Lecture 13
Sorting Algorithm

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Lecture Goals

- We introduce *binary heap* for priority queue data abstract, which leads to an efficient sorting algorithm known as ***heapsort***.
- We introduce and implement the *randomized* ***quicksort*** algorithm and analyze its performance.
- We study the ***mergesort*** algorithm and show that it guarantees to sort any array of n items with at most nlog(n) compares.

Short Videos of Sorting Algorithms

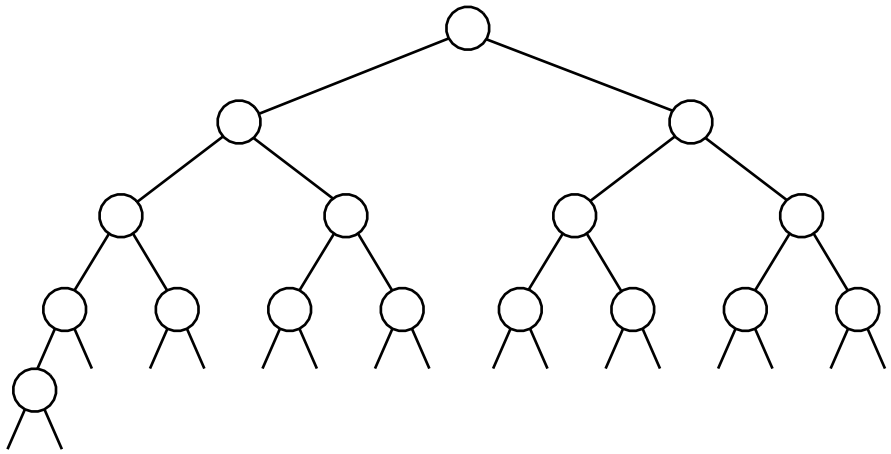
- Sort Algos // Michael Sambol Michael Sambol
  - https://www.youtube.com/playlist?list=PL9xmBV_5YoZOZSbGAXAPIq1BeUf4j20pl
  - Merge Sort, Quick Sort, Bubble Sort, Insertion Sort, Selection Sort, Heap Sort
- 10 Sorting Algorithms Easily Explained
  - https://www.youtube.com/watch?v=rbbTd-gkajw
  - Bubble Sort, Selection Sort, Insertion Sort, Merge Sort, Quick Sort, Heap Sort, Counting Sort, Shell Sort, Tim Sort, Radix Sort

Video Tutorials

- Heap Sort
  - Heaps // Michael Sambol
    - https://www.youtube.com/playlist?list=PL9xmBV_5YoZNsyqgPW-DNwUeT8F8uhWc6
  - HEAP SORT | Sorting Algorithms | DSA | GeeksforGeeks
    - https://www.youtube.com/watch?v=MtQL_ll5KhQ
- Quick Sort
  - Quick sort in 4 minutes
    - https://www.youtube.com/watch?v=Hoixgm4-P4M&t=19s
  - Quicksort Algorithm: A Step-by-Step Visualization
    - https://www.youtube.com/watch?v=bZkzH5x0SKU
  - Visualization of Quick sort (HD)
    - https://www.youtube.com/watch?v=aXXWXz5rF64
- Merge Sort
  - Merge sort in 3 minutes
  - https://www.youtube.com/watch?v=4VqmGXwpLqc

Heapsort: Binary Heap

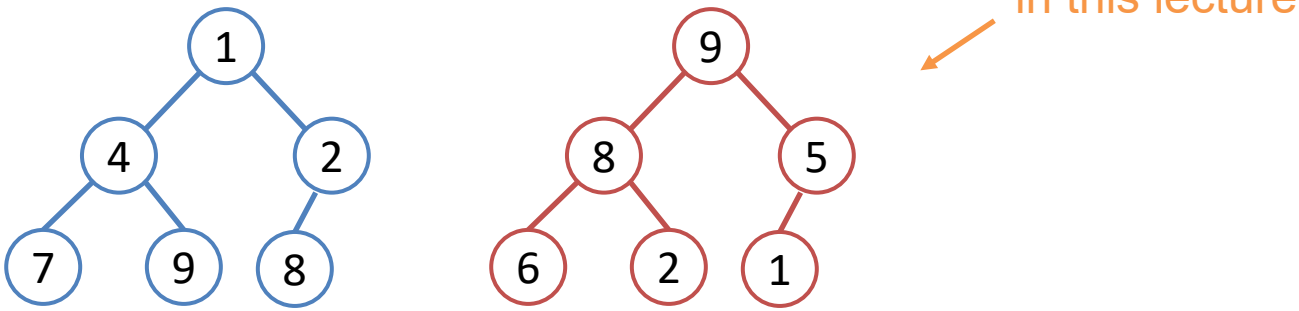
- In a **heap** the highest (or lowest) priority element is always stored at the root, hence the name "heap". A heap is useful data structure when you need to remove the object with the highest (or lowest) priority. A common use of a heap is to implement a **priority queue** and **heapsort**.
- A **binary heap** is a complete binary tree which is an efficient data structure satisfies the heap ordering property.
- In a *complete tree*, every level (except possibly the last) is completely filled; the last level is filled from left to right.



complete binary tree with n = 16 nodes (height = 4)

Heapsort: Binary Heap

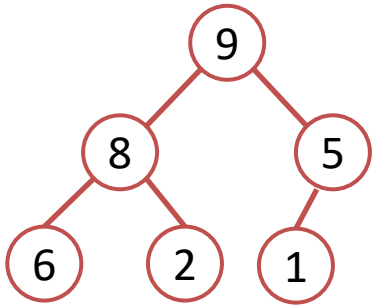
- The heap ordering can be one of two types:
- The **min-heap property**: the value of each node is greater than or equal to the value of its parent, with the minimum-value element at the root.
- The **max-heap property**: the value of each node is less than or equal to the value of its parent, with the maximum-value element at the root.



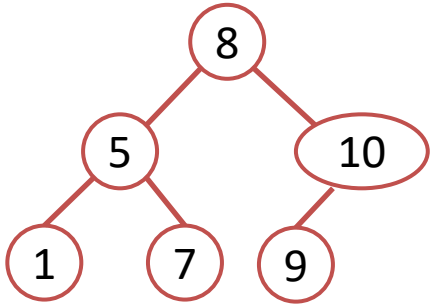
- A heap is not a sorted structure and can be regarded as partially ordered. As you see from the picture, there is no particular relationship among nodes on any given level, even among the siblings.
- Since a heap is a complete binary tree, it has a smallest possible height - a heap with n nodes has ***O(log n)*** height.

Binary Heap vs. Binary Search Tree

- Binary Heap is different from Binary Search Tree (e.g., red-black tree)
- Binary Heap: the max-heap property
  - Value of each node is less than or equal to the value of its parent, with the maximum-value element at the root.
- Binary Search Tree: Ordered, or sorted, binary trees
  - Items to the left of a given node are smaller.
  - Items to the right of a given node are larger.
- Both structures offer O(log n) time complexity for certain operations, they are used in different scenarios.
  - Heapsort is used for efficient sorting and simple priority queue implementations
  - Red-black trees are for maintaining ordered data with frequent updates and searches
  - Red-black trees can also be used for sorting, by insertions followed by in-order traversal, with O(n log(n)) complexity, but is less efficient in terms of memory and execution time than efficient sorting algorithms.



