Lecture 10
Basic Graph Algorithms

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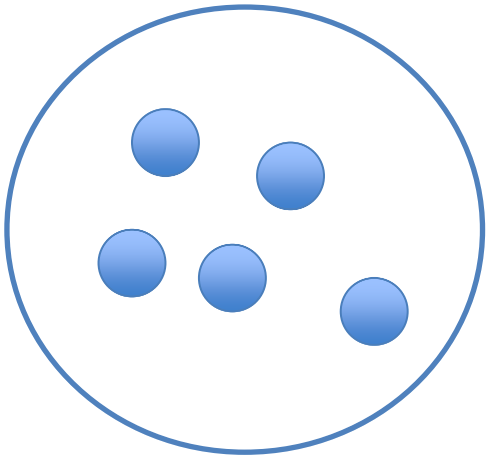
Lecture Goals

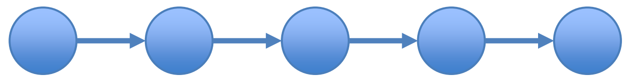
- Compare the Graph ADT with other ADTs
- Define basic notions associated with graphs
- Write classes in Java to implement graphs
- Implement graphs in Java using an adjacency matrix representation and an adjacency list representation
- Implement a method to find the neighbors of a vertex in two ways.
- We introduce two classic algorithms for searching a graph—depth-first search and breadth-first search.
- We also consider the problem of computing connected components and conclude with related problems and applications.
- we introduce a depth-first search based algorithm for computing the topological order of an acyclic digraph.

ADT of Graph



Unstructured structures

Sets

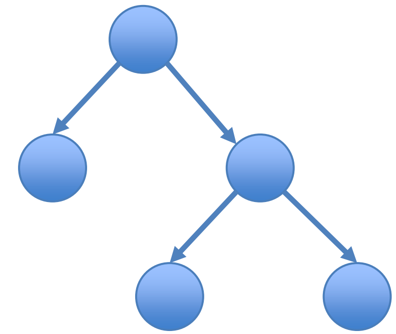


Sequential, linear structures

Arrays, linked lists

Useful for

- iterating over all elements,
- accessing via index



Hierarchical structures

Trees

Can indicate common structure in key

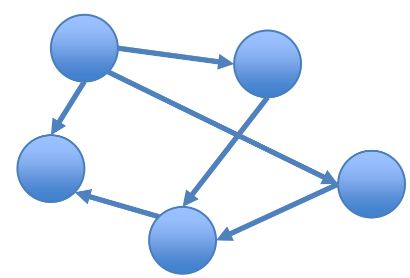
- for example, the prefix in tire

Principle: Basic objects & Relationships between them

Graph is a generalization of this principle

Basic objects: vertices, nodes

Relationships between them: edges, arcs, links



Examples of Graphs

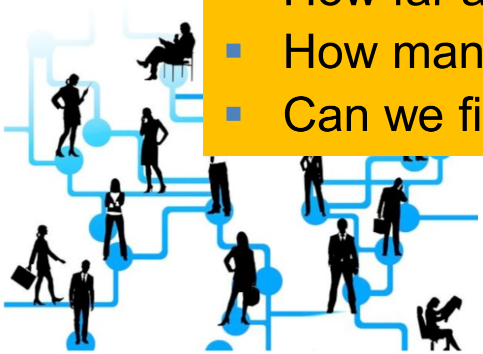
Some general questions related to graphs:

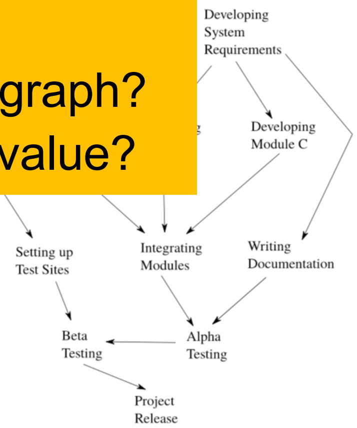
- How to create a graph?
- Are two vertices adjacent?
- Is the graph dense? sparse?
- How far are two vertices in the graph?
- How many components are there in the graph?
- Can we find a vertex with particular key value?

Basic objects: we

Relationships bet

hem: nonstop flights





Basic objects: people

Relationships between them: friends

Basic objects: tasks

Relationships between them: dependencies

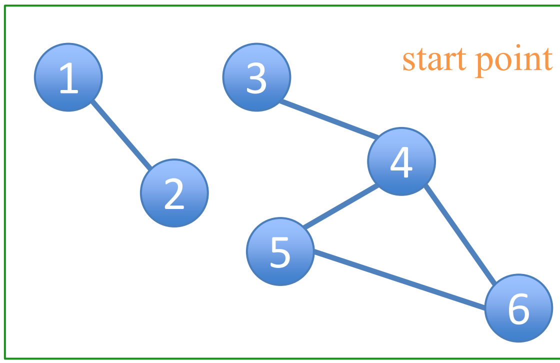
Graph Definitions

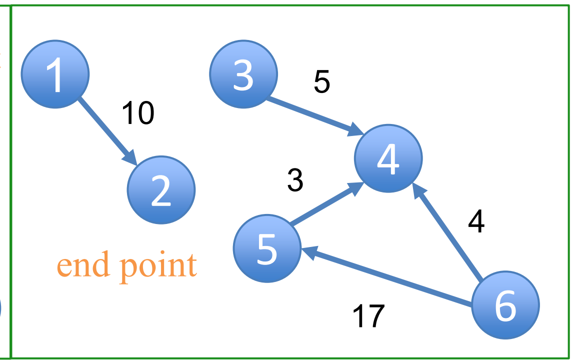
Basic objects: vertices, nodes $\boxed{V}$

Relationships between them: edges, arcs, links $\boxed{E}$

have a huge impact on the performance

Size of graph: $|V| + |E|$





Undirected

edges are symmetric

Directed

Weighted

cost

Neighbor: u is a neighbor of v if:

there is an edge from u to v

OR

there is an edge from v to u

What are the neighbors of the vertex 4?

A. 3,4,5,6
B. 3,5,6
C. 3,6
D. 5

Path: sequence of vertices and edges that depicts hopping along graph

For which pair of vertices is there a path in the graph starting at the first and ending at the second?

A. vertex 1 and vertex 3
B. vertex 4 and vertex 6
C. vertex 6 and vertex 5

What's the maximum number of edges in a directed and undirected graph with n vertices?

- $n*(n-1)$ e th $n*(n-1)/2$ elf-loops (i.e. edges from a node back to itself).
- Assume there there is at most one edge from a given start vertex to a given end vertex.

Implementing Graphs in Java

Basic objects: vertices, nodes ← Label by integers

Relationships between them: edges, arcs, links

```
public abstract class Graph {
        private int numVertices;
        private int numEdges;

        public Graph() {
                numVertices = numEdges = 0;
        }

        public int getNumVertices() {
                return numVertices;
        }

        public int getNumEdges() {
                return numEdges;
        }

        public void addVertex() {
                implementAddVertex();
                numVertices++;
        }

        public abstract void implementAddVertex();

        public abstract List<Integer> getNeighbors(int v);
}
```

size of a graph

For example, which cities we can reach with nonstop flight?

data associated with any graph

methods that ought to be available with any graph.

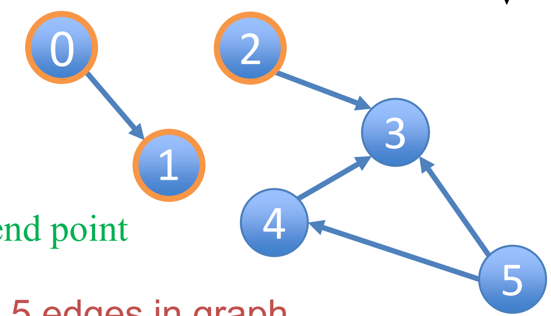
leave implementation of key functionalities to subclasses

Graph Representation: Adjacency Matrix

relabel vertices starting at 0 to use as index

start point

$V = \{0, 1, 2, 3, 4, 5\}$



end point

5 edges in graph

→ 5 nonzero entries in adjacency matrix

Column index

array entry > 1:
- multiple edges,
- or weighted edges

row index

Symmetric entry

	0	1	2	3	4	5
0	0	1	0	0	0	0
1	0	0	0	0	0	0
2	0	0	0	1	0	0
3	0	0	0	0	0	0
4	0	0	0	1	0	0
5	0	0	0	1	1	0

The grid (2-d array) is indexed by the vertices labels and stores information in a particular location based on whether these two vertices have an edge between them or not

```
public class GraphAdjMatrix extends Graph {
  private int[][] adjMatrix;

  public void implementAddEdge(int v, int w) {
    adjMatrix[v][w] = 1;
  }
  public void implementAddVertex() {
    int v = getNumVertices();
    if (v >= adjMatrix.length) {
        int[][] newAdjMatrix = new int[v * 2][v * 2];
      for (int i = 0; i < adjMatrix.length; i++) {
          for (int j = 0; j < adjMatrix.length; j++) {
              newAdjMatrix[i][j] = adjMatrix[i][j];
          }
      }
      adjMatrix = newAdjMatrix;
    }
}}
```

v*2 instead of v+1 to amortize cost of adding new vertices in the future.

expand the 2-d array

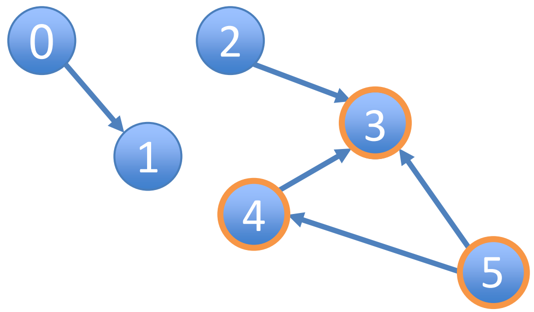
Graph Implementations
https://www.youtube.com/watch?v=2guA5uMEmZQ

How long does it take to test whether there is an edge between vertex v and vertex w in the graph?

O(1)

- Algebraic representation of graph structure.
- Fast to test for edges.
- Fast to add/remove edges.
- Slow to add/remove vertices.
- Requires a lot of memory. sparse

Graph Representation: Adjacency List



Neighbour can be reached by one hop

$0 \rightarrow \{1\}$
$1 \rightarrow$ null
$2 \rightarrow \{3\}$
$3 \rightarrow$ null
$4 \rightarrow \{3\}$
$5 \rightarrow \{3, 4\}$

- Easy to add vertices.
- Easy to add/remove edges.
- May use a lot less memory than adjacency matrices.

Motivation for new representation:
- want to avoid storing information on edges that aren't in the graph
- Edges connect a vertex to its neighbors

- Sparse graph: O(1) edges for each vertex
- most applications use sparse graphs

Is it also fast?

```
public class GraphAdjList extends Graph {
	private Map<Integer, ArrayList<Integer>> adjListsMap;
			vertex → {neighbors}

	public void implementAddVertex() {
		int v = getNumVertices();
		ArrayList<Integer> neighbors = new ArrayList<Integ
		adjListsMap.put(v, neighbors);
	}

	public void implementAddEdge(int v, int w) {
		(adjListsMap.get(v)).add(w);
	}
}
```

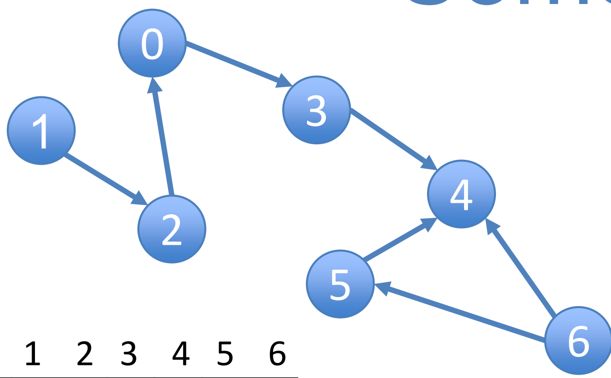
```
public class ArrayList<E>
extends AbstractList<E>
implements List<E>, RandomAccess, Cloneable, Serializable
```

Resizable-array implementation of the `List` interface. Implements all optional list operations, and permits all elements, including `null`. In addition to implementing the `List` interface, this class provides methods to manipulate the size of the array that is used internally to store the list. (This class is roughly equivalent to `Vector`, except that it is unsynchronized.)

