Lecture 5
Algorithm Performance Analysis

Department of Computer Science
Hofstra University

Lecture Goals

- Calculate the big-O class of complicated code snippets.
- Define worst case, average case, and best case performance and describe why each of these is used.
- State and justify the asymptotic performance for linear search, binary search, selection sort, insertion sort, merge sort, and quick sort.
- Recognize and avoid some common pitfalls in asymptotic analysis.
- Use Java timing libraries to measure execution time.
- Use runtimes from a real system to reason about performance.
- Identify components of real systems which impact execution time.

Motivation

Algorithm: a strategy for solving a problem.

Performance: how good that strategy is.

Algorithm with good performance can answer very hard questions in very short amount of time. We need to have a sense of how good our algorithm is without just running it.

There is hereby imposed on the taxable income of every individual (other than a surviving spouse as defined in section 2(a) or the head of a hou... as defined in section 2(b)) who is not a married individual (as defined in sectio... ...termined in accordance with the following table:

How long dose this take?

If you are single, neve... ...use, and not the head of a household, you pay taxes according to the followi... ...ble:

Use flesch score to measure of text readability

$$\text{FleschScore} = 206.835 - 1.015\left(\frac{\#\,\text{words}}{\#\,\text{sentences}}\right) - 84.6\left(\frac{\#\,\text{syllables}}{\#\,\text{words}}\right)$$

The time for running the specific code on a specific machine on a specific input

Problem with just looking at the “stopwatch” time.

- different computers
- different compilers
- different libraries/optimizations

Is NOT a good representation of how good our algorithm is.

Performance Analysis Overview

- What an algorithm can control?

The number of operations

#1: Count operations instead of time

Is data size all that matters?

#3: Go beyond input size

Start at first index of array/list
While current index is less than length:
    count syllables

- large input, more operations
- small input, less operations

We'd like our performance analysis to be able to capture not just the size of the input but also what might happen because of internal structure to the input.

#2: Focus on how performance scales

Worst, Best, and Average Performance Analysis

If list is **twice** as long,
how much **more time** does it take to search it?

Asymptotic Performance Analysis

Count Operations

```
public static boolean hasLetter(String word, char letter)
{                    initial step         final check
        for (int i = 0; i < word.length(); i++) {
                if (word.charAt(i) == letter) {
                        return true;
                }
        }
        return false;
}
```

Linear search

0	1	2	3	4
H	a	p	p	y

Is the number of operations the same every time we run `hasLetter(String word, char letter)`?

Search for the letter "a" in the word "Happy"

Search for the letter "x" in the word "Happy"

NO

How many operations get executed?

Each iteration(in the middle of the algorithm) contains 3 operations

```
hasLetter("happy", ”a");
hasLetter("happy", "x");
hasLetter("apple", "a");
```

Total operations so far: 7

Total iterations: 5

Total operations: 18

Introduction to Asymptotic Analysis

- What counts as an operation?

  Basic unit that doesn't change as the input changes

- We don't need to worry about anything irrelative to input size

  Implementations of specific operations

  Initialization time

- Focus on how performance scale with the increase of input size

If input is **twice** as big,
how many **more operations** do we need?

```java
if (word.charAt(i) == letter) {
    return true;
}
```

input

Constant time

irrelative to the input size

input of size n

```java
int count = 0;
for (int i = 0; i < word.length(); i++) {
    count++;
}
```

1

1

1

1

Linear time

relative to the input size

n times

3n + 3

count i n

Asymptotic Analysis

- Asymptotic analysis examines how functions behave as their input grows arbitrarily large. It focuses on the "tail behavior" or limiting behavior of functions rather than their exact values for specific inputs.
  - runtime as input size n gets large.
  - rate of growth determined by the dominating highest-order term.
  - leading coefficient and lower-order terms fell away.
  - e.g. we don’t care if the algorithm runs for 10 ms vs. 2 s with small input size n; we care if it runs for 100 s vs. 100 hours/days/years for very large n.

