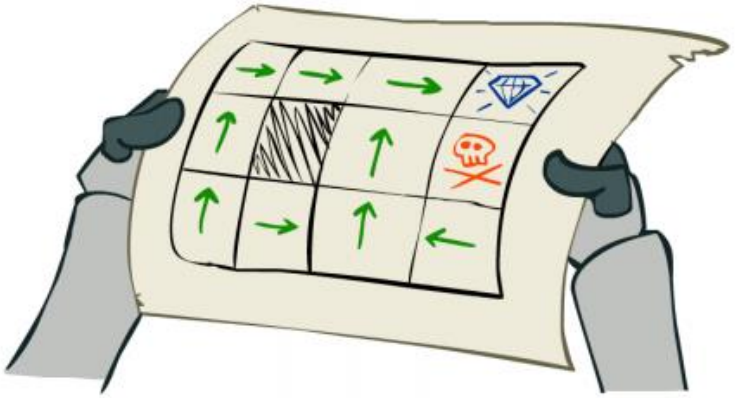
L7.1 MDP Planning

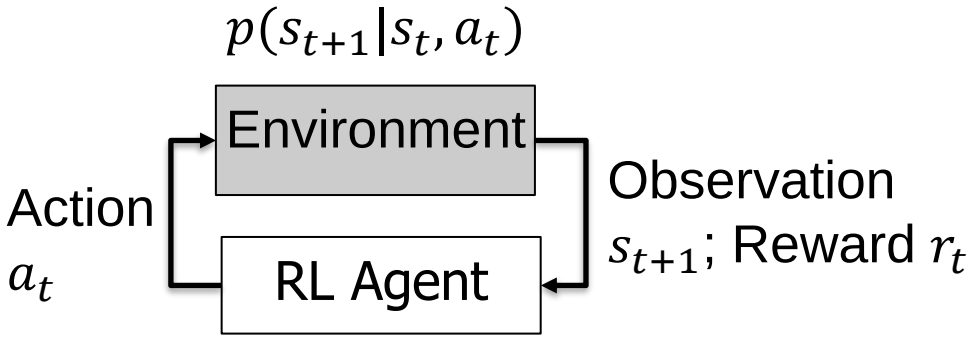
Zonghua Gu 2021



Acknowledgement: slides based on https://www.coursera.org/specializations/reinforcement-learning
And textbook by Sutton and Barto http://incompleteideas.net/book/the-book-2nd.html

Reinforcement Learning (RL)

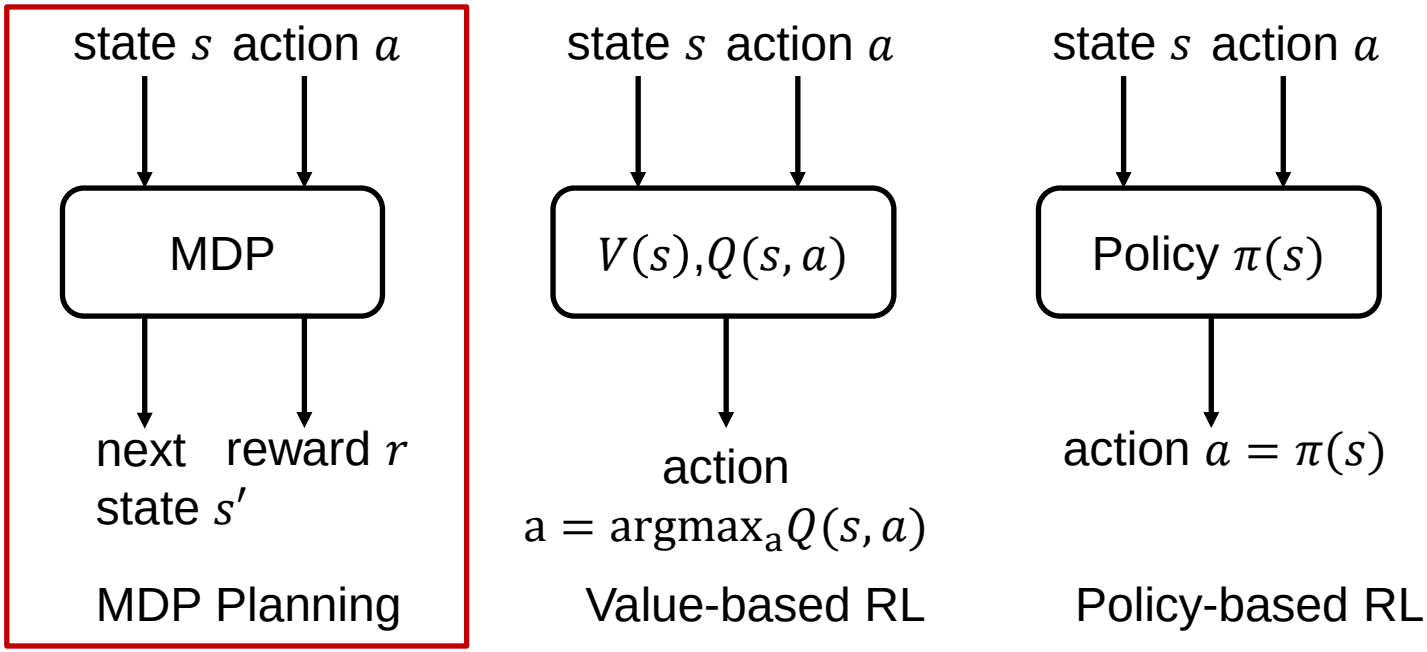
- The RL problem: find the optimal policy given the reward function.
  - Sometimes loss function is used, defined as negative of reward function, so minimizing loss is equivalent to maximizing reward.



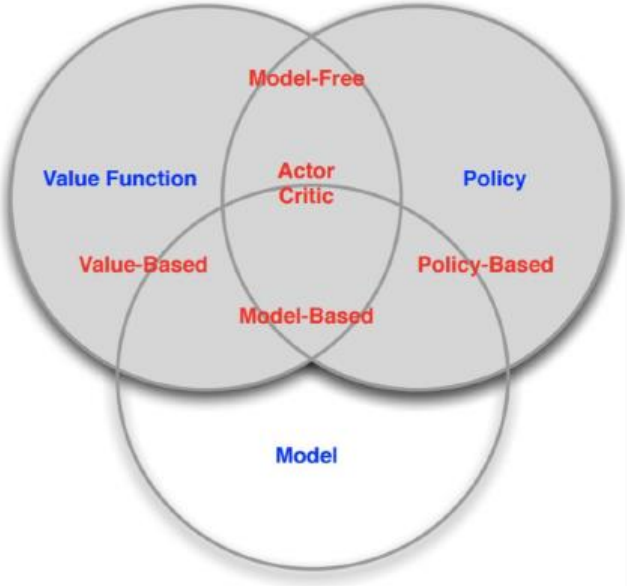
Goal: learn optimal policy $\pi_\theta(a_t|s_t)$ that maximizes cumulative reward $\max_\pi E(\sum_{t=0}^{T} \gamma^t r(s_t, a_t))$

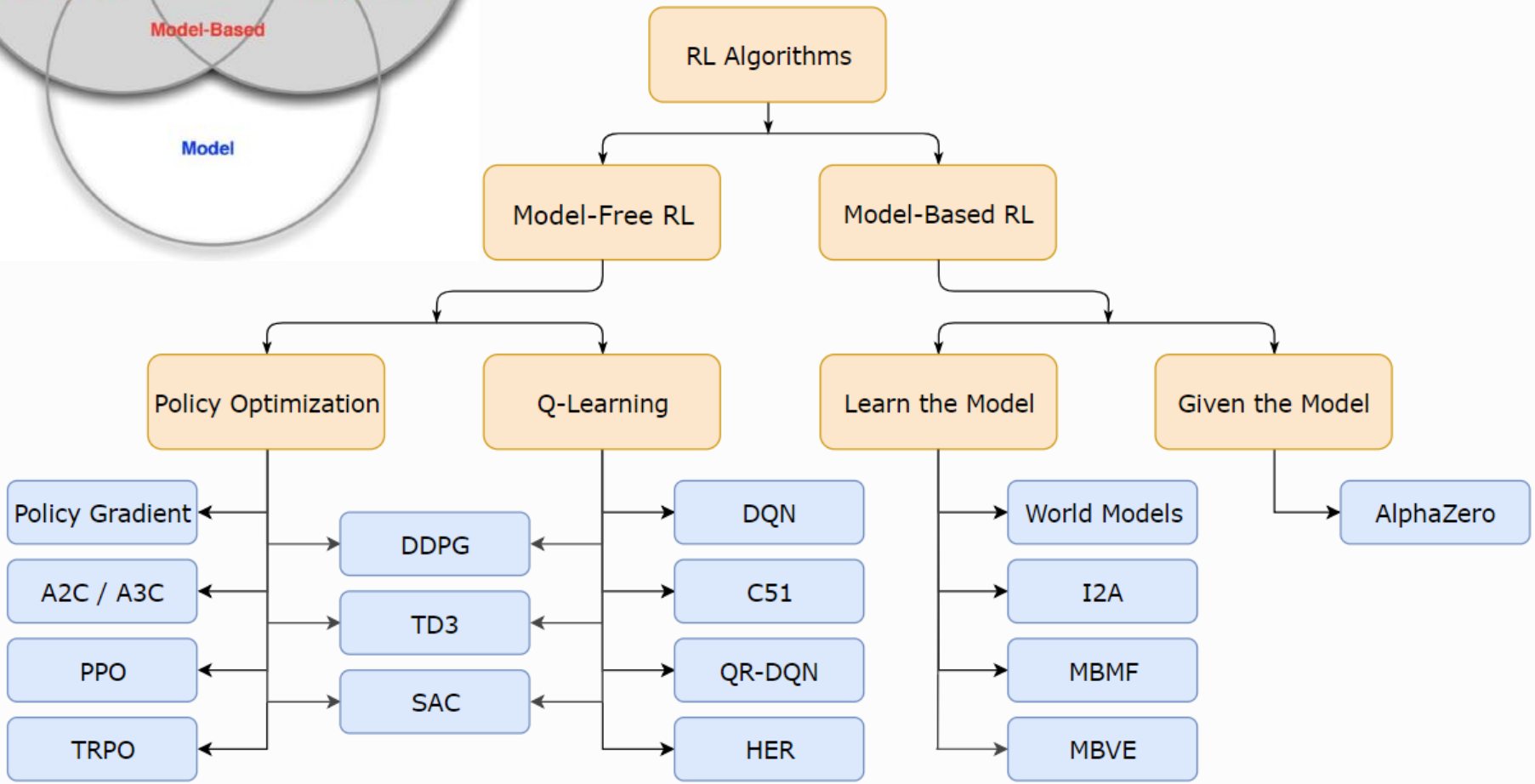
Model-Based vs. Model-Free

- Model-Based RL: MDP planning
  - Learn MDP $p(r, s'|s, a)$(given current state s and action a, returns prob distribution of current reward r and next state s'), then plan with Value Iteration or Policy Iteration
- Model-Free RL: Value-based and Policy-based
  - Learn value function $V(s)$ or $Q(s, a)$, or policy function $\pi(s)$ without learning MDP



Overview of RL Algorithms



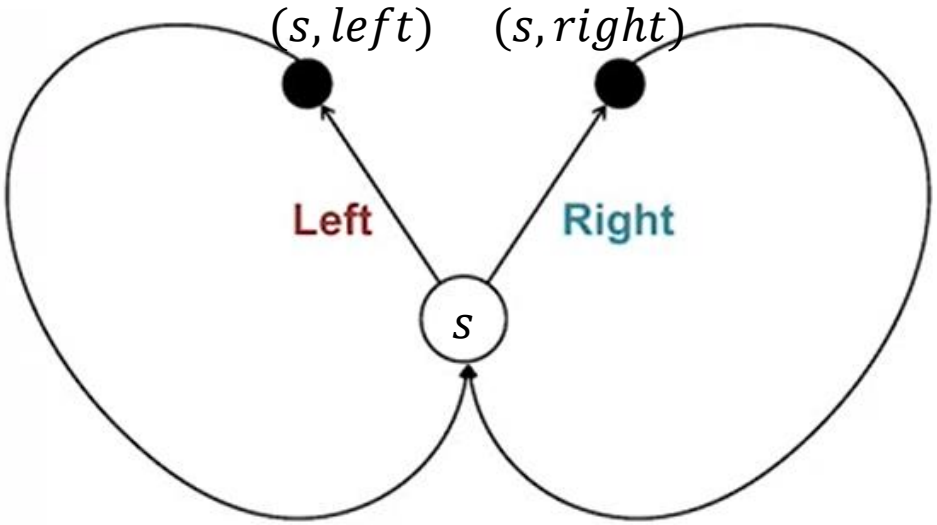


Markov Decision Process (MDP)

- An MDP consists of:
  - Set of states S
  - Start state s_0
  - Set of actions A
  - Transitions and rewards $p(r, s'|s, a)$ (w. discount γ)
- Policy maps from states to actions:
  - Deterministic policy $a = \pi(s)$ defines a deterministic action a for state s.
  - Stochastic policy $\pi(a|s)$ defines a probability distribution over possible actions a for state s.
- Markov means that next state only depends on current state
  - $P\left(S_{t+1} = s' \middle| S_t = s_t, A_t = a_t, S_{t-1} = s_{t-1}, A_{t-1} = a_{t-1,\dots}, S_0 = s_0, A_0 = a_0\right)$
  - $= P(S_{t+1} = s' | S_t = s_t, A_t = a_t)$
  - Given the present state, the future and the past are independent
  - e.g., for driving task, current vehicle position $\boldsymbol{x}$ as the state does not satisfy the Markov property, since the next state depends on not only $\boldsymbol{x}$, but also velocity $\dot{\boldsymbol{x}}$, acceleration $\ddot{\boldsymbol{x}}$, (assuming acceleration $\ddot{\boldsymbol{x}}$ stays constant within each step) If we redefine the state as vector $[\boldsymbol{x}, \dot{\boldsymbol{x}}, \ddot{\boldsymbol{x}}]^T$, then it satisfies the Markov property.
  - Or, current snapshot of front camera view can be used as the state (e.g., NVIDIA's PilotNet), but some works use past N video frames as the state to capture more dynamics (e.g., Waymo's ChauffeurNet).

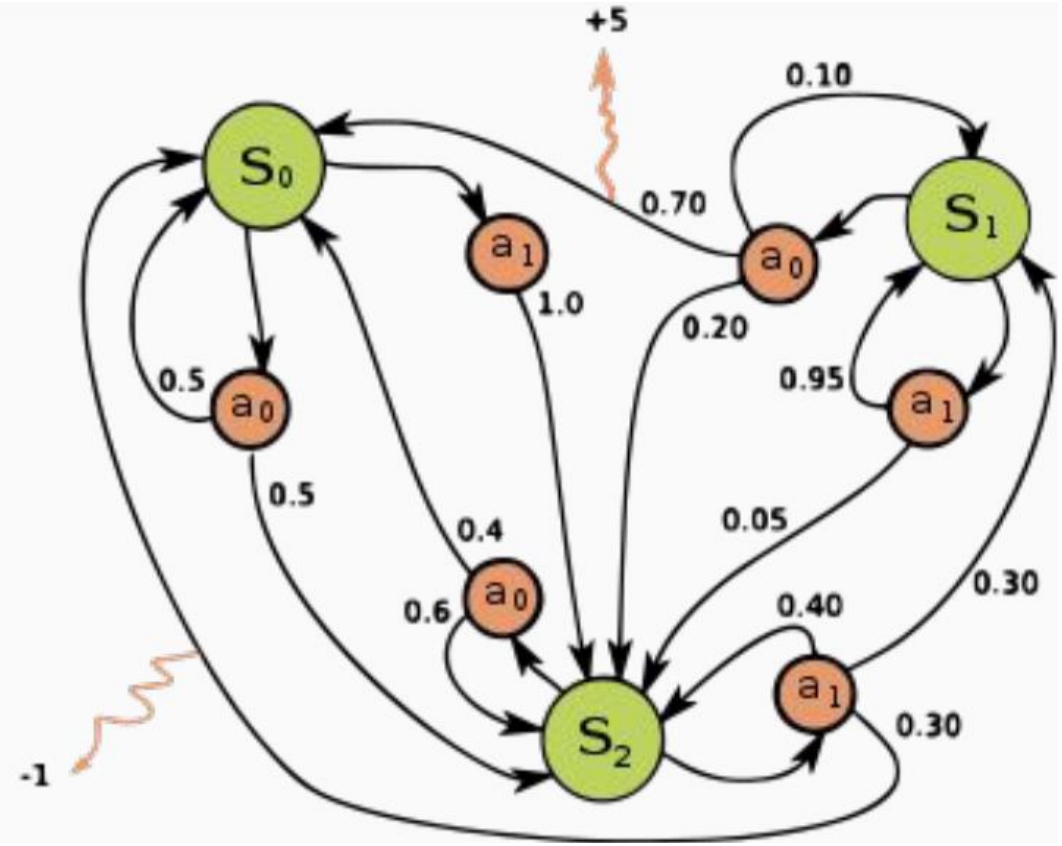
MDP Quiz

- For this MDP with a single state s and two possible actions $left$ and $right$. Are these valid policies?
  - 1) $\pi(left|s) = \pi(right|s) = 0.5$ (goes left or right with equal probability. uniform random policy)
  - 2) $\pi(left|s) = 1.0, \pi(right|s) = 0$ (always goes left)
  - 3) Alternating left and right, i.e., if previous action is left, then current action must be right, next action must be left, and so on.
  - ANS: 3) is not a valid policy, since it depends on the history of actions. To be a valid policy, the action must depend on the current state only.
- We can redefine the MDP' extended state to include the last action as part of it, then 3) is a valid policy for the new MDP.



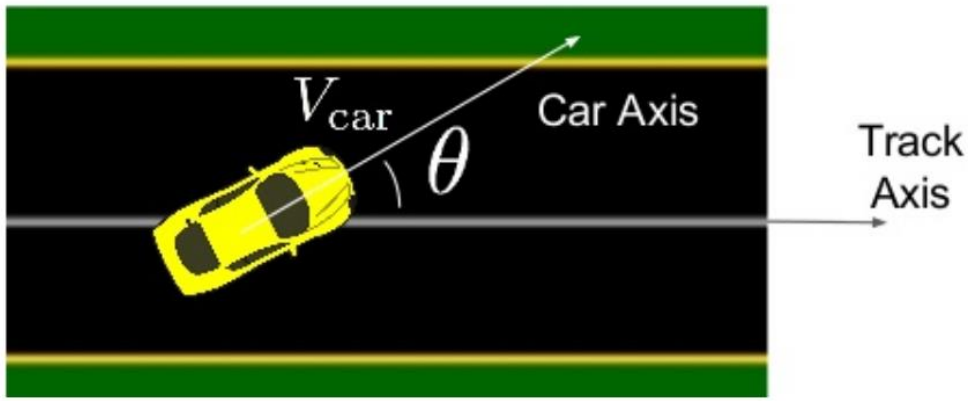
An Example MDP

- Green nodes denote 3 states s_0, s_1, s_2; Red nodes denote 2 possible actions a_0, a_1 in each state. Each red node can also be denoted as (s, a).
- Agent taking action a in state s may get different reward r and next state s', denoted as state transition (s, a, r, s'), due to environment uncertainty (all rewards are 0 expect $+5$ and -1 show in fig).
- State transition (s, a, r, s') is atomic. You can remove all red nodes and draw a single transition between green nodes.



