Lecture 9

Self-Balancing Trees

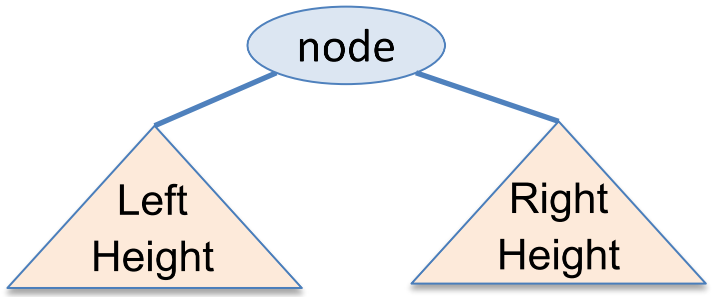
Department of Computer Science
Hofstra University

Acknowledgement: Lecture slides based on UofW Course on Data Structures

AVL Tree

AVL Tree: A balanced BST that maintains the invariant: |LeftHeight – RightHeight | <= 1 for all nodes in the tree.

- Named after Adelson-Velsky and Landis
- But also A Very Lovable Tree!

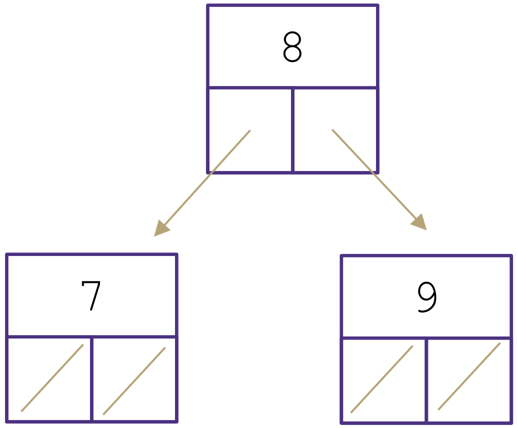


An AVL Tree has height ≈ log(n)

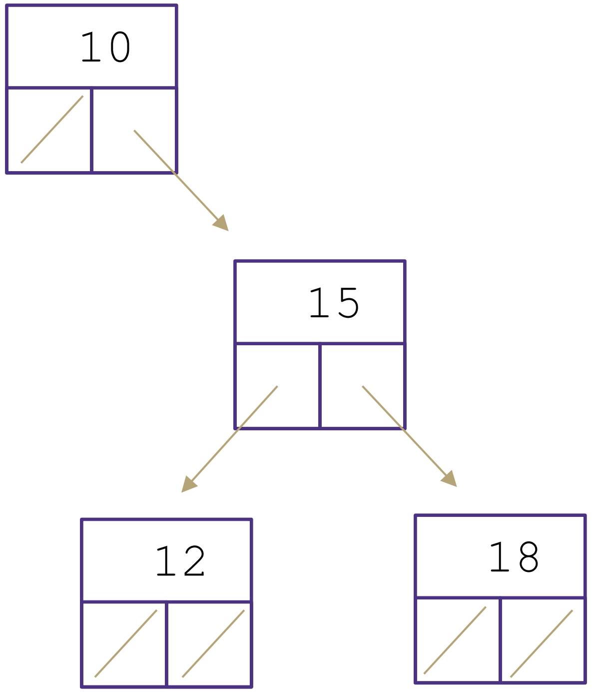
Measuring Balance

Measuring balance:

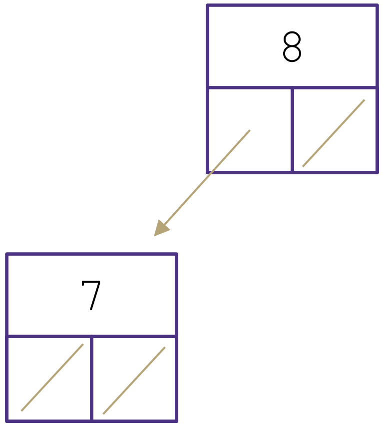
- For each node, compare the heights of its two subtrees
- Balanced when the difference in heights between subtrees is no greater than 1



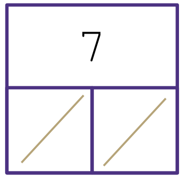
Balanced



Unbalanced



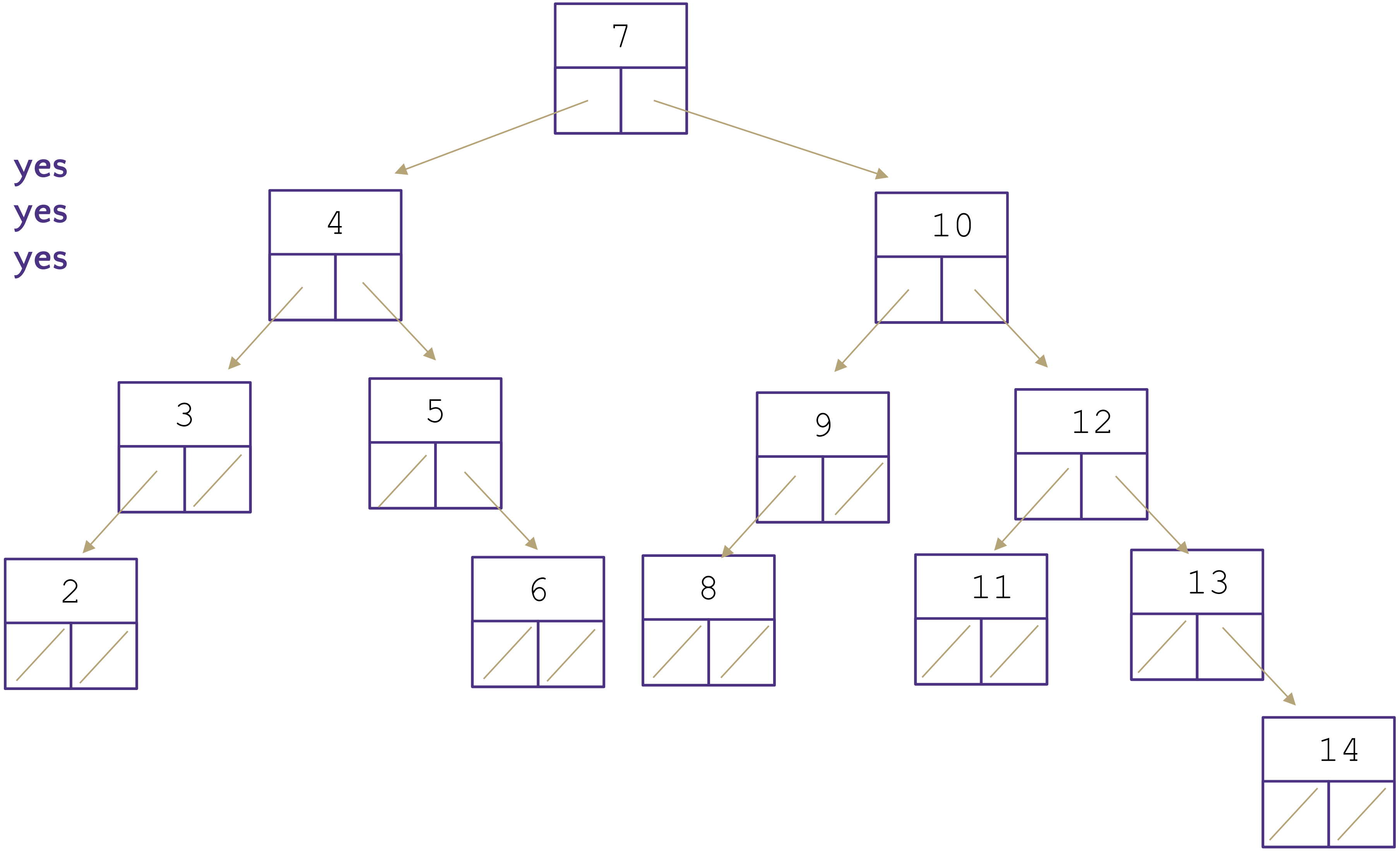
Balanced



Balanced

Is this a valid AVL tree?

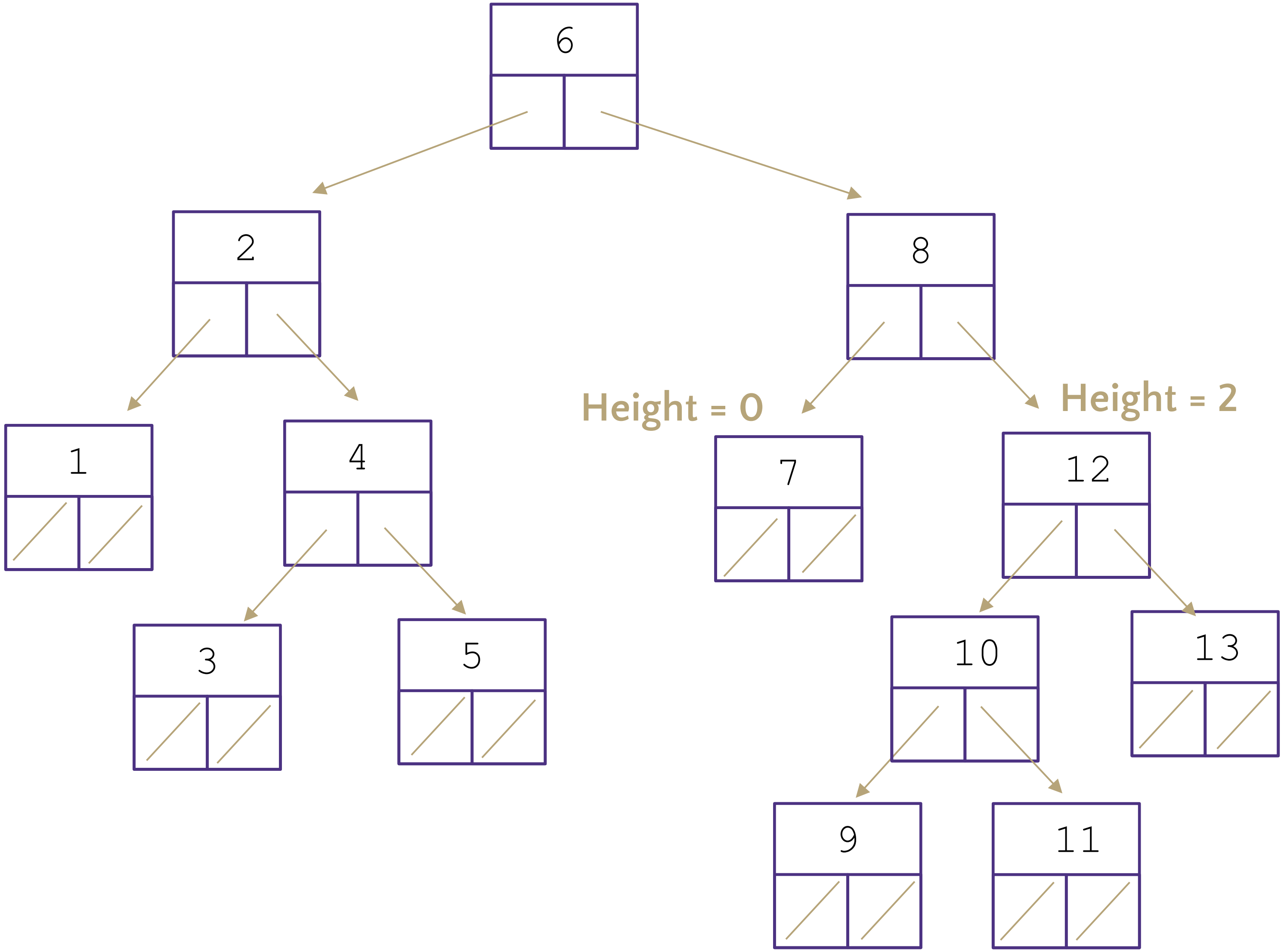
Is it...
- Binary **yes**
- BST **yes**
- Balanced? **yes**



Is this a valid AVL tree?

Is it…

- Binary yes
- BST yes
- Balanced? no



Time Complexity

✓ INVARIANT

```
public boolean containsKey(node, key) {
    // find key
}
```

✓ INVARIANT

`containsKey` benefits from invariant: worst-case O(log *n*) time

✓ INVARIANT

```
public boolean insert(node, key) {
    // find where key would go
    // insert
}
```

✓ INVARIANT

`Insert` benefits from invariant: worst-case O(log *n*) time to find location for key

How to maintain the invariant?

- Track heights of subtrees
- Detect any imbalance
- Restore balance

BST containsKey()

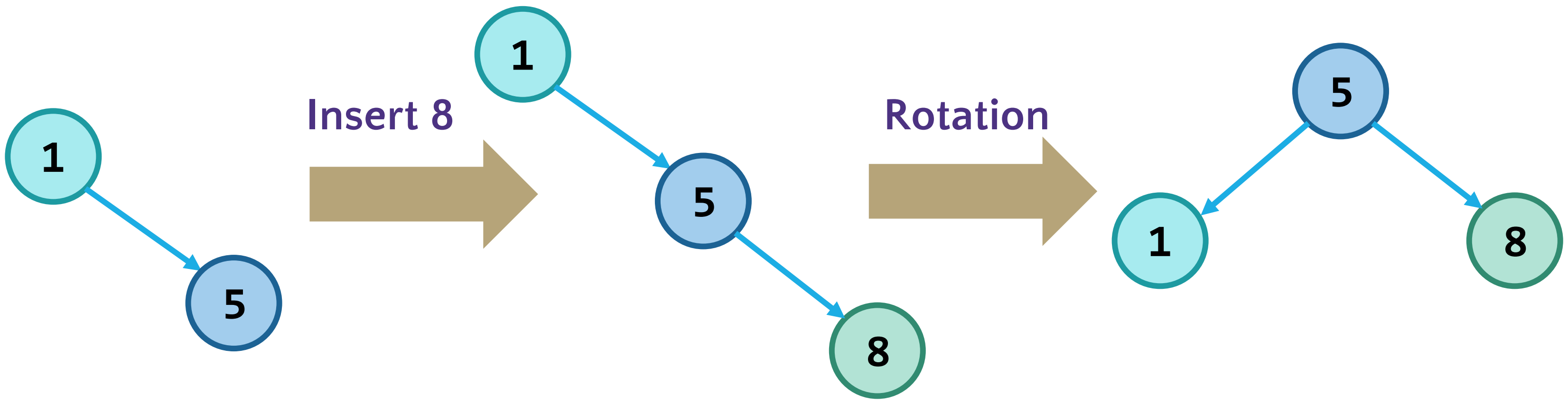
The AVL Invariant

Rotations

Insertion

What happens if an insertion breaks the AVL invariant?
The AVL rebalances itself by rotations. AVL is a type of “Self-Balancing Tree”

- A rotation alters the structure of a tree by rearranging subtrees.
- Goal is to decrease the height of the tree to maximum height of O(log n).
- Larger subtrees up, smaller subtrees down
- Does not affect the order of elements
- Time complexity O(1)

