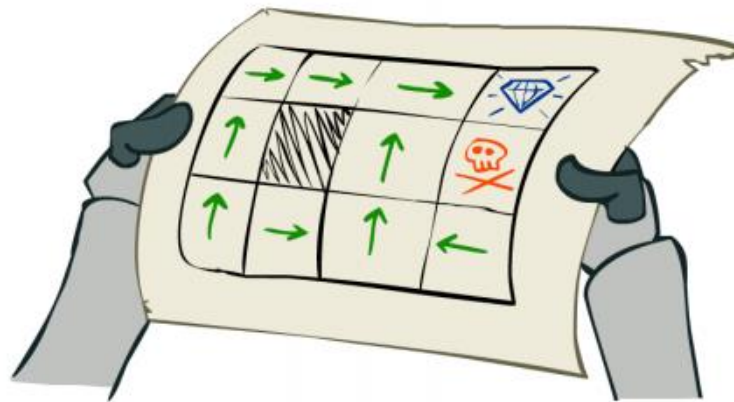


# L7.1 Markov Decision Process



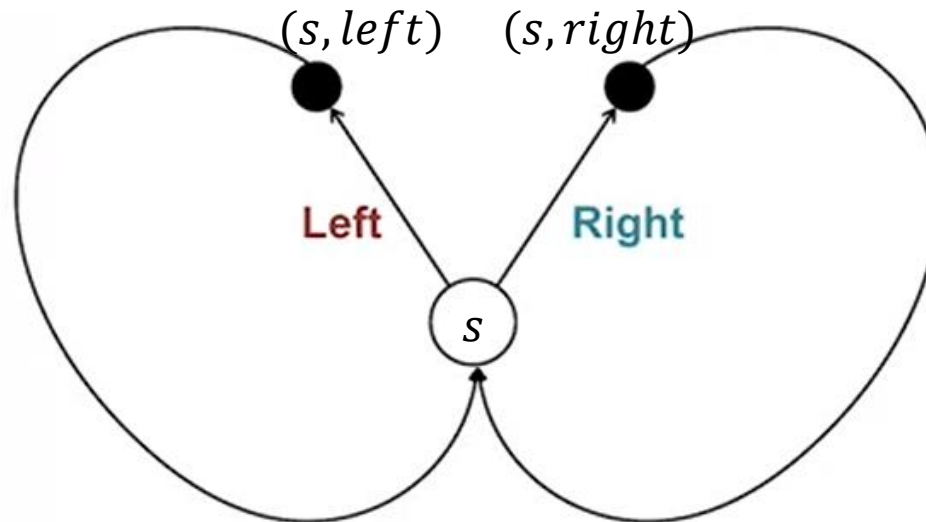
Zonghua Gu, Umeå University  
Nov. 2023

# Markov Decision Process (MDP)

- An MDP consists of:
  - Set of states  $S$
  - Start state  $s_0$
  - Set of actions  $A$
  - Transitions and rewards  $p(r, s' | s, a)$  (w. discount  $\gamma$ )
- Policy maps from states to actions:
  - Deterministic policy  $a = \pi(s)$  defines a deterministic action  $a$  for state  $s$ .
  - Stochastic policy  $\pi(a | s)$  defines a probability distribution over possible actions  $a$  for state  $s$ .
- Markov means that next state only depends on current state
  - $P(S_{t+1} = s' | S_t = s_t, A_t = a_t, S_{t-1} = s_{t-1}, A_{t-1} = a_{t-1}, \dots, S_0 = s_0, A_0 = a_0)$
  - $= P(S_{t+1} = s' | S_t = s_t, A_t = a_t)$
  - Given the present state, the future and the past are independent
  - e.g., for driving task, current vehicle position  $x$  as the state does not satisfy the Markov property, since the next state depends on not only  $x$ , but also velocity  $\dot{x}$ , acceleration  $\ddot{x}$ , (assuming acceleration  $\ddot{x}$  stays constant within each step) If we redefine the state as vector  $[x, \dot{x}, \ddot{x}]^T$ , then it satisfies the Markov property.
  - Or, current snapshot of front camera view can be used as the state (e.g., NVIDIA's PilotNet), but some works use past  $N$  video frames as the state to capture more dynamics (e.g., Waymo's ChauffeurNet).

