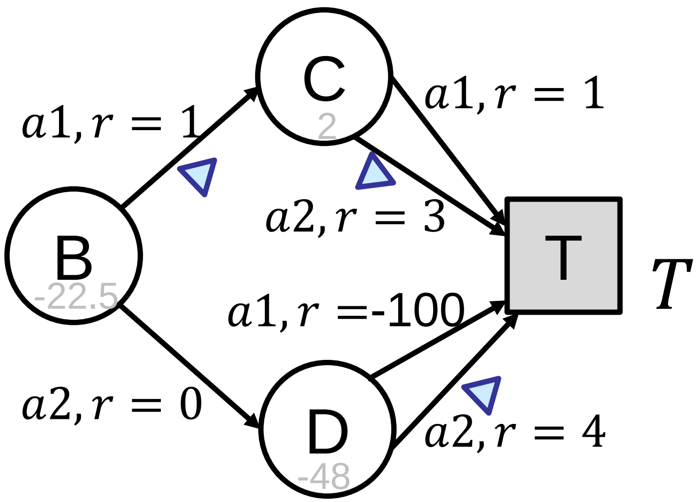
- Bellman Exp Equation: $v_\pi(s) = \sum_a \pi(a|s) q_\pi(s,a)$; $q_\pi(s,a) = R_s^a + \gamma v_\pi(s')$
- $v_\pi(B) = .5[q_\pi(B,a1) + q_\pi(B,a2)] = .5[1 + v_\pi(C) + v_\pi(D)]$
  - $q_\pi(B,a1) = 1 + v_\pi(C), q_\pi(B,a2) = 0 + v_\pi(D)$
- $v_\pi(C) = .5[q_\pi(C,a1) + q_\pi(C,a2)] = 2$
  - $q_\pi(C,a1) = 1, q_\pi(C,a2) = 3$
- $v_\pi(D) = .5[q_\pi(D,a1) + q_\pi(D,a2)] = -48$
  - $q_\pi(D,a1) = -100, q_\pi(D,a2) = 4$
- Solution: $v_\pi(B) = -22.5,\ v_\pi(C) = 2, v_\pi(D) = -48$



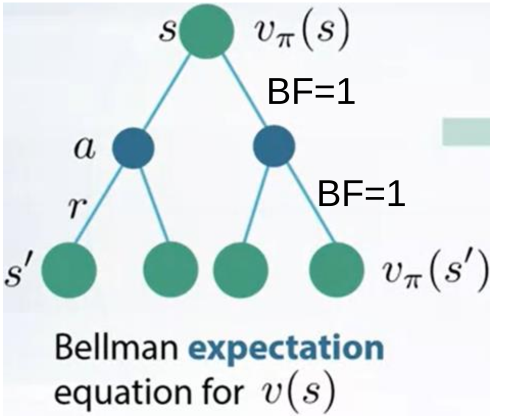
	$V_\pi(B)$	$V_\pi(C)$	$V_\pi(D)$
Iter1	-22.5	3	-48
Iter2	4	3	4
Iter3	4	3	4

1.2 Policy Improvement

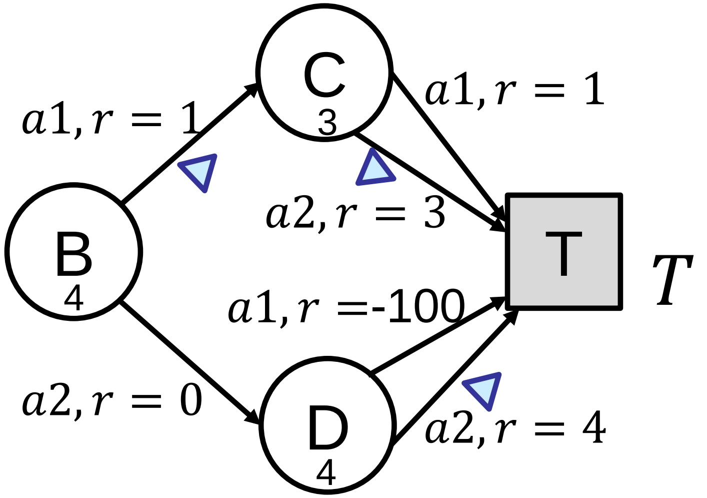
- Plug in values from PE to get new policy
- $\pi'(B) = \text{argmax}_\text{a}\big(q_\pi(B, a1), q_\pi(B, a2)\big) = a1$
  - $q_\pi(B, a1) = 1 + v_\pi(C) = 3, q_\pi(B, a2) = 0 + v_\pi(D) = -22.5$
- $\pi'(C) = \text{argmax}_\text{a}\big(q_\pi(C, a1), q_\pi(C, a2)\big) = a2$
  - $q_\pi(C, a1) = 1, q_\pi(C, a2) = 3$
- $\pi'(D) = \text{argmax}_\text{a}\big(q_\pi(D, a1), q_\pi(D, a2)\big) = a2$
  - $q_\pi(C, a1) = -100, q_\pi(C, a2) = 4$



2.1 Policy Evaluation of Det Policy



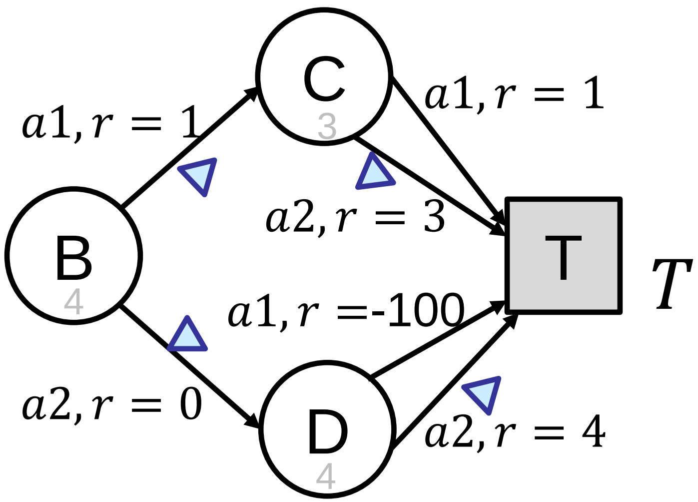
- Bellman Exp Equation: $v_\pi(s) = \sum_a \pi(a|s) q_\pi(s,a)$; $q_\pi(s,a) = R_s^a + \gamma v_\pi(s')$
- $v_\pi(B) = q_\pi(B, a1) = 1 + v_\pi(C)$
  - $q_\pi(B, a1) = 1 + v_\pi(C)$
- $v_\pi(C) = q_\pi(C, a2) = 3$
  - $q_\pi(C, a2) = 3$
- $v_\pi(D) = q_\pi(D, a2) = 4$
  - $q_\pi(D, a2) = 4$
- Solution: $v_\pi(B) = 4,\ v_\pi(C) = 3, v_\pi(D) = 4$



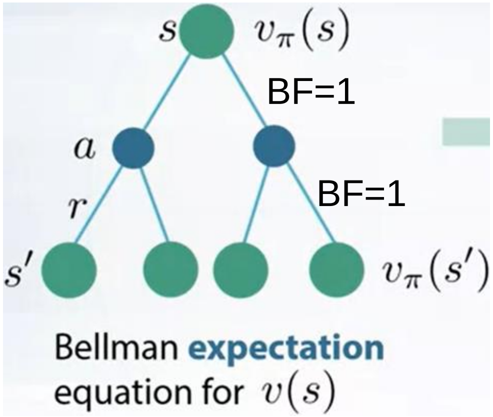
	$V_\pi(B)$	$V_\pi(C)$	$V_\pi(D)$
Iter1	-22.5	3	-48
Iter2	4	3	4
Iter3	4	3	4

2.2 Policy Improvement

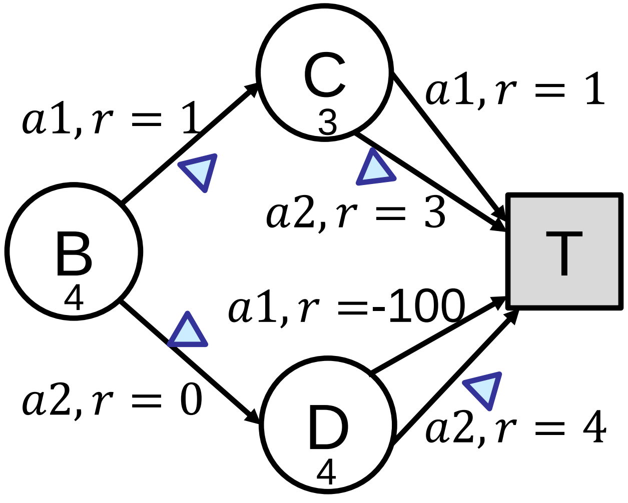
- Plug in values from PE to get new policy
- $\pi'(B) = \text{argmax}_\text{a}\big(q_\pi(B, a1), q_\pi(B, a2)\big) = a1$ or $a2$
  - $q_\pi(B, a1) = 1 + v_\pi(C) = 4, q_\pi(B, a2) = 0 + v_\pi(D) = 4$
- $\pi'(C) = \text{argmax}_\text{a}\big(q_\pi(C, a1), q_\pi(C, a2)\big) = a2$
  - $q_\pi(C, a1) = 1, q_\pi(C, a2) = 3$
- $\pi'(D) = \text{argmax}_\text{a}\big(q_\pi(D, a1), q_\pi(D, a2)\big) = a2$
  - $q_\pi(C, a1) = -100, q_\pi(C, a2) = 4$



3.1 Policy Evaluation



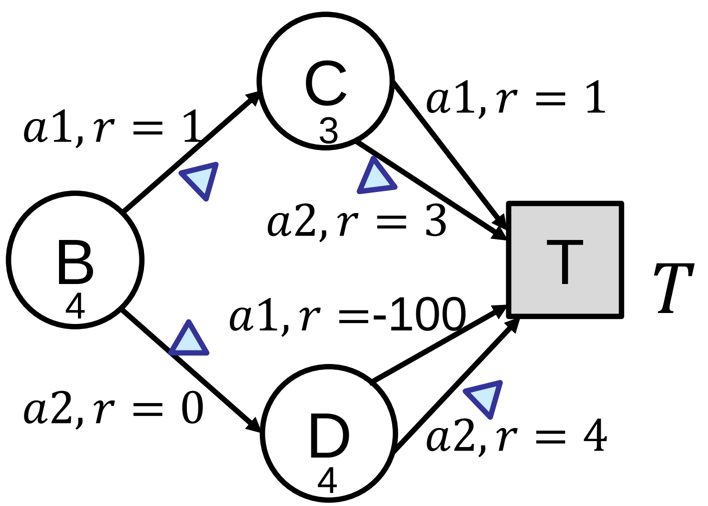
- Bellman Exp Equation: $v_\pi(s) = \sum_a \pi(a|s) q_\pi(s,a)$; $q_\pi(s,a) = R_s^a + \gamma v_\pi(s')$
- $v_\pi(B) = .5[q_\pi(B,a1) + q_\pi(B,a2)] = .5[1 + v_\pi(C) + v_\pi(D)]$
  - $q_\pi(B,a1) = 1 + v_\pi(C), q_\pi(B,a2) = 0 + v_\pi(D)$
- $v_\pi(C) = q_\pi(C,a2) = 3$
  - $q_\pi(C,a2) = 3$
- $v_\pi(D) = q_\pi(D,a2) = 4$
  - $q_\pi(D,a2) = 4$
- Solution: $v_\pi(B) = 4,\ v_\pi(C) = 3, v_\pi(D) = 4$



	$V_\pi(B)$	$V_\pi(C)$	$V_\pi(D)$
Iter1	-22.5	3	-48
Iter2	4	3	4
Iter3	4	3	4

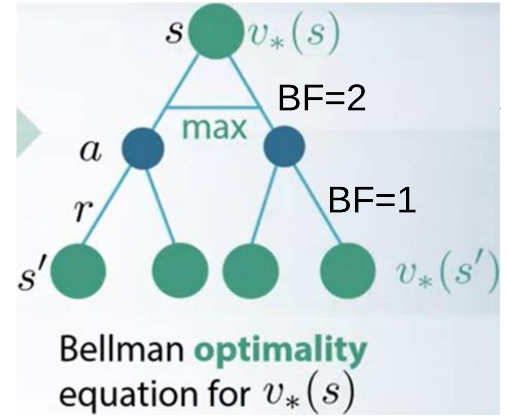
3.2 Policy Improvement

- Plug in values from PE to get new policy
- $\pi'(B) = \text{argmax}_\text{a}\big(q_\pi(B, a1), q_\pi(B, a2)\big) = a1$ or $a2$
  - $q_\pi(B, a1) = 1 + v_\pi(C) = 4, q_\pi(B, a2) = 0 + v_\pi(D) = 4$
- $\pi'(C) = \text{argmax}_\text{a}\big(q_\pi(C, a1), q_\pi(C, a2)\big) = a2$
  - $q_\pi(C, a1) = 1, q_\pi(C, a2) = 3$
- $\pi'(D) = \text{argmax}_\text{a}\big(q_\pi(D, a1), q_\pi(D, a2)\big) = a2$
  - $q_\pi(C, a1) = -100, q_\pi(C, a2) = 4$
- Policy has converged