Binary Heap

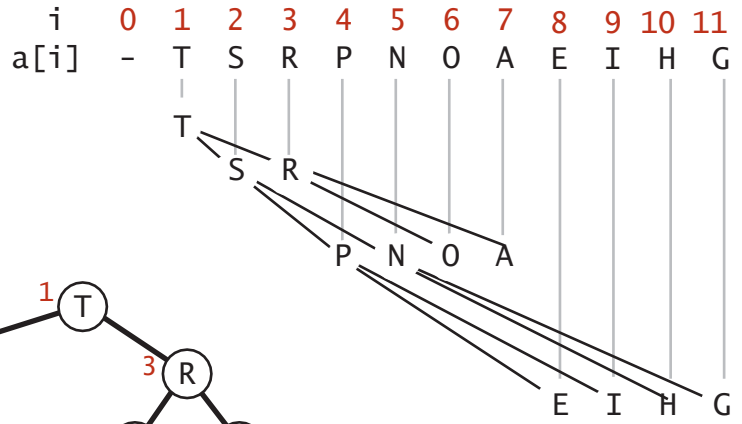


Binary Search Tree

Binary Heap: Array Representation

Array representation.

- Indices range in [1, n].
- Take nodes in level order.
- No explicit links needed!



Heap representations

Proposition. Largest key is a[1], which is root of binary tree.

- Leaf nodes have indices [floor(n/2)+1, n]
  - [floor(11/2)+1, n]=[6, 11]
- Non-leaf nodes have indices [1, floor(n/2)]
  - [1, floor(11/2)]=[1, 5]

Proposition. Can use array indices to move through tree.

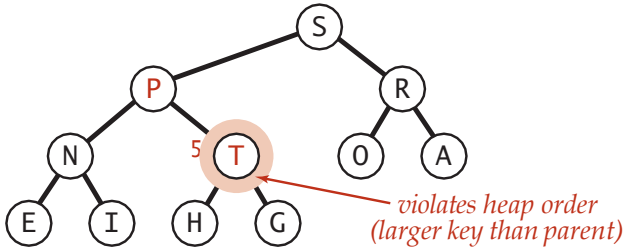
- Parent of node at k is at floor(k/2).
  - Parent of node at 8 or 9 is at floor(8/2)= floor(9/2)=4.
- Children of node at k are at 2k and 2k+1.
  - Children of node at 4 is at 2*4=8 and 2*4+1=9

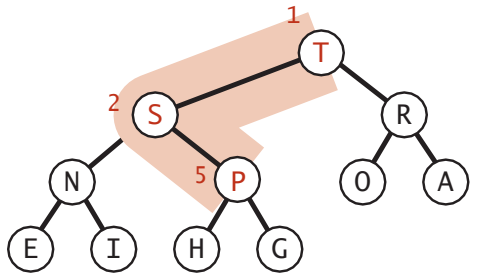
Binary Heap Operations: Promotion

- Scenario. A key becomes larger than its parent's key.
- To eliminate the violation:
- Exchange key in child with key in parent.
- Repeat until heap order restored.

```
private void swim(int k)
{
   while (k > 1 && less(k/2, k))
   {
      exch(k, k/2);
      k = k/2;
   }
}
```

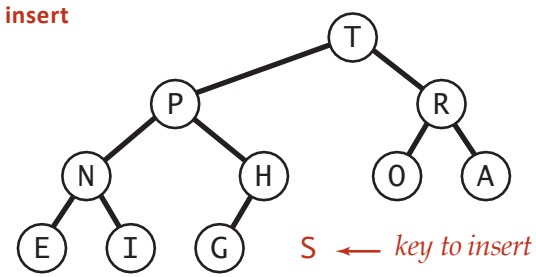
parent of node at k is at k/2

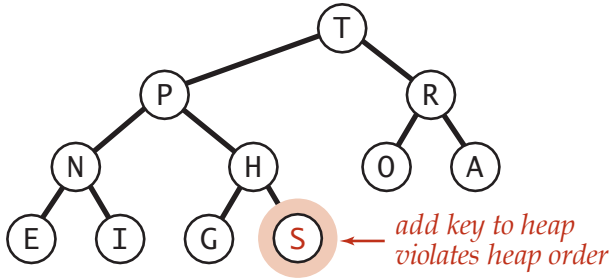


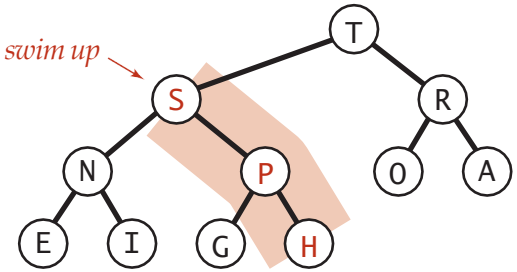


Binary Heap Operations: Insert

- Insert. Add node as leaf, then swim it up.
- Cost. O(log n) compares since tree height is O(log n).







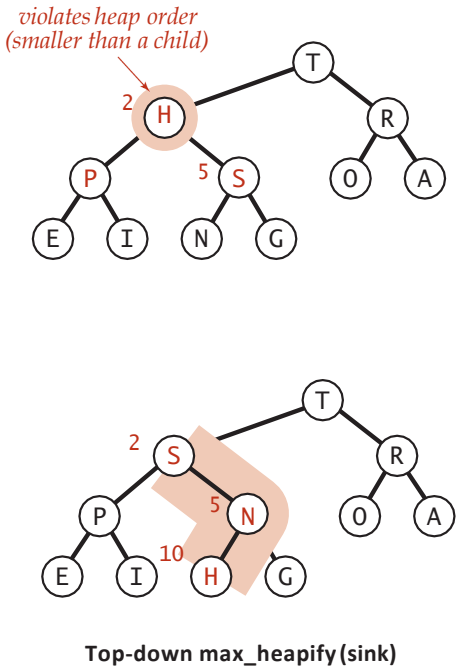

```
public void insert(Key x)
{
   pq[++n] = x;
   swim(n);
}
```

Binary Heap Operations: Demotion

- Scenario. A key becomes smaller than one (or both) of its children's.
- To eliminate the violation:
- Exchange key in parent with key in larger child.
- Repeat until heap order restored.
- (Called max_heapify in video)

```
private void sink(int k, int n)
{
   while (2*k <= n)
   {
      int j = 2*k;
      if (j < n && less(j, j+1)) j++;
      if (!less(k, j)) break;
      exch(k, j);
      k = j;
   }
}
```

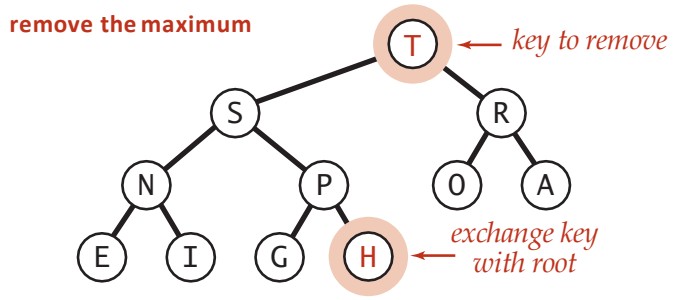
children of node at k are 2*k and 2*k+1

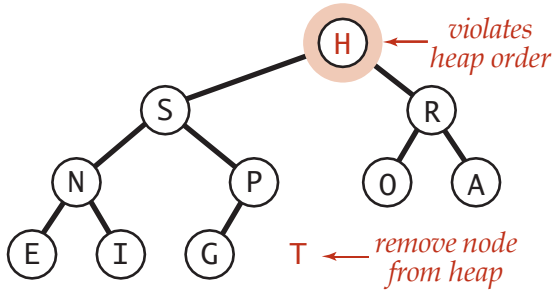


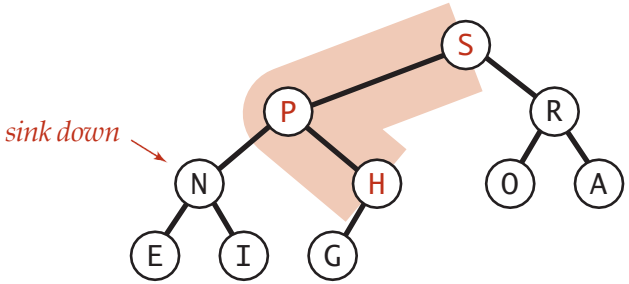
Top-down max_heapify (sink)

Binary Heap Operations: DeleteMax

- Delete max. Exchange root with node at end, then sink it down.
- Cost. O(log n) compares.



