Lecture 10
Basic Graph Algorithms

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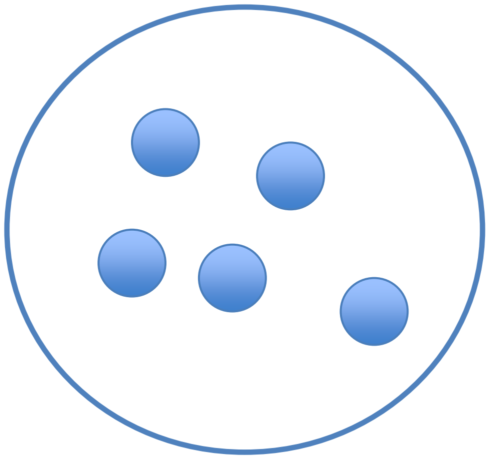
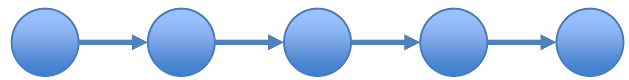
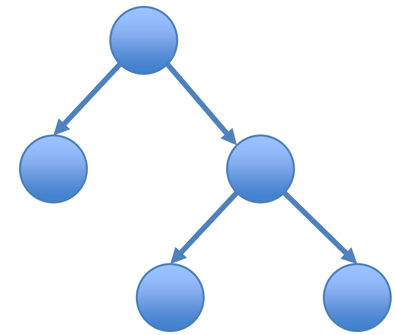
Lecture Goals

- Compare the Graph ADT with other ADTs
- Define basic notions associated with graphs
- Implement graphs in Java using an adjacency matrix representation and an adjacency list representation
- Implement a method to find the neighbors of a vertex in two ways.
- We introduce two classic algorithms for searching a graph—depth-first search and breadth-first search.
- we introduce a depth-first search based algorithm for computing the topological order of an acyclic digraph.

ADT of Graph







Sequential, linear structures

Arrays, linked lists

Unstructured structures

Sets

Hierarchical structures

Trees

Useful for

- iterating over all elements,
- accessing via index

Can indicate common structure in key

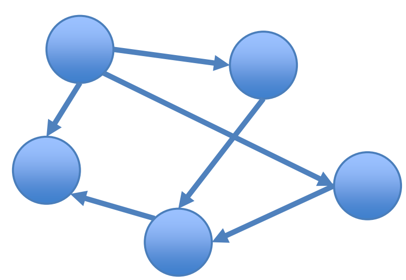
- for example, the prefix in tire

Principle: Basic objects & Relationships between them

Graph is a generalization of this principle

Basic objects: vertices, nodes

Relationships between them: edges, arcs, links



Examples of Graphs

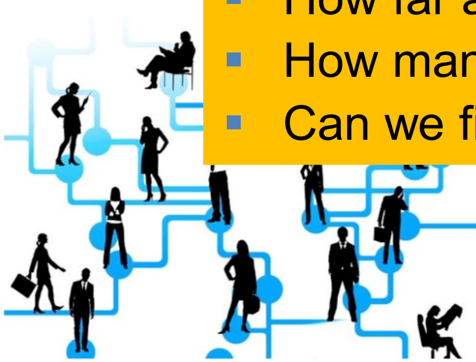
Some general questions related to graphs:

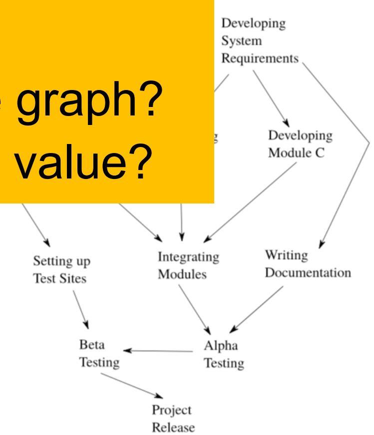
- How to create a graph?
- Are two vertices adjacent?
- Is the graph dense? sparse?
- How far are two vertices in the graph?
- How many components are there in the graph?
- Can we find a vertex with particular key value?

Basic objects: we

Relationships bet

hem: nonstop flights





Basic objects: people

Relationships between them: friends

Basic objects: tasks

Relationships between them: dependencies

Graph Definitions

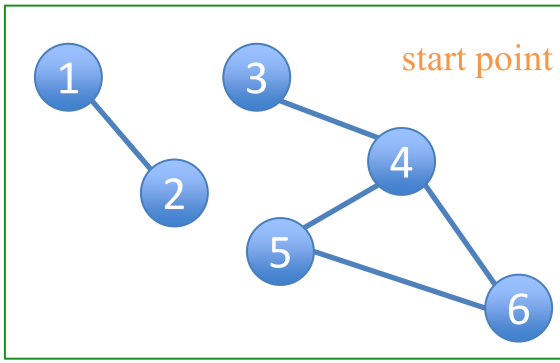
Basic objects: vertices, nodes

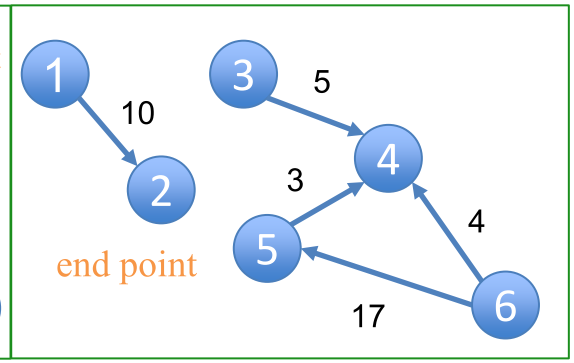
Relationships between them: edges, arcs, links

V

E

Size of graph: |V| + |E|
|V|: number of vertices
|E|: number of edges





Undirected

edges are symmetric

Directed

Weighted

cost

Neighbor: u is a neighbor of v if:
there is an edge from u to v
OR
there is an edge from v to u

What are the neighbors of the vertex 4?

A. 3,4,5,6
B. 3,5,6
C. 3,6
D. 5

Path: sequence of vertices and edges that depicts hopping along graph

For which pair of vertices is there a path in the graph starting at the first and ending at the second?

A. vertex 1 and vertex 3
B. vertex 4 and vertex 6
C. vertex 6 and vertex 5

What's the maximum number of edges in a directed and undirected graph with n vertices?

- n*(n-1) e th n*(n-1)/2 elf-loops (i.e. edges from a node back to itself).
- Assume there there is at most one edge from a given start vertex to a given end vertex.

Implementing Graphs in Java

Basic objects: vertices, nodes ← Label by integers

Relationships between them: edges, arcs, links

```
public abstract class Graph {
        private int numVertices;
        private int numEdges;

        public Graph() {
                numVertices = numEdges = 0;
        }

        public int getNumVertices() {
                return numVertices;
        }

        public int getNumEdges() {
                return numEdges;
        }

        public void addVertex() {
                implementAddVertex();
                numVertices++;
        }

        public abstract void implementAddVertex();

        public abstract List<Integer> getNeighbors(int v);
}
```

size of a graph

For example, which cities we can reach with nonstop flight?

data associated with any graph

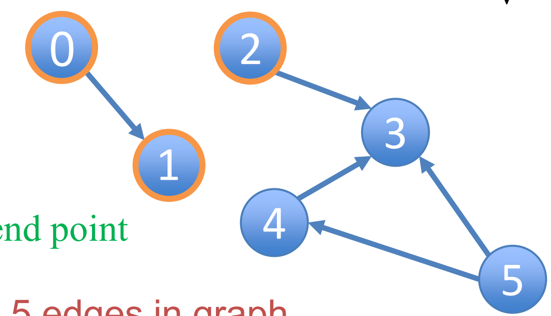
methods that ought to be available with any graph.

leave implementation of key functionalities to subclasses

Graph Representation: Adjacency Matrix

relabel vertices starting at 0 to use as index

start point



end point

$V = \{0, 1, 2, 3, 4, 5\}$

5 edges in graph

→ 5 nonzero entries in adjacency matrix

Column index

array entry > 1:
- multiple edges,
- or weighted edges

row index

Symmetric entry

	0	1	2	3	4	5
0	0	1	0	0	0	0
1	0	0	0	0	0	0
2	0	0	0	1	0	0
3	0	0	0	0	0	0
4	0	0	0	1	0	0
5	0	0	0	1	1	0

The grid (2-d array) is indexed by the vertices labels and stores information in a particular location based on whether these two vertices have an edge between them or not

```java
public class GraphAdjMatrix extends Graph {
  private int[][] adjMatrix;

  public void implementAddEdge(int v, int w) {
    adjMatrix[v][w] = 1;
  }
  public void implementAddVertex() {
    int v = getNumVertices();
    if (v >= adjMatrix.length) {
        int[][] newAdjMatrix = new int[v * 2][v * 2];
      for (int i = 0; i < adjMatrix.length; i++) {
          for (int j = 0; j < adjMatrix.length; j++) {
              newAdjMatrix[i][j] = adjMatrix[i][j];
          }
      }
      adjMatrix = newAdjMatrix;
  }
}}
```

v*2 instead of v+1 to amortize cost of adding new vertices in the future.

expand the 2-d array

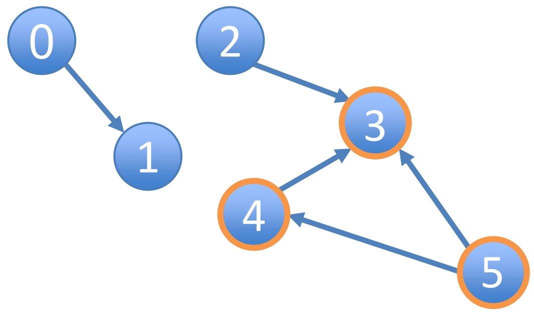
Graph Implementations
https://www.youtube.com/watch?v=2guA5uMEmZQ

How long does it take to test whether there is an edge between vertex v and vertex w in the graph?

O(1)

- Algebraic representation of graph structure.
- Fast to test for edges.
- Fast to add/remove edges.
- Slow to add/remove vertices.
- Requires a lot of memory. sparse

Graph Representation: Adjacency List



Neighbour can be reached by one hop

$0 \rightarrow \{1\}$

$1 \rightarrow$ null

$2 \rightarrow \{3\}$

$3 \rightarrow$ null

$4 \rightarrow \{3\}$

$5 \rightarrow \{3, 4\}$

- Easy to add vertices.
- Easy to add/remove edges.
- May use a lot less memory than adjacency matrices.

Motivation for new representation:
- want to avoid storing information on edges that aren't in the graph
- Edges connect a vertex to its neighbors

- Sparse graph: O(1) edges for each vertex
- most applications use sparse graphs

Is it also fast?

```
public class GraphAdjList extends Graph {
        private Map<Integer, ArrayList<Integer>> adjListsMap;

                        vertex → {neighbors}

        public void implementAddVertex() {
                int v = getNumVertices();
                ArrayList<Integer> neighbors = new ArrayList<Integ
                adjListsMap.put(v, neighbors);
        }

        public void implementAddEdge(int v, int w) {
                (adjListsMap.get(v)).add(w);
        }
}
```

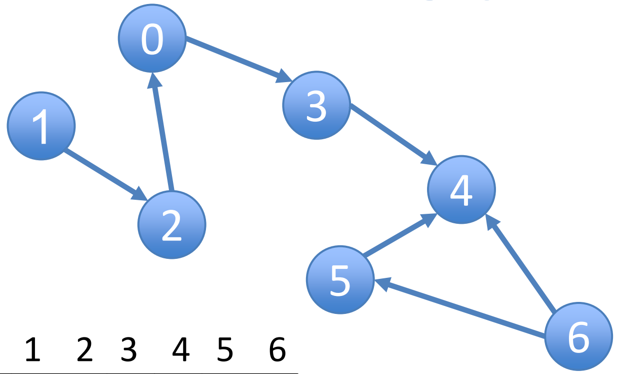
```
public class ArrayList<E>
extends AbstractList<E>
implements List<E>, RandomAccess, Cloneable, Serializable
```

Resizable-array implementation of the `List` interface. Implements all optional list operations, and permits all elements, including `null`. In addition to implementing the `List` interface, this class provides methods to manipulate the size of the array that is used internally to store the list. (This class is roughly equivalent to `Vector`, except that it is unsynchronized.)

The `size`, `isEmpty`, `get`, `set`, `iterator`, and `listIterator` operations run in constant time. The `add` operation runs in *amortized constant time*, that is, adding n elements requires O(n) time. All of the other operations run in linear time (roughly speaking). The constant factor is low compared to that for the

Yes. Operations are all O(1)

Some Practices



How much storage is required to represent a graph as a **matrix**? (Big-O, Tightest Bound)

A. |V| B. |E|
C. |V|+|E| D. $|V|^2$
E. $|E|^2$

	0	1	2	3	4	5	6
0	0	0	0	1	0	0	0
1	0	0	1	0	0	0	0
2	1	0	0	0	0	0	0
3	0	0	0	0	1	0	0
4	0	0	0	0	0	0	0
5	0	0	0	0	1	0	0
6	0	0	0	0	1	1	0

What would change if undirected?

Symmetric matrix, hence half of the matrix is redundant, but still $O(|V|^2)$

For dense graphs with lots of edges, |E| will be as large as $|V|^2$

	0	1	2	3	4	5	6
0	0	0	1	1	0	0	0
1	0	0	1	0	0	0	0
2	1	1	0	0	0	0	0
3	1	0	0	0	1	0	0
4	0	0	0	1	0	1	1
5	0	0	0	0	1	0	1
6	0	0	0	0	1	1	0

Symmetric matrix

O(|E|)

O(|V|)

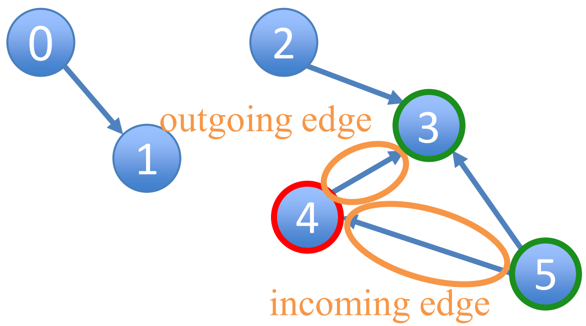
0 → {3}
1 → {2}
2 → {0}
3 → {4}
4 → null
5 → {4}
6 → {4, 5}

How much storage is required to represent a graph as **an adjacency list**? (Big-O, Tightest Bound)

A. |V| B. |E|
C. |V|+|E| D. $|V|^2$
E. $|E|^2$

Much more efficient for sparse graphs!

Find the Neighbors



Neighbors: vertices that are adjacent.

there is edge in between

Out degree: number of outgoing edges.

In degree: number of incoming edges.

0 → {1}
1 → null
2 → {3}
3 → null
4 → {3}
5 → {3, 4}

count the number of occurrences in all lists

return the size of list

	0	1	2	3	4	5
0	0	1	0	0	0	0
1	0	0	0	0	0	0
2	0	0	0	1	0	0
3	0	0	0	0	0	0
4	0	0	0	1	0	0
5	0	0	0	1	1	0

count the number of nonzero slots

Which implementation makes finding the in degree more efficient?

Matrix: O(|V|) List: O(|E| + |V|)

For dense graphs without multiple edges between pairs of vertices, |E| is O(|V|^2). so the adjacency matrix representation is faster. For sparse graphs, |E| = O(|V|) so both representations have the same performance.

Which implementation makes finding the out degree more efficient?

Matrix: O(|V|) List: O(1)

Coding getOutNeighbors (outgoing)

```
public class GraphAdjMatrix extends Graph {

    private int[][] adjMatrix;

    public List<Integer> getOutNeighbors(int v) {
        List<Integer> neighbors = new ArrayList<Integer>();
        for (int i = 0; i < getNumVertices(); i++) {
            for (int j=0; j< adjMatrix[v][i]; j++)
            if (adjMatrix[v][i] > 0)
                neighbors.add(i);
        }
        return neighbors;
    }
}
```

// list to return

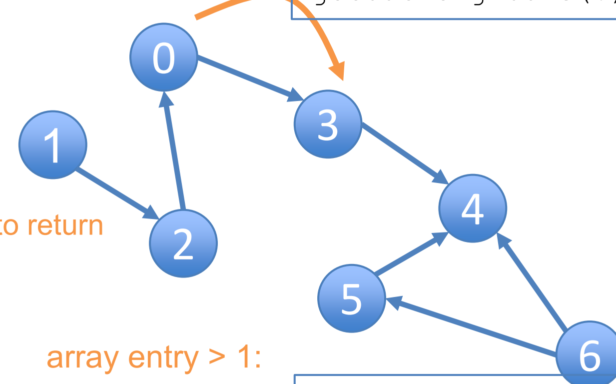
```
public class GraphAdjList extends Graph {

    private Map<Integer,ArrayList<Integer>> adjListsMap;

    public List<Integer> getOutNeighbors(int v) {
        return adjListsMap.get(v);
        return new ArrayList<Integer>(adjListsMap.get(v));
    }
}
```

// return v's list

// return a COPY of v's list

Returning the array pointer allows the caller to modify the list contents. Not good encapsulation.

getOutNeighbors(0)



getOutNeighbors(6)

array entry > 1:
- multiple edges

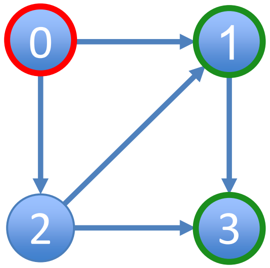
	0	1	2	3	4	5	6
0	0	0	0	2	0	0	0
1	0	0	1	0	0	0	0
2	1	0	0	0	0	0	0
3	0	0	0	0	1	0	0
4	0	0	0	0	0	0	0
5	0	0	0	0	1	0	0
6	0	0	0	0	1	1	0

- 0 → {3}
- 1 → {2}
- 2 → {0}
- 3 → {4}
- 4 → null
- 5 → {4}
- 6 → {4, 5}

What does this change do?