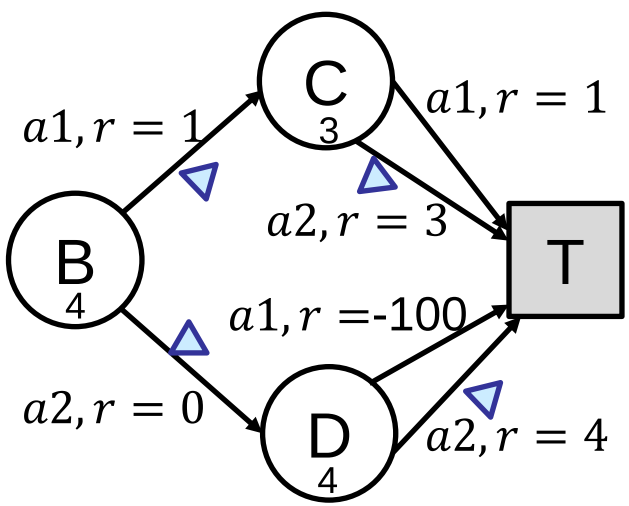


Value Iteration

Value Iteration



- Bellman Opt Equation: $v_*(s) = \max_a q_*(s,a); q_*(s,a) = R_s^a + \gamma v_*(s')$
- $v_*(B) = \max_a[q_*(B,a1),\ q_*(B,a2)] = \max[1 + v_*(C), v_*(D)]$
  - $q_*(B,a1) = 1 + v_*(C), q_*(B,a2) = 0 + v_*(D)$
- $v_*(C) = \max_a[q_*(C,a1),\ q_*(C,a2)] = q_*(C,a2) = 3$
  - $q_*(C,a1) = 1, q_*(C,a2) = 3$
- $v_*(D) = \max_a[q_*(D,a1), q_*(D,a2)] = 4$
  - $q_*(D,a1) = -100, q_*(D,a2) = 4$
- We use Value Iteration to solve it. Table shows the iteration process until convergence (not using in-place updates for clarity). Solution: $v_*(1) = -3,\ v_*(2) = -2, v_*(3) = -1$
- Optimal policy: $\pi_*(\mathrm{B}) = \operatorname{argmax}_{\mathrm{a}} q_*(B,a) = a1$ or $a2; \pi_*(\mathrm{C}) = \operatorname{argmax}_{\mathrm{a}} q_*(\mathrm{C},a) = a2; \pi_*(\mathrm{D}) = \operatorname{argmax}_{\mathrm{a}} q_*(D,a) = a2$

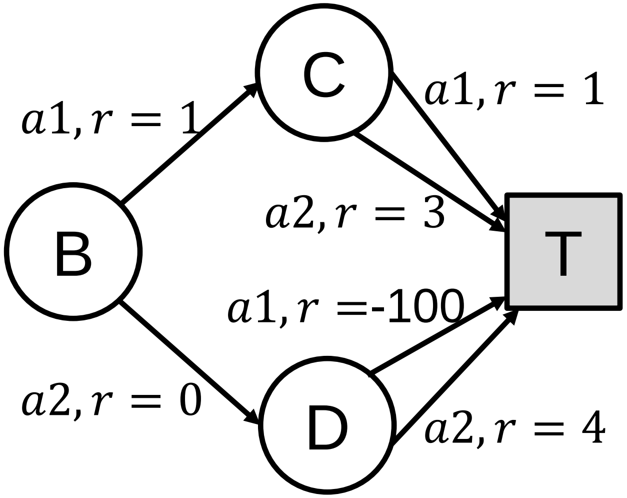


	$V_\pi(B)$	$V_\pi(C)$	$V_\pi(D)$
Init	0	0	0
Iter1	0	3	4
Iter2	4	3	4
Iter3	4	3	4

MC

MC, Episodes $3 \times (B, a2, 0, D, a1, -100, T)$

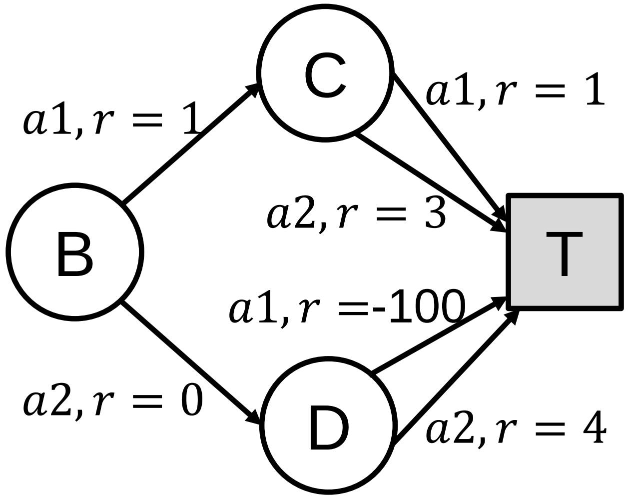
- MC update equation: $V(S_t) \leftarrow G_t$
- EP1:
- $G(D) = -100 + V(T) = -100, V(D) = G(D) = -100$
- $G(B) = 0 + G(D) = -100, V(D) = G(D) = -100$
- EP2:
- $G(D) = -100 + V(T) = -100, V(D) = G(D) = -100$
- $G(B) = 0 + G(D) = -100, V(D) = G(D) = -100$
- EP3:
- $G(D) = -100 + V(T) = -100, V(D) = G(D) = -100$
- $G(B) = 0 + G(D) = -100, V(D) = G(D) = -100$



	$\boldsymbol{V}(B)$	$\boldsymbol{V}(D)$
Init	0	0
EP1	-100	-100
EP2	-100	-100
EP3	-100	-100

MC, Episodes $n \times (B, a2, 0, D, a2, 4, T)$

- MC update equation: $V(S_t) \leftarrow G_t$
- EP1:
- $G(D) = -100 + V(T) = 4, V(D) = G(D) = 4$
- $G(B) = 0 + G(D) = 4, V(D) = G(D) = 4$
- EP2:
- $G(D) = -100 + V(T) = 4, V(D) = G(D) = 4$
- $G(B) = 0 + G(D) = 4, V(D) = G(D) = 4$
- EP3:
- $G(D) = -100 + V(T) = 4, V(D) = G(D) = 4$
- $G(B) = 0 + G(D) = 4, V(D) = G(D) = 4$

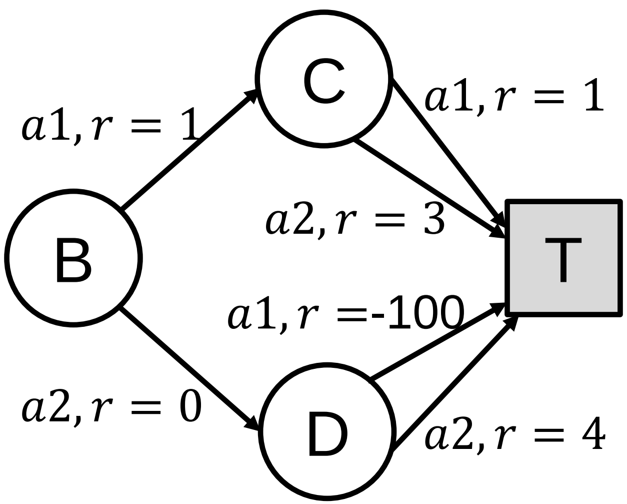


	$\boldsymbol{V}(B)$	$\boldsymbol{V}(D)$
Init	0	0
EP1	4	4
EP2	4	4
EP3	4	4

TD

TD, Episodes $n \times (B, a2, 0, D, a1, -100, T)$

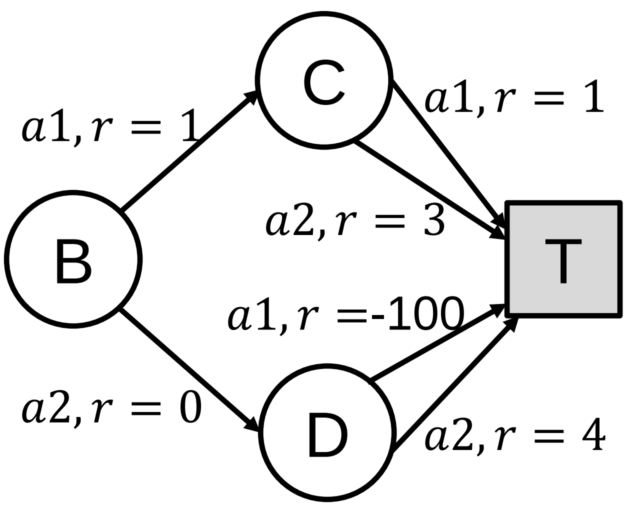
- TD update equation: $V(S_t) \leftarrow R_{t+1} + \gamma V(S_{t+1})$
- EP1:
- $V(B) \leftarrow R_{t+1} + \gamma V(D) = 0 + 0 = 0$
- $V(D) \leftarrow R_{t+1} + \gamma V(T) = -100 - 0 = -100$
- EP2:
- $V(B) \leftarrow R_{t+1} + \gamma V(D) = 0 - 100 = -100$
- $V(D) \leftarrow R_{t+1} + \gamma V(T) = -100 - 0 = -100$
- EP3:
- $V(B) \leftarrow R_{t+1} + \gamma V(D) = 0 - 100 = -100$
- $V(D) \leftarrow R_{t+1} + \gamma V(T) = -100 - 0 = -100$



	$\boldsymbol{V}(B)$	$\boldsymbol{V}(D)$
Init	0	0
EP1	0	-100
EP2	-100	-100
EP3	-100	-100

TD, Episodes $3 \times (B, a2, 0, D, a2, 4, T)$

- TD update equation: $V(S_t) \leftarrow R_{t+1} + \gamma V(S_{t+1})$
- EP1:
- $V(B) \leftarrow R_{t+1} + \gamma V(D) = 0 + 0 = 0$
- $V(D) \leftarrow R_{t+1} + \gamma V(T) = 4 - 0 = 4$
- EP2:
- $V(B) \leftarrow R_{t+1} + \gamma V(D) = 0 + 4 = 4$
- $V(D) \leftarrow R_{t+1} + \gamma V(T) = 4 - 0 = 4$
- EP3:
- $V(B) \leftarrow R_{t+1} + \gamma V(D) = 0 + 4 = 4$
- $V(D) \leftarrow R_{t+1} + \gamma V(T) = 4 - 0 = 4$

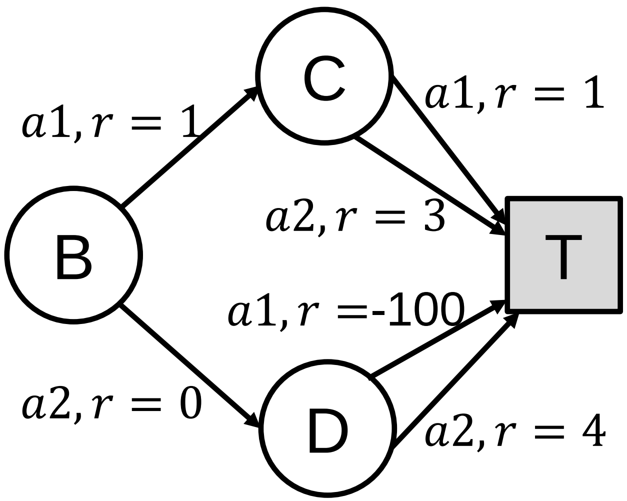


	$\boldsymbol{V}(B)$	$\boldsymbol{V}(D)$
Init	0	0
EP1	0	4
EP2	4	4
EP3	4	4

Sarsa

Sarsa, Episodes $3 \times (B, a2, 0, D, a1, -100, T)$

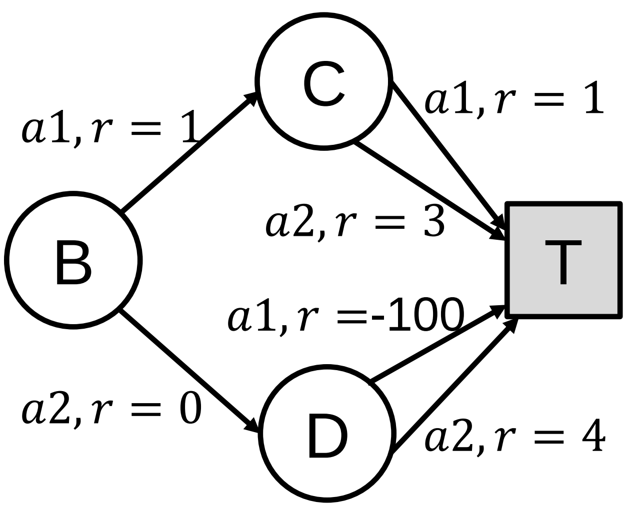
- Sarsa update equation: $Q(S_t, A_t) \leftarrow R_{t+1} + \gamma Q(S_{t+1}, A_{t+1})$
- EP1:
- $Q(B, a2) \leftarrow R_{t+1} + \gamma Q(D, a1) = 0 - 0 = 0$
- $Q(D, a1) \leftarrow R_{t+1} + \gamma Q(T, -) = -100 + 0 = -100$
- EP2:
- $Q(B, a2) \leftarrow R_{t+1} + \gamma Q(D, a1) = 0 - 100 = -100$
- $Q(D, a1) \leftarrow R_{t+1} + \gamma Q(T, -) = -100 + 0 = -100$
- EP3:
- $Q(B, a2) \leftarrow R_{t+1} + \gamma Q(D, a1) = 0 - 100 = -100$
- $Q(D, a1) \leftarrow R_{t+1} + \gamma Q(T, -) = -100 + 0 = -100$



	$Q(B, a1)$	$Q(B, a2)$	$Q(D, a1)$	$Q(D, a2)$
Init	0	0	0	0
EP1	0	0	-100	0
EP2	0	-100	-100	0
EP3	0	-100	-100	0

Sarsa, Episodes $3 \times (B, a2, 0, D, a2, 4, T)$

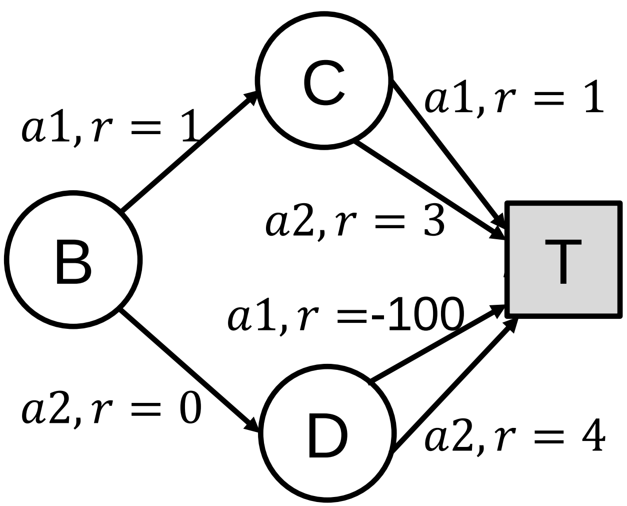
- Sarsa update equation: $Q(S_t, A_t) \leftarrow R_{t+1} + \gamma Q(S_{t+1}, A_{t+1})$
- EP1:
- $Q(B, a2) \leftarrow R_{t+1} + \gamma Q(D, a2) = 0 - 0 = 0$
- $Q(D, a2) \leftarrow R_{t+1} + \gamma Q(T, -) = 4 + 0 = 4$
- EP2:
- $Q(B, a2) \leftarrow R_{t+1} + \gamma Q(D, a2) = 0 + 4 = 4$
- $Q(D, a2) \leftarrow R_{t+1} + \gamma Q(T, -) = 4 + 0 = 4$
- EP3:
- $Q(B, a2) \leftarrow R_{t+1} + \gamma Q(D, a2) = 0 + 4 = 4$
- $Q(D, a2) \leftarrow R_{t+1} + \gamma Q(T, -) = 4 + 0 = 4$