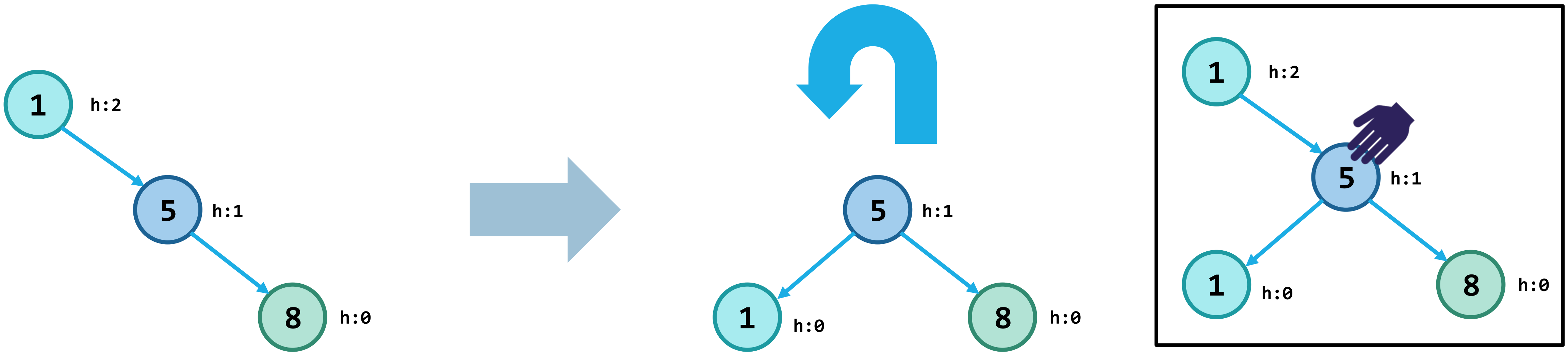


Fixing AVL Invariant: Left Rotation

We can fix the AVL invariant by performing rotations wherever an imbalance was created

Left Rotation

- Find the node that is violating the invariant (here, 1)
- Let it “fall” left to become a left child

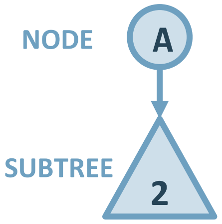


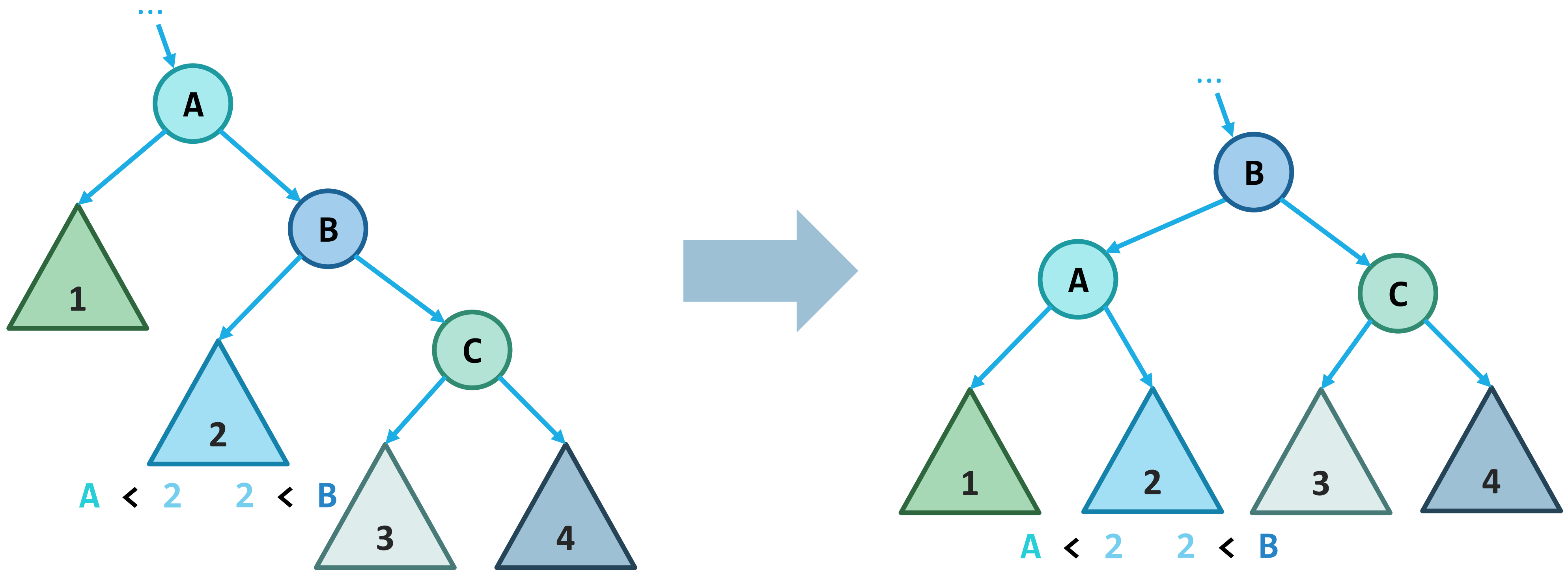
Apply a left rotation whenever the newly inserted node is located under the **right child of the right child**

Left Rotation: More Precisely

Subtrees are okay! They just come along for the ride.

- Subtree 2 changes from left child of B to right child of A – but notice that its relationship with nodes A and B doesn't change in the new position!

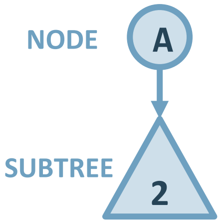


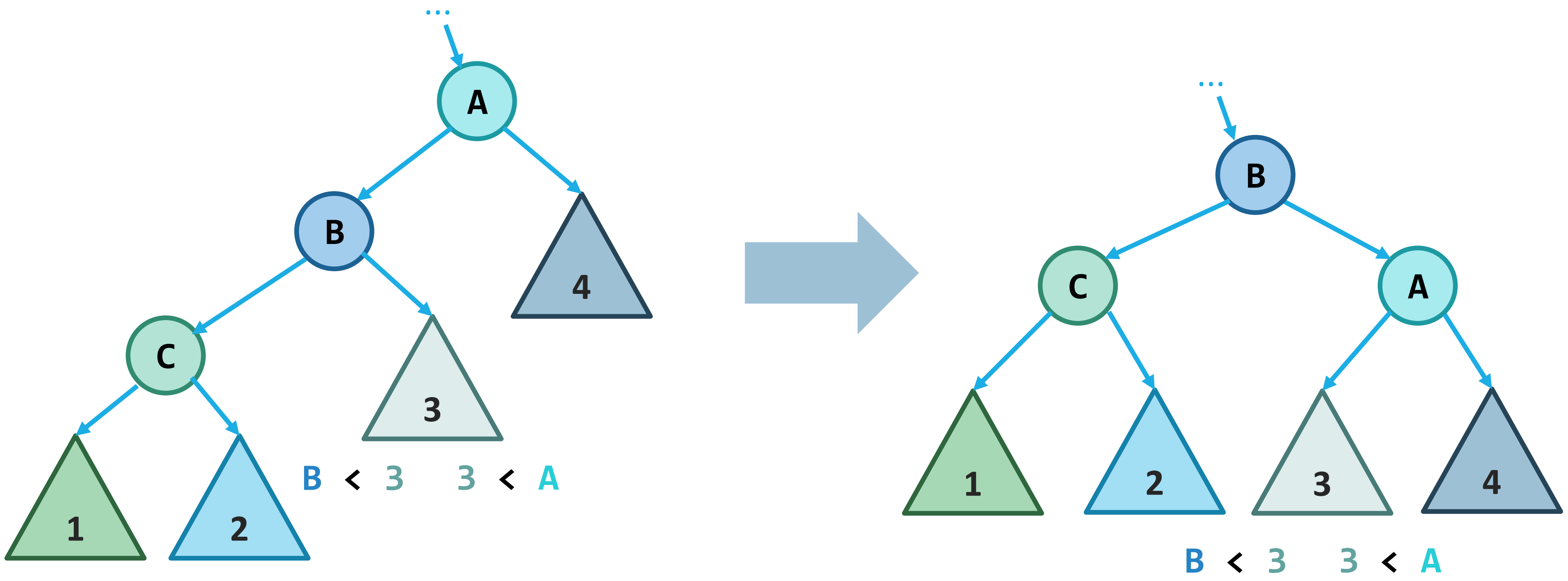


Right Rotation

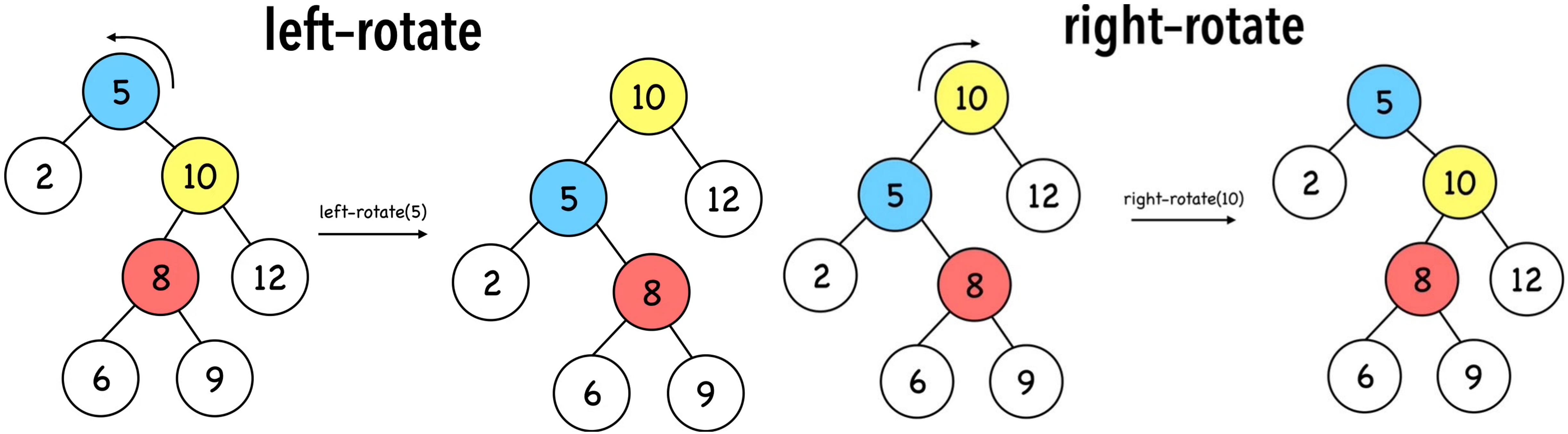
Right Rotation

- Mirror image of Left Rotation!





Left or Right Rotation Examples

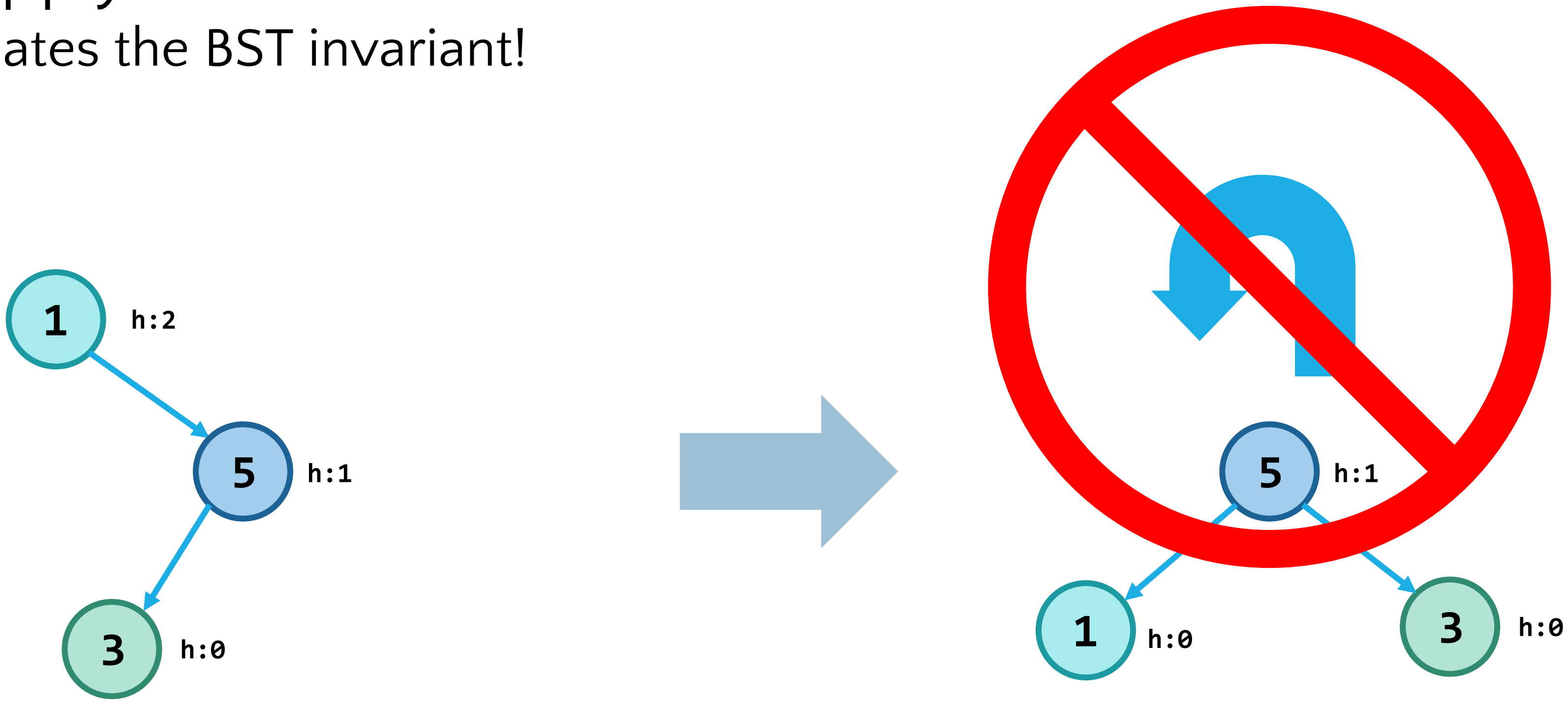


Not Quite as Straightforward

What if there's a "kink" in the tree where the insertion happened?

Can we apply a Left Rotation?

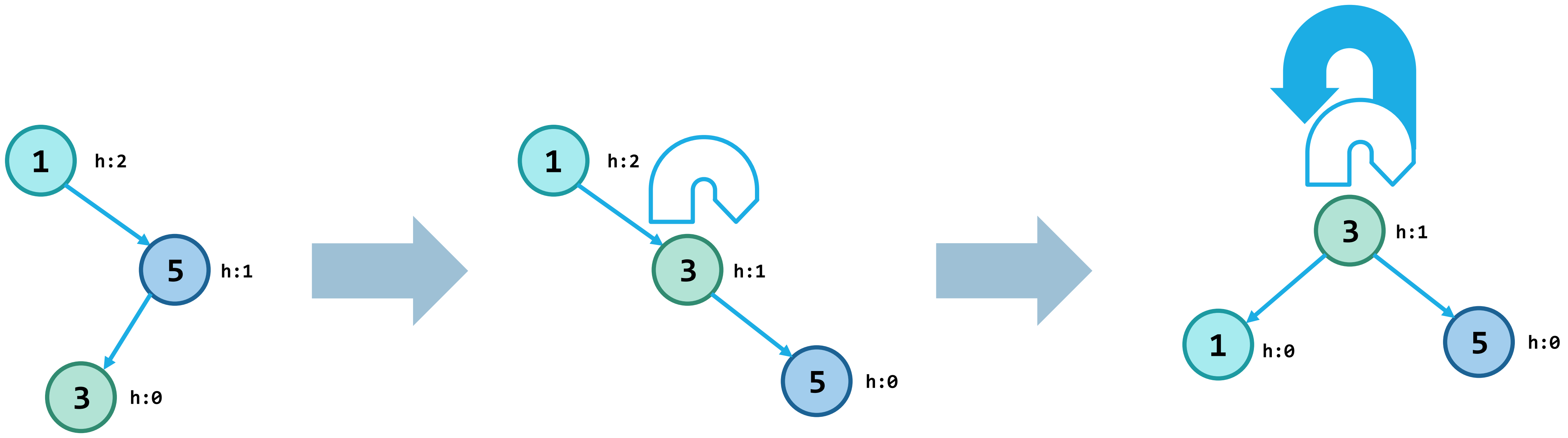
- No, violates the BST invariant!