A. It's a change in the code but will not materially affect the output.
B. It allows multiple edges between two vertices.
C. It will have some other effect on the code behavior.

Coding 2-Hop Neighbors (outgoing)



0 → {1, 2}

1 → {3}

2 → {1, 3}

3 → null

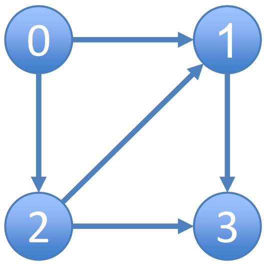
	0	1	2	3
0	0	1	1	0
1	0	0	0	1
2	0	1	0	1
3	0	0	0	0

Find all two-hop neighbors from given vertex

```
public class GraphAdjList extends Graph {

        private Map<Integer,ArrayList<Integer>> adjListsMap;

        public List<Integer> getDistance2 (int v) {
                List<Integer> distance2 = new ArrayList<>();

                // Loop through oneHop and get the neighbors of each
                for(int u : getOutNeighbors(v)){
                        distance2.addAll(getOutNeighbors(u));
                }
                return distance2;
        }
}
```

```
public class GraphAdjMatrix extends Graph {

        private int[][] adjMatrix;

        public List<Integer> getDistance2 (int v) {
                List<Integer> distance2 = new ArrayList<Integer>();

                // Loop through oneHop and get the neighbors of each
                for(int u : getOutNeighbors(v)){
                        distance2.addAll(getOutNeighbors(u));
                }
                return distance2;
        }
}
```

Coding 2-Hop Neighbors (Matrix Multiplication)



Matrix multiplication for finding two-hop neighbors

For all the vertices in the graph

$$\begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}^2 = \text{matrix whose entries are two-hop neighbors!}$$

	0	1	2	3
0	0	1	1	0
1	0	0	0	1
2	0	1	0	1
3	0	0	0	0

$$\begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \boxed{} & \boxed{} & \boxed{} & \boxed{} \\ \boxed{} & & & \\ & & & \\ & & & \end{pmatrix}$$

Dot product

$0*0 + 1*0 + 1*0 + 0*0 = 0$

$0*1 + 1*0 + 1*1 + 0*0 = 1$

$0*1 + 1*0 + 1*0 + 0*0 = 0$

$0*0 + 1*1 + 1*1 + 0*0 = 2$

$0*0 + 0*0 + 0*0 + 1*0 = 0$

Node 3 is a two-hop neighbor of node 0 along two different paths

	0	1	2	3
0	0	1	0	2
1	0	0	0	0
2	0	0	0	1
3	0	0	0	0

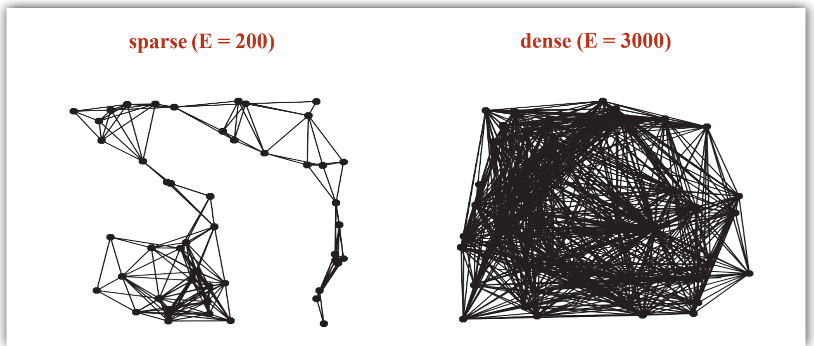
Matrix multiplication is well studied and optimized in software and hardware, and can be done very fast

Summary of Digraph Representations

In practice. Use adjacency-lists representation.

- Algorithms based on iterating over vertices adjacent from v.
- Real-world graphs tend to be sparse (not dense).

proportional to V   proportional to V^2



Two graphs (V = 50)

representation	space	insert edge from v to w	edge from v to w?	iterate over vertices adjacent from v?
adjacency matrix	V^2	1	1	V
adjacency lists	$E + V$	1	*outdegree(v)*	*outdegree(v)*

† disallows parallel edges

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- Define basic notions associated with graphs
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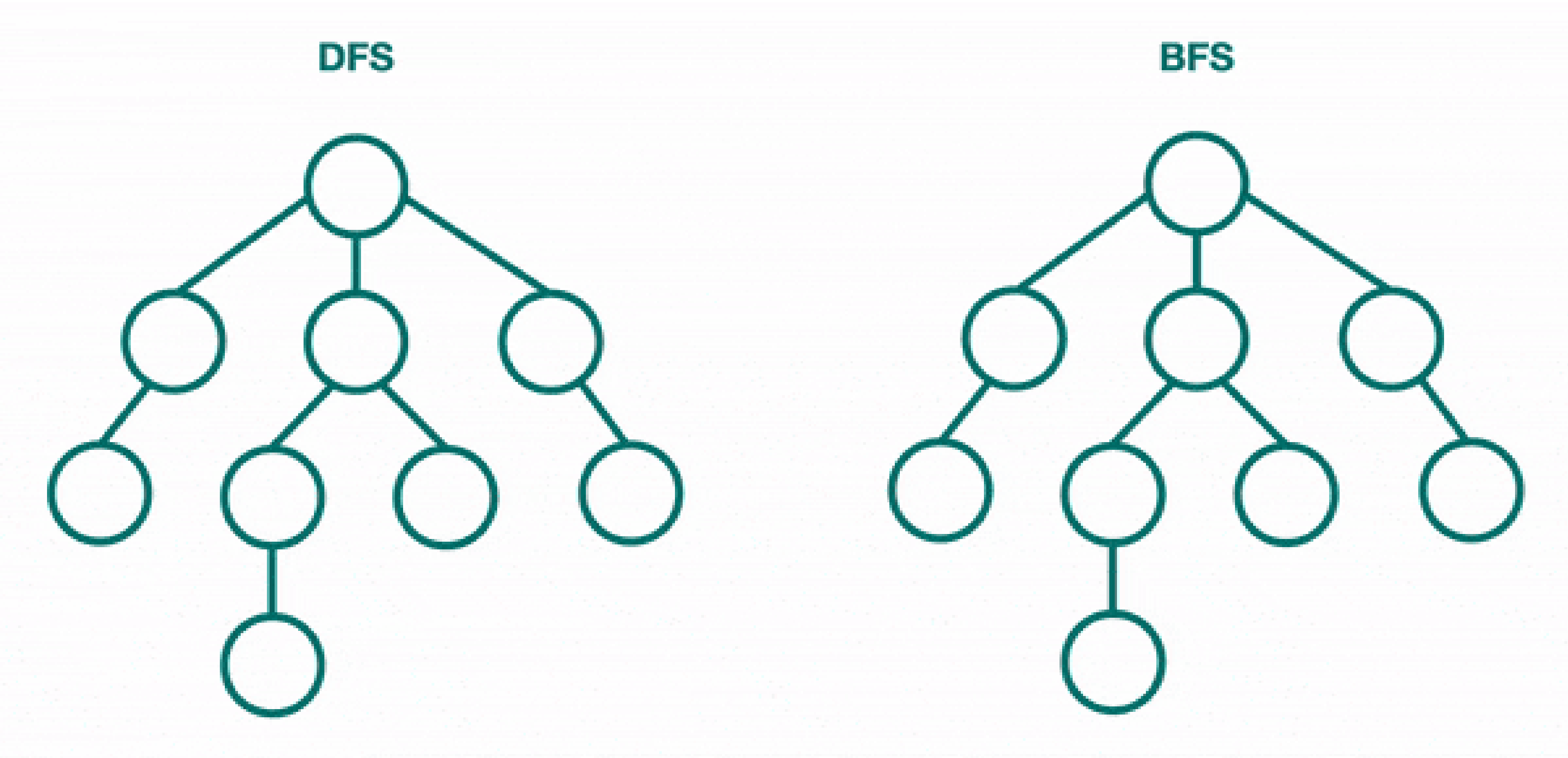
- We introduce two classic algorithms for searching a graph—depth-first search and breadth-first search.
- we introduce a depth-first search based algorithm for computing the topological order of an acyclic digraph.

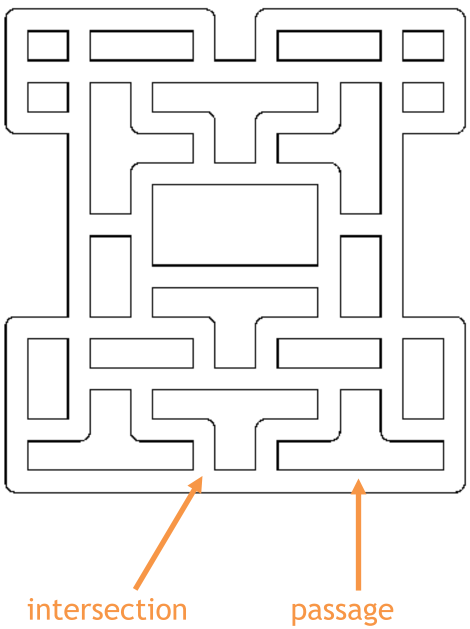
DFS vs. BFS

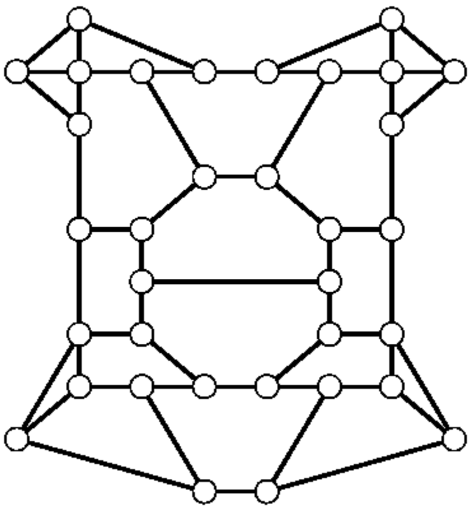


Represent Problems as Graphs: Maze Exploration

Goal. Explore every intersection in the maze.

Maze graph. Vertex = intersection. Edge = passage.





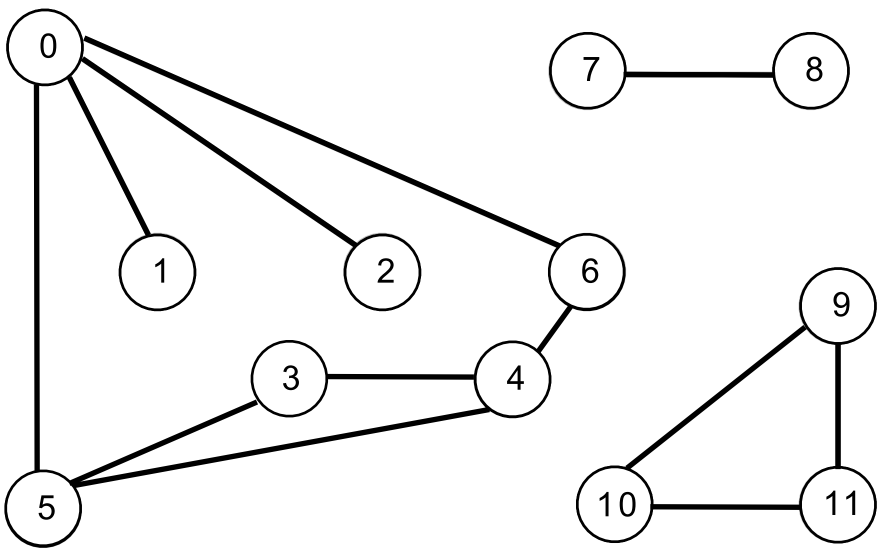
Depth-First Search (DFS)

Goal. Systematically traverse a graph.

Typical applications.
- Find all vertices connected to a given source vertex.
- Find a path between two vertices.

DFS (to visit a vertex v)

Mark vertex v.
Recursively visit all unmarked vertices w adjacent to v.



Java execution stack is used to keep track of where to search next

v	marked[ ]	edgeTo[ ]
0	T	–
1	T	0
2	T	0
3	T	5
4	T	6
5	T	4
6	T	0
7	F	–
8	F	–
9	F	–
10	F	–
11	F	–

```
dfs(0)
  dfs(6)
    dfs(4)
      dfs(5)
        dfs(3)
        3 done
      5 done
    4 done
  6 done
  dfs(2)
  2 done
  dfs(1)
  1 done
0 done
```

Data structures.
- Boolean array marked[] to mark vertices.
- Integer array edgeTo[] to keep track of paths.
  - (edgeTo[w] == v) means that edge v-w taken to discover vertex w

Class Design Pattern

Decouple graph data type from graph processing.

- Create a Graph object.
- Pass the Graph to a graph-processing routine.
- Query the graph-processing routine for information.

public class Paths	
Paths(Graph G, int s)	*//find paths in G from source s*
Boolean hasPathTo(int v)	*//is there a path from s to v?*
Iterable<Integer> pathTo(int v)	*//path from s to v; null if no such path*

```
Paths paths = new Paths(G, s);
    for (int v = 0; v < G.V(); v++)
        if (paths.hasPathTo(v))
            StdOut.println(v);
```

← print all vertices connected to s

Depth-First Search: Java Implementation

```
public class DepthFirstPaths {

    private boolean[] marked;
    private int[] edgeTo;
    private int s;

    public DepthFirstPaths(Graph G, int s) {
        ...
        dfs(G, s);
    }

    private void dfs(Graph G, int v) {
        marked[v] = true;
        for (int w : G.adj(v))
            if (!marked[w])
            {
                edgeTo[w] = v;
                dfs(G, w);
            }
    }
}
```

marked[v] = true if vconnected to s

edgeTo[v] = previous vertex on path from s to v

initialize data structures

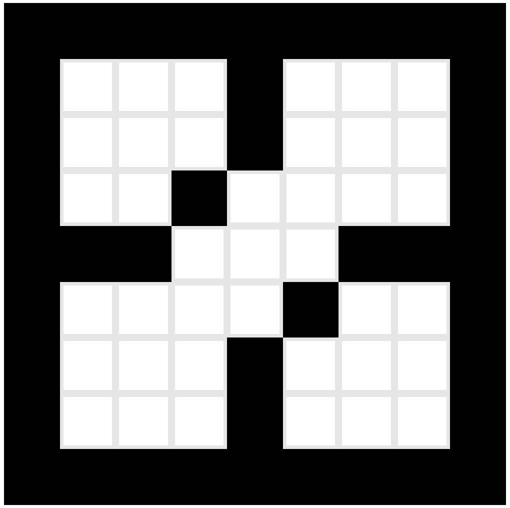
find vertices connected to s

recursive DFS does the work

- Code for directed graphs identical to undirected one.

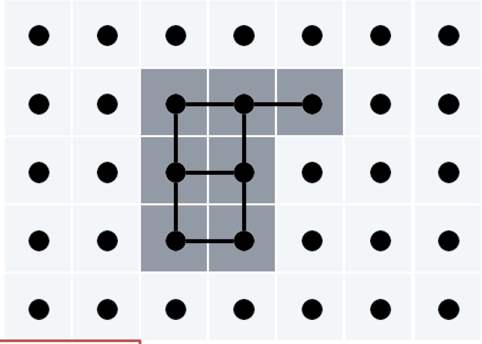
Depth-First Search Application: Flood Fill

Problem. Flood fill is a flooding algorithm that determines and alters the area connected to a given node in a multi-dimensional array with some matching attribute.



Solution.

- Build a grid graph.
- Vertex: pixel.
- Edge: between two adjacent gray pixels.
- Blob: all pixels connected to given pixel.



https://en.wikipedia.org/wiki/Flood_fill

Reachability Application: Mark–Sweep Garbage Collector

Every data structure is a digraph.

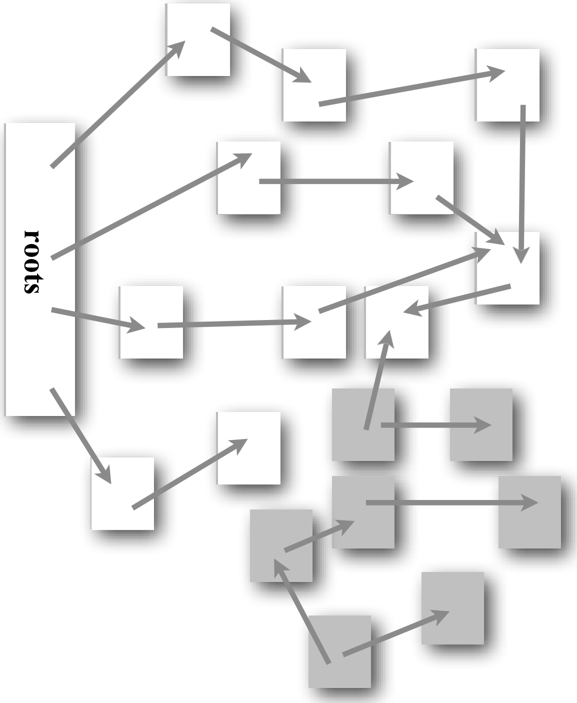
- Vertex = object.
- Edge = reference.
- Roots: Objects known to be directly accessible by program (e.g., stack).
- Reachable objects: Objects indirectly accessible by program (starting at a root and following a chain of pointers).

Mark–sweep algorithm. [McCarthy, 1960]

Mark: mark all reachable objects.

Sweep: if object is unmarked, it is garbage (so add to free list).