Big-O Classes

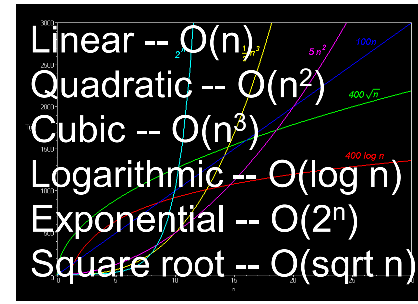
The goal is to look at the code and pick up its big-O classes. Don't worry about the formal definition too much.

f(n) = O(g(n)) means

f(n) is big-O of g(n) and they grow in same way as their input grows

there are constants N and c so that for each n > N, f(n) ≤ C g(n)

FORMAL



- We use big-O classes as a tool to phrase how algorithm performance scale.
- Two functions are in the same big-O class if they have the same rate of growth.
- Other notations represent a finer-grained asymptotic analysis, such as lower and upper bound. We focus on big-O for the tightest upper bound.
- How to compute big O?

Drop constants

Example: initialization cost, whose number of steps doesn't change with input size n

10000000 = O(1)

Keep only dominant term

Fastest growing

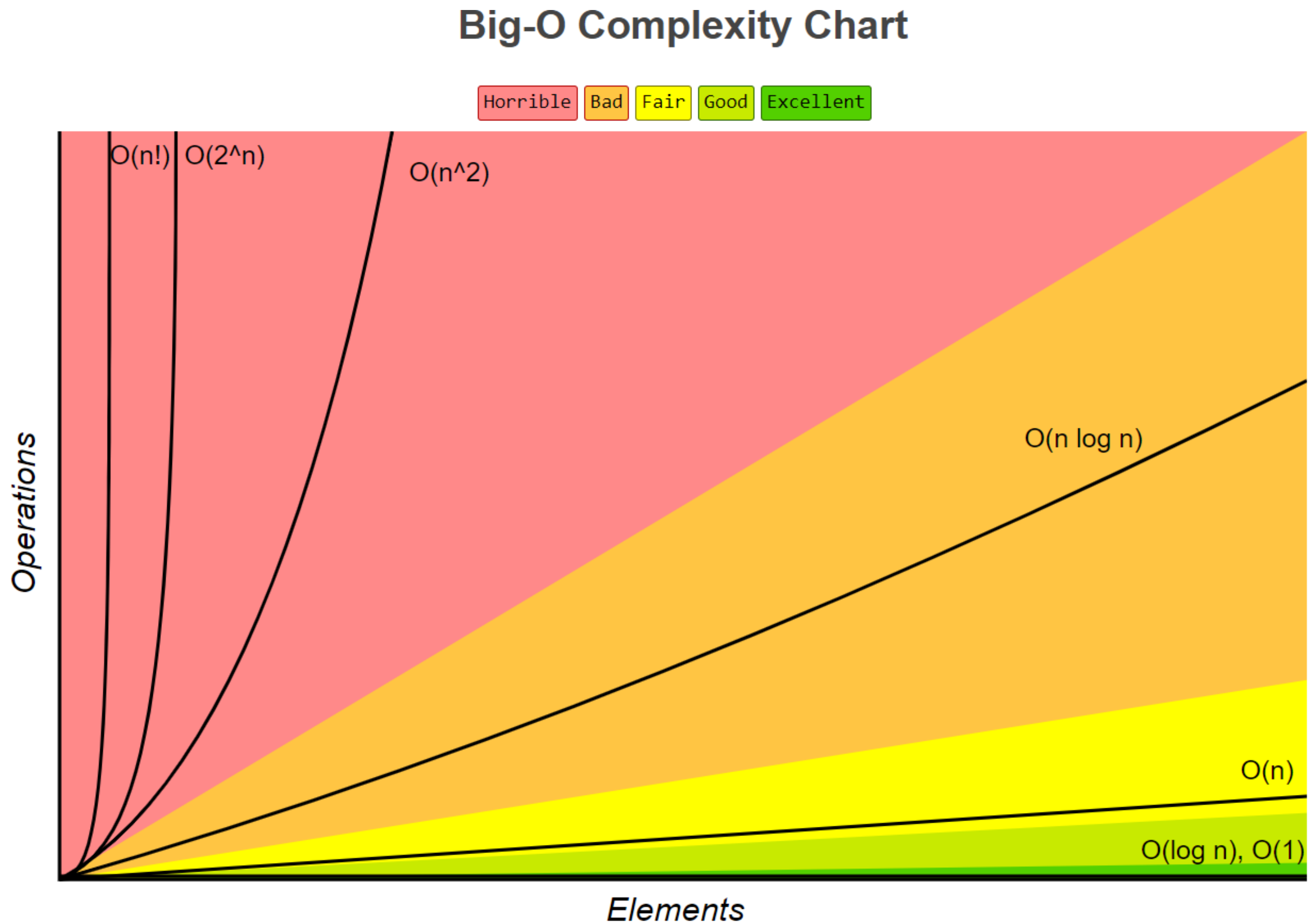
3n+3 = O(3n) = O(n)

