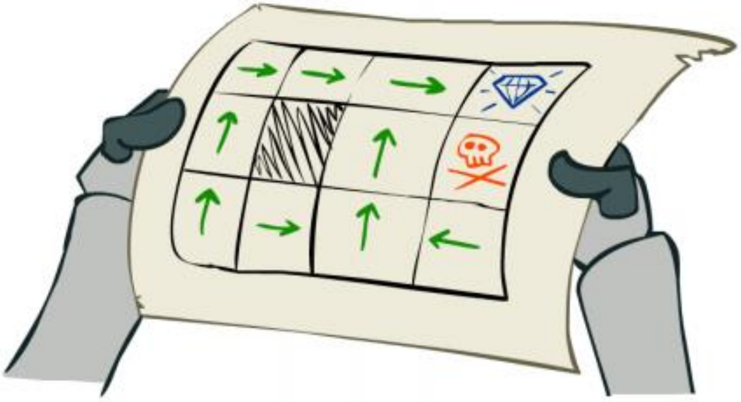
L7.1 MDP Planning



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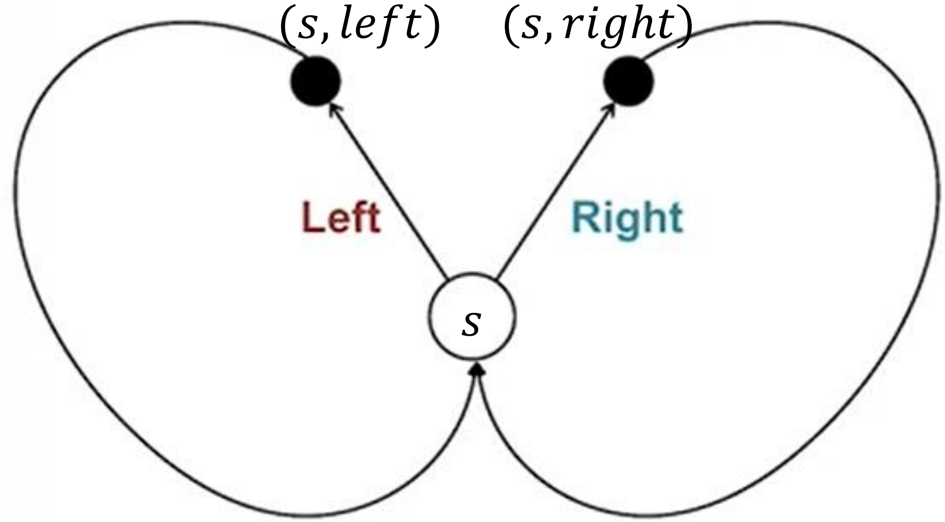
And textbook by Sutton and Barto http://incompleteideas.net/book/the-book-2nd.html

Markov Decision Process (MDP)

- An MDP consists of:
  - Set of states S
  - Start state s_0
  - Set of actions A
  - Transitions and rewards $p(r, s'|s, a)$ (w. discount γ)
- Policy maps from states to actions:
  - Deterministic policy $a = \pi(s)$ defines a deterministic action a for state s.
  - Stochastic policy $\pi(a|s)$ defines a probability distribution over possible actions a for state s.
- Markov means that next state only depends on current state
  - $P\left(S_{t+1} = s' \middle| S_t = s_t, A_t = a_t, S_{t-1} = s_{t-1}, A_{t-1} = a_{t-1,\dots,}S_0 = s_0, A_0 = a_0\right)$
  - $= P(S_{t+1} = s'|S_t = s_t, A_t = a_t)$
  - Given the present state, the future and the past are independent
  - e.g., for driving task, current vehicle position $\boldsymbol{x}$ as the state does not satisfy the Markov property, since the next state depends on not only $\boldsymbol{x}$, but also velocity $\dot{\boldsymbol{x}}$, acceleration $\ddot{\boldsymbol{x}}$, (assuming acceleration $\ddot{\boldsymbol{x}}$ stays constant within each step) If we redefine the state as vector $[\boldsymbol{x}, \dot{\boldsymbol{x}}, \ddot{\boldsymbol{x}}]^T$, then it satisfies the Markov property.
  - Or, current snapshot of front camera view can be used as the state (e.g., NVIDIA's PilotNet), but some works use past N video frames as the state to capture more dynamics (e.g., Waymo's ChauffeurNet).

