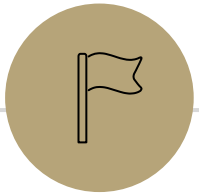
Lecture 11
Heaps

Department of Computer Science
Hofstra University

Acknowledgement: Lecture slides based on UofW Course on Data Structures



Priority Queue ADT

Binary Heap

Binary Heap Methods

Priority Queue ADT

Priority Queues are commonly used for sorting

If a Queue is "First-In-First-Out" (FIFO) Priority Queues are "Most-Important-Out-First"

Items in Priority Queue must be comparable – The data structure will maintain some amount of internal sorting, in a sort of similar way to BSTs/AVLs

Min Priority Queue ADT

state

Set of comparable values - Ordered based on "priority"

behavior

removeMin() – returns the element with the smallest priority, removes it from the collection

peekMin() – find, but do not remove the element with the smallest priority

add(value) – add a new element to the collection

Max Priority Queue ADT

state

Set of comparable values - Ordered based on "priority"

behavior

removeMax() – returns the element with the largest priority, removes it from the collection

peekMax() – find, but do not remove the element with the largest priority

add(value) – add a new element to the collection

Implementing Priority Queues: Take I

Maybe we already know how to implement a priority queue.
How long would removeMin and peek take with these data structures?

Implementation	add	removeMin	Peek
Unsorted Array	$\Theta(1)$	$\Theta(n)$	$\Theta(n)$
Linked List (sorted)	$\Theta(n)$	$\Theta(1)$	$\Theta(1)$
AVL Tree	$\Theta(\log n)$	$\Theta(\log n)$	$\Theta(\log n)$

For Array implementations, assume you do not need to resize.
Other than this assumption, do worst case analysis.

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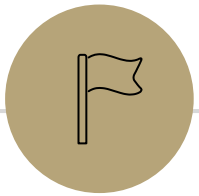
AVL Tree	$\Theta(\log n)$	$\Theta(\log n)$	~~$\Theta(\log n)$~~ $\Theta(1)$

Add a field to keep track of the min.
Update on every insert or remove.

AVL Trees are our baseline – let's look at what computer scientists came up with as an alternative, analyze that, and then come back to AVL Tree as an option later



Priority Queue ADT

Binary Heap

Binary Heap Methods

Heaps

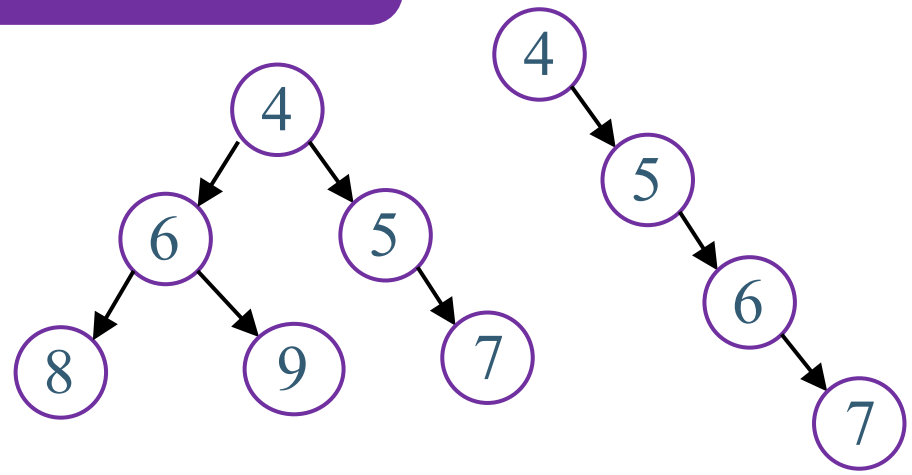
In a BST, we organized the data to find anything quickly. (go left or right to find a value deeper in the tree)

Now we just want to find the smallest item fast, so let's write a different invariant:

Heap invariant
Every node is less than or equal to both of its children.

In particular, the smallest node is at the root!

Do we need more invariants?



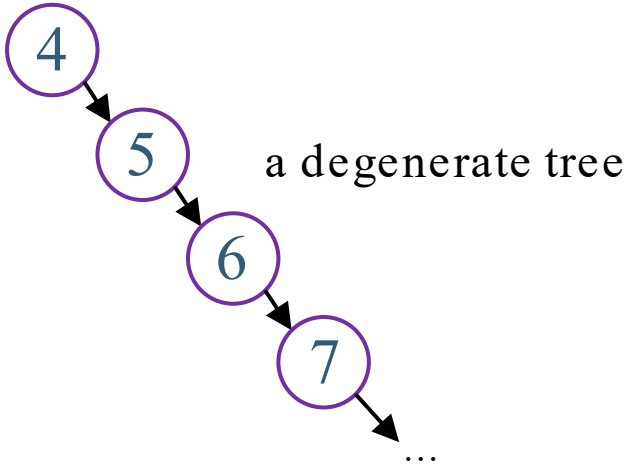
Heaps

We want to avoid degenerate trees (linear linked lists).

The heap invariant is less strict (looser) than the BST invariant, so we can impose stricter invariants on tree structure

- A BST is an ordered, or sorted, binary tree, with the following invariants:
- For every node with key k:
  - The left subtree has only keys smaller than k
  - The right subtree has only keys greater than k
  - This invariant applies recursively throughout tree

Recall: BST Invariant



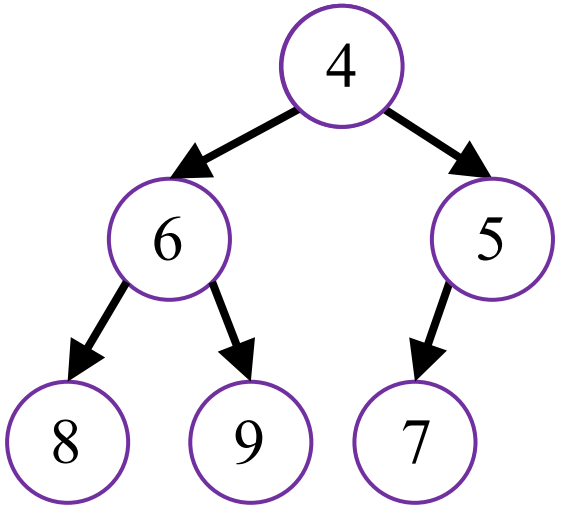
Heaps

Heap structure invariant:
A heap is always a complete tree.

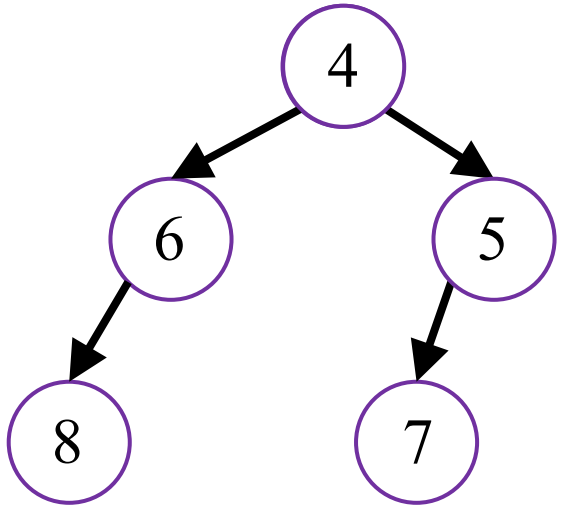
→ helps to avoid degenerate trees

A tree is complete if:

- Every row, except potentially the last, is completely full
- The last row is filled from left to right (no "gap")