The `size`, `isEmpty`, `get`, `set`, `iterator`, and `listIterator` operations run in constant time. The `add` operation runs in *amortized constant time*, that is, adding n elements requires O(n) time. All of the other operations run in linear time (roughly speaking). The constant factor is low compared to that for the

Yes. Operations are all O(1)

Some Practices



How much storage is required to represent a graph as a **matrix**? (Big-O, Tightest Bound)

A. |V| B. |E|
C. |V|+|E| D. $|V|^2$
E. $|E|^2$

	0	1	2	3	4	5	6
0	0	0	0	1	0	0	0
1	0	0	1	0	0	0	0
2	1	0	0	0	0	0	0
3	0	0	0	0	1	0	0
4	0	0	0	0	0	0	0
5	0	0	0	0	1	0	0
6	0	0	0	0	1	1	0

What would change if undirected?

Symmetric matrix, hence half of the matrix is redundant, but still $O(|V|^2)$

For dense graphs with lots of edges, |E| will be as large as $|V|^2$

	0	1	2	3	4	5	6
0	0	0	1	1	0	0	0
1	0	0	1	0	0	0	0
2	1	1	0	0	0	0	0
3	1	0	0	0	1	0	0
4	0	0	0	1	0	1	1
5	0	0	0	0	1	0	1
6	0	0	0	0	1	1	0

Symmetric matrix

O(|E|)

O(|V|)

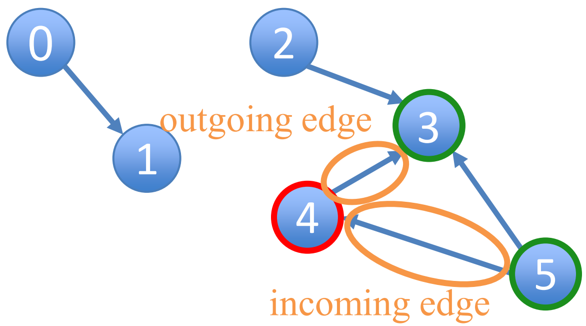
- 0 → {3}
- 1 → {2}
- 2 → {0}
- 3 → {4}
- 4 → null
- 5 → {4}
- 6 → {4, 5}

How much storage is required to represent a graph as **an adjacency list**? (Big-O, Tightest Bound)

A. |V| B. |E|
C. |V|+|E| D. $|V|^2$
E. $|E|^2$

Much more efficient for sparse graphs!

Find the Neighbors



Neighbors: vertices that are adjacent.

there is edge in between

Out degree: number of outgoing edges.

In degree: number of incoming edges.

count the number of occurrences in all lists

return the size of list

0 → {1}
1 → null
2 → {3}
3 → null
4 → {3}
5 → {3, 4}

	0	1	2	3	4	5
0	0	1	0	0	0	0
1	0	0	0	0	0	0
2	0	0	0	1	0	0
3	0	0	0	0	0	0
4	0	0	0	1	0	0
5	0	0	0	1	1	0

count the number of nonzero slots

Which implementation makes finding the in degree more efficient?

Matrix: O(|V|) List: O(|E| + |V|)

For dense graphs without multiple edges between pairs of vertices, |E| is O(|V|^2). so the adjacency matrix representation is faster. For sparse graphs, |E| = O(|V|) so both representations have the same performance.

Which implementation makes finding the out degree more efficient?

Matrix: O(|V|) List: O(1)

Coding getOutNeighbors (outgoing)

```java
public class GraphAdjMatrix extends Graph {

        private int[][] adjMatrix;

        public List<Integer> getOutNeighbors(int v) {
                List<Integer> neighbors = new ArrayList<Integer>();
                for (int i = 0; i < getNumVertices(); i++) {
                        for (int j=0; j< adjMatrix[v][i]; j++)
                        if (adjMatrix[v][i] > 0)
                                neighbors.add(i);
                }
                return neighbors;
        }
}
```

// list to return

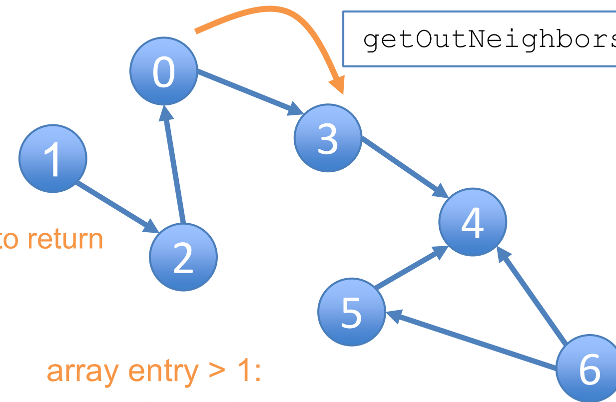
```java
public class GraphAdjList extends Graph {

        private Map<Integer,ArrayList<Integer>> adjListsMap;

        public List<Integer> getOutNeighbors(int v) {
                return adjListsMap.get(v);
                return new ArrayList<Integer>(adjListsMap.get(v));
        }
}
```

// return v's list

// return a COPY of v's list

From a software development point of view, is this a good implementation?

`getOutNeighbors(6)`



array entry > 1:
- multiple edges

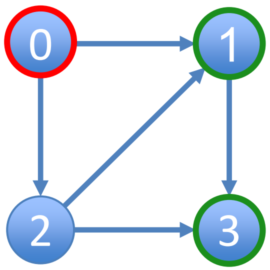
	0	1	2	3	4	5	6
0	0	0	0	2	0	0	0
1	0	0	1	0	0	0	0
2	1	0	0	0	0	0	0
3	0	0	0	0	1	0	0
4	0	0	0	0	0	0	0
5	0	0	0	0	1	0	0
6	0	0	0	0	1	1	0

- 0 → {3}
- 1 → {2}
- 2 → {0}
- 3 → {4}
- 4 → null
- 5 → {4}
- 6 → {4, 5}

What does this change do?

A. It's a change in the code but will not materially affect the output.
B. It will take multiple edges into account.
C. It will have some other effect on the code behavior.

Coding 2-Hop Neighbors (outgoing)



0 → {1, 2}

1 → {3}

2 → {1, 3}

3 → null

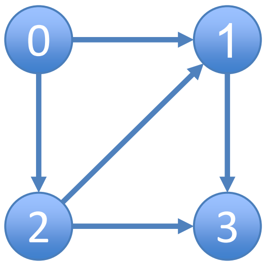
	0	1	2	3
0	0	1	1	0
1	0	0	0	1
2	0	1	0	1
3	0	0	0	0

Find all two-hop neighbors from given vertex

```
public class GraphAdjList extends Graph {

    private Map<Integer,ArrayList<Integer>> adjListsMap;

    public List<Integer> getDistance2 (int v) {
        List<Integer> distance2 = new ArrayList<>();

        // Loop through oneHop and get the neighbors of each
        for(int u : getOutNeighbors(v)){
            distance2.addAll(getOutNeighbors(u));
        }
        return distance2;
    }
}
```

```
public class GraphAdjMatrix extends Graph {

    private int[][] adjMatrix;

    public List<Integer> getDistance2 (int v) {
        List<Integer> distance2 = new ArrayList<Integer>();

        // Loop through oneHop and get the neighbors of each
        for(int u : getOutNeighbors(v)){
            distance2.addAll(getOutNeighbors(u));
        }
        return distance2;
    }
}
```

Coding 2-Hop Neighbors (Matrix Multiplication)



	0	1	2	3
0	0	1	1	0
1	0	0	0	1
2	0	1	0	1
3	0	0	0	0

Matrix multiplication for finding two-hop neighbors

For all the vertices in the graph

$$\begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}^2 = \text{matrix whose entries are two-hop neighbors!}$$

$$\begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \boxed{} & \boxed{} & \boxed{} & \boxed{} \\ \boxed{} & & & \\ & & & \\ & & & \end{pmatrix}$$

Dot product

$0*0 + 1*0 + 1*0 + 0*0 = 0$

$0*1 + 1*0 + 1*1 + 0*0 = 1$

$0*1 + 1*0 + 1*0 + 0*0 = 0$

$0*0 + 1*1 + 1*1 + 0*0 = 2$

$0*0 + 0*0 + 0*0 + 1*0 = 0$

	0	1	2	3
0	0	1	0	2
1	0	0	0	0
2	0	0	0	1
3	0	0	0	0

Matrix multiplication is well studied and optimized in software and hardware, and can be done very fast

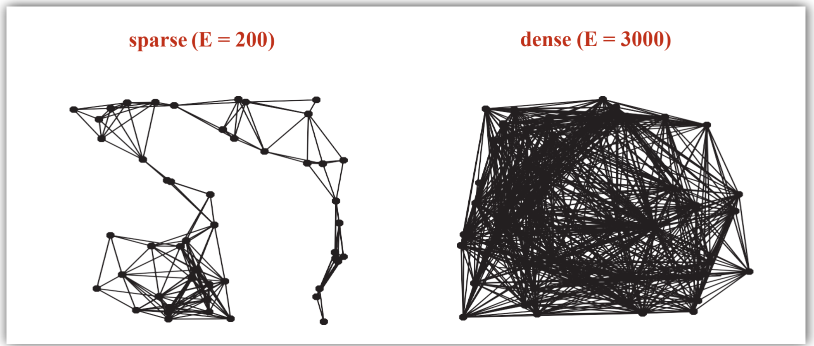
Summary of Digraph Representations

In practice. Use adjacency-lists representation.

- Algorithms based on iterating over vertices adjacent from v.
- Real-world graphs tend to be sparse (not dense).

proportional to V

proportional to V^2



Two graphs (V = 50)

representation	space	insert edge from v to w	edge from v to w?	iterate over vertices adjacent from v?
adjacency matrix	V^2	1 †	1	V
adjacency lists	$E + V$	1	*outdegree(v)*	*outdegree(v)*

† disallows parallel edges

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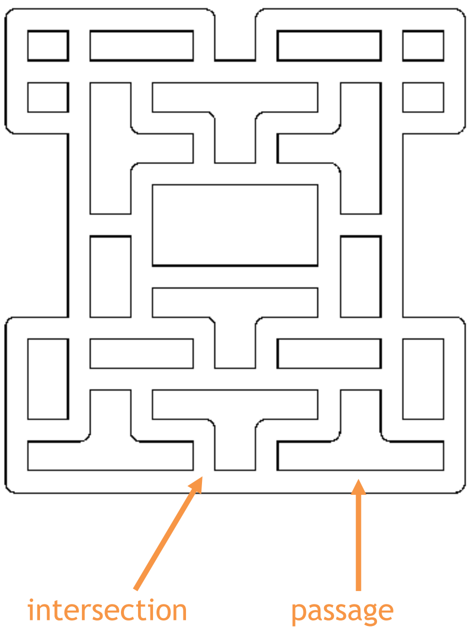
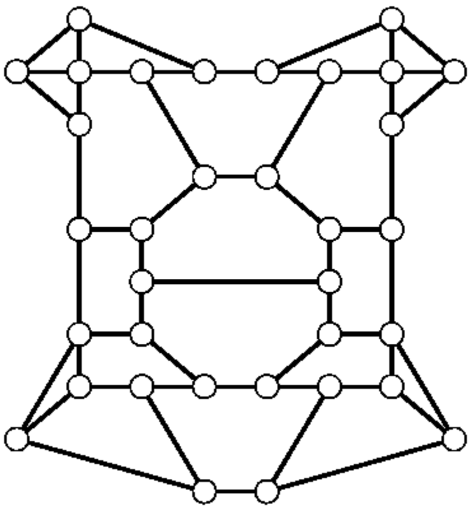
- Implement a method to find the neighbors of a vertex in two ways.
- We introduce two classic algorithms for searching a graph—depth-first search and breadth-first search.
- We also consider the problem of computing connected components and conclude with related problems and applications.
- we introduce a depth-first search based algorithm for computing the topological order of an acyclic digraph.

Represent Problems as Graphs: Maze Exploration

Goal. Explore every intersection in the maze.