	$Q(B, a1)$	$Q(B, a2)$	$Q(D, a1)$	$Q(D, a2)$
Init	0	0	0	0
EP1	0	0	0	4
EP2	0	4	0	4
EP3	0	4	0	4

QL, Episodes $3 \times (B, a2, 0, D, a1, -100, T)$

- QL update equation: $Q(S_t, A_t) \leftarrow R_{t+1} + \gamma \max_{a'} Q(S_{t+1}, a')$
- EP1:
- $Q(B, a2) \leftarrow R_{t+1} + \gamma \max_{a} Q(D, a) = 0 + \max(0, 0) \ = 0$
- $Q(D, a1) \leftarrow R_{t+1} + \gamma Q(T, -) = -100 + 0 = -100$
- EP2:
- $Q(B, a2) \leftarrow R_{t+1} + \gamma \max_{a} Q(D, a) = 0 + \max(-100, 0) \ = 0$
- $Q(D, a1) \leftarrow R_{t+1} + \gamma Q(T, -) = -100 + 0 = -100$
- EP3:
- $Q(B, a2) \leftarrow R_{t+1} + \gamma \max_{a} Q(D, a) = 0 + \max(-100, 0) \ = 0$
- $Q(D, a1) \leftarrow R_{t+1} + \gamma Q(T, -) = -100 + 0 = -100$

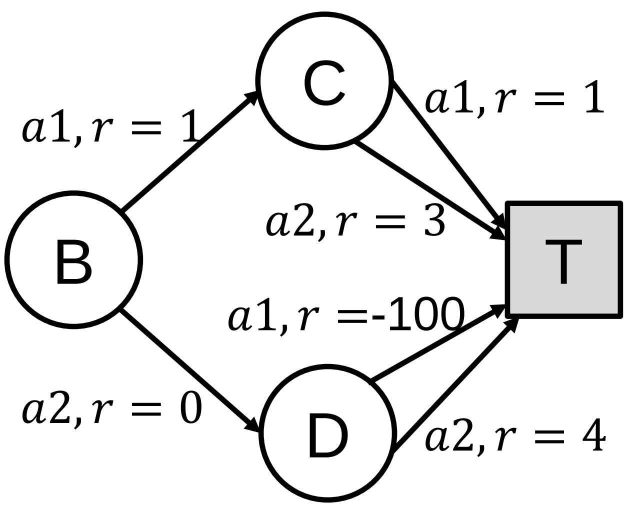


	$Q(B, a1)$	$Q(B, a2)$	$Q(D, a1)$	$Q(D, a2)$
Init	0	0	0	0
EP1	0	0	-100	0
EP2	0	0	-100	0
EP3	0	0	-100	0

Q Learning

QL, Episodes $3 \times (B, 2, 0, D, 2, 4, T)$

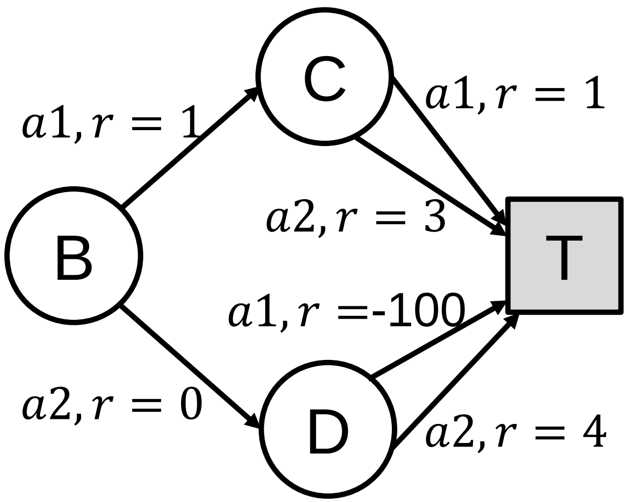
- QL update equation: $Q(S_t, A_t) \leftarrow R_{t+1} + \gamma \max_{a'} Q(S_{t+1}, a')$
- EP1:
- $Q(B, a2) \leftarrow R_{t+1} + \gamma \max_a Q(D, a) = 0 + \max(0, 0) \ = 0$
- $Q(D, a2) \leftarrow R_{t+1} + \gamma Q(T, -) = 4 + 0 = 4$
- EP2:
- $Q(B, a2) \leftarrow R_{t+1} + \gamma \max_a Q(D, a) = 0 + 4 \ = 4$
- $Q(D, a2) \leftarrow R_{t+1} + \gamma Q(T, -) = 4 + 0 = 4$
- EP3:
- $Q(B, a2) \leftarrow R_{t+1} + \gamma \max_a Q(D, a) = 0 + 4 \ = 4$
- $Q(D, a2) \leftarrow R_{t+1} + \gamma Q(T, -) = 4 + 0 = 4$



	$Q(B, a1)$	$Q(B, a2)$	$Q(D, a1)$	$Q(D, a2)$
Init	0	0	0	0
EP1	0	0	0	4
EP2	0	4	0	4
EP3	0	4	0	4

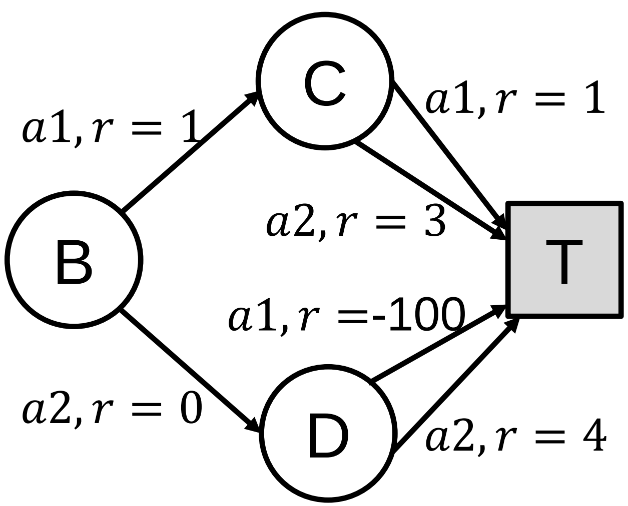
Comparisons

- MC and TD:
  - Transition $(D, a1, -100, T)$ drives $V(D) \to -100$; $V(D)$ drives $V(B) \to -100$.
  - Transition $(D, a2, 4, T)$ drives $V(D) \to 4$; $V(D)$ drives $V(B) \to 4$.
  - Final values of $V(B), V(D)$ depend on relative execution frequencies of the 2 transitions (e.g., ϵ-greedy).
- Sarsa:
  - Transition $(D, a1, -100, T)$ drives $Q(D, a1) \to -100$; $Q(D, 1)$ drives $Q(B, a2) \to -100$.
  - Transition $(D, a2, 4, T)$ drives $Q(D, a2) \to 4$; $Q(D, a2)$ drives $Q(B, a2) \to 4$.
  - Final value of $Q(B, 2)$ depends on relative execution frequencies of the 2 transitions (e.g., ϵ-greedy).
- QL:
  - Transition $(D, a1, -100, T)$ drives $Q(D, a1) \to -100$; $Q(D, a1)$ does not affect $Q(B, a2)$ since
  - $\max_a Q(D, a) = \max(Q(D, a1), Q(D, a2)) = 0$. (assuming $Q(D, a2)$ is initialized to 0 and it never updated)
  - Transition $(D, a2, 4, T)$ drives $Q(D, a2) \to 4$, which in turn drives $Q(B, a2) \to 4$.
- Our example here only illustrates the policy evaluation for a given set of episodes, not control. If we consider control, and Sarsa or QL uses ϵ-greedy policy with small ϵ, then the agent will likely avoid action $\mathrm{a}1$ in state D after taking it for the 1st time.



Sarsa w. ϵ-greedy

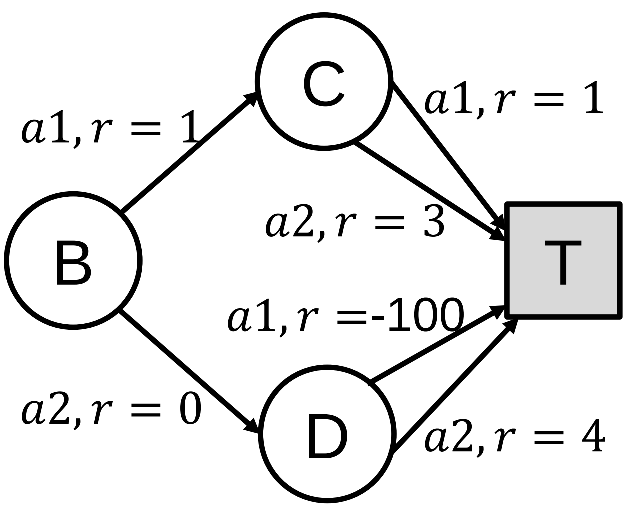
- Suppose EP1 is $(B, a2, 0, D, a1, -100, T)$. After EP1:
- $Q(B, a2) \leftarrow R_{t+1} + \gamma Q(D, a1) = 0 - 0 = 0$, $Q(D, a1) \leftarrow R_{t+1} + \gamma Q(T, -) = -100 + 0 = -100$
- Suppose EP2 starts with $(B, a2, 0, D)$, then in state D, the agent is likely to select action $\text{argmax}_a\{Q(D, a1) = -100, Q(D, a2) = 0\} = a2$ based on ϵ-greedy, so the episode is $(B, a2, 0, D, a2, 4, T)$. After EP2:
- $Q(B, a2) \leftarrow R_{t+1} + \gamma Q(D, a1) = 0 - 100 = -100$, $Q(D, a2) \leftarrow R_{t+1} + \gamma Q(T, -) = 4 + 0 = 4$
- In EP3, in initial state B, the agent is likely to select action $\text{argmax}_a\{Q(B, a1) = 0, Q(B, a2) = -100\} = a1$. Suppose the episode is $(B, a1, 1, C, a1, 1, T)$
- $Q(B, a1) \leftarrow R_{t+1} + \gamma Q(C, a1) = 1 + 0 = 1$, $Q(C, a1) \leftarrow R_{t+1} + \gamma Q(T, -) = 1 + 0 = 1$
- In EP4, in initial state B, the agent is likely to select action $\text{argmax}_a\{Q(B, a1) = 1, Q(B, a2) = -100\} = a1$. in state D, the agent is likely to select action $\text{argmax}_a\{Q(D, a1) = 1, Q(D, a2) = 0\} = a1$. Suppose the episode is again $(B, a1, 1, C, a1, 1, T)$
- $Q(B, a1) \leftarrow R_{t+1} + \gamma Q(C, a1) = 1 + 1 = 2$, $Q(C, a1) \leftarrow R_{t+1} + \gamma Q(T, -) = 1 + 0 = 1$
- if the agent always follows the greedy action, it will always follow the trajectory $(B, a1, 1, C, a1, 1, T)$ and never learn anything new, e.g., it will never experience the trajectories $(B, a1, 1, C, a2, 3, T), (B, a2, 0, D, a2, 4, T)$. It got scared when $Q(B, a2)$ was updated to -100 after EP2 and never wanted to take action a2 in state B, but if it were more adventurous and tried it, it will likely experience EP5 $(B, a2, 0, D, a2, 4, T)$:
- $Q(B, a2) \leftarrow R_{t+1} + \gamma Q(D, a2) = 0 + 4 = 4$, $Q(D, a2) \leftarrow R_{t+1} + \gamma Q(T, -) = 4 + 0 = 4$
- Now you can see the importance of exploration by selecting the non-greedy action occasionally.