complete

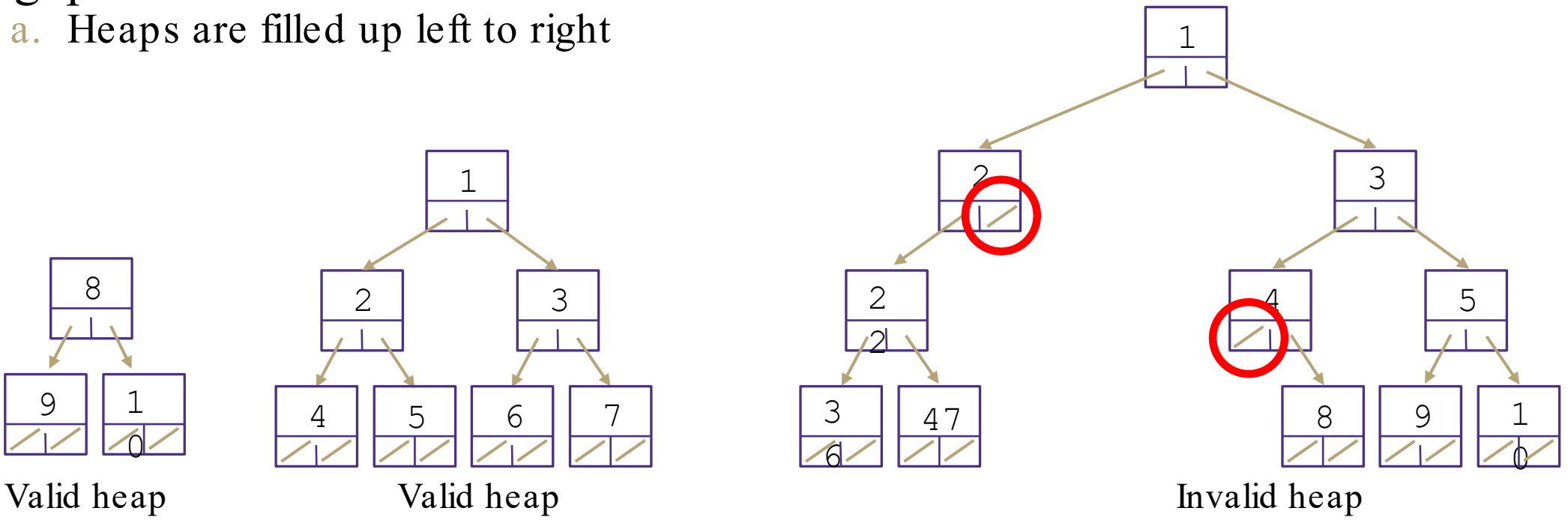


not complete

Binary Heap invariants

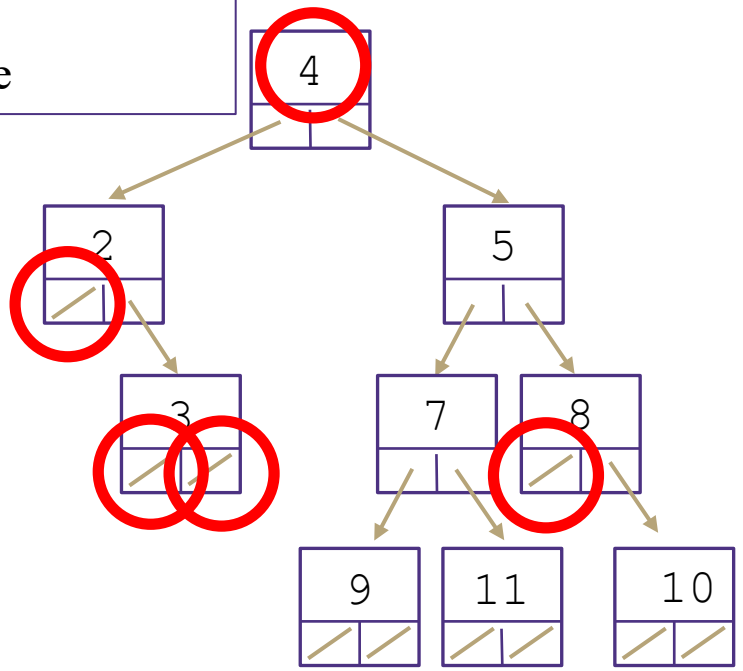
A binary heap satisfies the following invariants:

1. Binary Tree: every node has at most 2 children
2. Heap invariant: every node is smaller than (or equal to) its children
3. Heap structure invariant: each level is "complete" meaning it has no "gaps"
   a. Heaps are filled up left to right



Quiz - Are these valid heaps?

Binary Heap
Invariants:
1. Binary Tree
2. Heap
3. Complete



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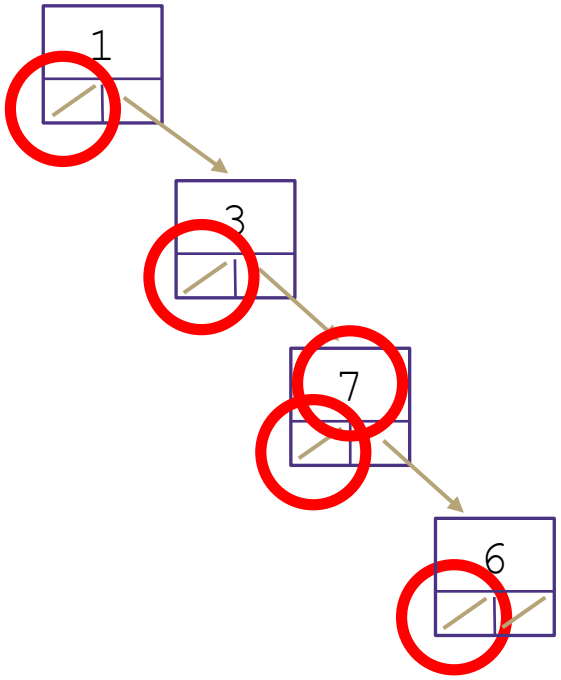

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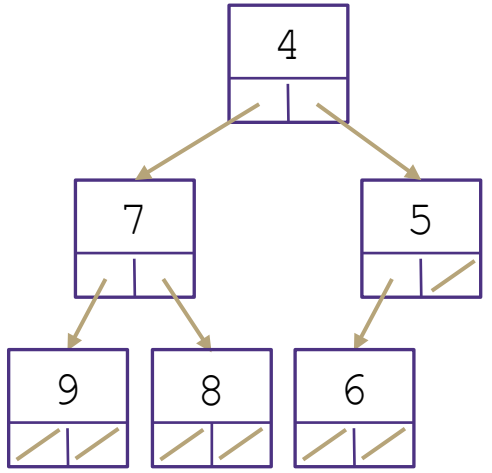

Valid

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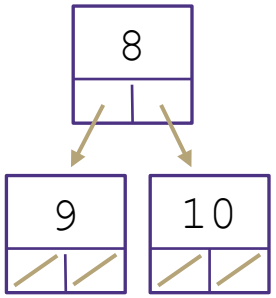

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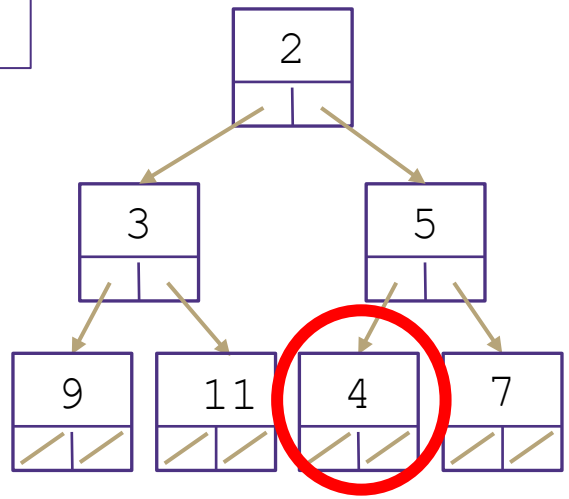

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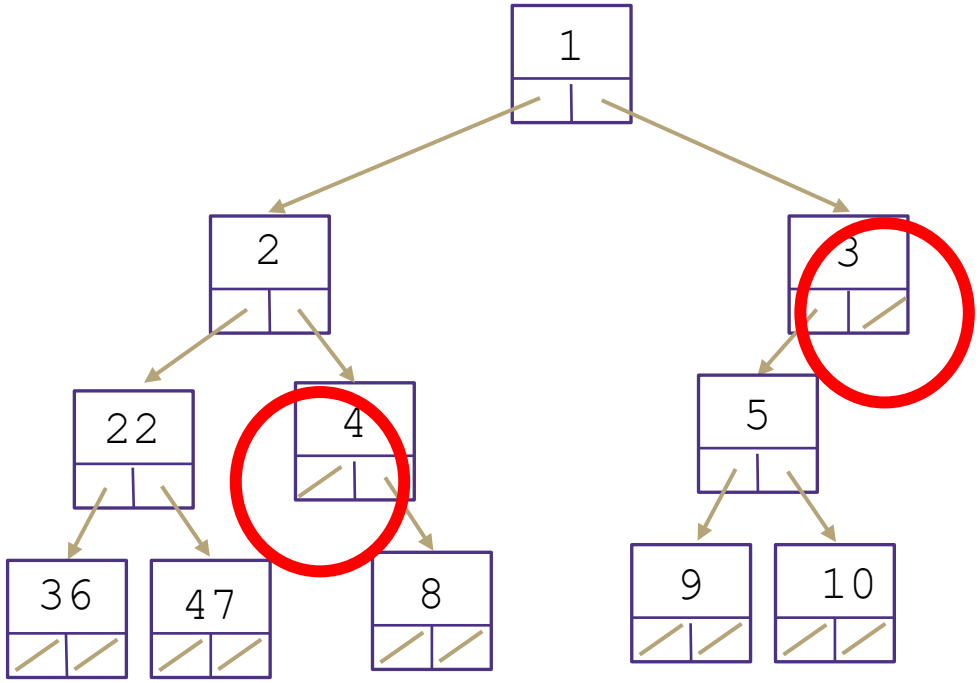
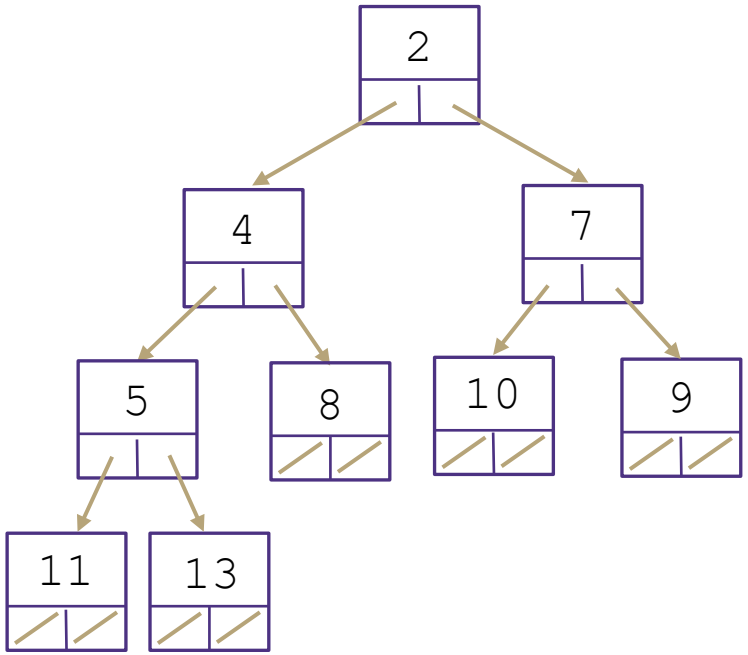


Invalid

Heap heights

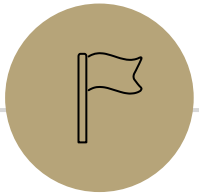
A binary heap bounds our height at O(log(n)) because it's complete – and it's actually a little stricter and better than AVL.