Asymptotic comparison operator	Numeric comparison operator
Our algorithm is o(something)	A number is < something
Our algorithm is O(something)	A number is ≤ something
Our algorithm is Θ(something)	A number is = something
Our algorithm is Ω(something)	A number is ≥ something
Our algorithm is ω(something)	A number is > something

Big-O notation in 5 minutes
https://www.youtube.com/watch?v=__vX2sjlpXU

Big-O Complexity Chart

- $O(1) < O(\log n) < O(n) < O(n \log n) < O(n^2) < O(2^n) < O(n!)$



https://www.bigocheatsheet.com/

Quiz

- 1. Suppose algorithm running time for input size n is $g(n) = 2^n + n^2 + 100$, what is its complexity in big-O notation?
- ANS: $O(2^n)$
- For $g(n) = 3n \log n + 4 \log n \ + n^2 + n$,
- ANS: $O(n^2)$
- For $g(n) = 3n \log n + 4 \log n \ + n$,
- ANS: $O(n \log n)$
- 2. If Algorithm 1 has complexity $O(\log n)$, Algorithm 2 has complexity $O(n^2)$, will Algorithm 1 always have fewer operations (shorter running time) than Algorithm 2?
- ANS: No. If Algorithm 1 has running time $100000 * \log n$, Algorithm 2 has running time $3n^2$, then $100000 * \log n > 3n^2$ for small n.

Quiz Con't

- 3. Suppose algorithm running time for input size n is
  - $n^2 + n + \log n$
  - $n * (n - i) + n + \log n$ (i is a loop iteration variable within 1 to n)
  - $0.001 * n^2 + 1000 * n + 10000 * \log n$
  - $(n + 100)^2 + (100 * (n + 100000)) + 100 * \log n$

  What is the big O notation for the algorithm complexity in each case?
- ANS: $O(n^2)$
  - Ignore all the constants, whether multiplied or added, and take the dominating term that grows the fastest
- What is the answer if 2^n is added to each term?
- ANS: $O(2^n)$

Compute Big O for Consecutive Code

```
public static void reduce (int[] vals) {
        int minIndex =0;                                O(1) +
        for (int i=0; i < vals.length; i++) {
                if (vals[i] < vals[minIndex]){
                        minIndex = i;                   O(n) +
                }}
        int minVal = vals[minIndex];                    O(1) +
        for (int i=0; i < vals.length; i++){
                vals[i] = vals[i] - minVal;             O(n) +
        }
}
```

[1,2,5,3] -> [0,1,4,2]

The first for loop finds the minimum value of the array. The second for loop reduces each value in the array by the minimum value.

- Run times are <u>independent</u>
- These <u>doesn't</u> depend on the input size(n)
- How the operations depend on the size of the input?
- There will be <u>n loop iterations</u>
- Each iteration will take <u>constant time</u>

$1 + n + 1 + n = 2n + 2 = \cancel{2}n + \cancel{2}$

Total: O(n)

Linear Algorithm

Compute Big O with Nested Operations

```
public static int maxDifference (int[] vals) {
    int max = 0;
    for (int i=0; i < vals.length; i++) {
        for (int j=0; j < vals.length; j++) {
            if (vals[i] – vals[j] > max) {
                max = vals[i] – vals[j];
            }
        }
    }
    return max;
}
```

$O(1)$ +

$O(1)$

$O(n)$ X

Multiplication

$O(1)$ + $O(n^2)$ +

$1 + n^2 + 1 = n^2 + \cancel{2}$

Total: $O(n^2)$

Quadratic Algorithm

[1,7,2,4,6,8] -> 7

The nested for loops look for the maximum difference between any two array elements. This biggest difference will be between 1 and 8.