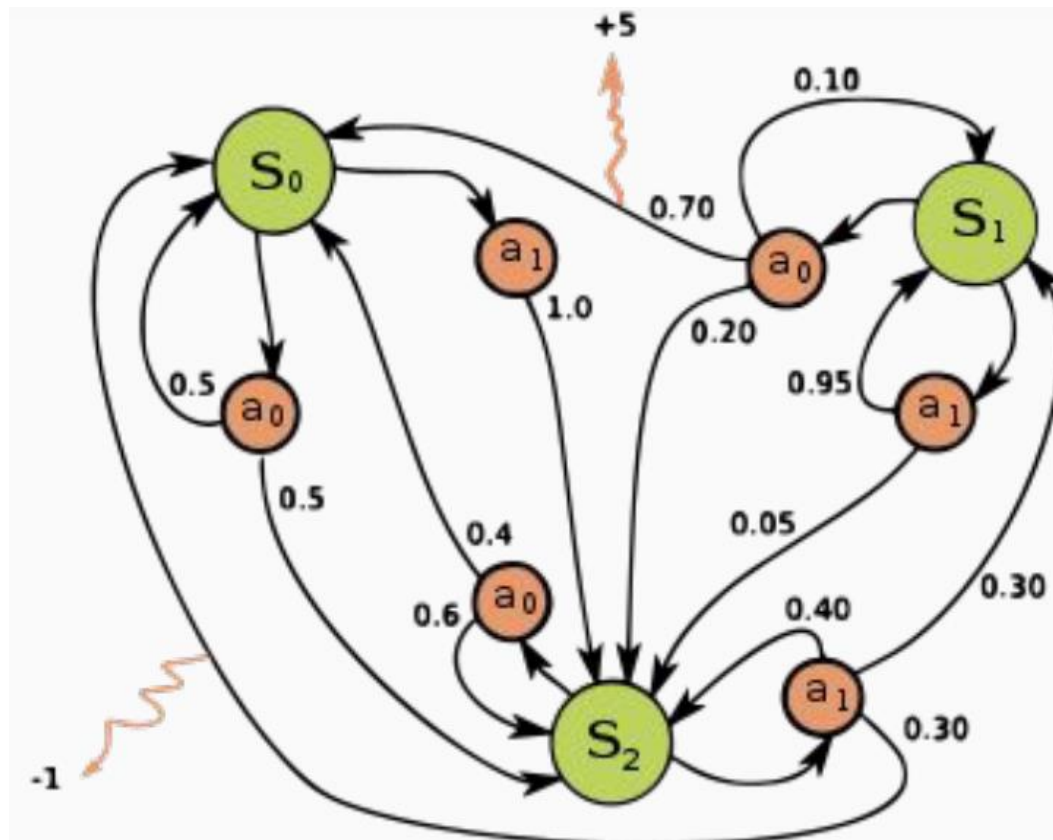
# MDP Quiz

- For this MDP with a single state  $s$  and two possible actions  $left$  and  $right$ . Are these valid policies?
  - 1)  $\pi(left|s) = \pi(right|s) = 0.5$  (goes left or right with equal probability. uniform random policy)
  - 2)  $\pi(left|s) = 1.0, \pi(right|s) = 0$  (always goes left)
  - 3) Alternating left and right, i.e., if previous action is left, then current action must be right, next action must be left, and so on.
  - ANS: 3) is not a valid policy, since it depends on the history of actions. To be a valid policy, the action must depend on the current state only
- We can redefine the MDP' extended state to include the last action as part of it, then 3) is a valid policy for the new MDP



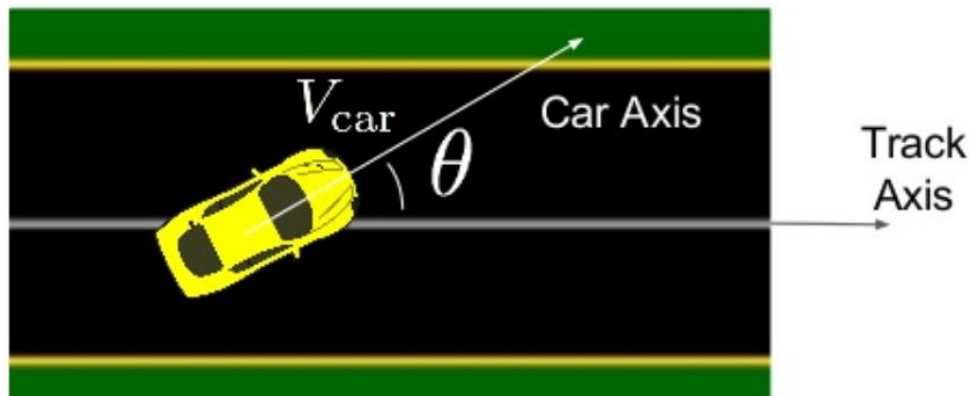
# An Example MDP

- Green nodes denote 3 states  $s_0, s_1, s_2$ ; Red nodes denote 2 possible actions  $a_0, a_1$  in each state. Each red node can also be denoted as  $(s, a)$ .
- Agent taking action  $a$  in state  $s$  may get different reward  $r$  and next state  $s'$ , denoted as state transition  $(s, a, r, s')$ , due to environment uncertainty (all rewards are 0 expect +5 and -1 show in fig).



# RL Reward Function

- For the vehicle in left fig:
  - state: Pose of ego-car ( $x, y, \theta$ ) and environment map; action: Steering wheel/brake/acceleration
- Possible reward function:  $R_t = w_1 V_{car} \cos \theta - w_2 |cte|$ 
  - Weighted sum to maximum longitudinal velocity (first term), and minimize cross-track error (distance to lane center)
  - This is an example of dense reward (e.g., at every time step), as opposed to sparse reward (e.g., only at the end of each episode)
- Compare with twiddle() :
  - twiddle() can be viewed as an RL algorithm (policy gradient), that learns PID parameters with sparse reward (cost function is average cross-track error (cte), computed at the end of each simulation episode, as sum of squares of ctes for N timesteps divided by N).
  - It does not use the numeric value of cte, only its relative size (if  $err < best\_err$ );
  - Cost function does not include heading angle  $\theta$ ;
  - if the track is very long and irregular, then we can make the reward denser, to adjust PID parameters every K timesteps instead of at the end of each episode.



```
if err < best_err:
    best_err = err
    dp[i] *= 1.1
else:
    p[i] -= 2 * dp[i]
    robot = make_robot()
    x_trajectory, y_trajectory, err = run(robot, p)

    if err < best_err:
        best_err = err
        dp[i] *= 1.1
    else:
        p[i] += dp[i]
        dp[i] *= 0.9
```

twiddle()

# Amazon DeepRacer

- Amazon Web Services (AWS) launched DeepRacer in 2018 for training AD algorithms with RL
  - <https://aws.amazon.com/deepracer/>
- You can train RL algorithm in the simulator on AWS cloud, but it costs money after some free time.
- They hold competitions, both online and in real-world. 1/10<sup>th</sup> scale race car costs USD \$349.