This means the runtime to traverse from root to leaf or leaf to root will be log(n) time.



Priority Queue ADT

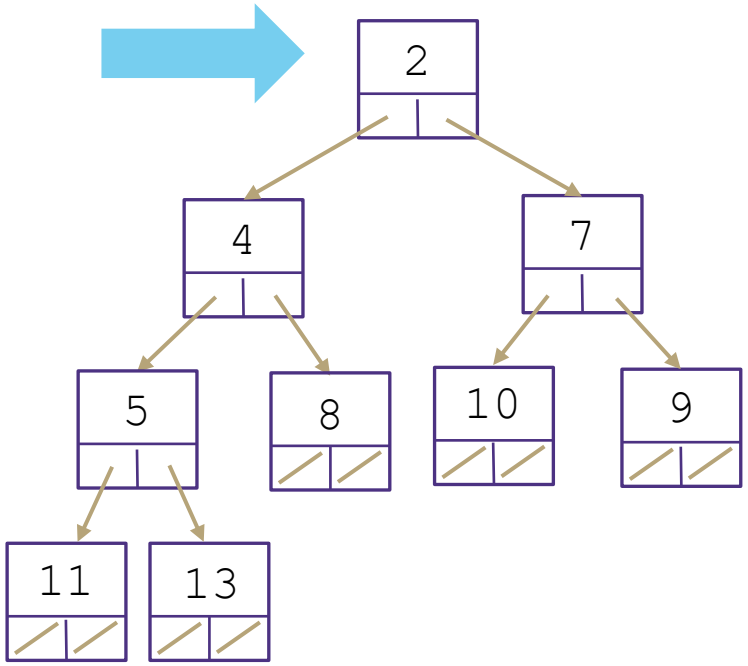
Binary Heap



Binary Heap Methods

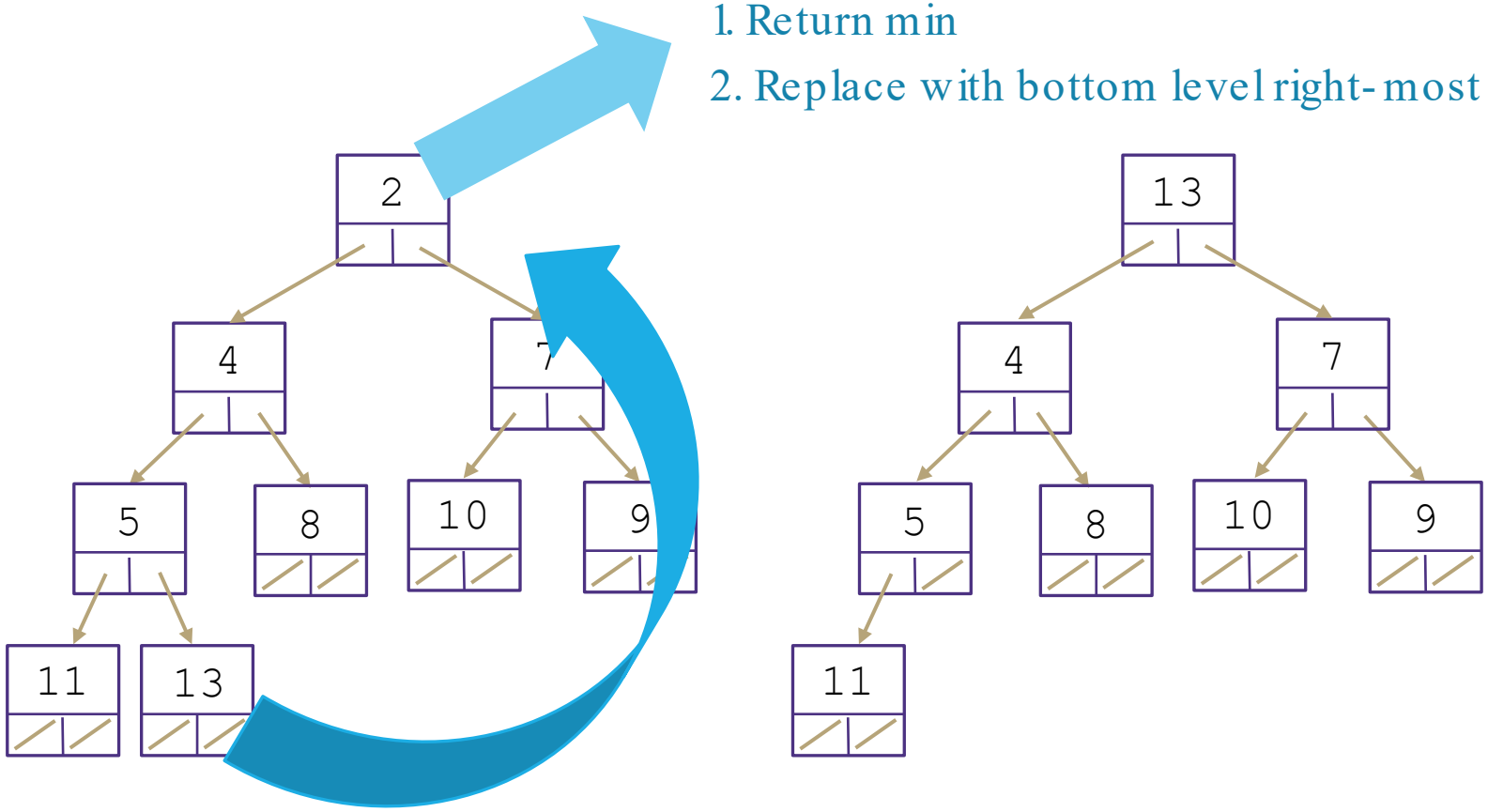
Implementing peekMin()

Runtime: $\Theta(1)$



Implementing removeMin()

1. Return min
2. Replace with bottom level right-most node



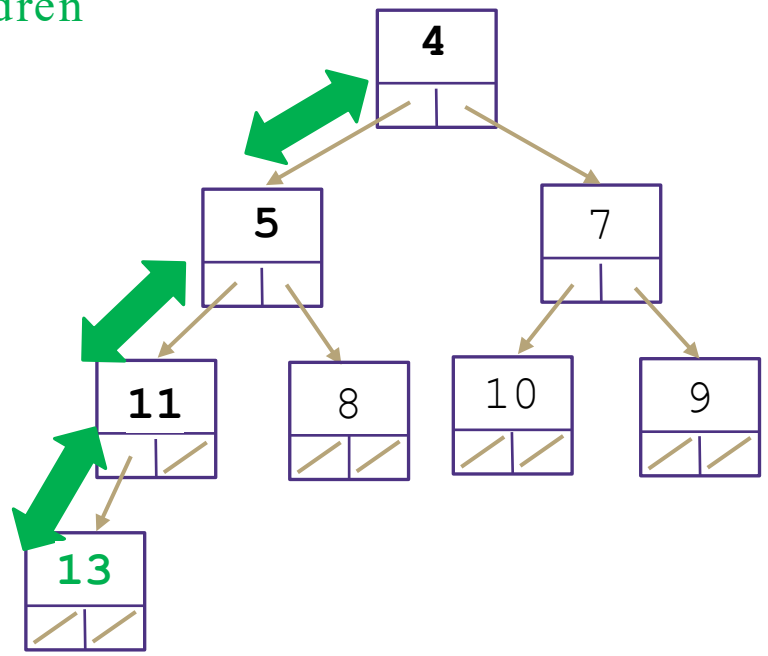
Structure invariant restored

Heap invariant broken

Implementing removeMin() - percolateDown

1. Return min
2. Replace with bottom level right-most node
3. percolateDown()

Recursively swap parent with smallest child until parent is smaller than both children (or we're at a leaf).



Structure invariant restored
Heap invariant restored

What's the worst-case running time?