Memory cost. Uses 1 extra mark bit per object (plus DFS stack).



Breadth-First Search (BFS)

BFS (from source vertex s)

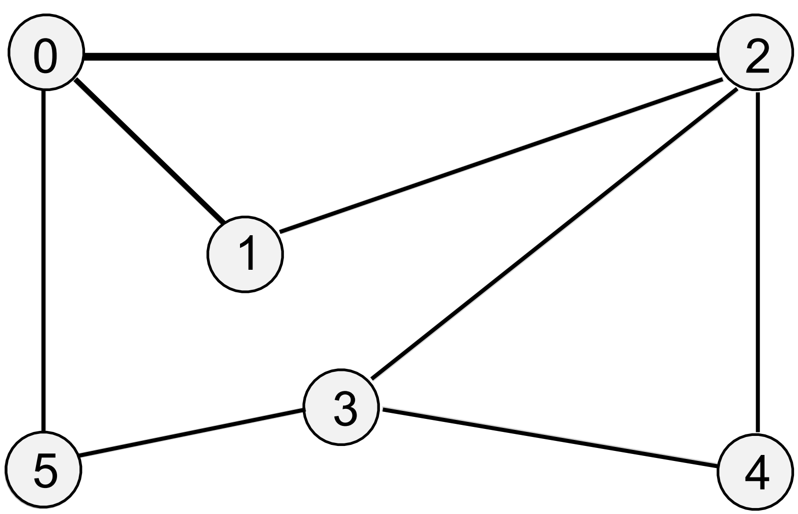
Put s onto a FIFO queue, and mark s as visited.

Repeat until the queue is empty:

- **remove the least recently added vertex v**
- **add each of v's unmarked neighbors to the queue, and mark them.**

Queue

4
3
5
1
2
0



v	marked[ ]	edgeTo[ ]	distTo[ ]
0	T	–	0
1	T	0	1
2	T	0	1
3	T	2	2
4	T	2	2
5	T	0	1

distTo[v] = distTo[edgeTo[v]] + 1;

s.distTo[v] stores the distance from s to v

Breadth-First Search: Java Implementation

```
public  class  BreadthFirstPaths {
    private  boolean[]  marked;
    private  int[]   edgeTo;
    private  int[]   distTo;
    ...
    private  void  bfs(Graph  G,  int  s)  {
        Queue<Integer> q  = new Queue<Integer>();
        q.enqueue(s);
        marked[s]  = true;
        distTo[s]    = 0;

        while  (!q.isEmpty())  {
            int v= q.dequeue();
            for  (int   w :  G.adj(v))   {
                if  (!marked[w]) {
                    q.enqueue(w);
                    marked[w]  = true;
                    edgeTo[w]  = v;
                    distTo[w]   = distTo[v]   + 1;
                }
            }
        }
    }
}
```

- DFS. Put unvisited vertices on a stack.
- BFS. Put unvisited vertices on a queue.

initialize FIFO queue of vertices to explore

found new vertex w via edge v–w

Every undirected graph is a digraph (with edges in both directions). BFS is a digraph algorithm.

- For directed graph, same method as for undirected graphs.
- Code for directed graphs identical to undirected one.

Breadth-First Search Properties

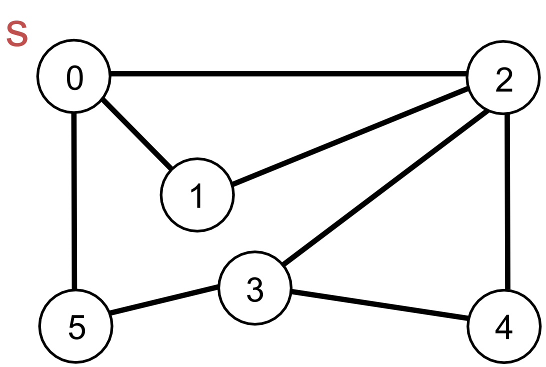
level-order

Proposition. BFS examines vertices in increasing distance (number of edges) from s.

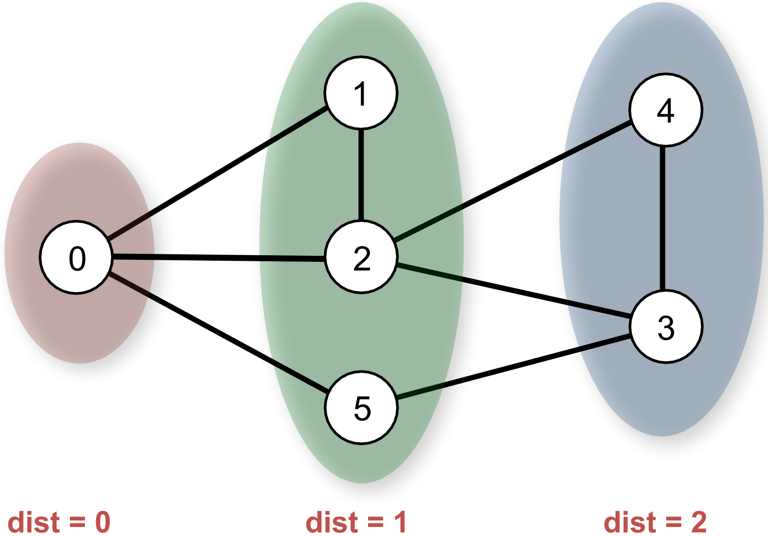
Proposition. In any connected graph, BFS computes shortest paths (fewest number of edges) from s to all other vertices in time proportional to E + V.

Pf. [correctness] Queue always consists of zero or more vertices of distance k from s, followed by zero or more vertices of distance k + 1.

Pf. [running time] Each vertex connected to s is visited once, and all its edges are checked.



graph G

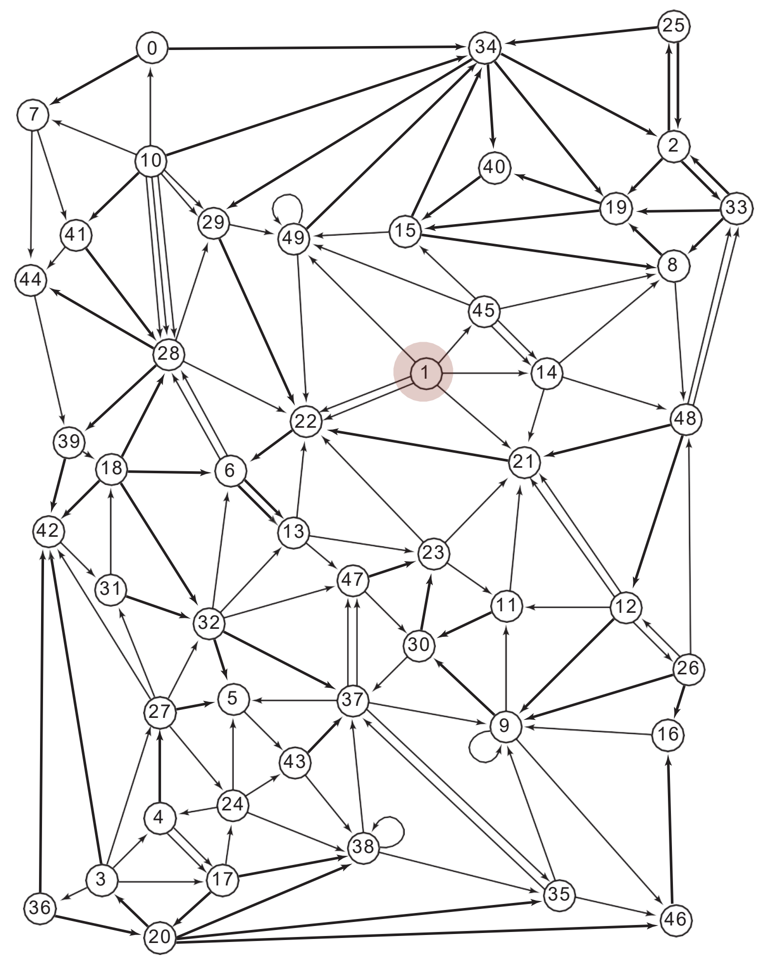


Breadth-First Search Application: Web Crawler

Goal. Crawl web, starting from some root web page, say www.hofstra.edu.

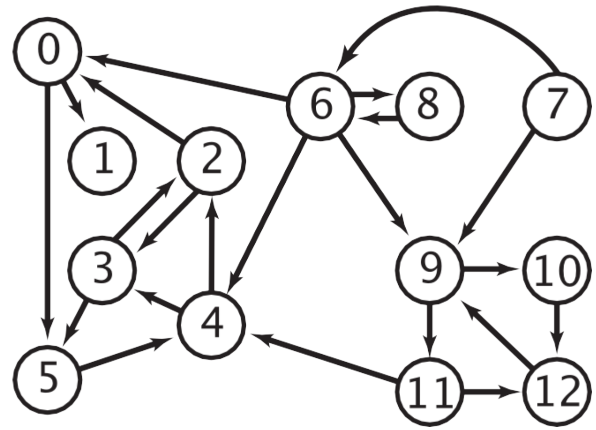
Solution. [BFS with implicit digraph]

- Choose root web page as source s.
- Maintain a Queue of websites to explore.
- Maintain a SET of marked websites.
- Dequeue the next website and enqueue any unmarked websites to which it links.



Multiple-Source Shortest Paths Problem

Given a digraph and a set of source vertices, find shortest path from any vertex in the set to each other vertex.

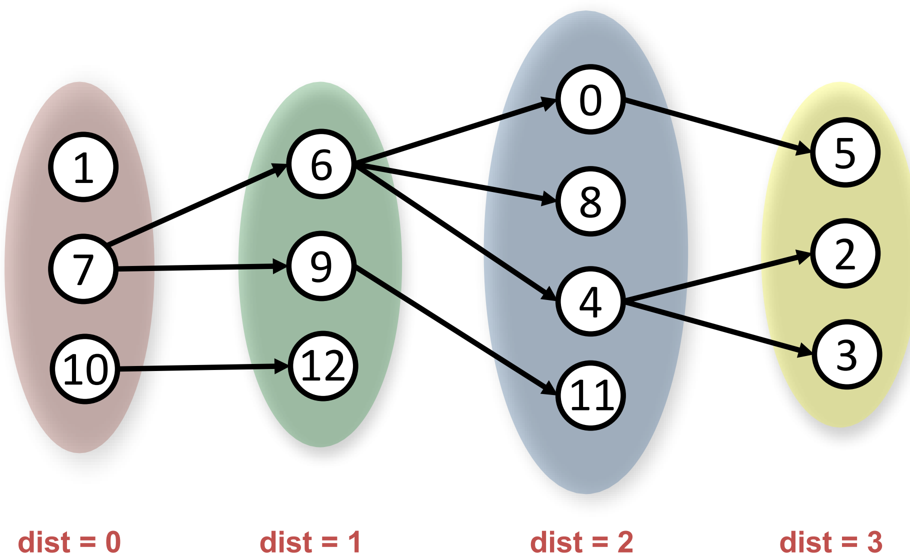


Ex. S = { 1, 7, 10 }.

- Shortest path to 4 is 7 → 6 → 4.
- Shortest path to 5 is 7 → 6 → 0 → 5.
- Shortest path to 12 is 10 → 12.
- …

How to implement multi-source shortest paths algorithm?

Use BFS, but initialize by enqueuing all source vertices.

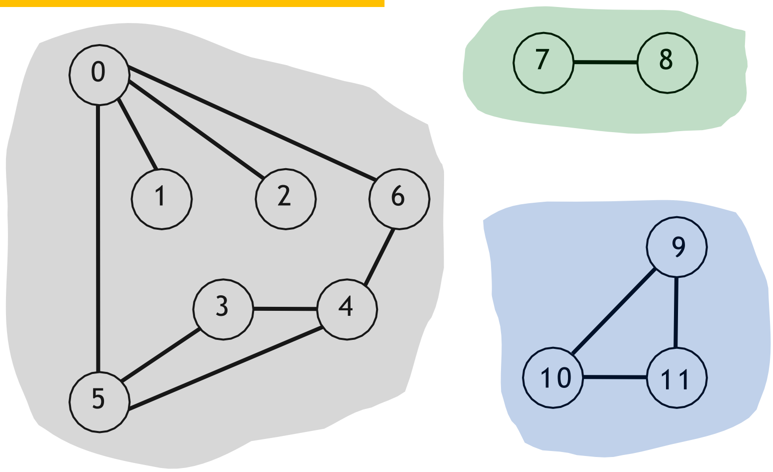


Connectivity Queries Problem

- Vertices v and w are connected if there is a path between them.
- In undirected graph, the relation "is connected to" is an equivalence relation:
  - Reflexive: v is connected to v.
  - Symmetric: if v is connected to w, then w is connected to v.
  - Transitive: if v connected to w and w connected to x, then v connected to x.
- Goal. Preprocess undirected graph to answer queries of the form *is v connected to w?* in constant time while using adjacency list.
- A connected component is a maximal set of connected vertices.
- Given connected components, can answer queries in constant time.

v	id[ ]
0	0
1	0
2	0
3	0
4	0
5	0
6	0
7	1
8	1
9	2
10	2
11	2

public class	CC	
	CC(Graph G)	*find connected components in G*
boolean	connected(int v, int w)	*are v and w connected?*
int	count()	*number of connected components*
int	id(int v)	*component identifier for v*



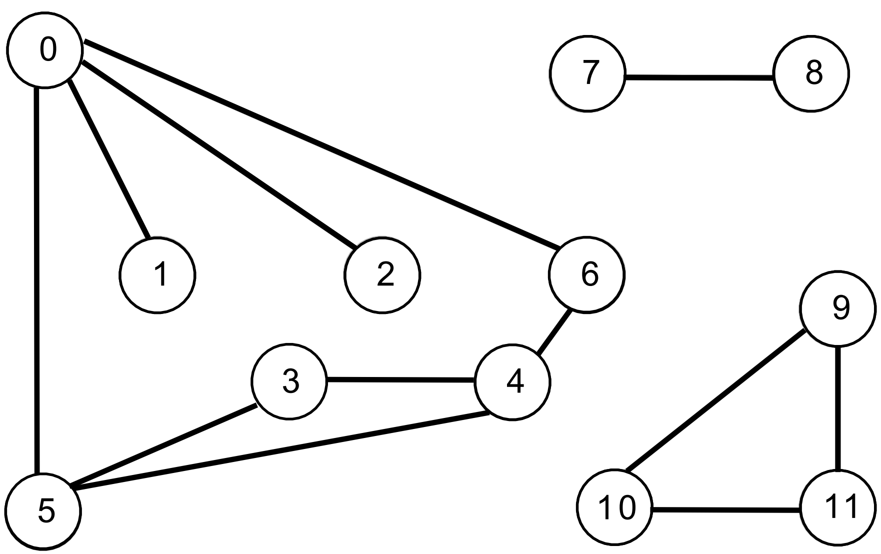
3 connected components

Finding Connected Components with DFS

Goal. Partition vertices into connected components.

Connected components

Initialize all vertices v as unmarked.
For each unmarked vertex v, run DFS to identify all vertices discovered as part of the same component.



v	marked[ ]	id[ ]
0	T	0
1	T	0
2	T	0
3	T	0
4	T	0
5	T	0
6	T	0
7	T	1
8	T	1
9	T	2
10	T	2
11	T	2

Java execution stack

```
dfs(0)
   dfs(6)
      dfs(4)
         dfs(5)
            dfs(3)
            3 done
         5 done
      4 done
   6 done
   dfs(2)
   2 done
   dfs(1)
   1 done
0 done
dfs(7)
   dfs(8)
   8 done
7 done
dfs(9)
   dfs(10)
      dfs(11)
      11 done
   10 done
9 done
```

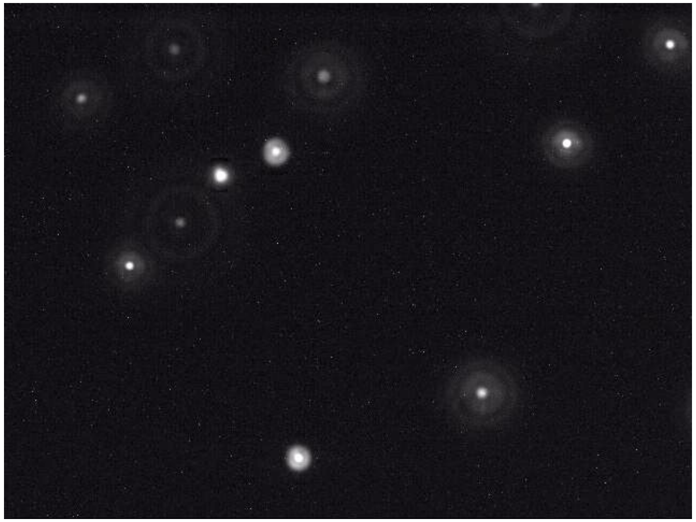
Can also use BFS

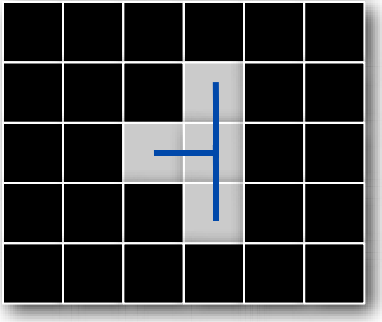
Connected Components Application: Particle Detection

Given grayscale image of particles, identify "blobs."

- Vertex: pixel.
- Edge: between two adjacent pixels with grayscale value > 70.
- Blob: connected component of 20-30 pixels.

black = 0
white = 255





Particle tracking. Track moving particles over time.