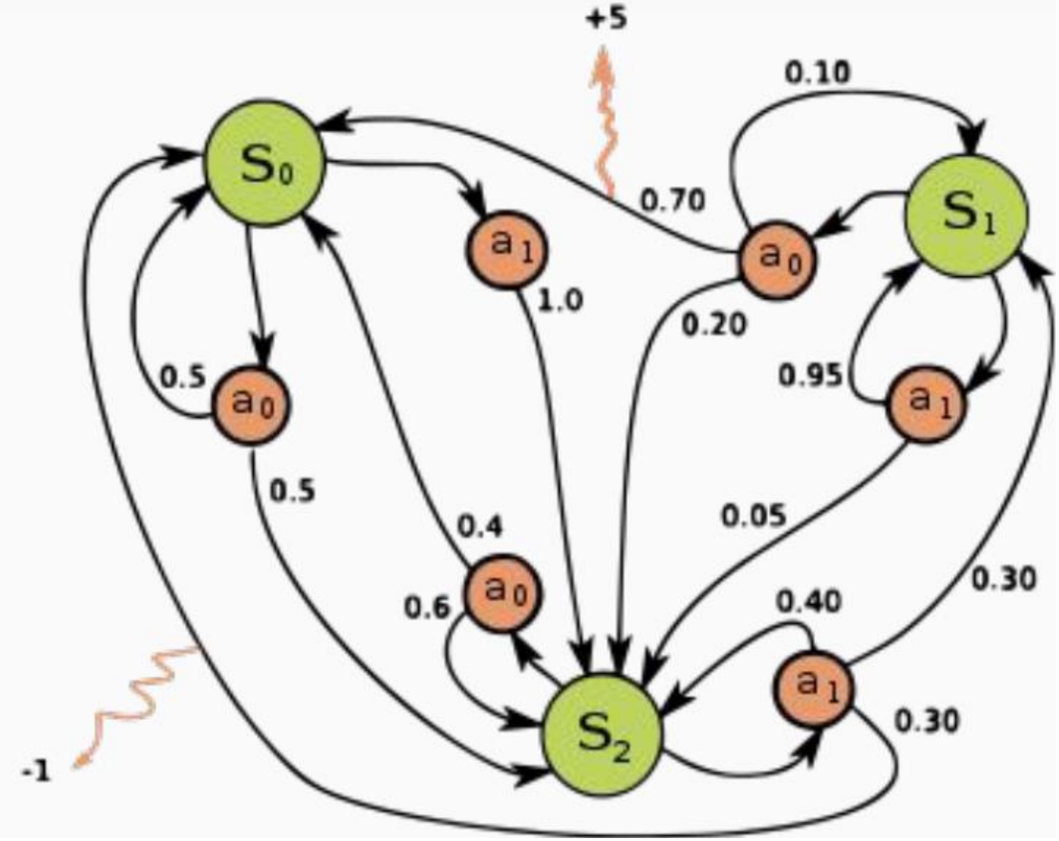
MDP Quiz

- For this MDP with a single state s and two possible actions $left$ and $right$. Are these valid policies?
  - 1) $\pi(left|s) = \pi(right|s) = 0.5$ (goes left or right with equal probability. uniform random policy)
  - 2) $\pi(left|s) = 1.0, \pi(right|s) = 0$ (always goes left)
  - 3) Alternating left and right, i.e., if previous action is left, then current action must be right, next action must be left, and so on.
  - ANS: 3) is not a valid policy, since it depends on the history of actions. To be a valid policy, the action must depend on the current state only
- We can redefine the MDP' extended state to include the last action as part of it, then 3) is a valid policy for the new MDP



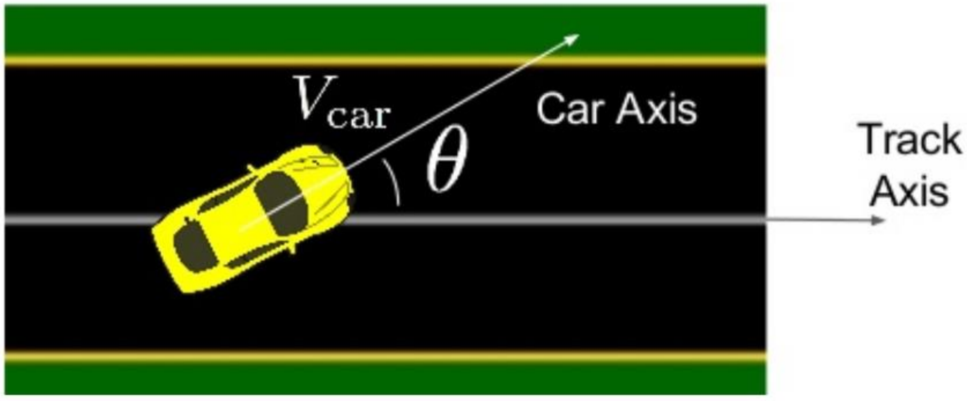
An Example MDP

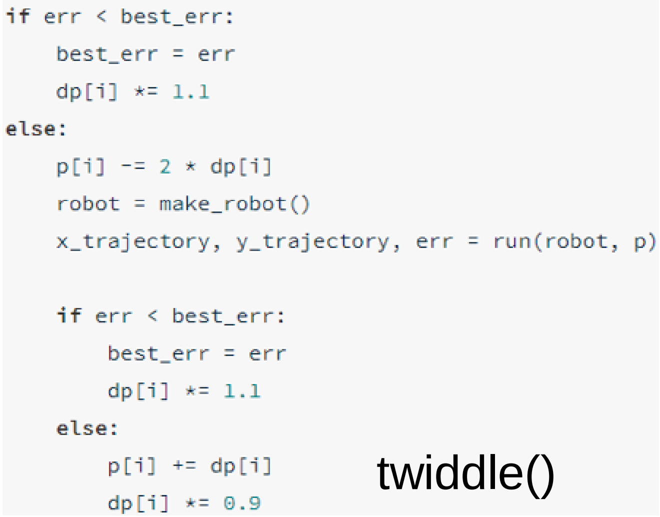
- Green nodes denote 3 states s_0, s_1, s_2; Red nodes denote 2 possible actions a_0, a_1 in each state. Each red node can also be denoted as (s, a).
- Agent taking action a in state s may get different reward r and next state s', denoted as state transition (s, a, r, s'), due to environment uncertainty (all rewards are 0 expect $+5$ and -1 show in fig).



RL Reward Function

- For the vehicle in left fig:
  - state: Pose of ego-car (x, y, θ) and environment map; action: Steering wheel/brake/acceleration
- Possible reward function: $R_t = w_1 V_{car} \cos\theta - w_2|cte|$
  - Weighted sum to maximum longitudinal velocity (first term), and minimize cross-track error (distance to lane center)
  - This is an example of dense reward (e.g., at every time step), as opposed to sparse reward (e.g., only at the end of each episode)
- Compare with twiddle() :
  - twiddle() can be viewed as an RL algorithm (policy gradient), that learns PID parameters with sparse reward (cost function is average cross-track error (cte), computed at the end of each simulation episode, as sum of squares of ctes for N timesteps divided by N.
  - It does not use the numeric value of cte, only its relative size (if err < best_err);
  - Cost function does not include heading angle θ;
  - if the track is very long and irregular, then we can make the reward denser, to adjust PID parameters every K timesteps instead of at the end of each episode.