	$Q(B,a1)$	$Q(B,a2)$	$Q(C,a1)$	$Q(C,a2)$	$Q(D,a1)$	$Q(D,a2)$
Init	0	0	0	0	0	0
EP1	0	0	0	0	-100	0
EP2	0	-100	0	0	-100	4
EP3	1	-100	1	0	-100	4
EP4	2	-100	1	0	-100	4
EP5	2	4	1	0	-100	4

QL w. ϵ-greedy

- Suppose EP1 is $(B, a2, 0, D, a1, -100, T)$. After EP1:
- $Q(B,a2) \leftarrow R_{t+1} + \gamma \max_a Q(D,a) = 0 + \max(0,0) = 0, Q(D,a1) \leftarrow R_{t+1} + \gamma Q(T,-) = -100 + 0 = -100$
- Suppose EP2 starts with $(B, a2, 0, D)$, then in state D, the agent is likely to select action $\text{argmax}_a\{Q(D,a1) = -100, Q(D,a2) = 0\} = a2$ based on ϵ-greedy, so the episode is $(B, a2, 0, D, a2, 4, T)$. After EP2:
- $Q(B,a2) \leftarrow R_{t+1} + \gamma \max_a Q(D,a) = 0 + \max(-100,0) = 0, Q(D,a2) \leftarrow R_{t+1} + \gamma Q(T,-) = 4 + 0 = 4$
- In EP3, in initial state B, the agent is equally likely to select action $a1$ and $a2$ since $Q(B,a1) = Q(B,a2) = 0$. Suppose the episode is $(B, a1, 1, C, a1, 1, T)$
- $Q(B,a1) \leftarrow R_{t+1} + \gamma \max_a Q(C,a) = 1 + \max(0,0) = 1$, $Q(C,a1) \leftarrow R_{t+1} + \gamma Q(T,-) = 1 + 0 = 1$
- In EP4, in initial state B, the agent is likely to select action $\text{argmax}_a\{Q(B,a1) = 1, Q(B,a2) = 0\} = a1$. in state D, the agent is likely to select action $\text{argmax}_a\{Q(D,a1) = 1, Q(D,a2) = 0\} = a1$. Suppose the episode is $(B, a1, 1, C, a1, 1, T)$
- $Q(B,a1) \leftarrow R_{t+1} + \gamma \max_a Q(C,a) = 1 + \max(1,0) = 2$, $Q(C,a1) \leftarrow R_{t+1} + \gamma Q(T,-) = 1 + 0 = 1$
- The difference from Sarsa lies in $Q(B,a2)$, which stays at 0 until the agent experienced EP4. So it got less scared than Sarsa (where $Q(B,a2)$ was updated to -100 after EP2), so QL agent is more likely to explore unseen states.
- Suppose EP5 is $(B, a2, 0, D, a2, 4, T)$:
- $Q(B,a2) \leftarrow R_{t+1} + \gamma \max_a Q(D,a) = 0 + \max(-100,4) = 4$, $Q(D,a2) \leftarrow R_{t+1} + \gamma Q(T,-) = 4 + 0 = 4$

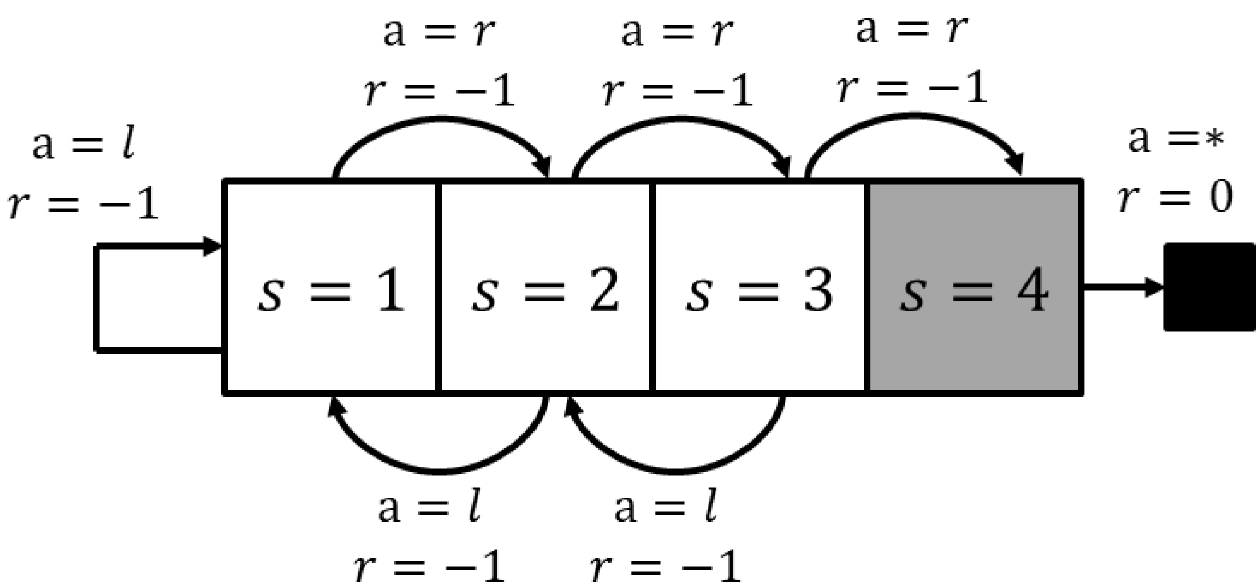


	$Q(B,a1)$	$\boldsymbol{Q(B,a2)}$	$Q(C,a1)$	$Q(C,a2)$	$Q(D,a1)$	$Q(D,a2)$
Init	0	**0**	0	0	0	0
EP1	0	**0**	0	0	-100	0
EP2	0	**0**	0	0	-100	4
EP3	1	**0**	1	0	-100	4
EP4	2	**0**	1	0	-100	4
EP5	2	**4**	1	0	-100	4

Linear Chain Example

Linear Chain Example

- Consider the following MDP. Environment is deterministic. In each state, there are two possible actions a∈{l,r}, where l corresponds to moving left, and r corresponds to moving right. Each movement incurs a reward of r=-1. State s=4 is the goal state: taking any action from s=4 results in reward of r=0 and ends the episode by going into the terminal state, hence $V(4) = 0, Q(4, a) = 0$ for any action a. (Alternatively, we can view state 4 as the terminal state itself.) Assume $\gamma = 1, \alpha = 1$. All value functions are initialized to 0.
- A. Use Policy Iteration, Value Iteration to derive optimal policy.



TD, Sarsa, QL

- B. Consider 8 given consecutive episodes in the form of (s,a,r) (we do not consider ϵ-greedy exploration here):

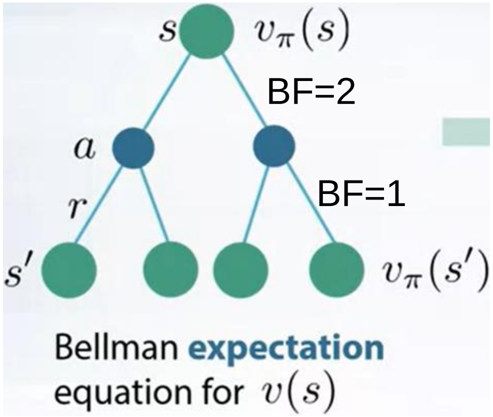
1. EP1: $(1, \mathrm{r}, -1), (2, \mathrm{r}, -1), (3, \mathrm{r}, -1), (4, \mathrm{r}, 0)$
2. EP2: $(1, \mathrm{r}, -1), (2, \mathrm{r}, -1), (3, \mathrm{r}, -1), (4, \mathrm{r}, 0)$
3. EP3: $(1, \mathrm{r}, -1), (2, \mathrm{r}, -1), (3, \mathrm{r}, -1), (4, \mathrm{r}, 0)$
4. EP4: $(3, \mathrm{l}, -1), (2, \mathrm{l}, -1), (1, \mathrm{l}, -1), (1, \mathrm{r}, -1), (2, \mathrm{r}, -1), (3, \mathrm{r}, -1), (4, \mathrm{r}, 0)$
5. EP5: $(3, \mathrm{l}, -1), (2, \mathrm{l}, -1), (1, \mathrm{l}, -1), (1, \mathrm{r}, -1), (2, \mathrm{r}, -1), (3, \mathrm{r}, -1), (4, \mathrm{r}, 0)$
6. EP6: $(3, \mathrm{l}, -1), (2, \mathrm{l}, -1), (1, \mathrm{l}, -1), (1, \mathrm{r}, -1), (2, \mathrm{r}, -1), (3, \mathrm{r}, -1), (4, \mathrm{r}, 0)$
7. EP7: $(3, \mathrm{l}, -1), (2, \mathrm{l}, -1), (1, \mathrm{l}, -1), (1, \mathrm{r}, -1), (2, \mathrm{r}, -1), (3, \mathrm{r}, -1), (4, \mathrm{r}, 0)$
8. EP8: $(3, \mathrm{l}, -1), (2, \mathrm{l}, -1), (1, \mathrm{l}, -1), (1, \mathrm{r}, -1), (2, \mathrm{r}, -1), (3, \mathrm{r}, -1), (4, \mathrm{r}, 0)$

- Derive the following:

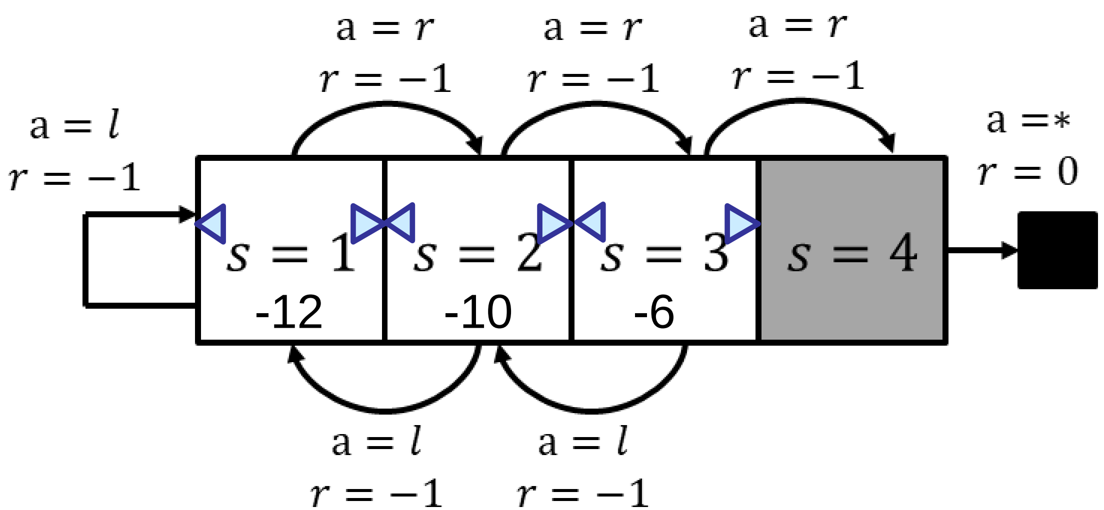
1. State value functions (V functions) after TD learning.
2. State-action value functions (Q Functions) after Sarsa, and the resulting policy.
3. State-action value functions (Q Functions) after Q learning, and the resulting policy.

Policy Iteration

1.1 Policy Evaluation of Random Policy



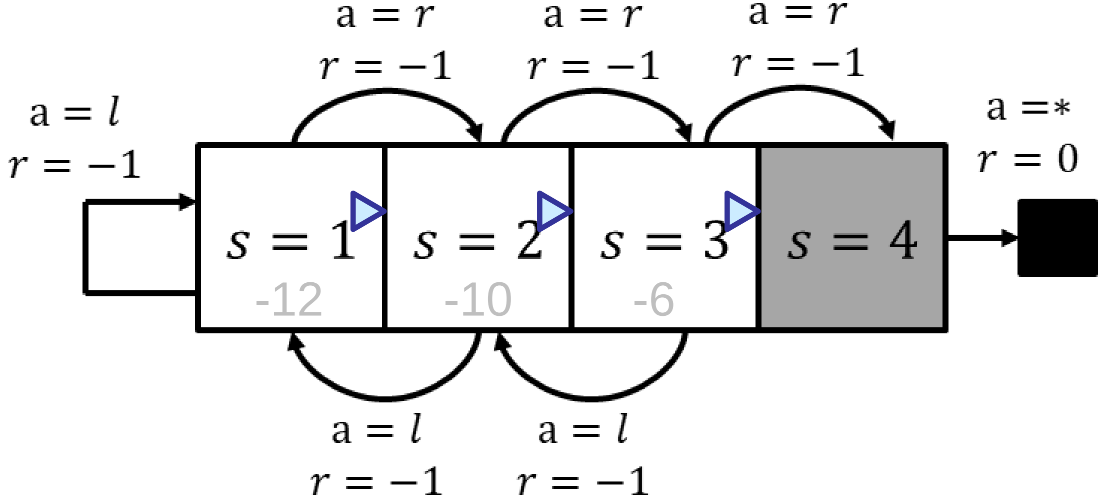
- Bellman Exp Equation: $v_\pi(s) = \sum_a \pi(a|s) q_\pi(s,a)\,;\, q_\pi(s,a) = R_s^a + \gamma v_\pi(s')$
- $v_\pi(1) = .5[q_\pi(1,l) + q_\pi(1,r)] = -1 + .5[v_\pi(1) + v_\pi(2)]$
  - $q_\pi(1,l) = -1 + v_\pi(1), q_\pi(1,r) = -1 + v_\pi(2)$
- $v_\pi(2) = .5[q_\pi(2,l) + q_\pi(2,r)] = -1 + .5[v_\pi(1) + v_\pi(3)]$
  - $q_\pi(2,l) = -1 + v_\pi(1), q_\pi(2,r) = -1 + v_\pi(3)$
- $v_\pi(3) = .5[q_\pi(3,l) + q_\pi(3,r)] = -\,1 + .5\, v_\pi(2)$
  - $q_\pi(3,l) = -1 + v_\pi(2), q_\pi(3,r) = -1 + v(4) = -1$
- Solution: $v_\pi(1) = -12,\; v_\pi(2) = -10, v_\pi(3) = -6$



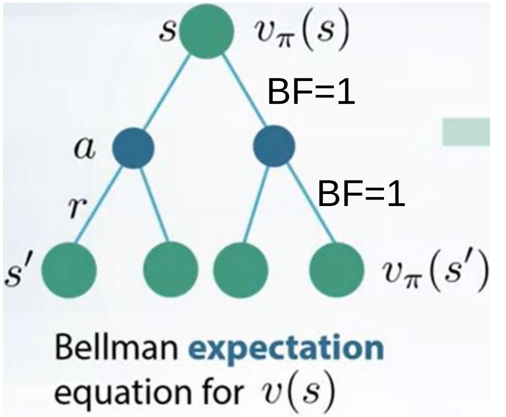
	$V_\pi(1)$	$V_\pi(2)$	$V_\pi(3)$
Iter1	-12	-10	-6
Iter2	4	3	4