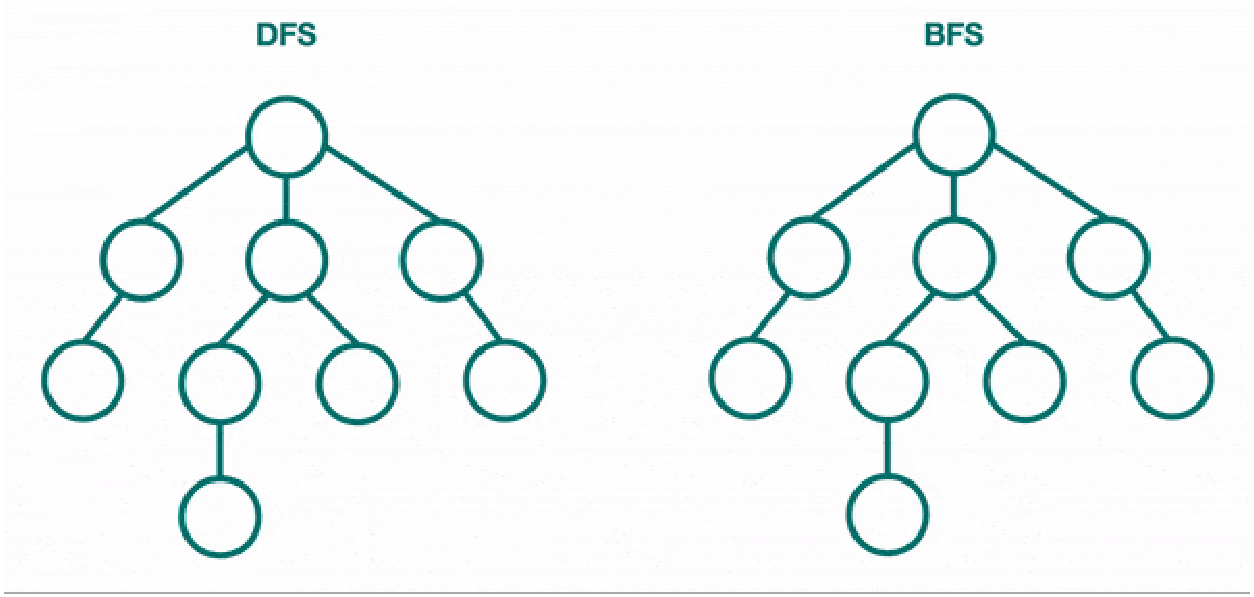
Maze graph. Vertex = intersection. Edge = passage.





DFS and BFS

- Breadth-first search in 4 minutes
  - https://www.youtube.com/watch?v=HZ5YTanv5QE
- Depth-first search in 4 minutes
  - https://www.youtube.com/watch?v=Urx87-NMm6c
- Graph Traversals - Breadth First and Depth First
  - https://www.youtube.com/watch?v=bIA8HEEUxZI



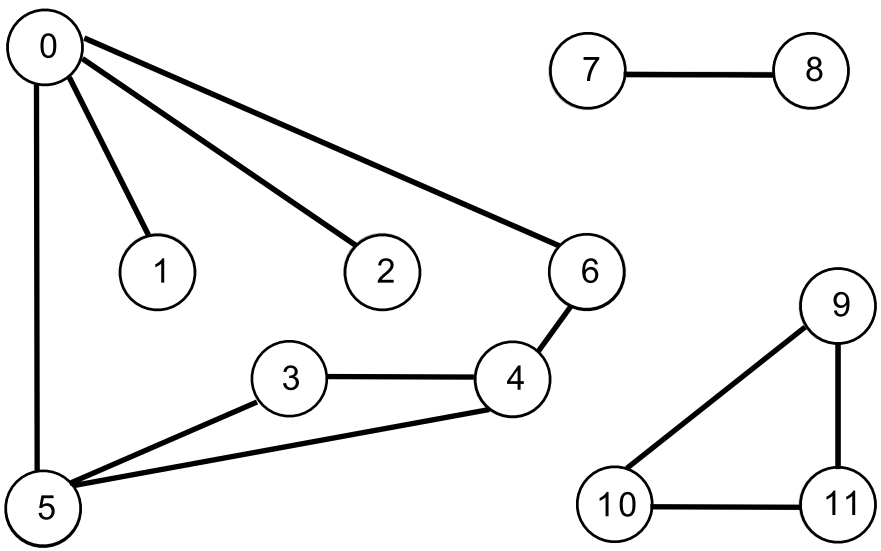
Depth-First Search (DFS)

Goal. Systematically traverse a graph.

Typical applications.
- Find all vertices connected to a given source vertex.
- Find a path between two vertices.

DFS (to visit a vertex v)

Mark vertex v.
Recursively visit all unmarked vertices w adjacent to v.



Data structures.
- Boolean array marked[] to mark vertices.
- Integer array edgeTo[] to keep track of paths.
  - (edgeTo[w] == v) means that edge v-w taken to discover vertex w

Java execution stack is used to keep track of where to search next

v	marked[ ]	edgeTo[ ]
0	T	–
1	T	0
2	T	0
3	T	5
4	T	6
5	T	4
6	T	0
7	F	–
8	F	–
9	F	–
10	F	–
11	F	–

```
dfs(0)
  dfs(6)
    dfs(4)
      dfs(5)
        dfs(3)
        3 done
      5 done
    4 done
  6 done
  dfs(2)
  2 done
  dfs(1)
  1 done
0 done
```

Class Design Pattern

Decouple graph data type from graph processing.

- Create a Graph object.
- Pass the Graph to a graph-processing routine.
- Query the graph-processing routine for information.

public class Paths	
Paths(Graph G, int s)	*//find paths in G from source s*
Boolean hasPathTo(int v)	*//is there a path from s to v?*
Iterable<Integer> pathTo(int v)	*//path from s to v; null if no such path*

```
Paths paths = new Paths(G, s);
        for (int v = 0; v < G.V(); v++)
                if (paths.hasPathTo(v))
                        StdOut.println(v);
```

print all vertices connected to s

Depth-First Search: Java Implementation

```
public   class   DepthFirstPaths {

  private   boolean[]   marked;
  private   int[] edgeTo;
  private   int s;

  public   DepthFirstPaths(Graph    G,  int   s) {
     ...
     dfs(G,   s);
  }

  private   void   dfs(Graph   G,  int   v) {
     marked[v]   = true;
     for   (int    w :  G.adj(v))
          if   (!marked[w])
          {
               edgeTo[w]   = v;
               dfs(G,   w);
          }
  }
}
```

marked[v] = true if vconnected to s

edgeTo[v] = previous vertex on path from s to v

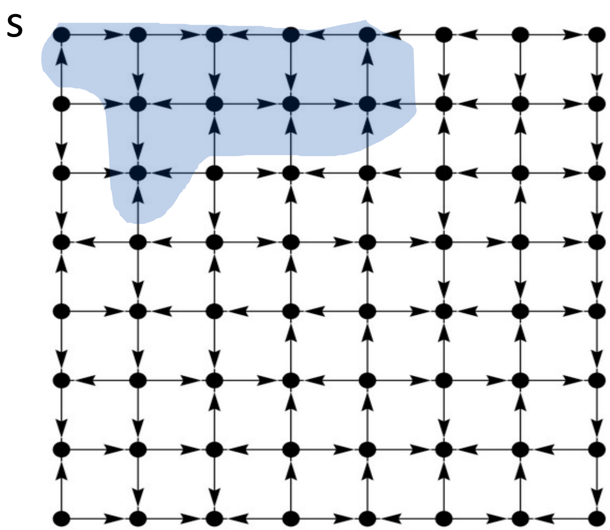
initialize data structures

find vertices connected to s

recursive DFS does the work

Depth-First Search For Directed Graph

Problem: Reachability - Find all vertices reachable from s along a directed path.



- Same method as for undirected graphs.
- Code for directed graphs identical to undirected one.

```
public class DirectedDFS {
    private boolean[] marked;                         // true if connected to s

    public DirectedDFS(Digraph G, int s) {
        marked = new boolean[G.V()];
        dfs(G, s);
    }                                                 // constructor marks vertices connected to s

    private void dfs(Digraph G, int v) {
        marked[v] = true;
        for (int w : G.adj(v))
            if (!marked[w])
                dfs(G, w);
    }                                                 // recursive DFS does the work

    public boolean visited(int v)
    { return marked[v]; }                             // client can ask whether any vertex is connected to s
}
```

Every undirected graph is a digraph (with edges in both directions). DFS is a digraph algorithm.

Depth-First Search: Properties

Proposition. DFS marks all vertices connected to s in time proportional to the sum of their degrees (plus time to initialize the `marked[ ]` array).

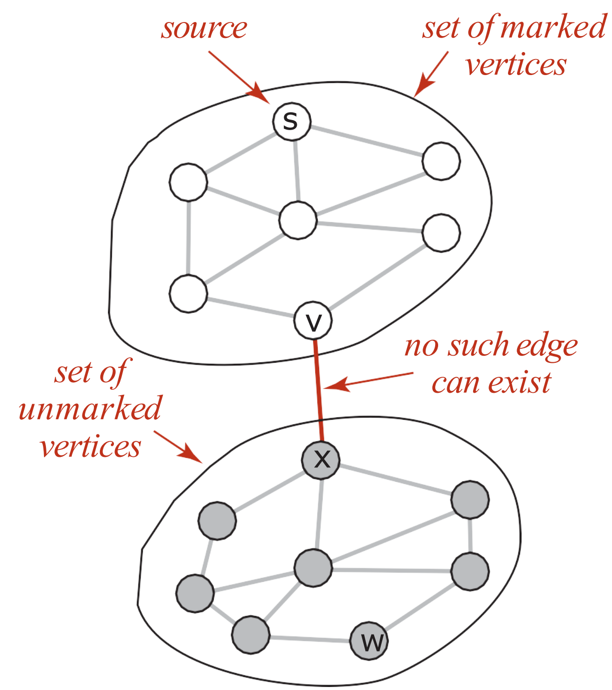
Pf. [correctness]
If w connected to s, then w marked. (if w unmarked, then consider last edge on a path from s to w that goes from a marked vertex to an unmarked one).

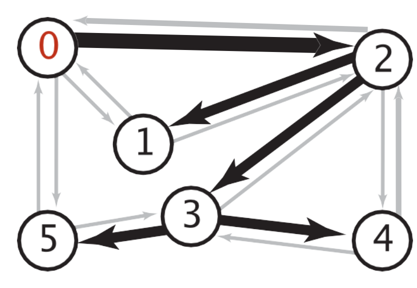
Pf. [running time]
Each vertex connected to s is visited once.

Proposition. After DFS, can check if vertex v is connected to s in constant time and can find v–s path (if one exists) in time proportional to its length.

Pf. `edgeTo[ ]` is parent-link representation of a tree rooted at vertex s.

```
public Boolean hasPathTo(int v)
{ return marked[v]; }
public Iterable<Integer> pathTo(int  v) {
      if (!hasPathTo(v)) return null;
      Stack<Integer> path = new Stack<Integer>();
      for (int x = v; x != s; x = edgeTo[x])
            path.push(x);
      path.push(s);
      return path;
}
```





edgeTo[]

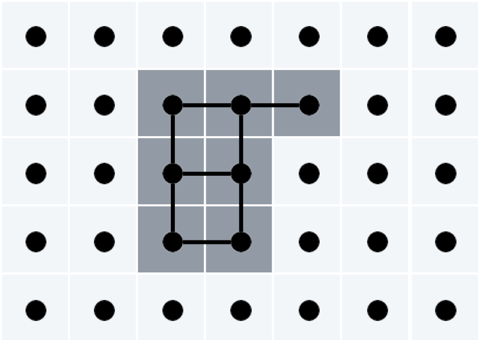
0	
1	2
2	0
3	2
4	3
5	3

Depth-First Search Application: Flood Fill

Problem. Flood fill (Photoshop magic wand).
Assumptions. Picture has millions to billions of pixels.

Solution.

- Build a grid graph.
- Vertex: pixel.
- Edge: between two adjacent gray pixels.
- Blob: all pixels connected to given pixel.



Reachability Application: Mark–Sweep Garbage Collector

Every data structure is a digraph.

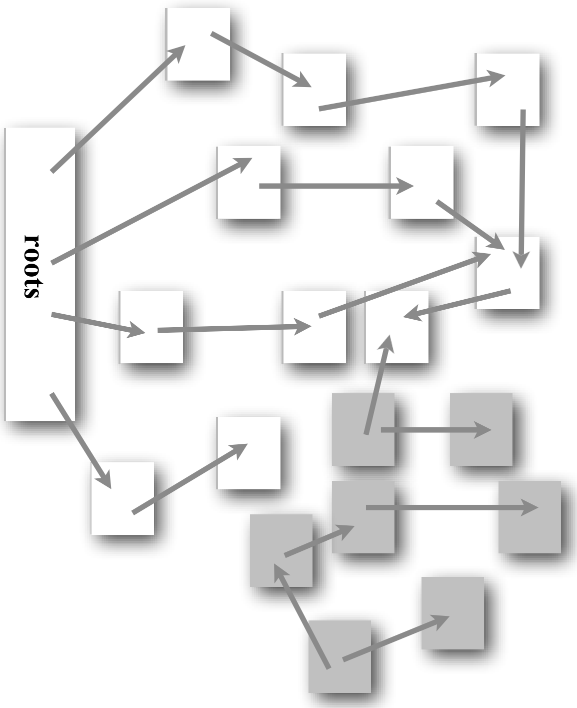
- Vertex = object.
- Edge = reference.
- Roots: Objects known to be directly accessible by program (e.g., stack).
- Reachable objects: Objects indirectly accessible by program (starting at a root and following a chain of pointers).