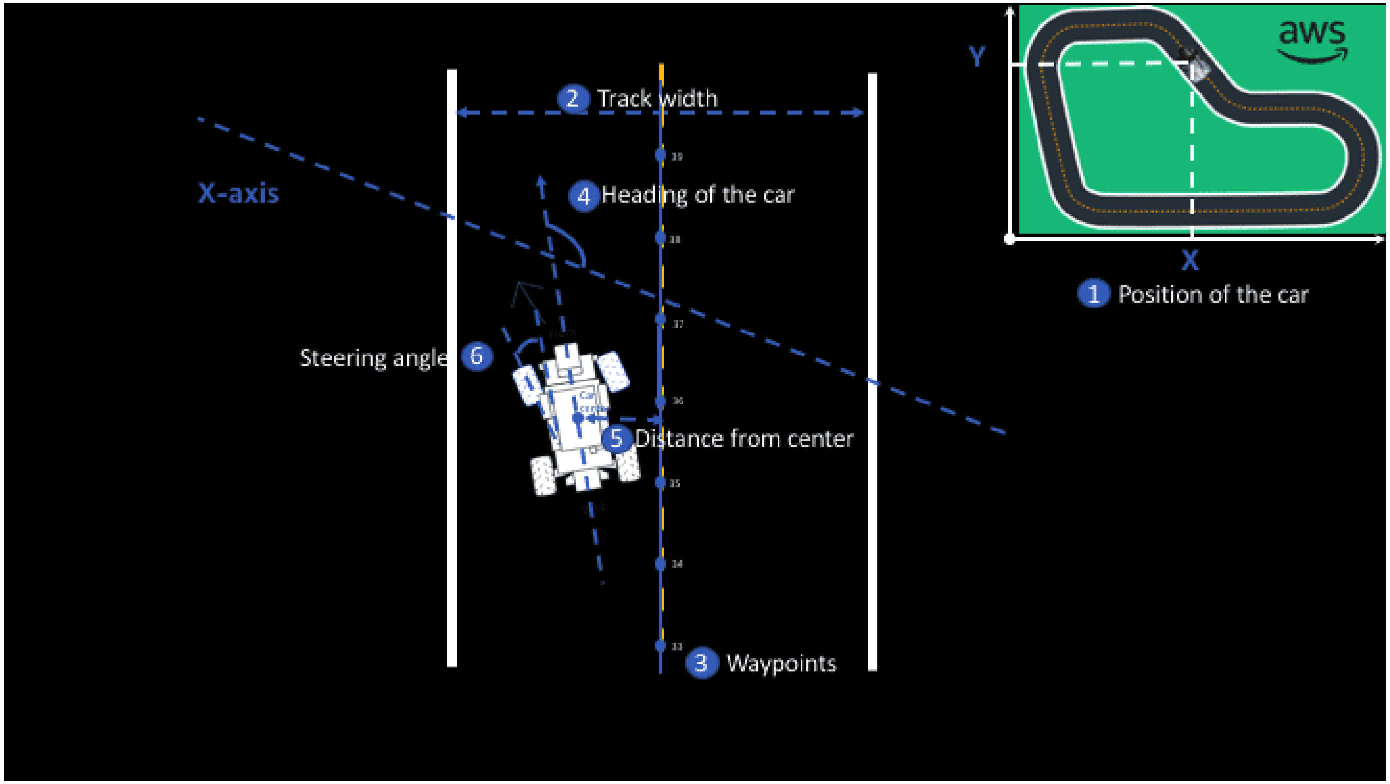
```
if err < best_err:
    best_err = err
    dp[i] *= 1.1
else:
    p[i] -= 2 * dp[i]
    robot = make_robot()
    x_trajectory, y_trajectory, err = run(robot, p)

    if err < best_err:
        best_err = err
        dp[i] *= 1.1
    else:
        p[i] += dp[i]
        dp[i] *= 0.9
```

twiddle()

Amazon DeepRacer

- Amazon Web Services (AWS) launched DeepRacer in 2018 for training AD algorithms with RL
  - https://aws.amazon.com/deepracer/
- You can train RL algorithm in the simulator on AWS cloud, but it costs money after some free time.
- They hold competitions, both online and in real-world. 1/10th scale race car costs USD $349.

Params for Writing Reward Function



- all_wheels_on_track
- x
- y
- distance_from_center
- is_left_of_center
- is_reversed
- heading
- progress
- steps
- speed
- steering_angle
- track_width
- waypoints
- closest_waypoints

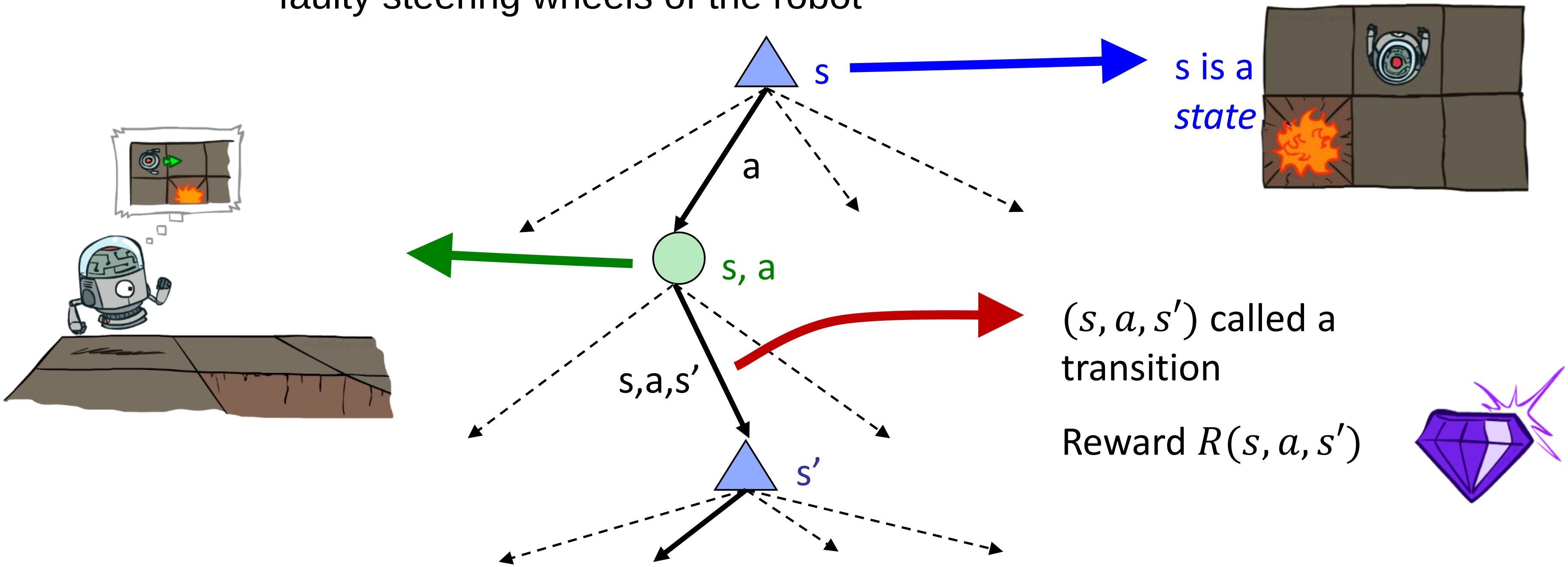
Example Reward Function

```
def reward_function(params):
    '''Example of penalize steering, which helps mitigate zig-zag behaviors'''
    # Read input parameters
    distance_from_center = params['distance_from_center']
    track_width = params['track_width']
    steering = abs(params['steering_angle']) # Only need the absolute steering angle

    # Calculate 3 markers that are at varying distances away from the center line
    marker_1 = 0.1 * track_width
    marker_2 = 0.25 * track_width
    marker_3 = 0.5 * track_width

    # Give higher reward if the agent is closer to center line and vice versa
    if distance_from_center <= marker_1:
        reward = 1
    elif distance_from_center <= marker_2:
        reward = 0.5
    elif distance_from_center <= marker_3:
        reward = 0.1
    else:
        reward = 1e-3  # likely crashed/ close to off track

    # Steering penalty threshold, change the number based on your action space setting
    ABS_STEERING_THRESHOLD = 15

    # Penalize reward if the agent is steering too much
    if steering > ABS_STEERING_THRESHOLD:
        reward *= 0.8

    return float(reward)
```