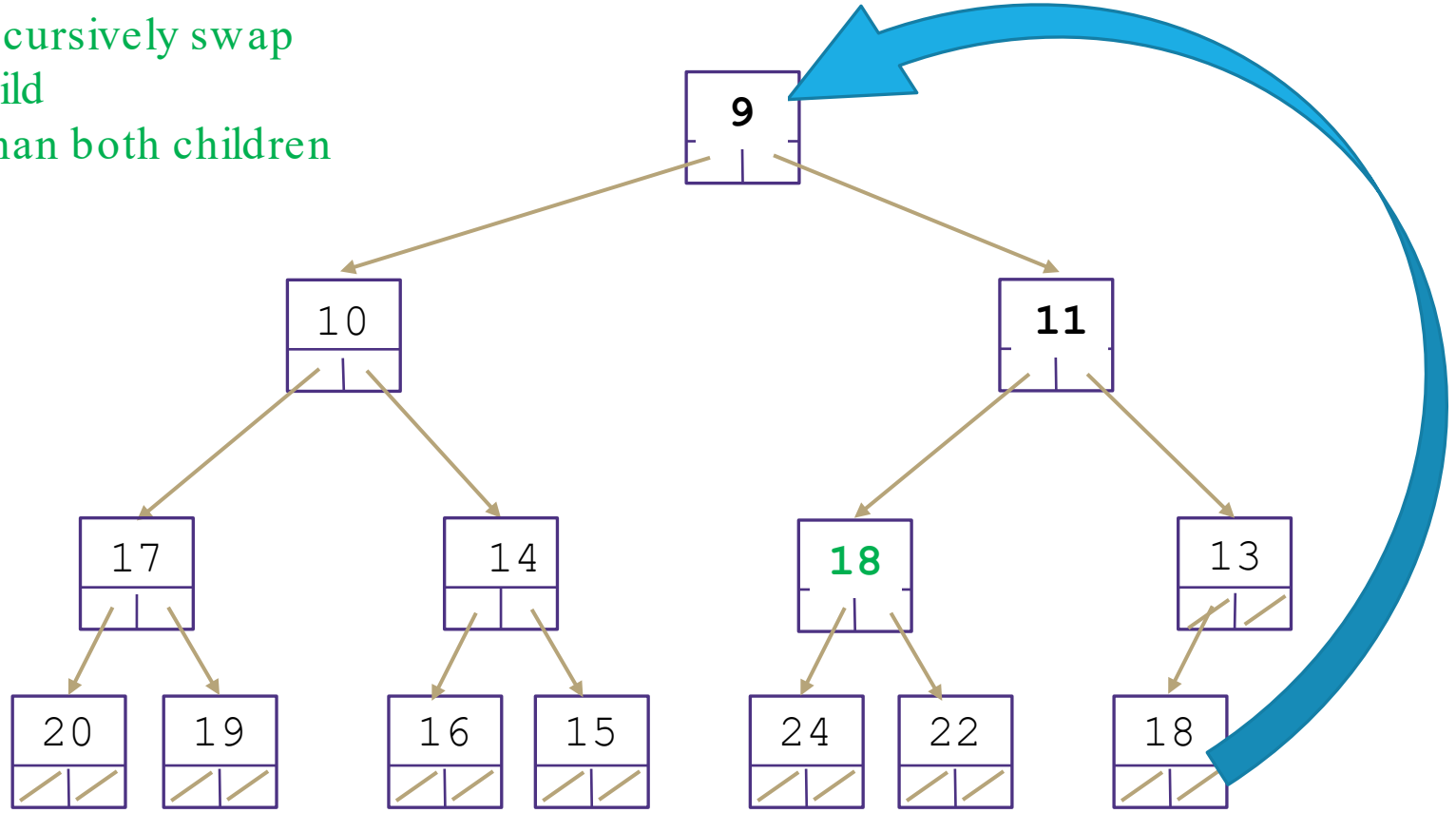
Have to:

- Find last element
- Move it to top spot
- Swap until invariant restored
- Number of swaps is O(TreeHeight)

Hence we want to keep tree height small, as tree height (BST, AVL, heaps) directly correlates with worst-case runtimes

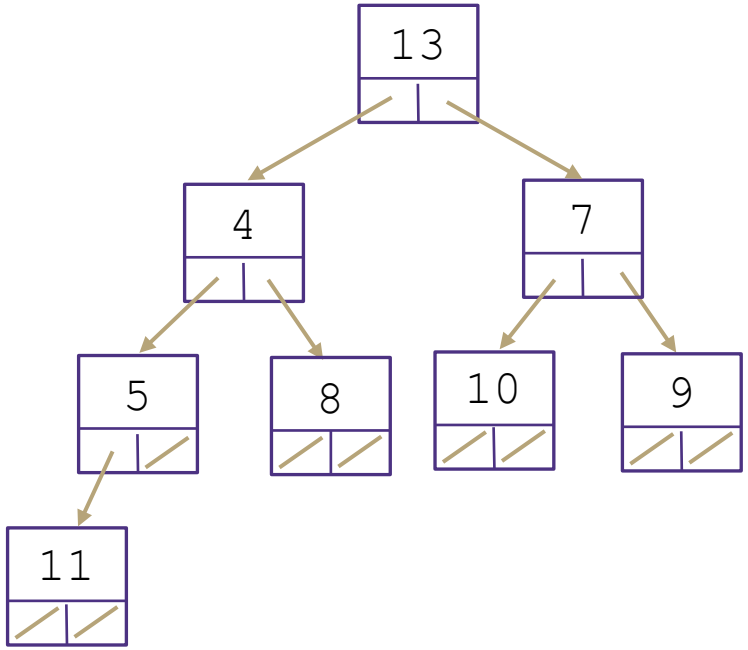
Practice: removeMin()

1.) Remove min node
2.) replace with bottom level right-most node
3.) percolateDown - Recursively swap parent with smallest child
until parent is smaller than both children (or we're at a leaf).



percolateDown()

Why does `percolateDown` swap with the smallest child instead of just any child?



If we swap 13 and 7, the heap invariant isn't restored!

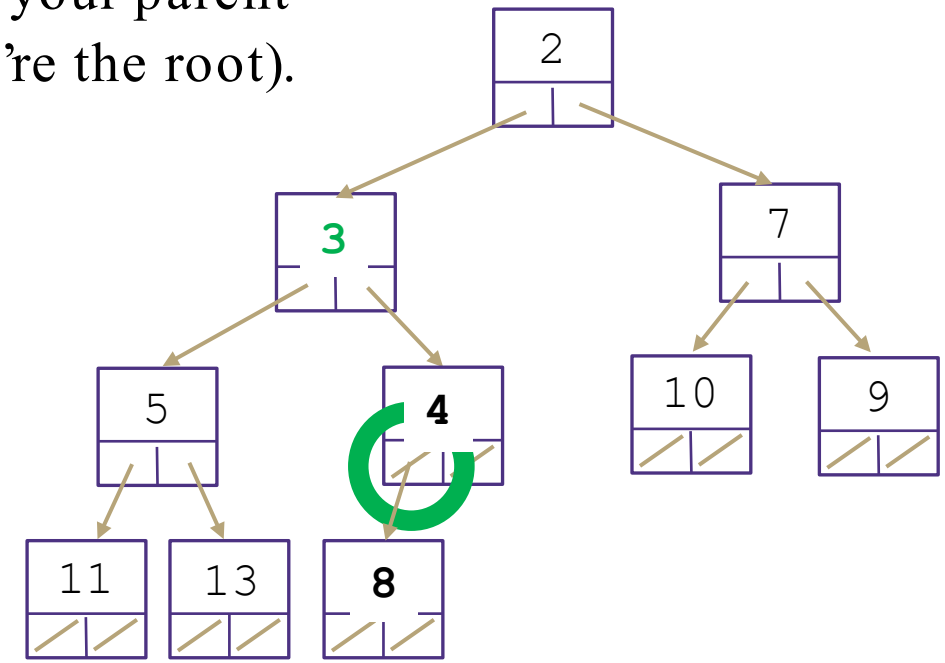
7 is greater than 4 (it's not the smallest child!) so it will violate the invariant.

Implementing add()

add() Algorithm:

1. Insert a node on the bottom level that ensure no gaps
2. Fix heap invariant by percolate UP

i.e. swap with parent, until your parent is smaller than you (or you're the root).

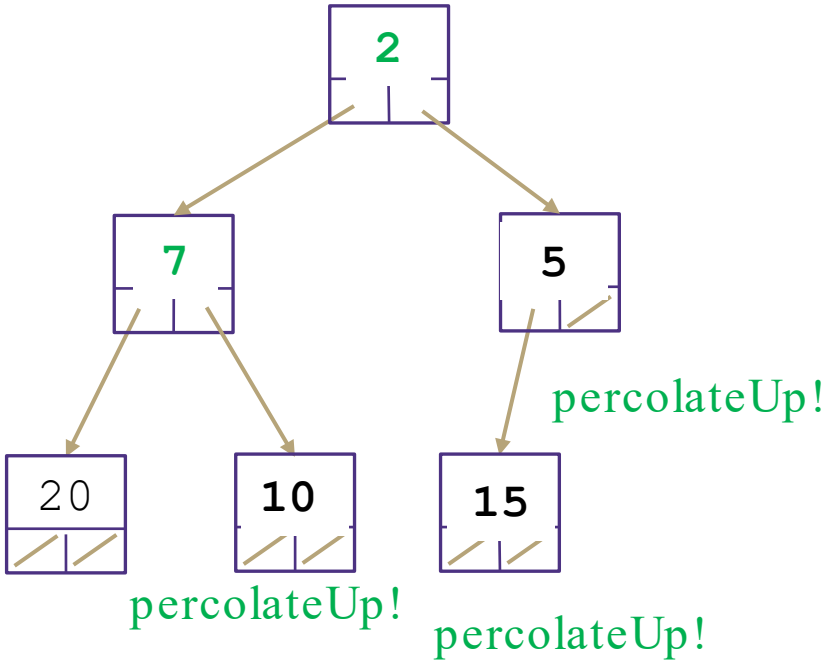


Worst case runtime is similar to `removeMin` and `percolateDown` – might have to do log(n) swaps, so the worst-case runtime is O(log(n))

Practice: Building a minHeap

Construct a Min Binary Heap by adding the following values in this order:

- 5, 10, 15, 20, 7, 2



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Min Binary Heap Invariants

1. Binary Tree – each node has at most 2 children
2. Min Heap – each node's children are larger than itself
3. Level Complete - new nodes are added from left to right completely filling each level before creating a new one

minHeap runtimes

removeMin():

- remove root node
- find last node in tree and swap to top level
- percolate down to fix heap invariant

add()

- insert new node into next available spot
- percolate up to fix heap invariant

Finding the last node/next available spot is the hard part.
You can do it in $\Theta(\log n)$ time on complete trees, with some extra class variants