Mark–sweep algorithm. [McCarthy, 1960]

Mark: mark all reachable objects.

Sweep: if object is unmarked, it is garbage (so add to free list).

Memory cost. Uses 1 extra mark bit per object (plus DFS stack).



Breadth-First Search (BFS)

BFS (from source vertex s)

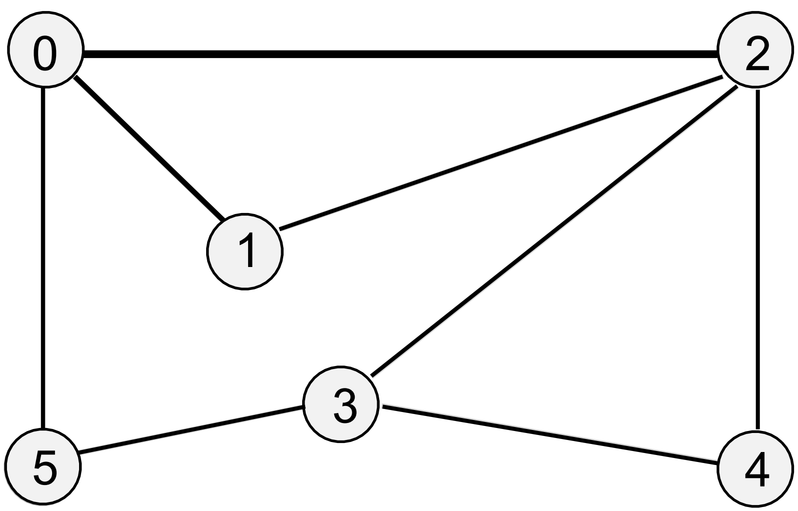
Put s onto a FIFO queue, and mark s as visited.

Repeat until the queue is empty:

- **remove the least recently added vertex v**
- **add each of v's unmarked neighbors to the queue, and mark them.**

Queue

4
3
5
1
2
0



v	marked[ ]	edgeTo[ ]	distTo[ ]
0	T	–	0
1	T	0	1
2	T	0	1
3	T	2	2
4	T	2	2
5	T	0	1

distTo[v] = distTo[edgeTo[v]] + 1;

s.distTo[v] stores the distance from s to v

Breadth-First Search: Java Implementation

```
public   class   BreadthFirstPaths {
    private   boolean[]   marked;
    private   int[]    edgeTo;
    private   int[]    distTo;
    ...
    private   void   bfs(Graph   G,  int   s)  {
        Queue<Integer> q  = new Queue<Integer>();
        q.enqueue(s);
        marked[s]  = true;
        distTo[s]     = 0;

        while   (!q.isEmpty())  {
            int v= q.dequeue();
            for   (int    w :  G.adj(v))    {
                if   (!marked[w]) {
                    q.enqueue(w);
                    marked[w]  = true;
                    edgeTo[w]  = v;
                    distTo[w]    = distTo[v]    + 1;
                }
            }
        }
    }
}
```

- DFS. Put unvisited vertices on a stack.
- BFS. Put unvisited vertices on a queue.

initialize FIFO queue of vertices to explore

found new vertex w via edge v–w

Every undirected graph is a digraph (with edges in both directions). BFS is a digraph algorithm.

- For directed graph, same method as for undirected graphs.
- Code for directed graphs identical to undirected one.

Breadth-First Search Properties

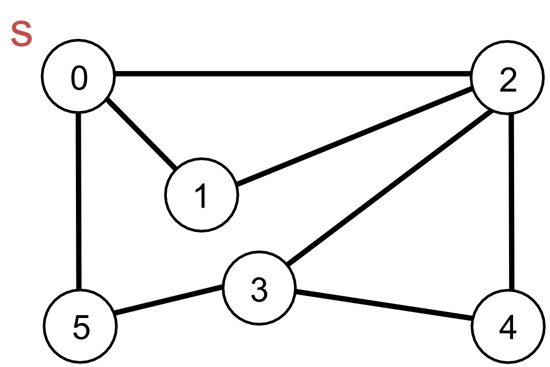
level-order

Proposition. BFS examines vertices in increasing distance (number of edges) from s.

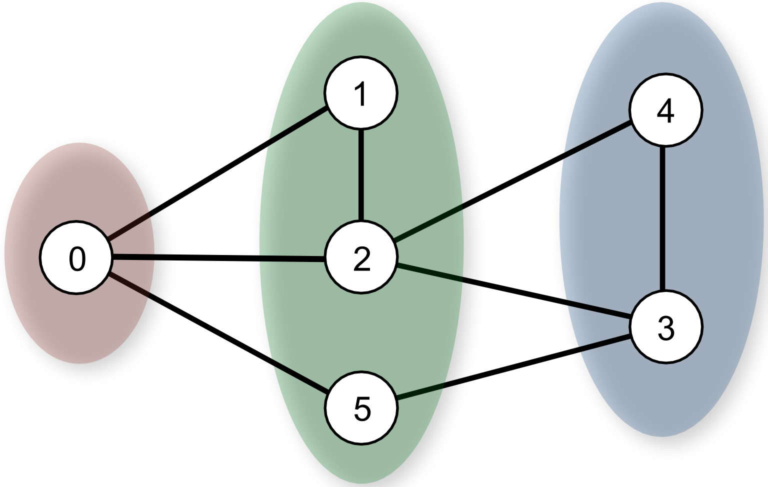
Proposition. In any connected graph, BFS computes shortest paths (fewest number of edges) from s to all other vertices in time proportional to E + V.

Pf. [correctness] Queue always consists of zero or more vertices of distance k from s, followed by zero or more vertices of distance k + 1.

Pf. [running time] Each vertex connected to s is visited once, and all its edges are checked.



graph G



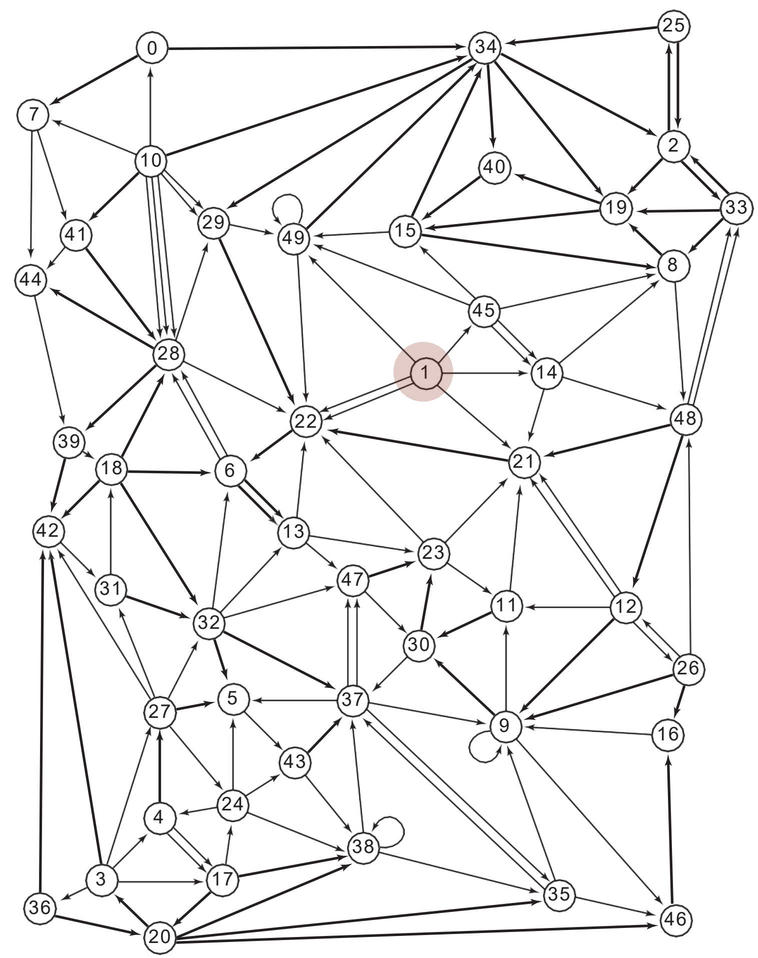
dist = 0 dist = 1 dist = 2

Breadth-First Search Application: Web Crawler

Goal. Crawl web, starting from some root web page, say www.hofstra.edu.

Solution. [BFS with implicit digraph]

- Choose root web page as source s.
- Maintain a Queue of websites to explore.
- Maintain a SET of marked websites.
- Dequeue the next website and enqueue any unmarked websites to which it links.

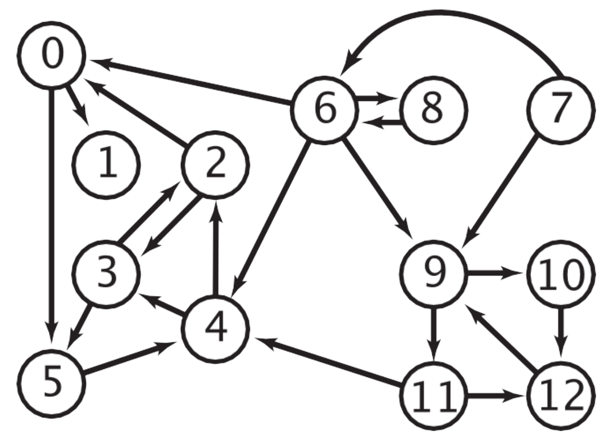


Why not use DFS?

Some web pages would **trap** the DFS search by creating new web pages and make links to them the first time that you visit them. DFS would always go to a new web page like that and it'd keep creating new ones and you wouldn't get very far.

Multiple-Source Shortest Paths Problem

Given a digraph and a set of source vertices, find shortest path from any vertex in the set to each other vertex.

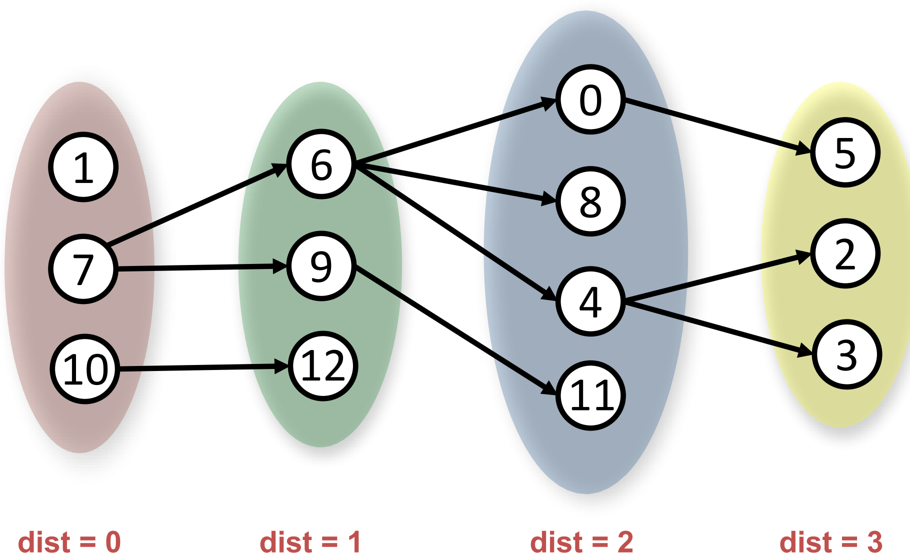


Ex. S = { 1, 7, 10 }.

- Shortest path to 4 is 7 → 6 → 4.
- Shortest path to 5 is 7 → 6 → 0 → 5.
- Shortest path to 12 is 10 → 12.
- …

How to implement multi-source shortest paths algorithm?

Use BFS, but initialize by enqueuing all source vertices.