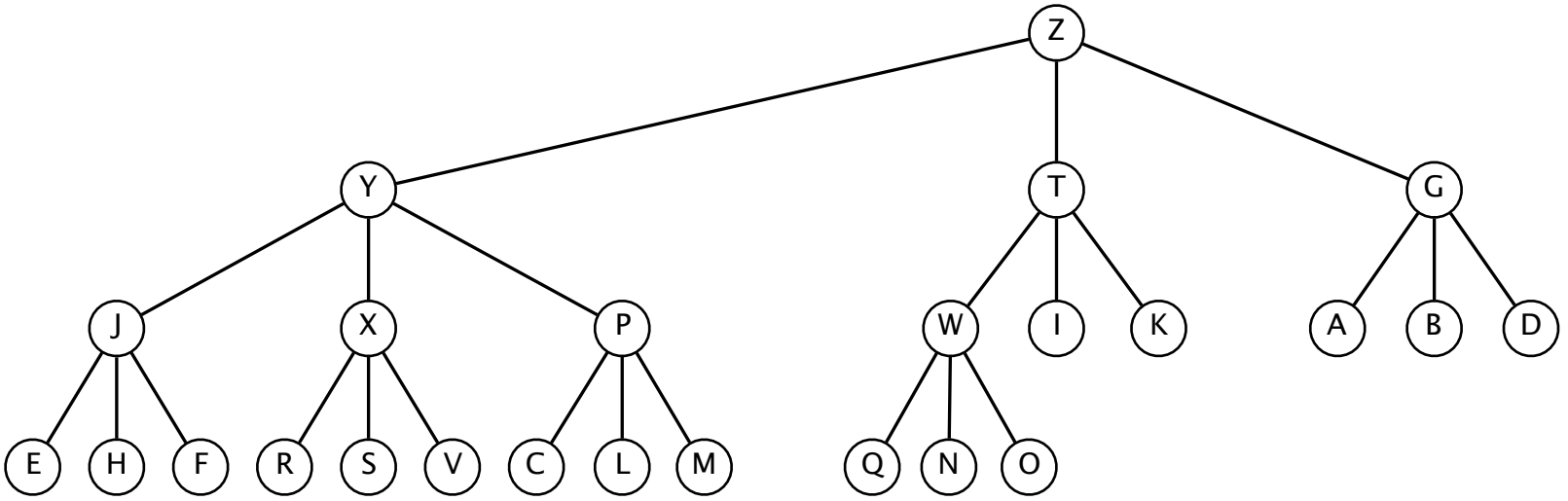





```
public Key delMax()
{
   Key max = pq[1];
   exch(1, n--);
   sink(1);
   pq[n+1] = null;
   return max;
}
```

Binary Heap: Practical improvements

- Multiway heaps. Complete d-way tree.
- Parent's key no smaller than its children's keys.
- Fact. Height of complete d-way tree on n nodes is ~ $\log_d n$.

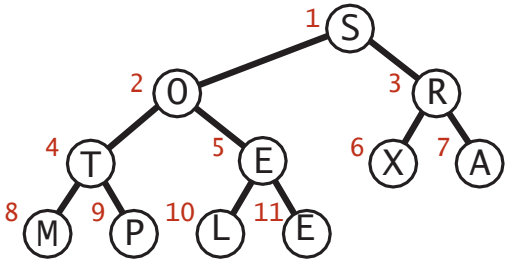


3-way heap

Heapsort Algorithm

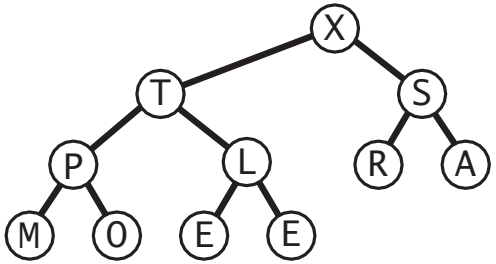
- Basic plan for in-place sort.
- View input array as a complete binary tree.
- Heap construction: build a max-heap with all n keys.
- Sortdown: repeatedly remove the maximum key.

keys in arbitrary order



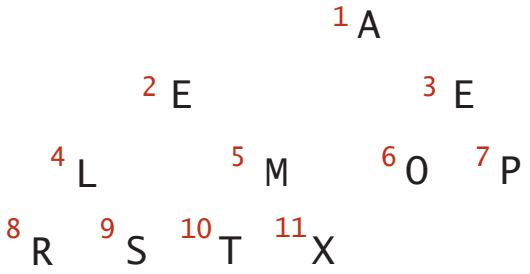
1	2	3	4	5	6	7	8	9	10	11
S	O	R	T	E	X	A	M	P	L	E

build max heap (in place)



1	2	3	4	5	6	7	8	9	10	11
X	T	S	P	L	R	A	M	O	E	E

sorted result (in place)

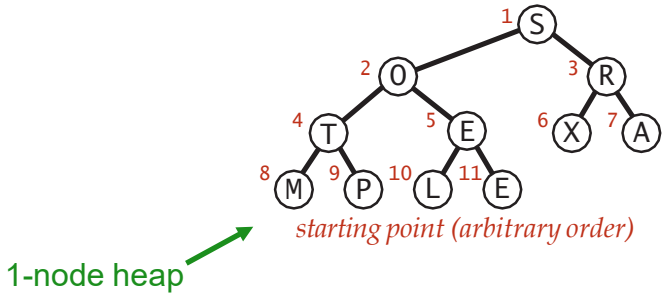


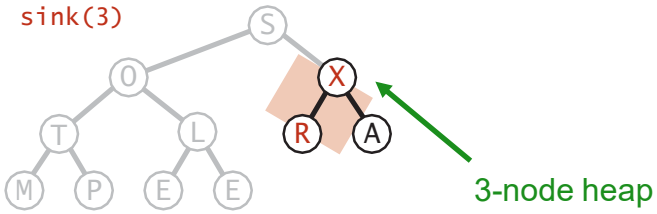
1	2	3	4	5	6	7	8	9	10	11
A	E	E	L	M	O	P	R	S	T	X

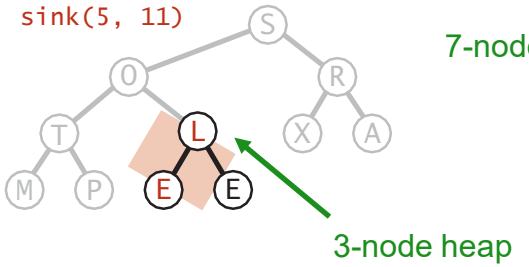
Heapsort: Heap Construction

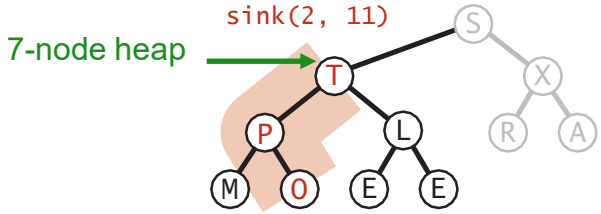
First pass. Build heap using bottom-up method.