- A more realistic and complex reward function: https://www.middleware-solutions.fr/2019/08/14/an-introduction-to-aws-deepracer

MDP Search Tree

- Each MDP state s projects a search tree starting from it
- Both policy and environment may be stochastic
  - Policy $\pi(a|s)$: probability distribution over possible actions a from state s
    - e.g., different actions may be taken for the same state
  - Environment $p(r, s'|s, a)$: if the agent takes action a in state s, env gives probability distribution over next state s' and reward r
    - e.g., due to non-determinism in the environment (sudden strong wind), or faulty steering wheels of the robot



s is a *state*

(s, a, s') called a transition

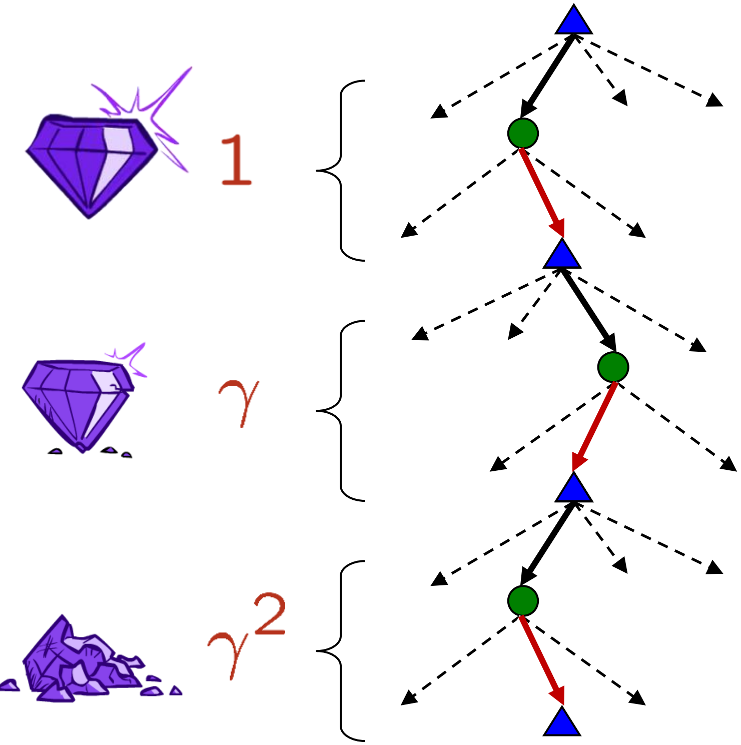
Reward $R(s, a, s')$

Preventing Infinite Rewards

- Problem: What if the game lasts forever? Do we get infinite rewards? No. Possible solutions:
- **Finite horizon**: (limit search tree depth)
  - Terminate episodes after a fixed T timesteps
- **Discount factor**: $0 < \gamma \leq 1$
  - Think of it as a $1 - \gamma$ chance of ending the episode at every step. Effective horizon (expected episode length): $\sum_{t=0}^{\infty} \gamma^t = \frac{1}{1-\gamma}$
  - Smaller γ leads to shorter horizon, and preference of short-term to long-term rewards, and vice versa
- (Can have both finite horizon and discount factor)
- **Absorbing state**: guarantee that for every policy, a terminal state will eventually be reached

Discount Factor Example

- Each time we descend a level in the search tree, we multiply in the discount once
- Example: $\gamma = 0.5$
  - $U([1,2,3]) = 1 * 1 + 0.5 * 2 + 0.25 * 3 < U([3,2,1]) = 1 * 3 + 0.5 * 2 + 0.25 * 1$



Discounting Example

- Given:
  - Actions: East, West, and Exit (only available in exit states a, e)
  - Transitions: deterministic
- For $\gamma = 1$, optimal policy in each state is always moving West
  - From state d, reward of going West is $0 + \gamma \cdot 0 + \gamma^2 \cdot 0 + \gamma^3 \cdot 10 = 10$, larger than reward of going East $0 + \gamma \cdot 1 = 1$
- For $\gamma = 0.1$, optimal policy in each state is shown below
  - From state d, reward from going West is $0 + \gamma \cdot 0 + \gamma^2 \cdot 0 + \gamma^3 \cdot 10 = 0.01$, less than reward from going East $0 + \gamma \cdot 1 = 0.1$.
- For which γ are West and East equally good when in state d?
  - $\gamma^3 \cdot 10 = \gamma \cdot 1 \Longrightarrow \gamma = \frac{1}{\sqrt{10}} \approx .32$