- Run times are independent
- These doesn't depend on the input size(n)
- How the operations depend on the size of the input?
- Count from inside out
- There will be n inner loop iterations and each takes constant time
- There will be n outer loop iterations and each takes linear time O(n)

Short Videos of Sorting Algorithms

- Sort Algos // Michael Sambol Michael Sambol
  - https://www.youtube.com/playlist?list=PL9xmBV_5YoZOZSbGAXAPIq1BeUf4j20pl
  - Merge Sort, Quick Sort, Bubble Sort, Insertion Sort, Selection Sort, Heap Sort
- 10 Sorting Algorithms Easily Explained
  - https://www.youtube.com/watch?v=rbbTd-gkajw
  - Bubble Sort, Selection Sort, Insertion Sort, Merge Sort, Quick Sort, Heap Sort, Counting Sort, Shell Sort, Tim Sort, Radix Sort

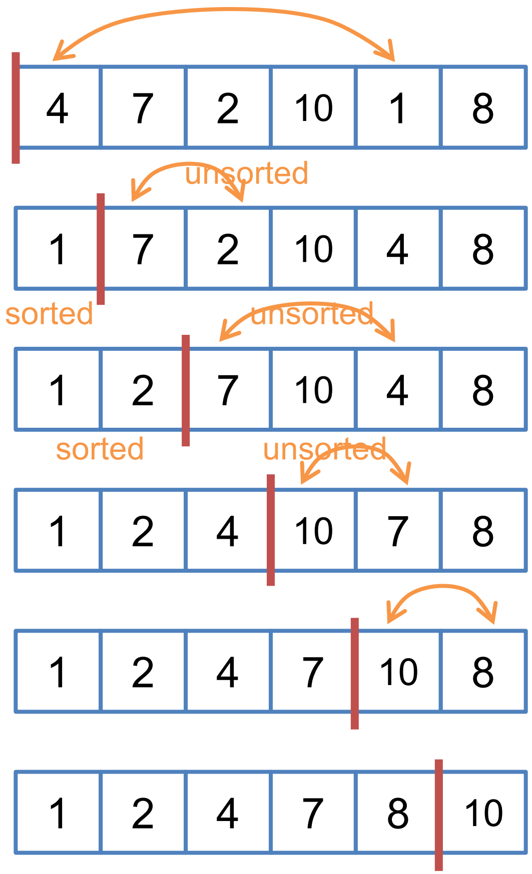
Practice: Analyze Big-O Class of Selection Sort

The idea is to find the smallest value in the remaining unsorted array and put that at the start. And then just keep repeating that process over and over.

$$n + (n-1) + (n-2) + \ldots + 1 = \frac{n \times (n+1)}{2}$$

(Gauss sum)

O(n-i)*O(n)? NO




```
public static void selectionSort(int[] vals) {
    int indexMin;                                   O(1)
    for(int i = 0; i < vals.length-1; i++) {
        indexMin = i;                               O(1)
        for(int j = i + 1; j < vals.length; j++) {
            if(vals[j] < vals[indexMin]) {
                indexMin = j ;
            }                                       O(1)
        }                                           O(n-i)
        swap (vals, indexMin , i);                  O(1)   O(n-i)
    }                                               O(n^2)
}
```

outer loop runs n times

nested loop

inner loop runs n - (i+1) times

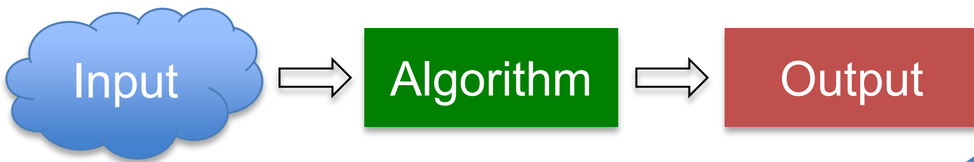
Total: $O(n^2)$

To swap:

```
temp = vals[indexMin];
vals[indexMin] = vals[i];
vals[i] = temp;
```

Best case, Average case, Worst case

- How does the algorithm behave for <u>all</u> inputs?



when will the largest amount of operations be executed?