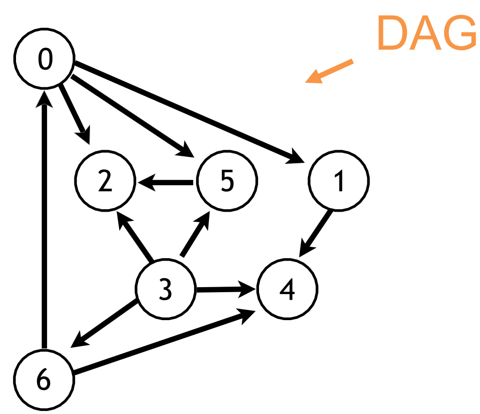
Precedence Scheduling Problem

Goal. Given a set of tasks to be completed with precedence constraints, in which order should we schedule the tasks?

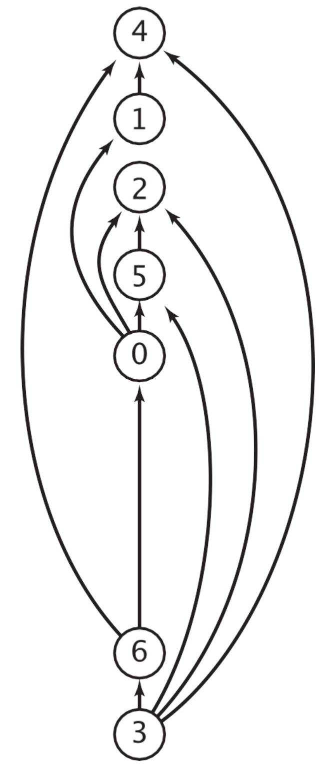
Digraph model. vertex = task; edge = precedence constraint.

0. Algorithms
1. Complexity Theory
2. Artificial Intelligence
3. Intro to CS
4. Cryptography
5. Scientific Computing
6. Advanced Programming

tasks



precedence constraint graph



feasible schedule

Topological sort. Redraw DAG(Directed acyclic graph) so all edges point upwards.

Graph traversal with DFS: pre-order, post-order

```
function preOrderTraversal(node) {
 if (node !== null) {
  visitNode(node);
  preOrderTraversal(node.left);
  preOrderTraversal(node.right);
 }
}
```

```
function postOrderTraversal(node) {
 if (node !== null) {
  postOrderTraversal(node.left);
  postOrderTraversal(node.right);
  visitNode(node);
 }
}
```

Recall: Binary Tree traversal with DFS: pre-order, post-order

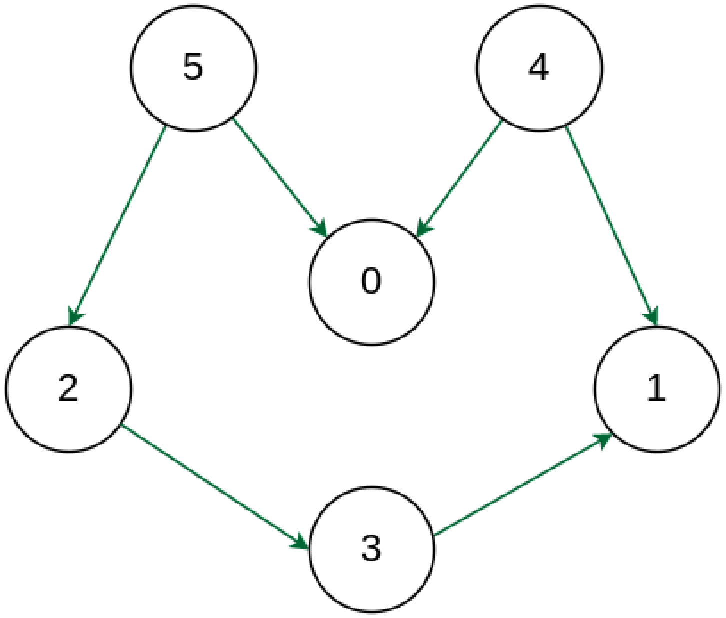
```
function preOrderTraversal(node) {
 if (node !== null) {
  visitNode(node);
  foreach(c ∈ node.children) {
     preOrderTraversal(c);}
  }
}
```

```
function postOrderTraversal(node) {
 if (node !== null) {
      foreach(c ∈ node.children) {
          postOrderTraversal(c);}
      visitNode(node);
 }
}
```

Graph traversal with DFS: pre-order, post-order

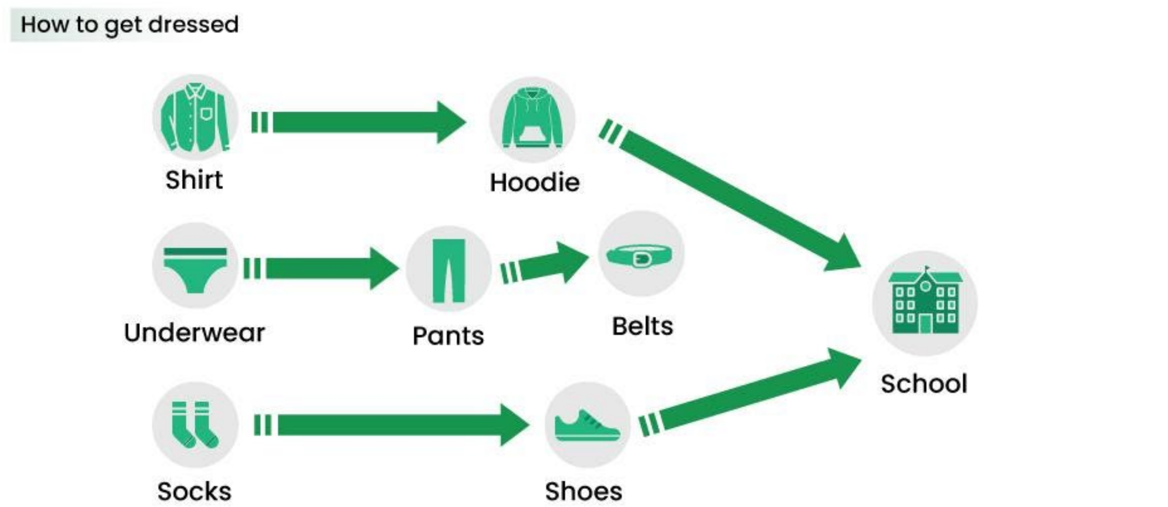
Topological Sort

- Topological sorting for Directed Acyclic Graph (DAG) is a linear ordering of vertices such that for every directed edge u-v, vertex u comes before v in the ordering, i.e., all pair-wise precedence constraints are satisfied.



The first vertex in topological sorting is always a vertex with an in-degree of 0 (a vertex with no incoming edges), i.e., 4 or 5. Possible topological sorts include "5 4 2 3 1 0", "4 5 2 3 1 0", "4 5 0 2 3 1", "5 2 3 4 1 0", etc.

https://www.geeksforgeeks.org/topological-sorting/



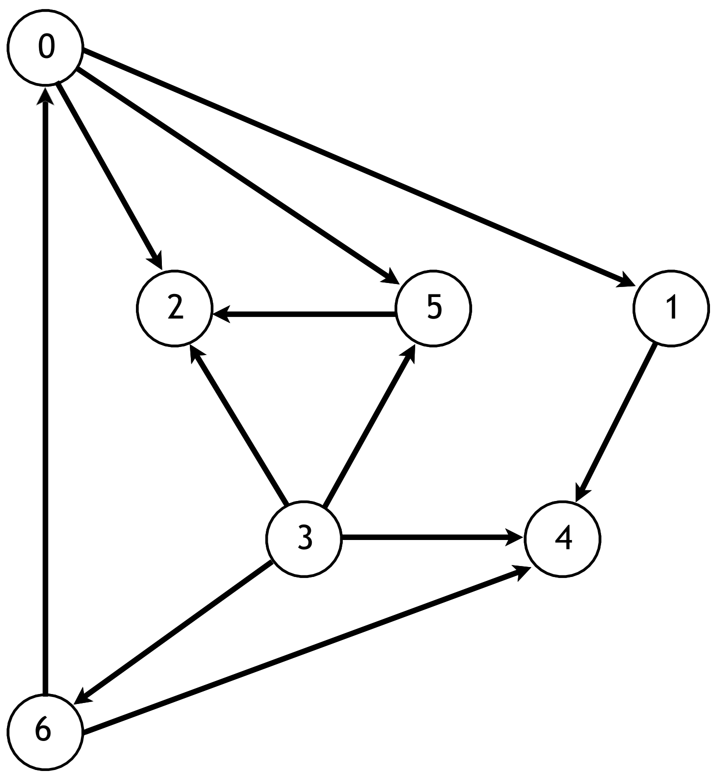
Multiple Topological ordering for a graph

Multiple Topological ordering for a graph

Topological Sort Details

- Run depth-first search
- Return vertices in reverse postorder.

not a reachability problem



stack top

Postorder

4	1	2	5	0	6	3

Topological order

3 6 0 5 2 1 4

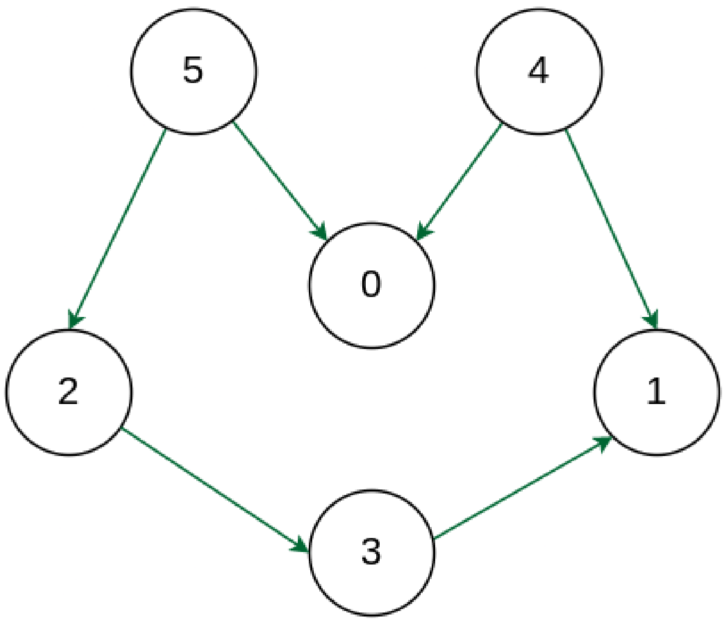
pop from the stack → reversed postorder

v	marked[ ]
0	T
1	T
2	T
3	T
4	T
5	T
6	T

Java execution stack

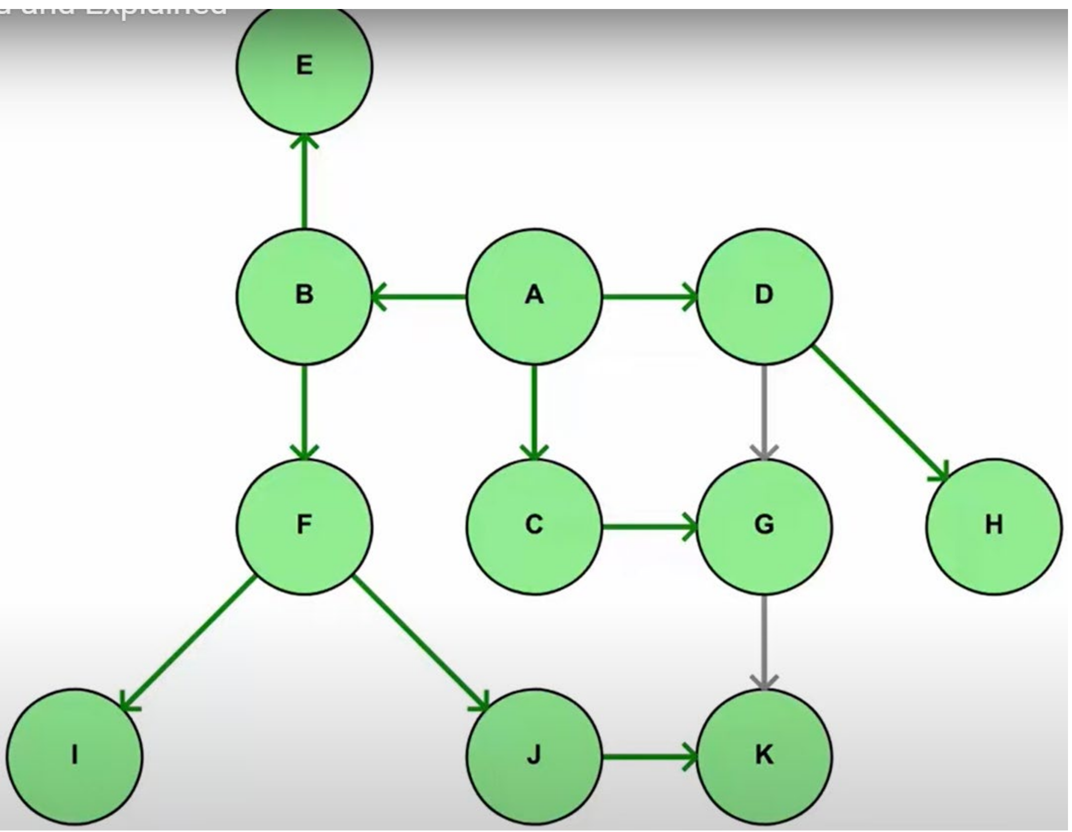
```
dfs(0)
  dfs(1)
    dfs(4)
    4 done
  1 done
  dfs(2)
  2 done
  dfs(5)
    check 2
  5 done
0 done
check 1
check 2
dfs(3)
  check 2
  check 4
  check 5
  dfs(6)
    check 0
    check 4
  6 done
3 done
check 4
check 5
check 6
done
```

Example 1: Reverse Post-order



- Starting from node 5, post-order traversal is “1 3 2 0 5 4”.
- Reverse order is “4 5 0 2 3 1”, which is a topological sort.
- Starting from node 4, post-order traversal is “0 1 4 3 2 5”.
- Reverse order is “5 2 3 4 1 0”, which is another topological sort.

Example 2: Reverse Post-order



- Starting from node A, post-order traversal is "I K J F E B G C H D A".
- Reverse order is "A D H C G B E F J K I", which is a topological sort.

Topological Sort Visualized and Explained
https://www.youtube.com/watch?v=7J3GadLzydI

Cycles and undirected edges

- Why Topological Sort is not possible for graphs having cycles?
  - Imagine a graph with 3 vertices and edges = {1 to 2 , 2 to 3, 3 to 1} forming a cycle. Now if we try to topologically sort this graph starting from any vertex, it will always create a contradiction to our definition. All the vertices in a cycle are indirectly dependent on each other hence topological sorting fails.
- Why Topological Sort is not possible for graphs with undirected edges?
  - Special case of a cycle. Undirected edge between two vertices u and v means, there is an edge from u to v as well as from v to u. Because of this both the nodes u and v depend upon each other and none of them can appear before the other in the topological ordering without creating a contradiction.

Topological Sort: Java Implementation

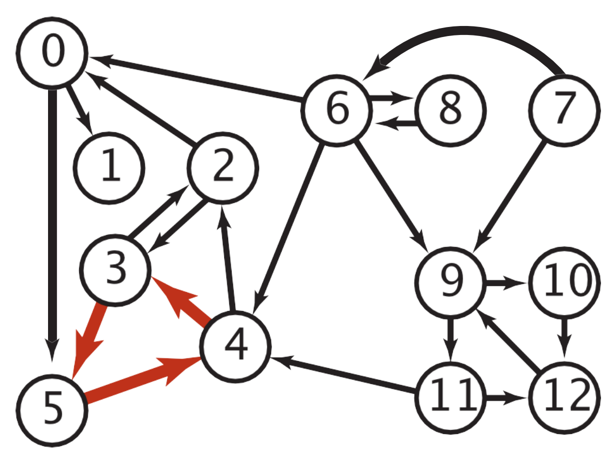
```
public class DepthFirstOrder {
   private boolean[] marked;
   private Stack<Integer> reversePostorder;

   public DepthFirstOrder(Digraph G) {
      reversePostorder = new Stack<Integer>();
      marked = new boolean[G.V()];
      for (int v = 0; v < G.V(); v++)
         if (!marked[v]) dfs(G, v);
   }

   private void dfs(Digraph G, int v) {
      marked[v] = true;
      for (int w : G.adj(v))
         if (!marked[w]) dfs(G, w);
      reversePostorder.push(v);
   }

   public Iterable<Integer> reversePostorder()
   { return reversePostorder; }

}
```

returns all vertices in "reverse DFS postorder"

Proposition. A digraph has a topological order iff no directed cycle.

Pf.

- If directed cycle, topological order impossible.
- If no directed cycle, DFS-based algorithm finds a topological order.



a digraph with a directed cycle

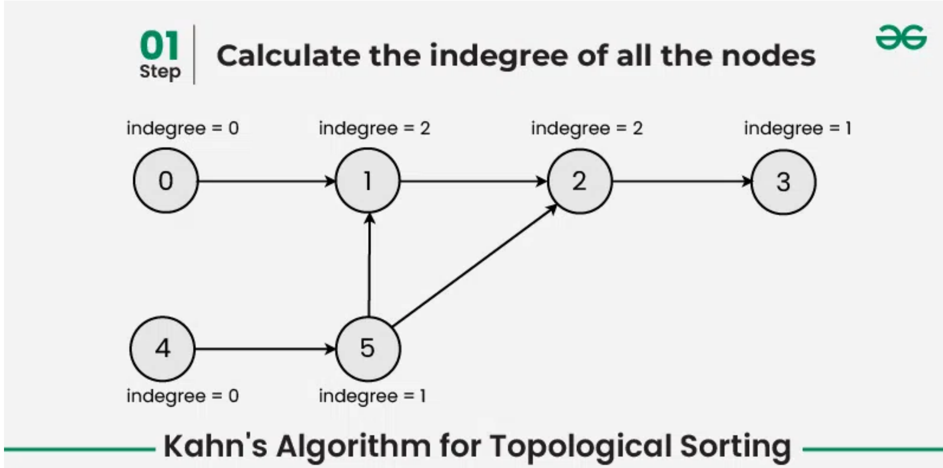
Goal. Given a digraph, find a directed cycle.