# Params for Writing Reward Function

all\_wheels\_on\_track

x

y

distance\_from\_center

is\_left\_of\_center

is\_reversed

heading

progress

steps

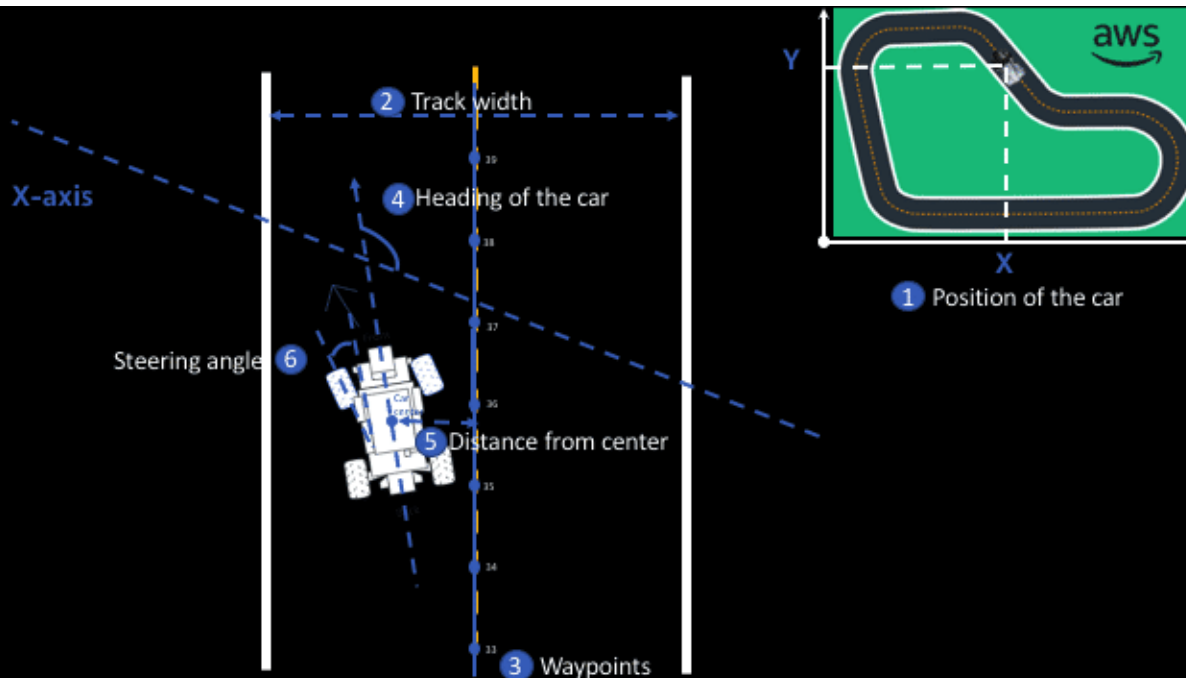
speed

steering\_angle

track\_width

waypoints

closest\_waypoints



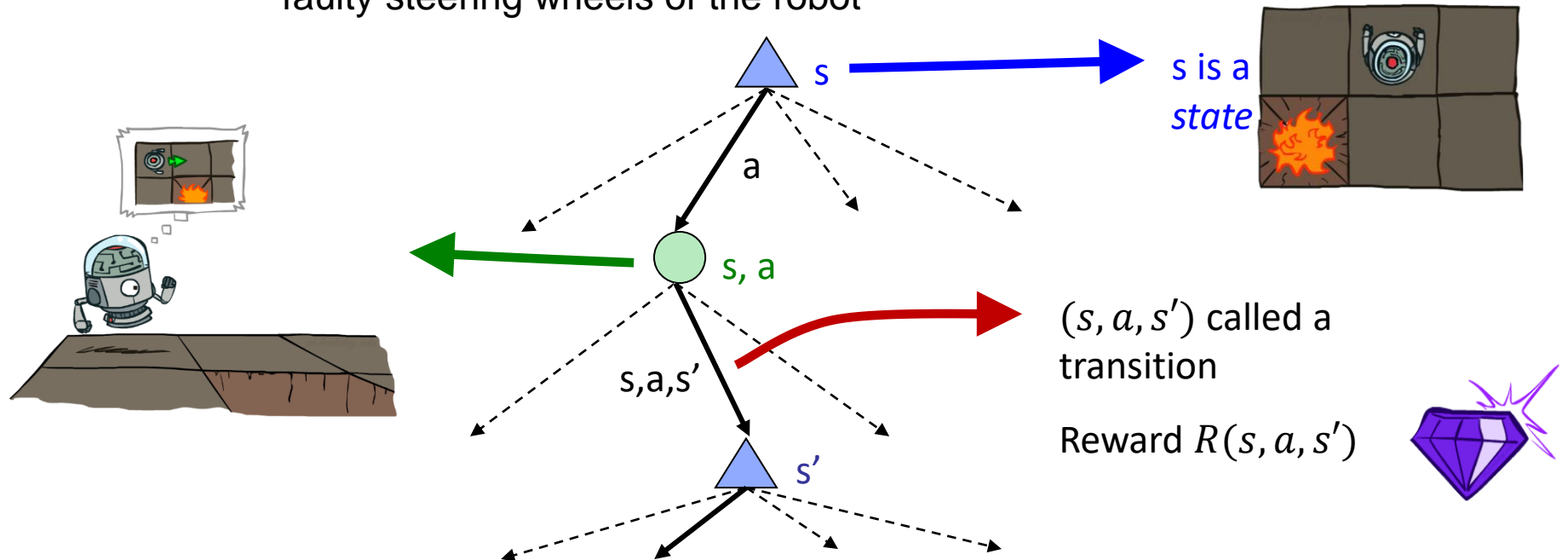
# Example Reward Function

- ```
def reward_function(params):  
    """Example of penalize steering, which helps mitigate zig-zag behaviors"""  
    # Read input parameters  
    distance_from_center = params['distance_from_center']  
    track_width = params['track_width']  
    steering = abs(params['steering_angle']) # Only need the absolute steering angle  
  
    # Calculate 3 markers that are at varying distances away from the center line  
    marker_1 = 0.1 * track_width  
    marker_2 = 0.25 * track_width  
    marker_3 = 0.5 * track_width  
  
    # Give higher reward if the agent is closer to center line and vice versa  
    if distance_from_center <= marker_1:  
        reward = 1  
    elif distance_from_center <= marker_2:  
        reward = 0.5  
    elif distance_from_center <= marker_3:  
        reward = 0.1  
    else:  
        reward = 1e-3 # likely crashed/ close to off track  
  
    # Steering penalty threshold, change the number based on your action space setting  
    ABS_STEERING_THRESHOLD = 15  
  
    # Penalize reward if the agent is steering too much  
    if steering > ABS_STEERING_THRESHOLD:  
        reward *= 0.8  
  
    return float(reward)
```
- A more realistic and complex reward function: <https://www.middleware-solutions.fr/2019/08/14/an-introduction-to-aws-deepracer>