(same input size)

```
public static boolean hasLetter(String word, char letter)
{
        for (int i = 0; i < word.length(); i++) {
                if (word.charAt(i) == letter) {
                        return true;
                }
        }
        return false;
}
```

input

output

when is the least number of operations be executed?

hasLetter("apple", "a");

Best case: word starts with letter O(1)

- <u>just</u> 1 loop iteration
- special case

hasLetter("happy", "x");

hasLetter("happy", ”y");

Worst case : letter at the end (or missing) O(n)

- n loop iterations

- algorithm performance depends on the combination of both inputs

- How can we account for this variability?

sandbox

(lower bound)		(upper bound)	(realistic,but too hard)
Best case	≤	**Worst case**	**Average case**
Best possible performance of algorithm for any input (of fixed size n)		Worst possible performance of algorithm for any input (of fixed size n)	Performance of algorithm on average, consider all possible inputs of size n

Analyze Search Algorithms

	Best Case	Worst Case
Linear Search	O(1)	O(n)
Binary Search*	O(1)	O(log(n))

* Assuming data is sorted

sorting cost?

times to half size?

How many times can we divide by 2 before we get to 1?

Linear Search: Basic Algorithm

Start at the first **index** in the array

while **index < length** of the array:
if toFind matches current value,
return true
increment index by 1

return false

E.g. hasLetter(String word, **char** letter)

Binary Search: Basic Algorithm

Initialize **low = 0, high = length of list**

while low <= high:
mid = (high+low)/2
if toFind matches value at mid,
return true
if toFind < value at mid
high = mid-1 first half
else low = mid+1 second half
return false

Cuts search base in half at each iteration, so the total # iterations is $\log_2(n)$

Worst case: don't find!

Binary search in 4 minutes
https://www.youtube.com/watch?v=fDKIpRe8GW4

$\log_{10}(n) = O(\log_2(n))$ since: $\log_{10}(n) = \dfrac{\log_2(n)}{\log_2(10)}$

Analyze Sorting Algorithms

	Best Case	Worst Case
Selection Sort	$O(n^2)$	$O(n^2)$
Insertion Sort	$O(n)$	$O(n^2)^*$

when already sorted

1	2	3	4	5	6

when in reverse order

6	5	4	3	2	1

* similar to selection sort analysis

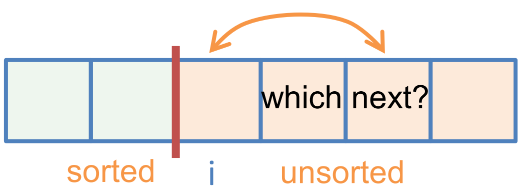
```
public static void insertionSort(int[] vals) {
        int currInd;
        for(int pos=1; pos < vals.length; pos++) {
                currInd = pos ;
                while (currInd > 0 &&
                        vals[currInd] < vals[currInd-1]) {
            swap(vals, currInd, currInd-1);
                currInd = currInd – 1;
                }
        }
}
```

- # of loop iterations depends on the values of the pairs of consecutive elements

Selection Sort: Basic Algorithm

For each **position i** from **0** to **length-2**

Find smallest element in **positions i** to **length-1**
Swap it with element in **position i**

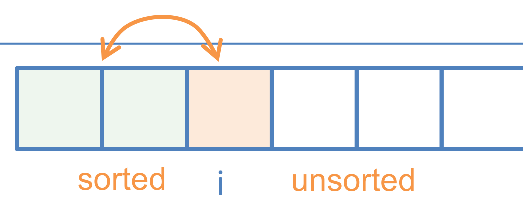


Best, average, worst?

Insertion Sort: Basic Algorithm

For each **position i** from **1** to **length-1**

Swap successive pairs to put value in **position i** in correct location relative to earlier values



1	8	4	3	7	2	pos 1
1	4	8	3	7	2	pos 2
1	3	4	8	7	2	pos 3
1	3	4	7	8	2	pos 4
1	2	3	4	7	8	pos 4

Analyze Merge Sort

<u>Merge Sort: Basic Algorithm</u>

If list has one element, return.

Divide list in half

Keep dividing by two until lists have size 1
$\log_2(n)$

Sort first half

Sort second half

Each time we divide, we call MergeSort on two (smaller) lists

HOW? Divide and conquer.
Recursion!