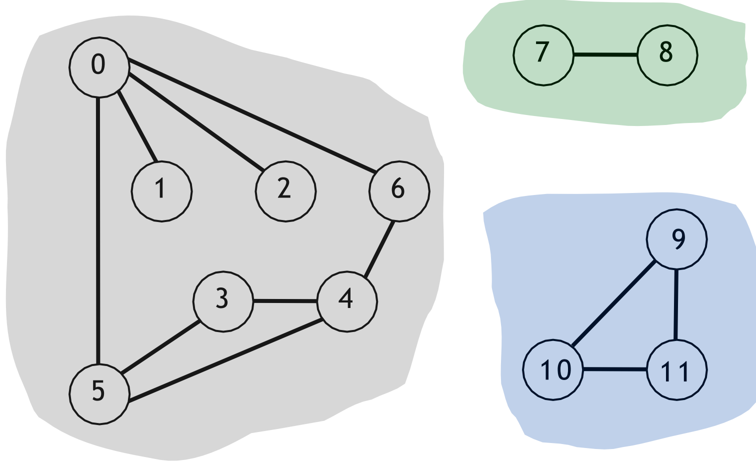


Connectivity Queries Problem

- Vertices v and w are connected if there is a path between them.
- In undirected graph, the relation "is connected to" is an equivalence relation:
  - Reflexive: v is connected to v.
  - Symmetric: if v is connected to w, then w is connected to v.
  - Transitive: if v connected to w and w connected to x, then v connected to x.
- Goal. Preprocess undirected graph to answer queries of the form *is v connected to w?* in constant time while using adjacency list.
- A connected component is a maximal set of connected vertices.
- Given connected components, can answer queries in constant time.

v	id[ ]
0	0
1	0
2	0
3	0
4	0
5	0
6	0
7	1
8	1
9	2
10	2
11	2

public class	CC	
	CC(Graph G)	*find connected components in G*
boolean	connected(int v, int w)	*are v and w connected?*
int	count()	*number of connected components*
int	id(int v)	*component identifier for v*



3 connected components

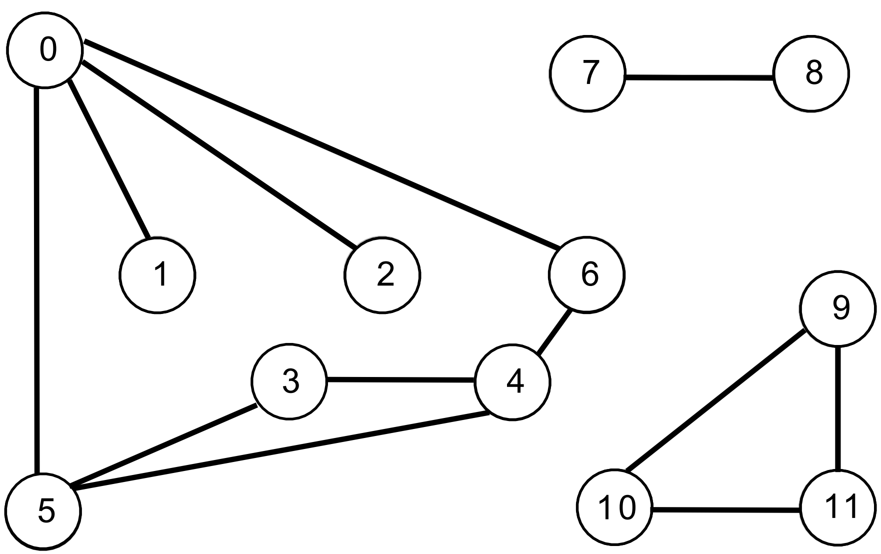
Finding Connected Components with DFS

Goal. Partition vertices into connected components.

Java execution stack

Connected components

Initialize all vertices v as unmarked.
For each unmarked vertex v, run DFS to identify all vertices discovered as part of the same component.



v	marked[ ]	id[ ]
0	T	0
1	T	0
2	T	0
3	T	0
4	T	0
5	T	0
6	T	0
7	T	1
8	T	1
9	T	2
10	T	2
11	T	2

```
dfs(0)
  dfs(6)
    dfs(4)
      dfs(5)
        dfs(3)
        3 done
      5 done
    4 done
  6 done
  dfs(2)
  2 done
  dfs(1)
  1 done
0 done
dfs(7)
  dfs(8)
  8 done
7 done
dfs(9)
  dfs(10)
    dfs(11)
    11 done
  10 done
9 done
```

Can also use BFS

Finding CCs with DFS: Java Implementation

```
public class CC {

private boolean[] marked;
private int[] id;
private int count;

public CC(Graph G) {
    marked = new boolean[G.V()];
    id = new int[G.V()];
    for (int v = 0; v < G.V(); v++) {
        if (!marked[v])
            dfs(G, v);
            count++;
    }
}
public int count() { return count; }
public int id(int v) { return id[v]; }

private void dfs(Graph G, int v) {
    marked[v] = true;
    id[v] = count;
    for (int w : G.adj(v));
        if (!marked[w]);
            dfs(G, w);
}
```

id[v] = id of component containing v

number of components

run DFS from one vertex in each component

number of components

id of component containing v

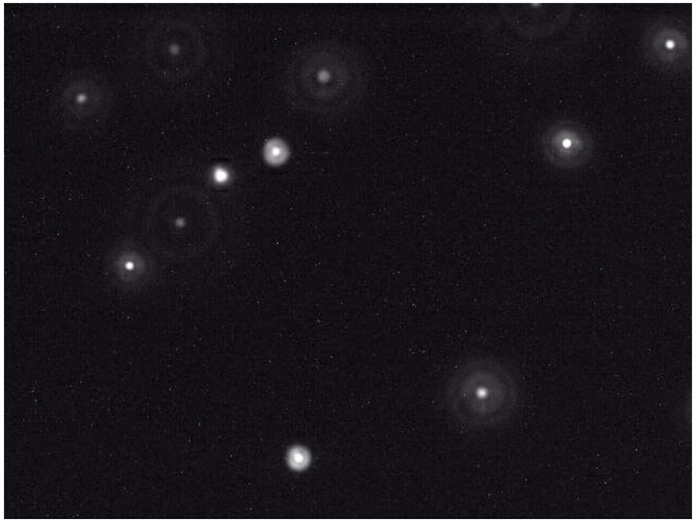
all vertices discovered in same call of dfs have same id

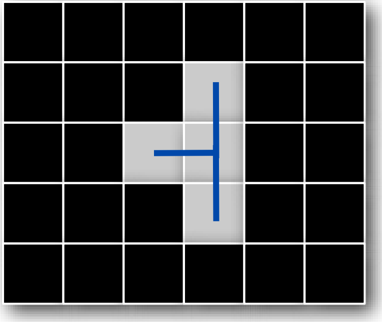
Connected Components Application: Particle Detection

Given grayscale image of particles, identify "blobs."

- Vertex: pixel.
- Edge: between two adjacent pixels with grayscale value > 70.
- Blob: connected component of 20-30 pixels.

black = 0
white = 255





Particle tracking. Track moving particles over time.

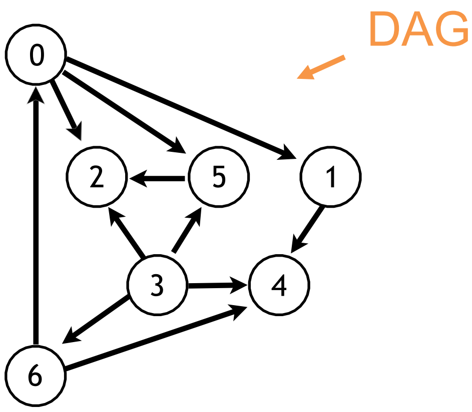
Precedence Scheduling Problem

Goal. Given a set of tasks to be completed with precedence constraints, in which order should we schedule the tasks?

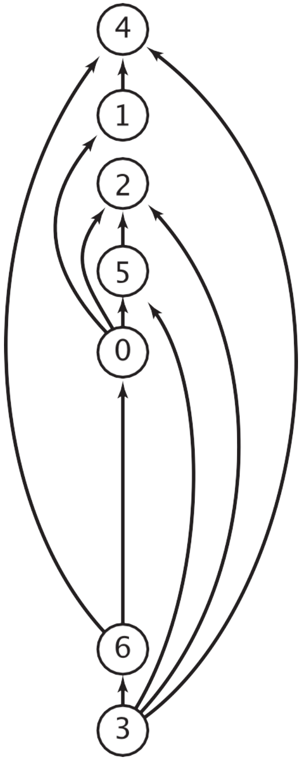
Digraph model. vertex = task; edge = precedence constraint.

0. Algorithms
1. Complexity Theory
2. Artificial Intelligence
3. Intro to CS
4. Cryptography
5. Scientific Computing
6. Advanced Programming

tasks



precedence constraint graph



feasible schedule

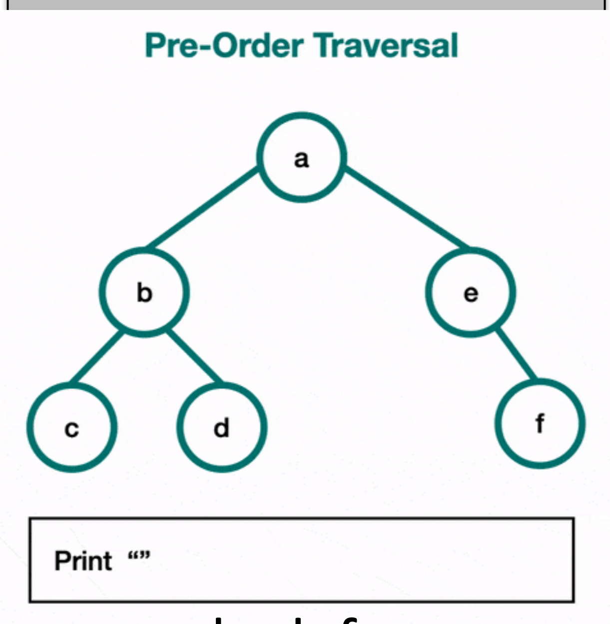
Topological sort. Redraw DAG(Directed acyclic graph) so all edges point upwards.

Graph traversal with DFS: pre-order, in-order, post-order

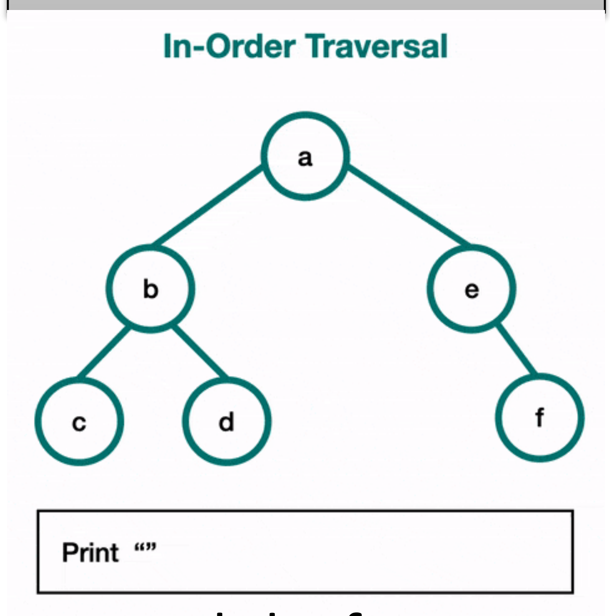
```
function preOrderTraversal(node) {
  if (node !== null) {
    visitNode(node);
    preOrderTraversal(node.left);
    preOrderTraversal(node.right);
  }
}
```

```
function inOrderTraversal(node) {
  if (node !== null) {
    inOrderTraversal(node.left);
    visitNode(node);
    inOrderTraversal(node.right);
  }
}
```

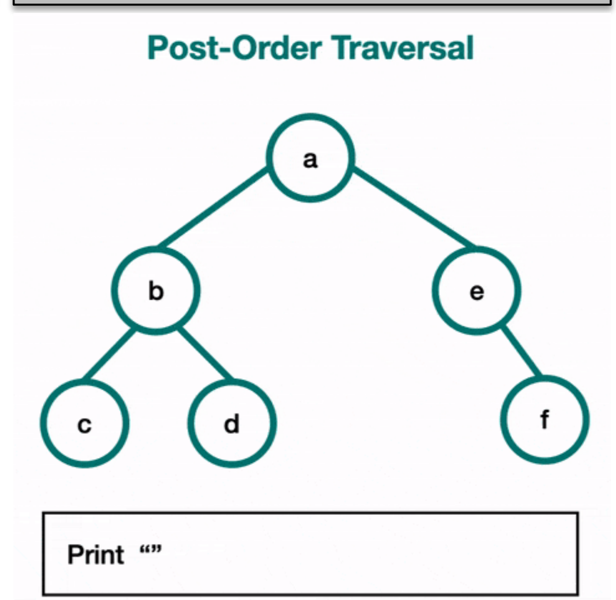
```
function postOrderTraversal(node) {
  if (node !== null) {
    postOrderTraversal(node.left);
    postOrderTraversal(node.right);
    visitNode(node);
  }
}
```



abcdef



cbdaef



cdbfea

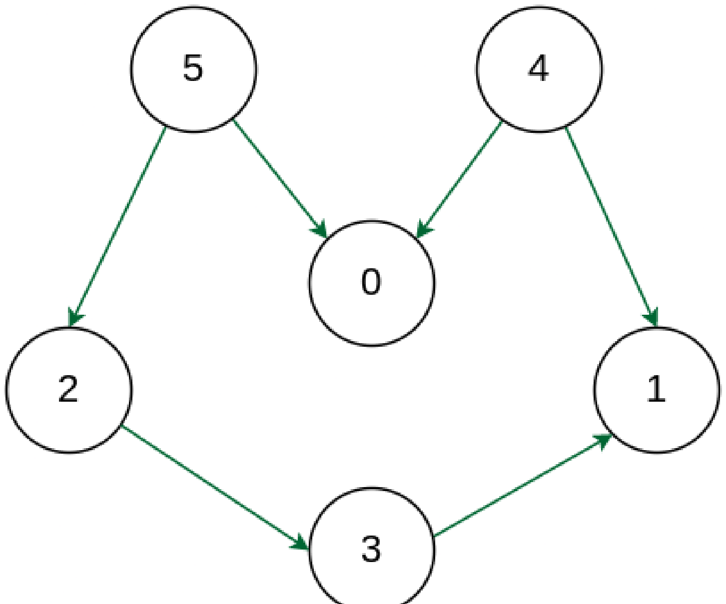
Preorder Traversal in Binary Tree Animations
https://www.youtube.com/watch?v=gLx7Px7IEzg