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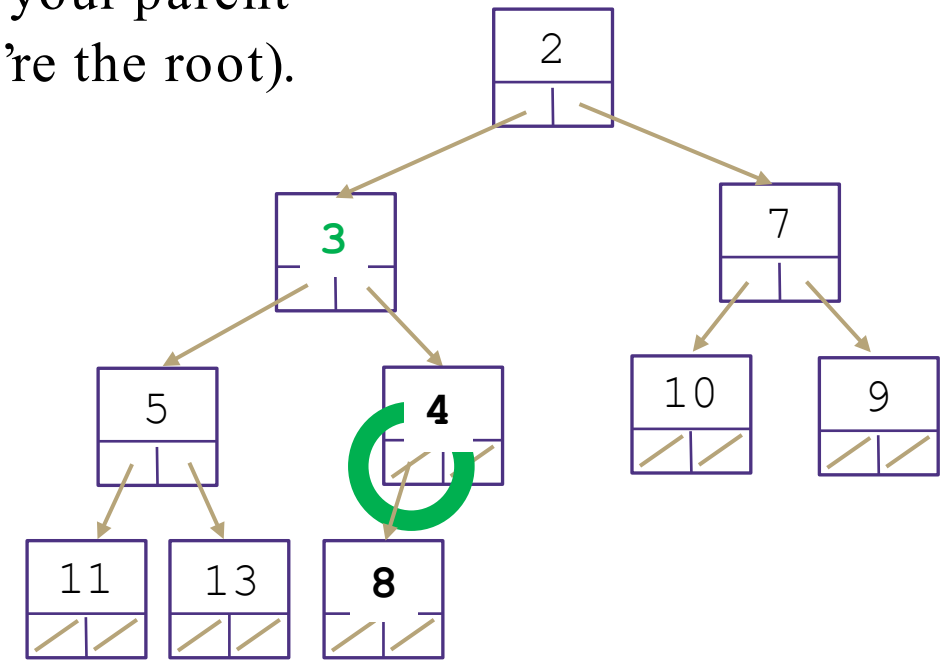

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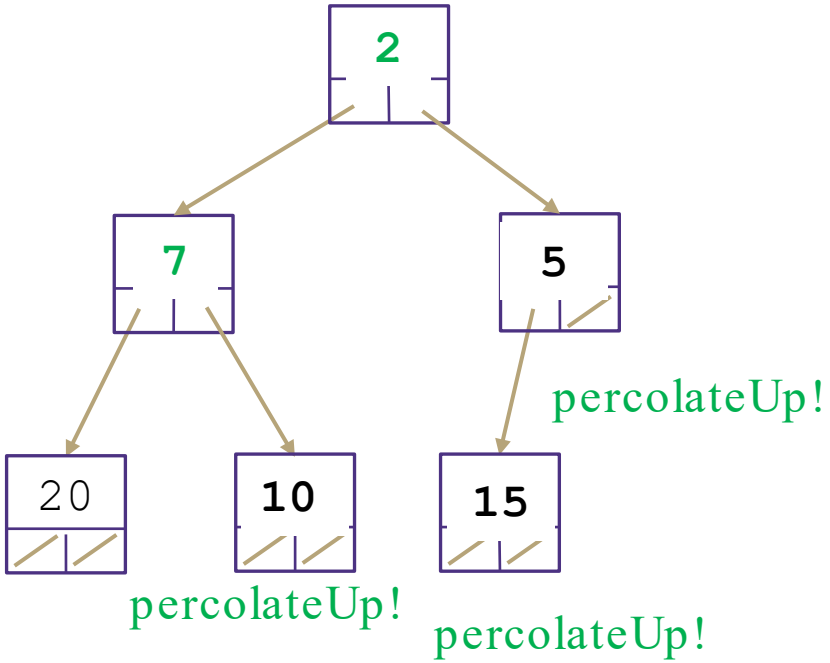

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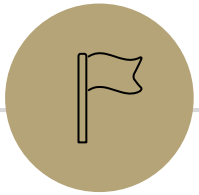
Finding the last node/next available spot is the hard part.
You can do it in $\Theta(\log n)$ time on complete trees, with some extra class variants

But it's NOT fun

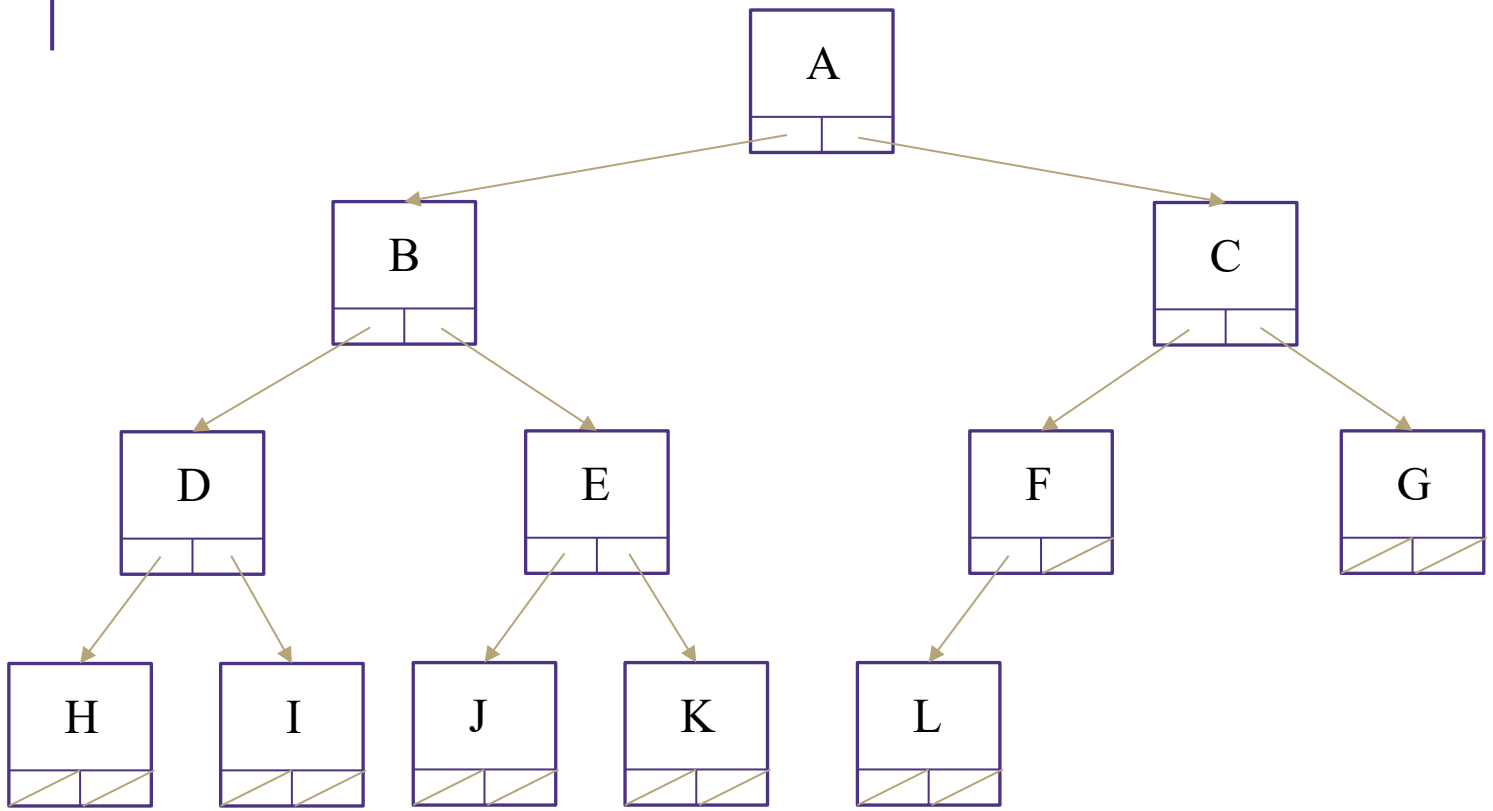
And there's a much better way (that we'll talk about Wednesday)!



Heap Array Implementation

More Priority Queue Operations

Implement Heaps with an array



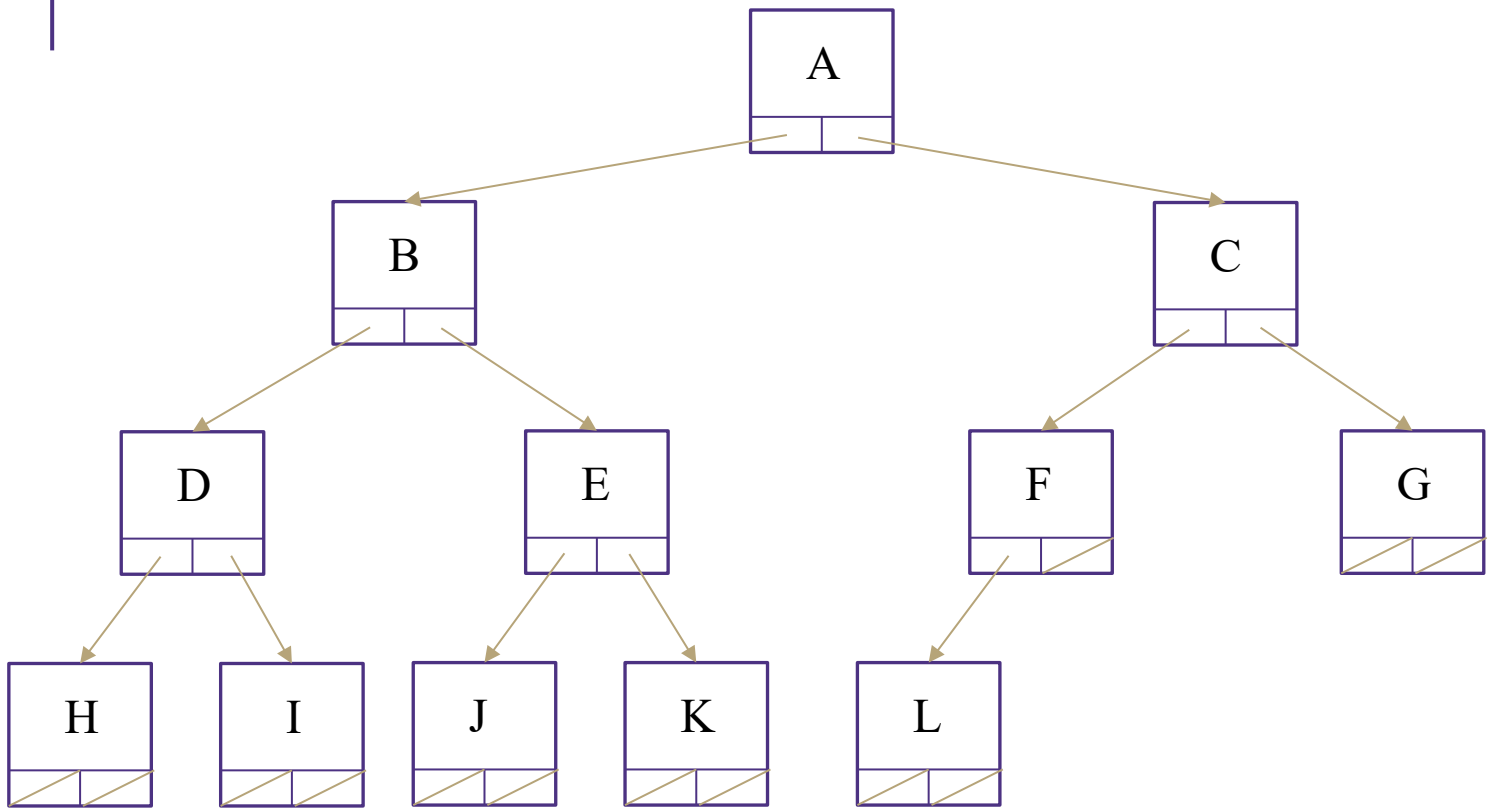
Fill array in level-order from left to right