RL Reward Function

- For the vehicle in left fig:
  - state: Pose of ego-car (x, y, θ) and environment map; action: Steering wheel/brake/acceleration
- Possible reward function: $R_t = w_1 V_{car} \cos\theta - w_2 |cte|$
  - Weight sum to maximum longitudinal velocity (first term), and minimize cross-track error (distance to lane center)
  - This is an example of dense reward (e.g., at every time step), as opposed to sparse reward (e.g., only at the end of each episode)
- Compare with twiddle() :
  - twiddle() can be viewed as an RL algorithm (policy gradient), that learns PID parameters with sparse reward (cost function is average cross-track error (cte), computed at the end of each simulation episode, as sum of squares of ctes for N timesteps divided by N. But it is very crude:
  - It does not use the numeric value of cte, only its relative size (if err < best_err);
  - Cost function does not include heading angle θ;
  - if the track is very long and irregular, then we can make the reward denser, to adjust PID parameters every K timesteps instead of at the end of each episode.



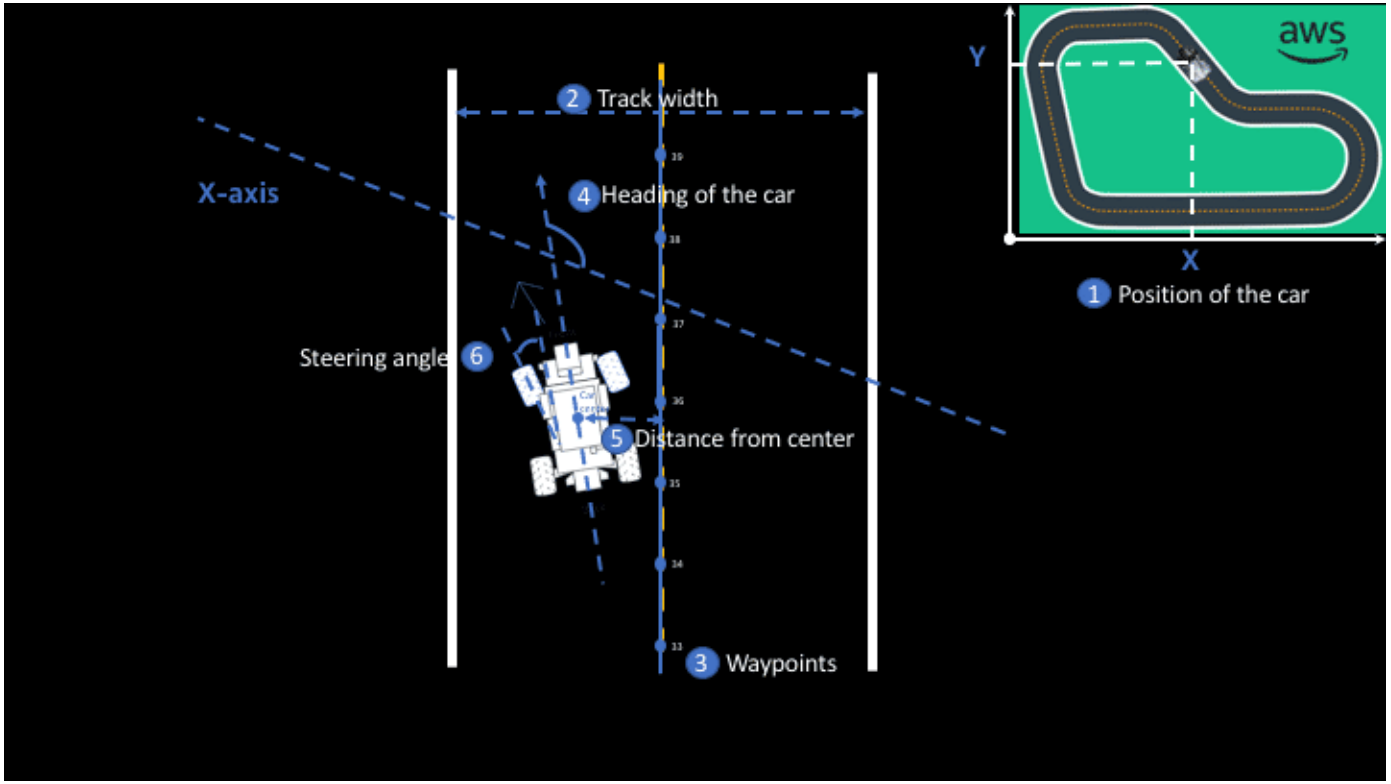

```
if err < best_err:
    best_err = err
    dp[i] *= 1.1
else:
    p[i] -= 2 * dp[i]
    robot = make_robot()
    x_trajectory, y_trajectory, err = run(robot, p)

    if err < best_err:
        best_err = err
        dp[i] *= 1.1
    else:
        p[i] += dp[i]
        dp[i] *= 0.9
```

twiddle()

Amazon DeepRacer

- Amazon Web Services (AWS) launched DeepRacer in 2018 for training AD algorithms with RL
  - https://aws.amazon.com/deepracer/
- You can train RL algorithm in the simulator on AWS cloud, but it costs money after some free time.
- They hold competitions, both online and in real-world. 1/10th scale race car costs USD $349.

Params for Writing Reward Function



- all_wheels_on_track
- x
- y
- distance_from_center
- is_left_of_center
- is_reversed
- heading
- progress
- steps
- speed
- steering_angle
- track_width
- waypoints
- closest_waypoints

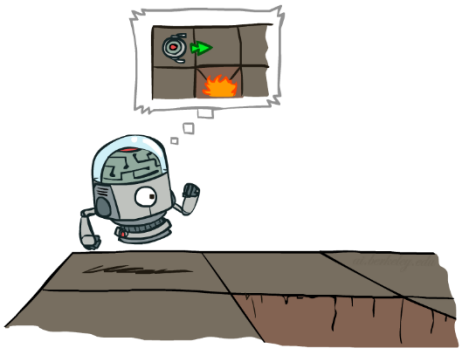
Example Reward Function

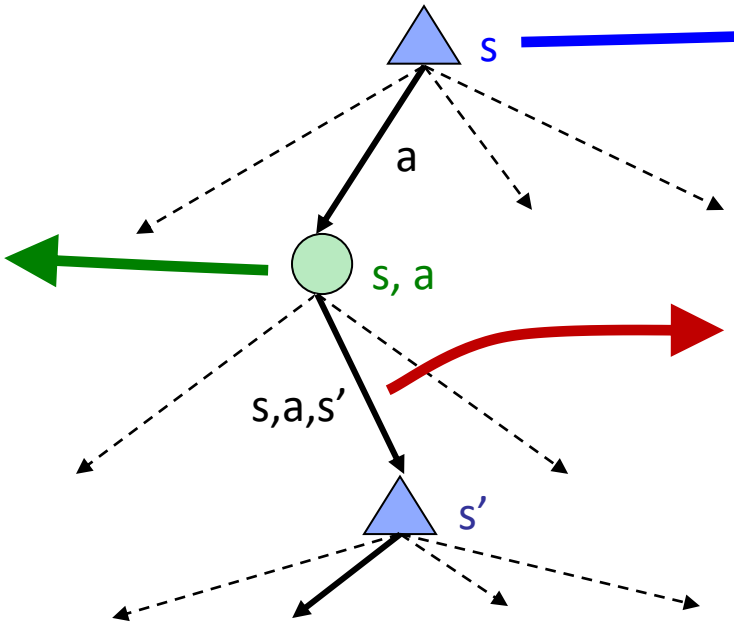
```
def reward_function(params):
    '''Example of penalize steering, which helps mitigate zig-zag behaviors'''
    # Read input parameters
    distance_from_center = params['distance_from_center']
    track_width = params['track_width']
    steering = abs(params['steering_angle']) # Only need the absolute steering angle

    # Calculate 3 markers that are at varying distances away from the center line
    marker_1 = 0.1 * track_width
    marker_2 = 0.25 * track_width
    marker_3 = 0.5 * track_width

    # Give higher reward if the agent is closer to center line and vice versa
    if distance_from_center <= marker_1:
        reward = 1
    elif distance_from_center <= marker_2:
        reward = 0.5
    elif distance_from_center <= marker_3:
        reward = 0.1
    else:
        reward = 1e-3  # likely crashed/ close to off track

    # Steering penalty threshold, change the number based on your action space setting
    ABS_STEERING_THRESHOLD = 15

    # Penalize reward if the agent is steering too much
    if steering > ABS_STEERING_THRESHOLD:
        reward *= 0.8

    return float(reward)
```

- A more realistic and complex reward function: https://www.middleware-solutions.fr/2019/08/14/an-introduction-to-aws-deepracer

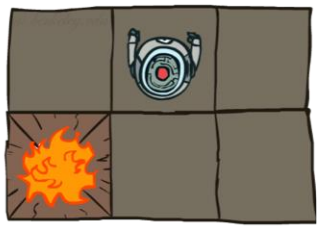
MDP Search Tree

- Each MDP state s projects a search tree starting from it.
- In general, both policy and environment may be stochastic
  - Policy $\pi(a|s)$: probability distribution over possible actions a from state s.
  - Environment $p(r, s'|s, a)$: if agent takes action a in state s, env gives probability distribution over reward r and next states s'.





s is a *state*



(s, a, s') called a transition

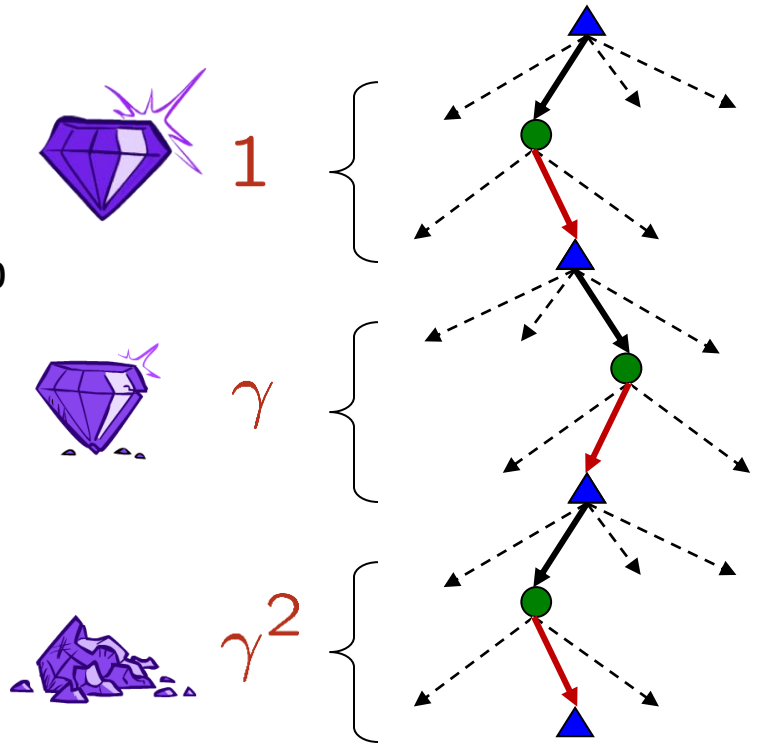
Reward $R(s, a, s')$

Preventing Infinite Rewards

- Problem: What if the game lasts forever? Do we get infinite rewards? No. Possible solutions:
- Finite horizon: (limit search tree depth)
  - Terminate episodes after a fixed T timesteps
- Discounting: use $0 \leq \gamma \leq 1$
  - $U([r_0, \ldots, r_\infty]) = \sum_{t=0}^{\infty} \gamma^t r_t \leq \sum_{t=0}^{\infty} \gamma^t r_{max} = \frac{r_{max}}{1-\gamma}$
  - Smaller γ means smaller "horizon" – shorter term focus
- Absorbing state: guarantee that for every policy, a terminal state will eventually be reached

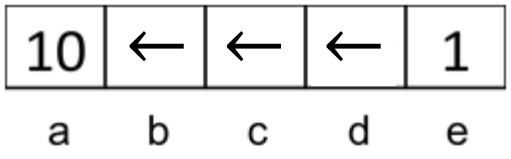
Discounting

- How to discount?
  - Each time we descend a level, we multiply in the discount once
- Why discount?
  - Think of it as a $1 - \gamma$ chance of ending the process at every step
  - Also helps our algorithms converge
- Example: $\gamma = 0.5$
  - $U([1,2,3]) = 1 * 1 + 0.5 * 2 + 0.25 * 3 < U([3,2,1])$

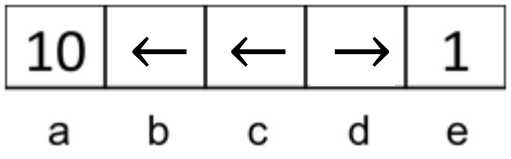


Discounting Example

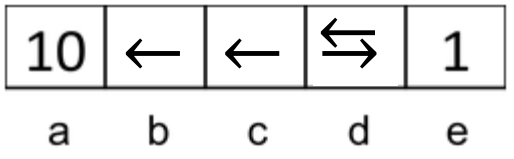
- Given:
  - Actions: East, West, and Exit (only available in exit states a, e)
  - Transitions: deterministic
- For $\gamma = 1$, optimal policy in each state is always moving West
  - From state d, reward of going West is $\gamma^3 \cdot 10 = 10$, larger than reward of going East $\gamma \cdot 1 = 1$
- For $\gamma = 0.1$, optimal policy in each state is below
  - From state d, reward from going West is $\gamma^3 \cdot 10 = 0.01$, less than reward from going East $\gamma \cdot 1 = 0.1$.
- For which γ are West and East equally good when in state d?
  - $\gamma^3 \cdot 10 = \gamma \cdot 1 \Longrightarrow \gamma = \frac{1}{\sqrt{10}} \approx .32$



$\gamma = 1$



$\gamma = 0.1$



$\gamma = \frac{1}{\sqrt{10}}$

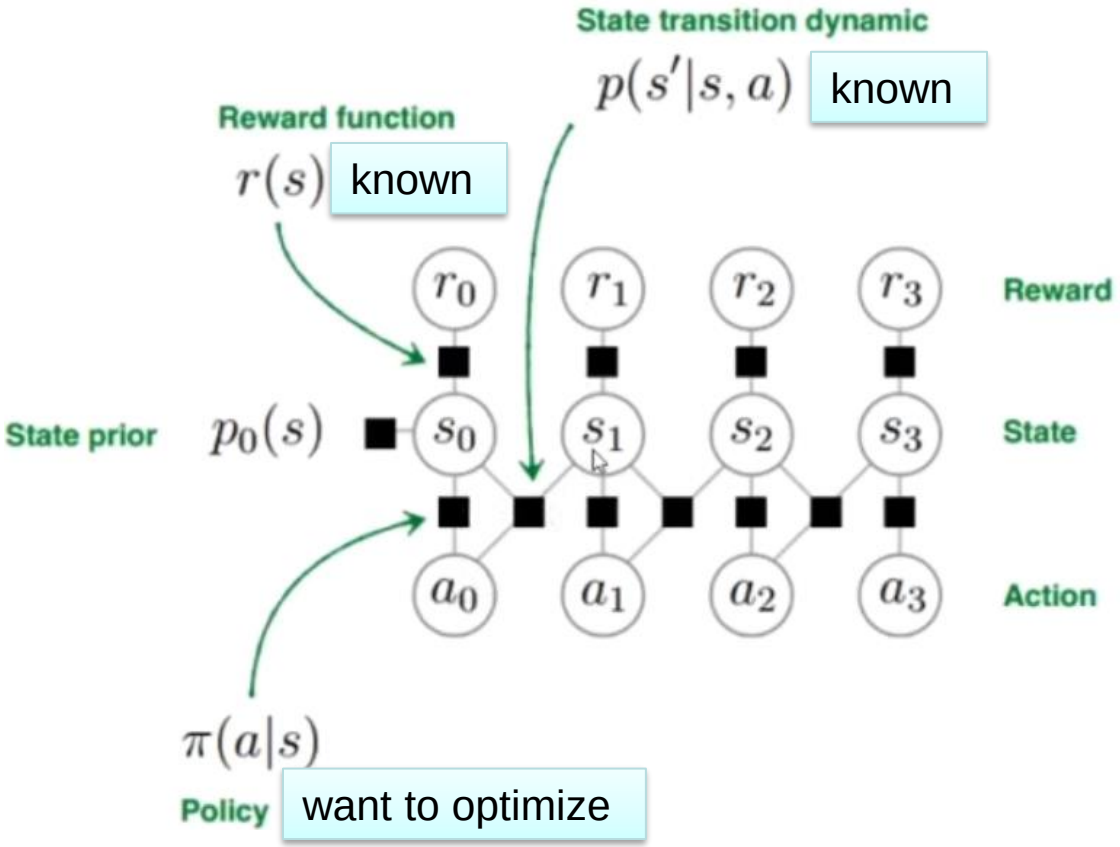
Important

The Big Picture

Problem	Bellman Equation	Algo (known MDP)	Algo (unknown MDP, sample-based)
Prediction (compute $v_\pi(s)$)	Bellman Exp. Equation for v	Policy Evaluation (PE)	MC Prediction, TD/TD(λ) (on-policy)
Control (compute $v_\pi(s)$, then $\pi(s) = \underset{a}{\text{argmax}}\, q(s,a)$) for known MDP	Bellman Exp. Equation for v + Greedy Policy Improvement (GPI)	Policy Iteration (PI=PE+GPI)	Cannot do GPI, since cannot get $Q(s,a)$ from $V(s)$ without MDP
Control (compute $v_*(s)$, then $\pi^*(s) = \underset{a}{\text{argmax}}\, q_*(s,a)$) for known MDP	Bellman Opt. Equation for v	Value Iteration (VI) (a form of Generalized PI)	Cannot compute $V^*(s)$ w. sample-based method due to $\max_a$ in front; cannot get $Q(s,a)$ from $V(s)$ without MDP
Control (compute $q_*(s,a)$, then $\pi^*(s) = \underset{a}{\text{argmax}}\, q_*(s,a)$)	Bellman Opt. Equation for q	Q Value Iteration (QVI)	MC control, Sarsa (on-policy) Q Learning, Expected Sarsa (off-policy)

Known MDP

- In this lecture, we assume known MDP, and use dynamic programming to solve Bellman Equations and find the optimal policy (no learning here).