10	←	←	←	1
a	b	c	d	e

$\gamma = 1$

10	←	←	→	1
a	b	c	d	e

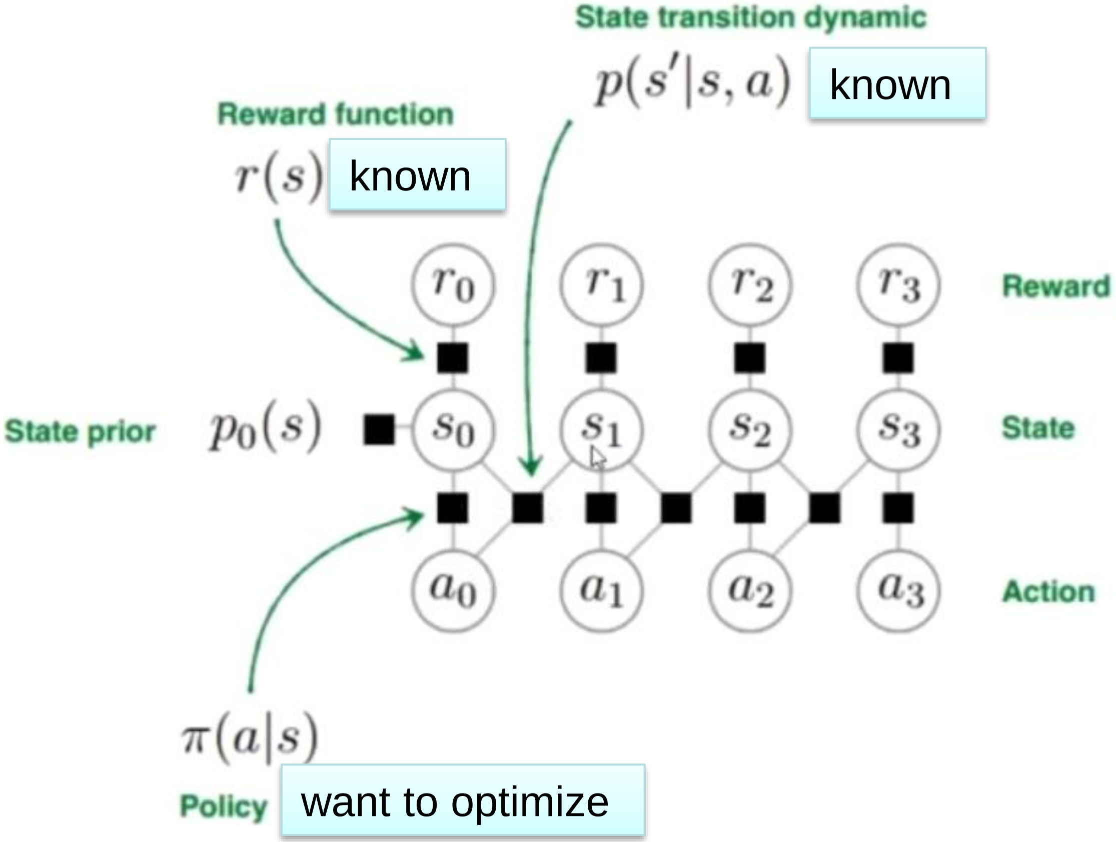
$\gamma = 0.1$

10	←	←	⇆	1
a	b	c	d	e

$\gamma = \frac{1}{\sqrt{10}}$

Known MDP

- In this lecture, we assume known MDP, and use dynamic programming to solve Bellman Equations and find the optimal policy (no learning here).



known

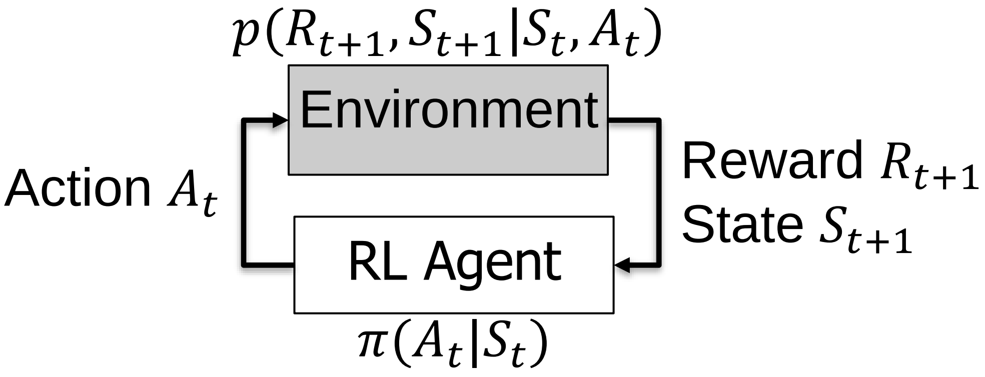
known

want to optimize

Important

Formal Definition of MDP

- Return (cumulative discounted reward) at time t: $G_t \doteq R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \cdots + \gamma^{T-1} R_T = \sum_{k=0}^{T-t-1} \gamma^k R_{t+k+1}$
  - At each step $t \in [0, T-1]$, agent takes an action A_t in state S_t; at step $t+1$, agent receives a reward R_{t+1} and transitions into the next state S_{t+1} with the trace $(S_t, A_t, R_{t+1}, S_{t+1})$
  - We assume episodic tasks, and this specific episode has length of T steps. ($T = \infty$ for continuous tasks)
- State Value Function: expected return under policy π: $v_\pi(s) \doteq \mathbb{E}_\pi[G_t | S_t = s]$
- Action Value Function: expected return from taking action a, then follow policy π: $q_\pi(s,a) \doteq \mathbb{E}_\pi[G_t | S_t = s, A_t = a]$
- The RL problem: find the optimal policy $\pi(a|s)$ that maximizes the expected return from each state (the state VF)