0	1	2	3	4	5	6	7	8	9	10	11	12	13
A	B	C	D	E	F	G	H	I	J	K	L		

We map our binary-tree representation of a heap into an array implementation where you fill in the array in level-order from left to right.

The implementation of a heap isan array, but the tree drawing is how to think of it conceptually.

Implement Heaps with an array



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0	1	2	3	4	5	6	7	8	9	10	11	12	13
A	B	C	D	E	F	G	H	I	J	K	L		

How do we find the minimum node?

$$peekMin() = arr[0]$$

How do we find the last node?

$$lastNode() = arr[size - 1]$$

How do we find the next open space?

$$openSpace() = arr[size]$$

How do we find a node's left child?

$$leftChild(i) = 2i + 1$$

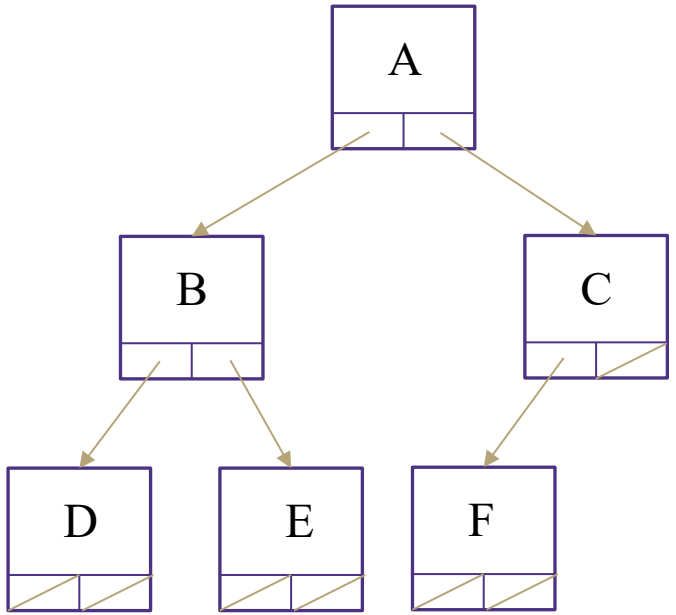
How do we find a node's right child?

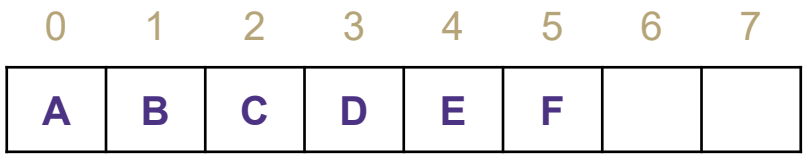
$$rightChild(i) = 2i + 2$$

How do we find a node's parent?

$$parent(i) = \frac{(i-1)}{2}$$

Heap Implementation Runtimes





Implementation	add	removeMin	Peek
Array-based heap	worst: O(log n) in-practice: O(1)	worst: O(log n) in-practice: O(log n)	O(1)

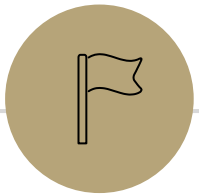
We've matched the asymptotic worst-case behavior of AVL trees.

But we're actually doing better!

- The constant factors for array accesses are better.
- The tree can be a constant factor shorter because of stricter height invariants.
- In-practice case for add is really good.
- A heap is MUCH simpler to implement.

Are heaps always better? AVL vs Heaps

- The really amazing things about heaps over AVL implementations are the constant factors (e.g. 1.2n instead of 2n) and the sweet sweet O(1) in-practice `add` time.
- The really amazing things about AVL implementations over heaps is that AVL trees are absolutely sorted, and they guarantee worst-case be able to find (contains/get) in O(log(n)) time.
- If heaps have to implement methods like contains/get/ (more generally: finding a particular value inside the data structure) – it pretty much just has to loop through the array and incur a worst case O(n) runtime.
- Heaps are stuck at O(n) runtime and we can't do anything more clever… aha, just kidding.. unless..?



Heap Array Implementation

More Priority Queue Operations

More Operations

Min Priority Queue ADT

state

Set of comparable values
- Ordered based on "priority"

behavior

add(value) – add a new element to the collection

removeMin() – returns the element with the smallest priority, removes it from the collection

peekMin() – find, but do not remove the element with the smallest priority

We'll use priority queues for lots of things later in the quarter.

Let's add them to our ADT now.

Some of these will be asymptotically faster for a heap than an AVL tree!

BuildHeap(elements $e_1, .., e_n$)

Given n elements, create a heap containing exactly those n elements.

Even More Operations

BuildHeap(elements $e_1, .., e_n$)

Given n elements, create a heap containing exactly those n elements.

Try 1: Just call insert n times.

- n calls, each with worst-case complexity O(log n), so overall worst-case complexity is O(n log n)
- Worst-case input: if we insert elements in decreasing order, every node will have to percolate all the way up to the root.
- Can we do better?

Can We Do Better?

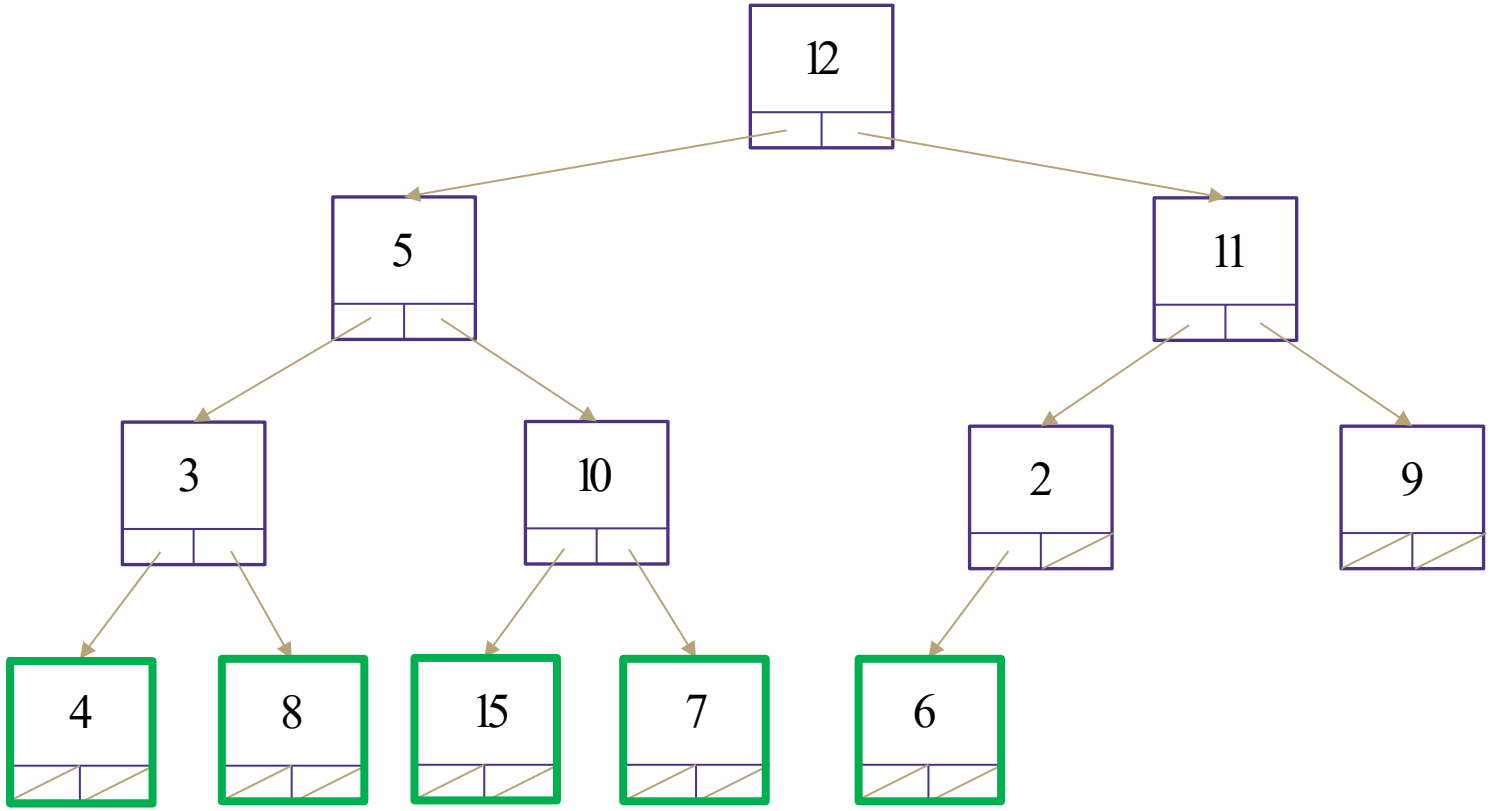
- What's causing the n add strategy to take so long?
  - Most nodes are near the bottom, and might need to percolate all the way up