1.2 Policy Improvement

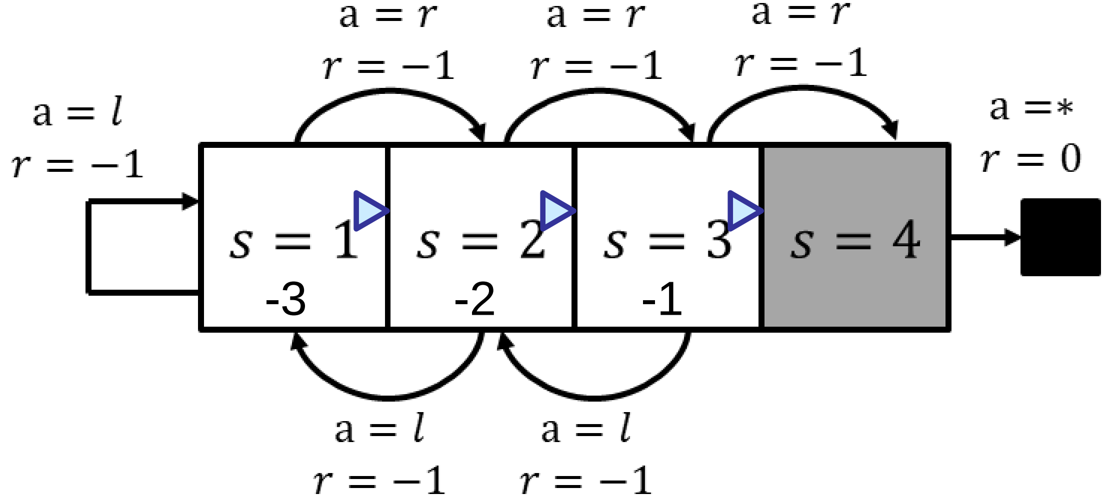
- Plug in values from PE to get new policy
- $\pi'(1) = \text{argmax}_\text{a}\big(q_\pi(1,l), q_\pi(1,r)\big) = r$
  - $q_\pi(1,l) = -1 + v_\pi(1) = -13, q_\pi(1,r) = -1 + v_\pi(2) = -11,$
- $\pi'(2) = \text{argmax}_\text{a}\big(q_\pi(2,l), q_\pi(2,r)\big) = r$
  - $q_\pi(2,l) = -1 + v_\pi(1) = -13, q_\pi(2,r) = -1 + v_\pi(3) = -7$
- $\pi'(3) = \text{argmax}_\text{a}\big(q_\pi(3,l), q_\pi(3,r)\big) = r$
  - $q_\pi(3,l) = -1 + v_\pi(2) = -11, q_\pi(3,r) = -1$



2.1 Policy Evaluation of Det Policy



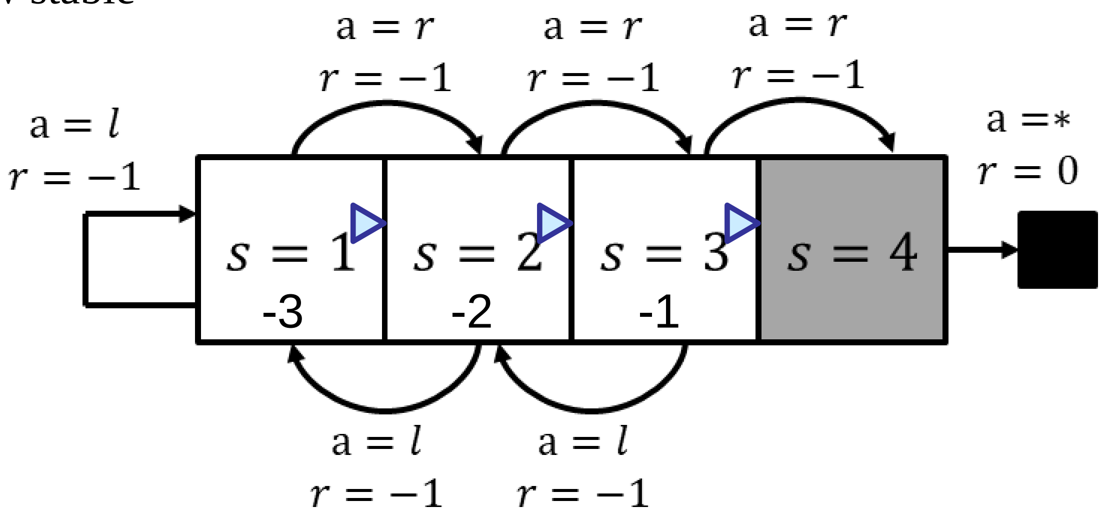
- $v_\pi(1) = 1.0q_\pi(1,r) = -1 + v_\pi(2)$
  - $q_\pi(1,r) = -1 + v_\pi(2)$
- $v_\pi(2) = 1.0q_\pi(2,r) = -1 + v_\pi(3)$
  - $q_\pi(2,r) = -1 + v_\pi(3)$
- $v_\pi(3) = 1.0q_\pi(3,r) = -1$
  - $q_\pi(3,r) = -1$
- Solution: $v_\pi(1) = -3,\ v_\pi(2) = -2, v_\pi(3) = -1$



	$V_\pi(1)$	$V_\pi(2)$	$V_\pi(3)$
Iter1	-12	-10	-6
Iter2	-3	-2	-1

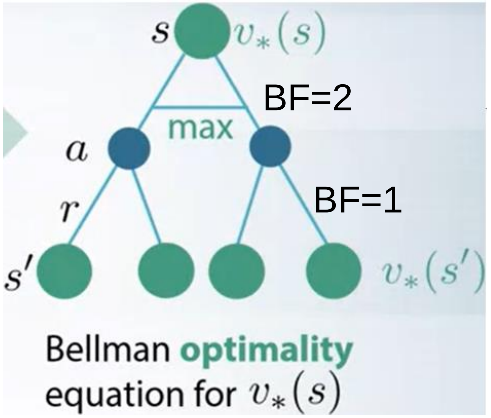
2.2 Policy Improvement

- Plug in values from PE to get new policy
- $\pi'(1) = \text{argmax}_{\text{a}}\big(q_\pi(1,l), q_\pi(1,r)\big) = r$
  - $q_\pi(1,l) = -1 - 3 = -4, q_\pi(1,r) = -1 - 2 = -3$
- $\pi'(2) = \text{argmax}_{\text{a}}\big(q_\pi(2,l), q_\pi(2,r)\big) = r$
  - $q_\pi(2,l) = -1 - 3 = -4, q_\pi(2,r) = -1 - 1 = -2$
- $\pi'(3) = \text{argmax}_{\text{a}}\big(q_\pi(3,l), q_\pi(3,r)\big) = r$
  - $q_\pi(3,l) = -1 - 2 = -3, q_\pi(3,r) = -1$
- Policy is now stable

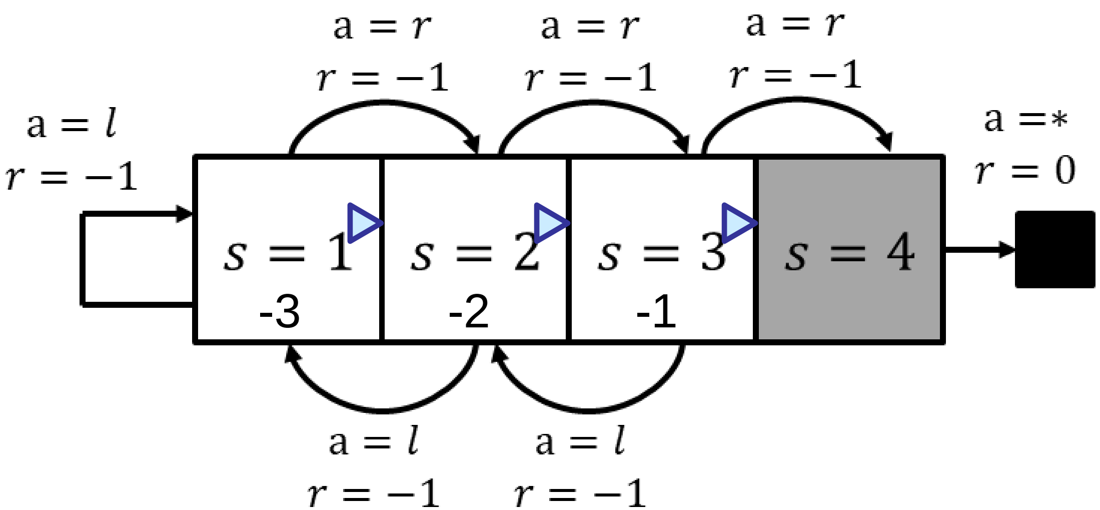


Value Iteration

Value Iteration



- Bellman Opt Equation: $v_*(s) = \max_a q_*(s,a); q_*(s,a) = R_s^a + \gamma v_*(s')$
- $v_*(1) = \max_a[q_*(1,l), q_*(1,r)] = \max_a[-1 + v_*(1), -1 + v_*(2)]$
  - $q_*(1,l) = -1 + v_*(1), q_*(1,r) = -1 + v_*(2)$
- $v_*(2) = \max_a[q_*(2,l), q_*(2,r)] = \max_a[-1 + v_*(1), -1 + v_*(3)]$
  - $q_*(2,l) = -1 + v_*(1), q_*(2,r) = -1 + v_*(3)$
- $v_*(3) = \max_a[q_*(3,l), q_*(3,r)] = \max_a[-1 + v_*(2), -1 + v(4)] = \max_a[-1 + v_*(2), -1]$
  - $q_*(3,l) = -1 + v_*(2), q_*(3,r) = -1 + v(4) = -1$
- We use Value Iteration to solve it. Table shows the iteration process until convergence (not using in-place updates for clarity). Solution: $v_*(1) = -3,\ v_*(2) = -2, v_*(3) = -1$
- Optimal policy: $\pi_*(1) = \operatorname{argmax}_a q_*(1,a) = r; \pi_*(2) = \operatorname{argmax}_a q_*(2,a) = r; \pi_*(3) = \operatorname{argmax}_a q_*(3,a) = r$



(The $V_*(4)$ column is omitted since it is always 0)

	$V_*(1)$	$V_*(2)$	$V_*(3)$
Init	0	0	0
Iter1	-1	-1	-1
Iter2	-2	-2	-1
Iter3	-3	-2	-1
Iter4	-3	-2	-1

MC

Recall: MC Prediction Details

- State s may be visited multiple times in the same episode; let us call the first time it is visited in an episode the first visit to s.
  - The first-visit MC method estimates $v_\pi(s)$ as the average of the returns following first visits to s,
  - The every-visit MC method averages the returns following all visits to s (show below).

MC prediction, for estimating $V \approx v_\pi$

Input: a policy π to be evaluated

Initialize:

  $V(s) \in \mathbb{R}$, arbitrarily, for all $s \in \mathcal{S}$

  $Returns(s) \leftarrow$ an empty list, for all $s \in \mathcal{S}$

Loop forever (for each episode):

  Generate an episode following π: $S_0, A_0, R_1, S_1, A_1, R_2, \ldots, S_{T-1}, A_{T-1}, R_T$

  $G \leftarrow 0$

  Loop for each step of episode, $t = T-1, T-2, \ldots, 0$:

    $G \leftarrow \gamma G + R_{t+1}$

    Append G to $Returns(S_t)$

    $V(S_t) \leftarrow \text{average}(Returns(S_t))$

Go backwards from last non-terminal S_{T-1}

Can compute running avg w. incremental update $V(S_t) \leftarrow_\alpha Returns(S_t)$

MC EP1-3

- TD update equation: $V(S_t) \leftarrow G_t$
- $V(4) \equiv 0$. Initialize $V(1) = V(2) = V(3) = 0$,
- EP1: $(1, \mathrm{r}, -1), (2, \mathrm{r}, -1), (3, \mathrm{r}, -1), (4, \mathrm{r}, 0)$

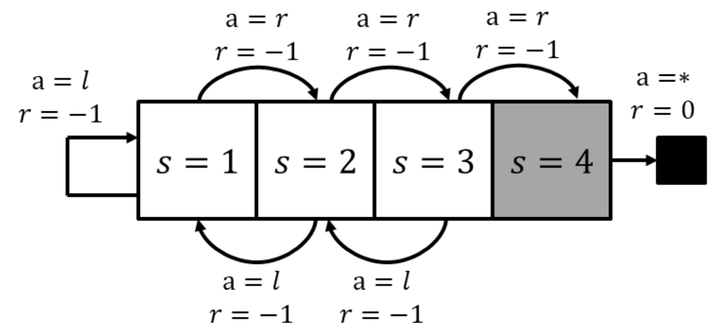
1. $G(3) \leftarrow -1, V(3) \leftarrow G(3) = -1$
2. $G(2) \leftarrow -1 + \gamma G(3) = -2,\ V(2) \leftarrow G(2) = -2$
3. $G(1) \leftarrow -1 + \gamma G(2) = -3,\ V(1) \leftarrow G(1) = -3$

- EP2: $(1, \mathrm{r}, -1), (2, \mathrm{r}, -1), (3, \mathrm{r}, -1), (4, \mathrm{r}, 0)$

1. $G(3) \leftarrow -1, V(3) \leftarrow G(3) = -1$
2. $G(2) \leftarrow -1 + \gamma G(3) = -2,\ V(2) \leftarrow G(2) = -2$
3. $G(1) \leftarrow -1 + \gamma G(2) = -3,\ V(1) \leftarrow G(1) = -3$

- EP3: $(1, \mathrm{r}, -1), (2, \mathrm{r}, -1), (3, \mathrm{r}, -1), (4, \mathrm{r}, 0)$

1. $G(3) \leftarrow -1, V(3) \leftarrow G(3) = -1$
2. $G(2) \leftarrow -1 + \gamma G(3) = -2,\ V(2) \leftarrow G(2) = -2$
3. $G(1) \leftarrow -1 + \gamma G(2) = -3,\ V(1) \leftarrow G(1) = -3$



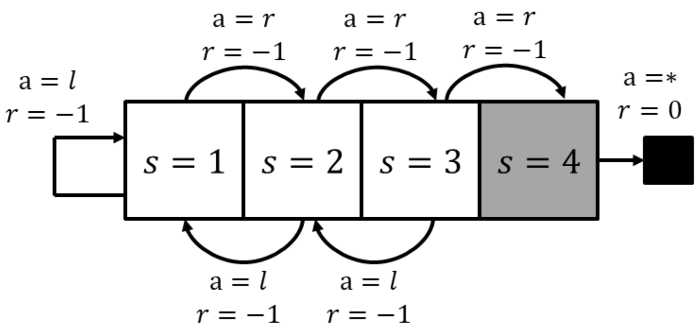
TD	$V(1)$	$V(2)$	$V(3)$
Init	0	0	0
After EP1	−3	−2	−1
After EP2	−3	−2	−1
After EP3	−3	−2	−1
After EP4	−3.5	−3.5	−3.5
After EP5	−3.5	−3.5	−3.5
After EP6	−3.5	−3.5	−3.5
After EP7	−3.5	−3.5	−3.5
After EP8	−3.5	−3.5	−3.5