Important

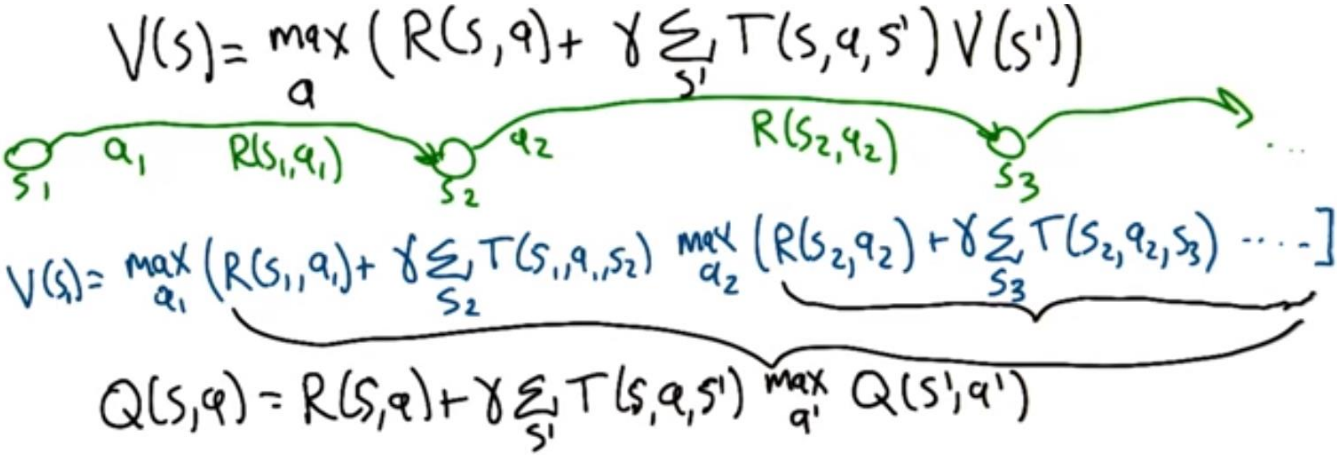
Bellman Expectation Equations

- Bellman Expectation Equation for State Value Function:
- $v_\pi(s) = \sum_a \pi(a|s) \sum_{r,s'} p(r,s'|s,a) \, [r + \gamma v_\pi(s')]$
  - Expected value starting from state s and following policy π.
- Bellman Expectation Equation for Action Value Function
- $q_\pi(s,a) = \sum_{r,s'} p(r,s'|s,a) \, [r$
  $+ \gamma \sum_{a'} \pi(a'|s') \, q_\pi(s',a')]$
  - Expected value starting from state s, taking action a, and thereafter following policy π.

Important

Bellman Optimality Equations

- Bellman Optimality Equation for State Value Function:
- $v_*(s) = \max_a \sum_{r,s'} p(r,s'|s,a)\,[r + \gamma v_*(s')]$
  - Max value starting from state s and following the greedy policy $\pi(s) = \operatorname{argmax}_a q_*(s,a)$
- Bellman Optimality Equation for Action Value Function
- $q_*(s,a) = \sum_{r,s'} p(r,s'|s,a)\left[r + \gamma \max_{a'} q_*(s',a')\right]$
  - Max value starting from state s, taking action a, and thereafter following the greedy policy $\pi(s) = \operatorname{argmax}_a q_*(s,a)$



$$V(s) = \max_a \left(R(s,a) + \gamma \sum_{s'} T(s,a,s')\,V(s')\right)$$

$$V(s_1) = \max_{a_1} \underbrace{\left(R(s_1,a_1) + \gamma \sum_{s_2} T(s_1,a_1,s_2) \max_{a_2} \left(\underbrace{R(s_2,a_2) + \gamma \sum_{s_3} T(s_2,a_2,s_3) \cdots}\right]\right.}$$

$$Q(s,a) = R(s,a) + \gamma \sum_{s'} T(s,a,s') \max_{a'} Q(s',a')$$

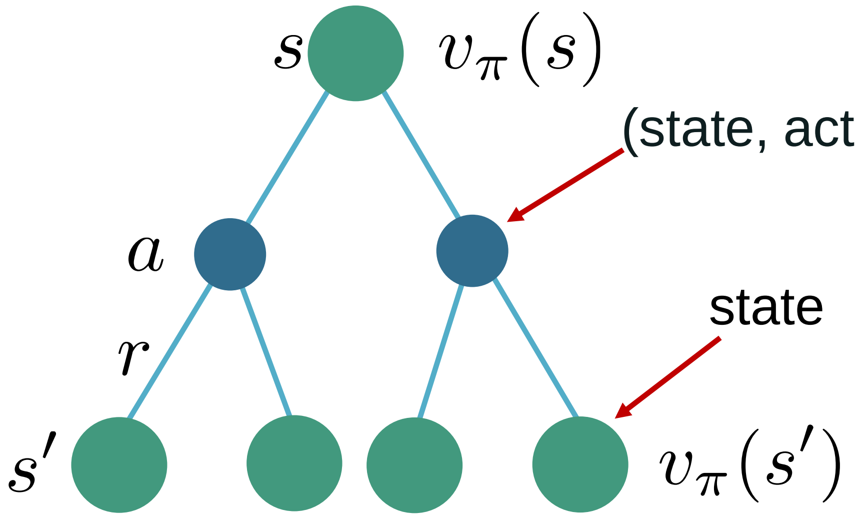
- Notations in left fig:
- $\sum_{s'} T(s,a,s')\,[\ldots]$ $= \sum_{r,s'} p(r,s'|s,a)\,[..$
- $R(s,a)$ $= \sum_{r,s'} p(r,s'|s,a)\,r$

Bellman Equations written with Expectation Symbols

- Bellman Exp Equations:
- $v_\pi(s) = \mathbb{E}_a \mathbb{E}_{r,\,s'}[r + \gamma v_\pi(s')]$
- $q_\pi(s,a) = \mathbb{E}_{r,\,s'}[r + \gamma \mathbb{E}_a q_\pi(s,a)]$
- Bellman Opt Equations:
- $v_*(s) = \max_a \mathbb{E}_{r,\,s'}[r + \gamma v_*(s')]$
- $q_*(s,a) = \mathbb{E}_{r,\,s'}\left[r + \gamma \max_a q_*(s,a)\right]$
- Detailed derivations:
  - $v_\pi(s) = \sum_a \pi(a|s)\, q_\pi(s,a) = \mathbb{E}_{a\sim\pi(a|s)} q_\pi(s,a)$
    $= \mathbb{E}_{a\sim\pi(a|s)}\, \mathbb{E}_{r,\,s'\sim p(r,s'|s,a)}[r + \gamma v_\pi(s')]$
  - $q_\pi(s,a) = \sum_{r,s'} p(r,s'|s,a)\,[r + \gamma v_\pi(s')] = \mathbb{E}_{r,\,s'\sim p(r,s'|s,a)}[r + \gamma v_\pi(s')]$
    $= \mathbb{E}_{r,\,s'\sim p(r,s'|s,a)}\left[r + \gamma \mathbb{E}_{a\sim\pi(a|s)}\, q_\pi(s,a)\right]$
  - $v_*(s) = \max_a q_*(s,a) = \max_a \mathbb{E}_{r,\,s'\sim p(r,s'|s,a)}[r + \gamma v_*(s')]$
  - $q_*(s,a) = \sum_{r,s'} p(r,s'|s,a)\,[r + \gamma v_*(s')] = \mathbb{E}_{r,\,s'\sim p(r,s'|s,a)}[r + \gamma v_*(s')]$