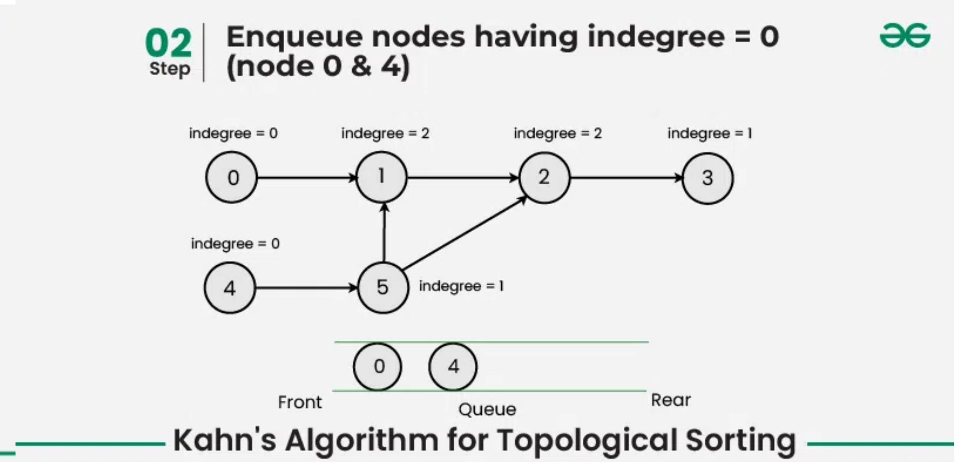
Solution. DFS. See next slide.

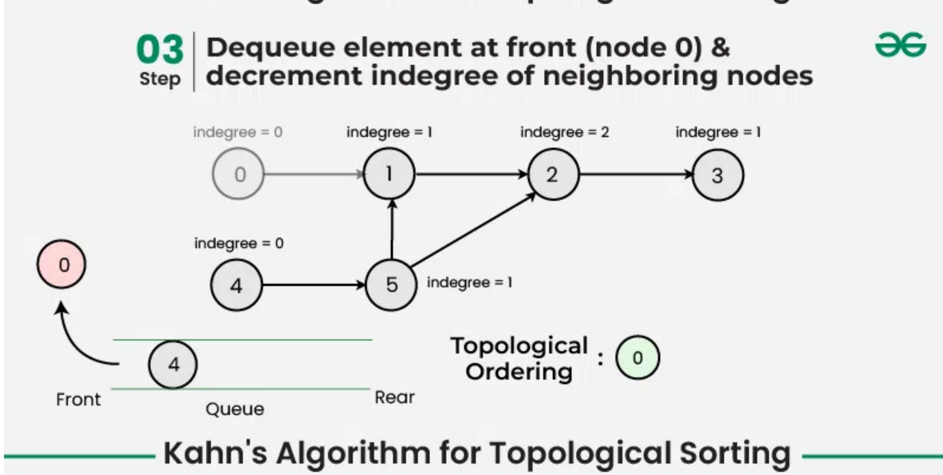
Kahn's algorithm for Topological Sorting

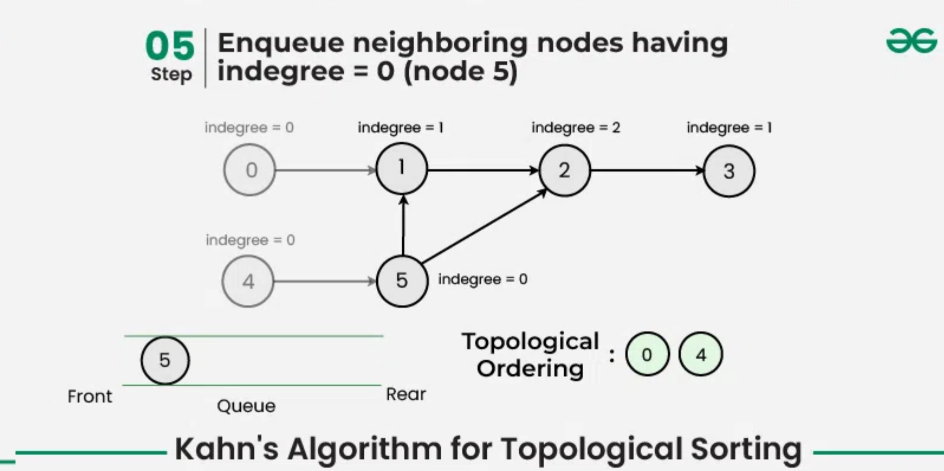
- The algorithm works by repeatedly finding vertices with no incoming edges, removing them from the graph, and updating the incoming edges of the remaining vertices. This process continues until all vertices have been ordered.
  - Add all nodes with in-degree 0 to a queue.
  - While the queue is not empty:
    - Remove a node from the queue.
    - For each outgoing edge from the removed node, decrement the in-degree of the destination node by 1.
    - If the in-degree of a destination node becomes 0, add it to the queue.
  - If the queue is empty and there are still nodes in the graph, the graph contains a cycle and cannot be topologically sorted.
  - The nodes in the queue represent the topological ordering of the graph.
- Time Complexity: O(V+E).
  - The outer for loop will be executed V number of times and the inner for loop will be executed E number of times.

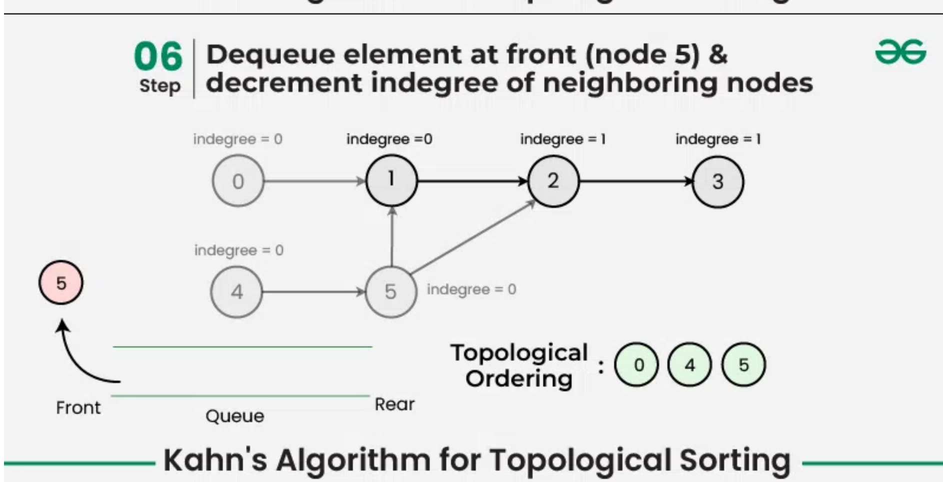
https://www.geeksforgeeks.org/topological-sorting-indegree-based-solution/

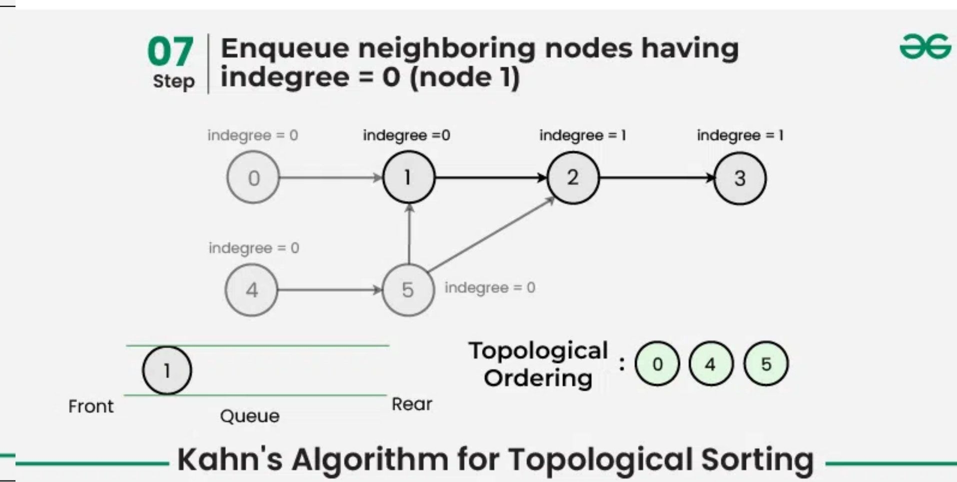

01 Step
Calculate the indegree of all the nodes
indegree = 0
indegree = 2
indegree = 2
indegree = 1
0
1
2
3
4
5
indegree = 0
indegree = 1
Kahn's Algorithm for Topological Sorting

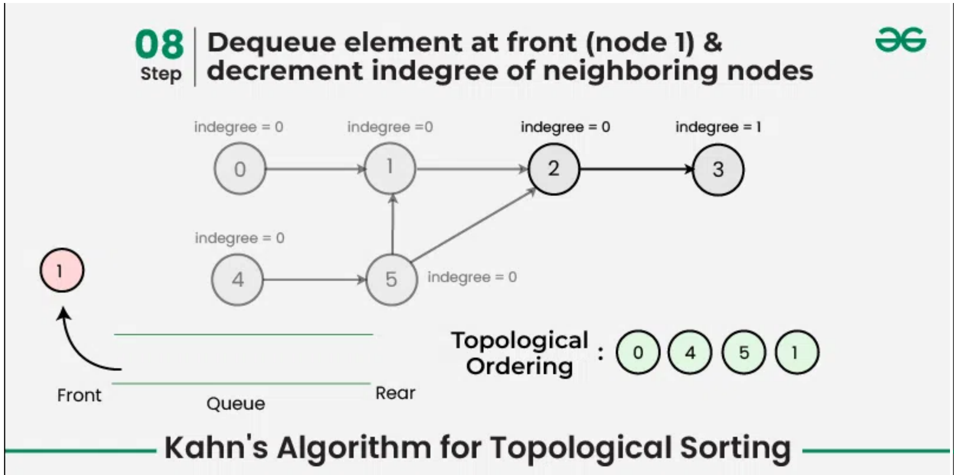

02 Step
Enqueue nodes having indegree = 0 (node 0 & 4)
indegree = 0
indegree = 2
indegree = 2
indegree = 1
indegree = 0
indegree = 1
0
1
2
3
4
5
0
4
Front
Queue
Rear
Kahn's Algorithm for Topological Sorting

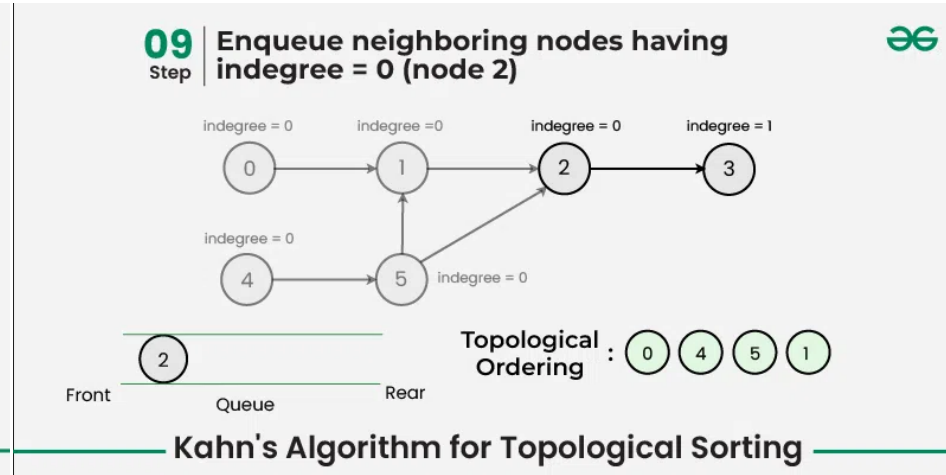

03 Step
Dequeue element at front (node 0) & decrement indegree of neighboring nodes
indegree = 0
indegree = 1
indegree = 2
indegree = 1
indegree = 0
indegree = 1
0
1
2
3
4
5
4
Front
Queue
Rear
Topological Ordering : 0
Kahn's Algorithm for Topological Sorting

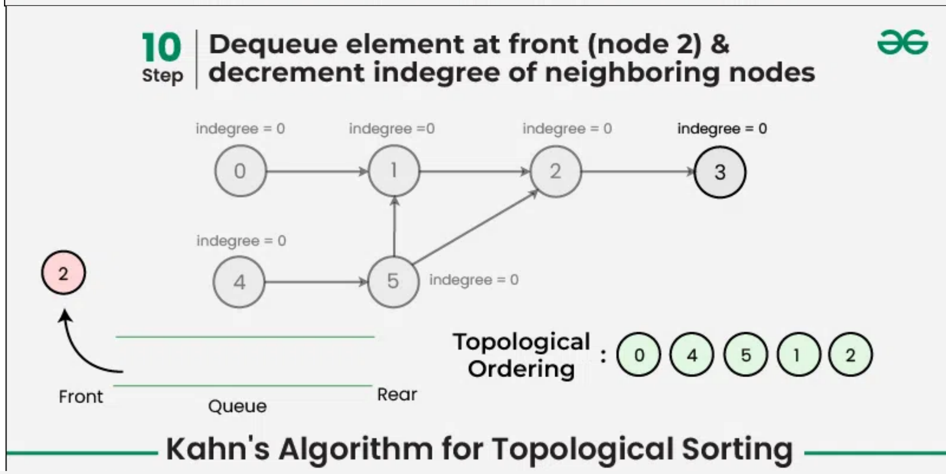

05 Step
Enqueue neighboring nodes having indegree = 0 (node 5)
indegree = 0
indegree = 1
indegree = 2
indegree = 1
indegree = 0
indegree = 0
0
1
2
3
4
5
5
Front
Queue
Rear
Topological Ordering : 0 4
Kahn's Algorithm for Topological Sorting

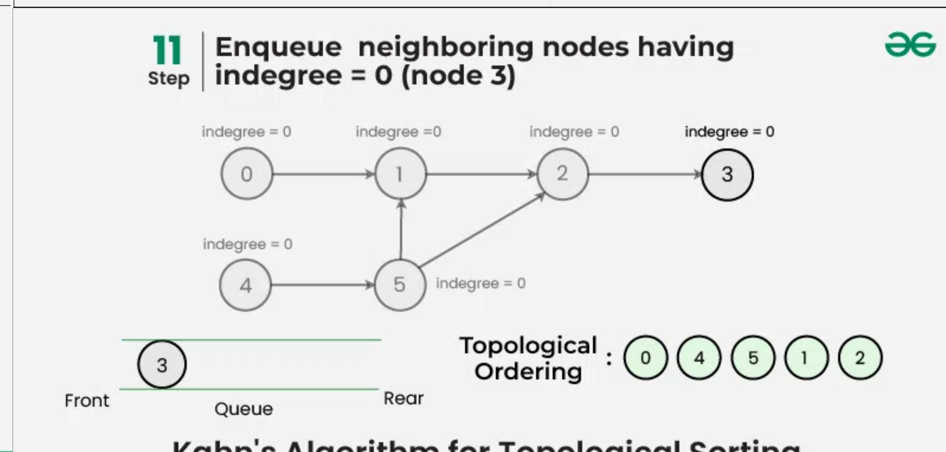

06 Step
Dequeue element at front (node 5) & decrement indegree of neighboring nodes
indegree = 0
indegree =0
indegree = 1
indegree = 1
indegree = 0
indegree = 0
0
1
2
3
4
5
Front
Queue
Rear
Topological Ordering : 0 4 5
Kahn's Algorithm for Topological Sorting

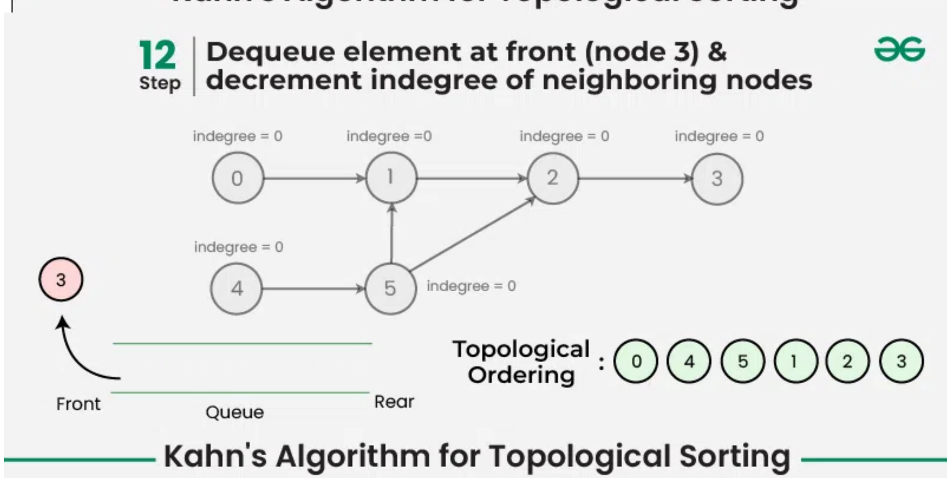

07 Step
Enqueue neighboring nodes having indegree = 0 (node 1)
indegree = 0
indegree =0
indegree = 1
indegree = 1
indegree = 0
indegree = 0
0
1
2
3
4
5
1
Front
Queue
Rear
Topological Ordering : 0 4 5
Kahn's Algorithm for Topological Sorting