Inorder Traversal in Binary Tree Animations
https://www.youtube.com/watch?v=ne5oOmYdWGw

Postorder Traversal in Binary Tree Animations
https://www.youtube.com/watch?v=a8kmbuNm8Uo

Topological Sort

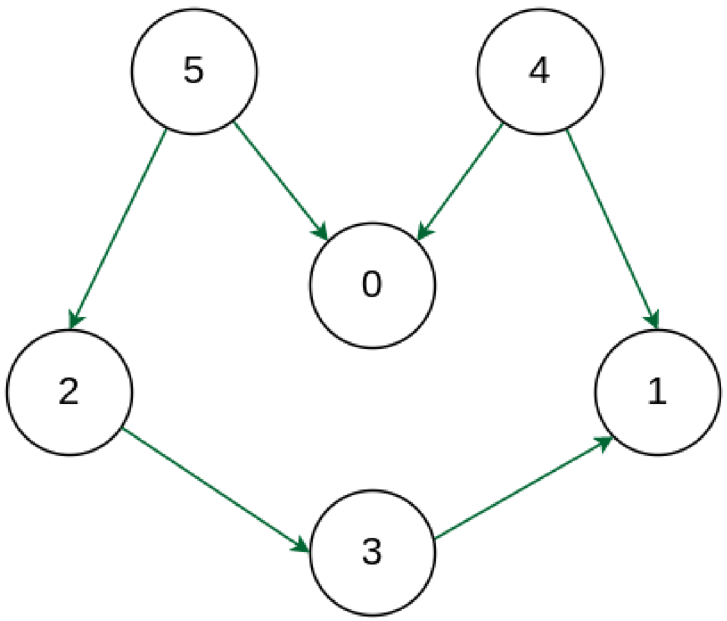
- Topological sorting for Directed Acyclic Graph (DAG) is a linear ordering of vertices such that for every directed edge u-v, vertex u comes before v in the ordering.



Output: *5 4 2 3 1 0*
Explanation: *The first vertex in topological sorting is always a vertex with an in-degree of 0 (a vertex with no incoming edges). A topological sorting of the following graph is "5 4 2 3 1 0". There can be more than one topological sorting for a graph. Another topological sorting of the following graph is "4 5 2 3 1 0".*

https://www.geeksforgeeks.org/topological-sorting/

Topological Sorting vs Preorder Traversal of DFS



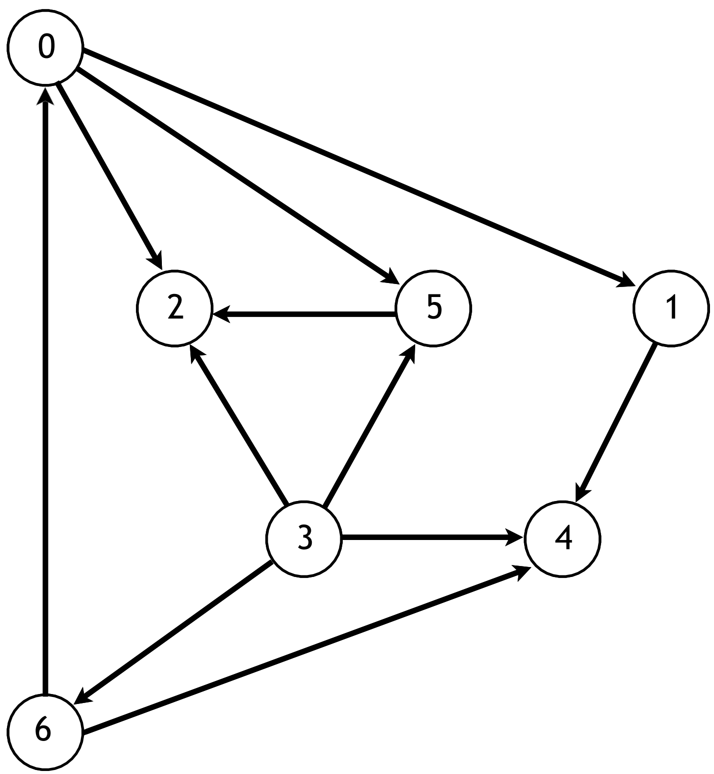
- In DFS, we print a vertex and then recursively call DFS for its adjacent vertices. In topological sorting, we need to print a vertex before its adjacent vertices.

  For example, In the above given graph, the vertex '5' should be printed before vertex '0', but unlike DFS, the vertex '4' should also be printed before vertex '0'. So Topological sorting is different from DFS pre-order traversal. For example, a DFS of the shown graph is "5 2 3 1 0 4", but it is not a topological sorting.

Topological Sort Details

- Run depth-first search
- Return vertices in reverse postorder.

not a reachability problem



stack top

Postorder

4	1	2	5	0	6	3

Topological order

3 6 0 5 2 1 4

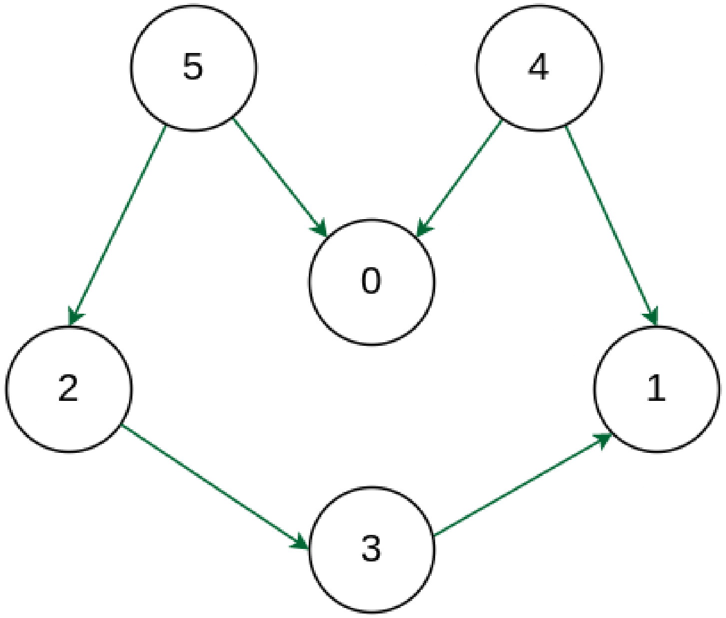
pop from the stack → reversed postorder

v	marked[ ]
0	T
1	T
2	T
3	T
4	T
5	T
6	T

Java execution stack

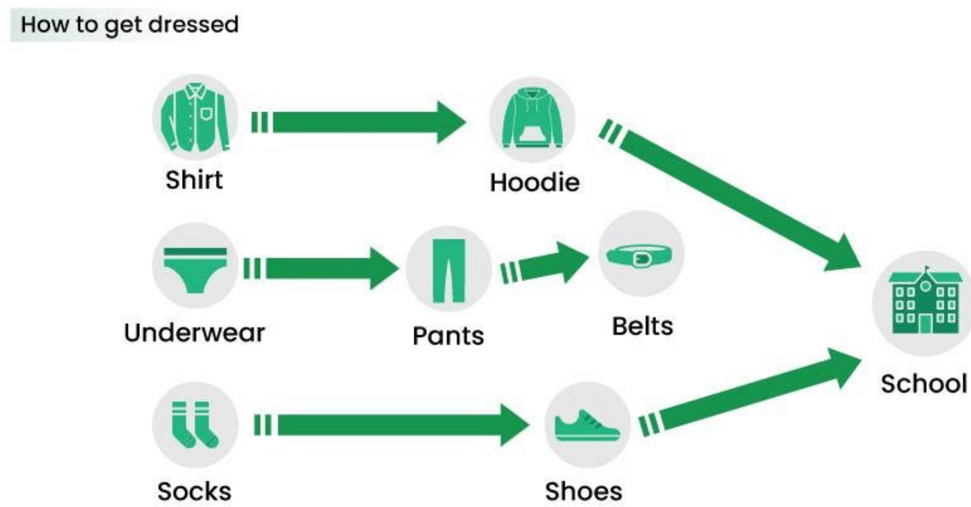
```
dfs(0)
  dfs(1)
    dfs(4)
    4 done
  1 done
  dfs(2)
  2 done
  dfs(5)
    check 2
  5 done
0 done
check 1
check 2
dfs(3)
  check 2
  check 4
  check 5
  dfs(6)
    check 0
    check 4
  6 done
3 done
check 4
check 5
check 6
done
```

Quiz: Reverse Post-order



- Post-order traversal of the shown graph is “1 3 2 0 5 4”.

  Reverse order is “4 5 0 2 3 1”, which is a topological sort.



Multiple Topological ordering for a graph

Multiple Topological ordering for a graph

Cycles and undirected edges

- Why Topological Sort is not possible for graphs having cycles?
  - Imagine a graph with 3 vertices and edges = {1 to 2 , 2 to 3, 3 to 1} forming a cycle. Now if we try to topologically sort this graph starting from any vertex, it will always create a contradiction to our definition. All the vertices in a cycle are indirectly dependent on each other hence topological sorting fails.
- Why Topological Sort is not possible for graphs with undirected edges?
  - Special case of a cycle. Undirected edge between two vertices u and v means, there is an edge from u to v as well as from v to u. Because of this both the nodes u and v depend upon each other and none of them can appear before the other in the topological ordering without creating a contradiction.

Topological Sort: Java Implementation

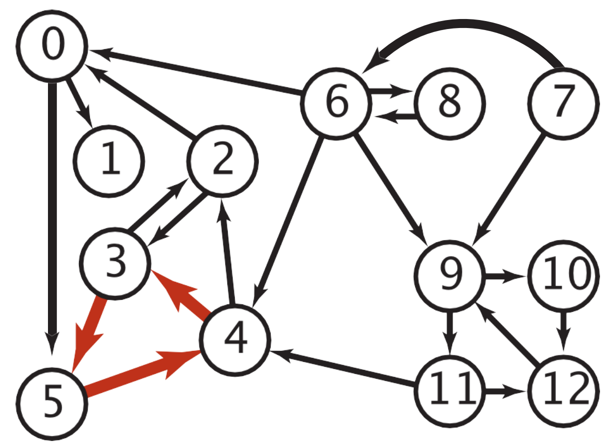
```
public class DepthFirstOrder {
    private boolean[] marked;
    private Stack<Integer> reversePostorder;

    public DepthFirstOrder(Digraph G) {
        reversePostorder = new Stack<Integer>();
        marked = new boolean[G.V()];
        for (int v = 0; v < G.V(); v++)
            if (!marked[v]) dfs(G, v);
    }

    private void dfs(Digraph G, int v) {
        marked[v] = true;
        for (int w : G.adj(v))
            if (!marked[w]) dfs(G, w);
        reversePostorder.push(v);
    }

    public Iterable<Integer> reversePostorder()
    { return reversePostorder; }
}
```

returns all vertices in "reverse DFS postorder"

Proposition. A digraph has a topological order iff no directed cycle.

Pf.