# MDP Search Tree

- Each MDP state  $s$  projects a search tree starting from it
- Both policy and environment may be stochastic
  - Policy  $\pi(a|s)$ : probability distribution over possible actions  $a$  from state  $s$ 
    - e.g., different actions may be taken for the same state
  - Environment  $p(r, s'|s, a)$ : if the agent takes action  $a$  in state  $s$ , env gives probability distribution over next state  $s'$  and reward  $r$ 
    - e.g., due to non-determinism in the environment (sudden strong wind), or faulty steering wheels of the robot



# Preventing Infinite Rewards

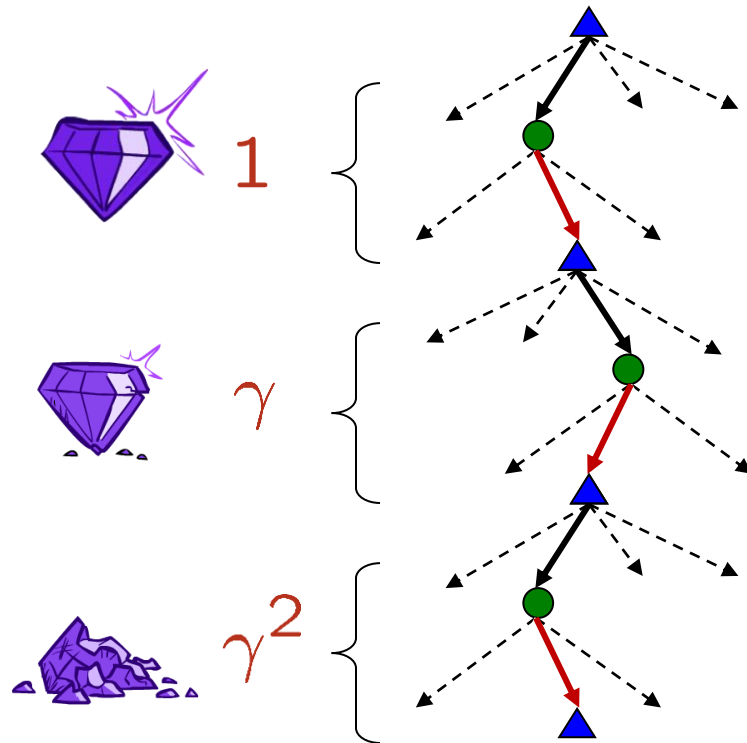
- Problem: What if the game lasts forever? Do we get infinite rewards? No. Possible solutions:
- **Finite horizon:** (limit search tree depth)
  - Terminate episodes after a fixed  $T$  timesteps
- **Discount factor:**  $0 < \gamma \leq 1$ 
  - Think of it as a  $1 - \gamma$  chance of ending the episode at every step. Effective horizon (expected episode length):  $\sum_{t=0}^{\infty} \gamma^t = \frac{1}{1-\gamma}$ 
    - $\sum_{t=0}^{\infty} 1^t = \frac{1}{1-.1} = 1.1, \sum_{t=0}^{\infty} .5^t = \frac{1}{1-.5} = 2, \sum_{t=0}^{\infty} .9^t = \frac{1}{1-.9} = 10$
  - Smaller  $\gamma$  leads to shorter horizon, and preference of short-term to long-term rewards, and vice versa
- (Can have both finite horizon and discount factor)

# Discount Factor Example

- Each time we descend a level in the search tree, we multiply in the discount once

- Example:  $\gamma = 0.5$

$$\begin{aligned} - U([1, 2, 3]) &= 1 * 1 + 0.5 * 2 + 0.25 * 3 < \\ U([3, 2, 1]) &= 1 * 3 + 0.5 * 2 + 0.25 * 1 \end{aligned}$$



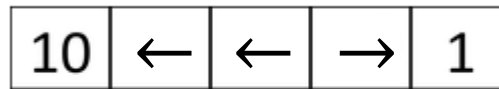
# Discounting Example

- Given:
  - Actions: East, West, and Exit (only available in exit states a, e)
  - Transitions: deterministic
- For  $\gamma = 1$ , optimal policy in each state is always moving West
  - From state d, reward of going West is  $0 + \gamma \cdot 0 + \gamma^2 \cdot 0 + \gamma^3 \cdot 10 = 10$ , larger than reward of going East  $0 + \gamma \cdot 1 = 1$
- For  $\gamma = 0.1$ , optimal policy in each state is shown below
  - From state d, reward from going West is  $0 + \gamma \cdot 0 + \gamma^2 \cdot 0 + \gamma^3 \cdot 10 = 0.01$ , less than reward from going East  $0 + \gamma \cdot 1 = 0.1$ .
- For which  $\gamma$  are West and East equally good when in state d?
  - $\gamma^3 \cdot 10 = \gamma \cdot 1 \Rightarrow \gamma = \frac{1}{\sqrt{10}} \approx .32$