Important

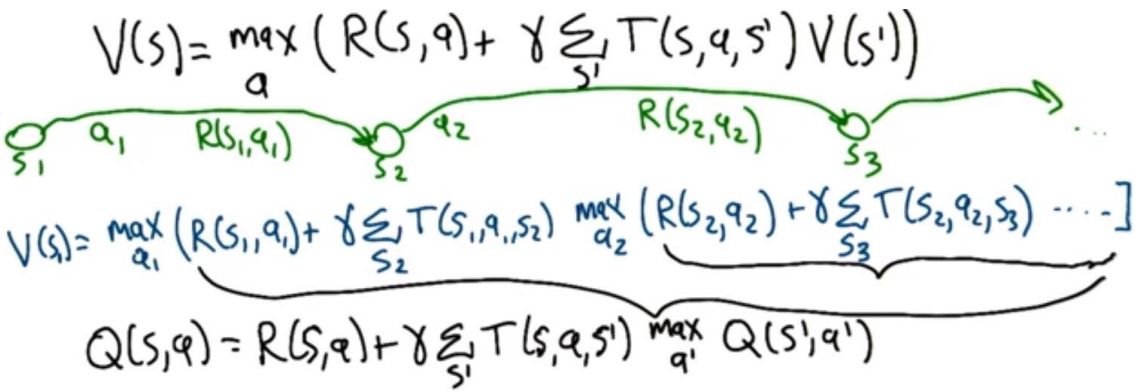
Bellman Expectation Equations

- Bellman Expectation Equation for State Value Function:
- $v_\pi(s) = \sum_a \pi(a|s) \sum_{r,s'} p(r, s'|s, a) \left[r + \gamma v_\pi(s')\right]$
  $= \mathbb{E}_\pi[R_{t+1} + \gamma v_\pi(S_{t+1})|S_t = s]$
  - Expected value starting from state s and following policy π.
- Bellman Expectation Equation for Action Value Function
- $q_\pi(s, a) = \sum_{r,s'} p(r, s'|s, a) \left[r + \gamma \sum_{a'} \pi(a'|s') \, q_\pi(s', a')\right]$
  $= \mathbb{E}_\pi[R_{t+1} + \gamma v_\pi(S_{t+1})|S_t = s, A_t = a]$
  - Expected value starting from state s, taking action a, and thereafter following policy π.
- Relating $v_\pi(s)$ and $q_\pi(s, a)$:
  - $v_\pi(s) = \sum_a \pi(a|s) \, q_\pi(s, a)$
  - $q_\pi(s, a) = \sum_{r,s'} p(r, s'|s, a) \left[r + \gamma v_\pi(s')\right]$

Important

Bellman Optimality Equations

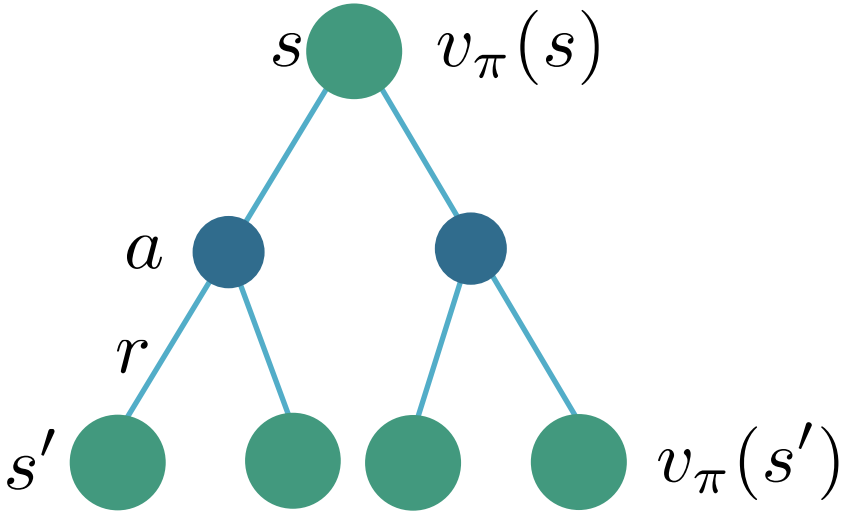
- Bellman Optimality Equation for Optimal State Value Function:
- $v_*(s) = \max_a \sum_{r,s'} p(r, s'|s, a)\,[r + \gamma v_*(s')] = \max_a \mathbb{E}_\pi[R_{t+1} + \gamma v_*(S_{t+1})|S_t = s, A_t = a]$
  - Max value starting from state s and following the greedy policy $\pi(s) = \operatorname{argmax}_a q_*(s, a)$
- Bellman Optimality Equation for Optimal Action Value Function
- $q_*(s, a) = \sum_{r,s'} p(r, s'|s, a)\left[r + \gamma \max_{a'} q_*(s', a')\right] = \mathbb{E}_\pi[R_{t+1} + \gamma v_*(S_{t+1})|S_t = s, A_t = a]$
  - Max value starting from state s, taking action a, and thereafter following the greedy policy $\pi(s) = \operatorname{argmax}_a q_*(s, a)$
- Relating $v_*(s)$ and $q_*(s, a)$:
  - $v_*(s) = \max_a q_*(s, a); q_*(s, a) = \sum_{r,s'} p(r, s'|s, a)\,[r + \gamma v_*(s')]$



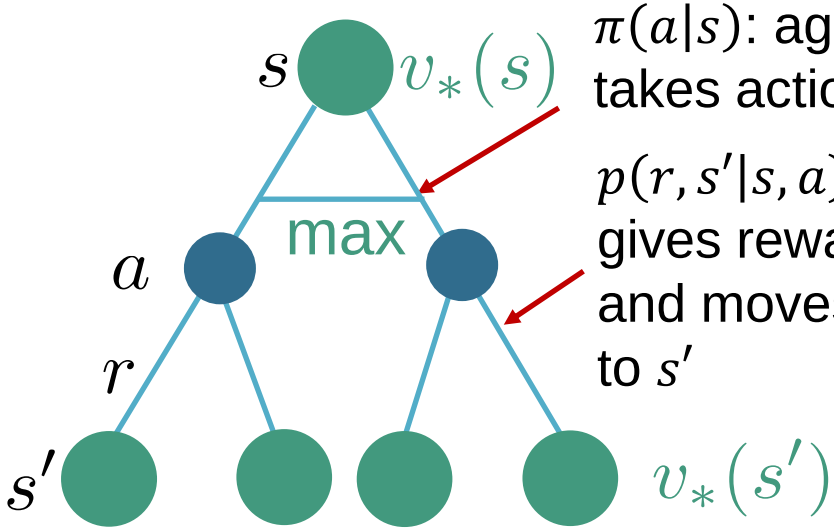
- Notations in left fig:
- $\sum_{s'} T(s, a, s')\,[\ldots] = \sum_{r,s'} p(r, s'|s, a)\,[\ldots]$
- $R(s, a) = R_t = \sum_{s'} T(s, a, s')\, r$

Important

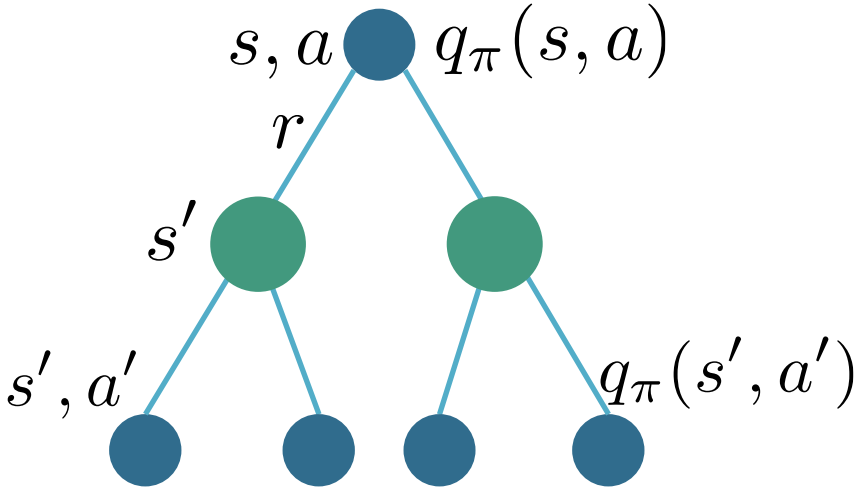
Backup Diagrams



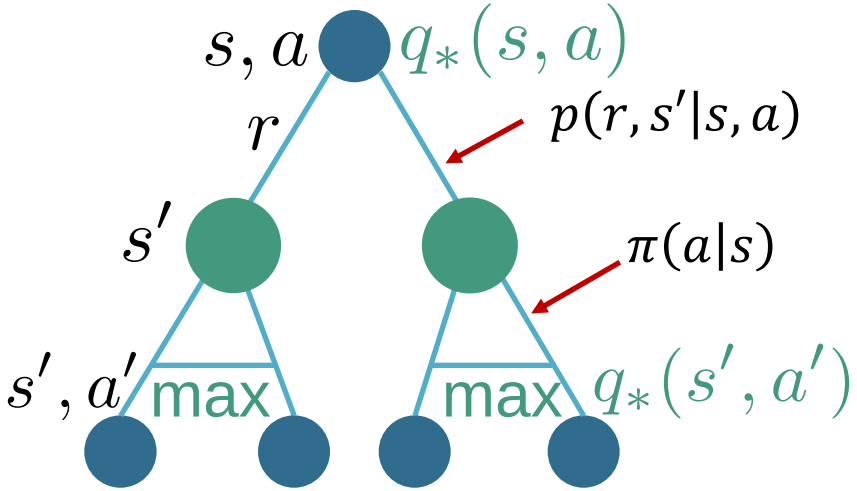
Bellman Exp Eqn for $v_\pi(s)$



Bellman Opt Eqn for $v_*(s)$



Bellman Exp Eqn for $q_\pi(s, a)$



Bellman Opt Eqn for $q_*(s, a)$

Important

$v(s)$ vs. $q(s,a)$

- $q(s,a)$ contains more information than $v(s)$. Given $q_*(s,a)$, optimal policy $\pi_*(\mathrm{s}) = \operatorname{argmax}_a q_*(s,a)$.
- Can always go from $q_\pi(s,a)$ to $v_\pi(s)$, or from $q_*(s,a)$ to $v_*(s)$:
  - $v_\pi(s) = \sum_a \pi(a|s)\, q_\pi(s,a); v_*(s) = \max_a q_*(s,a)$
- With known MDP ($p(r,s'|s,a)$, i.e., model-based): can go from $v_\pi(s)$ to $q_\pi(s,a)$, or from $v_*(s)$ to $q_*(s,a)$:
  - $q_\pi(s,a) = \sum_{r,s'} p(r,s'|s,a)\,[r + \gamma v_\pi(s')]$
  - $q_*(s,a) = \sum_{r,s'} p(r,s'|s,a)\,[r + \gamma v_*(s')]$
- With unknown MDP (unknown $p(r,s'|s,a)$, i.e., model-free) : cannot go from $v_\pi(s)$ to $q_\pi(s,a)$, or from $v_*(s)$ to $q_*(s,a)$

Policy Evaluation

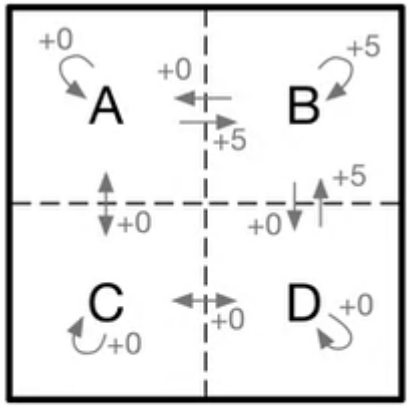
- Also called the Prediction Problem
- Predict Value Function for given policy π by solving Bellman Exp. Equation for State Value Function
  - $v_\pi(s) = \sum_a \pi(a|s) \sum_{r,s'} p(r, s'|s, a) [r + \gamma v_\pi(s')] = \mathbb{E}_\pi[R_{t+1} + \gamma v_\pi(S_{t+1})|S_t = s]$
- A set of linear equations that can be solved efficiently for small system
  - # unknowns = # equations = # states

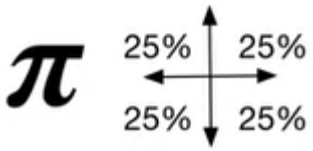
Simplfied Bellman Equations for Deterministic Env

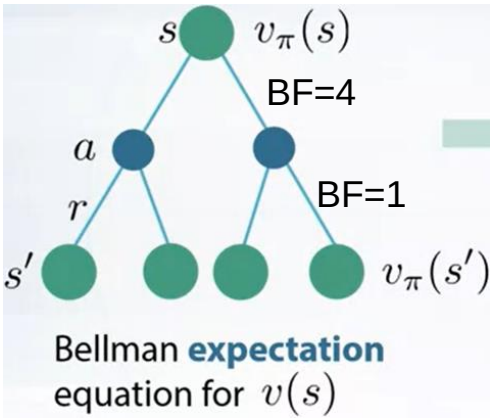
- Bellman Equations:
  - $v_\pi(s) = \sum_a \pi(a|s) q_\pi(s,a)\,; q_\pi(s,a) = \sum_{r,s'} p(r,s'|s,a)\,[r + \gamma v_\pi(s')]$
  - $v_*(s) = \max_a q_*(s,a); q_*(s,a) = \sum_{r,s'} p(r,s'|s,a)\,[r + \gamma v_*(s')]$
- For Deterministic Env: there is only one possible (r, s') for a given (s, a) (we use R_s^a to emphasize that reward r is specific to this (s, a)):
  - $q_\pi(s,a) = R_s^a + \gamma v_\pi(s')$
  - $q_*(s,a) = R_s^a + \gamma v_*(s')$

Grid World1: Policy Evaluation

- Deterministic env: Agent in state $s \in \{A, B, C, D\}$ taking action $a \in \{l, r, u, d\}$ always moves to the next state in the movement direction, unless it is blocked by the walls. Discount factor $\gamma = 0.7$.
- Random policy: Agent in state $s \in \{A, B, C, D\}$ takes a random action $a \in \{l, r, u, d\}$ with equal probability of 0.25 each.
- Bellman Exp. Equation for det env:
  - $v_\pi(s) = \sum_a \pi(a|s) q_\pi(s,a)\,;\; q_\pi(s,a) = R_s^a + \gamma v_\pi(s')$
- $v_\pi(A) = 0.25(q_\pi(A,l) + q_\pi(A,r) + q_\pi(A,u) + q_\pi(A,d)) = 0.5 \cdot 0.7 v_\pi(A) + 0.25 \cdot \big(5 + 0.7 v_\pi(B)\big) + 0.25 \cdot 0.7 v_\pi(C)$
  - $q_\pi(A,l) = q_\pi(A,u) = 0 + 0.7 v_\pi(A)$
  - $q_\pi(A,r) = 5 + 0.7 v_\pi(B)$
  - $q_\pi(A,d) = 0 + 0.7 v_\pi(C)$
- $v_\pi(B) = 0.25(q_\pi(B,l) + q_\pi(B,r) + q_\pi(B,u) + q_\pi(B,d)) = 0.25 \cdot 0.7 v_\pi(A) + 0.5 \cdot \big(5 + 0.7 v_\pi(B)\big) + 0.25 \cdot 0.7 v_\pi(D)$
  - $q_\pi(B,l) = 0 + 0.7 v_\pi(A)$
  - $q_\pi(B,r) = q_\pi(A,u) = 5 + 0.7 v_\pi(B)$
  - $q_\pi(B,d) = 0 + 0.7 v_\pi(D)$
- $v_\pi(C) = 0.25(q_\pi(C,l) + q_\pi(C,r) + q_\pi(C,u) + q_\pi(C,d)) = 0.25 \cdot 0.7 v_\pi(A) + 0.5 \cdot 0.7 v_\pi(C) + 0.25 \cdot 0.7 v_\pi(D)$
  - $q_\pi(C,l) = q_\pi(C,d) = 0 + 0.7 v_\pi(C)$
  - $q_\pi(C,r) = 0 + 0.7 v_\pi(D)$
  - $q_\pi(C,u) = 0 + 0.7 v_\pi(A)$
- $v_\pi(D) = 0.25(q_\pi(D,l) + q_\pi(D,r) + q_\pi(D,u) + q_\pi(D,d)) = 0.25 \cdot (5 + 0.7 v_\pi(B)) + 0.25 \cdot 0.7 v_\pi(C) + 0.5 \cdot 0.7 v_\pi(D)$
  - $q_\pi(D,l) = 0 + 0.7 v_\pi(C)$
  - $q_\pi(D,r) = q_\pi(D,d) = 0 + 0.7 v_\pi(D)$
  - $q_\pi(D,u) = 5 + 0.7 v_\pi(B)$
- Solution: $v_\pi(A) = 4.2, v_\pi(B) = 6.1, v_\pi(C) = 2.2, v_\pi(D) = 4.2$. $q_\pi(s,a)$ can also be obtained.