Important

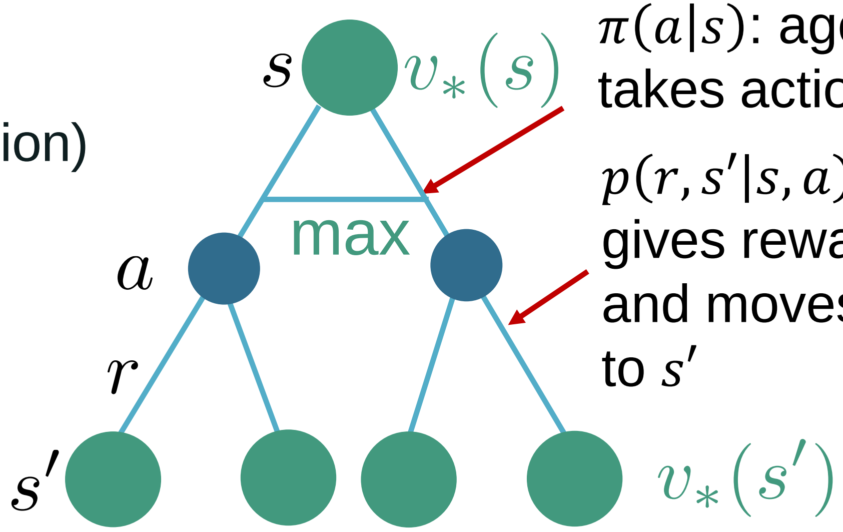
Backup Diagrams



(state, action)

state

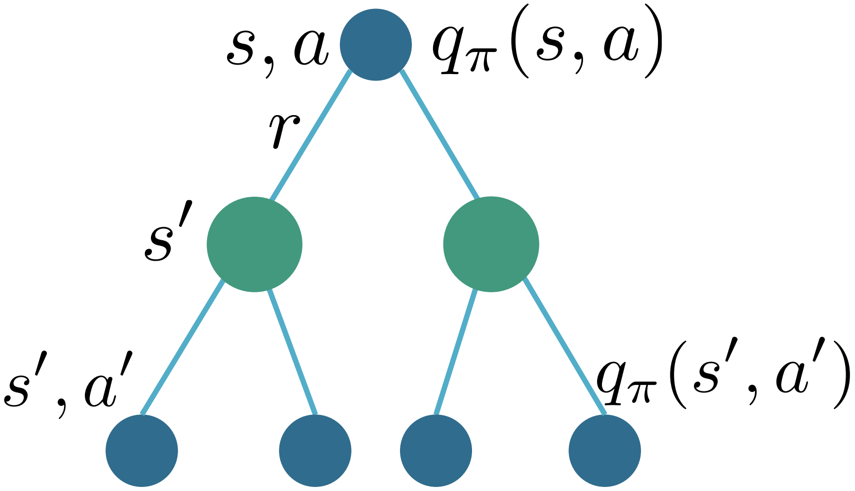
Bellman Exp Eqn for $v_\pi(s)$



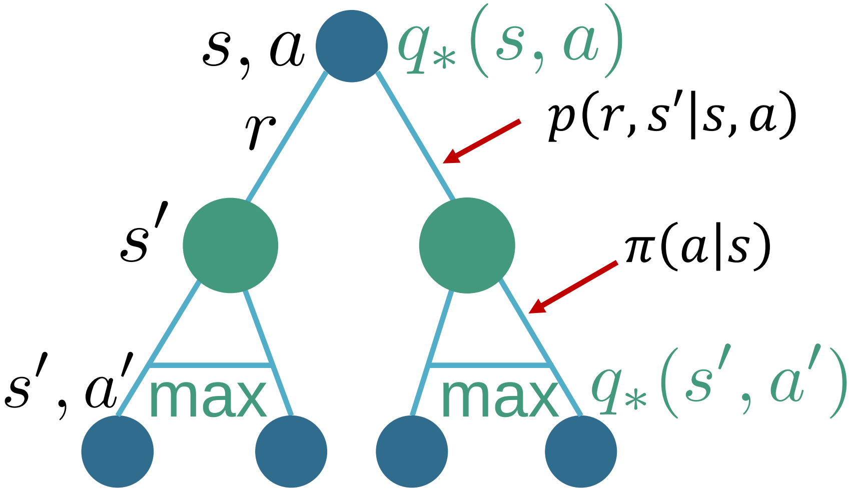
$\pi(a|s)$: agent takes action a

$p(r, s'|s, a)$: env gives reward r and moves agent to s'

Bellman Opt Eqn for $v_*(s)$



Bellman Exp Eqn for $q_\pi(s, a)$



$p(r, s'|s, a)$

$\pi(a|s)$

Bellman Opt Eqn for $q_*(s, a)$

Important

$v(s)$ vs. $q(s,a)$

- State-action Value Function $q(s,a)$ contains more information than State value function $v(s)$. Given $q_*(s,a)$, optimal policy $\pi_*(\mathrm{s}) = \operatorname{argmax}_a q_*(s,a)$.
- Can always go from $q_\pi(s,a)$ to $v_\pi(s)$, or from $q_*(s,a)$ to $v_*(s)$:
  - $v_\pi(s) = \sum_a \pi(a|s)\, q_\pi(s,a); v_*(s) = \max_a q_*(s,a)$
- With known MDP ($p(r,s'|s,a)$, i.e., model-based): can go from $v_\pi(s)$ to $q_\pi(s,a)$, or from $v_*(s)$ to $q_*(s,a)$:
  - $q_\pi(s,a) = \sum_{r,s'} p(r,s'|s,a)\,[r + \gamma v_\pi(s')]$
  - $q_*(s,a) = \sum_{r,s'} p(r,s'|s,a)\,[r + \gamma v_*(s')]$
- With unknown MDP (unknown $p(r,s'|s,a)$, i.e., model-free) : cannot go from $v_\pi(s)$ to $q_\pi(s,a)$, or from $v_*(s)$ to $q_*(s,a)$