Right/Left Rotation

Solution: **Right/Left Rotation**

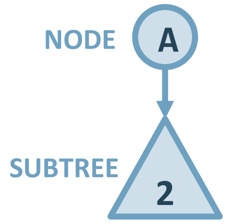
- Two steps: First do a right rotation for the right two nodes. then do a left rotation for the three nodes.
- Preserves BST invariant!

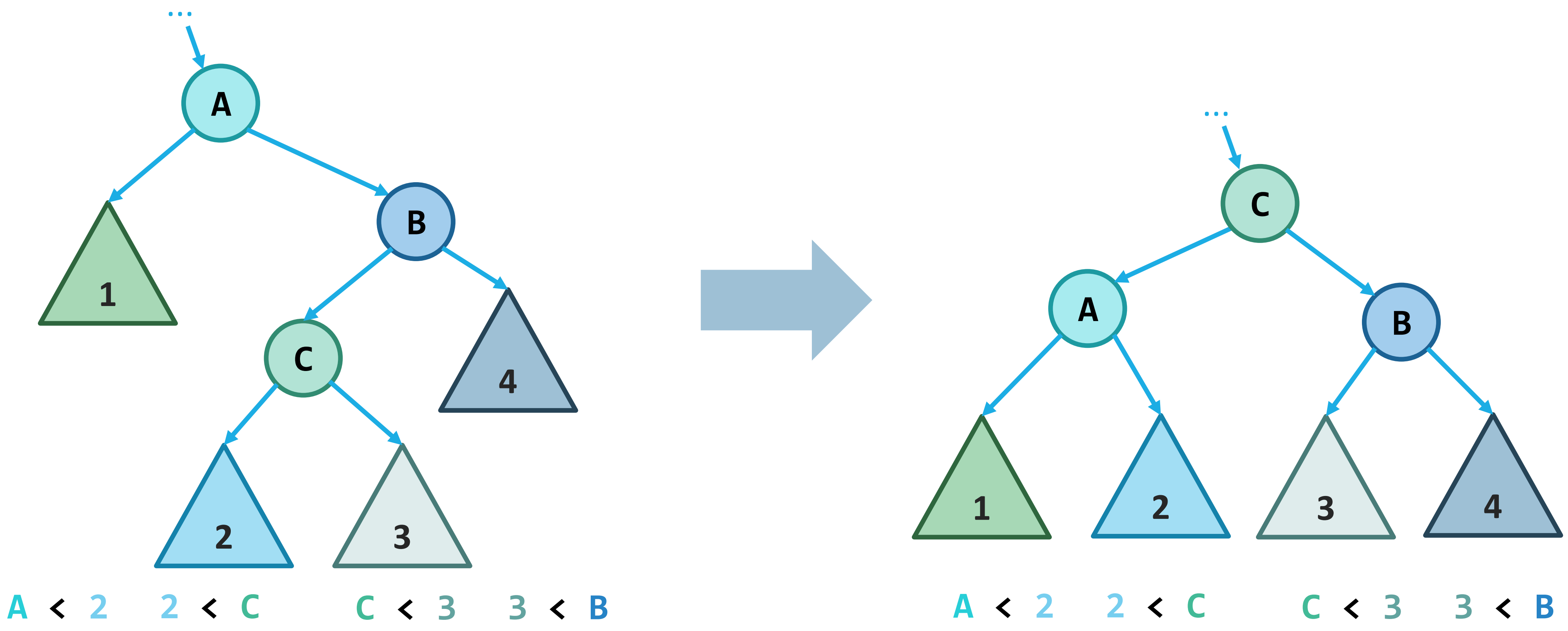


Right/Left Rotation: More Precisely

Again, subtrees are invited to come with

- Now 2 and 3 both have to hop, but all BST ordering properties are still preserved
- (Note that A, B and C denote some numerical value, not letters ‘A’, ’B’, ’C’)

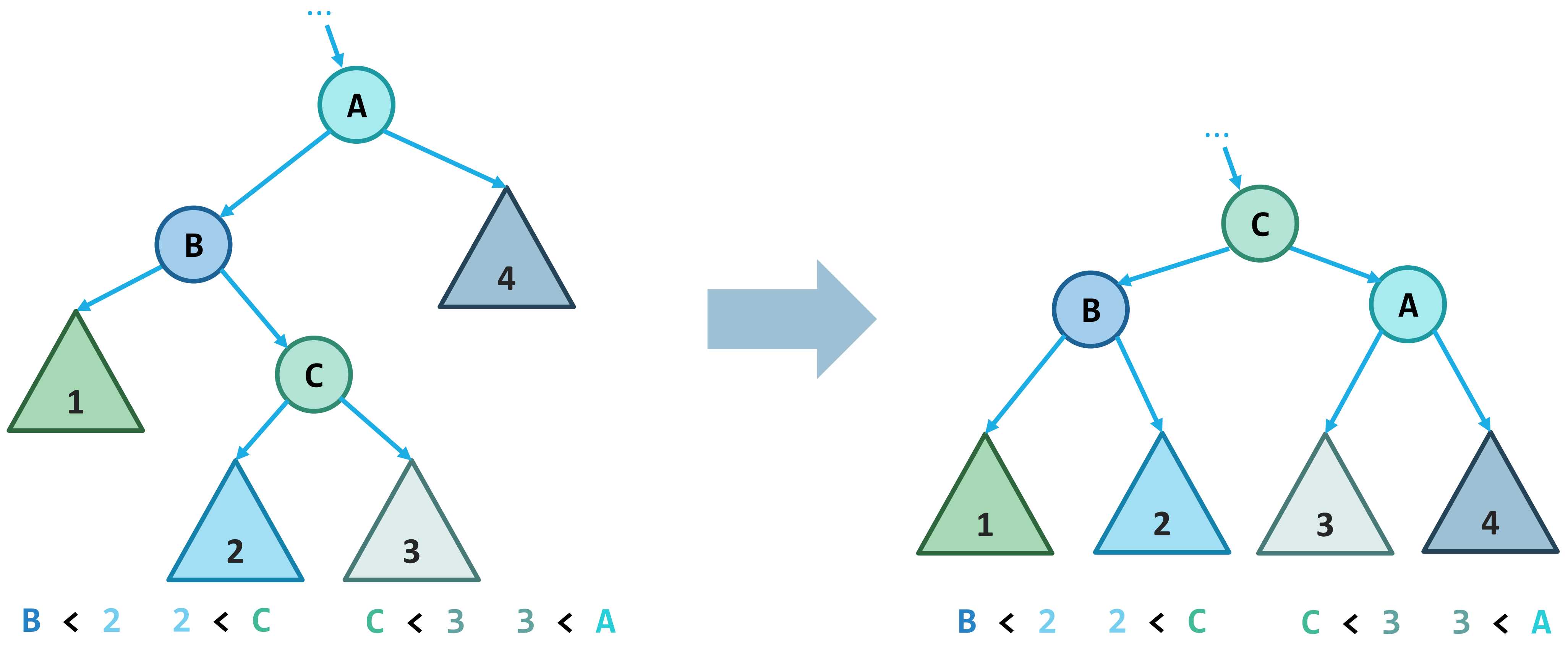


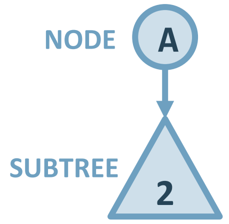


Left/Right Rotation

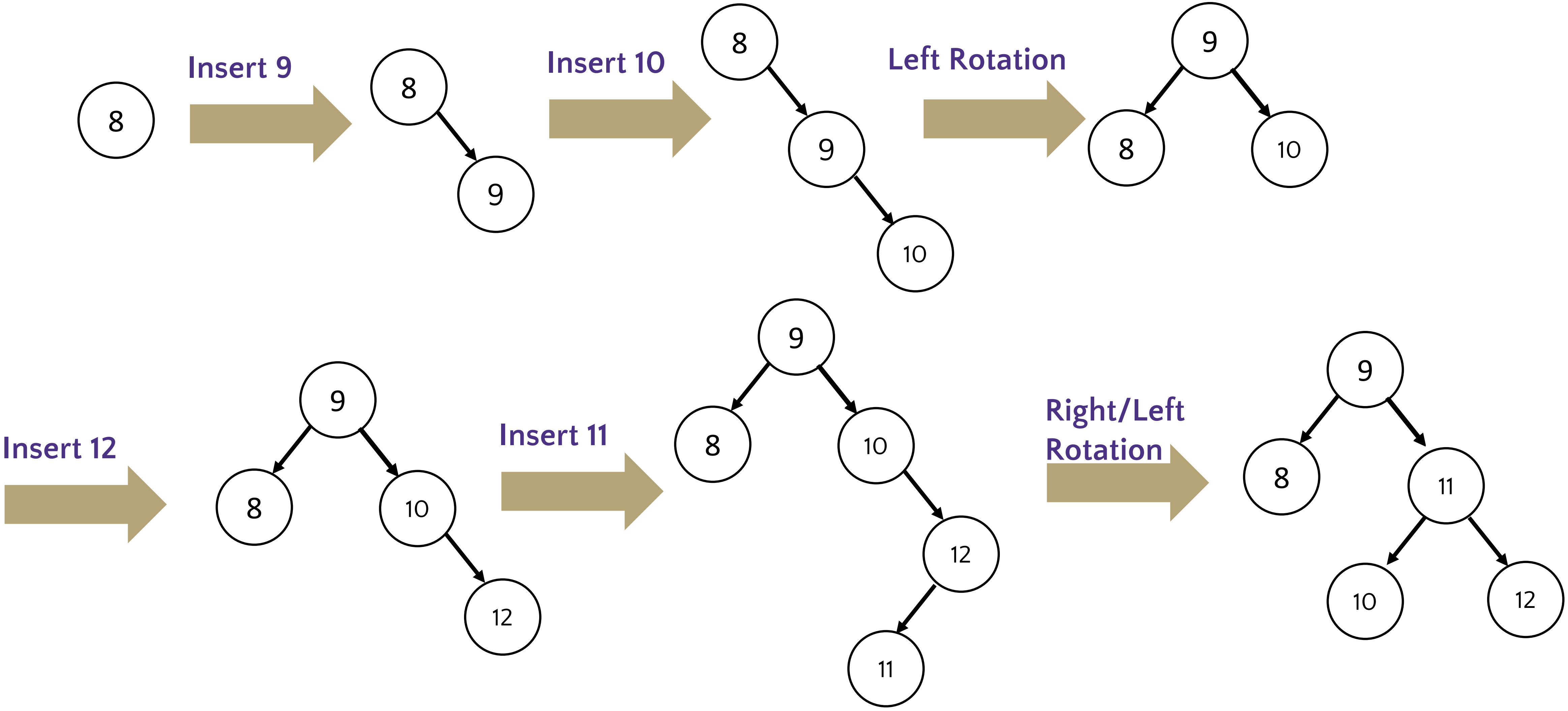
Left/Right Rotation

- Mirror image of Right/Left Rotation!