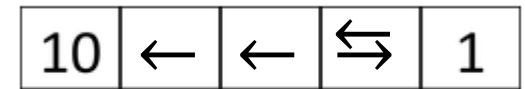
a    b    c    d    e

$\gamma = 1$



a    b    c    d    e

$\gamma = 0.1$

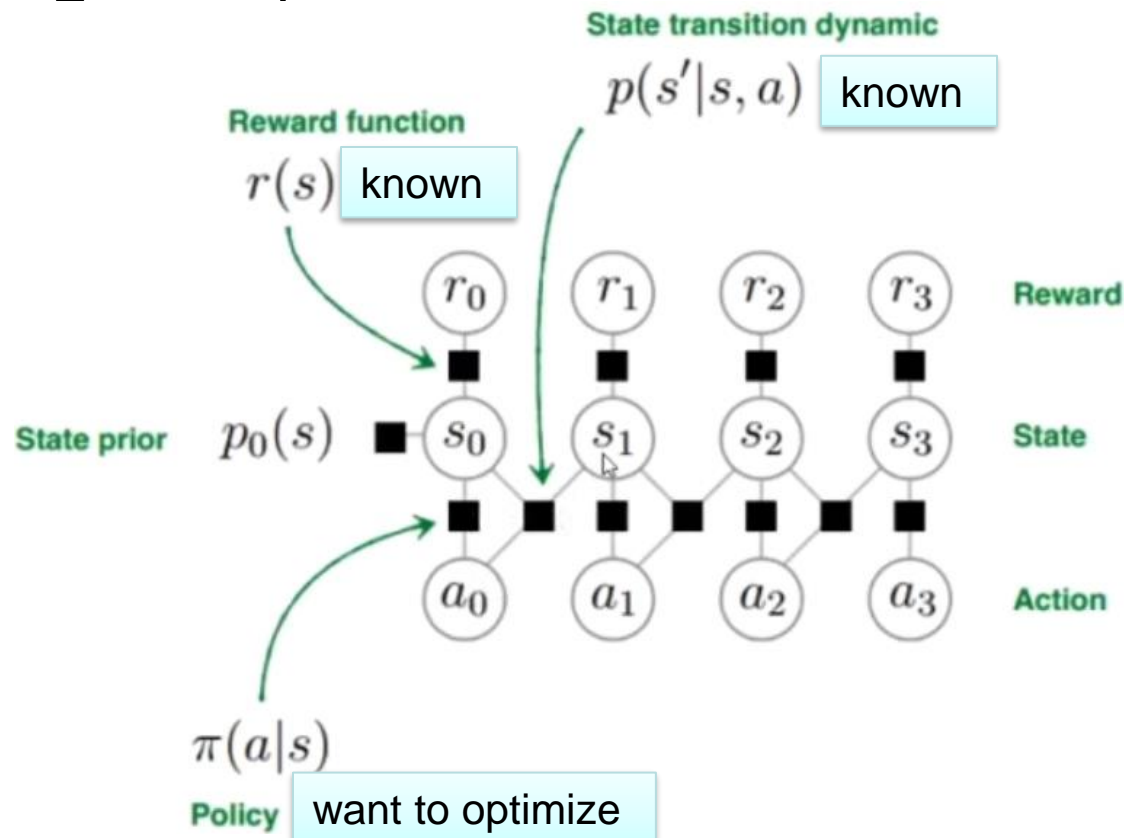


a    b    c    d    e

$\gamma = \frac{1}{\sqrt{10}}$

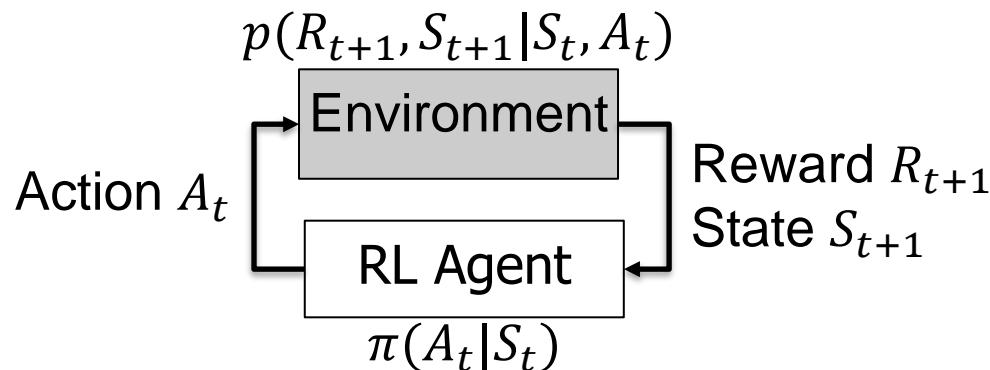
# Known MDP

- In this lecture, we assume known MDP, and use dynamic programming to solve Bellman Equations and find the optimal policy (no learning here).



# Formal Definition of MDP

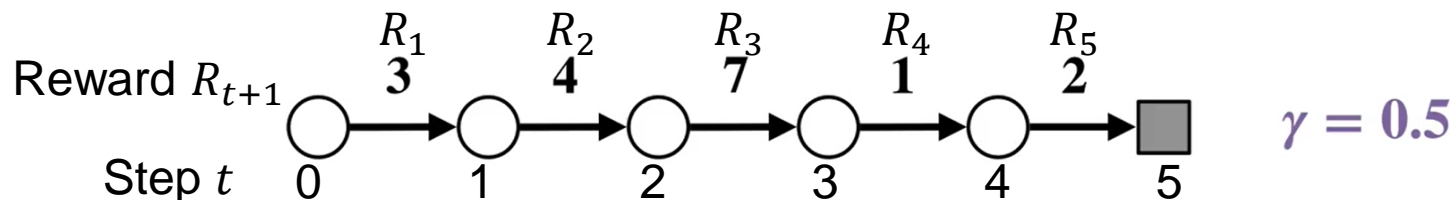
- **Return** (cumulative discounted reward) at time  $t$ :  $G_t \doteq R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \dots + \gamma^{T-t-1} R_T = \sum_{k=0}^{T-t-1} \gamma^k R_{t+k+1}$ 
  - At each step  $t \in [0, T-1]$ , agent takes an action  $A_t$  in state  $S_t$ ; at step  $t+1$ , agent receives a reward  $R_{t+1}$  and transitions into the next state  $S_{t+1}$  with the trace  $(S_t, A_t, R_{t+1}, S_{t+1})$
  - We assume episodic tasks, and this specific episode has length of  $T$  steps. ( $T = \infty$  for continuous tasks)
- **State Value Function**: expected return under policy  $\pi$ :  $v_\pi(s) \doteq \mathbb{E}_\pi[G_t | S_t = s]$
- **Action Value Function**: expected return from taking action  $a$ , then follow policy  $\pi$ :  $q_\pi(s, a) \doteq \mathbb{E}_\pi[G_t | S_t = s, A_t = a]$
- The RL problem: find the optimal policy  $\pi(a|s)$  that maximizes the expected return from each state



Important

# Example: Computing Returns for One Episode

- Working backward is more efficient than working forward as it avoids redundant computations.