Important

Simplified Bellman Equations for Deterministic Env

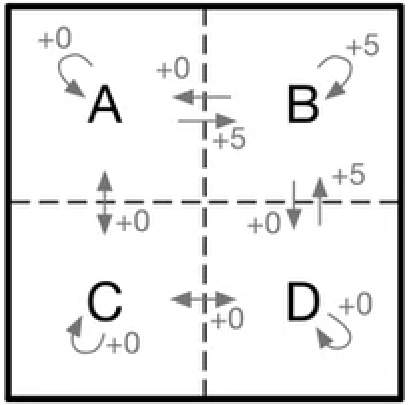
- Bellman Equations:
  - $v_\pi(s) = \sum_a \pi(a|s) q_\pi(s,a)$
  - $q_\pi(s,a) = \sum_{r,s'} p(r,s'|s,a)\,[r + \gamma v_\pi(s')]$
  - $v_*(s) = \max_a q_*(s,a)$
  - $q_*(s,a) = \sum_{r,s'} p(r,s'|s,a)\,[r + \gamma v_*(s')]$
- For Deterministic Env: there is only one possible (r,s') for a given (s,a) (we use R_s^a to emphasize that reward r is specific to this (s,a)):
  - $q_\pi(s,a) = R_s^a + \gamma v_\pi(s')$
  - $q_*(s,a) = R_s^a + \gamma v_*(s')$

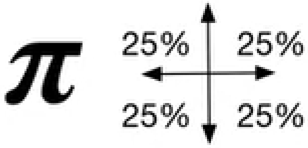
Policy Evaluation

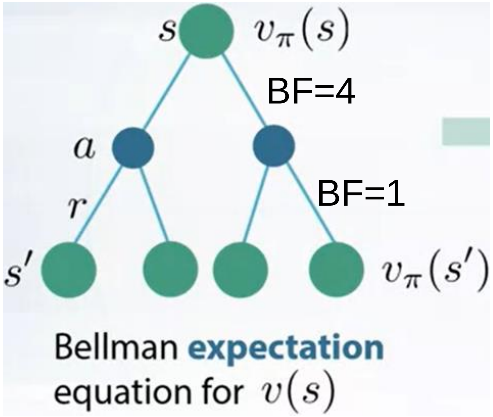
- The prediction problem: predict Value Function for given policy π by solving Bellman Exp. Equation for State Value Function
  - $v_\pi(s) = \sum_a \pi(a|s) \sum_{r,s'} p(r, s'|s, a) \left[r + \gamma v_\pi(s')\right]$
    $= \mathbb{E}_\pi[R_{t+1} + \gamma v_\pi(S_{t+1})|S_t = s]$
- Can also be written as:
  - $v_\pi(s) = \sum_a \pi(a|s) q_\pi(s, a)$
  - $q_\pi(s, a) = \sum_{r,s'} p(r, s'|s, a) \left[r + \gamma v_\pi(s')\right]$ denotes the State-Action Value Function for taking action a in state s, then follow policy π afterwards
- A set of linear equations that can be solved analytically for small system
  - # unknowns = # equations = # states

Grid World1: Policy Evaluation

- Non-episodic MDP w. deterministic env: Agent in state $s \in \{A, B, C, D\}$ taking action $a \in \{l, r, u, d\}$ always moves to the next state in the movement direction, unless it is blocked by the walls. Discount factor $\gamma = 0.7$.
- Random policy: Agent in state $s \in \{A, B, C, D\}$ takes a random action $a \in \{l, r, u, d\}$ with equal probability of 0.25 each.
- Bellman Exp. Equation for det env: $v_\pi(s) = \sum_a \pi(a|s) q_\pi(s, a)\,;\ q_\pi(s, a) = R_s^a + \gamma v_\pi(s')$
- $v_\pi(A) = 0.25(q_\pi(A, l) + q_\pi(A, r) + q_\pi(A, u) + q_\pi(A, d)) = 0.5 \cdot 0.7 v_\pi(A) + 0.25 \cdot \big(5 + 0.7 v_\pi(B)\big) + 0.25 \cdot 0.7 v_\pi(C)$
  - $q_\pi(A, l) = q_\pi(A, u) = 0 + 0.7 v_\pi(A)$
  - $q_\pi(A, r) = 5 + 0.7 v_\pi(B)$
  - $q_\pi(A, d) = 0 + 0.7 v_\pi(C)$
- $v_\pi(B) = 0.25(q_\pi(B, l) + q_\pi(B, r) + q_\pi(B, u) + q_\pi(B, d)) = 0.25 \cdot 0.7 v_\pi(A) + 0.5 \cdot \big(5 + 0.7 v_\pi(B)\big) + 0.25 \cdot 0.7 v_\pi(D)$
  - $q_\pi(B, l) = 0 + 0.7 v_\pi(A)$
  - $q_\pi(B, r) = q_\pi(A, u) = 5 + 0.7 v_\pi(B)$
  - $q_\pi(B, d) = 0 + 0.7 v_\pi(D)$
- $v_\pi(C) = 0.25(q_\pi(C, l) + q_\pi(C, r) + q_\pi(C, u) + q_\pi(C, d)) = 0.25 \cdot 0.7 v_\pi(A) + 0.5 \cdot 0.7 v_\pi(C) + 0.25 \cdot 0.7 v_\pi(D)$
  - $q_\pi(C, l) = q_\pi(C, d) = 0 + 0.7 v_\pi(C)$
  - $q_\pi(C, r) = 0 + 0.7 v_\pi(D)$
  - $q_\pi(C, u) = 0 + 0.7 v_\pi(A)$
- $v_\pi(D) = 0.25(q_\pi(D, l) + q_\pi(D, r) + q_\pi(D, u) + q_\pi(D, d)) = 0.25 \cdot (5 + 0.7 v_\pi(B)) + 0.25 \cdot 0.7 v_\pi(C) + 0.5 \cdot 0.7 v_\pi(D)$
  - $q_\pi(D, l) = 0 + 0.7 v_\pi(C)$
  - $q_\pi(D, r) = q_\pi(D, d) = 0 + 0.7 v_\pi(D)$
  - $q_\pi(D, u) = 5 + 0.7 v_\pi(B)$
- Solution: $v_\pi(A) = 4.2, v_\pi(B) = 6.1,\ v_\pi(C) = 2.2, v_\pi(D) = 4.2.$ $q_\pi(s, a)$ can also be obtained.







(BF: Branching Factor)