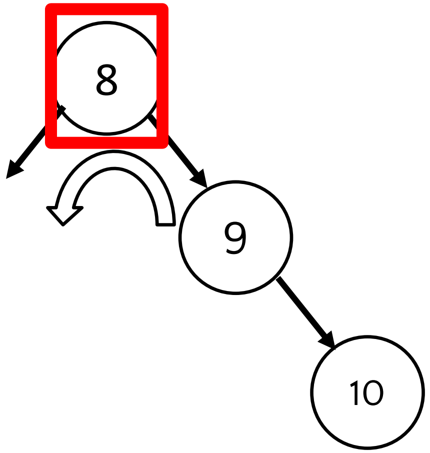




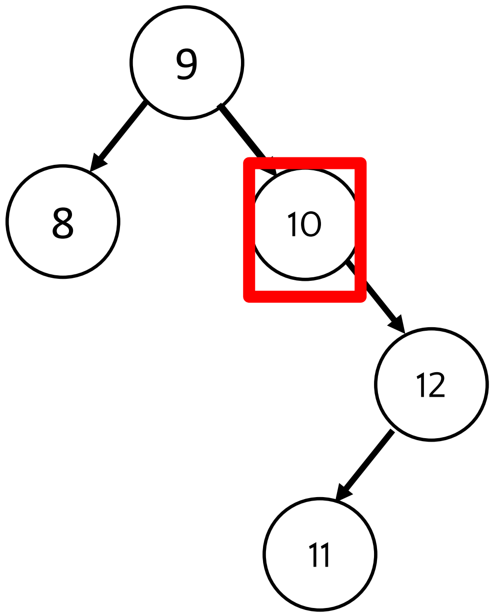
AVL Example: 8,9,10,12,11



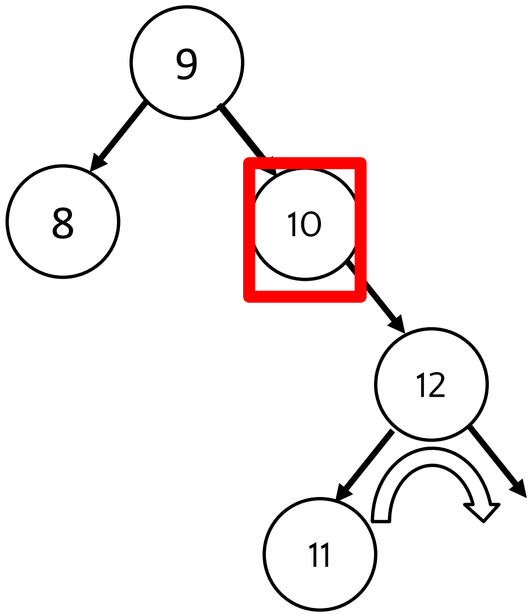
AVL Example: 8,9,10,12,11



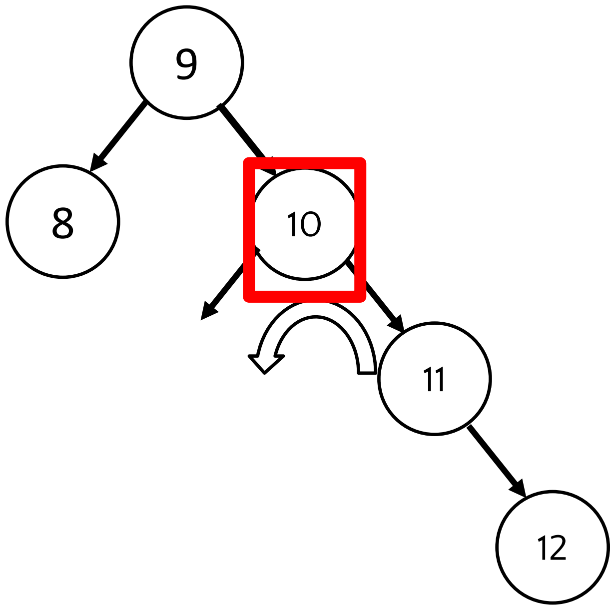
AVL Example: 8,9,10,12,11



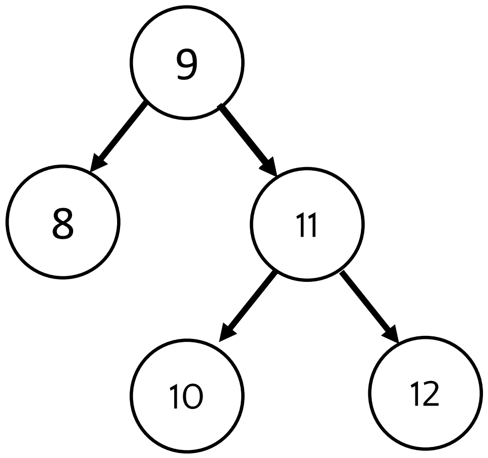
AVL Example: 8,9,10,12,11



AVL Example: 8,9,10,12,11



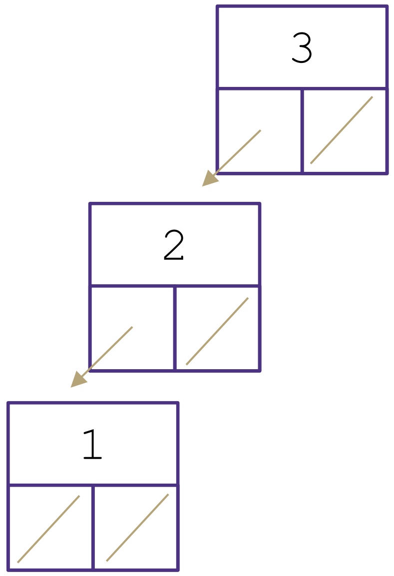
AVL Example: 8,9,10,12,11



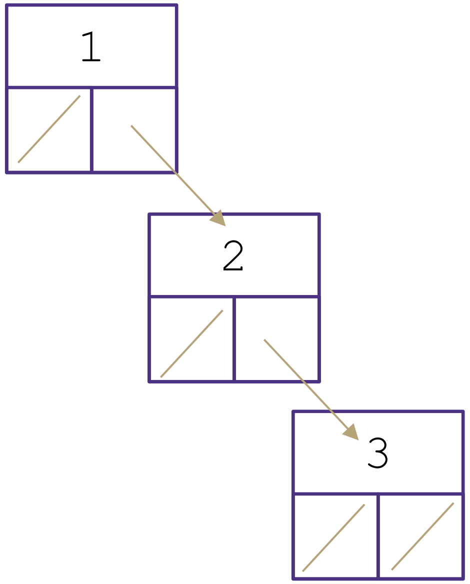
Two AVL Cases

Line Case

Solve with **1** rotation



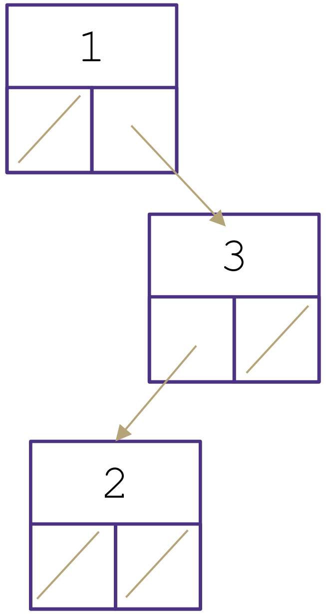
Rotate Right



Rotate Left

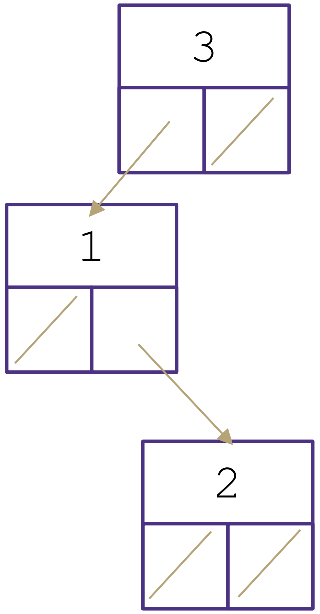
Kink Case

Solve with **2** rotations



Right Kink Resolution

Right/Left Rotation



Left Kink Resolution

Left/Right Rotation

How Long Does Rebalancing Take?

- Assume we store in each node the height of its subtree.
  - How do we find an unbalanced node?
  - Go back up the tree from where we inserted.
- How many rotations might we have to do?
  - Just a single or double rotation on the lowest unbalanced node.
  - A rotation will cause the subtree rooted where the rotation happens to have the same height it had before insertion
  - O(log n) time to traverse to a leaf of the tree
  - O(log n) time to find the imbalanced node
  - O(1) constant time to do the rotation(s)
  - Overall complexity: **O(log n)** time for adding a node.

AVL `insertion`: Approach

Overall algorithm:

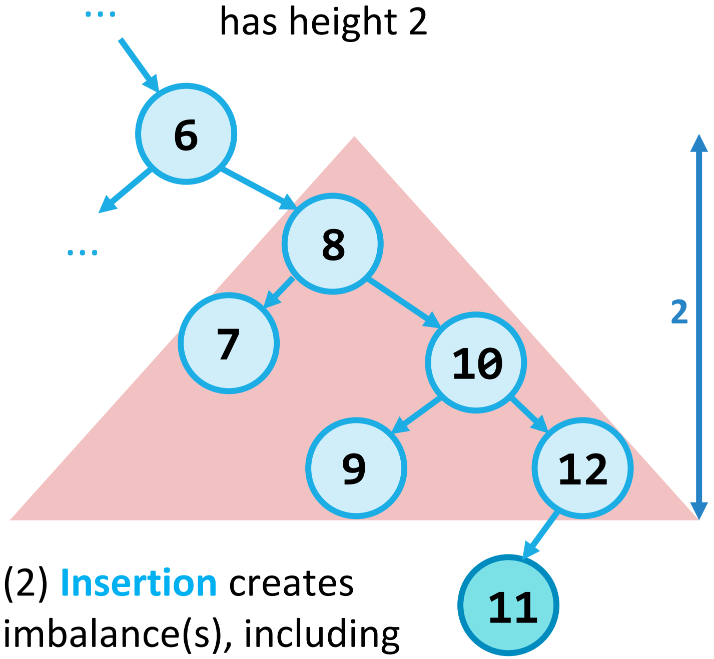
1. Insert the new node as in a BST (a new leaf)
2. For each node *on the path from the root to the new leaf*:
   - The insertion may (or may not) have changed the node's height
   - Detect height imbalance and perform a *rotation* to restore balance

Facts that make this easier:

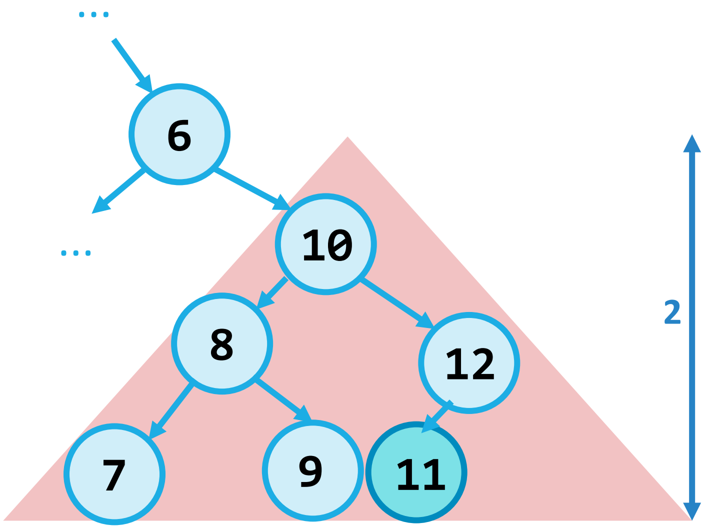
- Imbalances can only occur along the path from the new leaf to the root
- We only have to address the lowest unbalanced node
- Applying a rotation (or double rotation), restores the height of the subtree before the insertion -- when everything was balanced!
- Therefore, we need ***at most one rebalancing operation***

AVL `insertion`: Example

(1) Originally, whole tree balanced, and **this subtree** has height 2



(2) **Insertion** creates imbalance(s), including **the subtree** (8 is lowest unbalanced node)



(3) Left rotation on 8 will restore **the subtree** to height 2, whole tree balanced again!

AVL `deletion`

- Deletion involves a similar set of rotations that let you rebalance an AVL tree after deleting an element
  - Omitted since it is beyond scope of this course
- In the worst case, takes O(log n) time to rebalance after a deletion
  - Finding the node to delete is also O(log n), so total complexity is O(log n)

AVL Trees

PROS

- All operations on an AVL Tree (search, insertion, deletion) have worst-case complexity O(log n)
  - Because the tree is always balanced!

CONS

- Additional space for the height field
- Rebalancing does incur some overhead
  - May not may not be important depending on the application

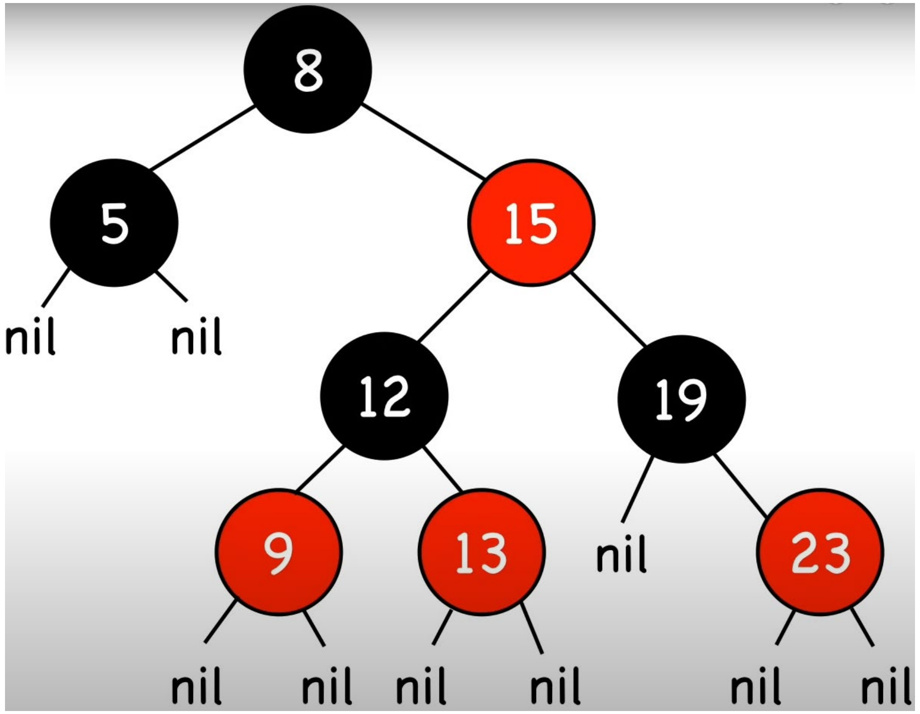
Video Tutorials

- AVL tree insertion, InvesTime
  - https://www.youtube.com/watch?v=vQptSYake4E&list=PLoW1nQhPBiz_h5wcmoZODnVnQK9Xja-Pi&index=5
  - In the middle of video (11 min) there is a typo where the root is written as 11, but it should be 14
- 14, 17, 11, 7, 53, 4, 13, 12, 8, 60, 19, 16, 20

Red Black Trees

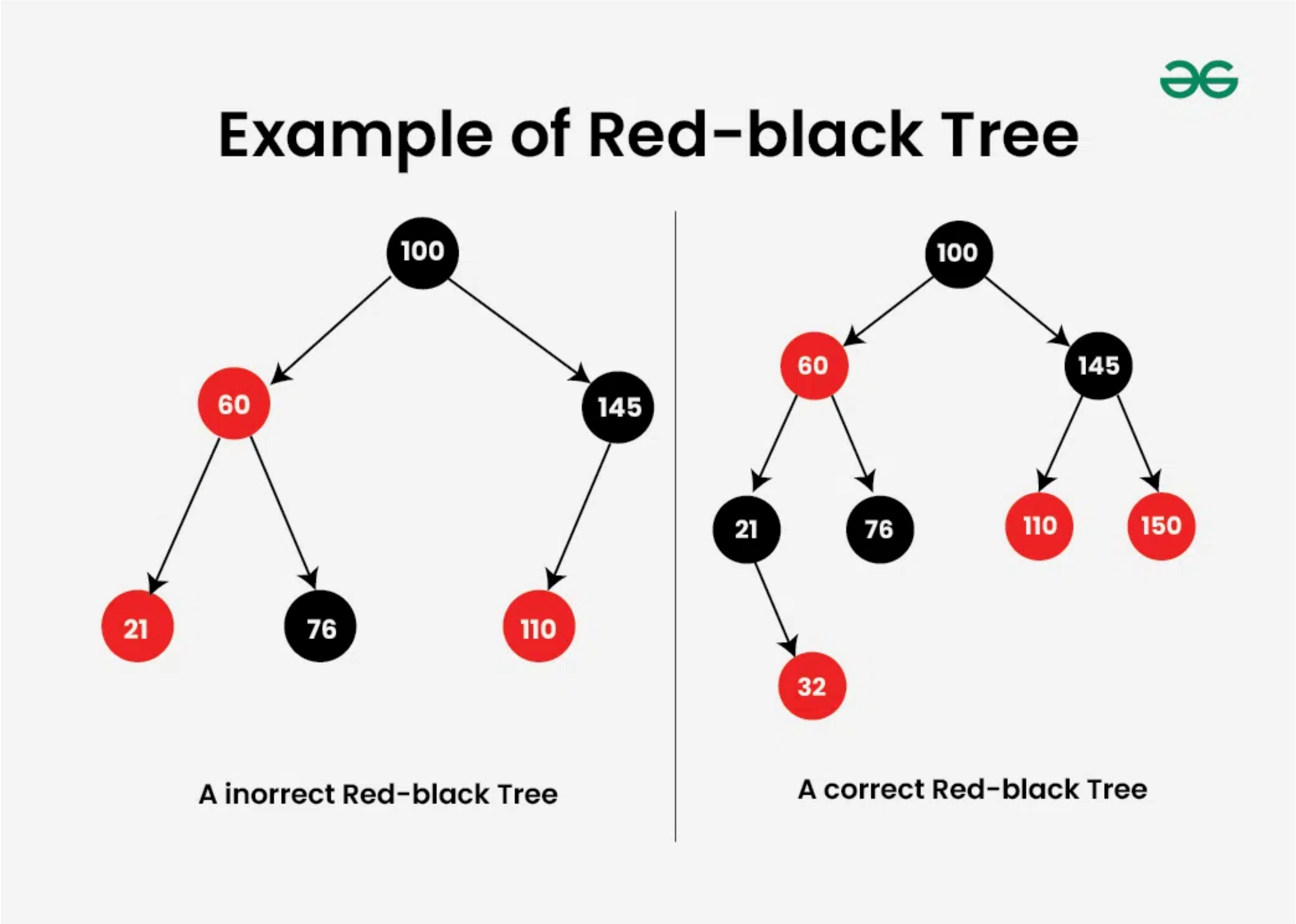
Red-Black Tree

- **Red-Black Tree:** A balanced BST that maintains the invariant: the longest path (root to farthest NIL) is no more than twice the length of the shortest path (root to nearest NIL).
  - Right figure: Shortest path: all black nodes (=2); Longest path: alternating red and black (=4)
  - This is NOT a legal AVL tree as it does not satisfy AVL invariant at root |LeftHeight – RightHeight | <= 1
  - (Path from root in RBT includes the NIL node, whereas height of AVL tree does not.)
- RBT uses color coding to maintain balance and reduce the number of node re-arrangements needed via rotations. It has four properties:
  - 1. Node Color: A node is either red or black.
  - 2. Root Property: The root and leaves (NIL) are black.
  - 3. Red Property: If a node is red, then its children are black. (no two adjacent red nodes)
  - 4. Black Property: All paths from a node to its NIL descendants contain the same number of black nodes.
- Node insertion and deletion may result in violation of these properties.
  - Use recoloring and rotations to maintain these properties.