$$G_0 = R_1 + \gamma G_1$$

$$G_1 = R_2 + \gamma G_2$$

$$G_2 = R_3 + \gamma G_3$$

$$G_3 = R_4 + \gamma G_4$$

$$G_4 = R_5 + \gamma G_5$$

$$G_5 = 0$$

$$G_0 = R_1 + \gamma R_2 + \gamma^2 R_3 + \gamma^3 R_4 + \gamma^4 R_5 = 7$$

$$G_1 = R_2 + \gamma R_3 + \gamma^2 R_4 + \gamma^3 R_5 = 8$$

$$G_2 = R_3 + \gamma R_4 + \gamma^2 R_5 = 8$$

$$G_3 = R_4 + \gamma R_5 = 2$$

$$G_4 = R_5 = 2$$

$$G_5 = 0$$

# Bellman Expectation Equations

- Bellman Expectation Equation (BEE) for State Value Function:
- $v_{\pi}(s) = \sum_a \pi(a|s) \sum_{r,s'} p(r, s'|s, a) [r + \gamma v_{\pi}(s')]$ 
  - Expected value starting from state  $s$  and following policy  $\pi$ .
- Bellman Expectation Equation for Action Value Function
- $q_{\pi}(s, a) = \sum_{r,s'} p(r, s'|s, a) [r + \gamma \sum_{a'} \pi(a'|s') q_{\pi}(s', a')]$ 
  - Expected value starting from state  $s$ , taking action  $a$ , and thereafter following policy  $\pi$ .



Important

# Bellman Optimality Equations

- Bellman Optimality Equation (BEE) for the Optimal State Value Function:
- $$v_*(s) = \max_a \sum_{r,s'} p(r, s'|s, a) [r + \gamma v_*(s')]$$
  - Max value starting from state  $s$  and following the greedy policy  $\pi(s) = \operatorname{argmax}_a q_*(s, a)$  (the optimal policy)
- Bellman Optimality Equation for the Optimal Action Value Function
- $$q_*(s, a) = \sum_{r,s'} p(r, s'|s, a) \left[ r + \gamma \max_{a'} q_*(s', a') \right]$$
  - Max value starting from state  $s$ , taking action  $a$ , and thereafter following the greedy policy  $\pi(s) = \operatorname{argmax}_a q_*(s, a)$  (the optimal policy)

$$V(s) = \max_a (R(s, a) + \gamma \sum_{s'} T(s, a, s') V(s'))$$

$$V(s) = \max_{a_1} (R(s_1, a_1) + \gamma \sum_{s_2} T(s_1, a_1, s_2) \max_{a_2} (R(s_2, a_2) + \gamma \sum_{s_3} T(s_2, a_2, s_3) \dots))$$

$$Q(s, a) = R(s, a) + \gamma \sum_{s'} T(s, a, s') \max_{a'} Q(s', a')$$

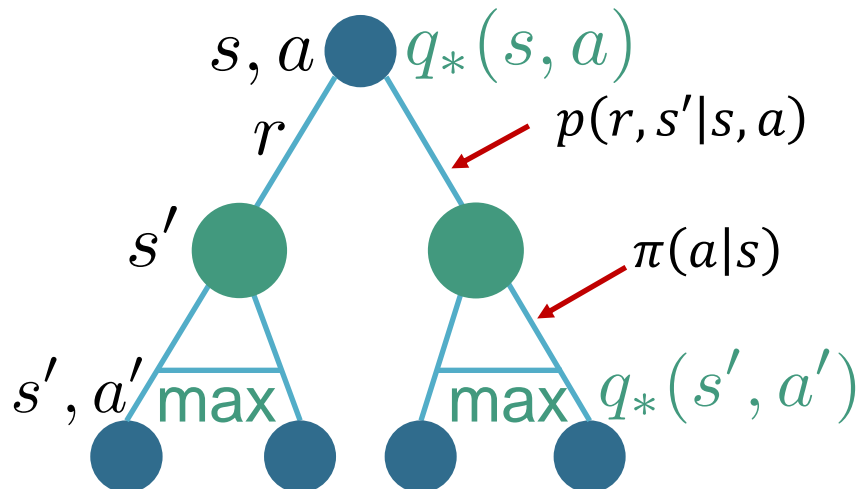
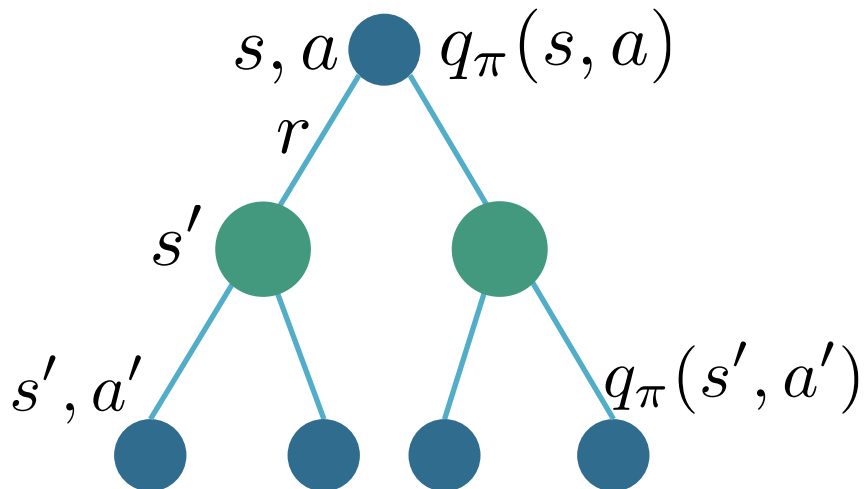
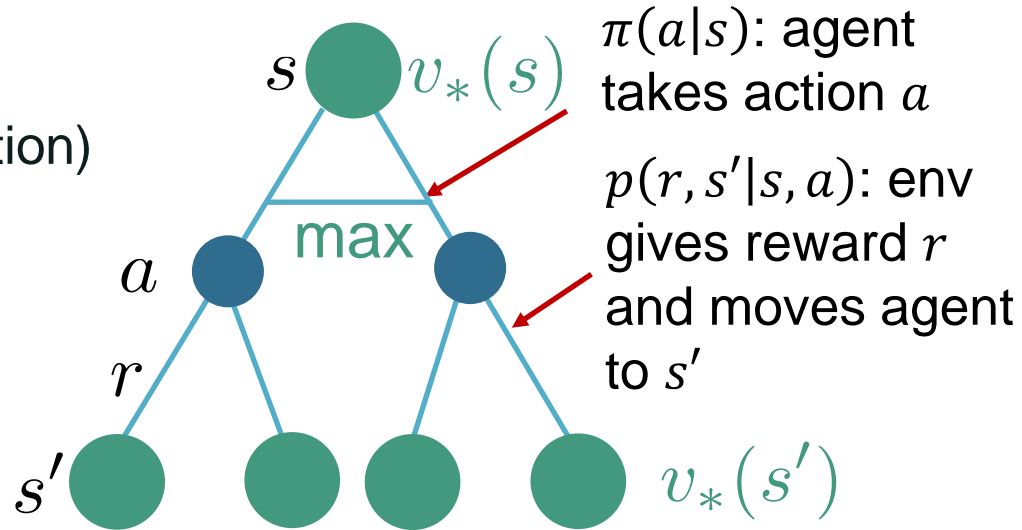
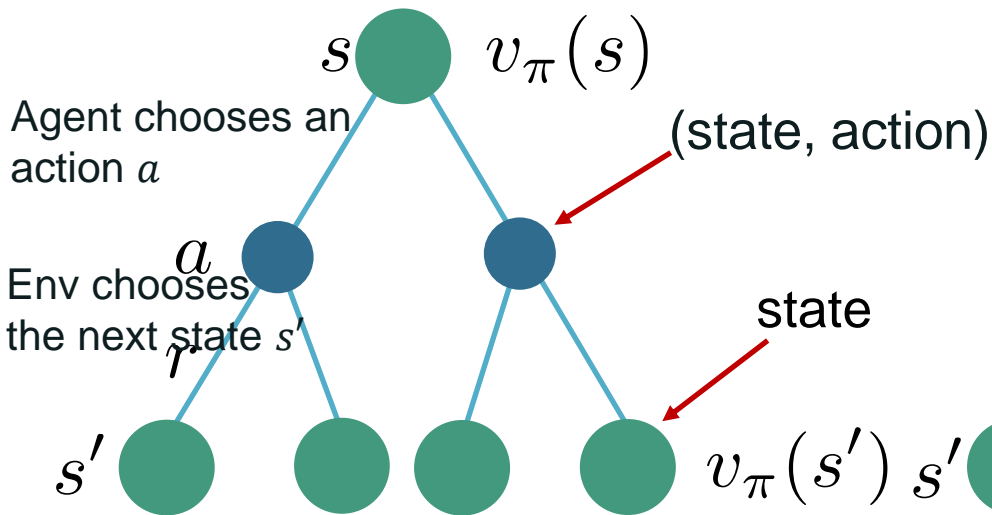
- Notations in left fig:
- $$\sum_{s'} T(s, a, s') [\dots] = \sum_{r,s'} p(r, s'|s, a) [\dots]$$
- $$R(s, a) = \sum_{r,s'} p(r, s'|s, a) r$$