MC EP4-8

- TD update equation: $V(S_t) \leftarrow G_t$
- EP4: $(3,\mathrm{l},-1),(2,\mathrm{l},-1),(1,\mathrm{l},-1),(1,\mathrm{r},-1),(2,\mathrm{r},-1),(3,\mathrm{r},-1),(4,\mathrm{r},0)$

1. $G(3) \leftarrow -1$ (2nd visit)
2. $G(2) \leftarrow -1+\gamma G(3) = -2$ (2nd visit)
3. $G(1) \leftarrow -1+\gamma G(2) = -3$ (2nd visit)
4. $G(1)' \leftarrow -1+\gamma G(1) = -4$ (1st visit)
5. $G(2)' \leftarrow -1+\gamma G(1)' = -5$ (1st visit)
6. $G(3)' \leftarrow -1+\gamma G(2)' = -6$ (1st visit)
7. (Every visit):
8. $V(3) = .5(G(3)+G(3)') = -3.5$
9. $V(2) = .5(G(2)+G(2)') = -3.5$
10. $V(1) = .5(G(1)+G(1)') = -3.5$
11. (First visit):
12. $V(3) = G(3)' = -6$
13. $V(2) = G(2)' = -5$
14. $V(1) = G(1)' = -4$

- EP5-8 omitted



TD	$V(1)$	$V(2)$	$V(3)$
Init	0	0	0
After EP1	-3	-2	-1
After EP2	-3	-2	-1
After EP3	-3	-2	-1
After EP4	-3.5	-3.5	-3.5
After EP5	-3.5	-3.5	-3.5
After EP6	-3.5	-3.5	-3.5
After EP7	-3.5	-3.5	-3.5
After EP8	-3.5	-3.5	-3.5

TD

TD EP1-3

- TD update equation: $V(S_t) \leftarrow R_{t+1} + V(S_{t+1})$
  - With $\gamma = 1, \alpha = 1$, each V(s) is completely replaced overwritten by the TD update
- $V(4) \equiv 0$. Initialize $V(1) = V(2) = V(3) = 0$,
- EP1: $(1, \mathrm{r}, -1), (2, \mathrm{r}, -1), (3, \mathrm{r}, -1), (4, \mathrm{r}, 0)$

1. $V(1) \leftarrow -1 + V(2) = -1 + 0 = -1$
2. $V(2) \leftarrow -1 + V(3) = -1 + 0 = -1$
3. $V(3) \leftarrow -1 + V(4) = -1 + 0 = -1$

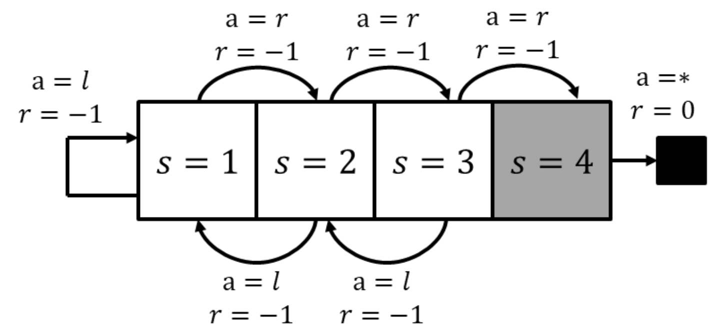
- EP2: $(1, \mathrm{r}, -1), (2, \mathrm{r}, -1), (3, \mathrm{r}, -1), (4, \mathrm{r}, 0)$

1. $V(1) \leftarrow -1 + V(2) = -1 - 1 = -2$
2. $V(2) \leftarrow -1 + V(3) = -1 - 1 = -2$
3. $V(3) \leftarrow -1 + V(4) = -1 + 0 = -1$

- EP3: $(1, \mathrm{r}, -1), (2, \mathrm{r}, -1), (3, \mathrm{r}, -1), (4, \mathrm{r}, 0)$

1. $V(1) \leftarrow -1 + V(2) = -1 - 2 = -3$
2. $V(2) \leftarrow -1 + V(3) = -1 - 1 = -2$
3. $V(3) \leftarrow -1 + V(4) = -1 + 0 = -1$

- Arrows denote bootstrap dependencies, e.g., $V(1)$ bootstraps off $V(2)$, $V(2)$ bootstraps off $V(3)$, $V(3)$ bootstraps off $V(4)$. They also denote direction of information flow during learning, e.g., $V(4) \equiv 0$ is the external learning signal, and info flows $V(4) \rightarrow V(3) \rightarrow V(2) \rightarrow V(1)$.



TD	$V(1)$	$V(2)$	$V(3)$
Init	0	0	0
After EP1	-1	-1	-1
After EP2	-2	-2	-1
After EP3	-3	-2	-1
After EP4	-5	-4	-1
After EP5	-7	-6	-1
After EP6	-9	-8	-1
After EP7	-11	-10	-1
After EP8	-13	-12	-1

TD EP4-8

- TD update equation: $V(S_t) \leftarrow R_{t+1} + V(S_{t+1})$

1. EP4:

   $(3, \mathrm{l}, -1), (2, \mathrm{l}, -1), (1, \mathrm{l}, -1), (1, \mathrm{r}, -1), (2, \mathrm{r}, -1), (3, \mathrm{r}, -1), (4, \mathrm{r}, 0)$

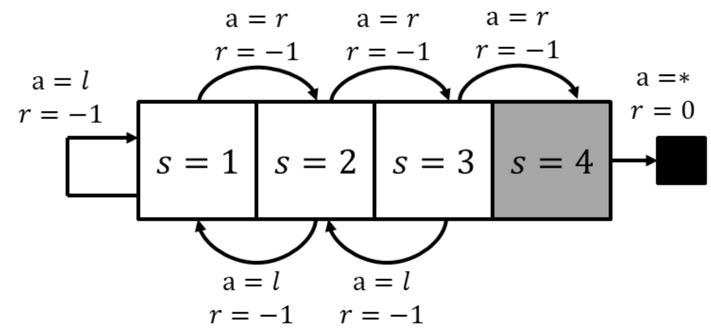
2. $V(3) \leftarrow -1 + V(2) = -1 - 2 = -3$
3. $V(2) \leftarrow -1 + V(1) = -1 - 3 = -4$
4. $V(1) \leftarrow -1 + V(1) = -1 - 3 = -4$
5. $V(1) \leftarrow -1 + V(2) = -1 - 4 = -5$
6. $V(2) \leftarrow -1 + V(3) = -1 - 3 = -4$
7. $V(3) \leftarrow -1 + V(4) = -1 + 0 = -1$

- EP5:

  $(3, \mathrm{l}, -1), (2, \mathrm{l}, -1), (1, \mathrm{l}, -1), (1, \mathrm{r}, -1), (2, \mathrm{r}, -1), (3, \mathrm{r}, -1), (4, \mathrm{r}, 0)$

1. $V(3) \leftarrow -1 + V(2) = -1 - 4 = -5$
2. $V(2) \leftarrow -1 + V(1) = -1 - 5 = -6$
3. $V(1) \leftarrow -1 + V(1) = -1 - 5 = -6$
4. $V(1) \leftarrow -1 + V(2) = -1 - 6 = -7$
5. $V(2) \leftarrow -1 + V(3) = -1 - 5 = -6$
6. $V(3) \leftarrow -1 + V(4) = -1 + 0 = -1$

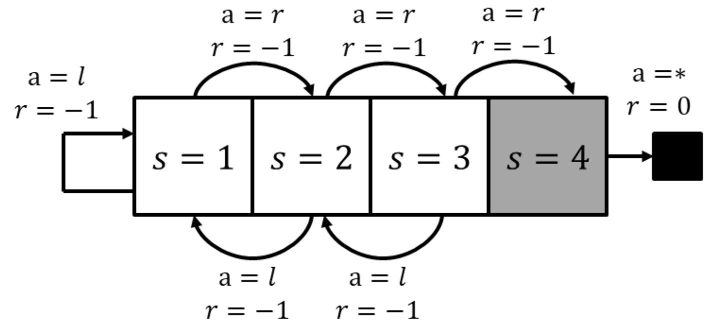
- EP6-8 omitted. Some arrows are omitted, since some values are overwritten within an episode.



TD	$V(1)$	$V(2)$	$V(3)$
Init	0	0	0
After EP1	-1	-1	-1
After EP2	-2	-2	-1
After EP3	-3	-2	-1
After EP4	-5	-4	-1
After EP5	-7	-6	-1
After EP6	-9	-8	-1
After EP7	-11	-10	-1
After EP8	-13	-12	-1

TD failed to converge

- TD failed to converge for this set of episodes. The sequence of TD updates cause all value functions to be increasingly negative.
  - For simplicity, consider the infinite sequence of $(2, \mathrm{l}, -1), (1, \mathrm{r}, -1), (2, \mathrm{l}, -1)$ ...
  - The sequence of TD updates: $V(2) = -1 + V(1), V(1) = -1 + V(2)$, ... So $V(1)$ and $V(2)$ bootstrap off each other and both go to $-\infty$.
  - An analogy is that two students $V(1)$ and $V(2)$ are copying from each other, but they never get any true reward feedback from the teacher ($V(4) = 0$)



TD	$V(1)$	$V(2)$	$V(3)$
Init	0	0	0
After EP1	-1	-1	-1
After EP2	-2	-2	-1
After EP3	-3	-2	-1
After EP4	-5	-4	-1
After EP5	-7	-6	-1
After EP6	-9	-8	-1
After EP7	-11	-10	-1
After EP8	-13	-12	-1

- $V(3)$ is bootstrapped off $V(2)$ when moving left, and is bootstrapped off $V(4) \equiv 0$ when moving right. Steps 1-5 form a bootstrap dependency cycle $V(3) \leftarrow V(2) \leftarrow V(1) \leftarrow V(2) \leftarrow V(3)$ that causes $V(1), V(2), V(3)$ to blow up. Even though $V(3)$ is updated to $V(3) = -1 + V(4) = -1$ when it moves right to state 4, the episode ends immediately afterwards, so $V(1)$ and $V(2)$ do not have a chance to bootstrap off the correct $V(3)$.

  1. $V(3) \leftarrow -1 + V(2) = -1 - 4 = -5$
  2. $V(2) \leftarrow -1 + V(1) = -1 - 5 = -6$
  3. $V(1) \leftarrow -1 + V(1) = -1 - 5 = -6$
  4. $V(1) \leftarrow -1 + V(2) = -1 - 6 = -7$
  5. $V(2) \leftarrow -1 + V(3) = -1 - 5 = -6$
  6. $V(3) \leftarrow -1 + V(4) = -1 + 0 = -1$

- If the episode does not end immediately, but the agent moves left again, then $V(1)$ and $V(2)$ will have a chance to bootstrap off the new $V(3)$, and they may converge to the correct values.

Sarsa

Sarsa EP1-3

- Sarsa update equation: $Q(S_t, A_t) \leftarrow Q(S_t, A_t) + \alpha\big(R_{t+1} + \gamma Q(S_{t+1}, A_{t+1}) - Q(S_t, A_t)\big) = R_{t+1} + Q(S_{t+1}, A_{t+1})$
  - With $\gamma = 1, \alpha = 1$, each Q(S, A) is completely replaced overwritten by the Sarsa update
- $Q(4, a) \equiv 0$. Initialize $Q(1,*) = Q(2,*) = Q(3,*) = 0$
- After EP1: $(1, \mathrm{r}, -1), (2, \mathrm{r}, -1), (3, \mathrm{r}, -1), (4, \mathrm{r}, 0)$

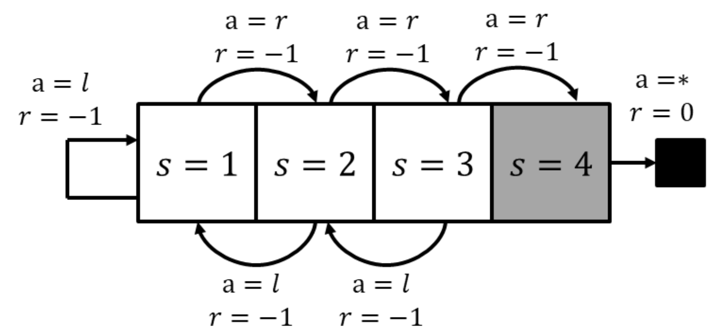
1. $Q(1, r) \leftarrow -1 + Q(2, r) = -1 + 0 = -1$
2. $Q(2, r) \leftarrow -1 + Q(3, r) = -1 + 0 = -1$
3. $Q(3, r) \leftarrow -1 + Q(4, r) = -1 + 0 = -1$

- After EP2: $(1, \mathrm{r}, -1), (2, \mathrm{r}, -1), (3, \mathrm{r}, -1), (4, \mathrm{r}, 0)$

1. $Q(1, r) \leftarrow -1 + Q(2, r) = -1 - 1 = -2$
2. $Q(2, r) \leftarrow -1 + Q(3, r) = -1 - 1 = -2$
3. $Q(3, r) \leftarrow -1 + Q(4, r) = -1 + 0 = -1$

- After EP3: $(1, \mathrm{r}, -1), (2, \mathrm{r}, -1), (3, \mathrm{r}, -1), (4, \mathrm{r}, 0)$

1. $Q(1, r) \leftarrow -1 + Q(2, r) = -1 - 2 = -3$
2. $Q(2, r) \leftarrow -1 + Q(3, r) = -1 - 1 = -2$
3. $Q(3, r) \leftarrow -1 + Q(4, r) = -1 + 0 = -1$