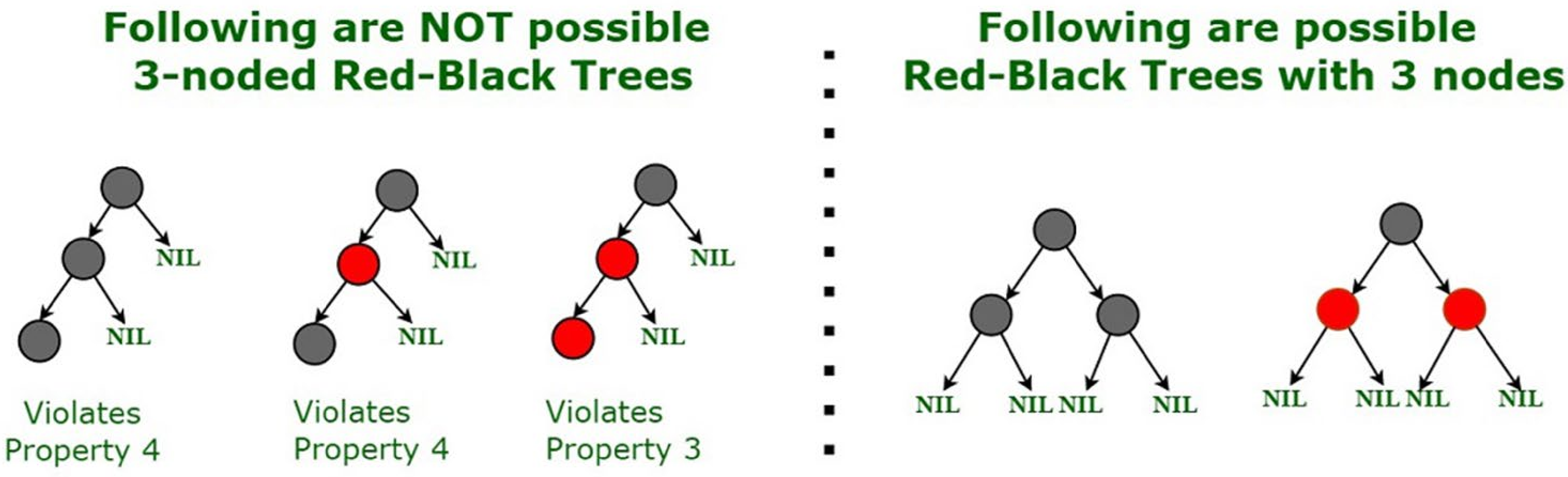
Examples

- Tree on the left: Incorrect Red Black Tree.
  - Two red nodes are adjacent to each other.
  - One of the paths to a leaf node has zero black nodes, whereas the other two paths contain 1 black node each.



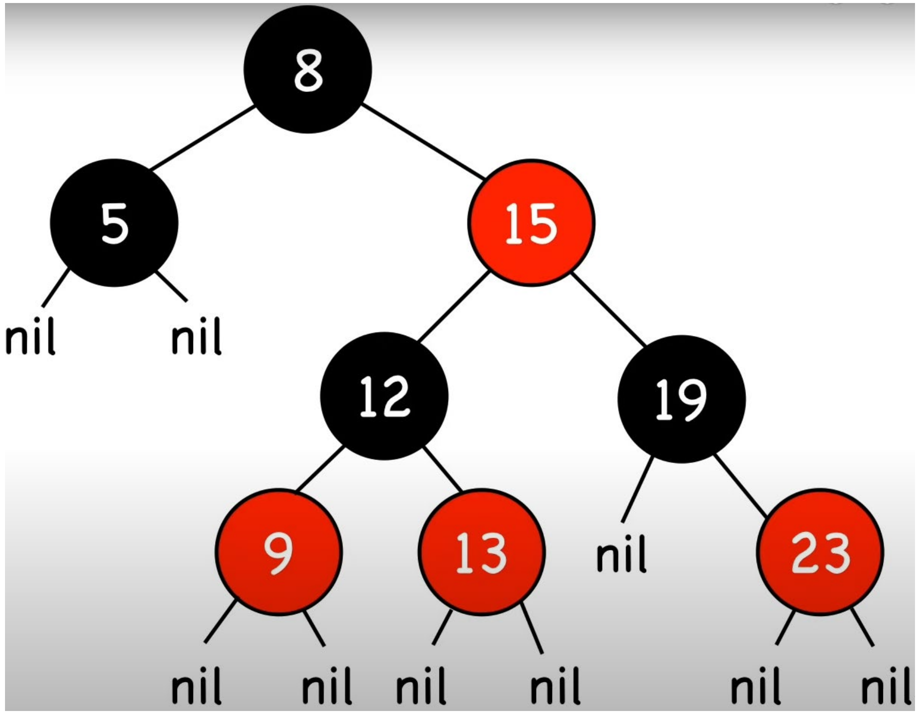
Red-Black tree ensures balancing

- A linear chain of 3 nodes is not possible in a Red-Black tree



Additional Properties

- **AVL Tree**: A balanced BST that maintains the invariant |LeftHeight – RightHeight | <= 1 for all nodes in the tree.



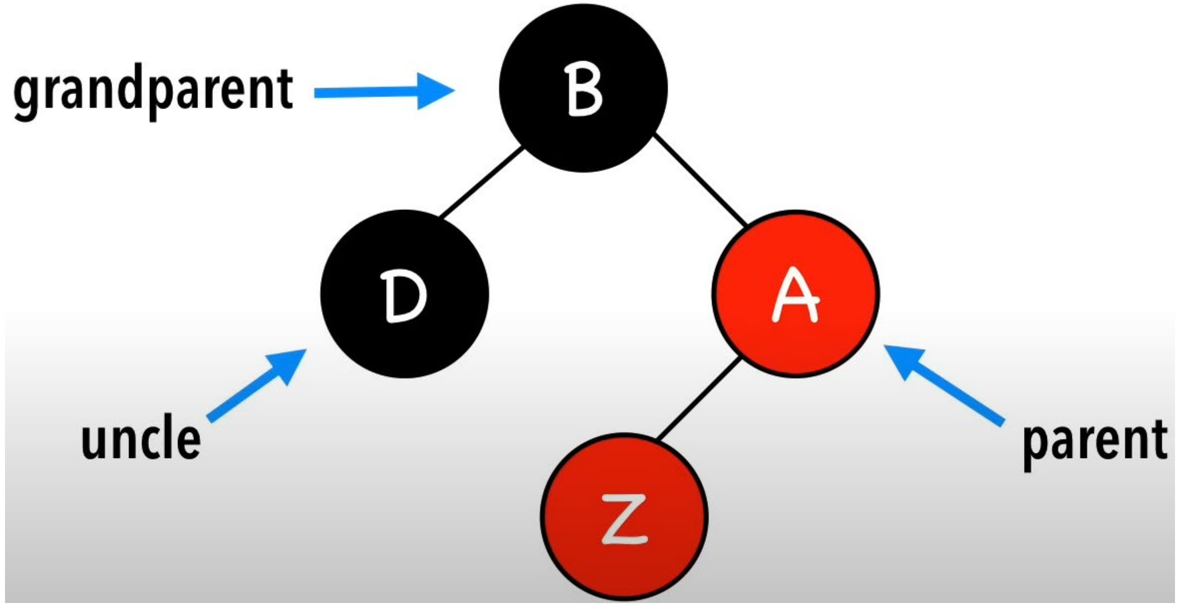
Insertion

- Inserting a new node in a Red-Black Tree involves a two-step process: performing a standard binary search tree (BST) insertion, followed by fixing any violations of Red-Black properties.
- **Insertion Steps**

1. **BST Insert**: Insert the new node into BST and color it red.
2. **Fix Violations**:
    2. If the parent of the new node is **black**, no properties are violated.
    3. If the parent is **red**, the tree might violate the Red Property, requiring fixes.

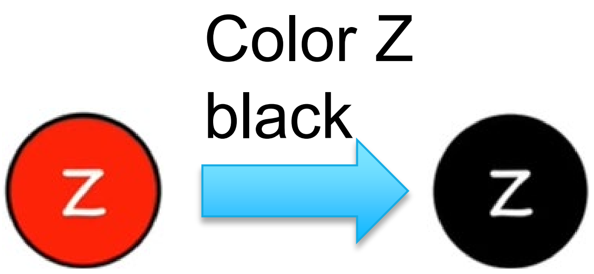
Insertion

- Step 1. Insert Z and color it red
- Step 2. Recolor and rotate nodes to fix violations
- 4 scenarios after inserting node Z
- Case 0. Z = root
  - Color Z black
- Case 1. Z.uncle = red
  - Recolor Z's parents and grandparent
- Case 2. Z.uncle = black (triangle)
  - Rotate Z.parent, turns into Case 3
- Case 3. Z.uncle = black (line)
  - Rotate Z.grandparent & Recolor Z's parents and grandparent