- If directed cycle, topological order impossible.
- If no directed cycle, DFS-based algorithm finds a topological order.



a digraph with a directed cycle

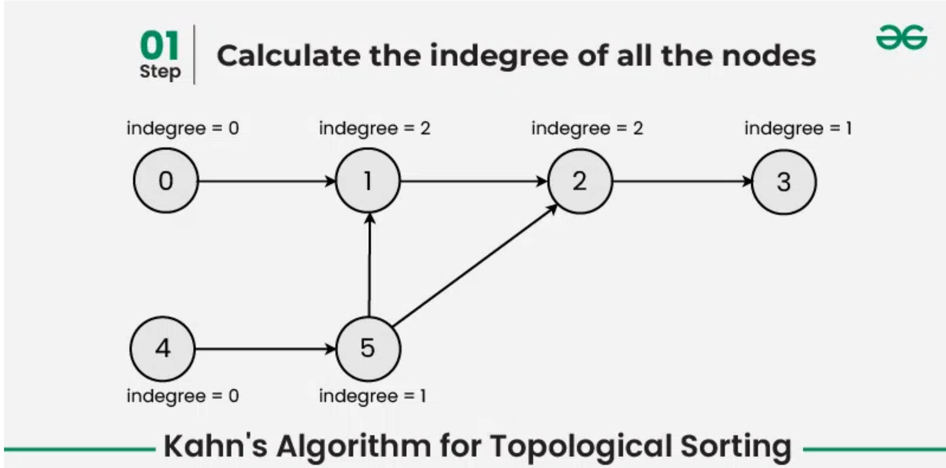
Goal. Given a digraph, find a directed cycle.

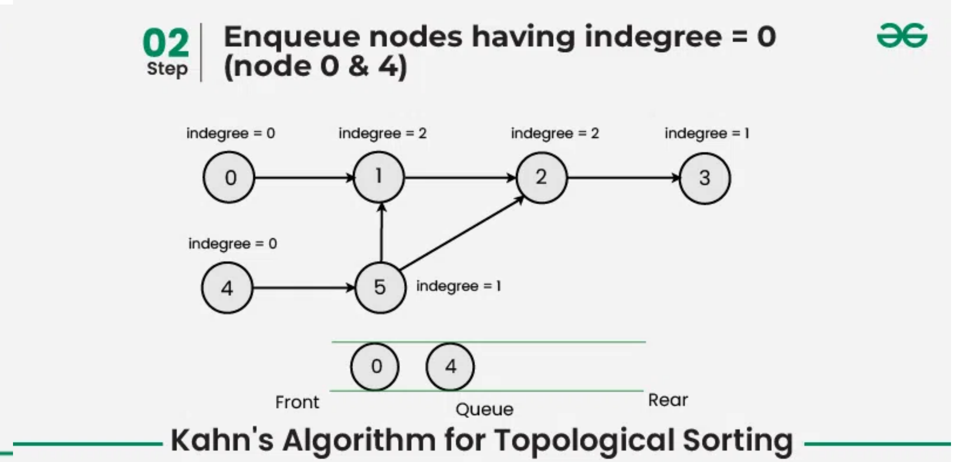
Solution. DFS. See next slide.

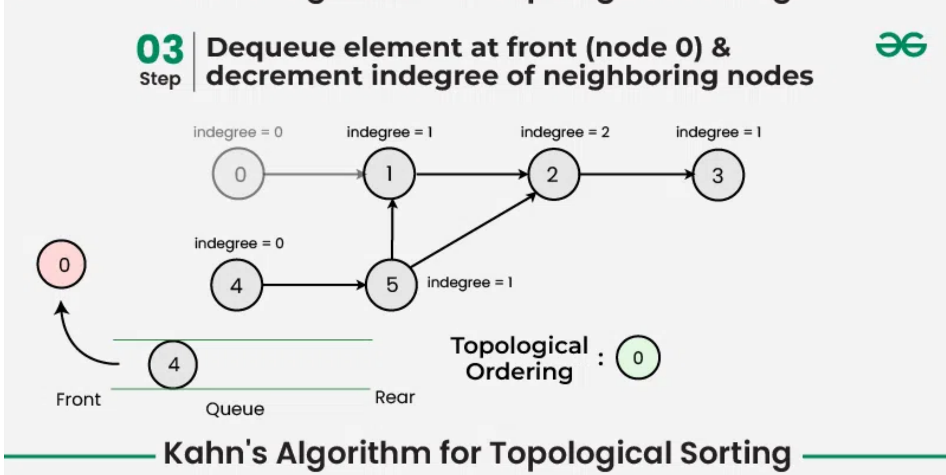
Kahn’s algorithm for Topological Sorting

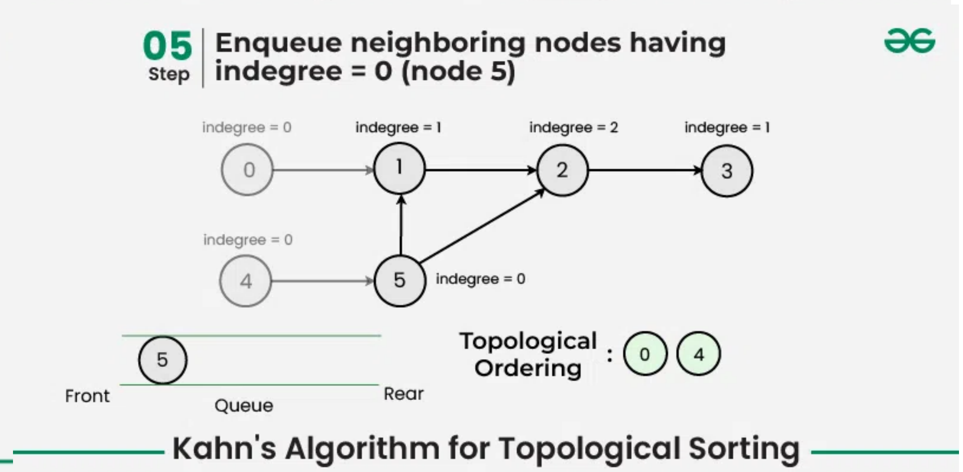
- The algorithm works by repeatedly finding vertices with no incoming edges, removing them from the graph, and updating the incoming edges of the remaining vertices. This process continues until all vertices have been ordered.
  - Add all nodes with in-degree 0 to a queue.
  - While the queue is not empty:
    - Remove a node from the queue.
    - For each outgoing edge from the removed node, decrement the in-degree of the destination node by 1.
    - If the in-degree of a destination node becomes 0, add it to the queue.
  - If the queue is empty and there are still nodes in the graph, the graph contains a cycle and cannot be topologically sorted.
  - The nodes in the queue represent the topological ordering of the graph.
- Time Complexity: O(V+E).
  - The outer for loop will be executed V number of times and the inner for loop will be executed E number of times.

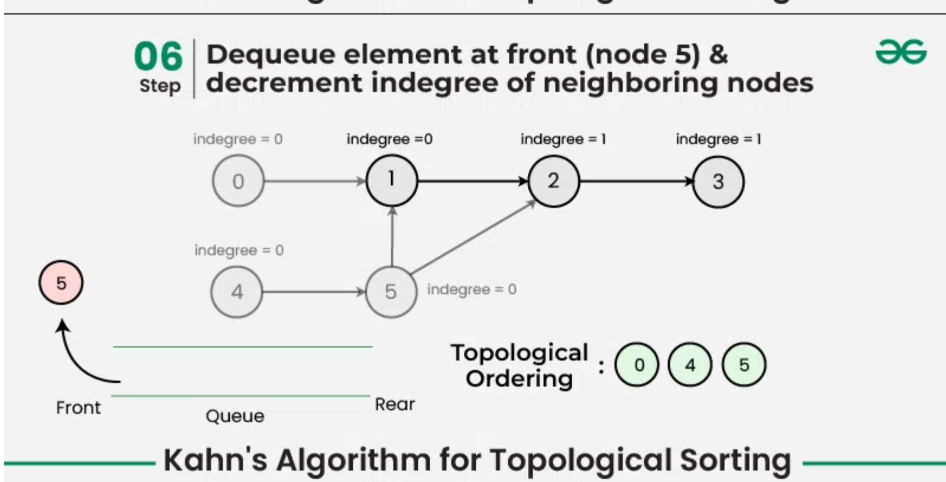
https://www.geeksforgeeks.org/topological-sorting-indegree-based-solution/

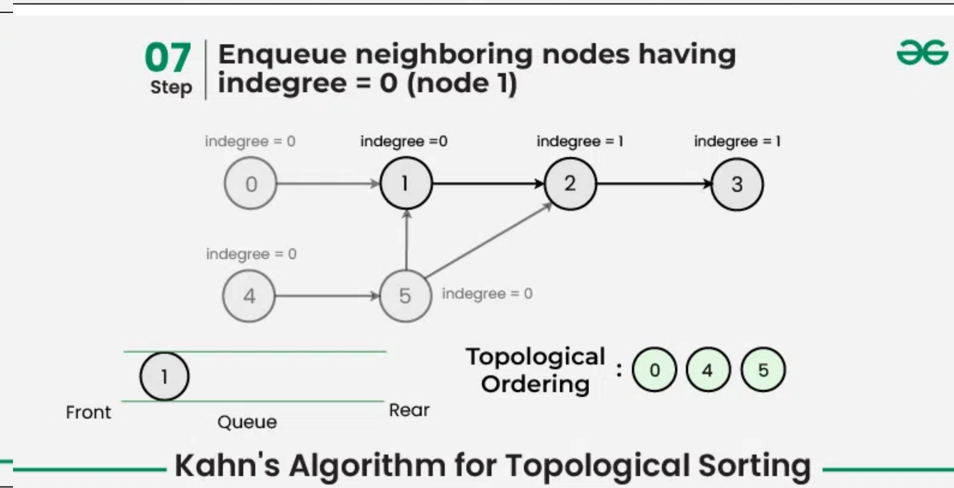

01 Step
Calculate the indegree of all the nodes
indegree = 0
indegree = 2
indegree = 2
indegree = 1
0
1
2
3
4
5
indegree = 0
indegree = 1
Kahn's Algorithm for Topological Sorting

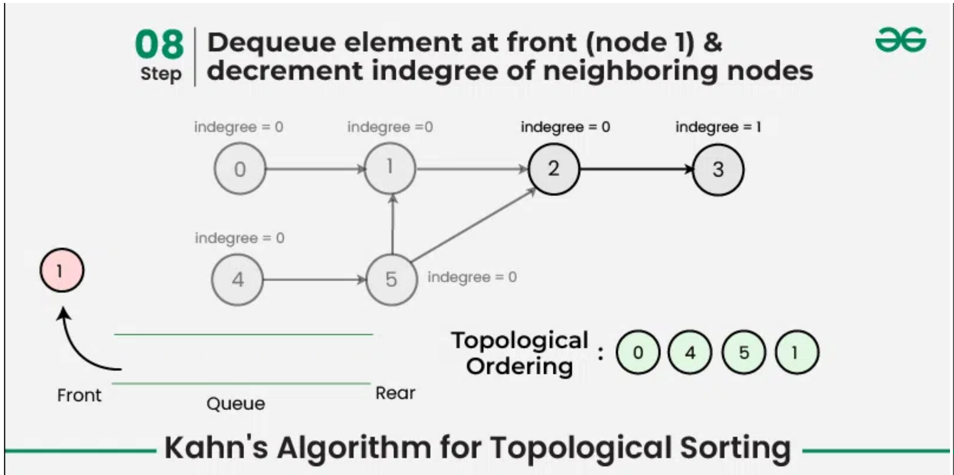

02 Step
Enqueue nodes having indegree = 0 (node 0 & 4)
indegree = 0
indegree = 2
indegree = 2
indegree = 1
0
1
2
3
indegree = 0
4
5
indegree = 1
0
4
Front
Queue
Rear
Kahn's Algorithm for Topological Sorting


03 Step
Dequeue element at front (node 0) & decrement indegree of neighboring nodes
indegree = 0
indegree = 1
indegree = 2
indegree = 1
0
1
2
3
indegree = 0
4
5
indegree = 1
0
4
Front
Queue
Rear
Topological Ordering : 0
Kahn's Algorithm for Topological Sorting

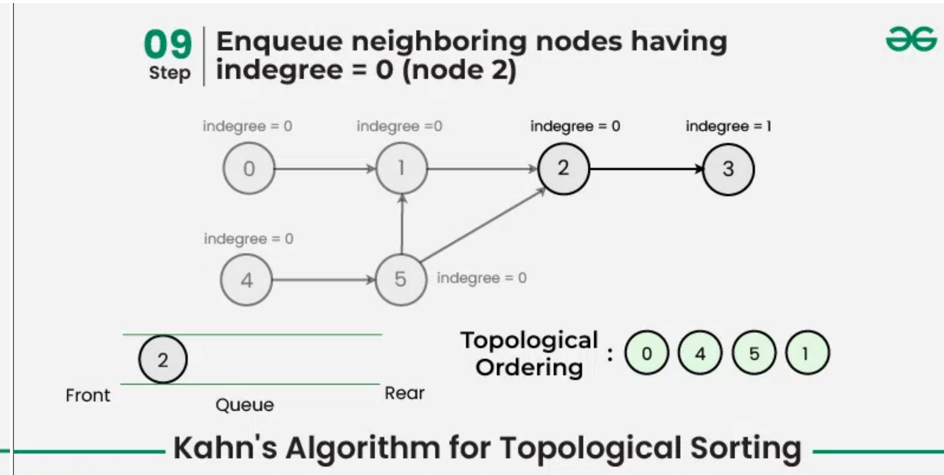

05 Step
Enqueue neighboring nodes having indegree = 0 (node 5)
indegree = 0
indegree = 1
indegree = 2
indegree = 1
0
1
2
3
indegree = 0
4
5
indegree = 0
5
Front
Queue
Rear
Topological Ordering : 0 4
Kahn's Algorithm for Topological Sorting