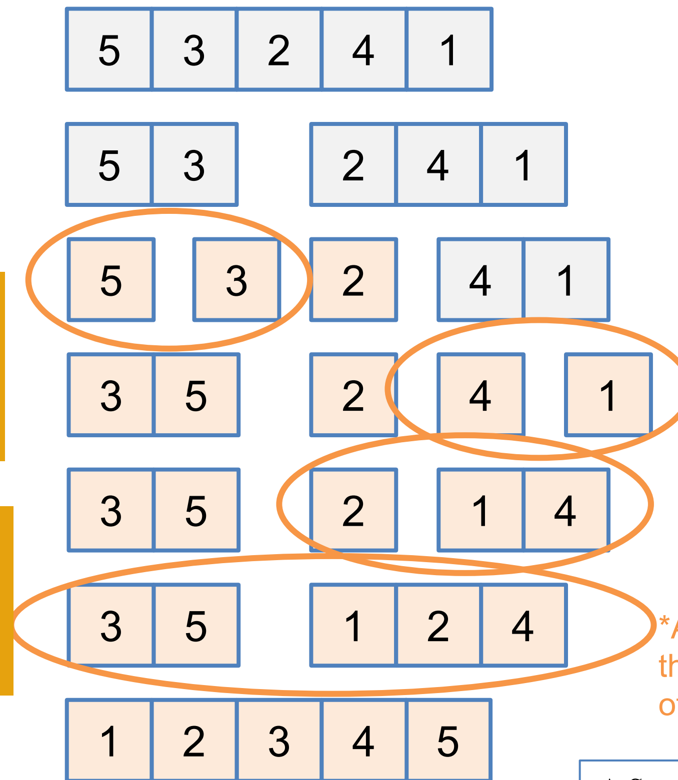
Merge sorted lists

O(n) work to merge all the lists on one level

compare the head of each list

Performance?

O(nlog(n))



*Asymptotics is not the only measure of performance

A Summary of Sorting

	Best	Average	Worst
Selection Sort	$O(n^2)$	$O(n^2)$	$O(n^2)$
Insertion Sort	O(n)	$O(n^2)$	$O(n^2)$
Merge Sort	O(nlog(n))	O(nlog(n))	O(nlog(n))
*Quick Sort	O(nlog(n))	O(nlog(n))	$O(n^2)$

Introduction to Benchmarking

www.speedtest.net

Your Java Code Version A

~10 seconds

Your Java Code Version B

~5 seconds

Times might not be consistent…

The running time of a program is influenced by many things!

Your Java Code ⇒ Java Compiler ⇒ Java Virtual Machine

bytecode

abstraction

Makes choices that affect performance

Operating System

Hardware

abstraction for hardware resource

Linux

Windows

android

Mac OS

These systems are MEANT to be hidden from you

So how do we reason about how long it takes for a program to run on real systems? Couldn't we just time how long our programs take? YES!

Details of Benchmarking (Using Java Timing API)

- Just means running programs on real machines and measuring performance
- For us, "performance" is just how long it takes for something to execute.
- Allows us to compare machines by running the same program
- Allows us to compare programs on a single machine

Same program

vs.

Program B

vs.

Program A

We'll do this, next!

java.lang

Class System

static long	nanoTime() Returns the current value of the running Java Virtual Machine's high-resolution time source, in nanoseconds.

```
public static void main(String [] args) {
        // set some size n
        double array[] = new double[n];
        // fill the array with contents (random)
        long startTime = System.nanoTime();
        selectionSort(array);
        long endTime = System.nanoTime();
        double estTime = (endTime-startTime) /
                                1000000000.0;
        System.out.println(estTime);
}
```

we just want this time

How long does selection sort run?