(BF: Branching Factor)

Iterative Policy Evaluation

- If # states are too large, it may be too expensive to solve the set of linear Bellman Exp. Equations. We can use Iterative Policy Evaluation and solve the recursive Bellman equations with iterative updates:
  - $v(s) = Bellman\big(v(s')\big) \Rightarrow v_k(s) \leftarrow Bellman(v_{k-1}(s'))$
- In-place updates:
  - For faster convergence, update $v_k(s)$ using the value of $v_k(s')$ that has been updated in the same iteration, instead of $v_{k-1}(s')$ from the previous iteration.
  - $v_k(s) \leftarrow Bellman(v_k(s'))$

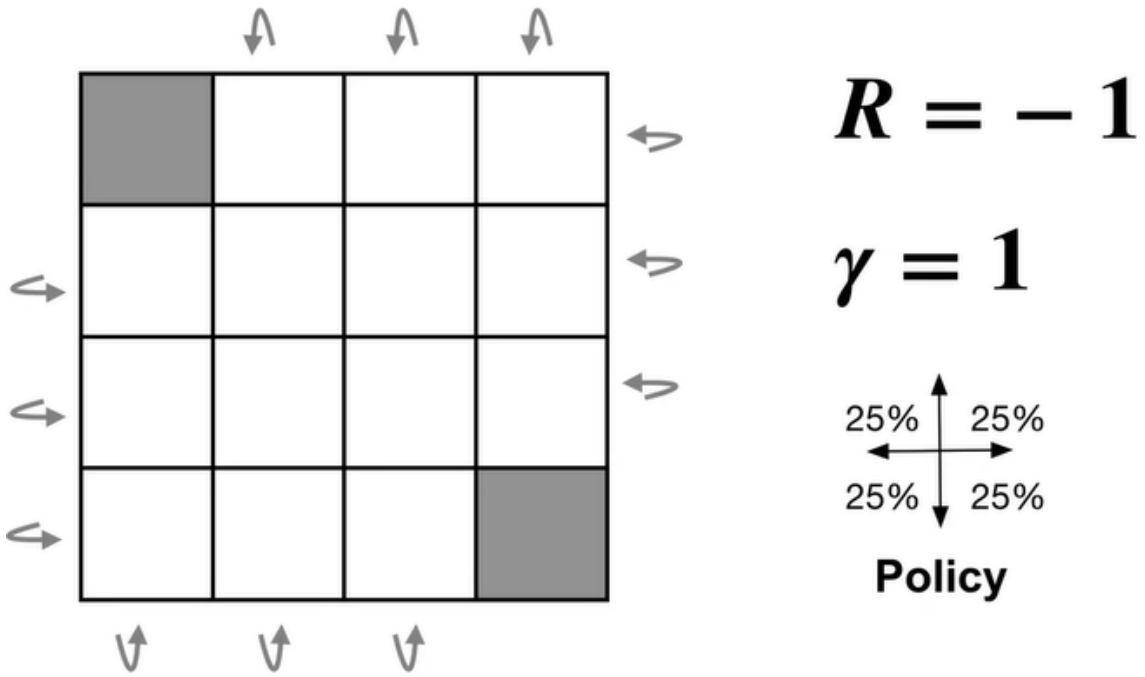
Iterative Policy Evaluation, for estimating $V \approx v_\pi$

Input π, the policy to be evaluated
Algorithm parameter: a small threshold $\theta > 0$ determining accuracy of estimation
Initialize $V(s)$, for all $s \in \mathcal{S}^+$, arbitrarily except that $V(terminal) = 0$

Loop:
  $\Delta \leftarrow 0$
  Loop for each $s \in \mathcal{S}$:
    $v \leftarrow V(s)$
    $V(s) \leftarrow \sum_a \pi(a|s) \sum_{s',r} p(s', r|s, a)\big[r + \gamma V(s')\big]$
    $\Delta \leftarrow \max(\Delta, |v - V(s)|)$
until $\Delta < \theta$

Iterative Policy Evaluation Example

- An episodic MDP with terminal states with $v(s) = 0$ located in the top left and bottom right corners. Reward $R = -1$ for every transition (agent is punished for delays before reaching the terminal state and ending the episode). Discount factor $\gamma = 1$. Four possible actions in each state: up, down, left, and right. Environment is deterministic. If the action would move the agent off the grid, it instead leaves the agent in the same state.
- Random policy: Agent in any state s takes a random action $a \in \{l, r, u, d\}$ with equal probability of 0.25 each.



Iterative Policy Evaluation Example

- For fixed random policy:
- Bellman Exp. Equation: $v_\pi(s) = \sum_a \pi(a|s) q_\pi(s,a)\,;\; q_\pi(s,a) = R_s^a + \gamma v_\pi(s')$
- 1st sweep:
  - for all states $v_\pi(s) = 0.25((-1+0)+(-1+0)+(-1+0)+(-1+0)) = -1$
- 2nd sweep:
  - For states marked -2: $v_\pi(s) = 0.25((-1-1)+(-1-1)+(-1-1)+(-1-1)) = -2$
  - For states marked -1.7: $v_\pi(s) = 0.25((-1+0)+(-1-1)+(-1-1)+(-1-1)) = -1.75$
- 3rd sweep:
  - For states marked -2.4: $v_\pi(s) = 0.25((-1+0)+(-1-2)+(-1-1.75)+(-1-2)) = -2.43$
  - For states marked -2.9: $v_\pi(s) = 0.25((-1-1.7)+(-1-2)+(-1-1.75)+(-1-2)) = -2.85$
  - For states marked -3: $v_\pi(s) = 0.25((-1-2)+(-1-2)+(-1-2)+(-1-2)) = -3$
- Influence of terminal states gradually spreads through the entire grid.

0	0	0	0
0	0	0	0
0	0	0	0
0	0	0	0

Initial

0	−1	−1	−1
−1	−1	−1	−1
−1	−1	−1	−1
−1	−1	−1	0

After 1st sweep

0	−1.7	−2	−2
−1.7	−2	−2	−2
−2	−2	−2	−1.7
−2	−2	−1.7	0

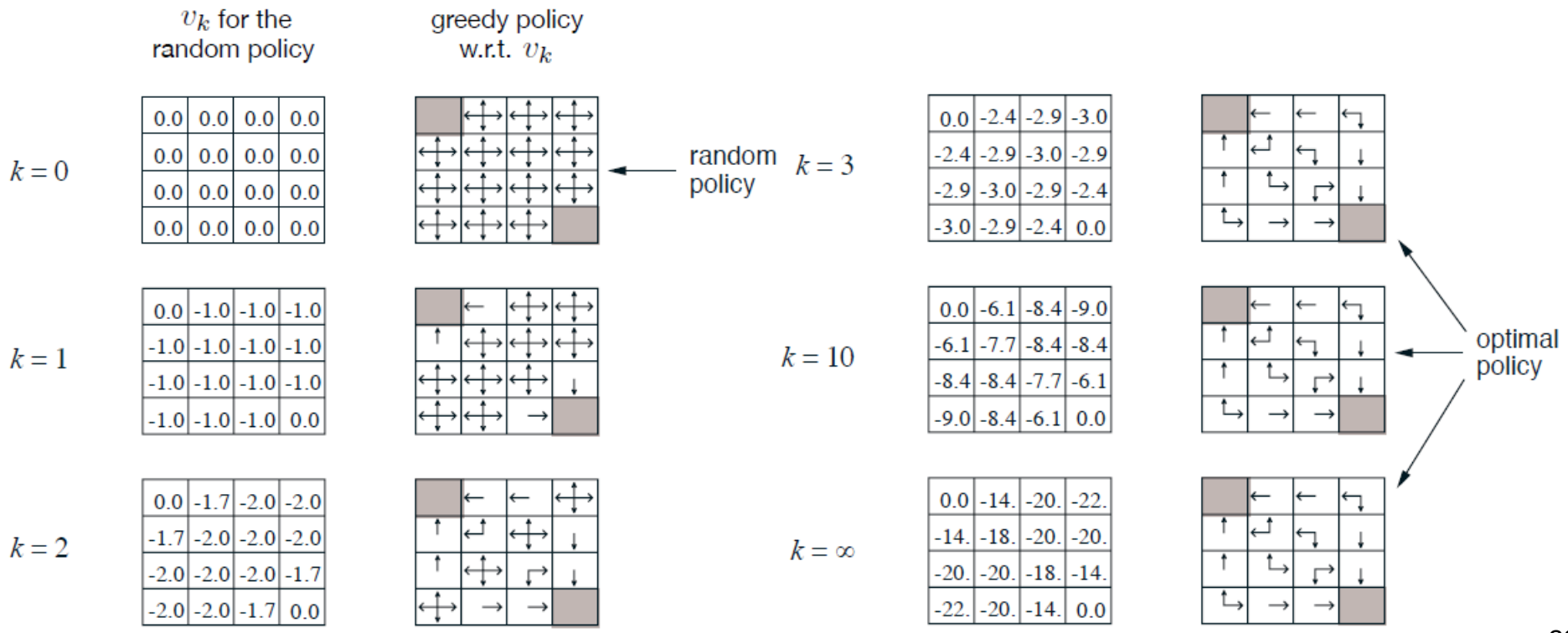
After 2nd sweep

0	−2.4	−2.9	−3
−2.4	−2.9	−3	−2.9
−2.9	−3	−2.9	−2.4
−3	−2.9	−2.4	0

After 3rd sweep

Iterative Policy Evaluation Results

- Figure 4.1: Convergence of iterative policy evaluation on a small gridworld with the random policy (all actions equally likely). The left column is the sequence of approximations of the state-value function. The right column is the sequence of greedy policies corresponding to the value function $v_\pi(s)$ estimates (arrows are shown for all actions achieving the max $v_\pi(s)$). All policies after $k = 3$ iterations are optimal.
- Note that we are not updating the policy (always the random policy) across iterations. If you follow the greedy action at the current step, then follow the random policy in the future, then it is better than following the random policy from the current step.



Important

Policy Iteration

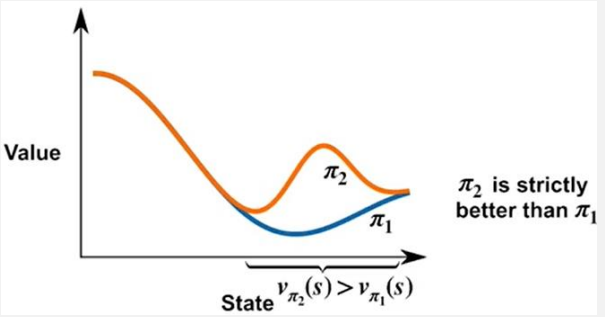
Policy Iteration (using iterative policy evaluation) for estimating $\pi \approx \pi_*$

1. Initialization
   $V(s) \in \mathbb{R}$ and $\pi(s) \in \mathcal{A}(s)$ arbitrarily for all $s \in \mathcal{S}$

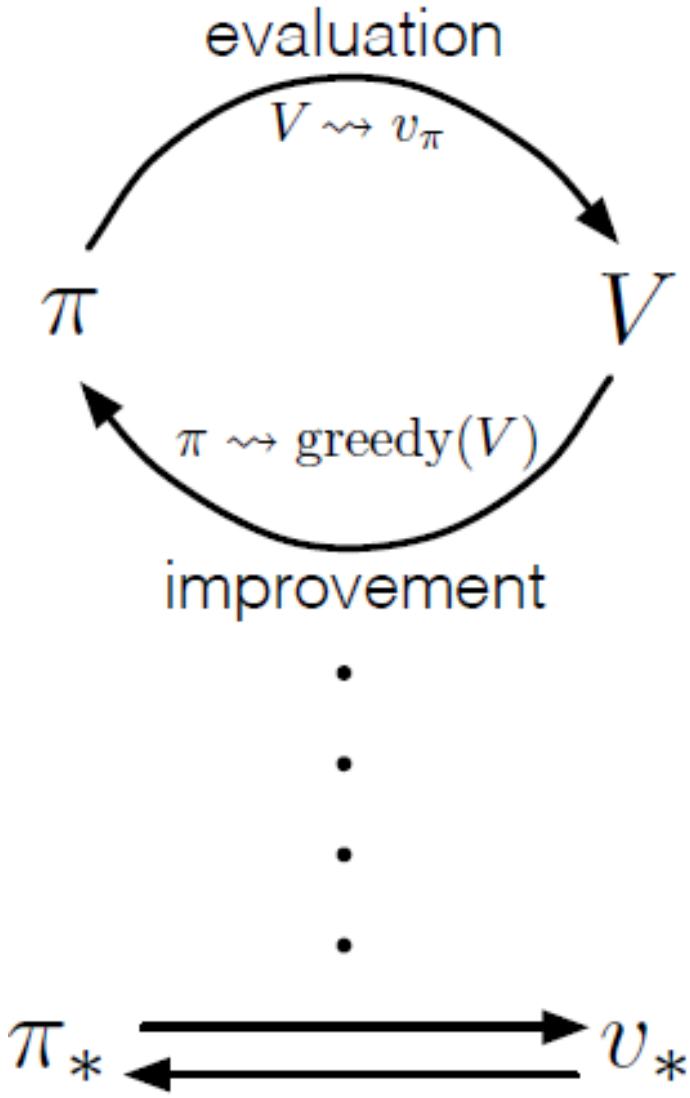
2. Policy Evaluation
   Loop:
   $\Delta \leftarrow 0$
   Loop for each $s \in \mathcal{S}$:
   $v \leftarrow V(s)$
   $V(s) \leftarrow \sum_{s',r} p(s', r \mid s, \pi(s)) \left[r + \gamma V(s') \right]$
   $\Delta \leftarrow \max(\Delta, |v - V(s)|)$
   until $\Delta < \theta$ (a small positive number determining the accuracy of estimation)

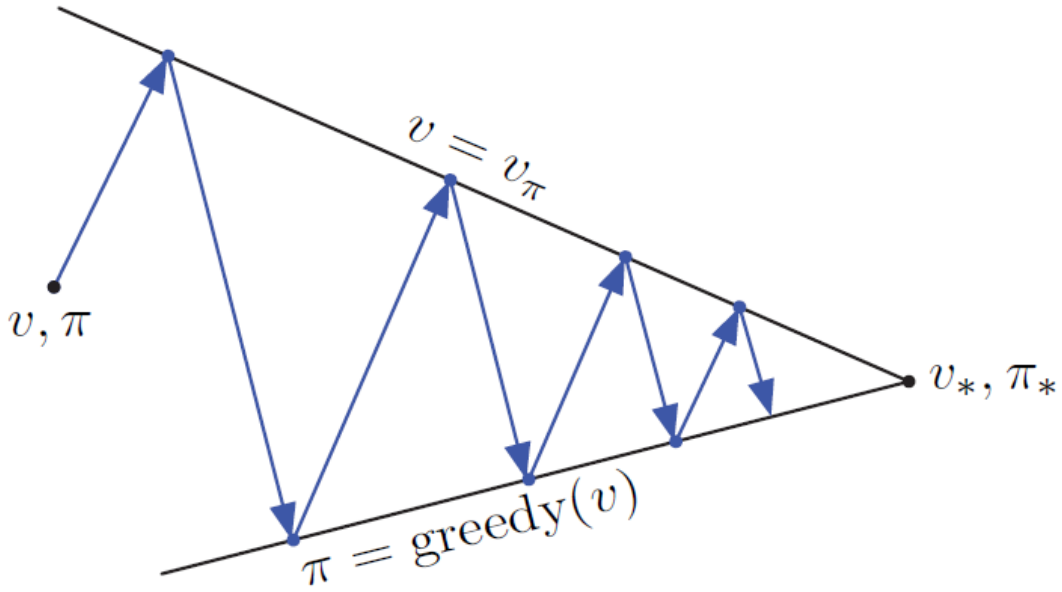
3. Policy Improvement
   policy-stable $\leftarrow$ *true*
   For each $s \in \mathcal{S}$:
   old-action $\leftarrow \pi(s)$
   $\pi(s) \leftarrow \arg\max_a \sum_{s',r} p(s', r \mid s, a) \left[r + \gamma V(s') \right]$
   If *old-action* $\neq \pi(s)$, then *policy-stable* $\leftarrow$ *false*
   If *policy-stable*, then stop and return $V \approx v_*$ and $\pi \approx \pi_*$; else go to 2

Repeat until policy converges:
Policy Evaluation: Estimate state value function v_π for some fixed policy π with Iterative Policy Evaluation (or solving linear equations).
Policy Improvement: generate new policy based on the newly estimated v_π: $\pi = \text{greedy}(v_\pi)$.



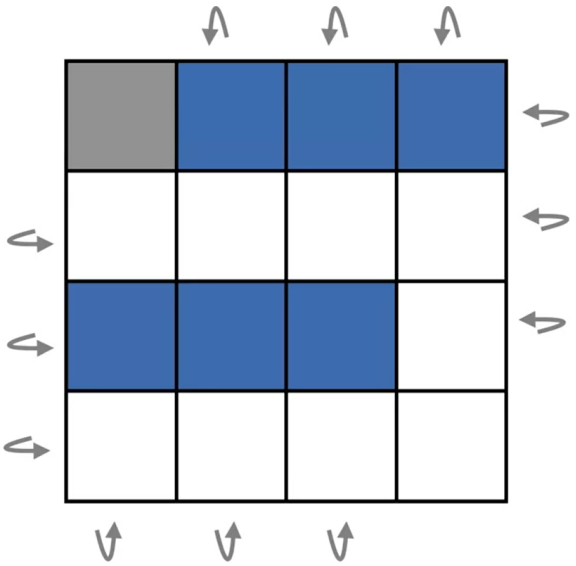
Policy Iteration Illustration





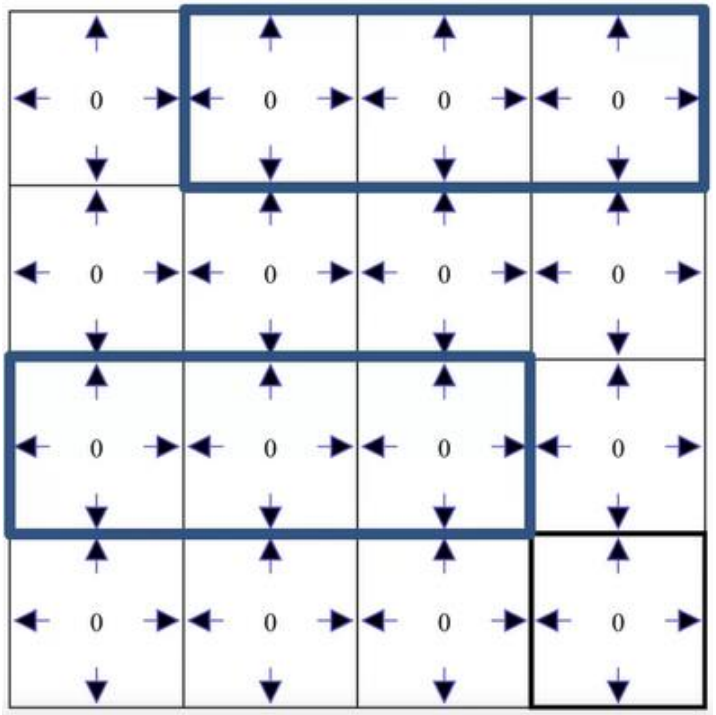
Policy Iteration Example

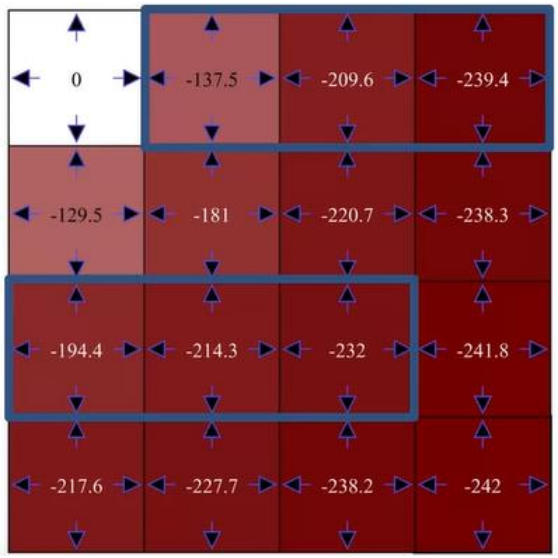
- An episodic MDP with terminal states with $v(s) = 0$ located in the top left corner. Blue states are bad states with large negative reward.
- Start with uniform random policy.
- Initialize $v(s) = 0$ for all s.



$$R = \begin{cases} -10 & \text{Blue states} \\ -1 & \text{Other states} \end{cases}$$

$$\gamma = 1$$

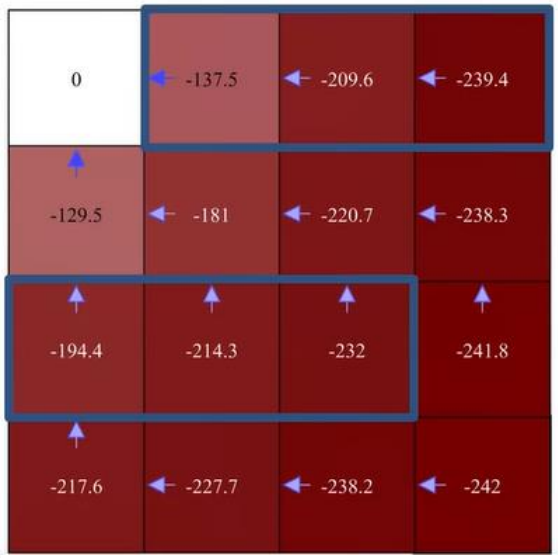




Policy Iteration

Evaluation

Improvement

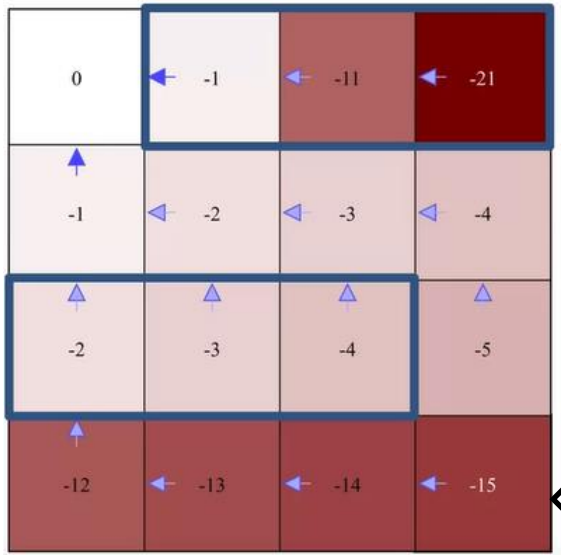


Policy Iteration

Evaluation

Improvement

After 1st sweep of PE+PI. Most state's policies are not good since they go towards blue states (darker colors denote low-value states).