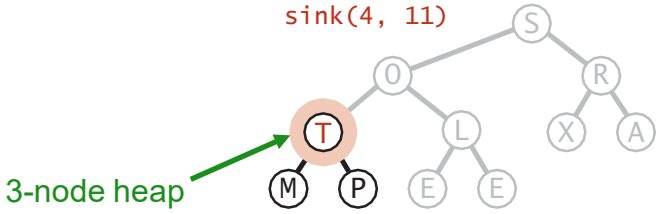
```
for (int k = floor(n/2); k >= 1; k--)
   //call sink(k) on all non-leaf
   nodes k from bottom up
   sink(k, n);
```

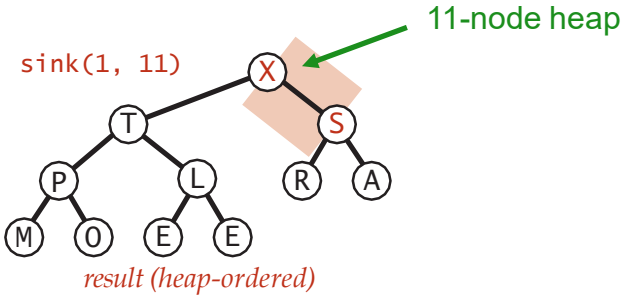










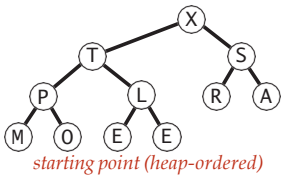


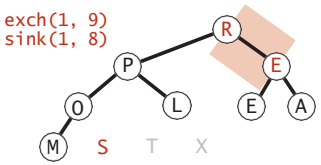
Heapsort: Sortdown

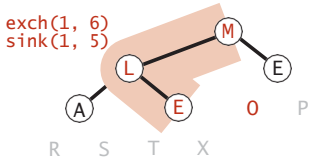
Second pass.

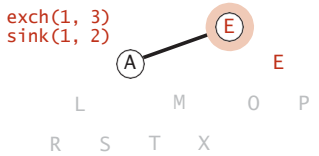
- Remove the maximum, one at a time.
- Leave in array, instead of nulling out.

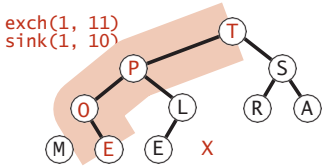
```
while (n > 1)
{
   exch(a, 1, n--);
   sink(a, 1, n);
}
```

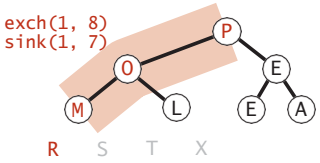


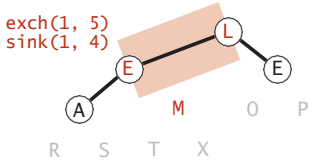


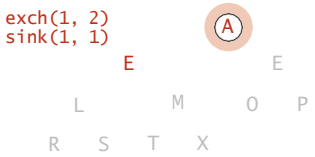


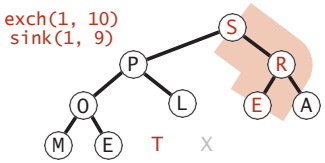


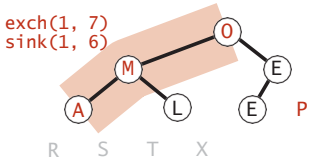


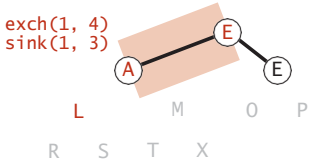


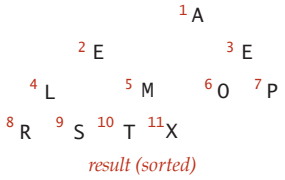












Heapsort: Java Implementation

```
public class Heap
{
   public static void sort(Comparable[] a)
   {
      int n = a.length;
      for (int k = n/2; k >= 1; k--)
          sink(a,  k,  n);
      while (n >   1)
      {
           exch(a,  1,  n);
           sink(a,  1,  --n);
      }
   }

   private static void sink(Comparable[] a, int k, int n)
   {  /* as before */  }

   private static boolean less(Comparable[] a, int i, int j)
   {  /* as before */  }

   private static void exch(Object[] a, int i, int j)
   {  /* as before */  }

}
```

O(nlogn)

but make static (and pass arguments)

but convert from 1-based indexing to 0-base indexing

Heapsort: Trace

sink(K, N)

N	k	a[i] 0	1	2	3	4	5	6	7	8	9	10	11
initial	*values*		S	O	R	T	E	X	A	M	P	L	E
11	5		S	O	R	T	L	X	A	M	P	E	E
11	4		S	O	R	T	L	X	A	M	P	E	E
11	3		S	O	X	T	L	R	A	M	P	E	E
11	2		S	T	X	P	L	R	A	M	O	E	E
11	1		X	T	S	P	L	R	A	M	O	E	E
heap-ordered			X	T	S	P	L	R	A	M	O	E	E
10	1		T	P	S	O	L	R	A	M	E	E	X
9	1		S	P	R	O	L	E	A	M	E	T	X
8	1		R	P	E	O	L	E	A	M	S	T	X
7	1		P	O	E	M	L	E	A	R	S	T	X
6	1		O	M	E	A	L	E	P	R	S	T	X
5	1		M	L	E	A	E	O	P	R	S	T	X
4	1		L	E	E	A	M	O	P	R	S	T	X
3	1		E	A	E	L	M	O	P	R	S	T	X
2	1		E	A	E	L	M	O	P	R	S	T	X
1	1		A	E	E	L	M	O	P	R	S	T	X
sorted	*result*		A	E	E	L	M	O	P	R	S	T	X

3-node heap

7-node heap

11-node heap

red: exchanged

black: compared

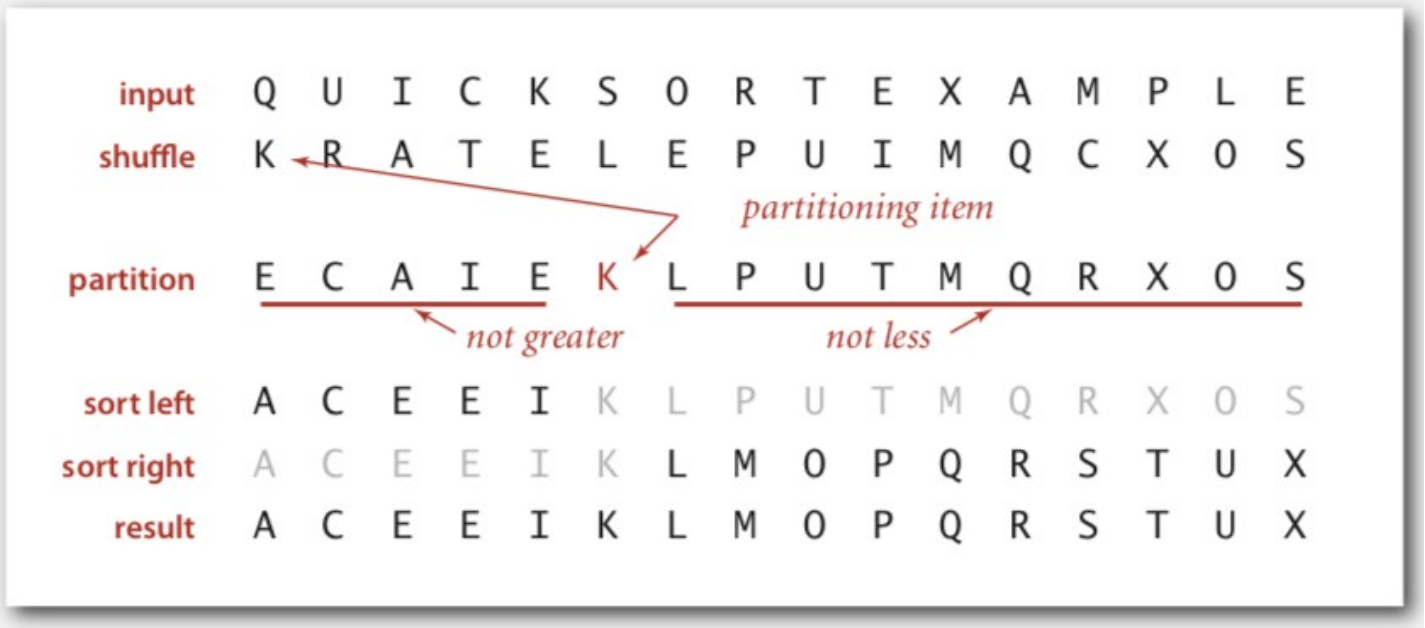
Heapsort trace (array contents just after each sink)