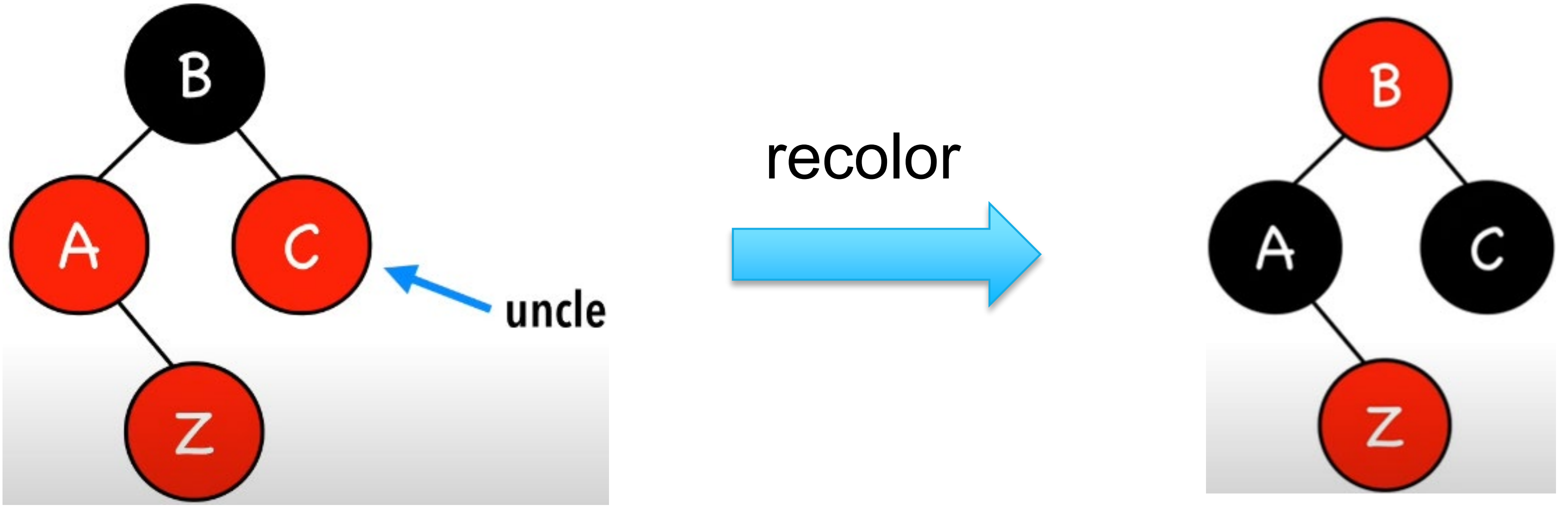
Case 0. Z = root

- Color Z black

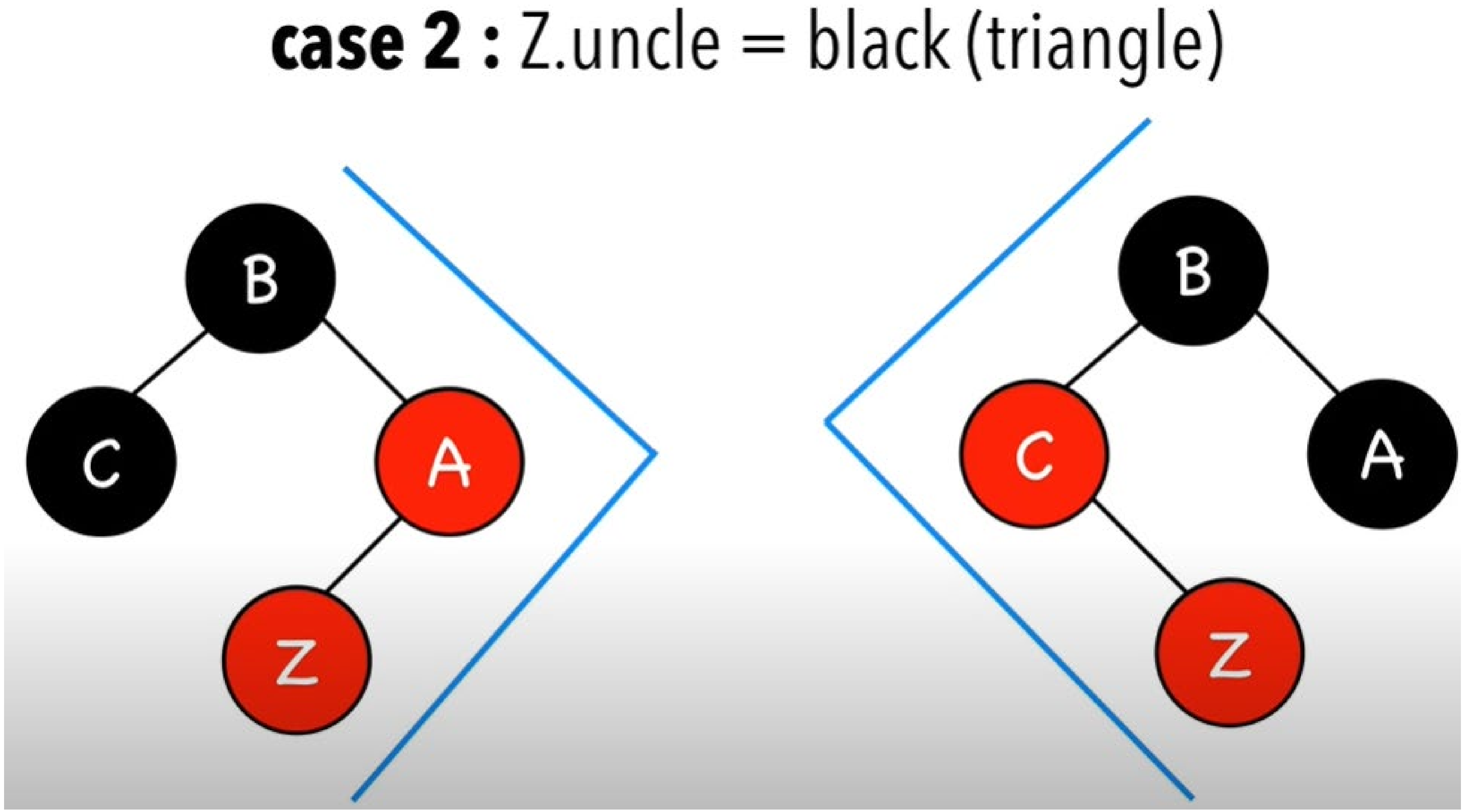


Case 1. Z.uncle = red

- Recolor Z’s parent, uncle, and grandparent

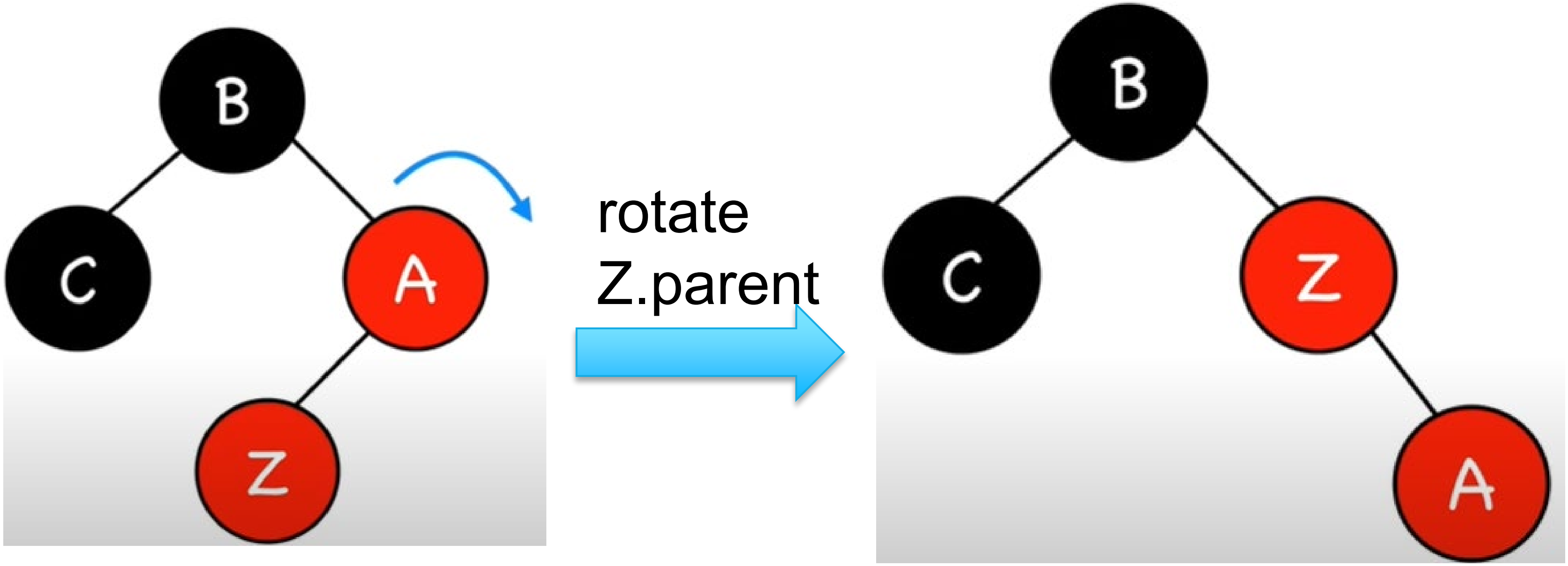


Case 2. Z.uncle = black (triangle)

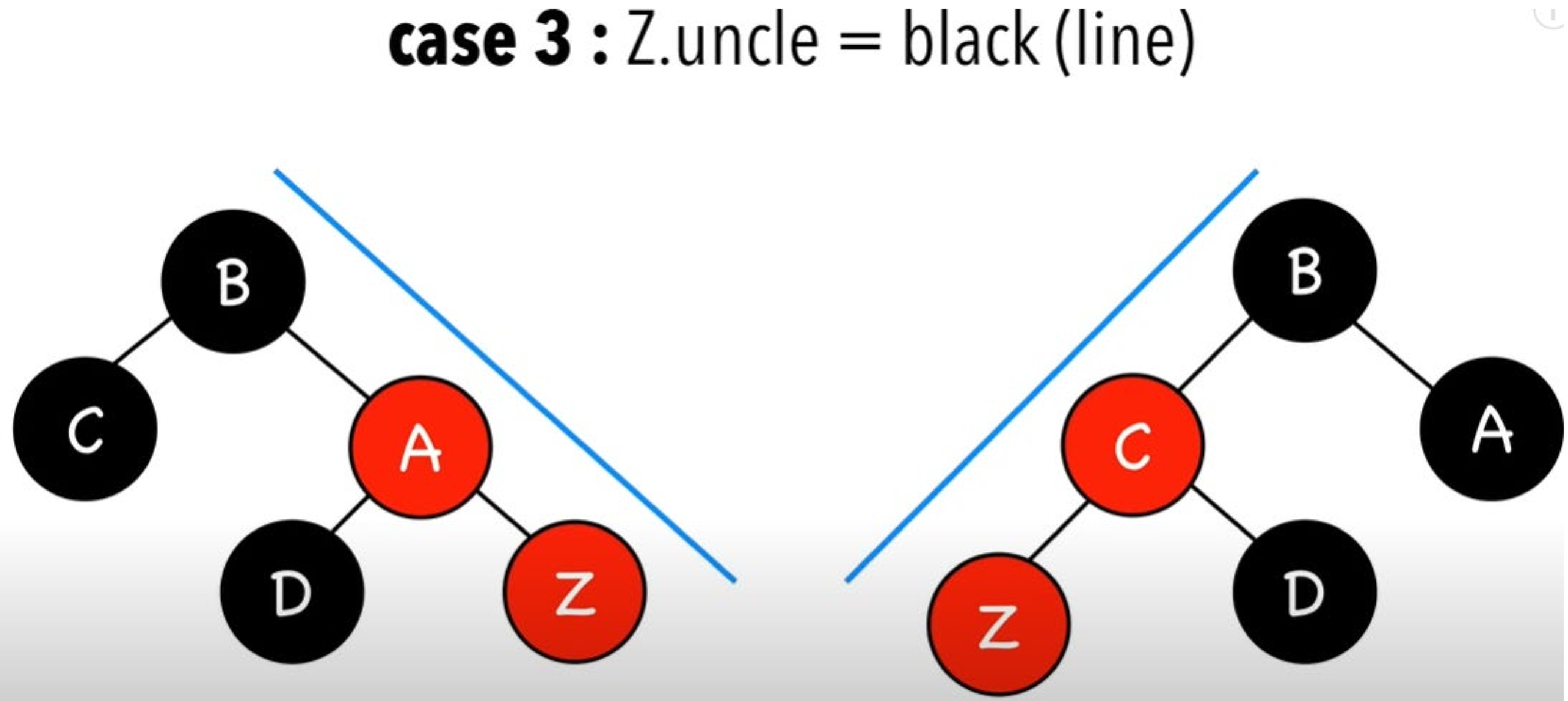


Case 2. Z.uncle = black (triangle)

- Rotate Z.parent
- Turns into Case 3

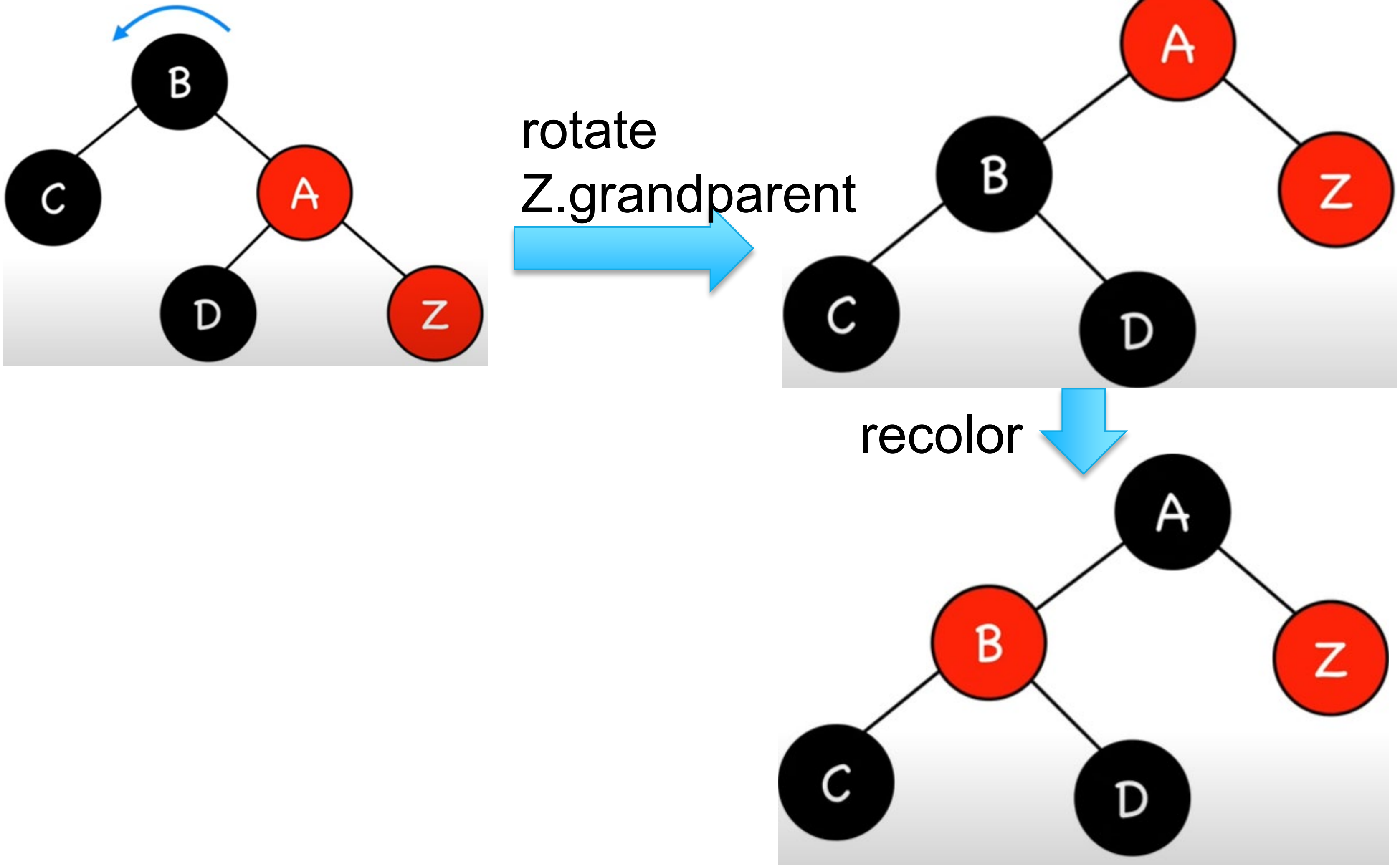


Case 3 Z.uncle = black (line)

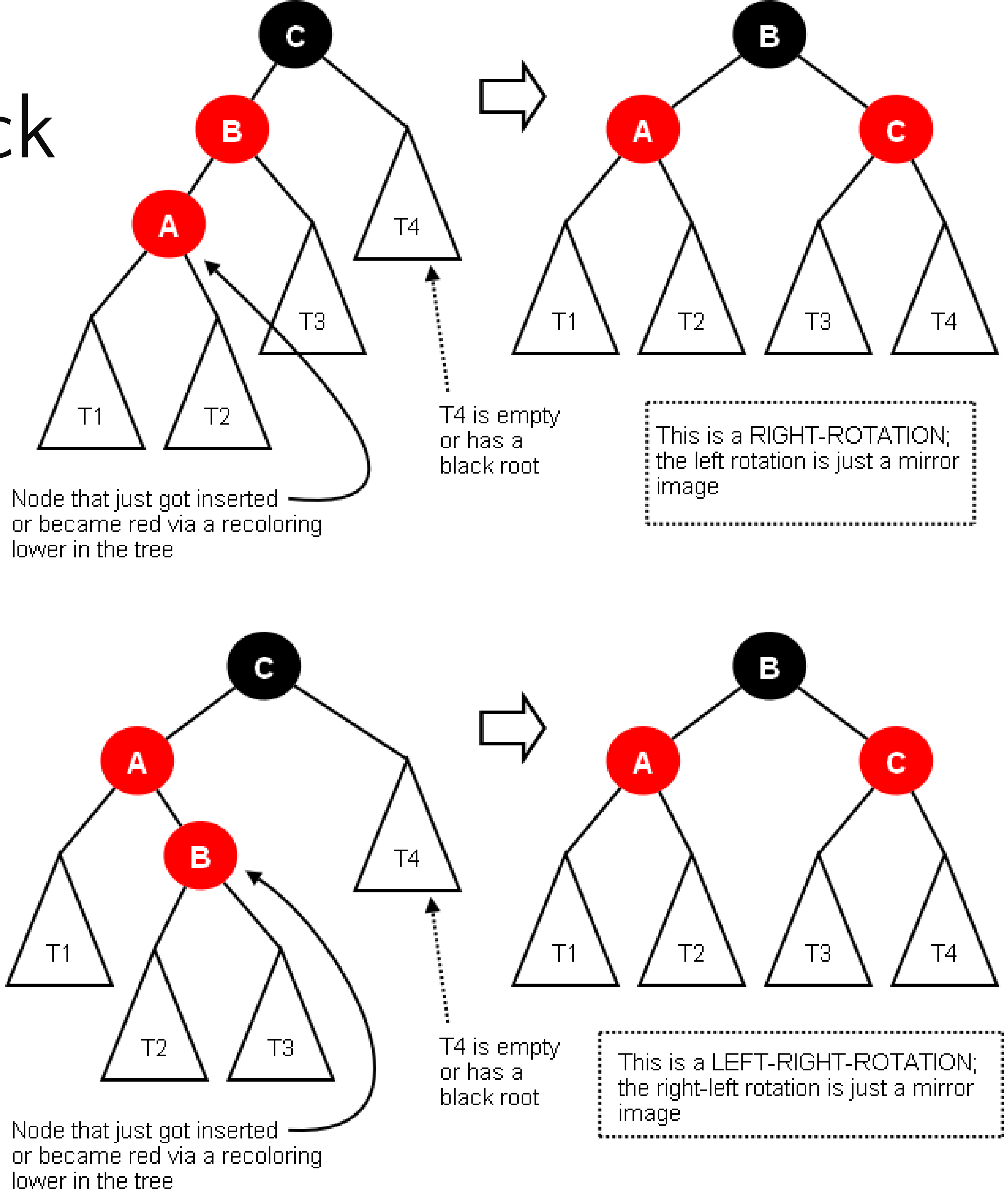


Case 3 Z.uncle = black (line)