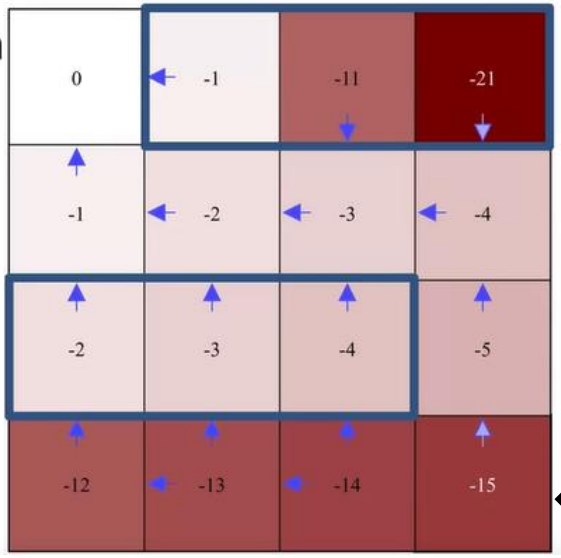


Policy Iteration

Evaluation

Improvement

Bad policy



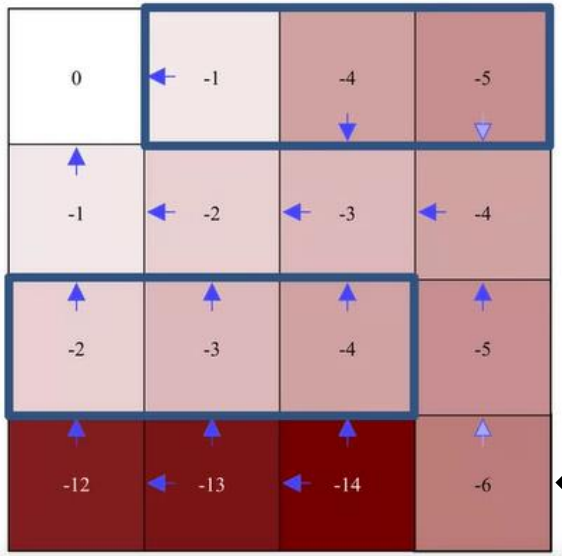
Policy Iteration

Evaluation

Improvement

Better policy

After 2nd sweep of PE+PI. Policies are much better, but a few states' policies still go towards blue states.

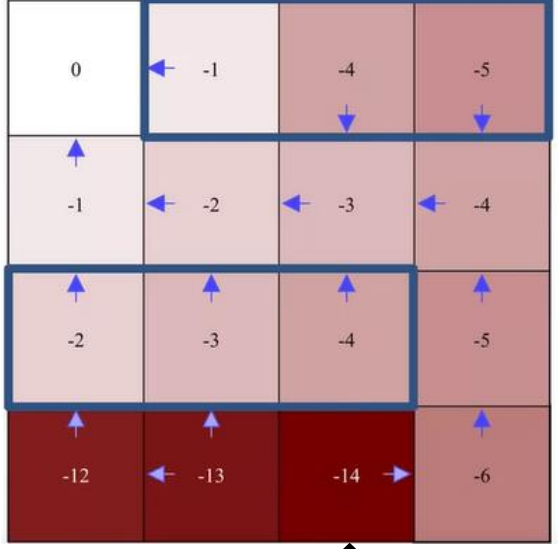


Policy Iteration

Evaluation

Improvement

Higher value

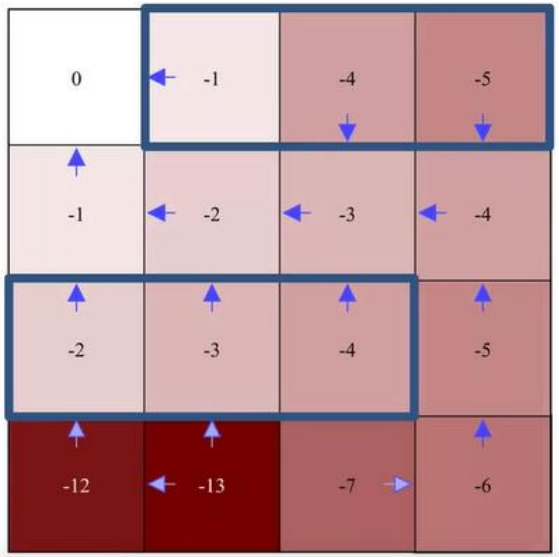


Policy Iteration

Evaluation

Improvement

Better policy

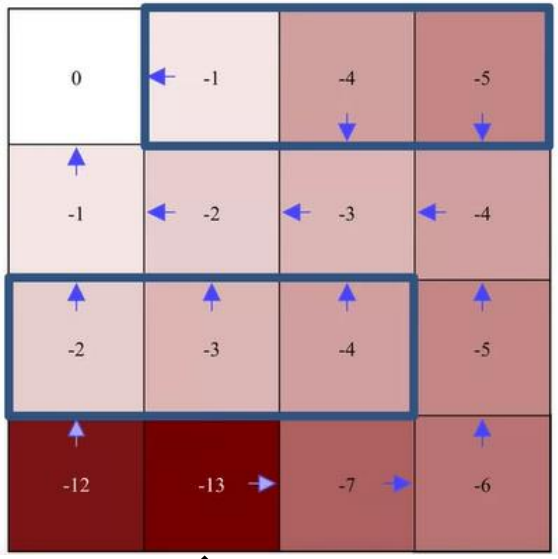


Policy Iteration

Evaluation

Improvement

Higher value

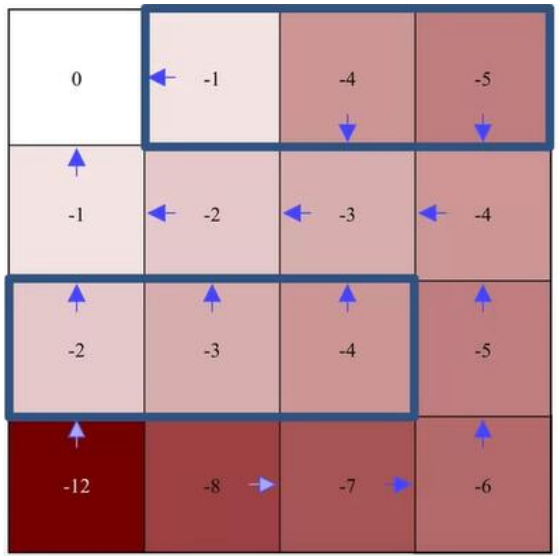


Policy Iteration

Evaluation

Improvement

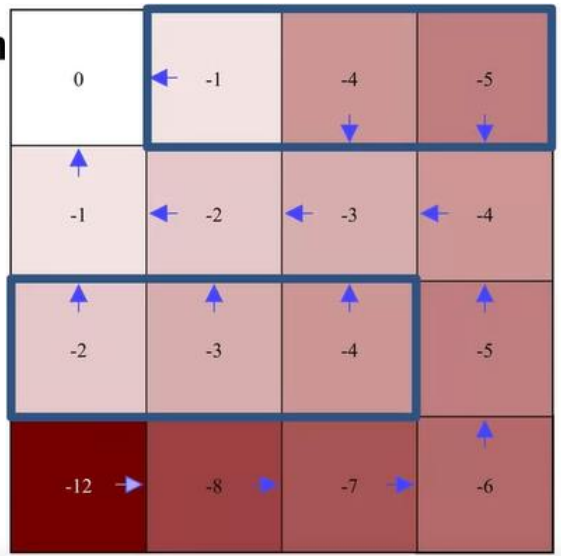
Better policy



Policy Iteration

Evaluation

Improvement



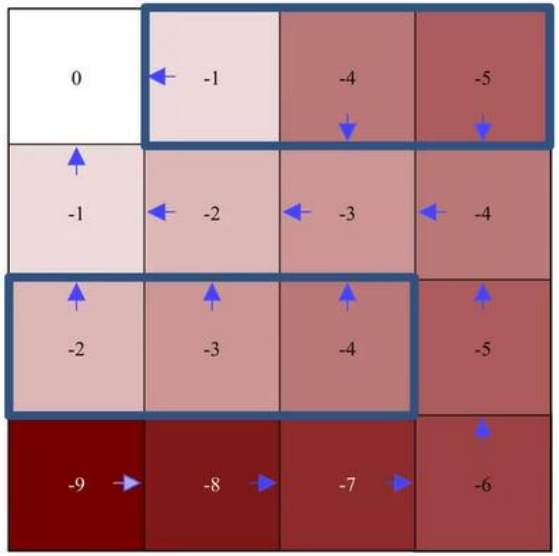
Policy Iteration

Evaluation

Improvement

Higher value

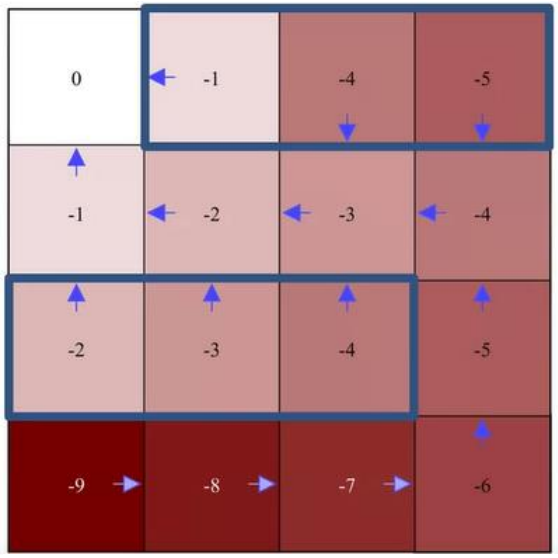
Better policy



Policy Iteration

Evaluation

Improvement



Policy Iteration

Evaluation

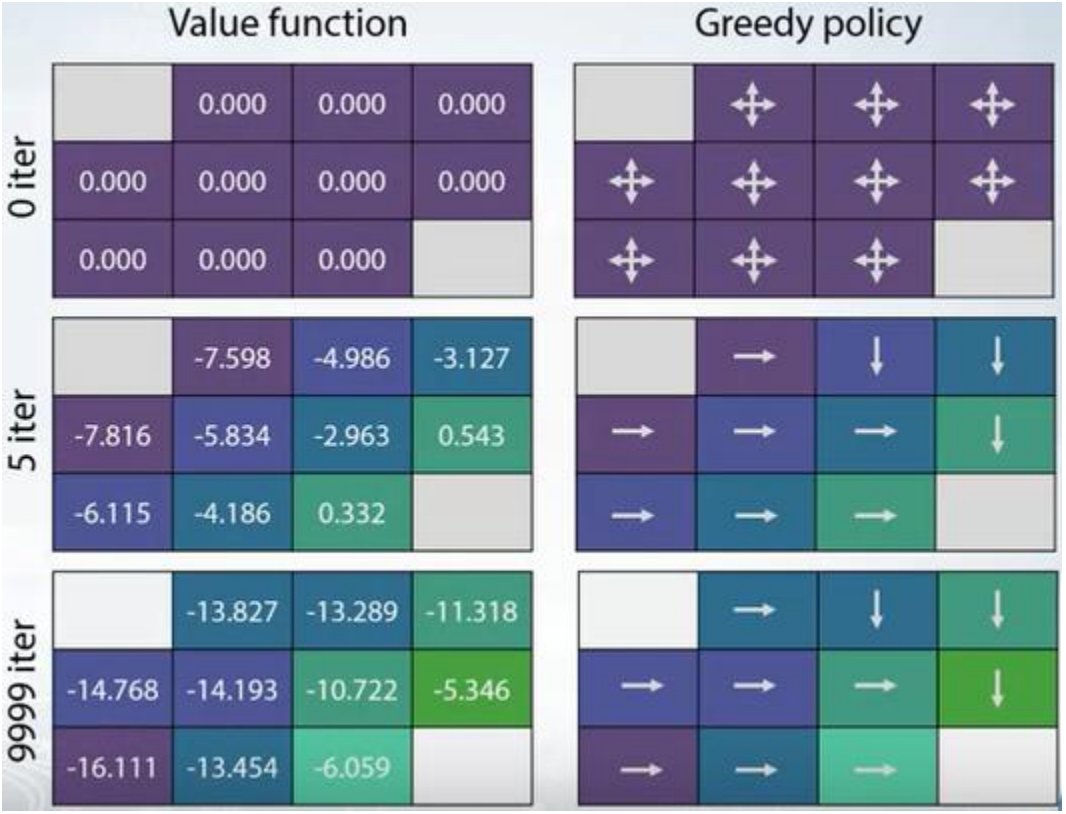
Improvement

Value functions converged
(May not converge in general. Policy often stable long before value functions converge.)

Policy stable, hence it is optimal policy)

Another PI Example

- The greedy policy converges to the optimal policy after 5th iteration of Policy Iteration.
- Precise values of $v(s)$ are not necessary for computing optimal policy with Policy Iteration.



Important

Value Iteration

- Obtain optimal value function by solving Bellman Opt. Equation for State Value Function
  - $v_*(s) = \max_a \sum_{r,s'} p(r, s'|s, a)\,[r + \gamma v_*(s')]$
- Optimal policy $\pi_*(s)$ is output at the end; intermediate value function $v(s)$ may not correspond to any valid policy.

Value Iteration, for estimating $\pi \approx \pi_*$

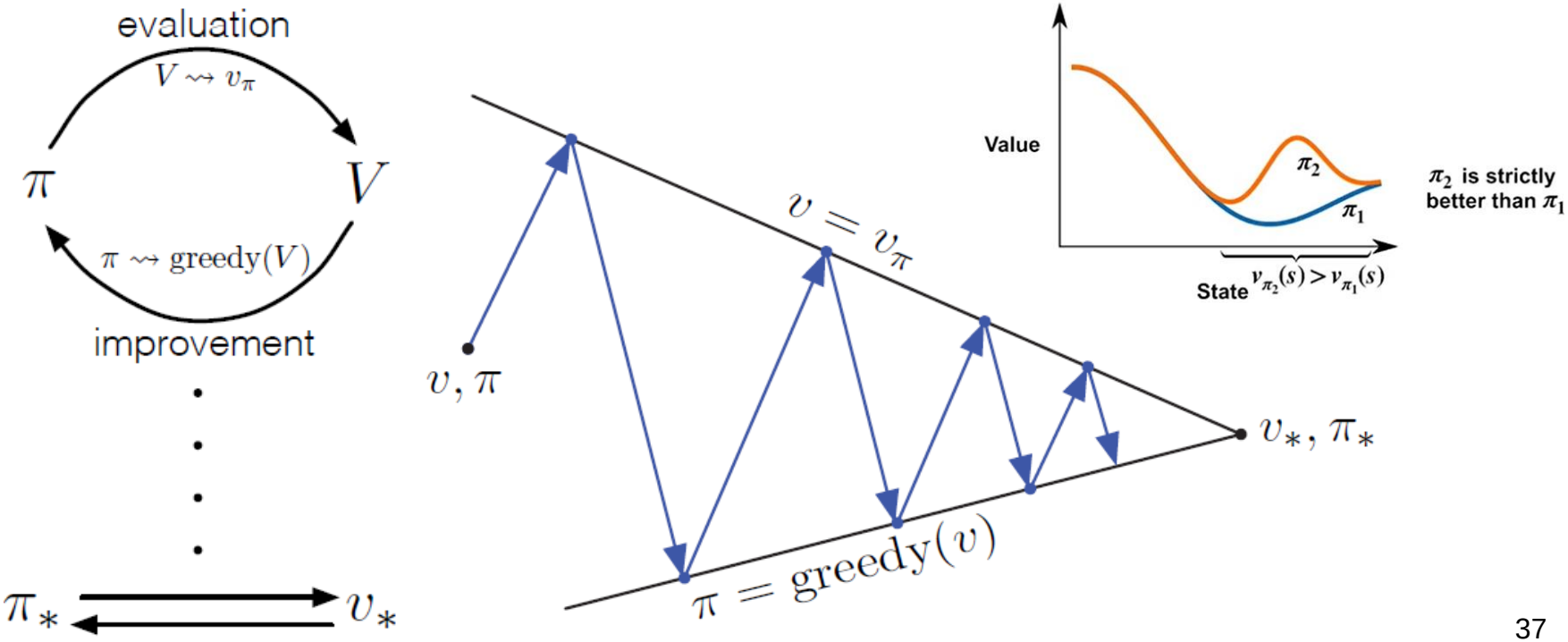
Algorithm parameter: a small threshold $\theta > 0$ determining accuracy of estimation
Initialize $V(s)$, for all $s \in \mathcal{S}^+$, arbitrarily except that $V(terminal) = 0$

Loop:
| $\Delta \leftarrow 0$
| Loop for each $s \in \mathcal{S}$:
| $v \leftarrow V(s)$
| $V(s) \leftarrow \max_a \sum_{s',r} p(s', r|s, a)\left[r + \gamma V(s')\right]$
| $\Delta \leftarrow \max(\Delta, |v - V(s)|)$
until $\Delta < \theta$