# Bellman Equations written with Expectation Symbols

- BEE:
- $v_{\pi}(s) = \mathbb{E}_a \mathbb{E}_{r, s'} [r + \gamma v_{\pi}(s')]$
- $q_{\pi}(s, a) = \mathbb{E}_{r, s'} [r + \gamma \mathbb{E}_a q_{\pi}(s, a)]$
- BOE:
- $v_*(s) = \max_a \mathbb{E}_{r, s'} [r + \gamma v_*(s')]$
- $q_*(s, a) = \mathbb{E}_{r, s'} \left[ r + \gamma \max_a q_*(s, a) \right]$
- Detailed derivations:
  - $v_{\pi}(s) = \sum_a \pi(a|s) q_{\pi}(s, a) = \mathbb{E}_{a \sim \pi(a|s)} q_{\pi}(s, a) = \mathbb{E}_{a \sim \pi(a|s)} \mathbb{E}_{r, s' \sim p(r, s'|s, a)} [r + \gamma v_{\pi}(s')]$
  - $q_{\pi}(s, a) = \sum_{r, s'} p(r, s'|s, a) [r + \gamma v_{\pi}(s')] = \mathbb{E}_{r, s' \sim p(r, s'|s, a)} [r + \gamma v_{\pi}(s')] = \mathbb{E}_{r, s' \sim p(r, s'|s, a)} [r + \gamma \mathbb{E}_{a \sim \pi(a|s)} q_{\pi}(s, a)]$
  - $v_*(s) = \max_a q_*(s, a) = \max_a \mathbb{E}_{r, s' \sim p(r, s'|s, a)} [r + \gamma v_*(s')]$
  - $q_*(s, a) = \sum_{r, s'} p(r, s'|s, a) [r + \gamma v_*(s')] = \mathbb{E}_{r, s' \sim p(r, s'|s, a)} [r + \gamma v_*(s')]$

Important

# Backup Diagrams



# Further Explanations

- Starting from state  $s$ , the agent and the env plays a game:
  - Agent chooses an action  $a$  based on its policy  $\pi(a|s)$ , e.g., the car has 10% prob turning left, 10% prob turning right, 80% prob going straight
  - Env chooses the next state  $s'$  based on the MDP  $p(r, s'|s, a)$ , e.g., if the agent (car) chose to turn left,  $p(r, L|s, TurnLeft) = 90\%$ ,  $p(r, R|s, TurnLeft) = 10\%$ , e.g., there is suddenly a gust of wind that turns the car to the right
  - Agent chooses an action  $a'$  in  $s'$
  - Env chooses the next state  $s''$
  - ...
- For BEE, take the expectation over all possible actions  $a$  in state  $s$  ( $\mathbb{E}_a$ )
- For BOE, take the max (greedy action) over all possible actions  $a$  in state  $s$  ( $\max_a$ )
- For both BEE and BOE, take the expectation over all possible next states  $s'$  if agent takes action  $a$  in state  $s$  ( $\mathbb{E}_{r, s'}$ )