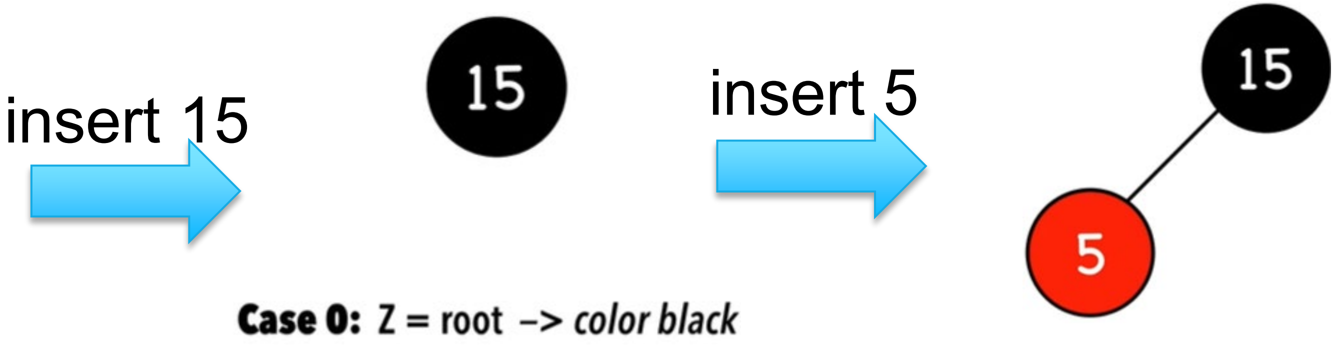
- Rotate Z.grandparent

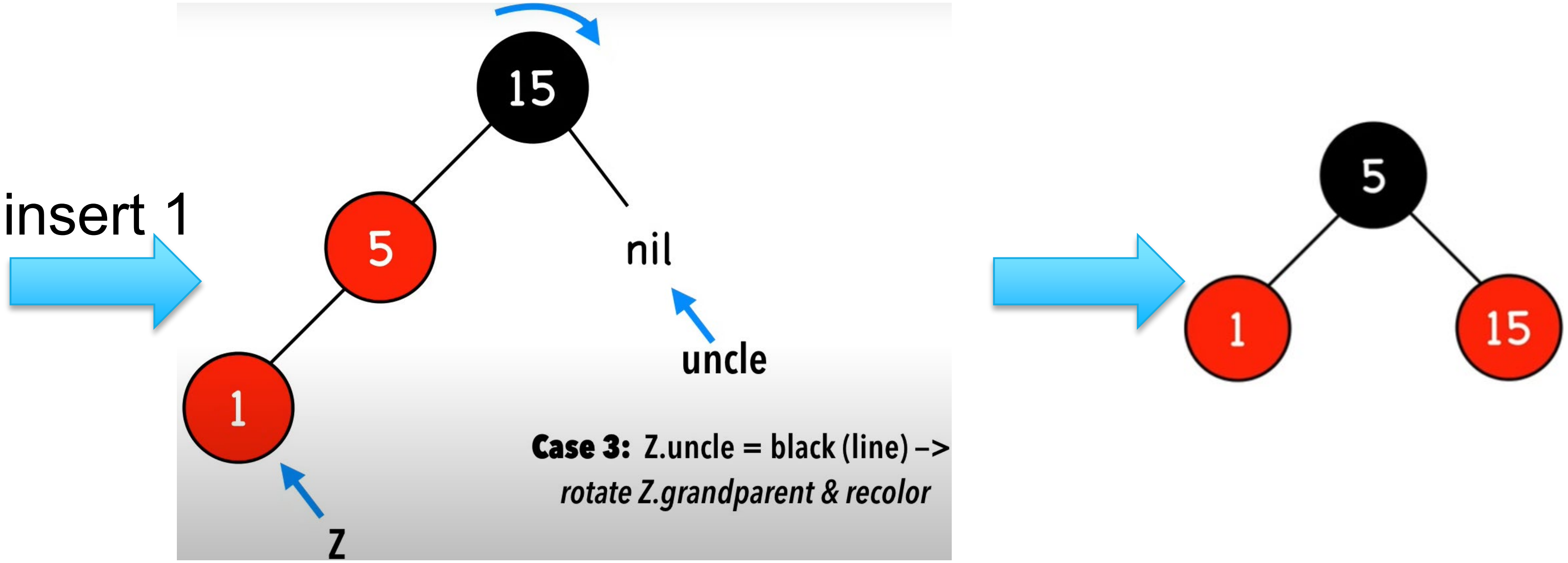


Case 3 Z.uncle = black

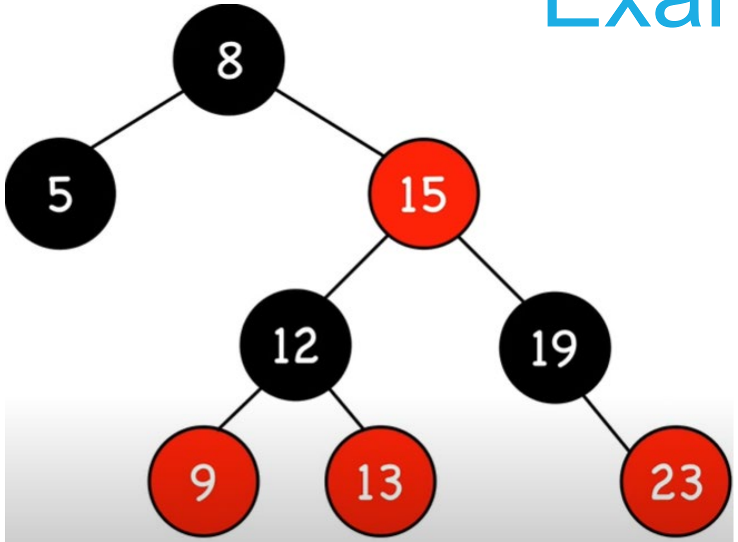


Example 1

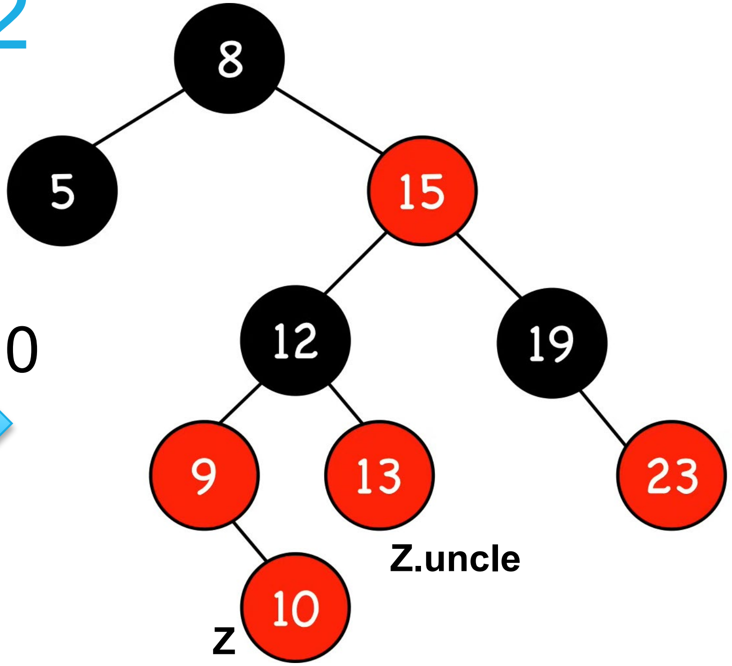

insert 15
15
insert 5
15
5
Case 0: Z = root –> color black


insert 1
15
5
1
nil
uncle
Z
Case 3: Z.uncle = black (line) –>
rotate Z.grandparent & recolor
5
1
15

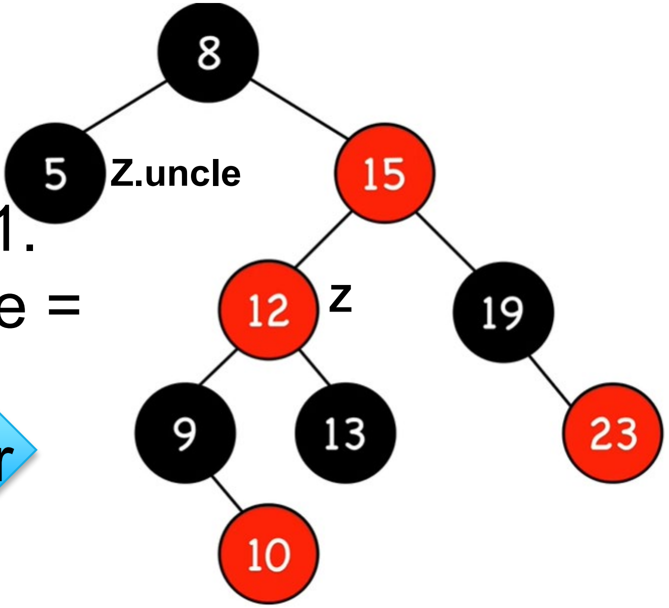
Example 2



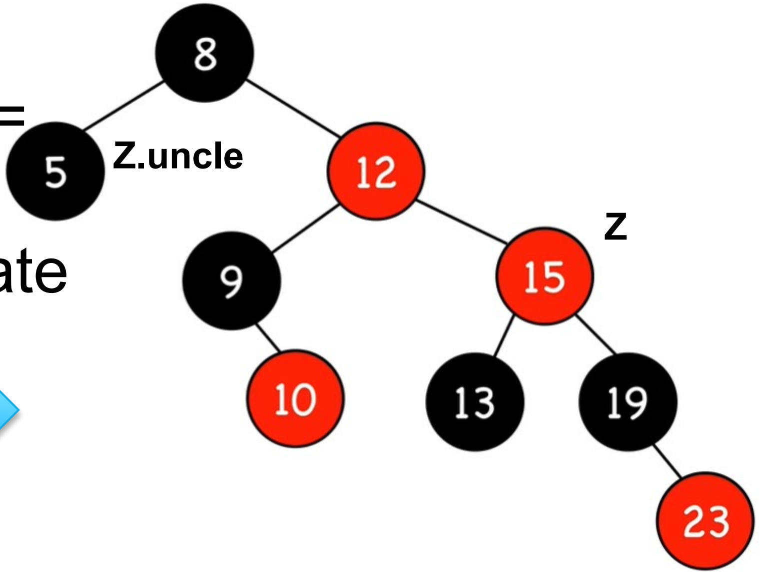
insert 10



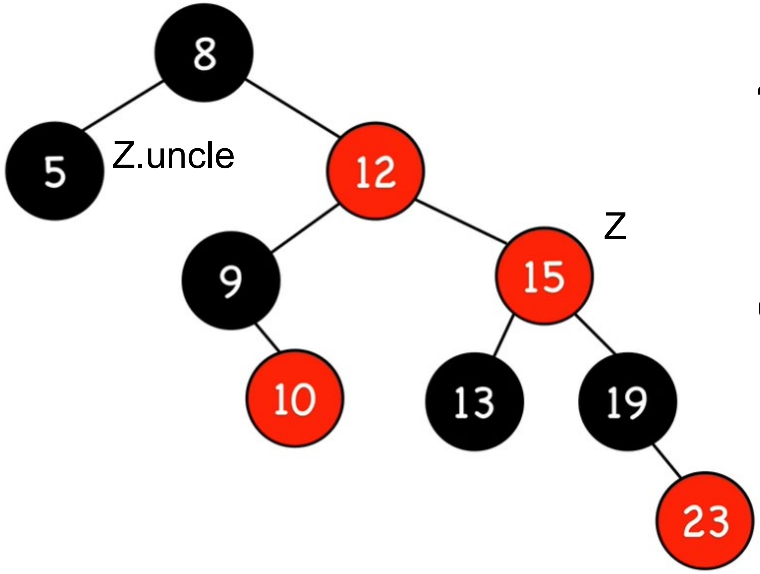
Case 1.
Z.uncle =
red
recolor



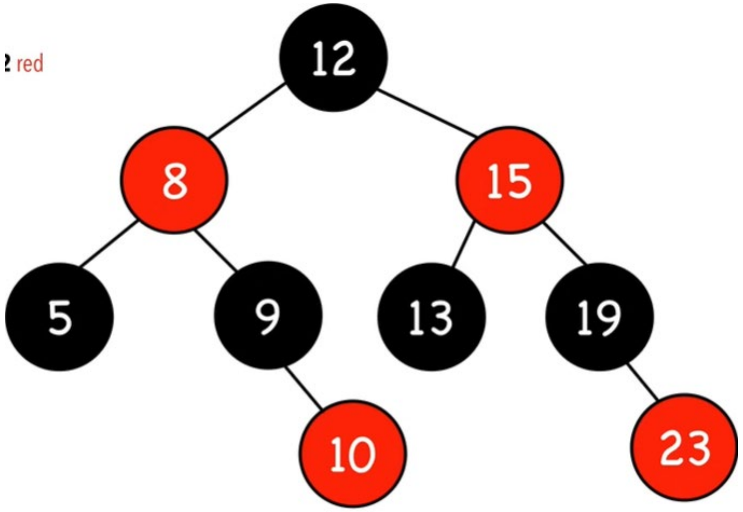
Case 2.
Z.uncle =
black
right rotate
on 15



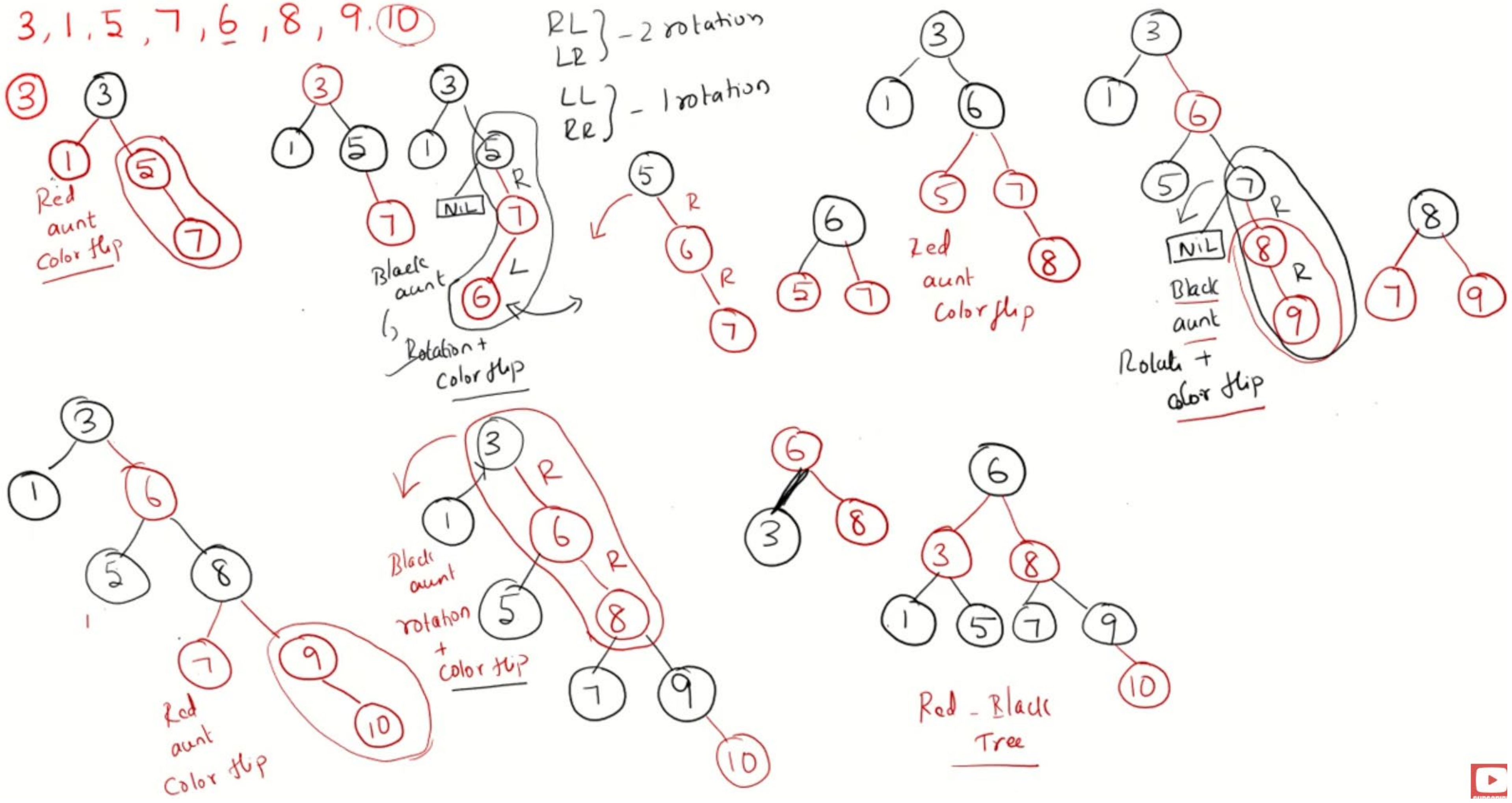
Example 2 Con't



Case 2.
Z.uncle = black
left rotate on 8 & recolor



Another Example



Red Black Tree – Insertion
https://www.youtube.com/watch?v=9ubIKipLpRU

Time Complexity

- 1. Insert : O(log(n))
  - maximum height of red-black trees
- 2. Color red : O(1)
- 3. Fix violations :
  - Constant # of:
  - a. Recolor : O(1)
  - b. Rotation: O(1)
- Overall time complexity: O(log(n))

AVL vs Red Black Trees

Red Black Tree:

- A balanced BST that maintains the (more relaxed) invariant: the longest path (root to farthest NIL) is no more than twice the length of the shortest path (root to nearest NIL).
- More efficient insertion and deletion operations because the balancing requirement is less strict than AVL Tree.