Output a deterministic policy, $\pi \approx \pi_*$, such that
$\pi(s) = \text{argmax}_a \sum_{s',r} p(s', r|s, a)\left[r + \gamma V(s')\right]$

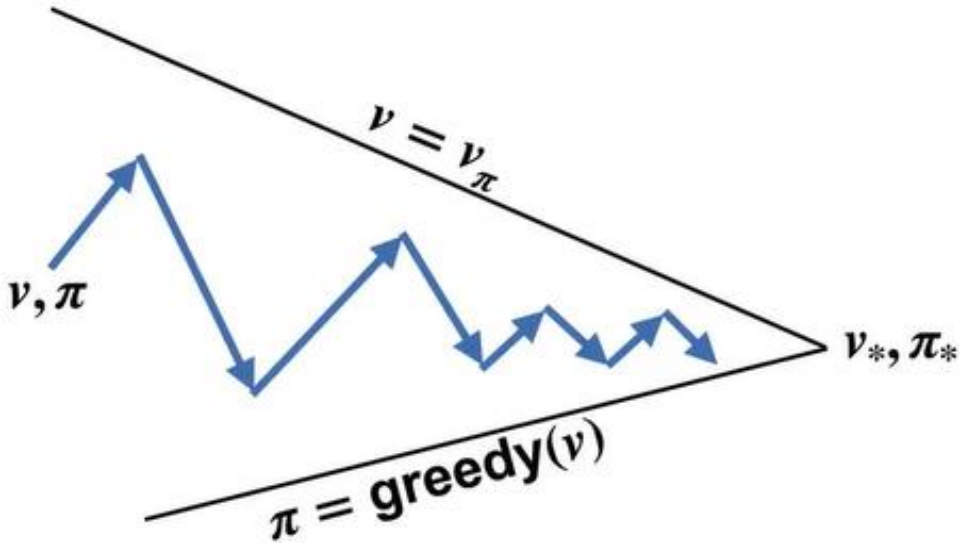
Generalized Policy Iteration (GPI)

- GPI: 1. evaluate given policy; 2. Improve policy by acting greedily w.r.t its value function. GPI is guaranteed to converge to the optimal deterministic policy π^*.
- Two special cases of GPI:
- Policy Iteration (PI) (shown below):
  - 1. Policy evaluation until convergence; 2. Improve policy
- Value Iteration (VI):
  - 1. Policy evaluation with one single iteration; 2. improve policy (implicitly)



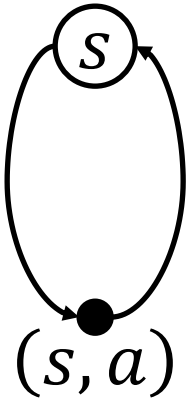
Modified Policy Iteration

- PI is slower per sweep (cycle), with complexity $O(|A||S|^2 + |S|^3)$, and requires fewer sweeps.
- VI is faster per sweep, with complexity $O(|A||S|^2)$, and requires more sweeps.
- Modified Policy Iteration: run Policy Evaluation for # steps between 1 for VI, and that needed for convergence of $v_\pi(s)$, may achieve best efficiency.



Toy Example

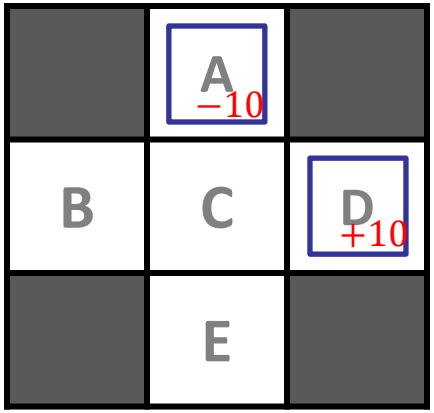
- An MDP with a single state s and single action a, with reward $r(s,a) = 1, \gamma = 0.9$
- From definition: $v_\pi(s) = v_*(s) = 1 + \gamma + \gamma^2 + \cdots = \frac{1}{1-\gamma} = 10$
- Policy Iteration:
  - Policy Evaluation:
  - Start with $\pi(s) = a$, $v_\pi(s) = \sum_a \pi(a|s)(1 + \gamma v_\pi(s)) = 1 + \gamma v_\pi(s) \Rightarrow v_\pi(s) = \frac{1}{1-\gamma} = 10$
  - $q_\pi(s,a) = \sum_{r,s'} p(r,s'|s,a)\,[r + \gamma v_\pi(s')] = R_s^a + \gamma v_\pi(s') = 1 + 0.9 \cdot 10 = 10$
  - Policy Improvement: policy $\pi(s) = a$ is now stable (there is only one possible action a).
- Value Iteration:
  - $v_*(s) = \max_a \left(1 + \gamma v_*(s)\right) = 1 + \gamma v_*(s) \Rightarrow v_*(s) = \frac{1}{1-\gamma} = 10$
  - $q_*(s,a) = \sum_{r,s'} p(r,s'|s,a)\,[r + \gamma v_*(s')] = R_s^a + \gamma v_*(s') = 1 + 0.9 \cdot 10 = 10$
  - Value function $q_*(s,a)$ is now stable, so $\pi_*(s) = \operatorname{argmax}_a q_*(s,a) = a$, (since there is only one possible action a).



MiniGW Example

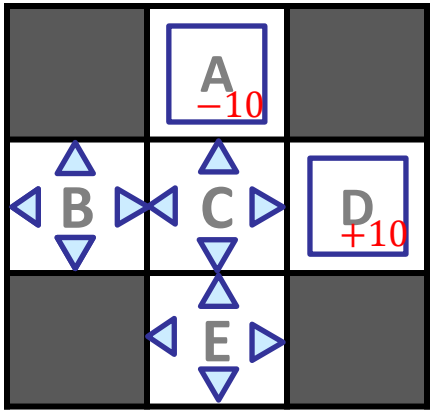
- Policy Iteration for Deterministic Environment
- Policy Iteration for Stochastic Environment
  - Policy Evaluation with analytic solution, not IPE
- Value Iteration for Deterministic Environment
- Value Iteration for Stochastic Environment
  - Bellman Opt. Equation solved iteratively w. in-place updates
- Model-based Learning

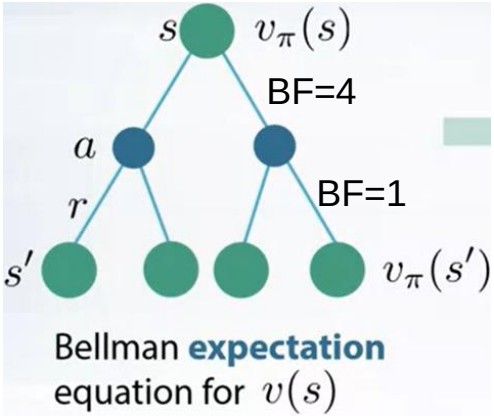
MiniGW Setup



- All transitions (s, a, r, s') for $s \in \{B, C, E\}, a \in \{l, r, u, d\}$ have the same reward $r = -1$.
- A and D are terminal states with fixed value function $v(A) = -10$, $v(D) = 10$
  - Or equivalently, you can think of an extra terminal state x with $v(x) = 0$, the only action in state A or D is $exit$ that leads to x with $R_A^a = -10, R_D^a = 10$.
- Dark squares denote obstacles which the agent cannot move into.
- Discount factor $\gamma = 1$.
- Deterministic env: Agent in state $s \in \{B, C, E\}$ taking action $a \in \{l, r, u, d\}$ always moves to the next state in the movement direction, unless it is blocked by an obstacle.
- Stochastic env: Agent in state $s \in \{B, C, E\}$ taking action $a \in \{l, r, u, d\}$ moves to the next state in the movement direction w. prob 0.8, or to the left or right side, each w. prob 0.1. If it is blocked by an obstacle in any direction, then it stays in the same state with prob of moving i the blocked direction.

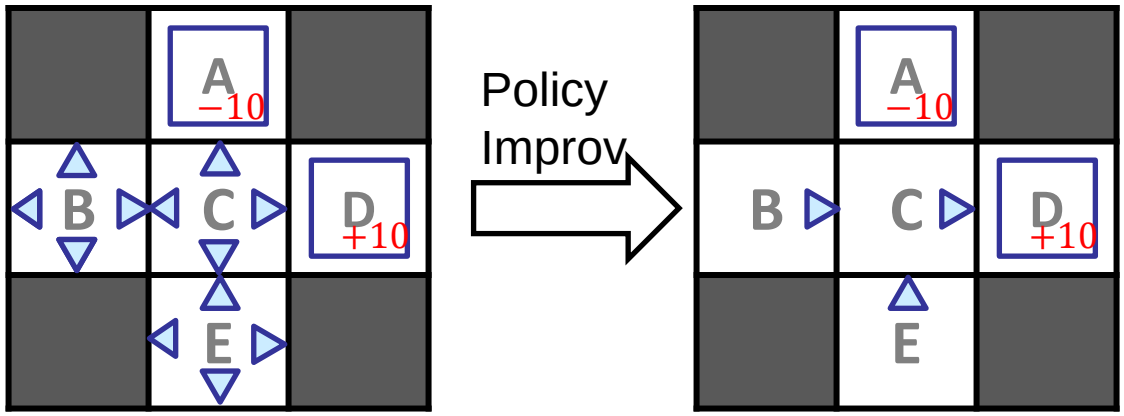
Iter1 Policy Evaluation of Random Policy

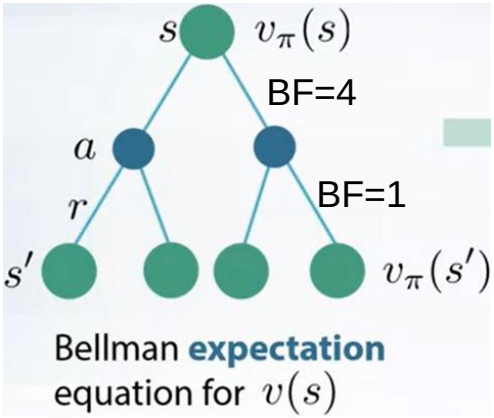




- Random policy: $\pi(l|s) = \pi(r|s) = \pi(u|s) = \pi(d|s) = 0.25$
- Policy Evaluation for Det Env: $v_\pi(s) = \sum_a \pi(a|s) q_\pi(s,a); q_\pi(s,a) = r + \gamma v_\pi(s')$
- Set of equations:
- $v_\pi(C) = 0.25\big(q_\pi(C,l) + q_\pi(C,r) + q_\pi(C,u) + q_\pi(C,d)\big) = 0.25(-4 + v_\pi(B) + v_\pi(E))$
  - $q_\pi(C,l) = -1 + 1.0 v_\pi(B)$
  - $q_\pi(C,r) = -1 + 1.0 v(D) = -1 + 10 = 9$
  - $q_\pi(C,u) = -1 + 1.0 v(A) = -1 - 10 = -11$
  - $q_\pi(C,d) = -1 + 1.0 v_\pi(E)$
- $v_\pi(B) = 0.25\big(q_\pi(B,l) + q_\pi(B,r) + q_\pi(B,u) + q_\pi(B,d)\big) = 0.25(-4 + 3.0 v_\pi(B) + v_\pi(C))$
  - $q_\pi(B,l) = q_\pi(B,u) = q_\pi(B,d) = -1 + 1.0 v_\pi(B)$
  - $q_\pi(B,r) = -1 + 1.0 v_\pi(C)$
- $v_\pi(E) = 0.25\big(q_\pi(E,l) + q_\pi(E,r) + q_\pi(E,u) + q_\pi(E,d)\big) = 0.25(-4 + 3.0 v_\pi(E) + v_\pi(C))$
  - $q_\pi(E,l) = q_\pi(E,r) = q_\pi(E,d) = -1 + 1.0 v_\pi(E)$
  - $q_\pi(E,u) = -1 + 1.0 v_\pi(C)$
- Solution: $v_\pi(C) = -6, v_\pi(B) = -10, v_\pi(E) = -10$

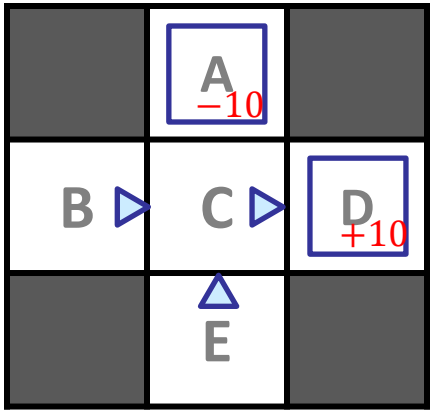
Iter1 Policy Improvement

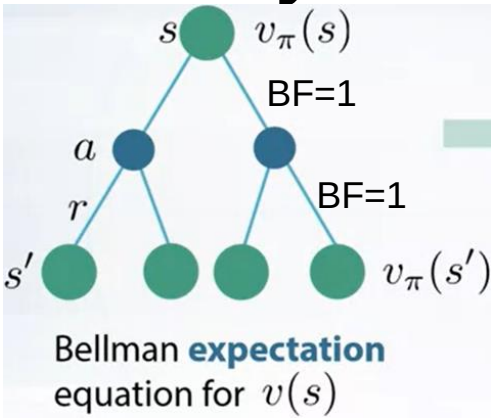




- Plug in values from PE: $v_\pi(C) = -6, v_\pi(B) = -10, v_\pi(E) = -10$, to get new policy π'
- $\pi'(C) = \underset{a \in (l,r,u,d)}{\text{argmax}}\ q_\pi(C,a) = r$
  - $q_\pi(C,l) = -1 + 1.0 v_\pi(B) = -1 - 10 = -11$
  - $q_\pi(C,r) = -1 + 1.0 v(D) = -1 + 10 = 9$
  - $q_\pi(C,u) = -1 + 1.0 v(A) = -1 - 10 = -11$
  - $q_\pi(C,d) = -1 + 1.0 v_\pi(E) = -1 - 10 = -11$
- $\pi'(B) = \underset{a \in (l,r,u,d)}{\text{argmax}}\ q_\pi(B,a) = r$
  - $q_\pi(B,l) = q_\pi(B,u) = q_\pi(B,d) = -1 + 1.0 v_\pi(B) = -1 - 10 = -11$
  - $q_\pi(B,r) = -1 + 1.0 v_\pi(C) = -1 - 6 = -7$
- $\pi'(E) = \underset{a \in (l,r,u,d)}{\text{argmax}}\ q_\pi(E,a) = u$
  - $q_\pi(E,l) = q_\pi(E,r) = q_\pi(E,d) = -1 + 1.0 v_\pi(E) = -1 - 10 = -11$
  - $q_\pi(E,u) = -1 + 1.0 v_\pi(C) = -1 - 6 = -7$

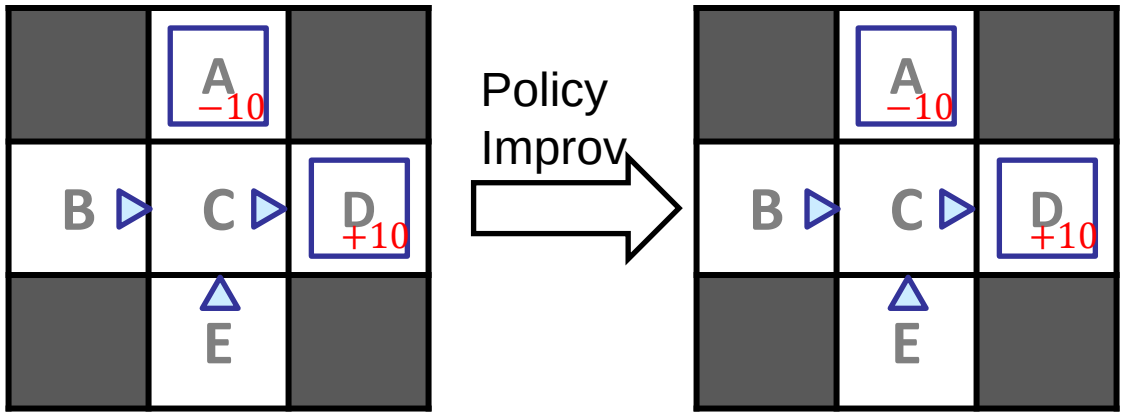
Iter2 Policy Evaluation of Det Policy

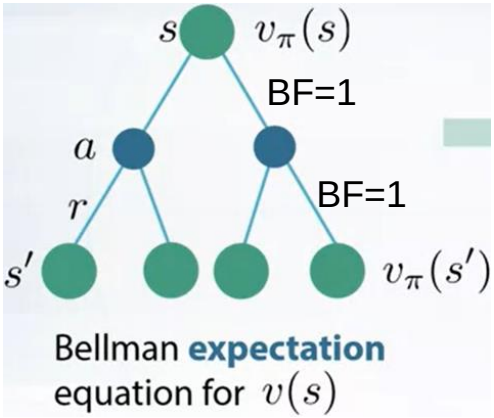




- Det policy: $\pi(r|B) = 1; \pi(r|C) = 1; \pi(u|C) = 1$
- Policy Evaluation for Det Env: $v_\pi(s) = \sum_a \pi(a|s) q_\pi(s,a); q_\pi(s,a) = r + \gamma v_\pi(s')$
- Set of equations:
- $v_\pi(C) = 1.0 q_\pi(C,r)$
  - $q_\pi(C,r) = -1 + 1.0 v(D) = -1 + 1.0 \cdot 10 = 9$
- $v_\pi(B) = 1.0 q_\pi(B,r)$
  - $q_\pi(B,r) = -1 + 1.0 v_\pi(C)$
- $v_\pi(E) = 1.0 q_\pi(E,u)$
  - $q_\pi(E,u) = -1 + 1.0 v_\pi(C)$
- Solution:
  - $v_\pi(C) = 9; v_\pi(B) = 8; v_\pi(E) = 8$

Iter2 Policy Improvement





- Plug in values from PE: $v_\pi(C) = 9, v_\pi(B) = 8, v_\pi(E) = 8$, to get new policy π'
- $\pi'(C) = \underset{a\in(l,r,u,d)}{\text{argmax}}\ q_\pi(C,a) = r$
  - $q_\pi(C,l) = -1 + 1.0v_\pi(B) = -1 + 8 = 7$
  - $q_\pi(C,r) = -1 + 1.0v(D) = -1 + 10 = 9$
  - $q_\pi(C,u) = -1 + 1.0v(A) = -1 - 10 = -11$
  - $q_\pi(C,d) = -1 + 1.0v_\pi(E) = -1 + 8 = 7$
- $\pi'(B) = \underset{a\in(l,r,u,d)}{\text{argmax}}\ q_\pi(B,a) = r$
  - $q_\pi(B,l) = q_\pi(B,u) = q_\pi(B,d) = -1 + 1.0v_\pi(B) = 7$
  - $q_\pi(B,r) = -1 + 1.0v_\pi(C) = -1 + 9 = 8$
- $\pi'(E) = \underset{a\in(l,r,u,d)}{\text{argmax}}\ q_\pi(E,a) = u$
  - $q_\pi(E,l) = q_\pi(E,r) = q_\pi(E,d) = -1 + 1.0v_\pi(E) = 7$
  - $q_\pi(E,u) = -1 + 1.0v_\pi(C) = -1 + 9 = 8$
- New policy is now stable ($\pi' = \pi$), so we have found the optimal policy.