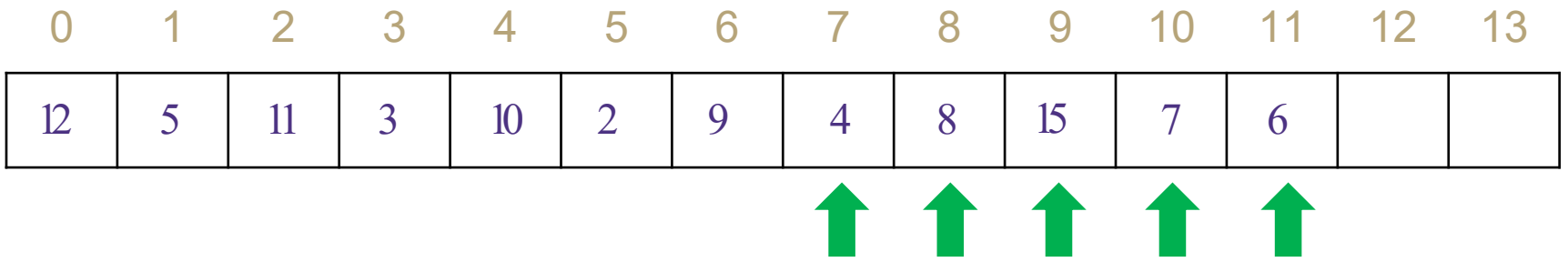
- Idea 2: Dump everything in the array, and percolate things down until the heap invariant is satisfied
  - The bottom two levels of the tree have Ω(n) nodes, the top two have 3 nodes
  - Maybe we can make "most of the nodes" at the bottom go only a constant distance

Floyd's buildHeap algorithm

Build a tree with the values:
12, 5, 11, 3, 10, 2, 9, 4, 8, 15, 7, 6

1. Add all values to back of array
2. percolateDown(parent) starting at last index
    1. percolateDown level 4





Floyd's buildHeap algorithm

Build a tree with the values:
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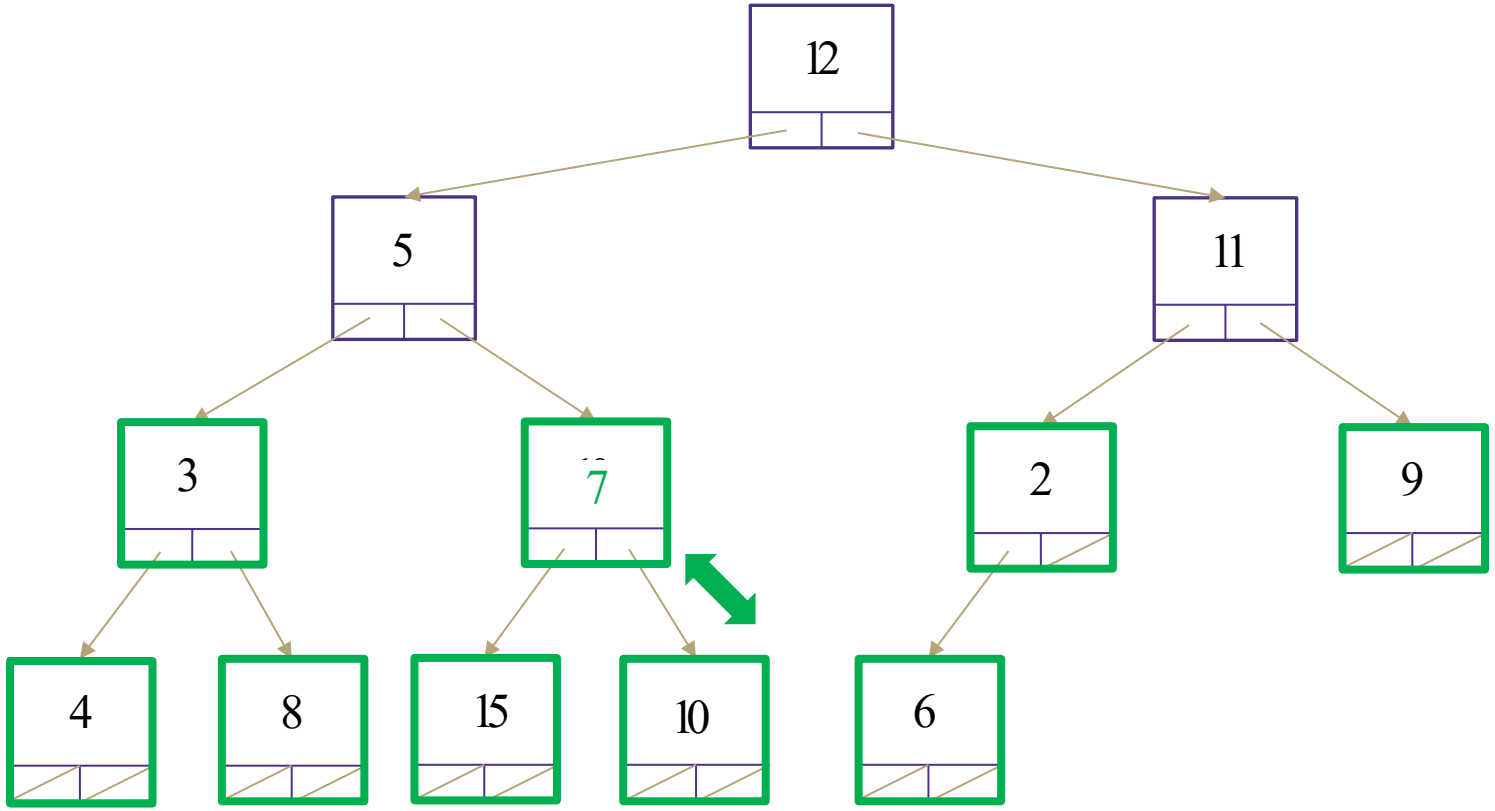
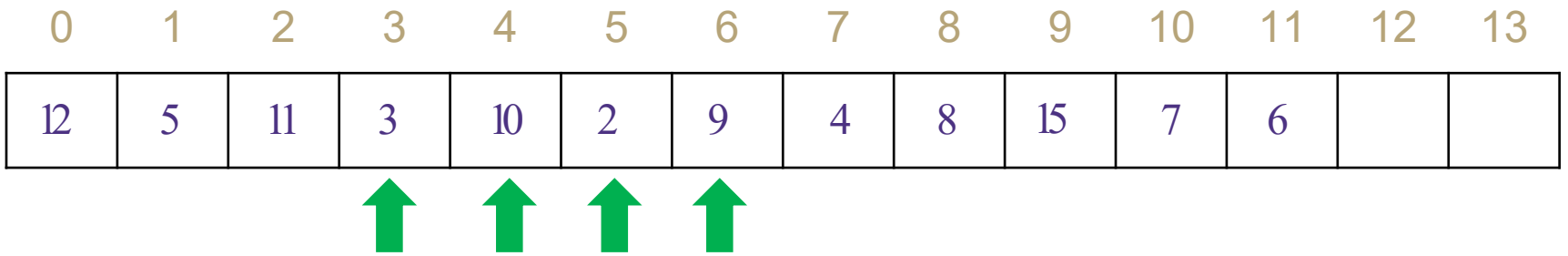
1. Add all values to back of array
2. percolateDown(parent) starting at last index
   1. percolateDown level 4
   2. percolateDown level 3