06 Step
Dequeue element at front (node 5) & decrement indegree of neighboring nodes
indegree = 0
indegree =0
indegree = 1
indegree = 1
0
1
2
3
indegree = 0
4
5
indegree = 0
5
Front
Queue
Rear
Topological Ordering : 0 4 5
Kahn's Algorithm for Topological Sorting

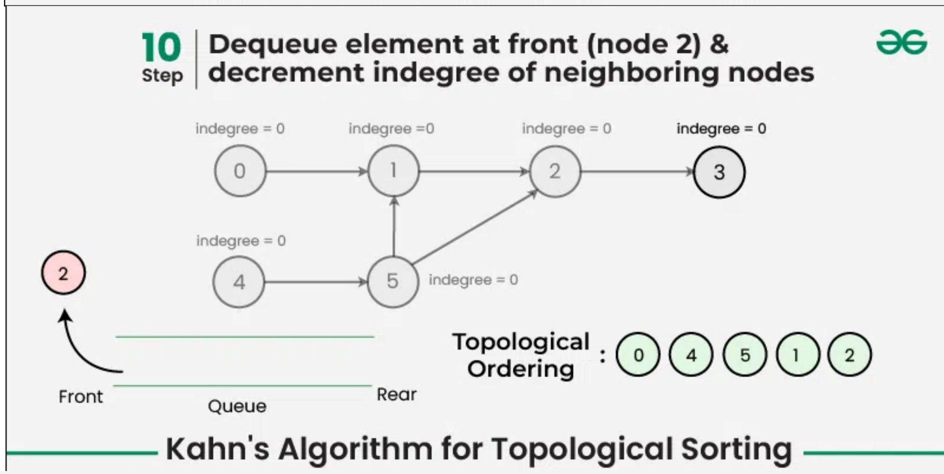

07 Step
Enqueue neighboring nodes having indegree = 0 (node 1)
indegree = 0
indegree =0
indegree = 1
indegree = 1
0
1
2
3
indegree = 0
4
5
indegree = 0
1
Front
Queue
Rear
Topological Ordering : 0 4 5
Kahn's Algorithm for Topological Sorting



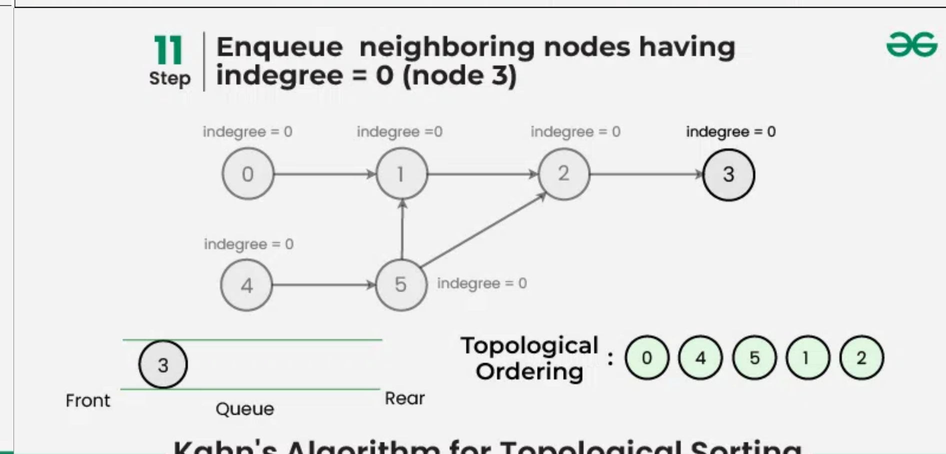
08 Step
Dequeue element at front (node 1) & decrement indegree of neighboring nodes
indegree = 0
indegree =0
indegree = 0
indegree = 1
indegree = 0
indegree = 0
0
1
2
3
4
5
1
Front
Queue
Rear
Topological Ordering : 0 4 5 1
Kahn's Algorithm for Topological Sorting



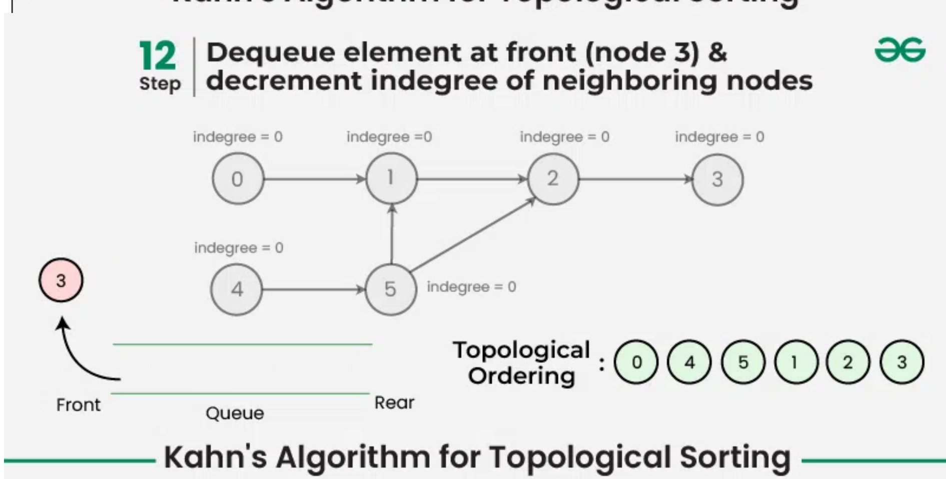
09 Step
Enqueue neighboring nodes having indegree = 0 (node 2)
indegree = 0
indegree =0
indegree = 0
indegree = 1
indegree = 0
indegree = 0
0
1
2
3
4
5
2
Front
Queue
Rear
Topological Ordering : 0 4 5 1
Kahn's Algorithm for Topological Sorting



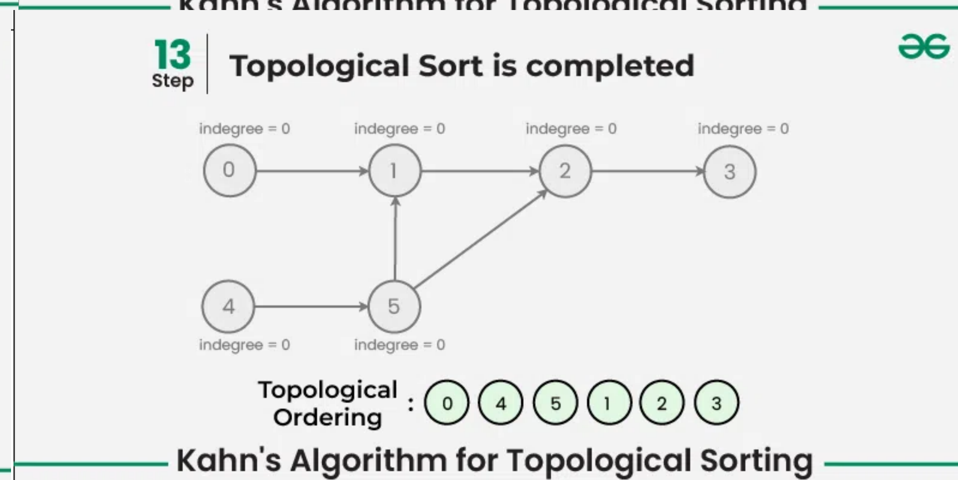
10 Step
Dequeue element at front (node 2) & decrement indegree of neighboring nodes
indegree = 0
indegree =0
indegree = 0
indegree = 0
indegree = 0
indegree = 0
0
1
2
3
4
5
2
Front
Queue
Rear
Topological Ordering : 0 4 5 1 2
Kahn's Algorithm for Topological Sorting



11 Step
Enqueue neighboring nodes having indegree = 0 (node 3)
indegree = 0
indegree =0
indegree = 0
indegree = 0
indegree = 0
indegree = 0
0
1
2
3
4
5
3
Front
Queue
Rear
Topological Ordering : 0 4 5 1 2



12 Step
Dequeue element at front (node 3) & decrement indegree of neighboring nodes
indegree = 0
indegree =0
indegree = 0
indegree = 0
indegree = 0
indegree = 0
0
1
2
3
4
5
3
Front
Queue
Rear
Topological Ordering : 0 4 5 1 2 3
Kahn's Algorithm for Topological Sorting



13 Step
Topological Sort is completed
indegree = 0
indegree = 0
indegree = 0
indegree = 0
0
1
2
3
4
5
indegree = 0
indegree = 0
Topological Ordering : 0 4 5 1 2 3
Kahn's Algorithm for Topological Sorting

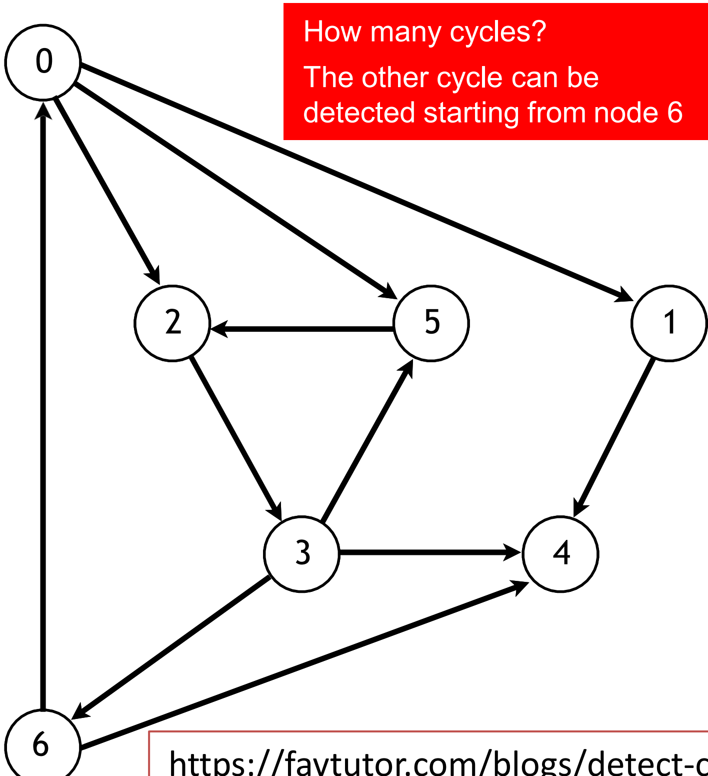
Topological Sort Applications

- Task scheduling and project management.
- In software deployment tools like Makefile.
- Dependency resolution in package management systems.
- Determining the order of compilation in software build systems.
- Deadlock detection in operating systems.
- Course scheduling in universities.
- It is used to find shortest paths in weighted directed acyclic graphs

Directed Cycle Detection

- Run depth-first search from every unmarked vertex.
- Keep track of vertices currently in recursion stack of function for DFS traversal with onStack[ ] array.
- If we reach a vertex that is already in the recursion stack, then we found a cycle in the tree, and we're done
- Retrieve the cycle using edgeTo[ ] array.

- set onStack[v] to T when dfs(v) is called
- set onStack[v] to F when dfs(v) returns



How many cycles?

The other cycle can be detected starting from node 6

v	marked[ ]	onStack[ ]	edgeTo[ ]
0	T	T	–
1	T	T	0
2	T	T	0
3	T	T	2
4	T	T	1
5	T	T	3
6	F	F	–

Java execution stack

dfs(0)
dfs(1)
dfs(4)
4 done
1 done
dfs(2)
dfs(3)
check 4
dfs(5)
check 2
done

stack top

5 3 2 5

- vertex is marked and onStack
- Found the cycle
- Save the cycle using edgeTo[ ] to a stack

https://favtutor.com/blogs/detect-cycle-in-directed-graph

Directed Cycle Detection Application: Cyclic Inheritance

The Java compiler does cycle detection.

```
public class A extends B
{
    ...
}
```

```
public class B extends C
{
    ...
}
```

```
public class C extends A
{
    ...
}
```

```
% javac A.java
A.java:1: cyclic inheritance
involving A
public class A extends B { }
             ^
1 error
```

Directed Cycle Detection Application: Spreadsheet Recalculation

Microsoft Excel does cycle detection.