MiniGW Example

- Policy Iteration for Deterministic Environment
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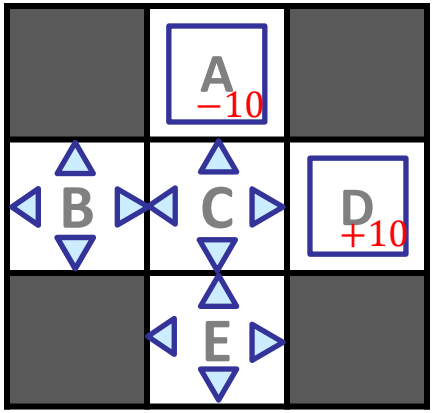
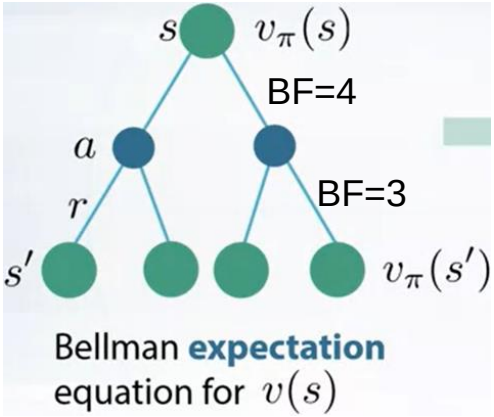
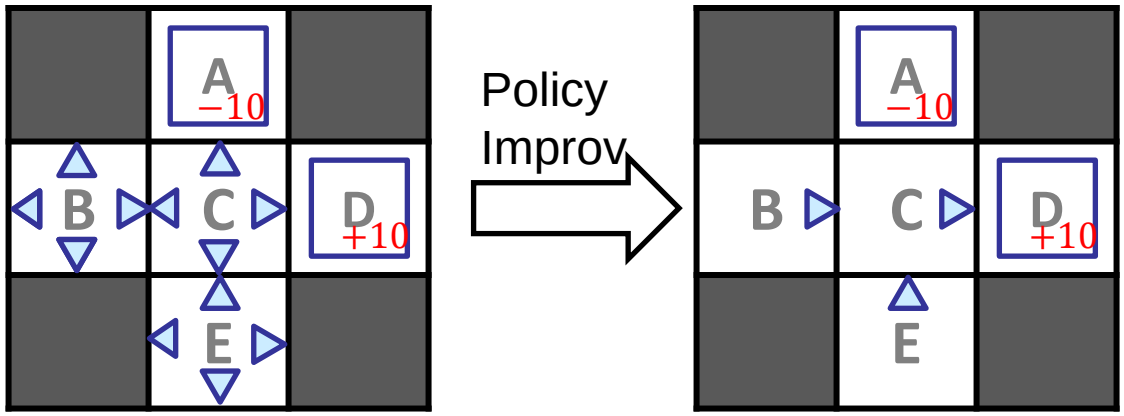
Iter1 Policy Evaluation of Random Policy

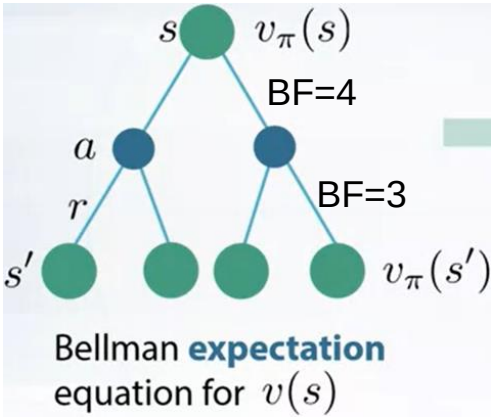




- Random policy: $\pi(l|s) = \pi(r|s) = \pi(u|s) = \pi(d|s) = 0.25$
- Policy Evaluation:$v_\pi(s) = \sum_a \pi(a|s) q_\pi(s,a)\,;\; q_\pi(s,a) = \sum_{r,s'} p(r,s'|s,a)\,[r + \gamma v_\pi(s')]$
- Set of equations:
- $v_\pi(C) = 0.25\big(q_\pi(C,l) + q_\pi(C,r) + q_\pi(C,u) + q_\pi(C,d)\big) = 0.25(-4 + v_\pi(B) + v_\pi(E))$
  - $q_\pi(C,l) = -1 + 0.8v_\pi(B) + 0.1v(A) + 0.1v_\pi(E) = -2 + 0.8v_\pi(B) + 0.1v_\pi(E)$
  - $q_\pi(C,r) = -1 + 0.8v(D) + 0.1v(A) + 0.1v_\pi(E) = 6 + 0.1v_\pi(E)$
  - $q_\pi(C,u) = -1 + 0.8v(A) + 0.1v_\pi(B) + 0.1v(D) = -8 + 0.1v_\pi(B)$
  - $q_\pi(C,d) = -1 + 0.8v_\pi(E) + 0.1v_\pi(B) + 0.1v(D) = 0.8v_\pi(E) + 0.1v_\pi(B)$
- $v_\pi(B) = 0.25\big(q_\pi(B,l) + q_\pi(B,r) + q_\pi(B,u) + q_\pi(B,d)\big) = 0.25(-4 + 3.0v_\pi(B) + 1.0v_\pi(C))$
  - $q_\pi(B,l) = -1 + 1.0v_\pi(B)$
  - $q_\pi(B,r) = -1 + 0.8v_\pi(C) + 0.2v_\pi(B)$
  - $q_\pi(B,u) = q_\pi(B,d) = -1 + 0.9v_\pi(B) + 0.1v_\pi(C)$
- $v_\pi(E) = 0.25(q_\pi(E,l) + q_\pi(E,r) + q_\pi(E,u) + q_\pi(E,d)) = 0.25(-4 + 3.0v_\pi(E) + 1.0v_\pi(C))$
  - $q_\pi(E,l) = q_\pi(B,r) = -1 + 0.9v_\pi(E) + 0.1V_\pi(C)$
  - $q_\pi(E,u) = -1 + 0.8v_\pi(C) + 0.2v_\pi(E)$
  - $q_\pi(E,d) = -1 + 1.0v_\pi(E)$
- Solution:
  - $V_\pi(C) = -6, V_\pi(B) = -10, V_\pi(E) = -10$

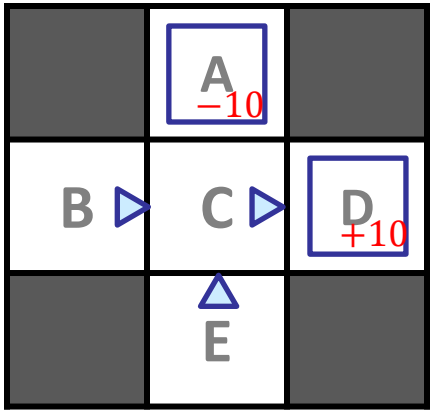
Iter1 Policy Improvement

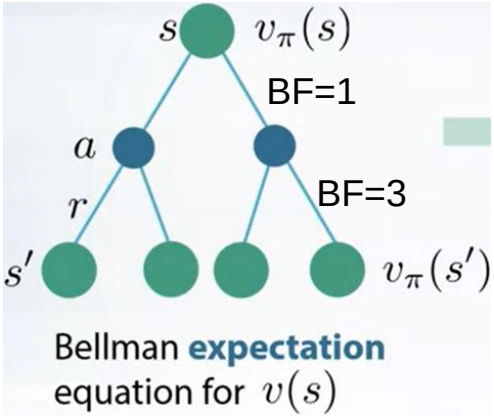




- Plug in values from PE: $v_\pi(C) = -6, v_\pi(B) = -10, v_\pi(E) = -10$, to get new policy π'
- $\pi'(C) = \underset{a \in (l,r,u,d)}{\text{argmax}}\ q_\pi(C,a) = r$
  - $q_\pi(C,l) = -1 + 1.0 v_\pi(B) = -1 - 10 = -11$
  - $q_\pi(C,r) = -1 + 1.0 v(D) = -1 + 10 = 9$
  - $q_\pi(C,u) = -1 + 1.0 v(A) = -1 - 10 = -11$
  - $q_\pi(C,d) = -1 + 1.0 v_\pi(E) = -1 - 10 = -11$
- $\pi'(B) = \underset{a \in (l,r,u,d)}{\text{argmax}}\ q_\pi(B,a) = r$
  - $q_\pi(B,l) = q_\pi(B,u) = q_\pi(B,d) = -1 + 1.0 v_\pi(B) = -11$
  - $q_\pi(B,r) = -1 + 1.0 v_\pi(C) = -1 - 6 = -7$
- $\pi'(E) = \underset{a \in (l,r,u,d)}{\text{argmax}}\ q_\pi(E,a) = u$
  - $q_\pi(E,l) = q_\pi(E,r) = q_\pi(E,d) = -1 + 1.0 v_\pi(E) = -11$
  - $q_\pi(E,u) = -1 + 1.0 v_\pi(C) = -1 - 6 = -7$

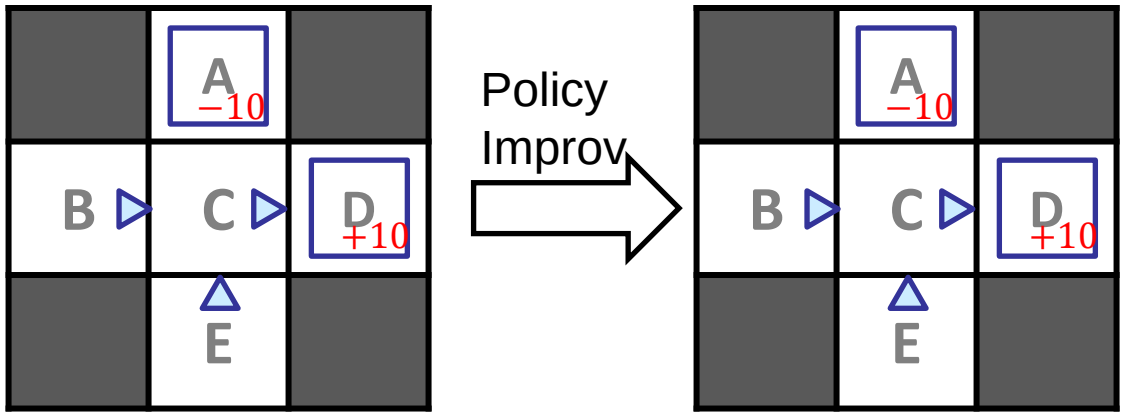
Iter2 Policy Evaluation of Det Policy

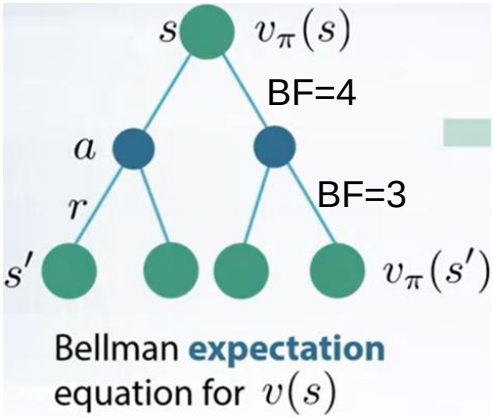




- Det policy: $\pi(r|B) = 1; \pi(r|C) = 1; \pi(u|C) = 1$
- Policy Evaluation:$v_\pi(s) = \sum_a \pi(a|s) q_\pi(s,a); q_\pi(s,a) = r + \gamma \sum_{s'} p(s'|s,a)\, v_\pi(s')$
- Set of equations:
- $v_\pi(C) = 1.0 q_\pi(C,r)$
  - $q_\pi(C,r) = -1 + 0.8v(D) + 0.1v(A) + 0.1v_\pi(E) = -1 + 0.8 \cdot 10 + 0.1(-10) + 0.1v_\pi(E) = 6 + 0.1v_\pi(E)$
- $v_\pi(B) = 1.0 q_\pi(B,r)$
  - $q_\pi(B,r) = -1 + 0.8v_\pi(C) + 0.2v_\pi(B)$
- $v_\pi(E) = 1.0 q_\pi(E,u)$
  - $q_\pi(E,u) = -1 + 0.8v_\pi(C) + 0.2v_\pi(E)$
- Solution:
  - $V_\pi(C) = 6.53, V_\pi(B) = 5.3, V_\pi(E) = 5.3$

Iter2 Policy Improvement





- Plugging in values from PE: $V_\pi(C) = 6.53, V_\pi(B) = 5.3, V_\pi(E) = 5.3$, we get new policy π'
- $\pi'(C) = \underset{a \in (l,r,u,d)}{\text{argmax}}\ q_\pi(C,a) = r$
  - $q_\pi(C,l) = -1 + 1.0 v_\pi(B) = -1 + 5.3 = 4.3$
  - $q_\pi(C,r) = -1 + 1.0 v(D) = -1 + 10 = 9$
  - $q_\pi(C,u) = -1 + 1.0 v(A) = -1 - 10 = -11$
  - $q_\pi(C,d) = -1 + 1.0 v_\pi(E) = -1 + 5.3 = 4.3$
- $\pi'(B) = \underset{a \in (l,r,u,d)}{\text{argmax}}\ q_\pi(B,a) = r$
  - $q_\pi(B,l) = q_\pi(B,u) = q_\pi(B,d) = -1 + 1.0 v_\pi(B) = -1 + 5.3 = 4.3$
  - $q_\pi(B,r) = -1 + 1.0 v_\pi(C) = -1 + 6.53 = 5.53$
- $\pi'(E) = \underset{a \in (l,r,u,d)}{\text{argmax}}\ q_\pi(E,a) = u$
  - $q_\pi(E,l) = q_\pi(E,r) = q_\pi(E,d) = -1 + 1.0 v_\pi(E) = 4.3$
  - $q_\pi(E,u) = -1 + 1.0 v_\pi(C) = -1 + 6.53 = 5.53$
- New policy is now stable ($\pi' = \pi$), so we have found the optimal policy.

MiniGW Example

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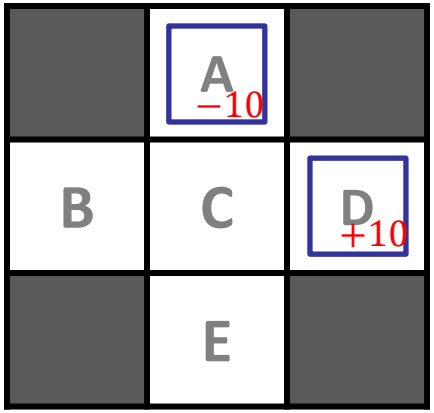
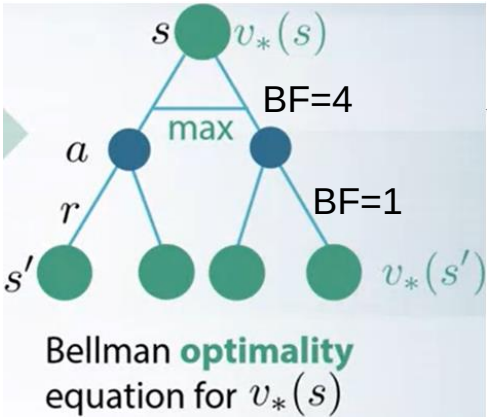
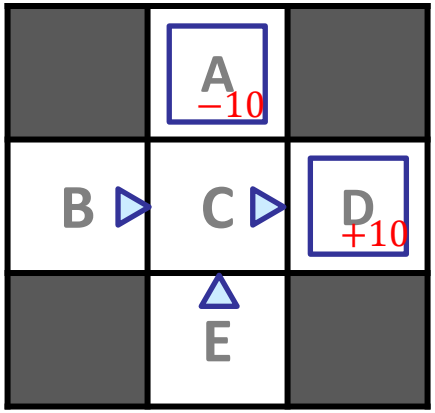
Value Iteration w. Det Env: 1st Iteration

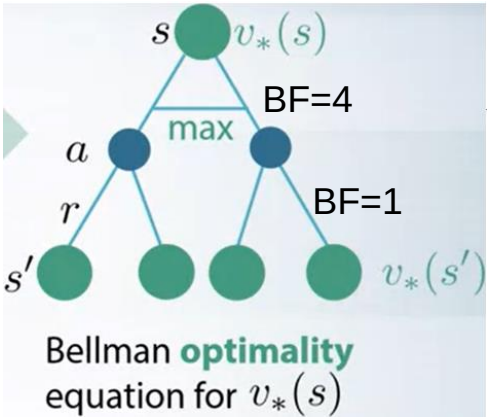




- Value Iteration: $v_{k+1}(s) = \max_a q_{k+1}(s,a) = \max_a[r + \gamma v_k(s')]$ w. in-place updates.
- Initialize $v_0(B) = v_0(C) = v_0(E) = 0$.
- $v_1(C) = \max_a q_1(C,a) = \max(-1, 9, -11, -1) = 9$
  - $q_1(C,l) = r + 1.0v_0(B) = -1 + 1.0 \cdot 0 = -1$
  - $q_1(C,r) = r + 1.0v(D) = -1 + 1.0 \cdot 10 = 9$
  - $q_1(C,u) = r + 1.0v(A) = -1 + 1.0(-10) = -11$
  - $q_1(C,d) = r + 1.0v_0(E) = -1 + 1.0 \cdot 0 = -1$
- $v_1(B) = \max_a q_1(B,a) = \max(-1, 8, -1, -1) = 8$
  - $q_1(B,l) = q_1(B,u) = q_1(B,d) = r + 1.0v_0(B) = -1 + 1.0 \cdot 0 = -1$
  - $q_1(B,r) = r + 1.0v_1(C) = -1 + 1.0 \cdot 9 = 8$ ($v_1(C)$ computed in the current iteration is used instead of $v_0(C)$)
- $v_1(E) = \max_a q_1(E,a) = \max(-1, 8, -1, -1) = 8$
  - $q_1(E,l) = q_1(E,r) = q_1(E,d) = r + 1.0v_0(E) = -1 + 1.0 \cdot 0 = -1$
  - $q_1(E,u) = r + 1.0v_1(C) = -1 + 1.0 \cdot 9 = 8$ ($v_1(C)$ is used)