Sarsa	$Q(1,l)$	$\boldsymbol{Q(1,r)}$	$Q(2,l)$	$\boldsymbol{Q(2,r)}$	$Q(3,l)$	$\boldsymbol{Q(3,r)}$
Init	0	**0**	0	**0**	0	**0**
After EP1	0	**−1**	0	**−1**	0	**−1**
After EP2	0	**−2**	0	**−2**	0	**−1**
After EP3	0	**−3**	0	**−2**	0	**−1**
After EP4	−4	**−3**	−1	**−2**	−1	**−1**
After EP5	−4	**−3**	−5	**−2**	−2	**−1**
After EP6	−4	**−3**	−5	**−2**	−6	**−1**
After EP7	−4	**−3**	−5	**−2**	−6	**−1**
After EP8	−4	**−3**	−5	**−2**	−6	**−1**

Sarsa EP4-8

- Sarsa update equation: $Q(S_t, A_t) \leftarrow R_{t+1} + Q(S_{t+1}, A_{t+1})$
- EP4: $(3, \mathrm{l}, -1), (2, \mathrm{l}, -1), (1, \mathrm{l}, -1), (1, \mathrm{r}, -1), (2, \mathrm{r}, -1), (3, \mathrm{r}, -1), (4, \mathrm{r}, 0)$

1. $Q(3, l) \leftarrow -1 + Q(2, l) = -1 + 0 = -1$
2. $Q(2, l) \leftarrow -1 + Q(1, l) = -1 + 0 = -1$
3. $Q(1, l) \leftarrow -1 + Q(1, r) = -1 - 3 = -4$
4. $Q(1, r) \leftarrow -1 + Q(2, r) = -1 - 2 = -3$
5. $Q(2, r) \leftarrow -1 + Q(3, r) = -1 - 1 = -2$
6. $Q(3, r) \leftarrow -1 + Q(4, r) = -1 + 0 = -1$

- EP5: $(3, \mathrm{l}, -1), (2, \mathrm{l}, -1), (1, \mathrm{l}, -1), (1, \mathrm{r}, -1), (2, \mathrm{r}, -1), (3, \mathrm{r}, -1), (4, \mathrm{r}, 0)$

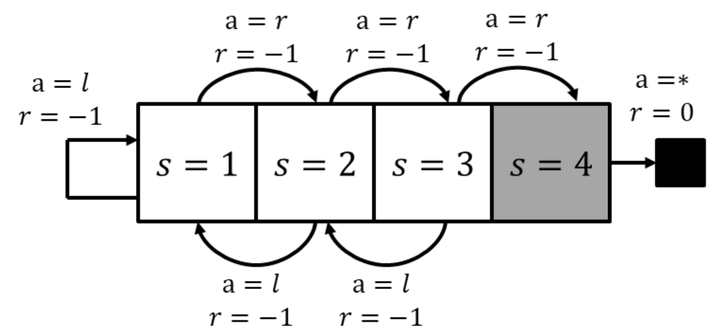
1. $Q(3, l) \leftarrow -1 + Q(2, l) = -1 - 1 = -2$
2. $Q(2, l) \leftarrow -1 + Q(1, l) = -1 - 4 = -5$
3. $Q(1, l) \leftarrow -1 + Q(1, r) = -1 - 3 = -4$
4. $Q(1, r) \leftarrow -1 + Q(2, r) = -1 - 2 = -3$
5. $Q(2, r) \leftarrow -1 + Q(3, r) = -1 - 1 = -2$
6. $Q(3, r) \leftarrow -1 + Q(4, r) = -1 + 0 = -1$

- EP6: $(3, \mathrm{l}, -1), (2, \mathrm{l}, -1), (1, \mathrm{l}, -1), (1, \mathrm{r}, -1), (2, \mathrm{r}, -1), (3, \mathrm{r}, -1), (4, \mathrm{r}, 0)$

1. $Q(3, l) \leftarrow -1 + Q(2, l) = -1 - 5 = -6$
2. $Q(2, l) \leftarrow -1 + Q(1, l) = -1 - 4 = -5$
3. $Q(1, l) \leftarrow -1 + Q(1, r) = -1 - 3 = -4$
4. $Q(1, r) \leftarrow -1 + Q(2, r) = -1 - 2 = -3$
5. $Q(2, r) \leftarrow -1 + Q(3, r) = -1 - 1 = -2$
6. $Q(3, r) \leftarrow -1 + Q(4, r) = -1 + 0 = -1$

- EP7: $(3, \mathrm{l}, -1), (2, \mathrm{l}, -1), (1, \mathrm{l}, -1), (1, \mathrm{r}, -1), (2, \mathrm{r}, -1), (3, \mathrm{r}, -1), (4, \mathrm{r}, 0)$

1. $Q(3, l) \leftarrow -1 + Q(2, l) = -1 - 5 = -6$
2. $Q(2, l) \leftarrow -1 + Q(1, l) = -1 - 4 = -5$
3. $Q(1, l) \leftarrow -1 + Q(1, r) = -1 - 3 = -4$
4. $Q(1, r) \leftarrow -1 + Q(2, r) = -1 - 2 = -3$
5. $Q(2, r) \leftarrow -1 + Q(3, r) = -1 - 1 = -2$
6. $Q(3, r) \leftarrow -1 + Q(4, r) = -1 + 0 = -1$ (EP8 omitted)

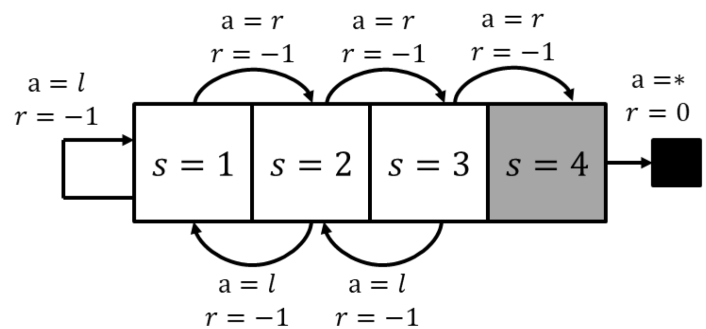


Sarsa	$Q(1, l)$	$\boldsymbol{Q(1, r)}$	$Q(2, l)$	$\boldsymbol{Q(2, r)}$	$Q(3, l)$	$\boldsymbol{Q(3, r)}$
Init	0	**0**	0	**0**	0	**0**
After EP1	0	**−1**	0	**−1**	0	**−1**
After EP2	0	**−2**	0	**−2**	0	**−1**
After EP3	0	**−3**	0	**−2**	0	**−1**
After EP4	−4	**−3**	−1	**−2**	−1	**−1**
After EP5	−4	**−3**	−5	**−2**	−2	**−1**
After EP6	−4	**−3**	−5	**−2**	−6	**−1**
After EP7	−4	**−3**	−5	**−2**	−6	**−1**
After EP8	−4	**−3**	−5	**−2**	−6	**−1**

Q values have converged at EP6. Bootstrap dependency arrows are omitted for EP7-8, since they are the same as EP6. Red arrows denote the stable set of dependencies that keep the Q values stable after EP6.

Comments on Sarsa

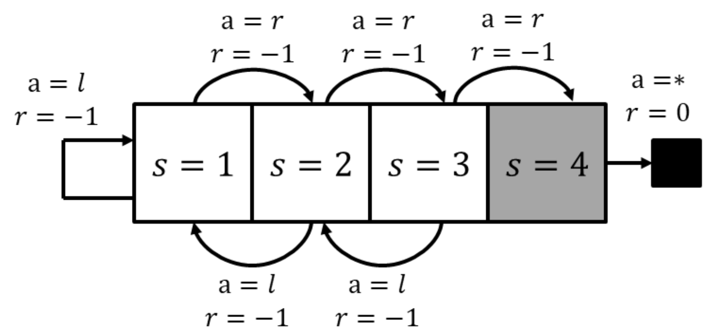
- State-action value functions for moving right look reasonable:$Q(1,r) = -3, Q(2,r) = -2, Q(3,r) = -1$.
- State-action value functions for moving left look unreasonable: $Q(1,l) = -4, Q(2,l) = -5, Q(3,l) = -6$. This is because the only trajectory with move left actions are $3 \rightarrow 2 \rightarrow 1 \rightarrow 1 \rightarrow 2 \rightarrow 3 \rightarrow 4$, the Q values are updated based on only this episode (on-policy), i.e., from state 3 taking action left, it can only take the above trajectory, and reach the goal in 6 steps, hence $Q(3,l) = -6$. If we had collected more trajectories like $3 \rightarrow 2 \rightarrow 3 \rightarrow 4$, then Sarsa could learn the more accurate Q value $Q(3,l) = -1 + Q(2,r) = -3$.
- Even though the Q values for left actions are inaccurate, the greedy policy is still optimal since right action is always better than left:
- $\pi_*(1) = \text{argmax}_\text{a}\big(Q(1,l), Q(1,r)\big) = r$
- $\pi_*(2) = \text{argmax}_\text{a}\big(Q(2,l), Q(2,r)\big) = r$
- $\pi_*(3) = \text{argmax}_\text{a}\big(Q(3,l), Q(3,r)\big) = r$



Sarsa	$Q(1,l)$	$\boldsymbol{Q(1,r)}$	$Q(2,l)$	$\boldsymbol{Q(2,r)}$	$Q(3,l)$	$\boldsymbol{Q(3,r)}$
Init	0	**0**	0	**0**	0	**0**
After EP1	0	**−1**	0	**−1**	0	**−1**
After EP2	0	**−2**	0	**−2**	0	**−1**
After EP3	0	**−3**	0	**−2**	0	**−1**
After EP4	−4	**−3**	−1	**−2**	−1	**−1**
After EP5	−4	**−3**	−5	**−2**	−2	**−1**
After EP6	−4	**−3**	−5	**−2**	−6	**−1**
After EP7	−4	**−3**	−5	**−2**	−6	**−1**
After EP8	−4	**−3**	−5	**−2**	−6	**−1**

Why Sarsa did not blow up

- TD: $V(s)$ is updated regardless if agent moves left or right. $V(3)$ is bootstrapped off $V(2)$ when moving left, and is bootstrapped off $V(4) \equiv 0$ when moving right. bootstrap dependency cycle $V(3) \leftarrow V(2) \leftarrow V(1) \leftarrow V(2) \leftarrow V(3)$ that causes $V(1), V(2), V(3)$ to blow up
- Sarsa: when agent moves left, $Q(s,l)$ is updated; when agent moves right, $Q(s,r)$ is updated. $Q(3,r)$ is always bootstrapped off $Q(4,r) \equiv 0$. So there is no bootstrap dependency cycle like TD. The linear dependency chain from $Q(4,r)$ to $Q(3,l)$ determines the stable values:
  1. $Q(3,l) \leftarrow -1 + Q(2,l) = -1 - 5 = -6$
  2. $Q(2,l) \leftarrow -1 + Q(1,l) = -1 - 4 = -5$
  3. $Q(1,l) \leftarrow -1 + Q(1,r) = -1 - 3 = -4$
  4. $Q(1,r) \leftarrow -1 + Q(2,r) = -1 - 2 = -3$
  5. $Q(2,r) \leftarrow -1 + Q(3,r) = -1 - 1 = -2$
  6. $Q(3,r) \leftarrow -1 + Q(4,r) \ = -1 + 0 = -1$



Sarsa	$Q(1,l)$	$\boldsymbol{Q(1,r)}$	$Q(2,l)$	$\boldsymbol{Q(2,r)}$	$Q(3,l)$	$\boldsymbol{Q(3,r)}$
Init	0	**0**	0	**0**	0	**0**
After EP1	0	**−1**	0	**−1**	0	**−1**
After EP2	0	**−2**	0	**−2**	0	**−1**
After EP3	0	**−3**	0	**−2**	0	**−1**
After EP4	−4	**−3**	−1	**−2**	−1	**−1**
After EP5	−4	**−3**	−5	**−2**	−2	**−1**
After EP6	−4	**−3**	−5	**−2**	−6	**−1**
After EP7	−4	**−3**	−5	**−2**	−6	**−1**
After EP8	−4	**−3**	−5	**−2**	−6	**−1**

Q Learning

QL EP1-3

- QL update equation: $(S_t, A_t) \leftarrow Q(S_t, A_t) + \alpha\left(R_{t+1} + \gamma \max_{a'} Q(S_{t+1}, a') - Q(S_t, A_t)\right) = R_{t+1} + \gamma \max_{a'} Q(S_{t+1}, a')$
  - With $\gamma = 1, \alpha = 1$, each Q(S, A) is completely replaced overwritten by the Q update
- After EP1: $(1, \mathrm{r}, -1), (2, \mathrm{r}, -1), (3, \mathrm{r}, -1), (4, \mathrm{r}, 0)$

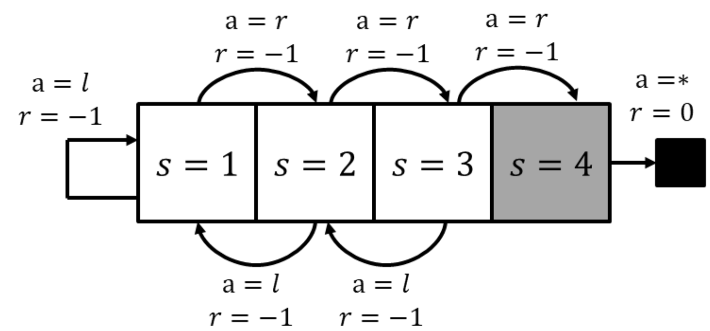
1. $Q(1, r) \leftarrow -1 + \max_{a'} Q(2, a') = -1 + \max(0\,, 0) = -1$
2. $Q(2, r) \leftarrow -1 + \max_{a'} Q(3, a') = -1 + \max(0, 0) = -1$
3. $Q(3, r) \leftarrow -1 + \max_{a'} Q(4, a') = -1 + 0 = -1$

- After EP2: $(1, \mathrm{r}, -1), (2, \mathrm{r}, -1), (3, \mathrm{r}, -1), (4, \mathrm{r}, 0)$

1. $Q(1, r) \leftarrow -1 + \max_{a'} Q(2, a') = -1 + \max(-1, 0) = -1$
2. $Q(2, r) \leftarrow -1 + \max_{a'} Q(3, a') = -1 + \max(-1, 0) = -1$
3. $Q(3, r) \leftarrow -1 + \max_{a'} Q(4, a') = -1 + 0 = -1$

- After EP3: : $(1, \mathrm{r}, -1), (2, \mathrm{r}, -1), (3, \mathrm{r}, -1), (4, \mathrm{r}, 0)$

1. $Q(1, r) \leftarrow -1 + \max_{a'} Q(2, a') = -1 + \max(-1, 0) = -1$
2. $Q(2, r) \leftarrow -1 + \max_{a'} Q(3, a') = -1 + \max(-1, 0) = -1$
3. $Q(3, r) \leftarrow -1 + \max_{a'} Q(4, a') = -1 + 0 = -1$



QL	$Q(1, l)$	$\boldsymbol{Q(1, r)}$	$Q(2, l)$	$\boldsymbol{Q(2, r)}$	$Q(3, l)$	$\boldsymbol{Q(3, r)}$
Init	0	**0**	0	**0**	0	**0**
After EP1	0	**−1**	0	**−1**	0	**−1**
After EP2	0	**−1**	0	**−1**	0	**−1**
After EP3	0	**−1**	0	**−1**	0	**−1**
After EP4	−1	**−2**	−1	**−2**	−1	**−1**
After EP5	−2	−3	−2	**−2**	−2	**−1**
After EP6	−3	**−3**	−3	**−2**	−3	**−1**
After EP7	−4	**−3**	−4	**−2**	−3	**−1**
After EP8	−4	**−3**	−4	**−2**	−3	**−1**

QL EP4-6

- EP4: $(3,\mathrm{l},-1),(2,\mathrm{l},-1),(1,\mathrm{l},-1),(1,\mathrm{r},-1),(2,\mathrm{r},-1),(3,\mathrm{r},-1),(4,\mathrm{r},0)$

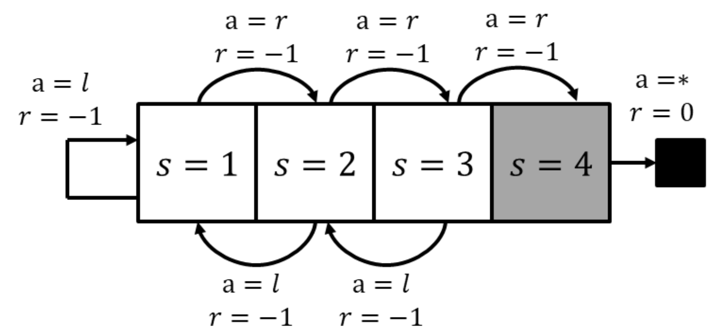
1. $Q(3,l) \leftarrow -1 + \max_{a'} Q(2,a') = -1 + \max(0,-1) = -1$
2. $Q(2,l) \leftarrow -1 + \max_{a'} Q(1,a') = -1 + \max(0,-1) = -1$
3. $Q(1,l) \leftarrow -1 + \max_{a'} Q(1,a') = -1 + \max(0,-1) = -1$
4. $Q(1,r) \leftarrow -1 + \max_{a'} Q(2,a') = -1 + \max(-1,-1) = -2$
5. $Q(2,r) \leftarrow -1 + \max_{a'} Q(3,a') = -1 + \max(-1,-1) = -2$
6. $Q(3,r) \leftarrow -1 + \max_{a'} Q(4,a') = -1 + 0 = -1$

- EP5: $(3,\mathrm{l},-1),(2,\mathrm{l},-1),(1,\mathrm{l},-1),(1,\mathrm{r},-1),(2,\mathrm{r},-1),(3,\mathrm{r},-1),(4,\mathrm{r},0)$

1. $Q(3,l) \leftarrow -1 + \max_{a'} Q(2,a') = -1 + \max(-1,-2) = -2$
2. $Q(2,l) \leftarrow -1 + \max_{a'} Q(1,a') = -1 + \max(-1,-2) = -2$
3. $Q(1,l) \leftarrow -1 + \max_{a'} Q(1,a') = -1 + \max(-1,-2) = -2$
4. $Q(1,r) \leftarrow -1 + \max_{a'} Q(2,a') = -1 + \max(-2,-2) = -3$
5. $Q(2,r) \leftarrow -1 + \max_{a'} Q(3,a') = -1 + \max(-2,-1) = -2$
6. $Q(3,r) \leftarrow -1 + \max_{a'} Q(4,a') = -1 + 0 = -1$

- EP6: $(3,\mathrm{l},-1),(2,\mathrm{l},-1),(1,\mathrm{l},-1),(1,\mathrm{r},-1),(2,\mathrm{r},-1),(3,\mathrm{r},-1),(4,\mathrm{r},0)$

1. $Q(3,l) \leftarrow -1 + \max_{a'} Q(2,a') = -1 + \max(-2,-2) = -3$
2. $Q(2,l) \leftarrow -1 + \max_{a'} Q(1,a') = -1 + \max(-2,-3) = -3$
3. $Q(1,l) \leftarrow -1 + \max_{a'} Q(1,a') = -1 + \max(-2,-3) = -3$
4. $Q(1,r) \leftarrow -1 + \max_{a'} Q(2,a') = -1 + \max(-3,-2) = -3$
5. $Q(2,r) \leftarrow -1 + \max_{a'} Q(3,a') = -1 + \max(-3,-1) = -2$
6. $Q(3,r) \leftarrow -1 + \max_{a'} Q(4,a') = -1 + 0 = -1$



QL	$Q(1,l)$	$\boldsymbol{Q(1,r)}$	$Q(2,l)$	$\boldsymbol{Q(2,r)}$	$Q(3,l)$	$\boldsymbol{Q(3,r)}$
Init	0	**0**	0	**0**	0	**0**
After EP1	0	**−1**	0	**−1**	0	**−1**
After EP2	0	**−1**	0	**−1**	0	**−1**
After EP3	0	**−1**	0	**−1**	0	**−1**
After EP4	−1	**−2**	−1	**−2**	−1	**−1**
After EP5	−2	−3	−2	−2	−2	−1
After EP6	−3	**−3**	−3	−2	−3	−1
After EP7	−4	**−3**	−4	**−2**	−3	**−1**
After EP8	−4	**−3**	−4	**−2**	−3	**−1**

QL EP7-8

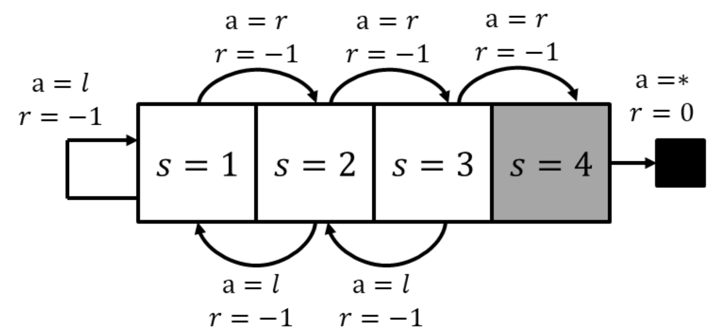
- EP7: $(3, \mathrm{l}, -1), (2, \mathrm{l}, -1), (1, \mathrm{l}, -1), (1, \mathrm{r}, -1), (2, \mathrm{r}, -1), (3, \mathrm{r}, -1), (4, \mathrm{r}, 0)$

1. $Q(3, l) \leftarrow -1 + \max_{a'} Q(2, a') = -1 + \max(-3, -2) = -3$
2. $Q(2, l) \leftarrow -1 + \max_{a'} Q(1, a') = -1 + \max(-3, -3) = -4$
3. $Q(1, l) \leftarrow -1 + \max_{a'} Q(1, a') = -1 + \max(-3, -3) = -4$
4. $Q(1, r) \leftarrow -1 + \max_{a'} Q(2, a') = -1 + \max(-4, -2) = -3$
5. $Q(2, r) \leftarrow -1 + \max_{a'} Q(3, a') = -1 + \max(-3, -1) = -2$
6. $Q(3, r) \leftarrow -1 + \max_{a'} Q(4, a') = -1 + 0 = -1$

- EP8: $(3, \mathrm{l}, -1), (2, \mathrm{l}, -1), (1, \mathrm{l}, -1), (1, \mathrm{r}, -1), (2, \mathrm{r}, -1), (3, \mathrm{r}, -1), (4, \mathrm{r}, 0)$

1. $Q(3, l) \leftarrow -1 + \max_{a'} Q(2, a') = -1 + \max(-4, -2) = -3$
2. $Q(2, l) \leftarrow -1 + \max_{a'} Q(1, a') = -1 + \max(-4, -3) = -4$
3. $Q(1, l) \leftarrow -1 + \max_{a'} Q(1, a') = -1 + \max(-4, -3) = -4$
4. $Q(1, r) \leftarrow -1 + \max_{a'} Q(2, a') = -1 + \max(-4, -2) = -3$
5. $Q(2, r) \leftarrow -1 + \max_{a'} Q(3, a') = -1 + \max(-3, -1) = -2$
6. $Q(3, r) \leftarrow -1 + \max_{a'} Q(4, a') = -1 + 0 = -1$

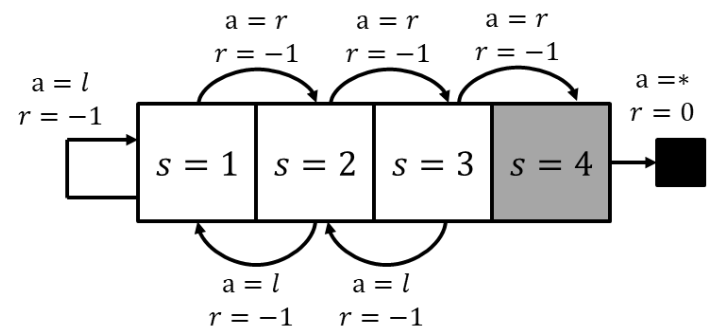
- Q values have converged at EP7. Bootstrap dependency arrows are omitted for EP8, since they are the same as EP7. Red arrows denote the stable set of dependencies that keep the Q values stable after EP7.



QL	$Q(1, l)$	$\boldsymbol{Q(1, r)}$	$Q(2, l)$	$\boldsymbol{Q(2, r)}$	$Q(3, l)$	$\boldsymbol{Q(3, r)}$
Init	0	**0**	0	**0**	0	**0**
After EP1	0	**−1**	0	**−1**	0	**−1**
After EP2	0	**−1**	0	**−1**	0	**−1**
After EP3	0	**−1**	0	**−1**	0	**−1**
After EP4	−1	**−2**	−1	**−2**	−1	**−1**
After EP5	−2	**−3**	−2	**−2**	−2	**−1**
After EP6	−3	**−3**	−3	**−2**	−3	**−1**
After EP7	−4	**−3**	−4	**−2**	−3	**−1**
After EP8	−4	**−3**	−4	**−2**	−3	**−1**

Comments on QL

- QL converges. All state-action value functions look reasonable.
- $Q(1,r) = -3, Q(2,r) = -2, Q(3,r) = -1$. The optimal path can be derived from bootstrap dependencies, e.g., dependency chain $Q(1,r) \leftarrow Q(2,r) \leftarrow Q(3,r)$ corresponds to the optimal path $1 \rightarrow 2 \rightarrow 3 \rightarrow 4$.
- $Q(1,l) = -4$: If agent moves left in state 1, dependency chain $Q(1,l) \leftarrow Q(1,r) \leftarrow Q(2,r) \leftarrow Q(3,r)$ corresponds to the optimal path $1 \rightarrow 1 \rightarrow 2 \rightarrow 3 \rightarrow 4$ w. 4 steps to reach goal state 4.
- $Q(2,l) = -4$: If agent moves left in state 2, dependency chain $Q(2,l) \leftarrow Q(1,r) \leftarrow Q(2,r) \leftarrow Q(3,r)$ corresponds to the optimal path $2 \rightarrow 1 \rightarrow 2 \rightarrow 3 \rightarrow 4$ w. 4 steps to reach goal state 4.
- $Q(3,l) = -3$: If agent moves left in state 3, dependency chain $Q(3,l) \leftarrow Q(2,r) \leftarrow Q(3,r)$ corresponds to the optimal path $3 \rightarrow 2 \rightarrow 3 \rightarrow 4$ w. 3 steps to reach goal state 4.
- QL is smarter than Sarsa: since it is off-policy, agent can learn the correct Q value functions that correspond to trajectories that it has never actually experienced, e.g., if If agent moves left in state 3, even though it has never experienced the trajectory $3 \rightarrow 2 \rightarrow 3 \rightarrow 4$, the bootstrap dependency $Q(3,l) \leftarrow Q(2,r)$ lead to that trajectory instead of the experienced trajectory $3 \rightarrow 2 \rightarrow 1 \rightarrow 1 \rightarrow 2 \rightarrow 3 \rightarrow 4$.
- The intermediate Q values before convergence may not correspond to a valid policy, e.g., before EP7, $\text{argmax}_a\, Q(1,a) = l$, so the agent would be stuck in state 1 trying to go left forever.
- Q values learned by QL are accurate, and the greedy policy is optimal:
- $\pi_*(1) = \text{argmax}_a\big(Q(1,l), Q(1,r)\big) = r$
- $\pi_*(2) = \text{argmax}_a\big(Q(2,l), Q(2,r)\big) = r$
- $\pi_*(3) = \text{argmax}_a\big(Q(3,l), Q(3,r)\big) = r$



QL	$Q(1,l)$	$\boldsymbol{Q(1,r)}$	$Q(2,l)$	$\boldsymbol{Q(2,r)}$	$Q(3,l)$	$\boldsymbol{Q(3,r)}$
Init	0	**0**	0	**0**	0	**0**
After EP1	0	**−1**	0	**−1**	0	**−1**
After EP2	0	**−1**	0	**−1**	0	**−1**
After EP3	0	**−1**	0	**−1**	0	**−1**
After EP4	−1	**−2**	−1	**−2**	−1	**−1**
After EP5	−2	−3	−2	−2	−2	−1
After EP6	−3	**−3**	−3	−2	−3	−1
After EP7	−4	**−3**	−4	**−2**	−3	**−1**
After EP8	−4	**−3**	−4	**−2**	−3	**−1**