Idea for Analyzing Our Sorts

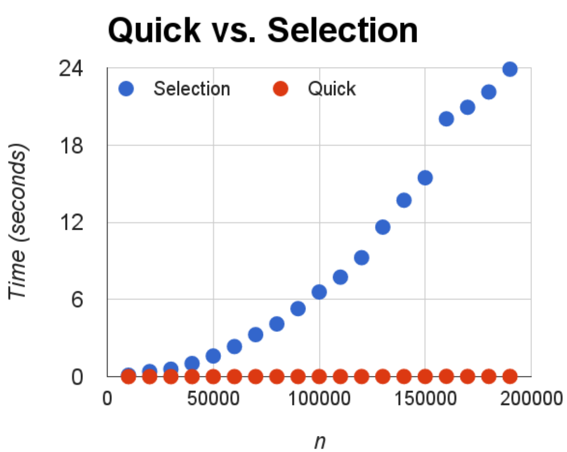
For increasing sizes of n

Print n

Create a randomized array of size n
Time **selection sort**, print outcome

Create a randomized array of size n
Time **quick sort**, print outcome

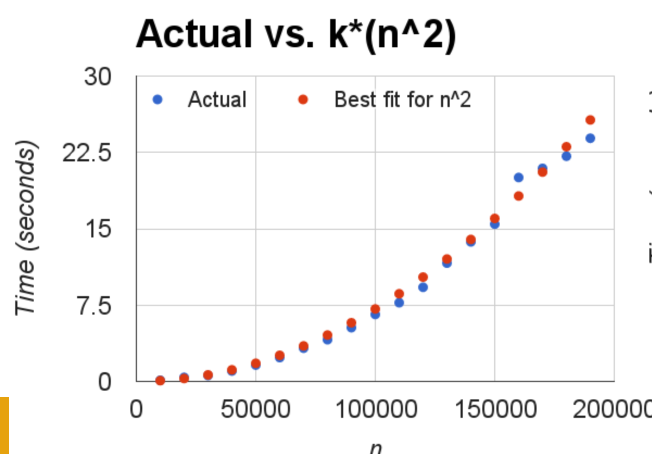
n	Selection (s)	Quick (s)
10000	0.112887621	0.001323534
20000	0.397227565	0.001568662
30000	0.580318935	0.002420492
40000	1.020979179	0.003304295
50000	1.605557659	0.004232703
60000	2.340087449	0.004983088
70000	3.264979954	0.006035047
80000	4.097073897	0.006989112
90000	5.285101776	0.007900941
100000	6.57904119	0.008538038



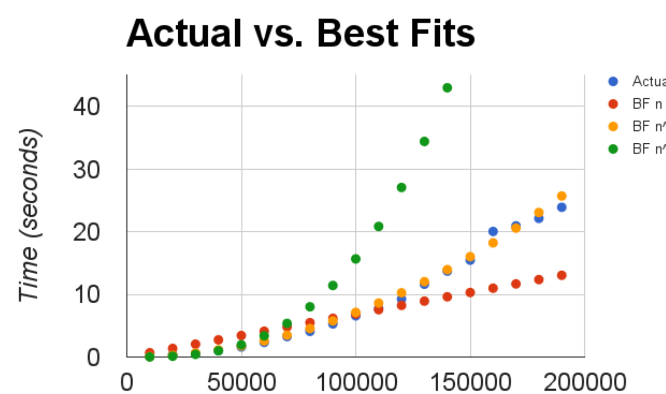
Quicksort is fantastic

Select: Looks like n^2 growth

By “best fit” I just found a good value for constant “k”

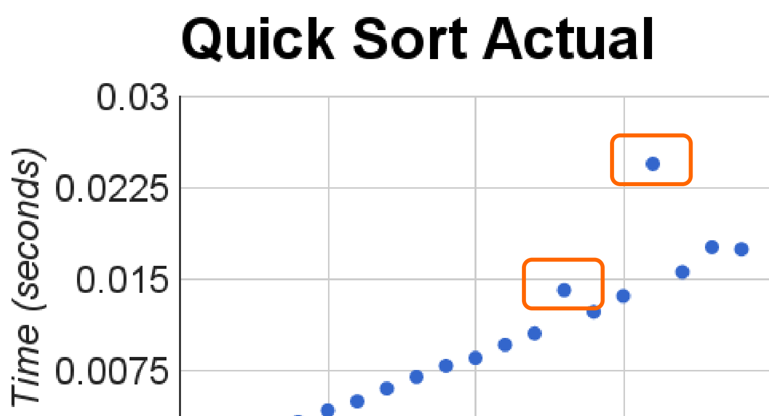


Won’t all “best fits” look really good?



n^2 is best, matching our high-level analysis

Idea for Analyzing Our Sorts (Contd.)