Value Iteration w. Det Env: 2nd Iteration



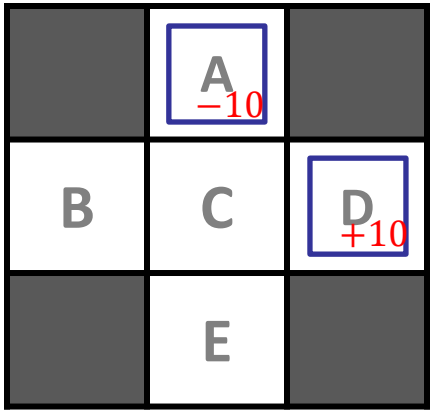


- Value Iteration: $v_{k+1}(s) = \max_a q_{k+1}(s,a) = \max_a [r + \gamma v_k(s')]$ w. in-place updates.
- We have now $v_1(B) = v_1(E) = 8, v_1(C) = 9$.
- $v_2(C) = \max_a q_2(C,a) = \max(7, 9, -11, 7) = 9$
  - $q_2(C,l) = r + 1.0 v_1(B) = -1 + 1.0 \cdot 8 = 7$
  - $q_2(C,r) = r + 1.0 v(D) = -1 + 1.0 \cdot 10 = 9$
  - $q_2(C,u) = r + 1.0 v(A) = -1 + 1.0(-10) = -11$
  - $q_2(C,d) = r + 1.0 v_1(E) = -1 + 1.0 \cdot 8 = 7$
- $v_2(B) = \max_a q_2(B,a) = \max(7, 8, 7, 7) = 8$
  - $q_2(B,l) = q_2(B,u) = q_2(B,d) = r + 1.0 v_1(B) = -1 + 1.0 \cdot 8 = 7$
  - $q_2(B,r) = r + 1.0 v_2(C) = -1 + 1.0 \cdot 9 = 8$ ($v_2(C)$ is used instead of $v_1(C)$)
- $v_2(E) = \max_a q_2(E,a) = \max(7, 8, 7, 7) = 8$
  - $q_2(E,l) = q_2(E,r) = q_2(E,d) = r + 1.0 v_1(E) = -1 + 1.0 \cdot 0 = -1$
  - $q_2(E,u) = r + 1.0 v_2(C) = -1 + 1.0 \cdot 9 = 8$ ($v_2(C)$ is used)
- Value iteration has converged, so $v_*(\cdot) = v_2(\cdot)$. We can get the optimal policy:
  - $\pi_*(C) = \underset{a \in (l,r,u,d)}{\text{argmax}}\ q_*(C,a) = r$; $\pi_*(B) = \underset{a \in (l,r,u,d)}{\text{argmax}}\ q_*(B,a) = r$; $\pi_*(E) = \underset{a \in (l,r,u,d)}{\text{argmax}}\ q_*(E,a) = u$

MiniGW Example

- Policy Iteration for Deterministic Environment
- Policy Iteration for Stochastic Environment
  - Policy Evaluation with analytic solution, not IPE
- Value Iteration for Deterministic Environment
- Value Iteration for Stochastic Environment
  - Bellman Opt. Equation solved iteratively w. in-place updates
- Model-based Learning

Value Iteration w. Sto Env: 2nd Iteration



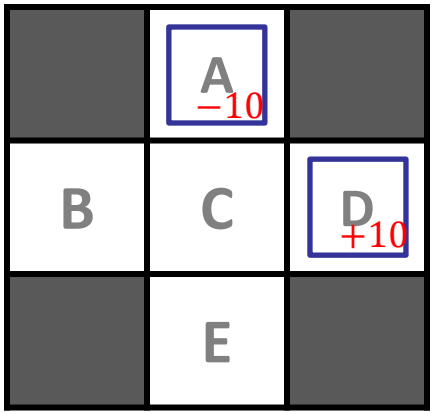
BF=4

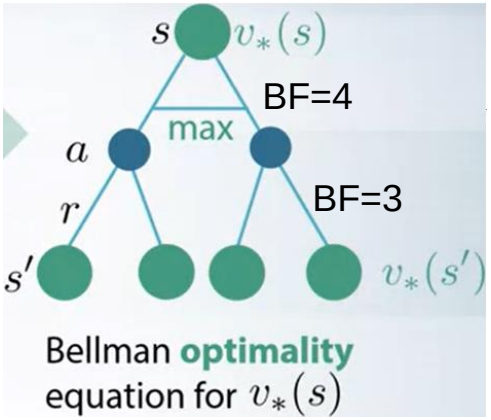
−10

BF=3

- Value Iteration: $v_{k+1}(s) = \max_a q_{k+1}(s,a) = \max_a \left[r + \gamma \sum_{s'} p(s'|s,a)\, v_k(s')\right]$ w. in-place updates.
- We have now: $v_1(B) = v_1(E) = 3.8, v_1(C) = 6$.
- $v_2(C) = \max_a q_2(C,a) = \max(-2, 6, -8, 0) = 6$
  - $q_2(C,l) = r + .8v_1(B) + .1v(A) + .1v_1(E) = -1 + .8 \cdot 3.8 + .1(-10) + .1 \cdot 3.8 = 1.42$
  - $q_2(C,r) = r + .8v(D) + .1v(A) + .1v_0(E) = -1 + .8 \cdot 10 + .1(-10) + .1 \cdot 3.8 = 6$
  - $q_2(C,u) = r + .8v(A) + .1v_1(B) + .1v(D) = -1 + .8(-10) + .1 \cdot 0 + .1 \cdot 10 = -8$
  - $q_2(C,d) = r + .8v_1(E) + .1v_1(B) + .1v(D) = -1 + .8 \cdot 3.8 + .1 \cdot 0 + .1 \cdot 10 = 3.04$
- $v_2(B) = \max_a q_2(B,a) = \max(2.8, 4.56, 3.02, 3.02) = 4.56$
  - $q_2(B,l) = r + 1.0v_1(B) = -1 + 1.0 \cdot 3.8 = 2.8$
  - $q_2(B,r) = r + .8v_2(C) + .2v_1(B) = -1 + .8 \cdot 6 + .2 \cdot 3.8 = 4.56$
  - $q_2(B,u) = q_2(B,d) = r + .9v_1(B) + .1v_1(C) = -1 + .9 \cdot 3.8 + .1 \cdot 6 = 3.02$
- $v_2(E) = \max_a q_2(E,a) = \max(2.8, 4.56, 3.02, 3.02) = 4.56$
  - $q_1(E,d) = r + 1.0v_1(E) = -1 + 1.0 \cdot 3.8 = 2.8$
  - $q_2(E,u) = r + 0.8v_2(C) + 0.2v_1(E) = -1 + 0.8 \cdot 6 + 0.2 \cdot 3.8 = 4.56$
  - $q_2(E,l) = q_2(E,r) = r + .9v_1(E) + .1v_2(C) = -1 + 0.9 \cdot 3.8 + .1 \cdot 6 = 3.02$
- 1st sweep has converged, so $q_*(\cdot) = q_1(\cdot)$. We can get the optimal policy:
  - $\pi_*(C) = \underset{a\in(l,r,u,d)}{\text{argmax}}\, q_*(C,a) = r;\ \pi_*(B) = \underset{a\in(l,r,u,d)}{\text{argmax}}\, q_*(B,a) = r; \pi_*(E) = \underset{a\in(l,r,u,d)}{\text{argmax}}\, q_*(E,a) = u$

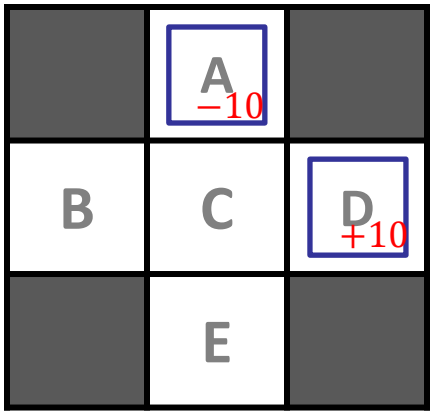
Value Iteration w. Sto Env: 1st Iteration

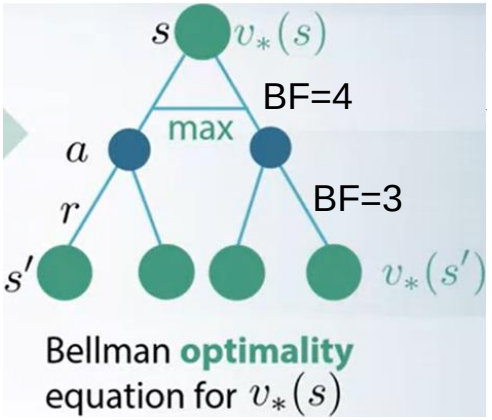




- Value Iteration: $v_{k+1}(s) = \max_a q_{k+1}(s,a) = \max_a \left[r + \gamma \sum_{s'} p(s'|s,a)\, v_k(s')\right]$ w. in-place updates.
- Initialize $v_0(B) = v_0(C) = v_0(E) = 0$.
- $v_1(C) = \max_a q_1(C,a) = \max(-2, 6, -8, 0) = 6$
  - $q_1(C,l) = r + 0.8v_0(B) + 0.1v_0(A) + 0.1v_0(E) = -1 + 0.8 \cdot 0 + 0.1(-10) + 0.1 \cdot 0 = -2$
  - $q_1(C,r) = r + 0.8v_0(D) + 0.1v_0(A) + 0.1v_0(E) = -1 + 0.8 \cdot 10 + 0.1(-10) + 0.1 \cdot 0 = 6$
  - $q_1(C,u) = r + 0.8v_0(A) + 0.1v_0(B) + 0.1v_0(D) = -1 + 0.8(-10) + 0.1 \cdot 0 + 0.1 \cdot 10 = -8$
  - $q_1(C,d) = r + 0.8v_0(E) + 0.1v_0(B) + 0.1v_0(D) = -1 + 0.8 \cdot 0 + 0.1 \cdot 0 + 0.1 \cdot 10 = 0$
- $v_1(B) = \max_a q_1(B,a) = \max(-1, 3.8, -0.4, -0.4) = 3.8$
  - $q_1(B,l) = r + 1.0v_0(B) = -1 + 1.0 \cdot 0 = -1$
  - $q_1(B,r) = r + 0.8v_1(C) + 0.2v_0(B) = -1 + 0.8 \cdot 6 + 0.2 \cdot 0 = 3.8$
  - $q_1(B,u) = q_1(B,d) = r + 0.9v_0(B) + 0.1v_1(C) = -1 + 0.9 \cdot 0 + 0.1 \cdot 6 = -0.4$
- $v_1(E) = \max_a q_1(E,a) = \max(-1, 3.8, -0.4, -0.4) = 3.8$
  - $q_1(E,l) = q_1(E,r) = r + 0.9v_0(E) + 0.1v_1(C) = -1 + 0.9 \cdot 0 + 0.1 \cdot 6 = -0.4$
  - $q_1(E,u) = r + 0.8v_1(C) + 0.2v_0(E) = -1 + 0.8 \cdot 6 + 0.2 \cdot 0 = 3.8$
  - $q_1(E,d) = r + 1.0v_0(E) = -1 + 1.0 \cdot 0 = -1$

Value Iteration w. Sto Env: 2nd Iteration





- Value Iteration: $v_{k+1}(s) = \max_a q_{k+1}(s,a) = \max_a \left[r + \gamma \sum_{s'} p(s'|s,a)\, v_k(s')\right]$ w. in-place updates.
- We have now: $v_1(B) = v_1(E) = 3.8, v_1(C) = 6.$
- $v_2(C) = \max_a q_2(C,a) = \max(1.42, 6.38, -7.62, 3.42) = 6.38$
  - $q_2(C,l) = r + .8v_1(B) + .1v(A) + .1v_1(E) = -1 + .8 \cdot 3.8 + .1(-10) + .1 \cdot 3.8 = 1.42$
  - $q_2(C,r) = r + .8v(D) + .1v(A) + .1v_1(E) = -1 + .8 \cdot 10 + .1(-10) + .1 \cdot 3.8 = 6.38$
  - $q_2(C,u) = r + .8v(A) + .1v_1(B) + .1v(D) = -1 + .8(-10) + .1 \cdot 3.8 + .1 \cdot 10 = -7.62$
  - $q_2(C,d) = r + .8v_1(E) + .1v_1(B) + .1v(D) = -1 + .8 \cdot 3.8 + .1 \cdot 3.8 + .1 \cdot 10 = 3.42$
- $v_2(B) = \max_a q_2(B,a) = \max(2.8, 4.864, 3.058, 3.058) = 4.864$
  - $q_2(B,l) = r + 1.0v_1(B) = -1 + 1.0 \cdot 3.8 = 2.8$
  - $q_2(B,r) = r + .8v_2(C) + .2v_1(B) = -1 + .8 \cdot 6.38 + .2 \cdot 3.8 = 4.864$
  - $q_2(B,u) = q_2(B,d) = r + .9v_1(B) + .1v_2(C) = -1 + .9 \cdot 3.8 + .1 \cdot 6.38 = 3.058$
- $v_2(E) = \max_a q_2(E,a) = \max(2.8, 4.864, 3.058\ 3.058) = 4.864$
  - $q_1(E,d) = r + 1.0v_1(E) = -1 + 1.0 \cdot 3.8 = 2.8$
  - $q_2(E,u) = r + 0.8v_2(C) + 0.2v_1(E) = -1 + 0.8 \cdot 6.38 + 0.2 \cdot 3.8 = 4.864$
  - $q_2(E,l) = q_2(E,r) = r + .9v_1(E) + .1v_2(C) = -1 + 0.9 \cdot 3.8 + .1 \cdot 6.38 = 3.058$

Value Iteration w. Sto Env: after Convergence

- Value functions $v_*(s)$ converge after 6 iterations (shown below), with $v_*(C) = 6.53, v_*(B) = v_*(C) = 5.28$. We can get the optimal policy:
  - $\pi_*(C) = \underset{a \in (l,r,u,d)}{\text{argmax}}\ q_*(C,a) = \underset{a \in (l,r,u,d)}{\text{argmax}} \sum_{r,s'} p(r,s'|s,a)\,[r + \gamma v_*(s')] = r$
    - We can obtain $q_*(s,a)$ from $v_*(s)$, but we can also keep track of $q_*(s,a)$, which are computed during the process of computing $v_*(s)$.
  - $\pi_*(B) = \underset{a \in (l,r,u,d)}{\text{argmax}}\ q_*(B,a) = r$;
  - $\pi_*(E) = \underset{a \in (l,r,u,d)}{\text{argmax}}\ q_*(E,a) = u$

```
values:
--------------------
 0.00|-10.00| 0.00|
--------------------
 3.80| 6.00| 10.00|
--------------------
 0.00| 3.80| 0.00|
```

```
values:
----------------------
 0.00|-10.00| 0.00|
----------------------
 4.86| 6.38| 10.00|
----------------------
 0.00| 4.86| 0.00|
```

```
values:
----------------------
 0.00|-10.00| 0.00|
----------------------
 5.16| 6.49| 10.00|
----------------------
 0.00| 5.16| 0.00|
```

```
values:
----------------------
 0.00|-10.00| 0.00|
----------------------
 5.25| 6.52| 10.00|
----------------------
 0.00| 5.25| 0.00|
```

```
values:
----------------------
 0.00|-10.00| 0.00|
----------------------
 5.27| 6.52| 10.00|
----------------------
 0.00| 5.27| 0.00|
```

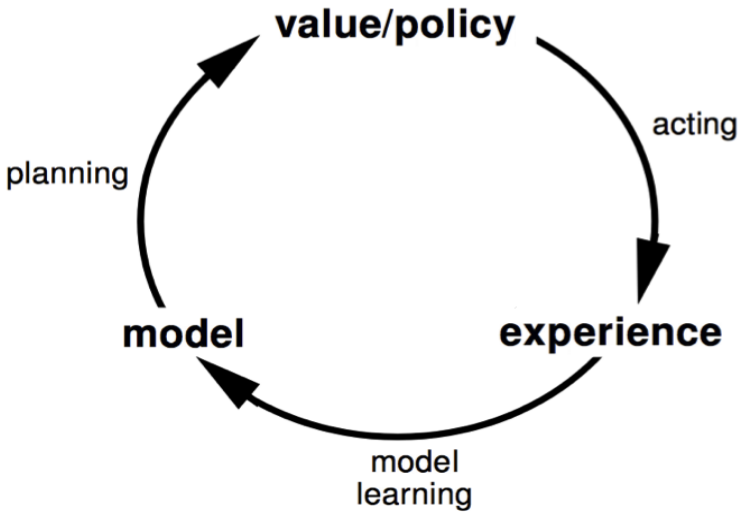
```
values:
----------------------
 0.00|-10.00| 0.00|
----------------------
 5.28| 6.53| 10.00|
----------------------
 0.00| 5.28| 0.00|
```

MiniGW Example

- Policy Iteration for Deterministic Environment
- Policy Iteration for Stochastic Environment
  - Policy Evaluation with analytic solution, not IPE
- Value Iteration for Deterministic Environment
- Value Iteration for Stochastic Environment
  - Bellman Opt. Equation solved iteratively w. in-place updates
- Model-based Learning

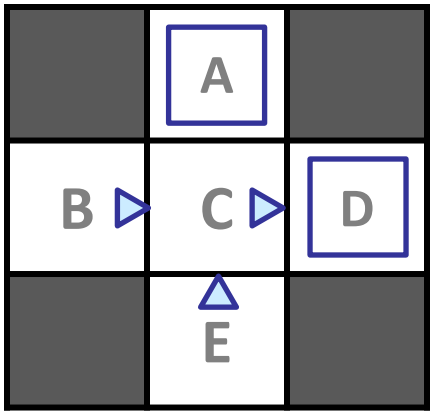
Model-Based RL

- If MDP is not available, we can use Model-Based RL:
- Step 1: Learn empirical MDP model
  - Estimate the model $p(r, s'|s, a)$ by executing some policy π (may be random), and keeping track of outcomes r, s' for each s, a in the observed episodes.
- Step 2: Do planning w. the learned MDP for the optimal policy
  - Dynamic Programming w. Value Iteration or Policy Iteration



MiniGW: Model Learning

Input Policy π



Assume: $\gamma = 1$

Observed Episodes (Training)

Episode 1

$(B, r, C, -1)$
$(C, r, D, -1)$
$(D, exit, x, 10)$

Episode 2

$(B, r, C, -1)$
$(C, r, D, -1)$
$(D, exit, x, +10)$

Episode 3

$(E, u, C, -1)$
$(C, r, D, -1)$
$(D, exit, x, 10)$

Episode 4

$(E, u, C, -1)$
$(C, r, A, -1)$
$(A, exit, x, -10)$

Learned Model

$p(s'|s, a)$

$p(-1, C|B, r) = 1.0$
$p(-1, D|C, r) = 0.75$
$p(-1, A|C, r) = 0.25$
$p(10, x|D, exit) = 1.0$
$p(-10, x|A, exit) = 1.0$

- In the 4 episodes, we see 4 transitions from $(s = C, a = r)$. 3 of them go to next state $s' = D$, and one goes to next state $s' = A$, each w. reward -1. Hence $p(-1, D|C, r) = 0.75; p(-1, A|C, r) = 0.25$.