AVL Tree:

- A balanced BST that maintains the (more strict) invariant: |LeftHeight – RightHeight | <= 1 for all nodes in the tree.
- More efficient look up operation because of the strict balance requirement.

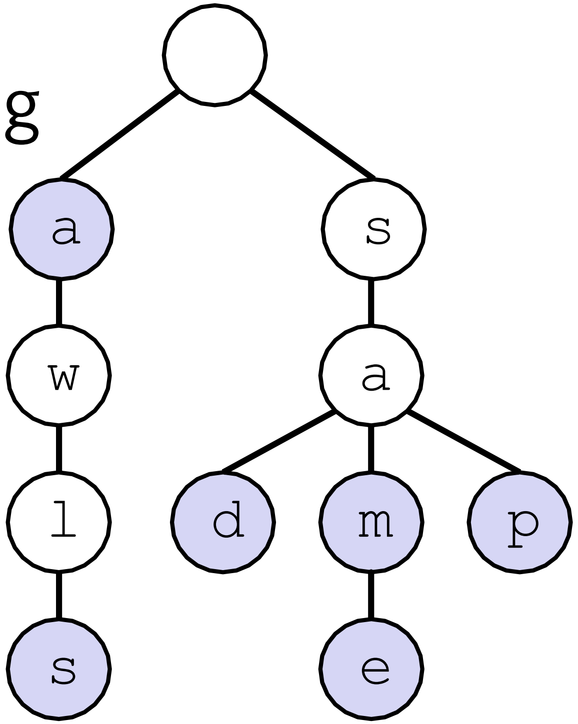
Video Tutorials

- Red-Black Trees // Michael Sambol
  - https://www.youtube.com/playlist?list=PL9xmBV_5YoZNqDI8qfOZgzbqahCUmUEin
  - Lecture slides based in this video series
- Red Black Tree – Insertion, InvesTime
  - https://www.youtube.com/watch?v=9ubIKipLpRU&list=PLoW1nQhPBiz_h5wcmoZODnVnQK9Xja-Pi&index=4
- Introduction to Red-Black Tree
  - https://www.geeksforgeeks.org/introduction-to-red-black-tree/

Tries

Trie: An Introduction

- Tries view its keys as:
  - a **sequence of characters**
  - some (hopefully many!) sequences share common prefixes
- Each level of the tree represents an index in the string
  - Children at that level represent possible
  - characters at that index
- This abstract trie stores the set of strings:
  - `awls, a, sad, same, sap, sam`
- How to deal with `a` and `awls`?
  - Mark which nodes *complete* a string (shown in purple)

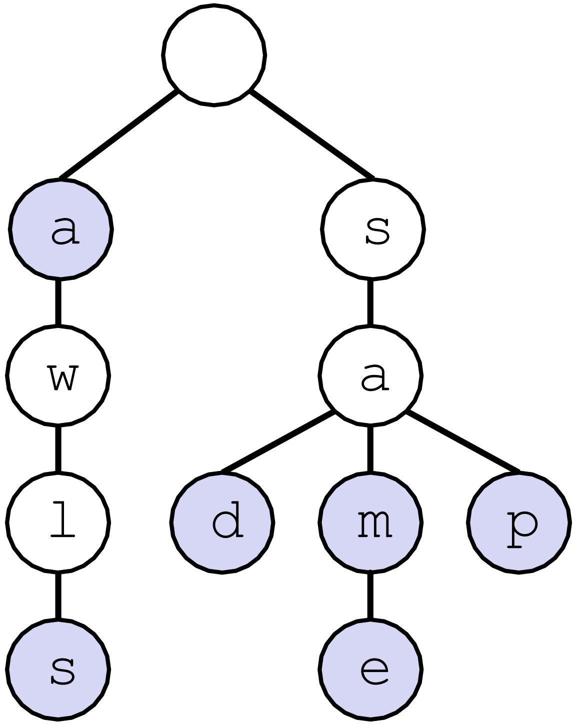


Searching in Tries

Two ways to fail a contains() check:

1. If we fall off the tree
2. If the final node isn't purple (not a key)

Input String	Fall Off? / Is Key?	Result
`contains("sam")`	hit / purple	True
`contains("sa")`	hit / white	False
`contains("a")`	hit / purple	True
`contains("saq")`	fell off / n/a	False



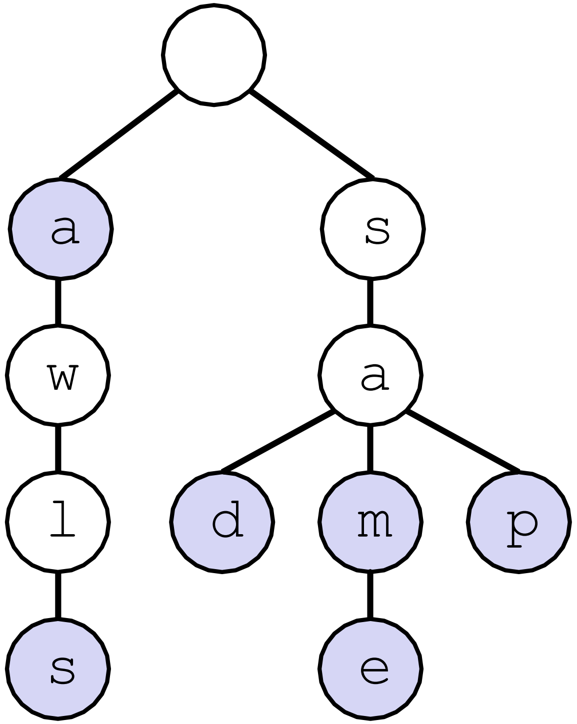
Keys as “a sequence of characters”

- Most dictionaries treat their keys as an “atomic blob”: you can’t disassemble the key into smaller components
- Tries take the opposite view: keys are a **sequence of characters**
  - `Strings` are made of `Characters`
- Tries are defined by 3 types:
  - An “alphabet”: the domain of the characters
  - A “key”: a sequence of “characters” from the alphabet
  - A “value”: the usual Dictionary value

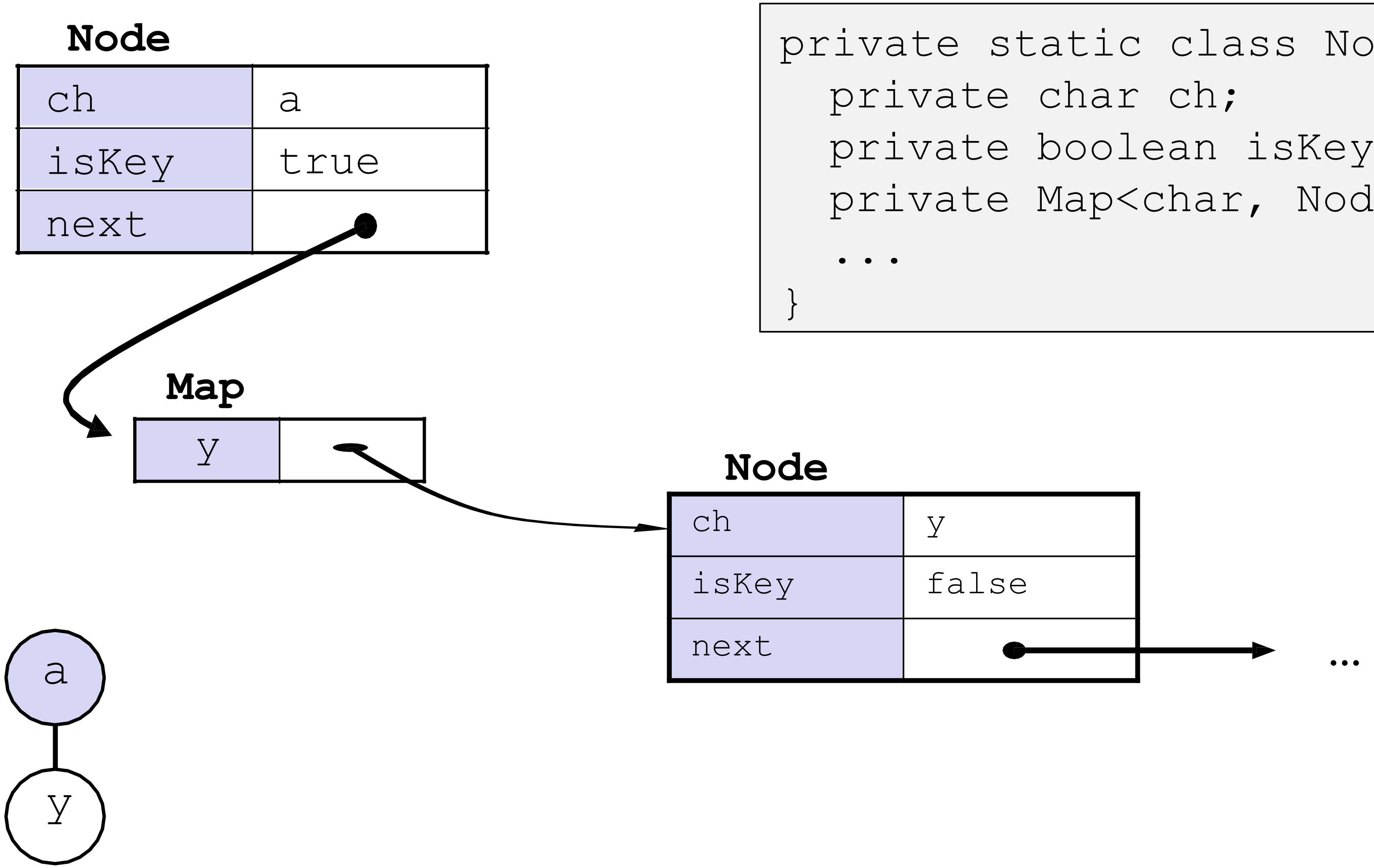
Simple Trie Implementation

```
public class TrieSet {
   private Node root;

   private static class Node {
      private char ch;
      private boolean isKey;
      private Map<char, Node> next;
      private Node(char c, boolean b) {
         ch = c;
         isKey = b;
         next = new HashMap();
      }
   }
}
```



Simple Trie Node Implementation

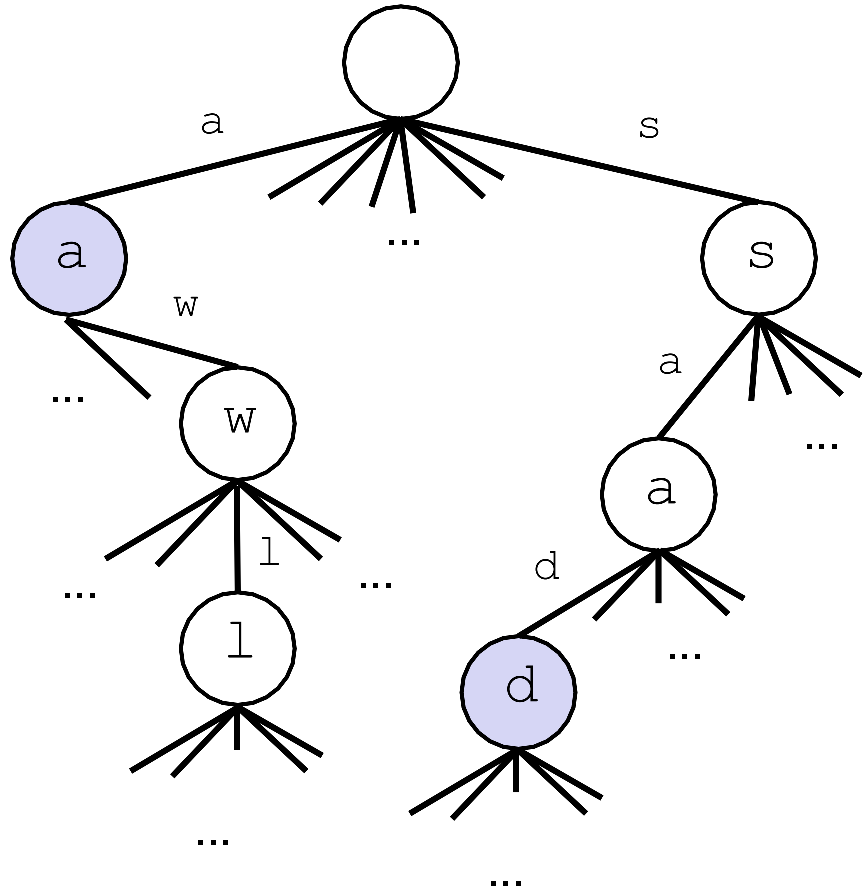



```
private static class Node {
  private char ch;
  private boolean isKey;
  private Map<char, Node> next;
  ...
}
```

Simple Trie Implementation

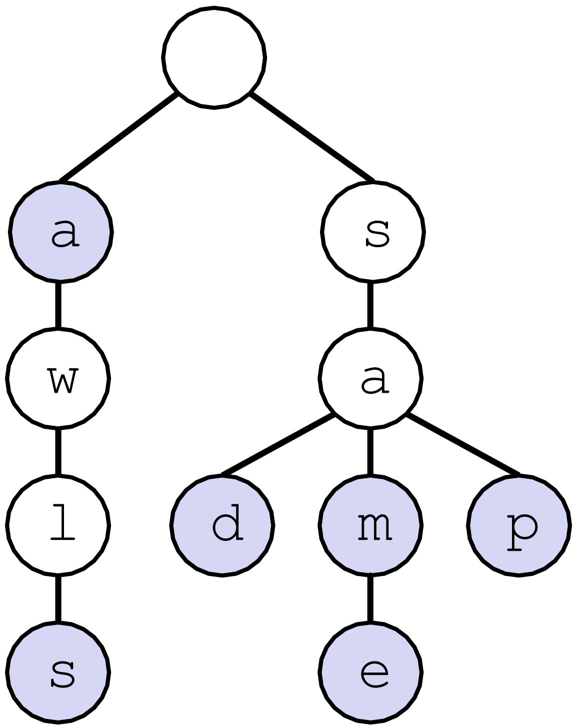
```
  public class TrieSet {
    private Node root;

   private static class Node {
     private char ch;
     private boolean isKey;  private
     Map<char, Node> next;
     private Node(char c, boolean b) {
       ch = c;
       isKey = b;
       next = new HashMap();
     }
   }
}
```



Trie-Specific Operations

- Prefix matching
  - Keys are sequences that can have prefixes
- **Longest prefix**
  - `longestPrefixOf("sample")`
  - Want: `{"sam"}`
- **Prefix match**
  - `findPrefix("sa")`
  - Want: `{"sad", "sam", "same", "sap"}`



Summary

- A trie data structure implements the Dictionary and Set
- Tries store sequential keys
  - … which enables very efficient prefix operations like `findPrefix`
- Tries have many different implementations
  - Could store HashMap/TreeMap/any-dictionary within nodes

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