Zoom in on quick sort

Real data…

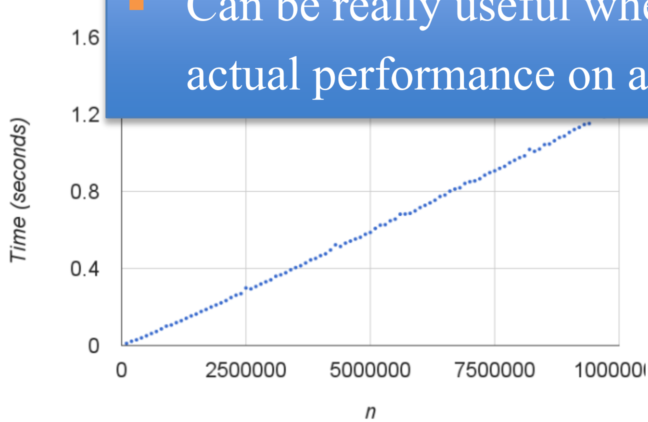
- input
- system task

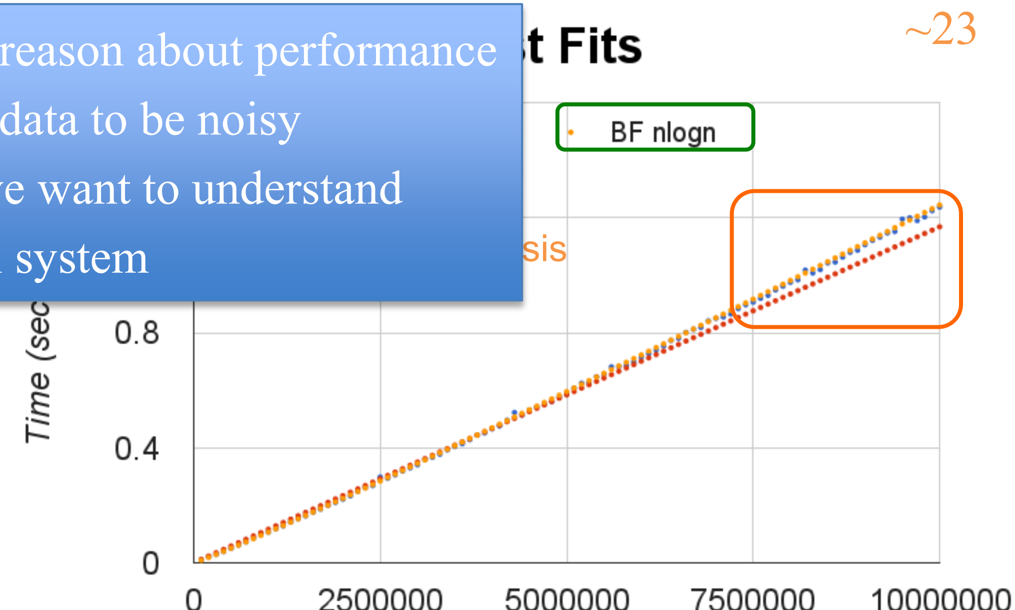
Looks linear?

n	$\log_2 n$
10,000,000	?

~23

- We can use real runtimes to reason about performance
- Be prepared for real system data to be noisy
- Can be really useful when we want to understand actual performance on a real system





Get more data…

Still appears linear

log(n) is just really small relative to n

Additional Resources

- Big-O analysis
  - http://web.mit.edu/16.070/www/lecture/big_o.pdf -- Big O handout from MIT
  - https://www.interviewcake.com/article/java/big-o-notation-time-and-space-complexity -- explanation of Big O with examples
  - http://discrete.gr/complexity/ -- "A Gentle Introduction to Algorithm Complexity Analysis" GIves a lot more detail than what we provided.
- Sorting algorithms
  - http://www.java2novice.com/java-sorting-algorithms/ -- 5 different sort algorithm explanation with codes
  - https://www.cs.cmu.edu/~adamchik/15-121/lectures/Sorting%20Algorithms/sorting.html -- different search algrotihms with solid examples
- Timing code in Java
  - http://stackoverflow.com/questions/180158/how-do-i-time-a-methods-execution-in-java -- many ways offered by many people