# $v(s)$ vs. $q(s, a)$

- State-action Value Function  $q(s, a)$  contains more information than State value function  $v(s)$ . Given  $q_*(s, a)$ , optimal policy  $\pi_*(s) = \underset{a}{\operatorname{argmax}} q_*(s, a)$ .
- **Can always** go from  $q_\pi(s, a)$  to  $v_\pi(s)$ , or from  $q_*(s, a)$  to  $v_*(s)$ :
  - $v_\pi(s) = \sum_a \pi(a|s) q_\pi(s, a)$ ;  $v_*(s) = \max_a q_*(s, a)$
- With known MDP ( $p(r, s'|s, a)$ , i.e., model-based): **can** go from  $v_\pi(s)$  to  $q_\pi(s, a)$ , or from  $v_*(s)$  to  $q_*(s, a)$ :
  - $q_\pi(s, a) = \sum_{r, s'} p(r, s'|s, a) [r + \gamma v_\pi(s')]$
  - $q_*(s, a) = \sum_{r, s'} p(r, s'|s, a) [r + \gamma v_*(s')]$
- With unknown MDP (unknown  $p(r, s'|s, a)$ , i.e., model-free) : **cannot** go from  $v_\pi(s)$  to  $q_\pi(s, a)$ , or from  $v_*(s)$  to  $q_*(s, a)$

# Simplified Bellman Equations for Deterministic Env

- Bellman Equations:
  - $v_{\pi}(s) = \sum_a \pi(a|s) q_{\pi}(s, a)$
  - $q_{\pi}(s, a) = \sum_{r,s'} p(r, s'|s, a) [r + \gamma v_{\pi}(s')]$
  - $v_*(s) = \max_a q_*(s, a)$
  - $q_*(s, a) = \sum_{r,s'} p(r, s'|s, a) [r + \gamma v_*(s')]$
- For Deterministic Env: there is only one possible  $(r, s')$  for a given  $(s, a)$  (we use  $R_s^a$  to emphasize that reward  $r$  is specific to this  $(s, a)$ ):
  - $q_{\pi}(s, a) = R_s^a + \gamma v_{\pi}(s')$
  - $q_*(s, a) = R_s^a + \gamma v_*(s')$

# Policy Evaluation

- The prediction problem: predict Value Function for given policy  $\pi$  by solving Bellman Exp. Equation for State Value Function
  - $v_{\pi}(s) = \sum_a \pi(a|s) \sum_{r,s'} p(r, s'|s, a) [r + \gamma v_{\pi}(s')] = \mathbb{E}_{\pi}[R_{t+1} + \gamma v_{\pi}(S_{t+1}) | S_t = s]$
- Can also be written as:
  - $v_{\pi}(s) = \sum_a \pi(a|s) q_{\pi}(s, a)$
  - $q_{\pi}(s, a) = \sum_{r,s'} p(r, s'|s, a) [r + \gamma v_{\pi}(s')]$  denotes the State-Action Value Function for taking action  $a$  in state  $s$ , then follow policy  $\pi$  afterwards
- A set of linear equations that can be solved analytically for small system
  - # unknowns = # equations = # states

# Grid World1: Policy Evaluation

- Non-episodic MDP w. **deterministic env**: Agent in state  $s \in \{A, B, C, D\}$  taking action  $a \in \{l, r, u, d\}$  always moves to the next state in the movement direction, unless it is blocked by the walls. Discount factor  $\gamma = 0.7$ .
- Random policy**: Agent in state  $s \in \{A, B, C, D\}$  takes a random action  $a \in \{l, r, u, d\}$  with equal probability of 0.25 each.
- Bellman Exp. Equation for det env:  $v_\pi(s) = \sum_a \pi(a|s) q_\pi(s, a)$ ;  $q_\pi(s, a) = R_s^a + \gamma v_\pi(s')$
- $v_\pi(A) = 0.25(q_\pi(A, l) + q_\pi(A, r) + q_\pi(A, u) + q_\pi(A, d)) = 0.5 \cdot 0.7v_\pi(A) + 0.25 \cdot (5 + 0.7v_\pi(B)) + 0.25 \cdot 0.7v_\pi(C)$ 
  - $q_\pi(A, l) = q_\pi(A, u) = 0 + 0.7v_\pi(A)$
  - $q_\pi(A, r) = 5 + 0.7v_\pi(B)$
  - $q_\pi(A, d) = 0 + 0.7v_\pi(C)$
- $v_\pi(B) = 0.25(q_\pi(B, l) + q_\pi(B, r) + q_\pi(B, u) + q_\pi(B, d)) = 0.25 \cdot 0.7v_\pi(A) + 0.5 \cdot (5 + 0.7v_\pi(B)) + 0.25 \cdot 0.7v_\pi(D)$ 
  - $q_\pi(B, l) = 0 + 0.7v_\pi(A)$
  - $q_\pi(B, r) = q_\pi(A, u) = 5 + 0.7v_\pi(B)$
  - $q_\pi(B, d) = 0 + 0.7v_\pi(D)$
- $v_\pi(C) = 0.25(q_\pi(C, l) + q_\pi(C, r) + q_\pi(C, u) + q_\pi(C, d)) = 0.25 \cdot 0.7v_\pi(A) + 0.5 \cdot 0.7v_\pi(C) + 0.25 \cdot 0.7v_\pi(D)$ 
  - $q_\pi(C, l) = q_\pi(C, d) = 0 + 0.7v_\pi(C)$
  - $q_\pi(C, r) = 0 + 0.7v_\pi(D)$
  - $q_\pi(C, u) = 0 + 0.7v_\pi(A)$
- $v_\pi(D) = 0.25(q_\pi(D, l) + q_\pi(D, r) + q_\pi(D, u) + q_\pi(D, d)) = 0.25 \cdot (5 + 0.7v_\pi(B)) + 0.25 \cdot 0.7v_\pi(C) + 0.5 \cdot 0.7v_\pi(D)$ 
  - $q_\pi(D, l) = 0 + 0.7v_\pi(C)$
  - $q_\pi(D, r) = q_\pi(D, d) = 0 + 0.7v_\pi(D)$
  - $q_\pi(D, u) = 5 + 0.7v_\pi(B)$
- Solution**:  $v_\pi(A) = 4.2, v_\pi(B) = 6.1, v_\pi(C) = 2.2, v_\pi(D) = 4.2$ .  $q_\pi(s, a)$  can also be obtained.