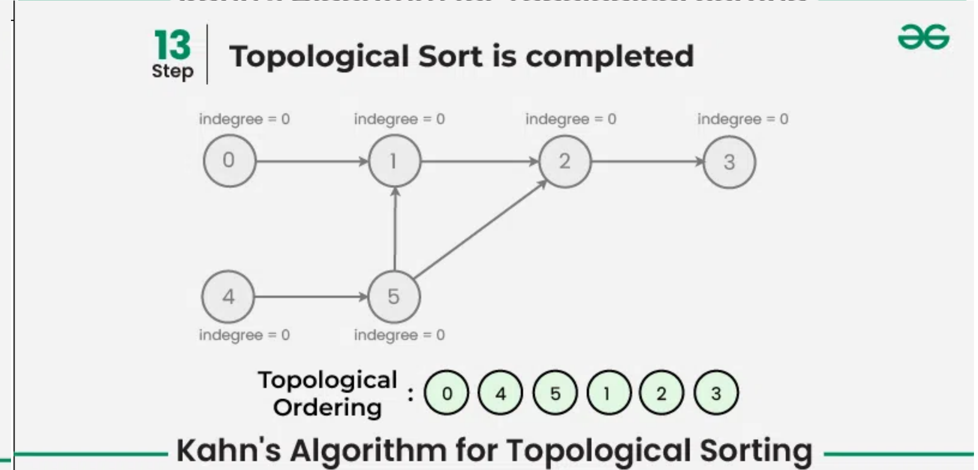

08 Step
Dequeue element at front (node 1) & decrement indegree of neighboring nodes
indegree = 0
indegree =0
indegree = 0
indegree = 1
0
1
2
3
indegree = 0
4
5
indegree = 0
1
Topological Ordering : 0 4 5 1
Front
Queue
Rear
Kahn's Algorithm for Topological Sorting


09 Step
Enqueue neighboring nodes having indegree = 0 (node 2)
indegree = 0
indegree =0
indegree = 0
indegree = 1
0
1
2
3
indegree = 0
4
5
indegree = 0
2
Topological Ordering : 0 4 5 1
Front
Queue
Rear
Kahn's Algorithm for Topological Sorting


10 Step
Dequeue element at front (node 2) & decrement indegree of neighboring nodes
indegree = 0
indegree =0
indegree = 0
indegree = 0
0
1
2
3
indegree = 0
4
5
indegree = 0
2
Topological Ordering : 0 4 5 1 2
Front
Queue
Rear
Kahn's Algorithm for Topological Sorting


11 Step
Enqueue neighboring nodes having indegree = 0 (node 3)
indegree = 0
indegree =0
indegree = 0
indegree = 0
0
1
2
3
indegree = 0
4
5
indegree = 0
3
Topological Ordering : 0 4 5 1 2
Front
Queue
Rear
Kahn's Algorithm for Topological Sorting


12 Step
Dequeue element at front (node 3) & decrement indegree of neighboring nodes
indegree = 0
indegree =0
indegree = 0
indegree = 0
0
1
2
3
indegree = 0
4
5
indegree = 0
3
Topological Ordering : 0 4 5 1 2 3
Front
Queue
Rear
Kahn's Algorithm for Topological Sorting


13 Step
Topological Sort is completed
indegree = 0
indegree = 0
indegree = 0
indegree = 0
0
1
2
3
4
5
indegree = 0
indegree = 0
Topological Ordering : 0 4 5 1 2 3
Kahn's Algorithm for Topological Sorting

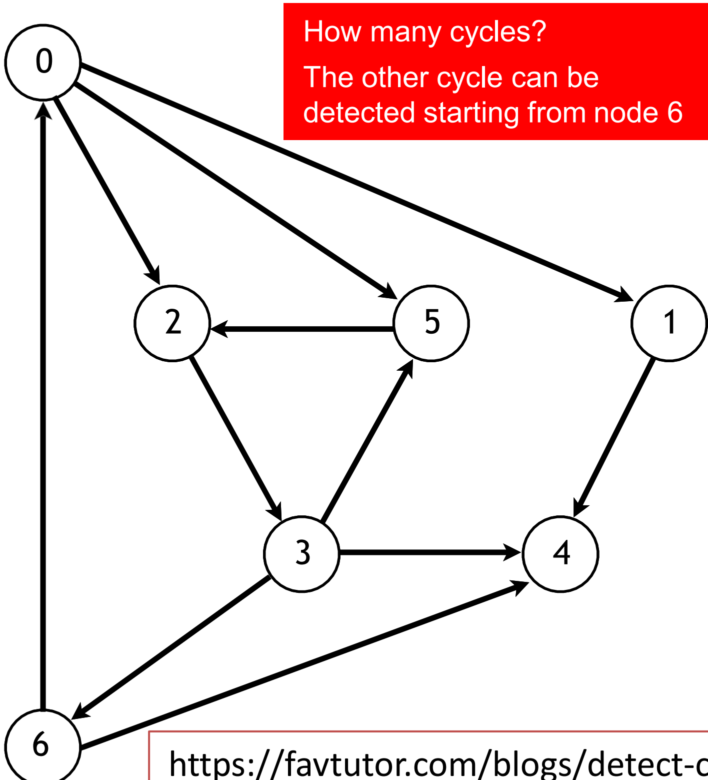
Topological Sort Applications

- Task scheduling and project management.
- In software deployment tools like Makefile.
- Dependency resolution in package management systems.
- Determining the order of compilation in software build systems.
- Deadlock detection in operating systems.
- Course scheduling in universities.
- It is used to find shortest paths in weighted directed acyclic graphs

Directed Cycle Detection

- Run depth-first search from every unmarked vertex.
- Keep track of vertices currently in recursion stack of function for DFS traversal with onStack[ ] array.
- If we reach a vertex that is already in the recursion stack, then we found a cycle in the tree, and we're done
- Retrieve the cycle using edgeTo[ ] array.

- set onStack[v] to T when dfs(v) is called
- set onStack[v] to F when dfs(v) returns



How many cycles?

The other cycle can be detected starting from node 6

Java execution stack

v	marked[ ]	onStack[ ]	edgeTo[ ]
0	T	T	–
1	T	T	0
2	T	T	0
3	T	T	2
4	T	T	1
5	T	T	3
6	F	F	–

dfs(0)
dfs(1)
dfs(4)
4 done
1 done
dfs(2)
dfs(3)
check 4
dfs(5)
check 2
done

stack top

2	3	5	2

- vertex is marked and onStack
- Found the cycle
- Save the cycle using edgeTo[ ] to a stack

https://favtutor.com/blogs/detect-cycle-in-directed-graph

Directed Cycle Detection Application: Cyclic Inheritance

The Java compiler does cycle detection.

```
public class A extends B
{
    ...
}
```

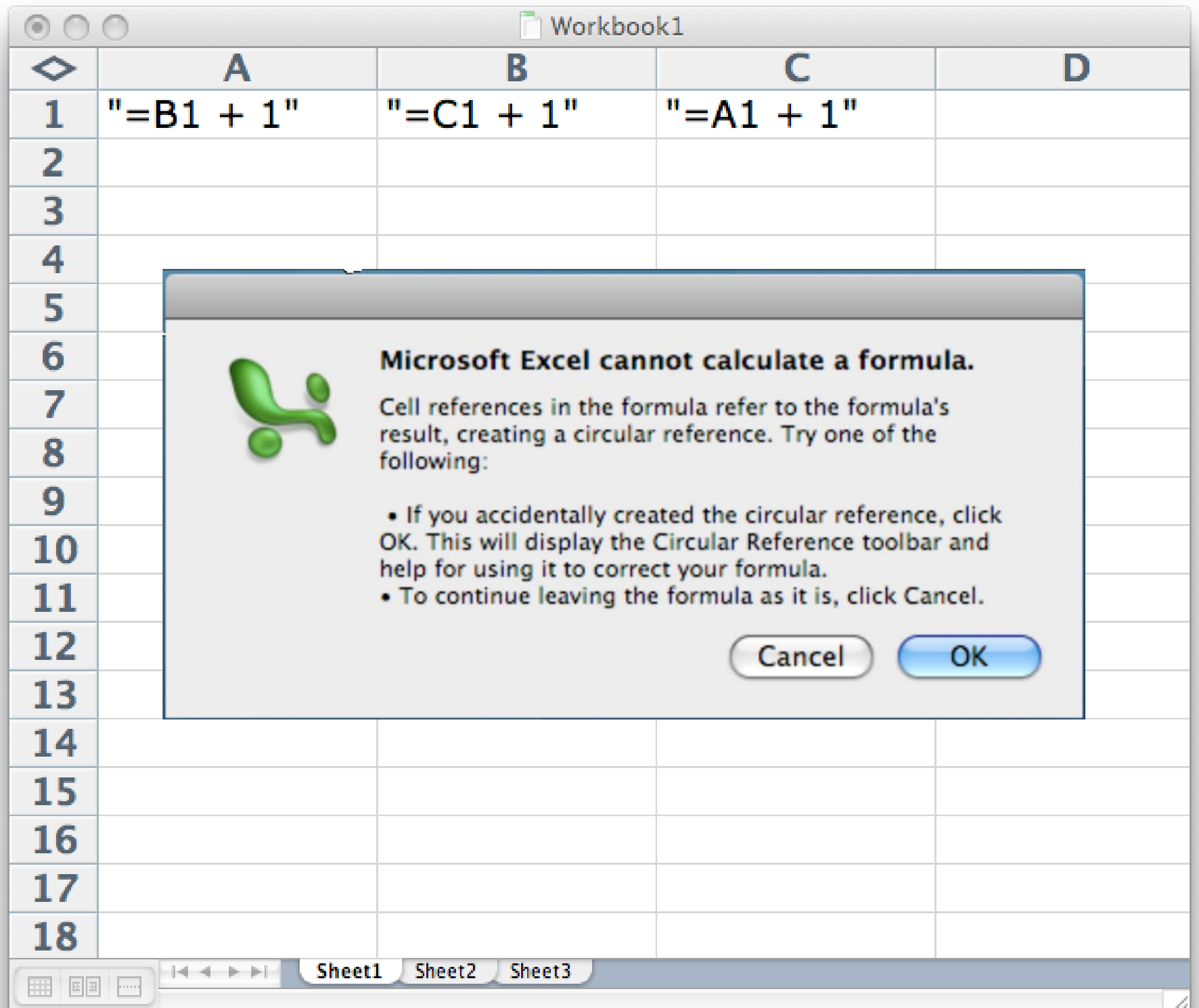
```
public class B extends C
{
    ...
}
```

```
public class C extends A
{
    ...
}
```

```
% javac A.java
A.java:1: cyclic inheritance
involving A
public class A extends B { }
             ^
1 error
```

Directed Cycle Detection Application: Spreadsheet Recalculation

Microsoft Excel does cycle detection.

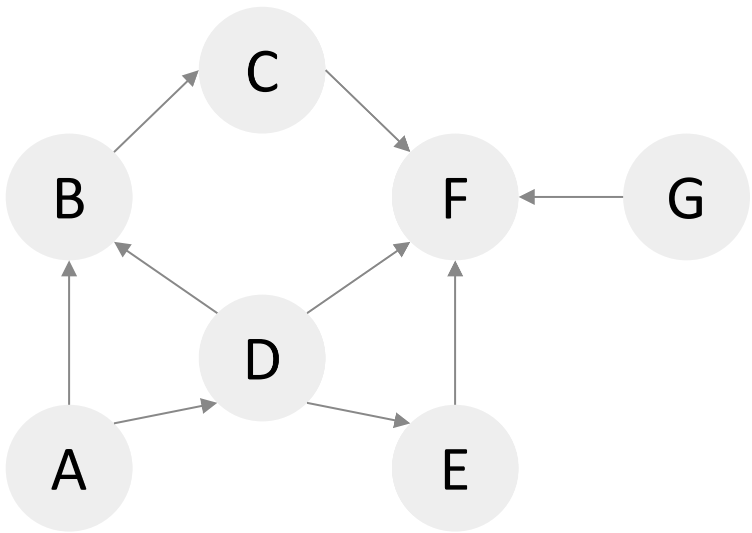


Video Tutorials: DFS and BFS

- Breadth-first search in 4 minutes
  - https://www.youtube.com/watch?v=HZ5YTanv5QE
- Depth-first search in 4 minutes
  - https://www.youtube.com/watch?v=Urx87-NMm6c
- Graph Traversals - Breadth First and Depth First
  - https://www.youtube.com/watch?v=bIA8HEEUxZI

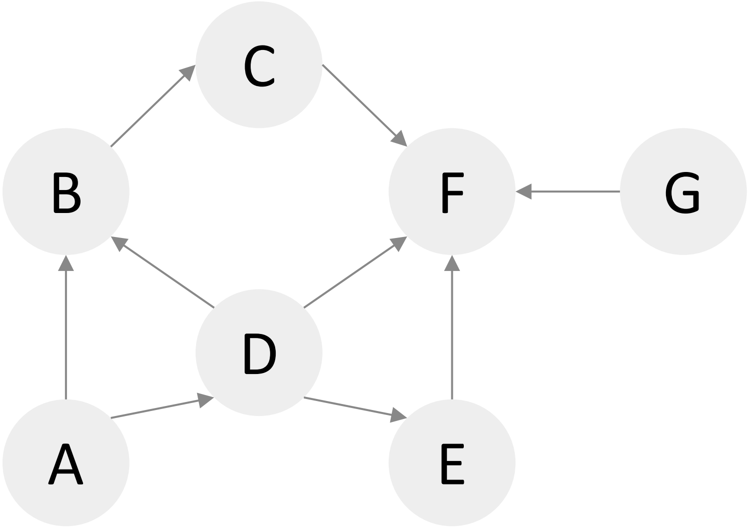
Quiz 1

- Write out the adjacency matrix and adjacency list for the directed graph.



Acknowledgement: https://sp24.datastructur.es/

Adjacency Matrix

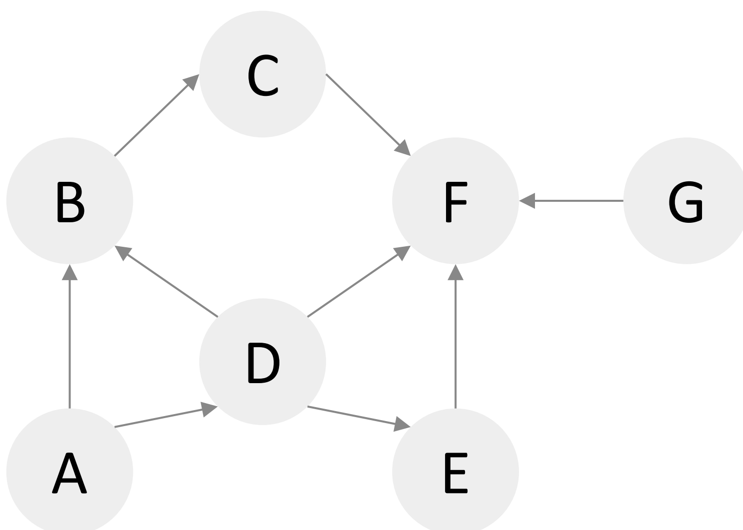


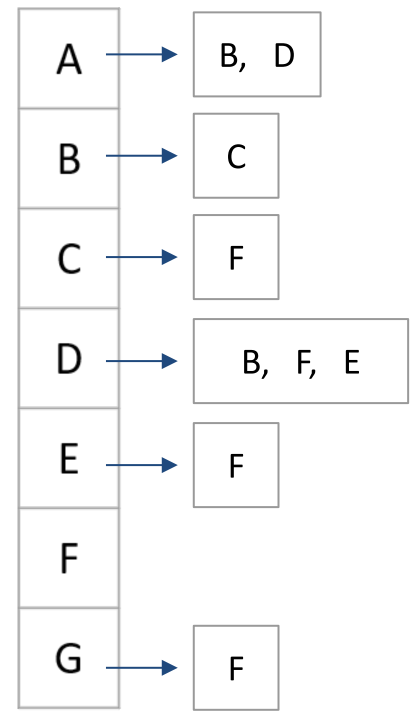
	A	B	C	D	E	F	G
A		✓		✓			
B			✓				
C						✓	
D		✓			✓	✓	
E						✓	
F							
G						✓	

To

From

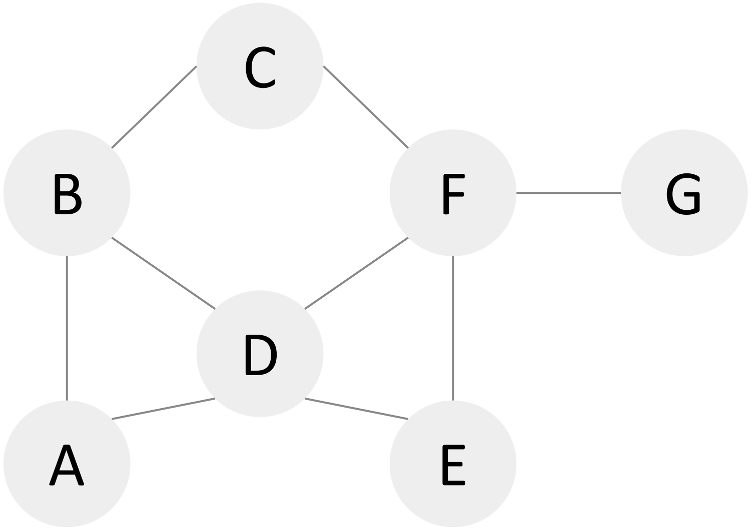
Adjacency List



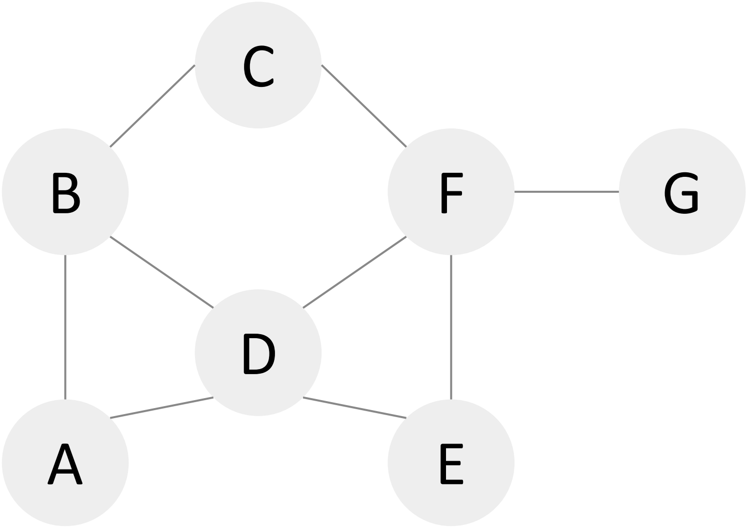


Quiz 2

- Write out the adjacency matrix and adjacency list for the undirected graph.