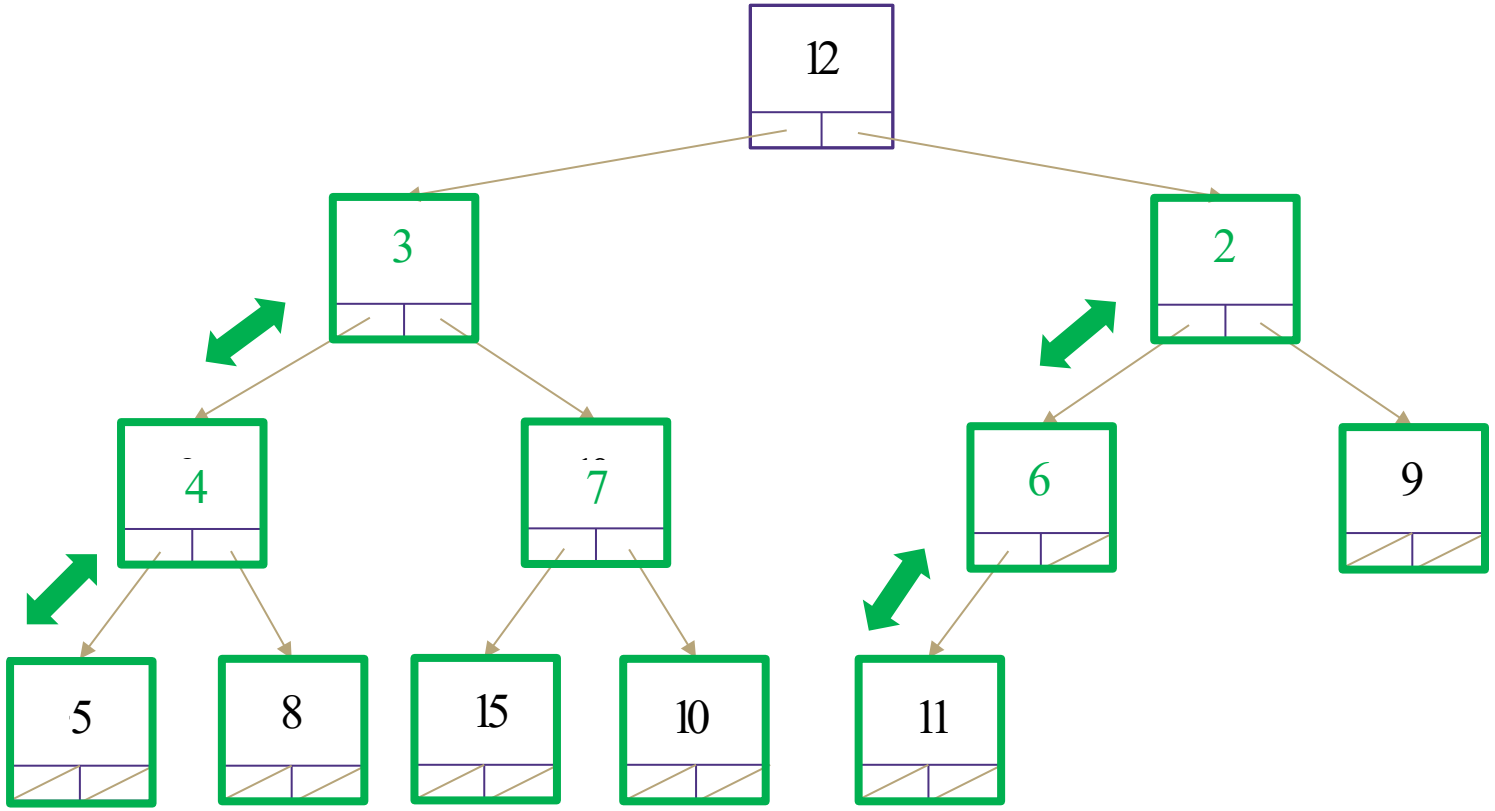




Floyd's buildHeap algorithm

Build a tree with the values:
12, 5, 11, 3, 10, 2, 9, 4, 8, 15, 7, 6

1. Add all values to back of array
2. percolateDown(parent) starting at last index
   1. percolateDown level 4
   2. percolateDown level 3
   3. percolateDown level 2

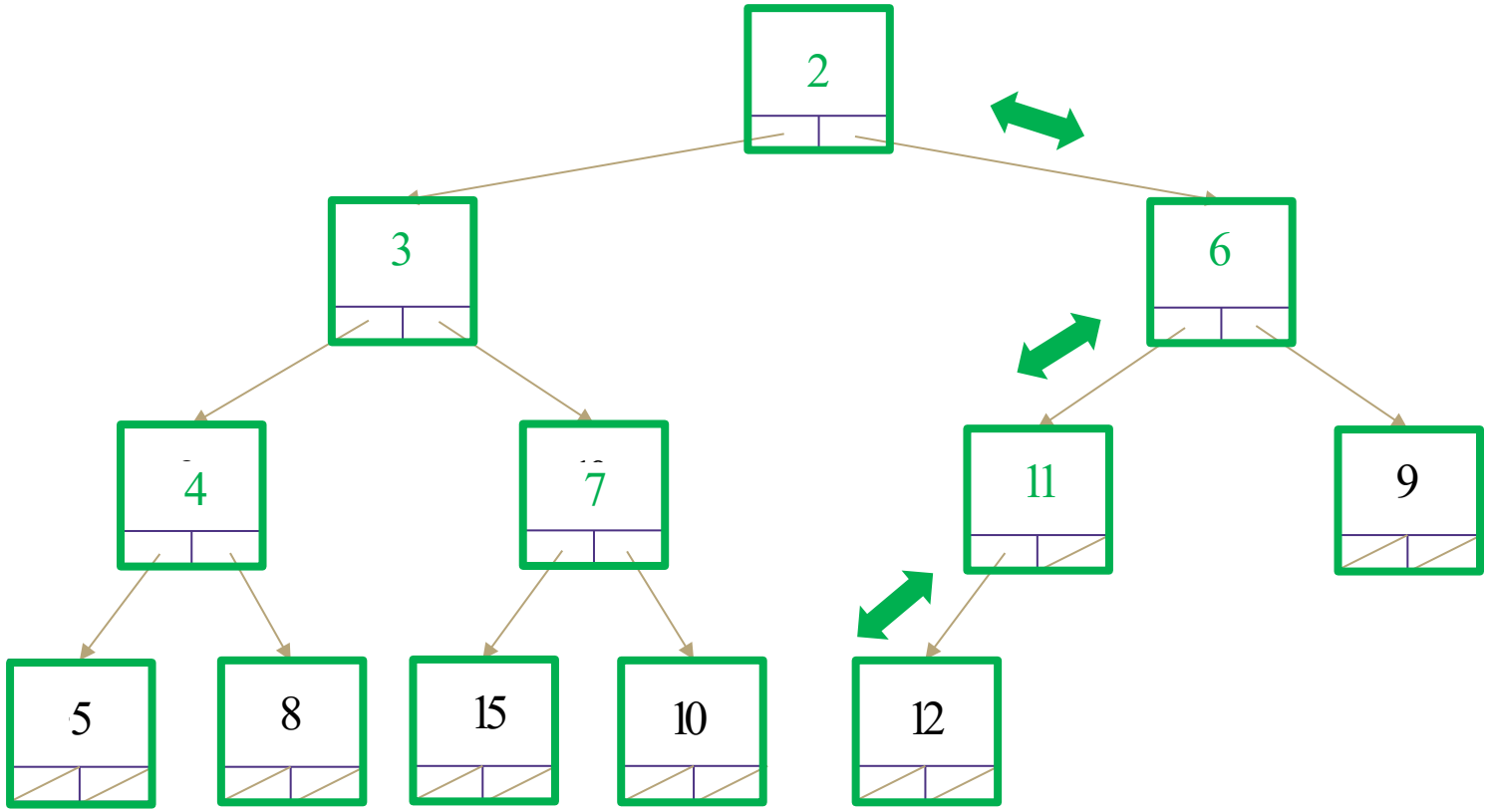


0	1	2	3	4	5	6	7	8	9	10	11	12	13
12	5	11	3	10	2	9	4	8	15	7	6		

Floyd's buildHeap algorithm

Build a tree with the values:
12, 5, 11, 3, 10, 2, 9, 4, 8, 15, 7, 6

1. Add all values to back of array
2. percolateDown(parent) starting at last index
    1. percolateDown level 4
    2. percolateDown level 3
    3. percolateDown level 2
    4. percolateDown level 1



0	1	2	3	4	5	6	7	8	9	10	11	12	13
12	5	11	3	10	2	9	4	8	15	7	6		

Is It Really Faster?

Floyd's buildHeap runs in O(n) time!

percolateDown() has worst case log n in general, but for most of these nodes, it has a much smaller worst case!

- n/2 nodes in the tree are leaves, have 0 levels to travel
- n/4 nodes have at most 1 level to travel
- n/8 nodes have at most 2 levels to travel
- etc…

$$\text{worst-case-work(n)} \approx \underbrace{\frac{n}{2} \cdot 1}_{\text{much of the work}} + \underbrace{\frac{n}{4} \cdot 2}_{\text{a little less}} + \underbrace{\frac{n}{8} \cdot 3}_{\text{a little less}} + \cdots + \underbrace{1 \cdot (\log n)}_{\text{barely anything}}$$

Intuition: Even though there are log n levels, each level does a smaller and smaller amount of work. Even with infinite levels, as we sum smaller and smaller values (think $1/2^i$) we converge to a constant factor of n.

Optional Slide Floyd's buildHeap Summation

- $n/2 \cdot 1 + n/4 \cdot 2 + n/8 \cdot 3 + \cdots + 1 \cdot (\log n)$

factor out n

work(n) $\approx n\left(\frac{1}{2}+\frac{2}{4}+\frac{3}{8}+\cdots+\frac{\log n}{n}\right)$ find a pattern ->powers of 2 work(n) $\approx n\left(\frac{1}{2^1}+\frac{2}{2^2}+\frac{3}{2^3}+\cdots+\frac{\log n}{2^{\log n}}\right)$ Summation!

$$work(n) \approx n\sum_{i=1}^{?}\frac{i}{2^i}$$? = upper limit should give last term

We don't have a summation for this! Let's make it look more like a summation we do know.

Infinite geometric series

$$work(n) \leq n\sum_{i=1}^{\log n}\frac{\left(\frac{3}{2}\right)^i}{2^i} \quad if\ -1 < x < 1\ then \sum_{i=0}^{\infty} x^i = \frac{1}{1-x} = x \quad work(n) \approx n\sum_{i=1}^{\log n}\frac{i}{2^i} \leq n\sum_{i=0}^{\infty}\left(\frac{3}{4}\right)^i = n * 4$$

Floyd's buildHeap runs in O(n) time!