Quicksort

- Quicksort is a widely used sorting algorithm based on the divide-and-conquer approach.
- ## Pivot Selection in Quicksort
- The pivot is an element chosen from the array that serves as a reference point for partitioning the array into two subarrays. There are several strategies for selecting the pivot:
  - ###First or Last Element as Pivot. One common approach is to choose the last element of the array as the pivot. This method is simple to implement but can lead to poor performance if the array is already sorted or nearly sorted.
    - We use the first element as Pivot in the examples.
  - ### Random Element as Pivot. Selecting a random element as the pivot can help avoid worst-case scenarios and provide more consistent performance across different input distributions.
  - ### Median-of-Three. This method selects the median of the first, middle, and last elements of the array as the pivot. It often provides a good balance between simplicity and performance.
- ## Partitioning Process
- Once the pivot is selected, the partitioning process begins:
  - 1. The pivot is compared with each element in the array.
  - 2. Elements smaller than the pivot are moved to its left.
  - 3. Elements larger than the pivot are moved to its right.
  - 4. The pivot is placed in its final sorted position.

Quicksort

1. Shuffle the array.
2. Partition so that, for some pivot j
   - entry a[j] is in place
   - no larger entry to the left of j
   - no smaller entry to the right of j
3. Sort each piece recursively.



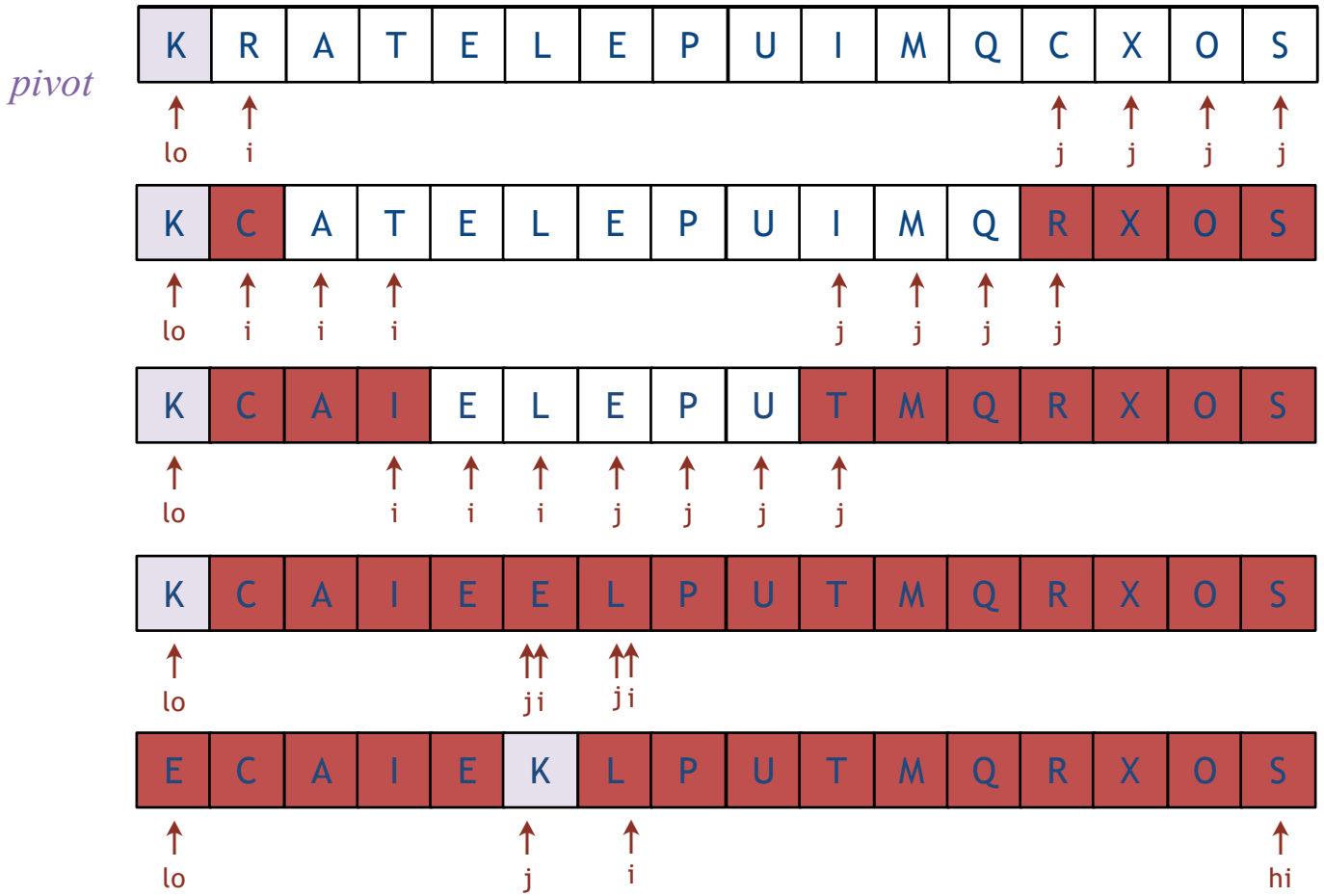
Partition Operation

Repeat until i and j pointers cross.

- Scan i from left to right so long as (a[i] < a[lo]).
- Scan j from right to left so long as (a[j] > a[lo]).
- Exchange a[i] with a[j].

When pointers cross.

- Exchange a[lo] with a[j].



Partition Operation: Java Implementation

```
private static int partition(Comparable[] a, int lo, int hi)
{
    int i = lo, j = hi+1;
    while (true)
    {
        while (less(a[++i], a[lo]))
        if (i == hi) break;

        while (less(a[lo], a[--j]))
        if (j == lo) break;

        if (i >= j) break;
        exch(a, i, j);
    }

    exch(a, lo, j);
    return j;
}
```

find item on left to swap

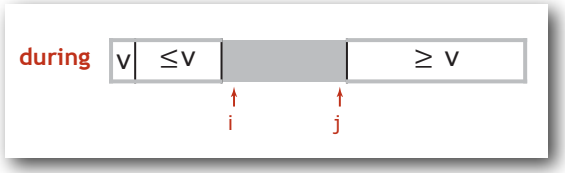
find item on right to swap

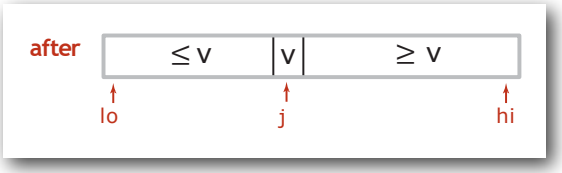
check if pointers cross

swap

swap with partitioning item

return index of item now known to be in place





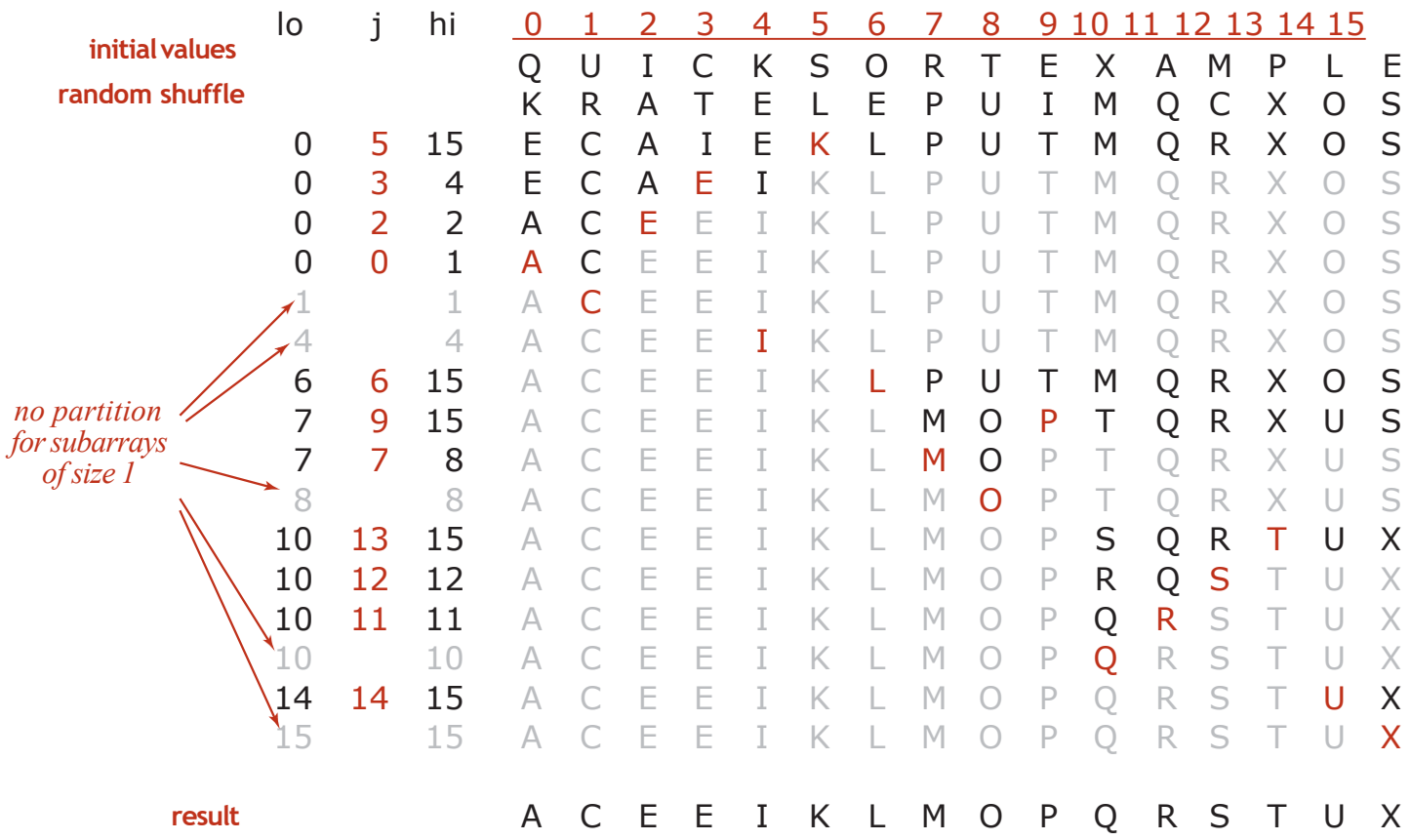
Quicksort: Java Implementation

```
public class Quick
{
   private static int     partition(Comparable[]  a,  int   lo,  int   hi)
      {/* see previous slide /     } *

    public static void sort(Comparable[] a)
    {
        StdRandom.shuffle(a);
        sort(a, 0, a.length - 1);
    }

    private static void sort(Comparable[] a, int lo, int hi)
    {
        if (hi <= lo) return;
        int j = partition(a, lo, hi);
        sort(a, lo, j-1);
        sort(a, j+1, hi);
   }
}
```

Shuffle needed for performance guarantee

Pivot selection

Quicksort: Trace



Quicksort trace (array contents after each partition)

Quicksort: Best-case Analysis

Best case. Number of compares is ~ N log N.

			a[ ]													
lo	j	hi	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
initial values			H	A	C	B	F	E	G	D	L	I	K	J	N	M	O
random shuffle			H	A	C	B	F	E	G	D	L	I	K	J	N	M	O
0	7	14	D	A	C	B	F	E	G	H	L	I	K	J	N	M	O
0	3	6	B	A	C	D	F	E	G	H	L	I	K	J	N	M	O
0	1	2	A	B	C	D	F	E	G	H	L	I	K	J	N	M	O
0		0	A	B	C	D	F	E	G	H	L	I	K	J	N	M	O
2		2	A	B	C	D	F	E	G	H	L	I	K	J	N	M	O
4	5	6	A	B	C	D	E	F	G	H	L	I	K	J	N	M	O
4		4	A	B	C	D	E	F	G	H	L	I	K	J	N	M	O
6		6	A	B	C	D	E	F	G	H	L	I	K	J	N	M	O
8	11	14	A	B	C	D	E	F	G	H	J	I	K	L	N	M	O
8	9	10	A	B	C	D	E	F	G	H	I	J	K	L	N	M	O
8		8	A	B	C	D	E	F	G	H	I	J	K	L	N	M	O
10		10	A	B	C	D	E	F	G	H	I	J	K	L	N	M	O
12	13	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
12		12	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
14		14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
			A	B	C	D	E	F	G	H	I	J	K	L	M	N	O

Quicksort: Worst-case Analysis

Worst case. Number of compares is ~ ½ N^2 .

			a[ ]													
lo	j	hi	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
initial values			A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
random shuffle			A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
0	0	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
1	1	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
2	2	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
3	3	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
4	4	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
5	5	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
6	6	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
7	7	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
8	8	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
9	9	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
10	10	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
11	11	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
12	12	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
13	13	14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
14		14	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
			A	B	C	D	E	F	G	H	I	J	K	L	M	N	O

Quicksort: Practical Improvements

Insertion sort small subarrays.

- Even quicksort has too much overhead for tiny subarrays.
- Cutoff to insertion sort for ≈ 10 items.
- Note: could delay insertion sort until one pass at end.

```
private static void sort(Comparable[] a, int lo, int hi)
{
    if (hi <= lo + CUTOFF - 1)
    {
        Insertion.sort(a, lo, hi);
        return;
    }
    int j = partition(a, lo, hi);
    sort(a, lo, j-1);
    sort(a, j+1, hi);
}
```

Quicksort: Summary of Performance Characteristics

Worst case. Number of compares is quadratic.

- $N + (N - 1) + (N - 2) + \ldots + 1 \sim \frac{1}{2} N^2$.
- More likely that your computer is struck by lightning bolt.

Average case. Number of compares is $\sim 1.39 N \lg N$.

- 39% more compares than mergesort.
- But faster than mergesort in practice because of less data movement.

Random shuffle.

- Probabilistic guarantee against worst case.
- Basis for math model that can be validated with experiments.

Mergesort Algorithm

Basic plan.

1. Divide array into two halves.
2. Recursively sort each half.
3. Merge two halves.

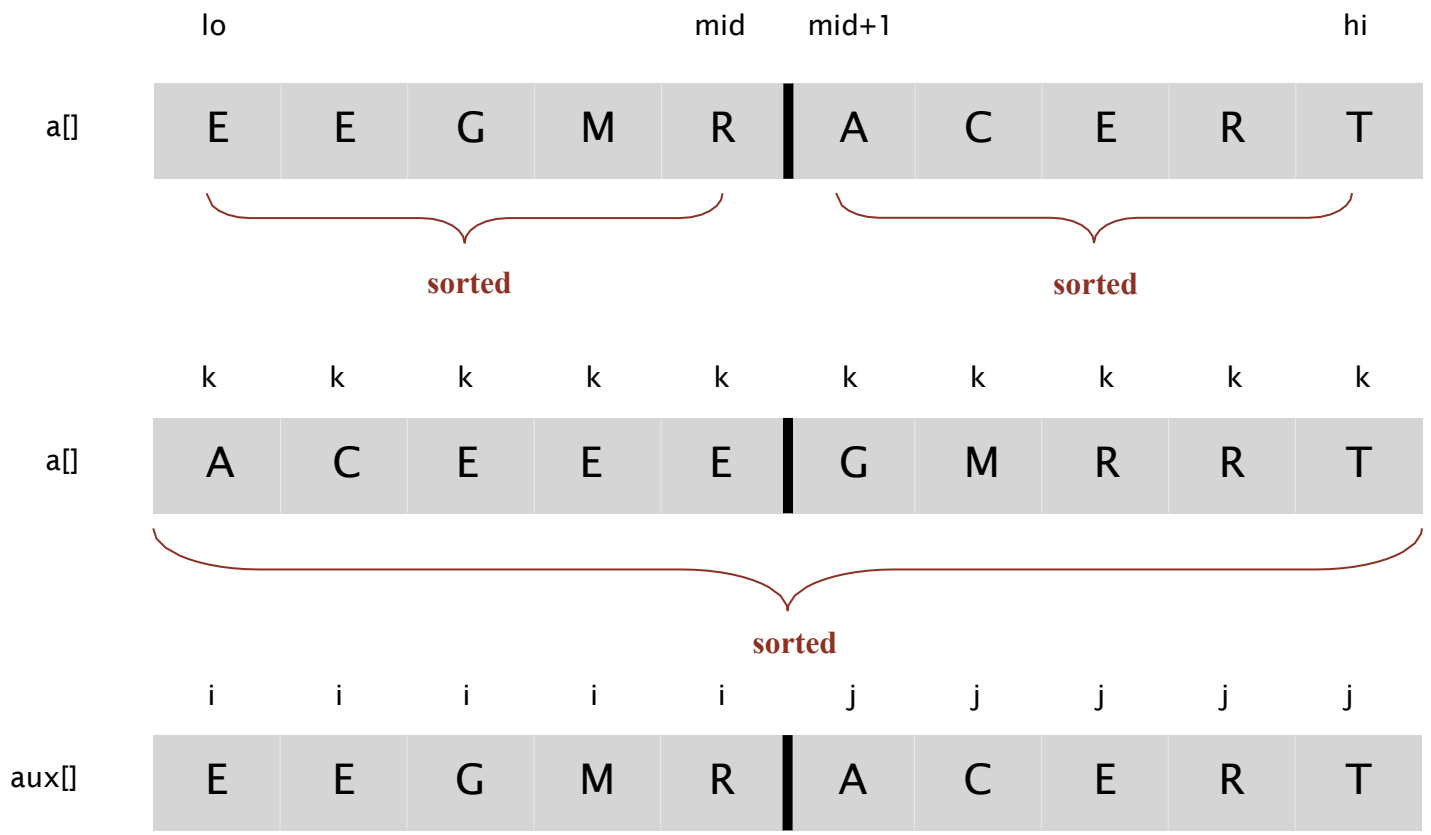
Mergesort overview

Merge Sort Algorithm: A Step-by-Step Visualization
https://www.youtube.com/watch?v=ho05egqcPl4

Merge Sort vs Quick Sort
https://www.youtube.com/watch?v=es2T6KY45cA

Merge Operation

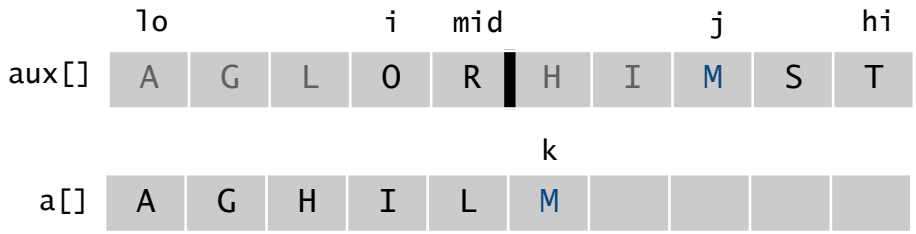
Goal. Given two sorted subarrays a[lo] to a[mid] and a[mid+1] to a[hi], replace with sorted subarray a[lo] to a[hi].



one subarray exhausted, take from other

Merge Operation: Java Implementation

```
private static void merge(Comparable[] a, Comparable[] aux, int lo, int mid, int hi)
{
   assert isSorted(a, lo, mid);    // precondition: a[lo..mid]   sorted
   assert isSorted(a, mid+1, hi);  // precondition: a[mid+1..hi] sorted

   for (int k = lo; k <= hi; k++)                                          copy
      aux[k] = a[k];

   int i = lo, j = mid+1;
   for (int k = lo; k <= hi; k++)                                          merge
   {
      if      (i > mid)              a[k] = aux[j++];
      else if (j > hi)               a[k] = aux[i++];
      else if (less(aux[j], aux[i])) a[k] = aux[j++];
      else                           a[k] = aux[i++];
   }

   assert isSorted(a, lo, hi);     // postcondition: a[lo..hi] sorted
}
```



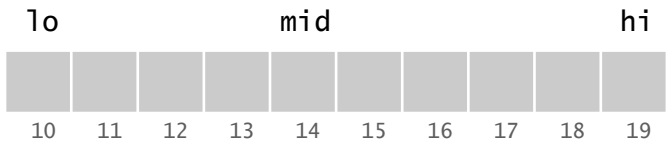
Can enable or disable at runtime.

⇒ No cost in production code.

```
java -ea MyProgram  // enable assertions
java -da MyProgram  // disable assertions (default)
```

Mergesort: Java implementation

```
public class Merge
{
   private static void merge(...)
   {  /* as before */  }

   private static void sort(Comparable[] a, Comparable[] aux, int lo, int hi)
   {
      if (hi <= lo) return;
      int mid = lo + (hi - lo) / 2;
      sort(a, aux, lo, mid);
      sort(a, aux, mid+1, hi);
      merge(a, aux, lo, mid, hi);
   }

   public static void sort(Comparable[] a)
   {
      aux = new Comparable[a.length];
      sort(a, aux, 0, a.length - 1);
   }
}
```



Mergesort: Trace

	a[] 0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
	M	E	R	G	E	S	O	R	T	E	X	A	M	P	L	E
merge(a, aux, 0, 0, 1)	E	M	R	G	E	S	O	R	T	E	X	A	M	P	L	E
merge(a, aux, 2, 2, 3)	E	M	G	R	E	S	O	R	T	E	X	A	M	P	L	E
merge(a, aux, 0, 1, 3)	E	G	M	R	E	S	O	R	T	E	X	A	M	P	L	E
merge(a, aux, 4, 4, 5)	E	G	M	R	E	S	O	R	T	E	X	A	M	P	L	E
merge(a, aux, 6, 6, 7)	E	G	M	R	E	S	O	R	T	E	X	A	M	P	L	E
merge(a, aux, 4, 5, 7)	E	G	M	R	E	O	R	S	T	E	X	A	M	P	L	E
merge(a, aux, 0, 3, 7)	E	E	G	M	O	R	R	S	T	E	X	A	M	P	L	E
merge(a, aux, 8, 8, 9)	E	E	G	M	O	R	R	S	E	T	X	A	M	P	L	E
merge(a, aux, 10, 10, 11)	E	E	G	M	O	R	R	S	E	T	A	X	M	P	L	E
merge(a, aux, 8, 9, 11)	E	E	G	M	O	R	R	S	A	E	T	X	M	P	L	E
merge(a, aux, 12, 12, 13)	E	E	G	M	O	R	R	S	A	E	T	X	M	P	L	E
merge(a, aux, 14, 14, 15)	E	E	G	M	O	R	R	S	A	E	T	X	M	P	E	L
merge(a, aux, 12, 13, 15)	E	E	G	M	O	R	R	S	A	E	T	X	E	L	M	P
merge(a, aux, 8, 11, 15)	E	E	G	M	O	R	R	S	A	E	E	L	M	P	T	X
merge(a, aux, 0, 7, 15)	A	E	E	E	E	G	L	M	M	O	P	R	R	S	T	X

lo

hi

result after recursive call

Mergesort: Practical Improvement

Use insertion sort for small subarrays.

- Mergesort has too much overhead for tiny subarrays.
- Cutoff to insertion sort for ≈ 7 items.

```
private static void sort(Comparable[] a, Comparable[] aux, int lo, int hi)
{
   if (hi <= lo + CUTOFF - 1)
   {
      Insertion.sort(a, lo, hi);
      return;
   }
   int mid = lo + (hi - lo) / 2;
   sort (a, aux, lo, mid);
   sort (a, aux, mid+1, hi);
   merge(a, aux, lo, mid, hi);
}
```

Mergesort: Practical Improvement

Stop if already sorted.

- Is biggest item in first half ≤ smallest item in second half?
- Helps for partially-ordered arrays.

```
A B C D E F G H I J M N O P Q R S T U V

A B C D E F G H I J M N O P Q R S T U V
```

```
private static void sort(Comparable[] a, Comparable[] aux, int lo, int hi)
{
   if (hi <= lo) return;
   int mid = lo + (hi - lo) / 2;
   sort (a, aux, lo, mid);
   sort (a, aux, mid+1, hi);
   if (!less(a[mid+1], a[mid])) return;
   merge(a, aux, lo, mid, hi);
}
```

Bottom-up Mergesort

Basic plan.

1. Pass through array, merging subarrays of size 1.
2. Repeat for subarrays of size 2, 4, 8, 16, ...

Simple and non-recursive version of mergesort. but about 10% slower than recursive, top-down mergesort on typical systems

	a[i]															
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
	M	E	R	G	E	S	O	R	T	E	X	A	M	P	L	E
sz = 1																
merge(a, aux, 0, 0, 1)	E	M	R	G	E	S	O	R	T	E	X	A	M	P	L	E
merge(a, aux, 2, 2, 3)	E	M	G	R	E	S	O	R	T	E	X	A	M	P	L	E
merge(a, aux, 4, 4, 5)	E	M	G	R	E	S	O	R	T	E	X	A	M	P	L	E
merge(a, aux, 6, 6, 7)	E	M	G	R	E	S	O	R	T	E	X	A	M	P	L	E
merge(a, aux, 8, 8, 9)	E	M	G	R	E	S	O	R	E	T	X	A	M	P	L	E
merge(a, aux, 10, 10, 11)	E	M	G	R	E	S	O	R	E	T	A	X	M	P	L	E
merge(a, aux, 12, 12, 13)	E	M	G	R	E	S	O	R	E	T	A	X	M	P	L	E
merge(a, aux, 14, 14, 15)	E	M	G	R	E	S	O	R	E	T	A	X	M	P	E	L
sz = 2																
merge(a, aux, 0, 1, 3)	E	G	M	R	E	S	O	R	E	T	A	X	M	P	E	L
merge(a, aux, 4, 5, 7)	E	G	M	R	E	O	R	S	E	T	A	X	M	P	E	L
merge(a, aux, 8, 9, 11)	E	G	M	R	E	O	R	S	A	E	T	X	M	P	E	L
merge(a, aux, 12, 13, 15)	E	G	M	R	E	O	R	S	A	E	T	X	E	L	M	P
sz = 4																
merge(a, aux, 0, 3, 7)	E	E	G	M	O	R	R	S	A	E	T	X	E	L	M	P
merge(a, aux, 8, 11, 15)	E	E	G	M	O	R	R	S	A	E	E	L	M	P	T	X
sz = 8																
merge(a, aux, 0, 7, 15)	A	E	E	E	E	G	L	M	M	O	P	R	R	S	T	X

Stability of Sorting Algorithms

A stable sort preserves the relative order of items with equal keys.

i	min	0	1	2
0	2	B_1	B_2	A
1	1	A	B_2	B_1
2	2	A	B_2	B_1
		A	B_2	B_1

selectsort is not stable

i	j	0	1	2	3
		B_1	C_1	C_2	A_1
1	3	B_1	C_1	C_2	A_1
1	3	B_1	A_1	C_2	C_1
0	1	A_1	B_1	C_2	C_1

quicksort is not stable

i	j	0	1	2	3	4
0	0	B_1	A_1	A_2	A_3	B_2
1	0	A_1	B_1	A_2	A_3	B_2
2	1	A_1	A_2	B_1	A_3	B_2
3	2	A_1	A_2	A_3	B_1	B_2
4	4	A_1	A_2	A_3	B_1	B_2
		A_1	A_2	A_3	B_1	B_2

insertsort is stable

0	1	2	3	4	5	6	7	8	9	10
A_1	A_2	A_3	B	D	A_4	A_5	C	E	F	G

mergesort is stable

0	1	2	3
B	A	C_1	C_2
B	C_2	C_1	A
C_1	C_2	B	A
C_2	A	B	C_1
B	A	C_2	C_1
A	B	C_2	C_1

heapsort is not stable

Summary

	inplace?	stable?	best	average	worst	remarks
selection	✔		$\frac{1}{2} n^2$	$\frac{1}{2} n^2$	$\frac{1}{2} n^2$	n exchanges
insertion	✔	✔	n	$\frac{1}{4} n^2$	$\frac{1}{2} n^2$	use for small n
merge		✔	$\frac{1}{2} n \lg n$	$n \lg n$	$n \lg n$	n log n guarantee; stable
quick	✔		$n \lg n$	$2 n \ln n$	$\frac{1}{2} n^2$	n log n probabilistic guarantee; fastest in practice
heap	✔		$3 n$	$2 n \lg n$	$2 n \lg n$	n log n guarantee; in-place
?	✔	✔	n	$n \lg n$	$n \lg n$	holy sorting grail

Additional Resources

- Additional sorting algorithms (not covered in exam)