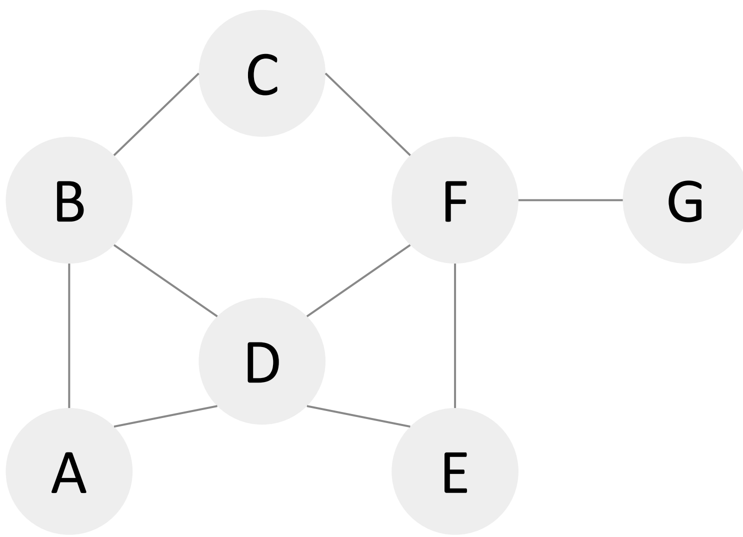


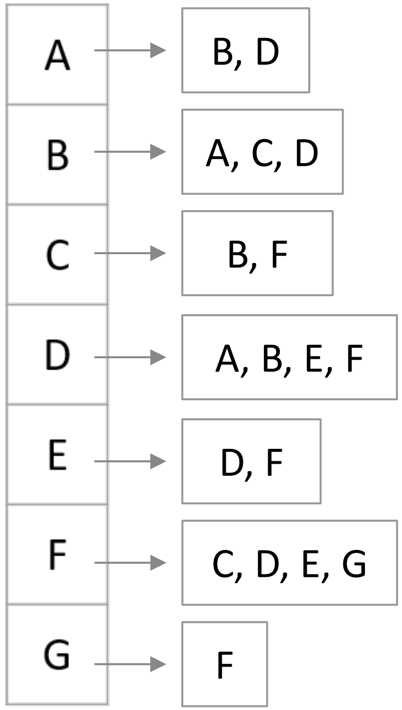
Adjacency Matrix



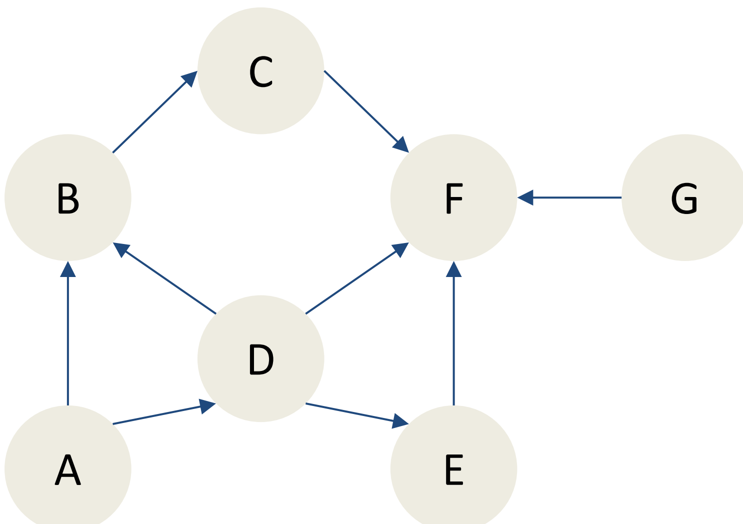
	A	B	C	D	E	F	G
A		✓		✓			
B	✓		✓	✓			
C		✓				✓	
D	✓	✓			✓	✓	
E				✓		✓	
F			✓	✓	✓		✓
G						✓	

Adjacency List





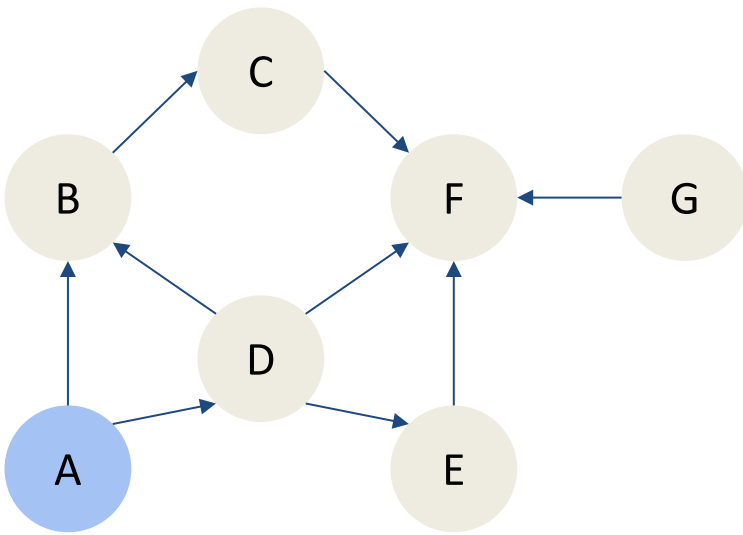
Quiz 3: Pre-Order & Post-Order Traversals



DFS Pre-Order:

DFS Post-Order:

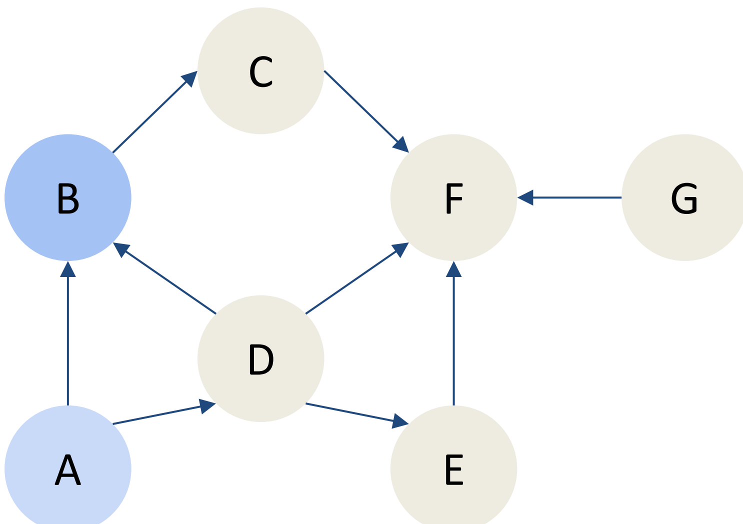
Stack:



DFS Pre-Order:
A

DFS Post-Order:

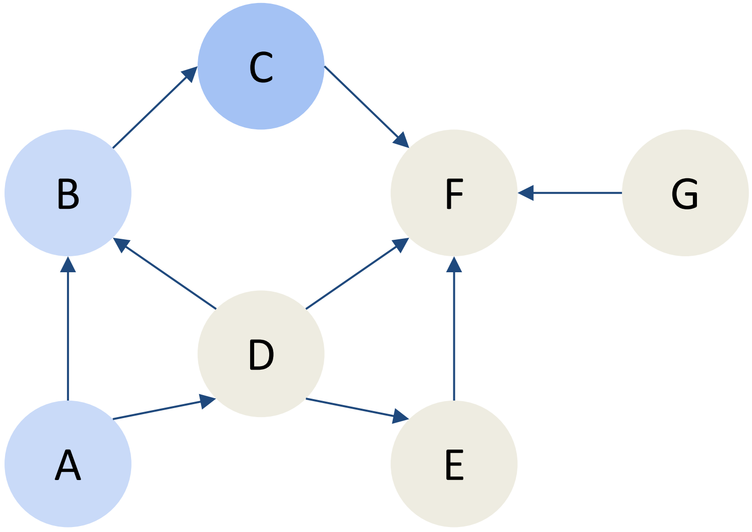
Stack: A



DFS Pre-Order:
A, B

DFS Post-Order:

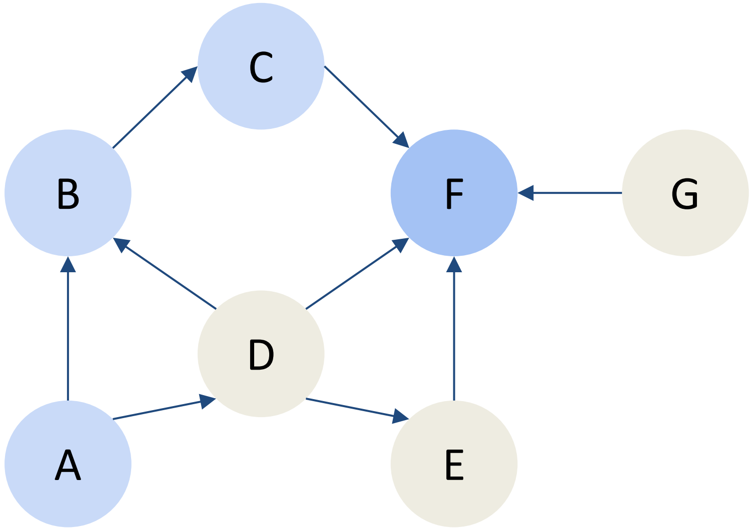
Stack: A, B



DFS Pre-Order:
A, B, C

DFS Post-Order:

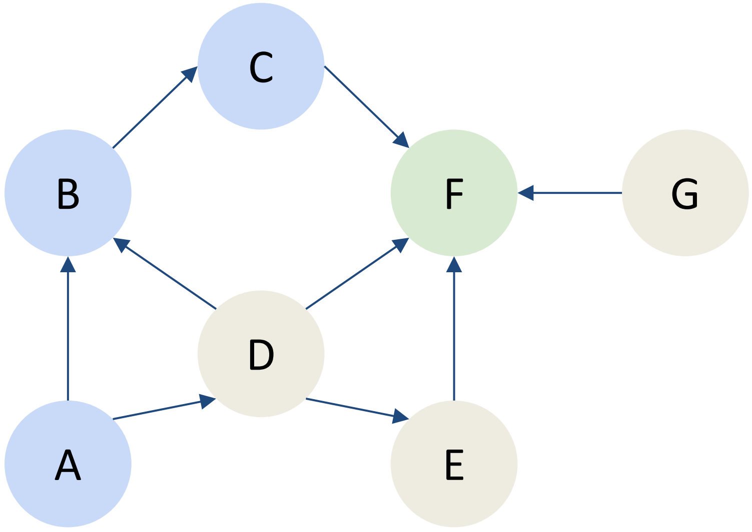
Stack: A, B, C



DFS Pre-Order:
A, B, C, F

DFS Post-Order:

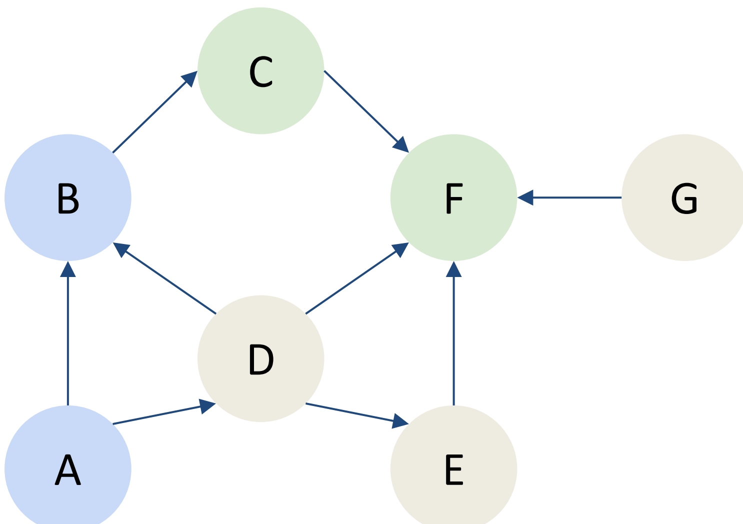
Stack: A, B, C, F



DFS Pre-Order:
A, B, C, F

DFS Post-Order:
F

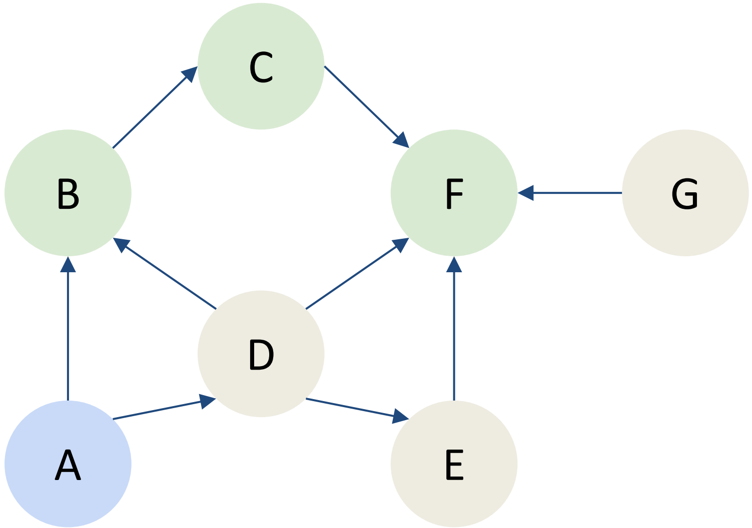
Stack: A, B, C



DFS Pre-Order:
A, B, C, F

DFS Post-Order:
F, C

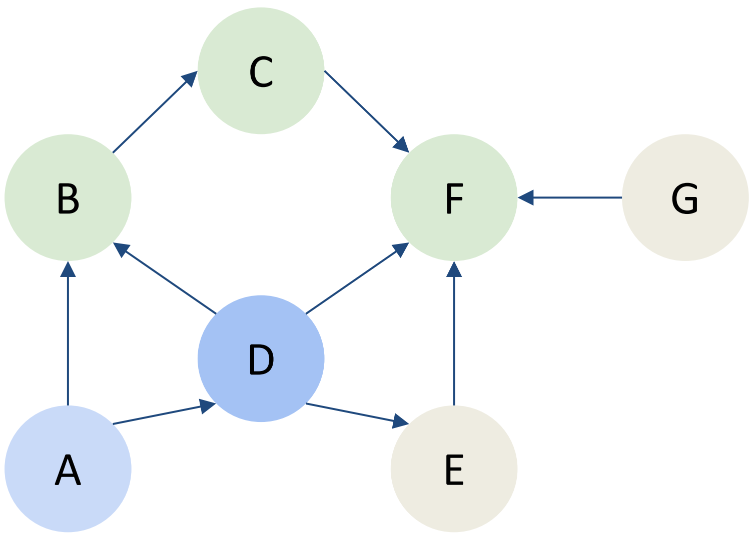
Stack: A, B



DFS Pre-Order:
A, B, C, F

DFS Post-Order:
F, C, B

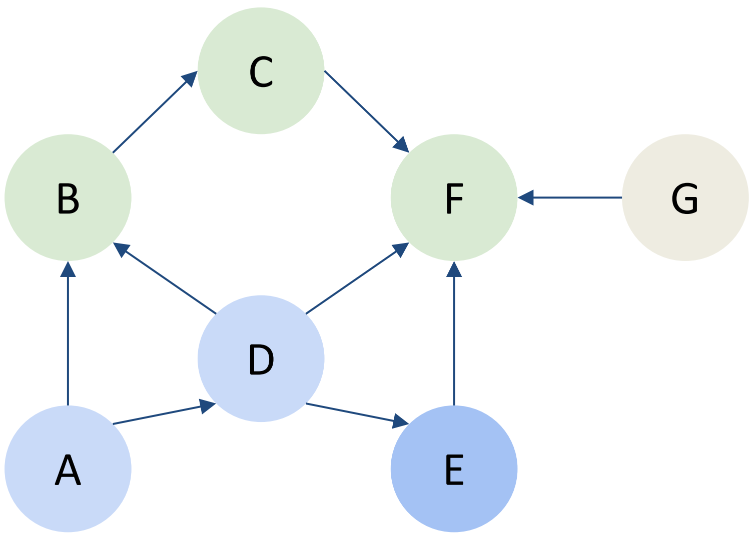
Stack: A



DFS Pre-Order:
A, B, C, F, D

DFS Post-Order:
F, C, B

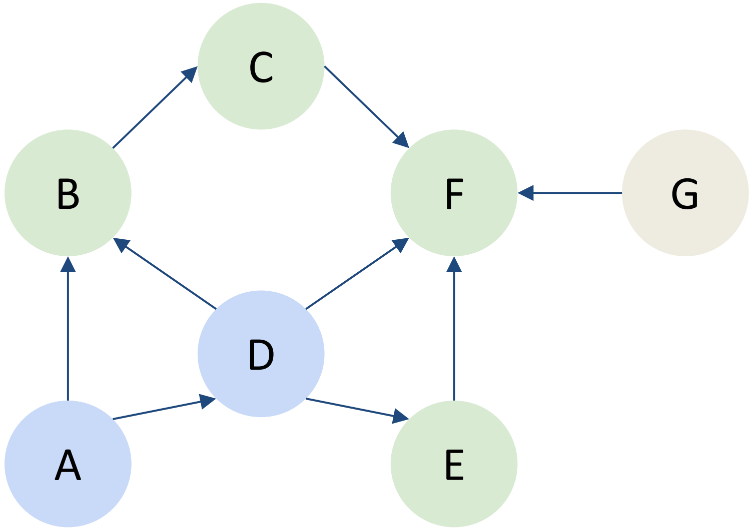
Stack: A, D



DFS Pre-Order:
A, B, C, F, D, E

DFS Post-Order:
F, C, B,

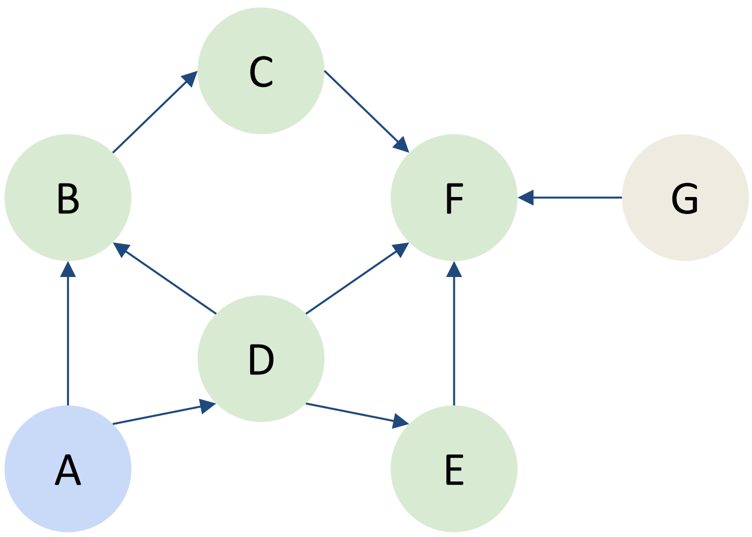
Stack: A, D, E



DFS Pre-Order:
A, B, C, F, D, E

DFS Post-Order:
F, C, B, E

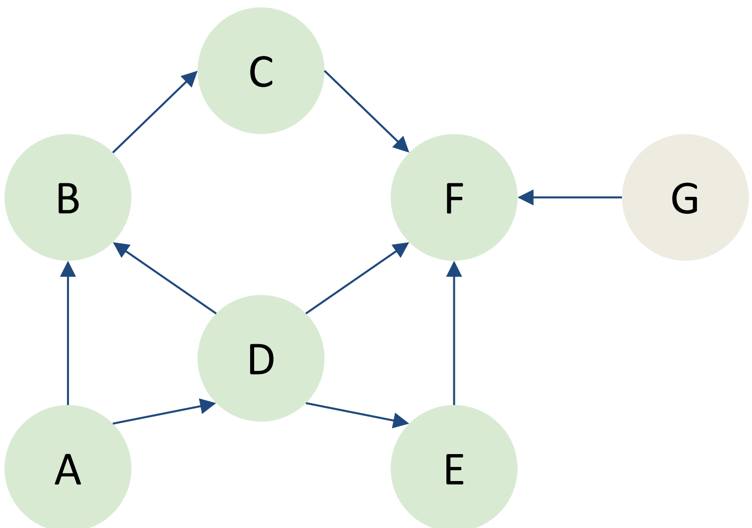
Stack: A, D



DFS Pre-Order:
A, B, C, F, D, E

DFS Post-Order:
F, C, B, E, D

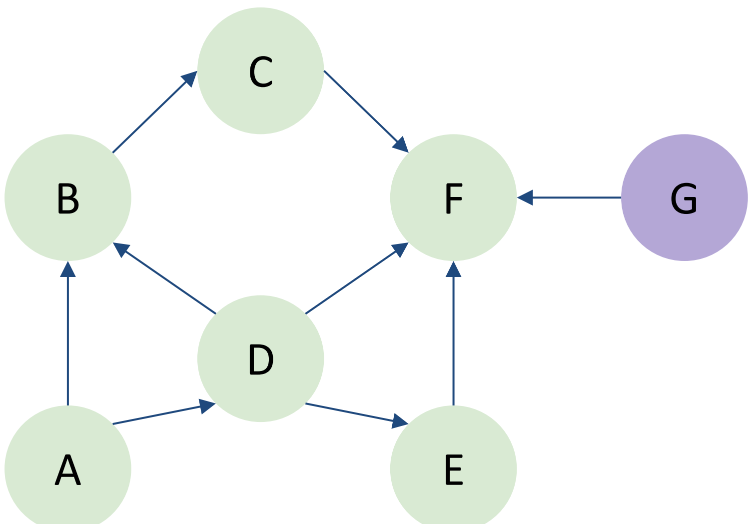
Stack: A,



DFS Pre-Order:
A, B, C, F, D, E

DFS Post-Order:
F, C, B, E, D, A

Stack:



Stack:

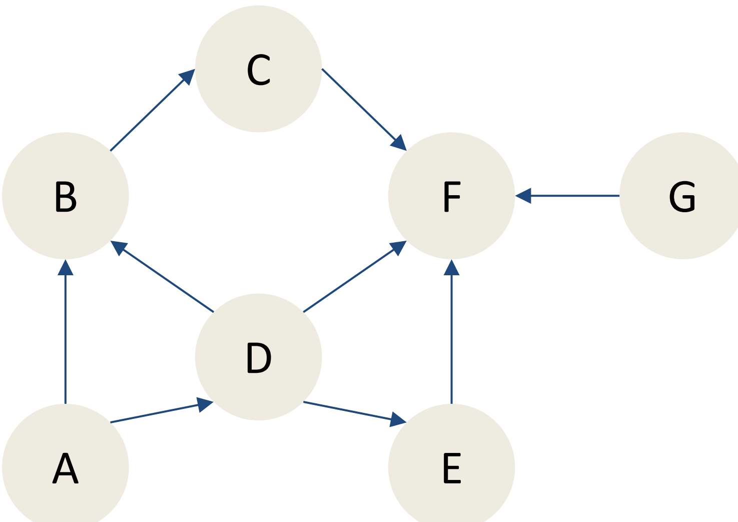
DFS Pre-Order:
A, B, C, F, D, E, G

DFS Post-Order:
F, C, B, E, D, A, G

Toplogical Sort:
G, A, D, E, B, C, F

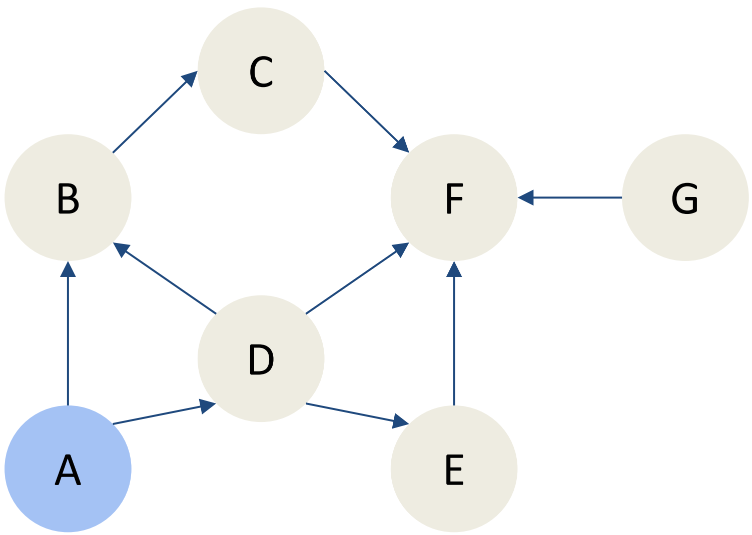
* if we allow DFS to restart on unmarked nodes, G would be added to the stack (and ultimately the last element in both the preorder and postorder traversals)

Quiz 4: BFS



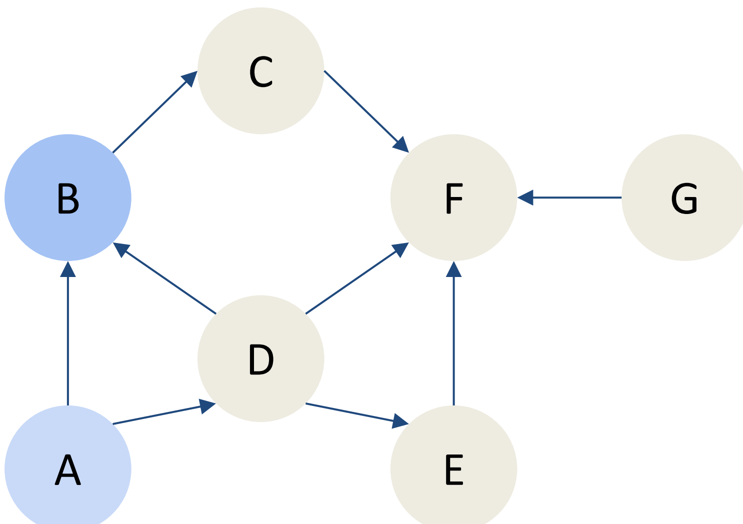
BFS:

Queue: A



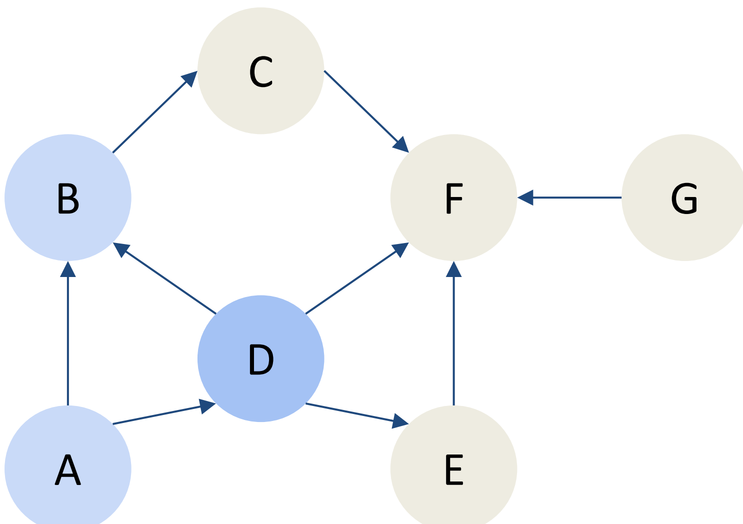
BFS:
A

Queue: B D



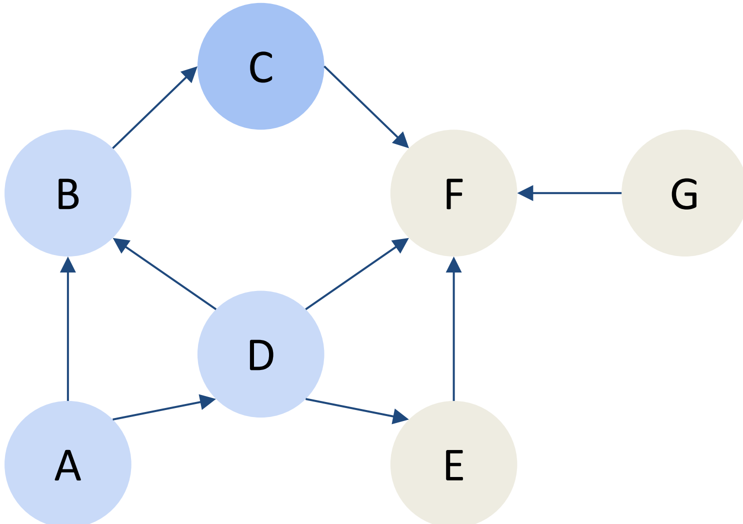
BFS:
A B

Queue: D C



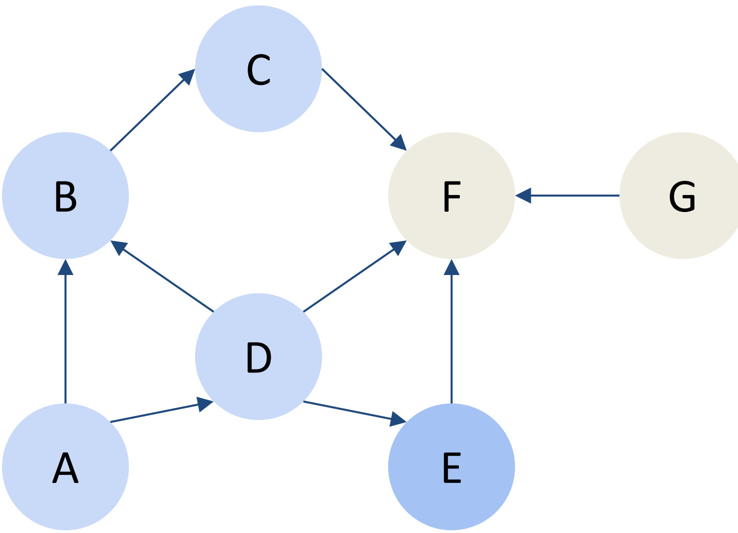
BFS:
A B D

Queue: C E F



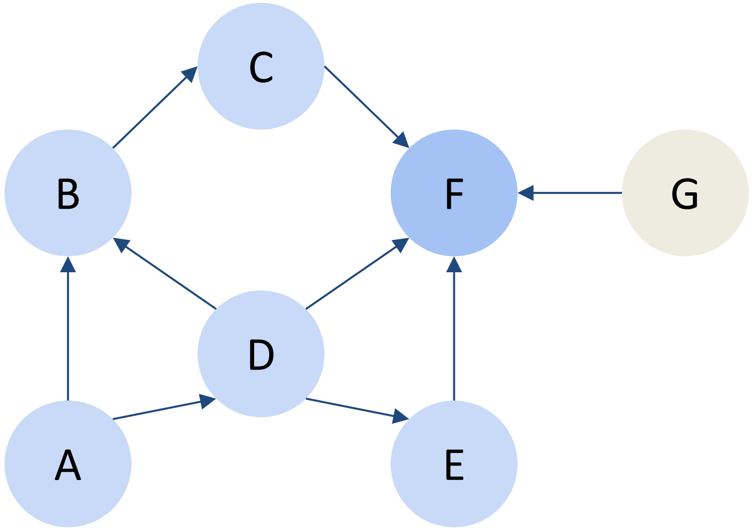
BFS:
A B D C

Queue: E F



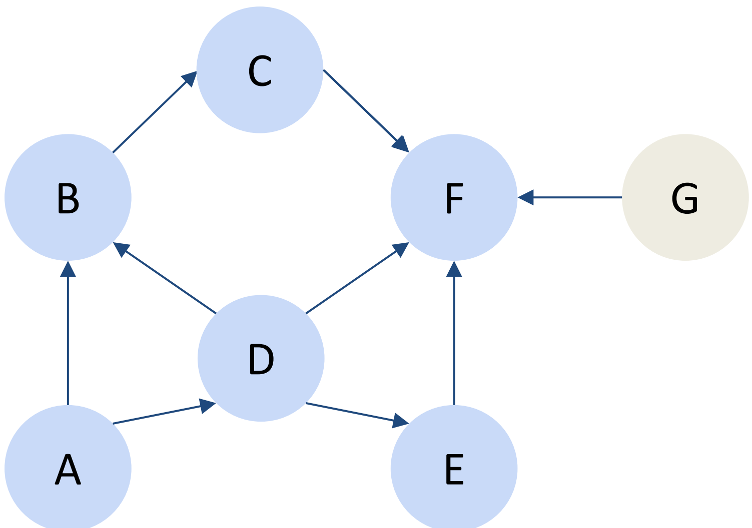
BFS:
A B D C E

Queue: F



BFS:
A B D C E F

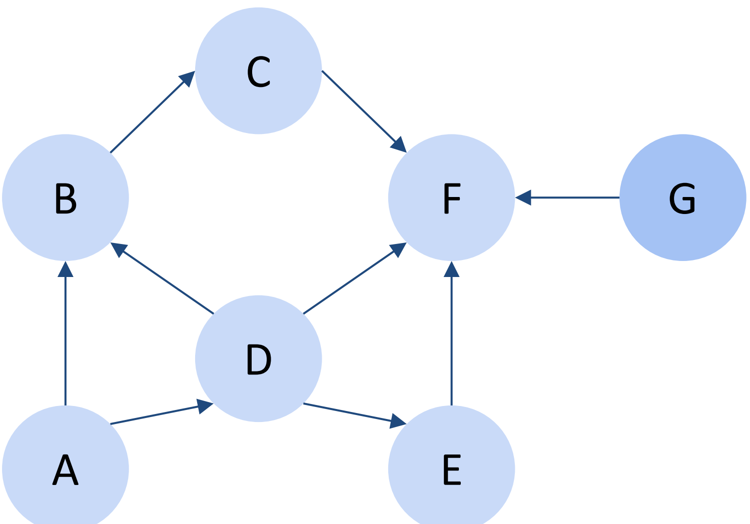
Queue:



BFS:

A B D C E F

Queue: G



BFS:
A B D C E F G

Queue: