

# Lecture 5.0

## Shortest Paths

Department of Computer Science  
Hofstra University

# Lecture Goals

- In this lecture we study **shortest-paths** problems. We begin by analyzing some basic properties of shortest paths and a generic algorithm for the problem.
- We introduce and analyze **Dijkstra's algorithm** for shortest-paths problems with nonnegative weights.
- We conclude with the **Bellman–Ford** algorithm for edge-weighted digraphs with no negative cycles.

# Shortest Paths in an Edge-weighted Digraph

Given an edge-weighted digraph, find the shortest path from  $s$  to  $t$ .

edge-weighted digraph

4→5	0.35
5→4	0.35
4→7	0.37
5→7	0.28
7→5	0.28
5→1	0.32
0→4	0.38
0→2	0.26
7→3	0.39
1→3	0.29
2→7	0.34
6→2	0.40
3→6	0.52
6→0	0.58
6→4	0.93



shortest path from 0 to 6

0→2	0.26
2→7	0.34
7→3	0.39
3→6	0.52

## Variants

### ❖ Which vertices?

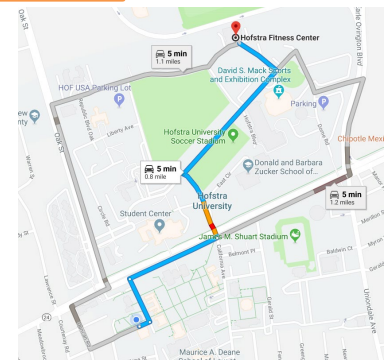
- Single source: from one vertex  $s$  to every other vertex.
- Source-sink: from one vertex  $s$  to another  $t$ .
- All pairs: between all pairs of vertices.

### ❖ Nonnegative weights?

### ❖ Cycles?

- Negative cycles.

Can we use BFS?



Simplifying assumption: Each vertex is reachable from  $s$ .

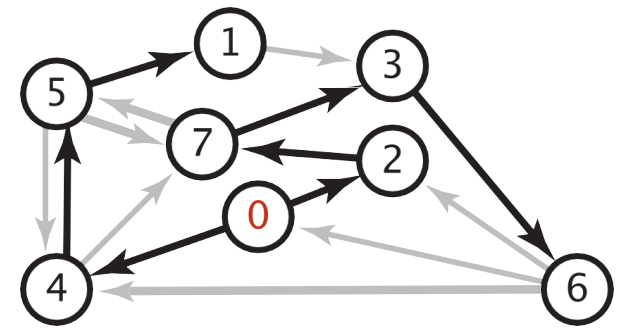
# Data Structures for Single-source Shortest Paths

**Goal.** Find the shortest path from  $s$  to every other vertex.

**Observation.** A shortest-paths tree (SPT) solution exists.

**Consequence.** Can represent the SPT with two vertex-indexed arrays:

- $\text{distTo}[v]$  is length of shortest path from  $s$  to  $v$ .
- $\text{edgeTo}[v]$  is last edge on shortest path from  $s$  to  $v$ .



shortest-paths tree from 0

	edgeTo[]	distTo[]
0	null	0
1	5->1 0.32	1.05
2	0->2 0.26	0.26
3	7->3 0.37	0.97
4	0->4 0.38	0.38
5	4->5 0.35	0.73
6	3->6 0.52	1.49
7	2->7 0.34	0.60

parent-link representation

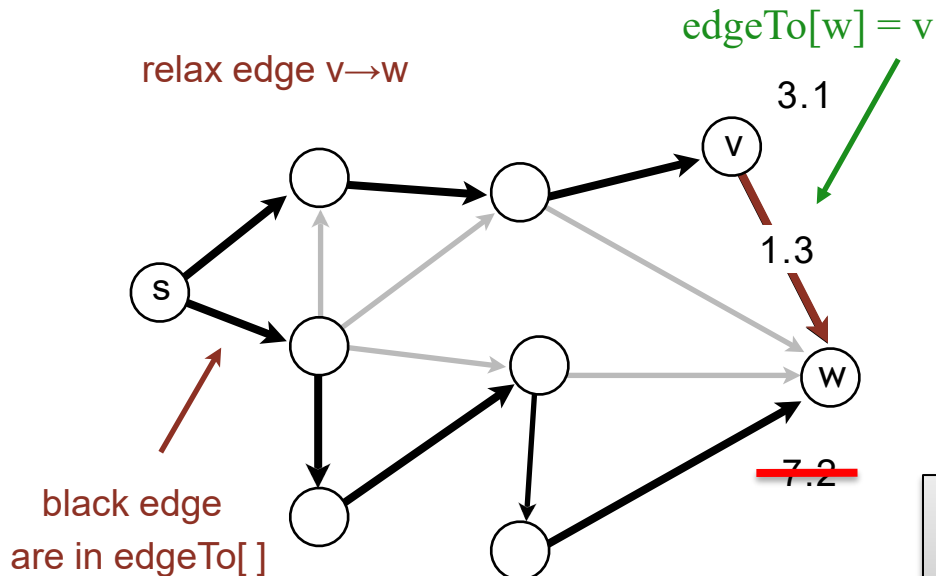
```
public double distTo(int v)
{ return distTo[v]; }

public Iterable<DirectedEdge> pathTo(int v)
{
    Stack<DirectedEdge> path = new Stack<DirectedEdge>();
    for (DirectedEdge e = edgeTo[v]; e != null; e = edgeTo[e.from()])
        path.push(e);
    return path;
}
```

# Edge Relaxation

Relax edge  $e = v \rightarrow w$ . (basic of building SPT)

- distTo[v] is length of shortest **known** path from s to v.
- distTo[w] is length of shortest **known** path from s to w.
- edgeTo[w] is last edge on shortest **known** path from s to w.
- If  $e = v \rightarrow w$  gives shorter path to w through v, update distTo[w] and edgeTo[w].
  - distTo[w] = min(distTo[w], distTo[v] + e.weight());



```
private void relax(DirectedEdge e)
{
    int v = e.from(), w = e.to();
    if (distTo[w] > distTo[v] + e.weight())
    {
        distTo[w] = distTo[v] + e.weight();
        edgeTo[w] = e;
    }
}
```

OLD distTo[w] = 7.2 > distTo[v] + e.weight() = 3.1 + 1.3 = 4.4  
 NEW distTo[w] ← distTo[v] + e.weight() = 4.4

# Generic Shortest-paths Algorithm

---

## Generic algorithm (to compute SPT from $s$ )

---

For each vertex  $v$ :  $\text{distTo}[v] = \infty$ .

For each vertex  $v$ :  $\text{edgeTo}[v] = \text{null}$ .

$\text{distTo}[s] = 0$ .

Repeat until done:

- Relax any edge.

---

**Proposition.** Generic algorithm computes SPT (if it exists) from  $s$ .

**Pf.**

- Throughout algorithm,  $\text{distTo}[v]$  is the length of a simple path from  $s$  to  $v$  (and  $\text{edgeTo}[v]$  is last edge on path).
- Each successful relaxation decreases  $\text{distTo}[v]$  for some  $v$ .
- The entry  $\text{distTo}[v]$  can decrease at most a finite number of times.

**Efficient implementations.** How to choose which edge to relax?

- Ex 1. Dijkstra's algorithm. (**nonnegative weights, directed cycles**).
- Ex 2. Bellman–Ford algorithm. (**no negative cycles**).



# Dijkstra's Algorithm

- Initialization:
  - Set the distance to the source node as 0 and to all other nodes as infinity.
  - Mark all nodes as unvisited and store them in a priority queue.
- Main Loop:
  - Extract the **unvisited node** with **the smallest known distance** from the queue.
  - For each neighboring node, calculate its tentative distance through the current node. **If this distance is smaller than the previously recorded distance, update it with relaxation.**
  - Mark the current node as visited once all its neighbors are processed.
- Termination:
  - The algorithm continues until all reachable nodes are visited, or until the shortest path to a specific destination is found.

# Dijkstra's Algorithm: Correctness Proof

**Proposition.** Dijkstra's algorithm computes a SPT in any edge-weighted digraph with nonnegative weights.

**Pf.**

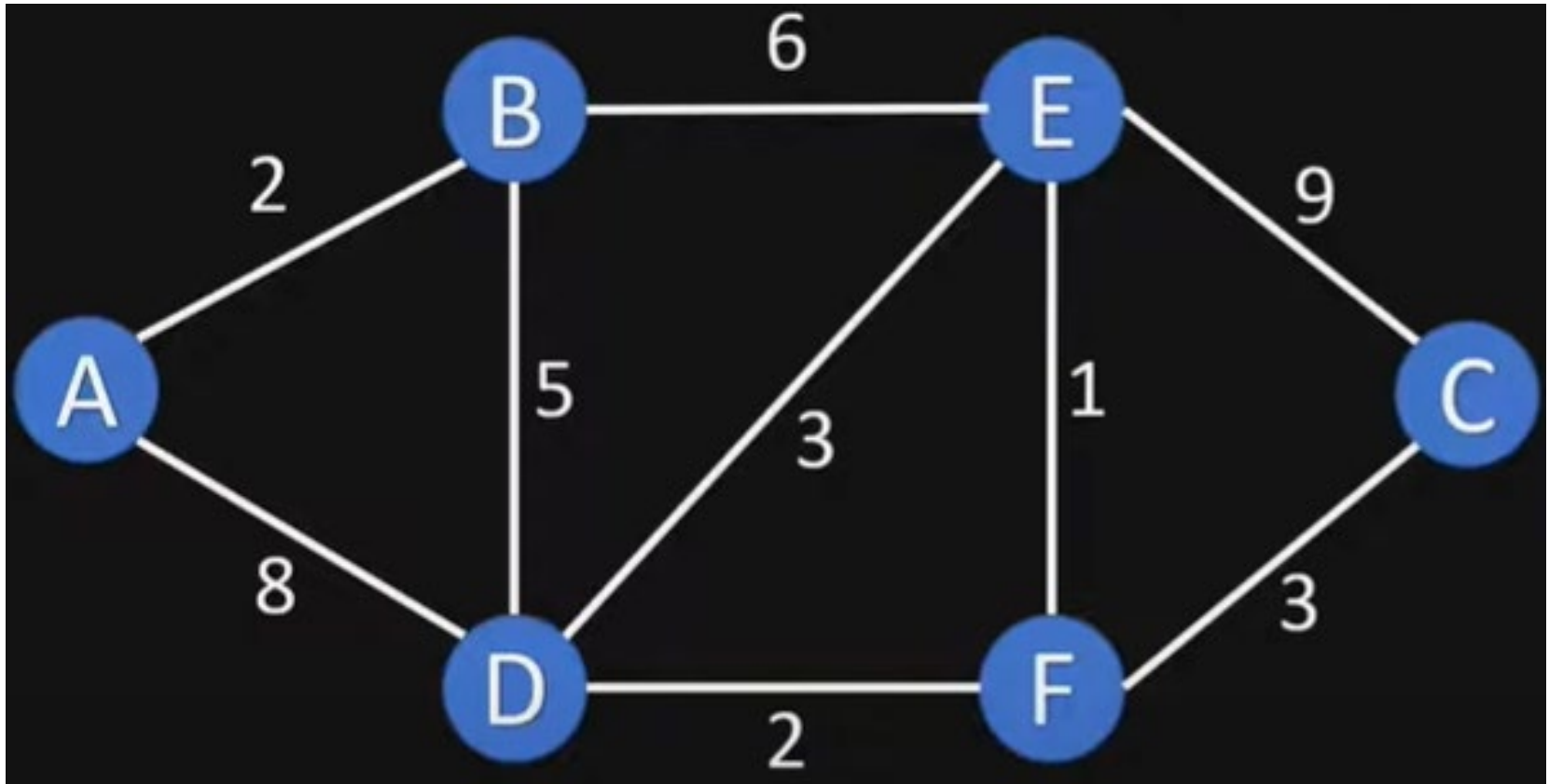
- Each edge  $e = v \rightarrow w$  is relaxed exactly once (when  $v$  is relaxed),
  - leaving  $\text{distTo}[w] \leq \text{distTo}[v] + e.\text{weight}()$ .
- Inequality holds until algorithm terminates because:
  - $\text{distTo}[w]$  cannot increase   $\text{distTo}[\ ]$  values are monotone decreasing
  - $\text{distTo}[v]$  will not change  we choose lowest  $\text{distTo}[\ ]$  value at each step (and edge weights are nonnegative)
- Thus, upon termination, shortest-paths optimality conditions hold.



# Video Tutorial

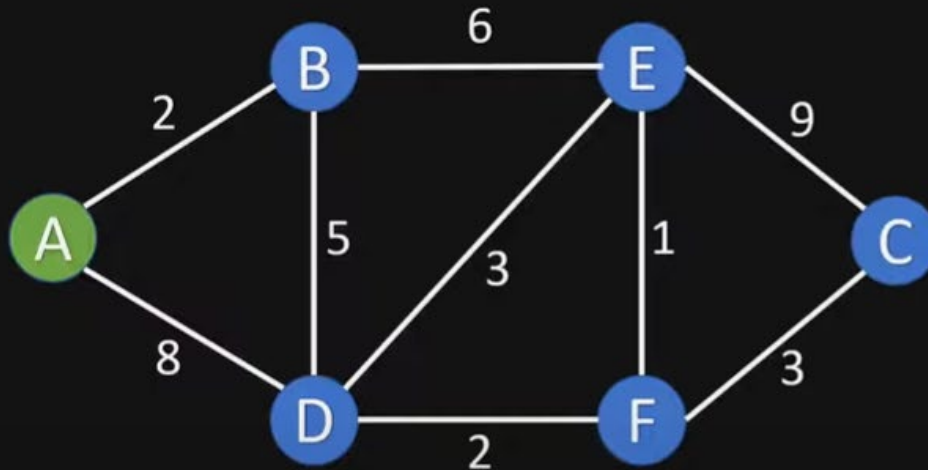
- The following lecture slides are based on this video:
- Dijkstras Shortest Path Algorithm Explained | With Example | Graph Theory
  - <https://www.youtube.com/watch?v=bZkzH5x0SKU>

# Example Graph



# Initialize

2. Assign to all nodes a tentative distance value



Visited Nodes: []

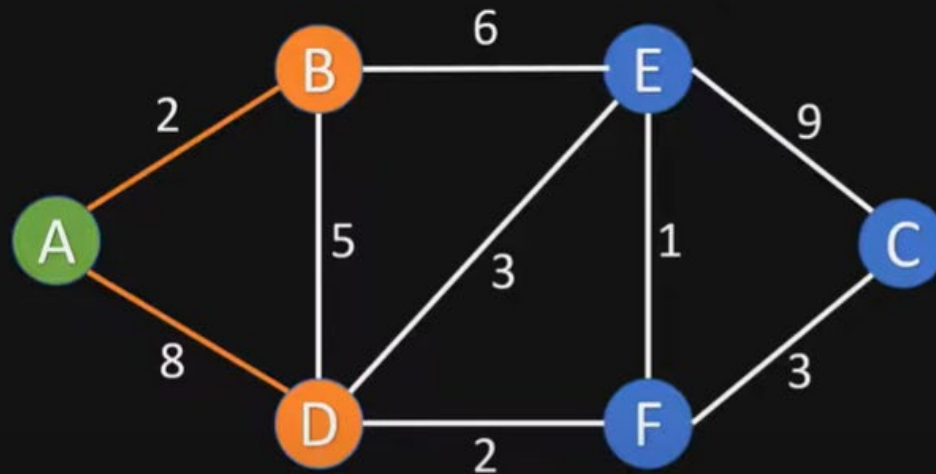
Unvisited Nodes: [A, B, C, D, E, F]

Node	Shortest Distance	Previous Node
A	0	
B	$\infty$	
C	$\infty$	
D	$\infty$	
E	$\infty$	
F	$\infty$	

# Visit Node A

3. For the current node calculate the distance to all unvisited neighbours

3.1. Update shortest distance, if new distance is shorter than old distance



Node	Shortest Distance	Previous Node
A	0	
B	2	A
C	$\infty$	
D	8	A
E	$\infty$	
F	$\infty$	

Visited Nodes: []

Unvisited Nodes: [A, B, C, D, E, F]

OLD  $\text{distTo}[B] = \infty > \text{distTo}[A] + e[A][B].\text{weight}() = 0 + 2 = 2$

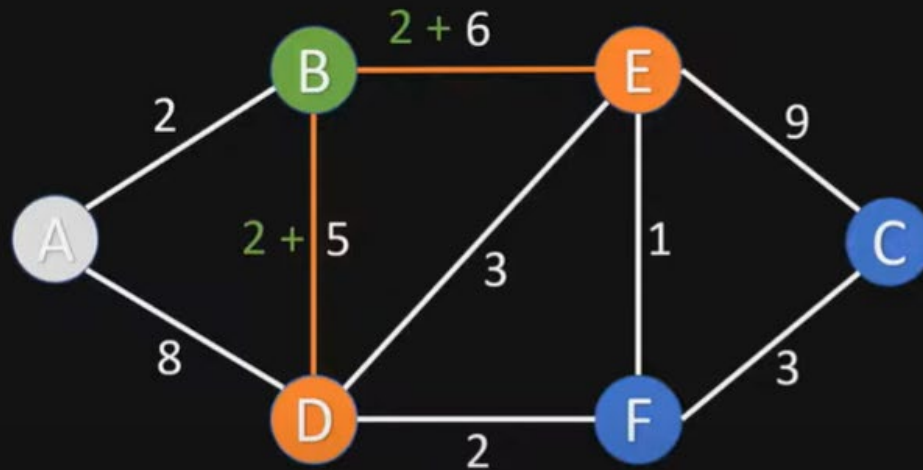
NEW  $\text{distTo}[B] \leftarrow \text{distTo}[A] + e[A][B].\text{weight}() = 2$

OLD  $\text{distTo}[D] = \infty > \text{distTo}[A] + e[A][D].\text{weight}() = 0 + 8 = 8$

NEW  $\text{distTo}[D] \leftarrow \text{distTo}[A] + e[A][D].\text{weight}() = 8$

# Visit Node B

3. For the current node calculate the distance to all unvisited neighbours  
 3.1. Update shortest distance, if new distance is shorter than old distance



Visited Nodes: [A]

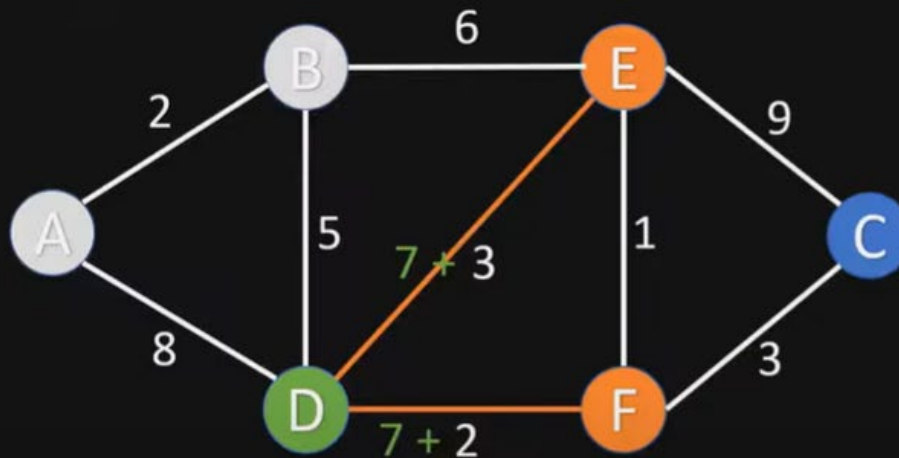
Unvisited Nodes: [B, C, D, E, F]

Node	Shortest Distance	Previous Node
A	0	
B	2	A
C	$\infty$	
D	7	B
E	8	B
F	$\infty$	

OLD  $\text{distTo}[D] = 8 > \text{distTo}[B] + e[B][D].\text{weight}() = 2+5 = 7$   
 NEW  $\text{distTo}[D] \leftarrow \text{distTo}[B] + e[B][D].\text{weight}() = 7$   
 OLD  $\text{distTo}[E] = \infty > \text{distTo}[B] + e[B][E].\text{weight}() = 2+6 = 8$   
 NEW  $\text{distTo}[E] \leftarrow \text{distTo}[B] + e[B][E].\text{weight}() = 8$

# Visit Node D

3. For the current node calculate the distance to all unvisited neighbours  
 3.1. Update shortest distance, if new distance is shorter than old distance



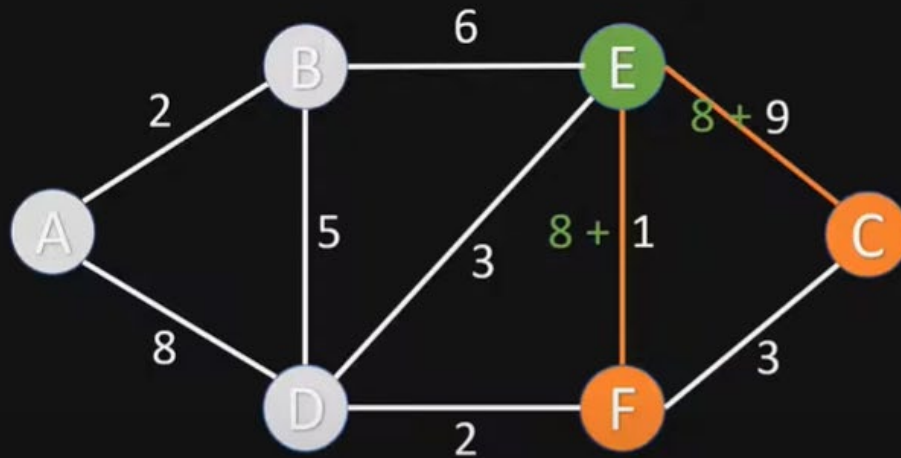
Visited Nodes: [A, B]    Unvisited Nodes: [C, D, E, F]

Node	Shortest Distance	Previous Node
A	0	
B	2	A
C	$\infty$	
D	7	B
E	8	B
F	9	D

$\text{OLD distTo}[E] = 8 < \text{distTo}[D] + e[D][E].\text{weight}() = 7+3 = 10$   
 No update,  $\text{distTo}[E]$  stays 8  
 $\text{OLD distTo}[F] = \infty > \text{distTo}[D] + e[D][F].\text{weight}() = 7+2 = 9$   
 $\text{NEW distTo}[F] \leftarrow \text{distTo}[D] + e[D][F].\text{weight}() = 9$

# Visit Node E

3. For the current node calculate the distance to all unvisited neighbours
- 3.1. Update shortest distance, if new distance is shorter than old distance



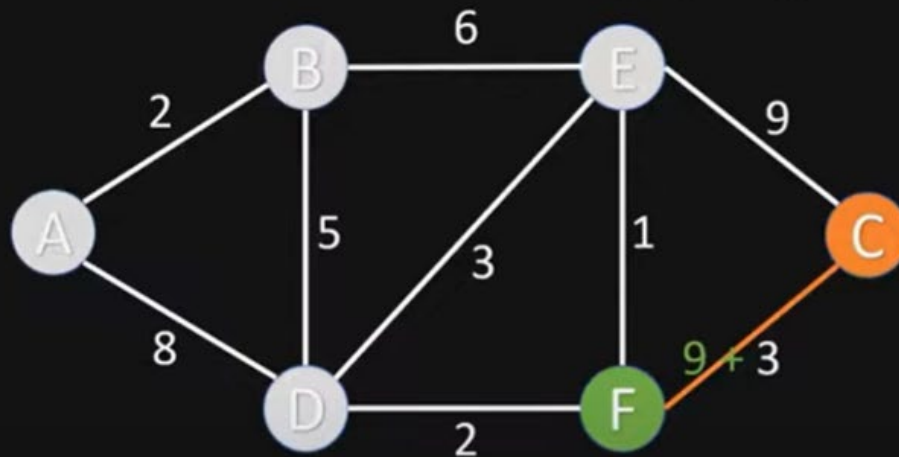
Visited Nodes: [A, B, D] Unvisited Nodes: [C, E, F]

Node	Shortest Distance	Previous Node
A	0	
B	2	A
C	17	E
D	7	B
E	8	B
F	9	D

$\text{OLD distTo}[C] = \infty > \text{distTo}[E] + e[E][C].\text{weight}() = 8 + 9 = 17$   
 $\text{NEW distTo}[C] \leftarrow \text{distTo}[E] + e[E][C].\text{weight}() = 17$   
 $\text{OLD distTo}[F] = 9 = \text{distTo}[E] + e[E][F].\text{weight}() = 8 + 1 = 9$   
 No update,  $\text{distTo}[F]$  stays 9

# Visit Node F

3. For the current node calculate the distance to all unvisited neighbours  
 3.1. Update shortest distance, if new distance is shorter than old distance



Visited Nodes: [A, B, D, E]    Unvisited Nodes: [C, F]

Node	Shortest Distance	Previous Node
A	0	
B	2	A
C	12	F
D	7	B
E	8	B
F	9	D

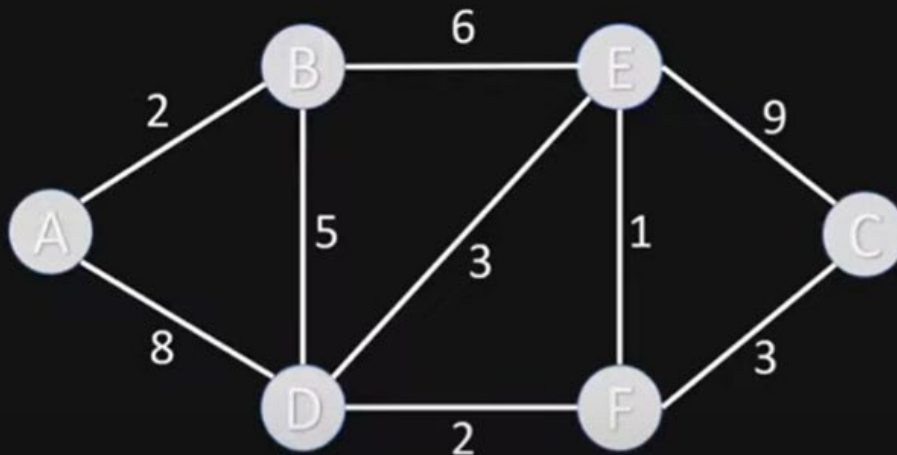
OLD  $\text{distTo}[C] = 17 > \text{distTo}[F] + e[F][C].\text{weight}() = 9 + 3 = 12$   
 NEW  $\text{distTo}[C] \leftarrow \text{distTo}[F] + e[F][C].\text{weight}() = 12$



# End of Algorithm

- Table contains the shortest distance to each node N from the source node A, and its previous node in the shortest path

4. Mark current node as visited



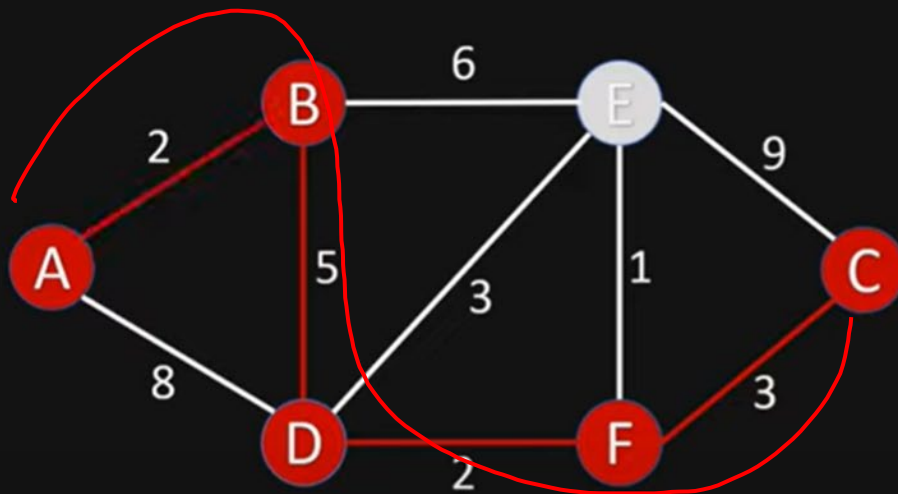
Visited Nodes: [A, B, D, E, F, C] Unvisited Nodes: []

Node	Shortest Distance	Previous Node
A	0	
B	2	A
C	12	F
D	7	B
E	8	B
F	9	D

# Getting the Shortest Path from A to C

- C's previous node is F; F's previous node is D; D's previous node is B; B's previous node is A
- Shortest Path from A to C is ABDFC

Get shortest path from A to C

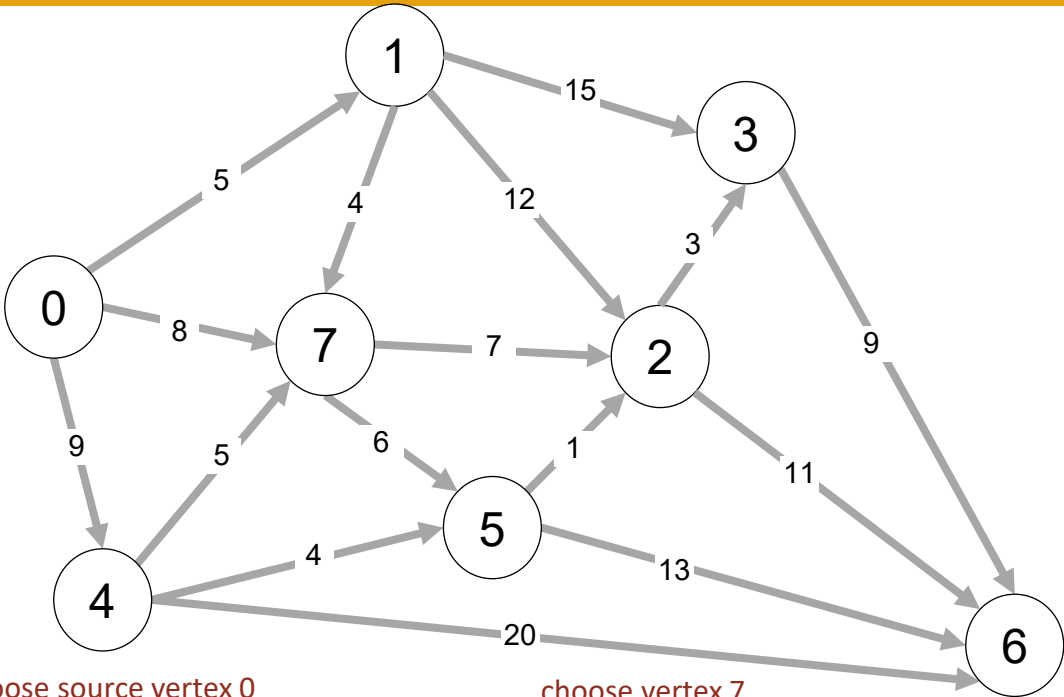


Node	Shortest Distance	Previous Node
A	0	
B	2	A
C	12	F
D	7	B
E	8	B
F	9	D

# Dijkstra's Algorithm Example 2

- Consider vertices in increasing order of distance from s
  - (non-tree vertex with the lowest distTo[ ] value).
- Add vertex to tree and relax all edges pointing from that vertex.

choose vertex 5  
 relax all edges adjacent from 5  
 choose vertex 2  
 relax all edges adjacent from 2  
 choose vertex 3  
 relax all edges adjacent from 3  
 choose vertex 6  
 relax all edges adjacent from 6



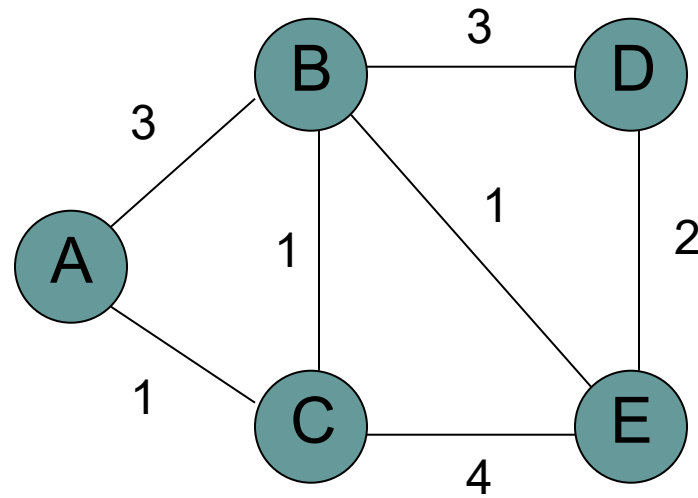
choose source vertex 0  
 relax all edges adjacent from 0  
 choose vertex 1  
 relax all edges adjacent from 1

choose vertex 7  
 relax all edges adjacent from 7  
 choose vertex 4  
 relax all edges adjacent from 4

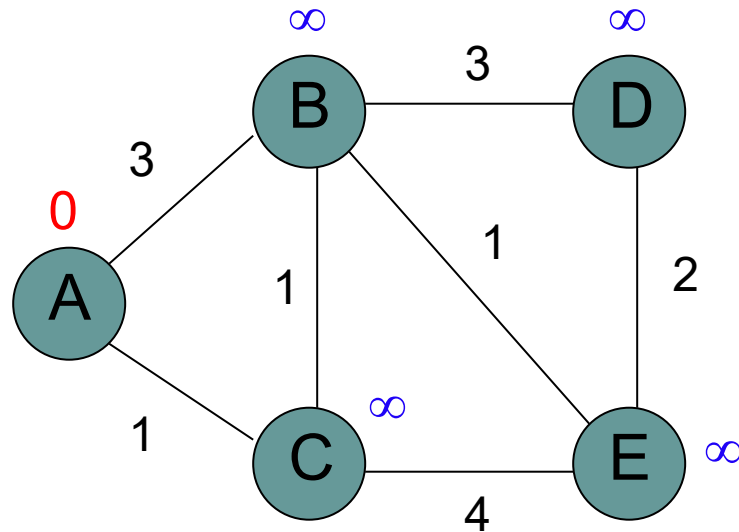
v distTo[]			
→ 0	<del>∞</del>	0	
→ 1	<del>∞</del>	5	
→ 2	<del>∞</del>	<del>17</del>	<del>15</del> 14
→ 3	<del>∞</del>	<del>20</del>	17
→ 4	<del>∞</del>	9	
→ 5	<del>∞</del>	<del>14</del>	13
→ 6	<del>∞</del>	<del>29</del>	<del>26</del> 25
→ 7	<del>∞</del>	8	

v edgeTo[]			
0	-		
1	<del>-</del>	0	
2	<del>-</del>	<del>1</del>	<del>7</del> 5
3	<del>-</del>	<del>1</del>	2
4	<del>-</del>	0	
5	<del>-</del>	<del>7</del>	4
6	<del>-</del>	<del>4</del>	<del>5</del> 2
7	<del>-</del>	0	

# Dijkstra's Algorithm Example 3

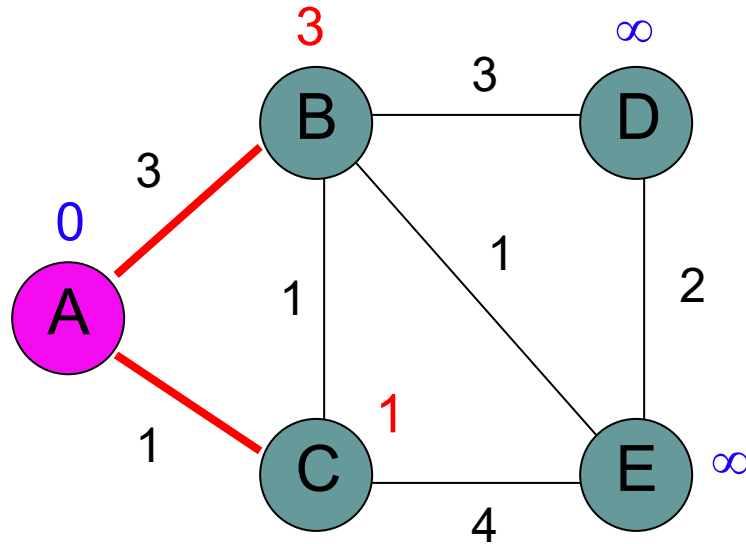


# Initialize



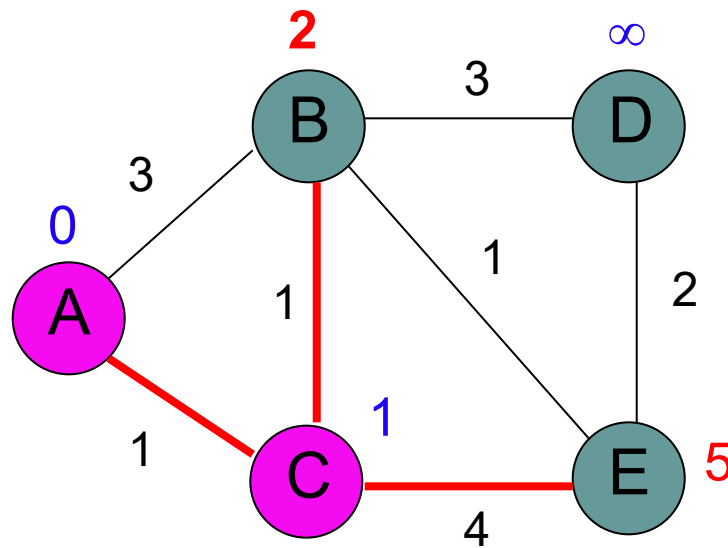
N	SD	PN
A	0	
B	$\infty$	
C	$\infty$	
D	$\infty$	
E	$\infty$	

# Visit Node A



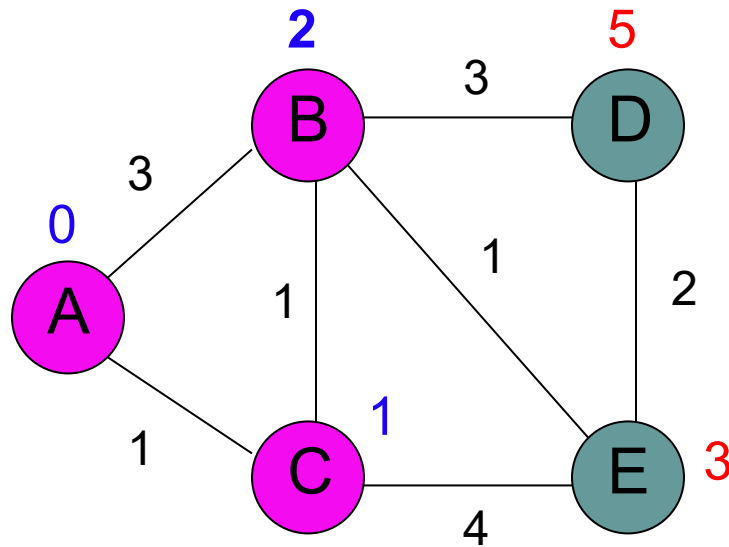
N	SD	PN
A	0	
B	3	A
C	1	A
D	$\infty$	
E	$\infty$	

# Visit Node C



N	SD	PN
A	0	
B	2	C
C	1	A
D	$\infty$	
E	5	C

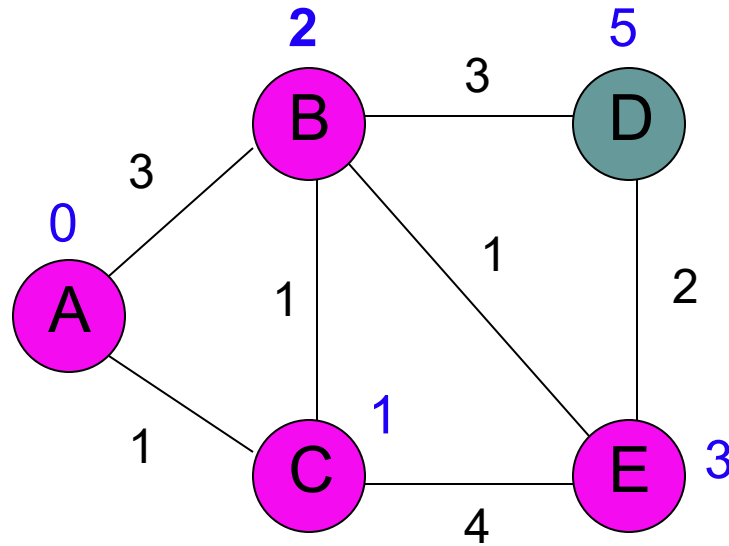
# Visit Node B



N	SD	PN
A	0	
B	2	C
C	1	A
D	5	B
E	3	B



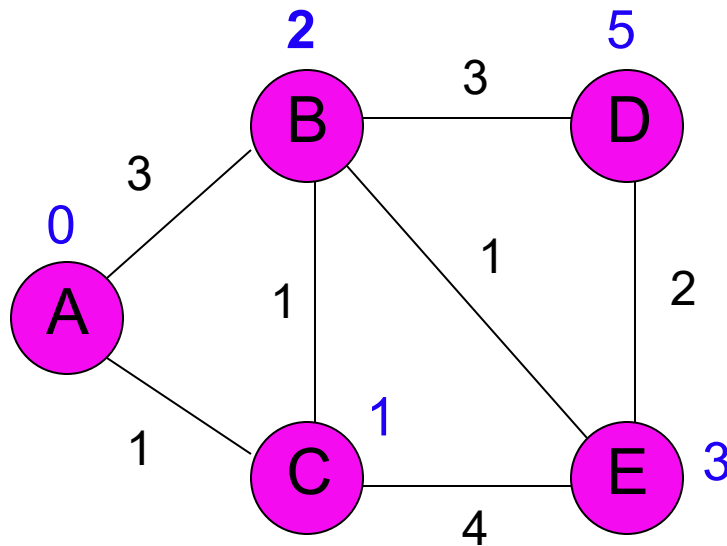
# Visit Node E



N	SD	PN
A	0	
B	2	C
C	1	A
D	5	B
E	3	B

Nothing changes

# Visit Node D

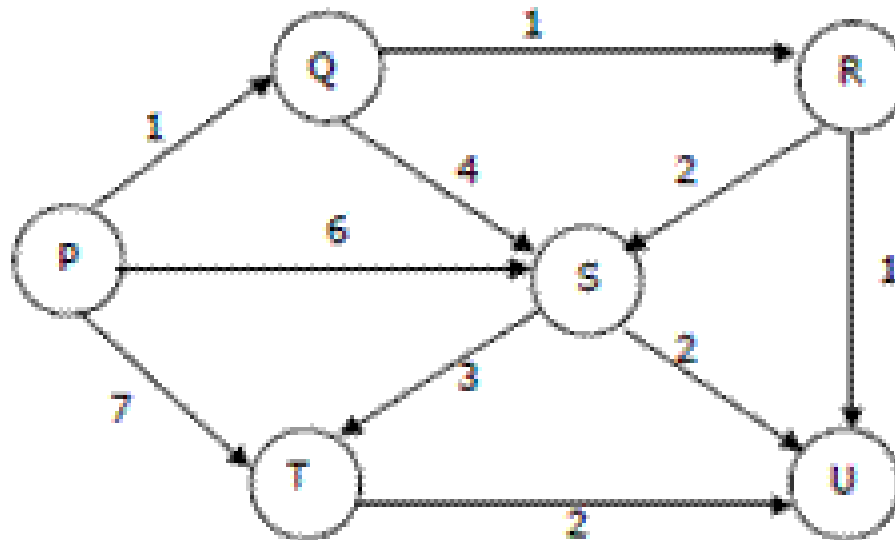


N	SD	PN
A	0	
B	2	C
C	1	A
D	5	B
E	3	B

Nothing changes

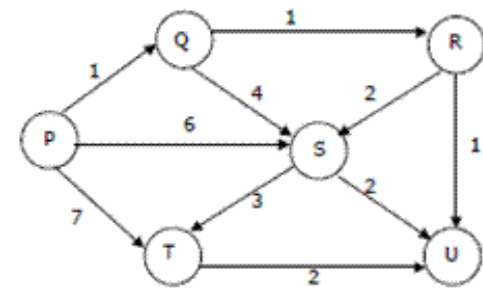
## Quiz: Dijkstra's Algorithm

- Suppose we run Dijkstra's single source shortest-path algorithm on the following edge weighted directed graph with vertex P as the source. In what order do the nodes get included into the set of vertices for which the shortest path distances are finalized?
- ANS: P, Q, R, U, S, T



# SD: Shortest Distance

## PN: Previous Node



N	SD	PN
P	0	
Q	$\infty$	
R	$\infty$	
S	$\infty$	
T	$\infty$	
U	$\infty$	

Visit P  
→

N	SD	PN
P	0	
Q	1	P
R	$\infty$	
S	6	P
T	7	P
U	$\infty$	

Visit Q  
→

N	SD	PN
P	0	
Q	1	P
R	2	Q
S	5	Q
T	7	P
U	$\infty$	

Visit R  
→

N	SD	PN
P	0	
Q	1	P
R	2	Q
S	4	Q
T	7	P
U	3	R

← Visit U (nothing changes)

N	SD	PN
P	0	
Q	1	P
R	2	Q
S	4	Q
T	7	P
U	3	R

Visit S  
(nothing changes)  
→

N	SD	PN
P	0	
Q	1	P
R	2	Q
S	4	Q
T	7	P
U	3	R

Visit T  
(nothing changes)  
→

N	SD	PN
P	0	
Q	1	P
R	2	Q
S	4	Q
T	7	P
U	3	R

Finished  
→

N	SD	PN
P	0	
Q	1	P
R	2	Q
S	4	Q
T	7	P
U	3	R

# Bellman-Ford Algorithm

- Initialize distance array `distTo[]` for each vertex `v` as **`distTo[v] = ∞`**, and **`distTo[s] = 0`** to source vertex `s`.
- Relax all **edges**  $|V|-1$  times.

```
private void relax(DirectedEdge e)
{
    int v = e.from(), w = e.to();
    if (distTo[w] > distTo[v] + e.weight())
    {
        distTo[w] = distTo[v] +
            e.weight();
        edgeTo[w] = e;
    }
}
```

## Recall:

### Generic algorithm (to compute SPT from `s`)

For each vertex `v`: `distTo[v] = ∞`.

For each vertex `v`: `edgeTo[v] = null`.

`distTo[s] = 0`.

Repeat until done:

- Relax any edge.

### Bellman-Ford algorithm

For each vertex `v`: `distTo[v] = ∞`.

For each vertex `v`: `edgeTo[v] = null`.

`distTo[s] = 0`.

Repeat  $|V| - 1$  times:

- Relax each edge.

# Bellman-Ford Algorithm Proof of Correctness

- Relaxing edges  $|V|-1$  times in the Bellman-Ford Algorithm guarantees that the algorithm has explored all possible paths of length up to  $|V|-1$ , which is the maximum possible length of a shortest path in a graph with  $|V|$  vertices. This allows the algorithm to correctly calculate the shortest paths from the source vertex to all other vertices, **given that there are no negative-weight cycles.**

# Bellman-Ford Algorithm with Negative Cycle Detection

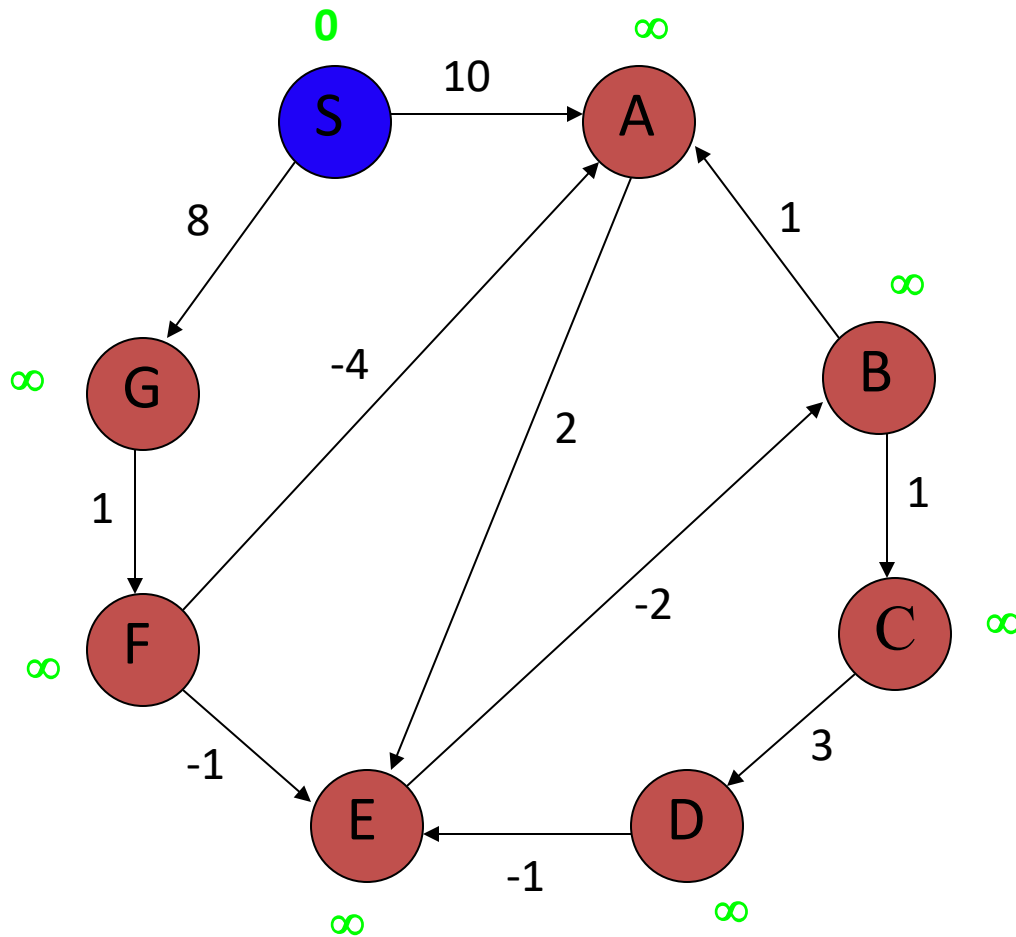
- Initialize distance array `distTo[]` for each vertex `v` as **`distTo[v] = ∞`**, and **`distTo[s] = 0`** to source vertex `s`.
- Relax all **edges**  $|V|-1$  times.
- Relax all the edges one more time i.e. the **N-th** time:
  - Case 1 (Negative cycle exists): if any edge can be further relaxed, i.e., for any **edge e**, if **`distTo[w] > distTo[v] + e.weight()`**
  - Case 2 (No Negative cycle) : case 1 fails for all the edges.

# Time Complexity of Bellman-Ford Algorithm

- Time complexity for connected graph:
- Best Case:  $O(|E|)$ , when distance array after 1st and 2nd relaxation are same, we can simply stop further processing after one iteration
- Average Case:  $O(|V|*|E|)$
- Worst Case:  $O(|V|*|E|)$ 
  - If the graph is complete, the value of  $E$  becomes  $O(|V|^2)$ . So overall time complexity becomes  $O(|V|^3)$

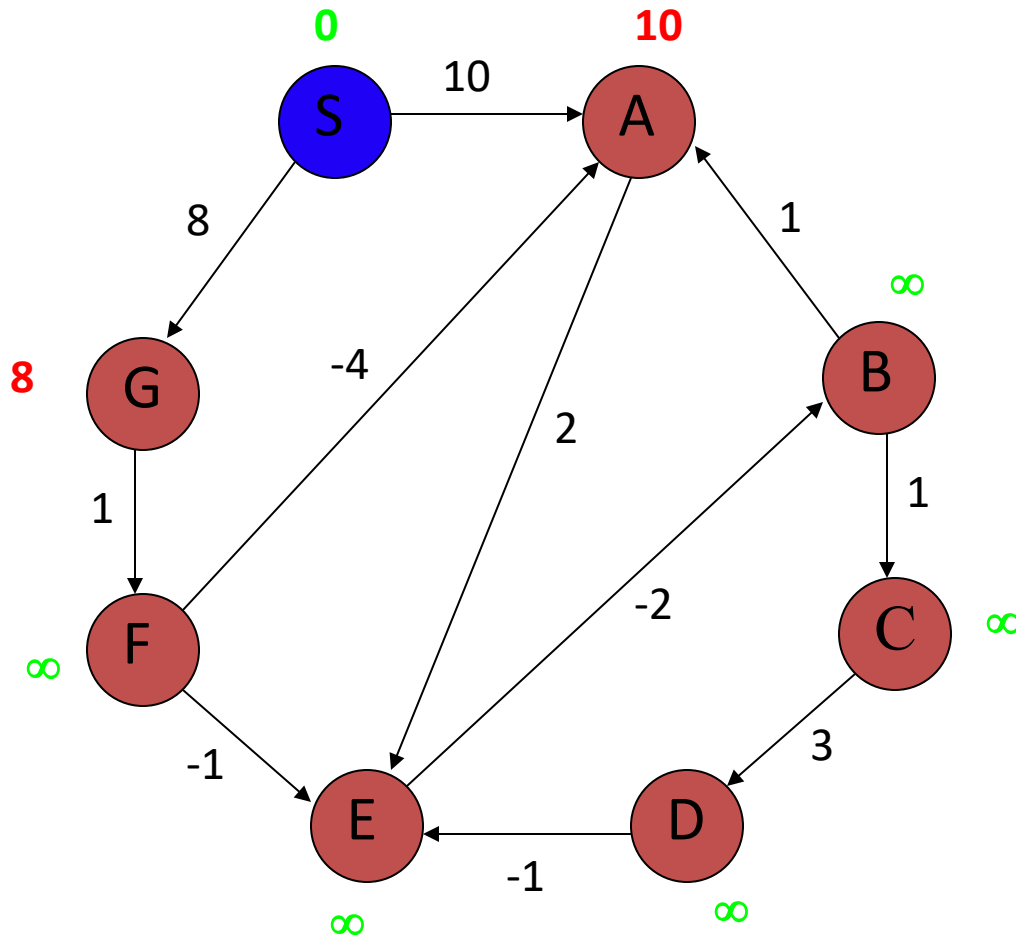


# Bellman-Ford Algorithm Example 1



Iteration: 0

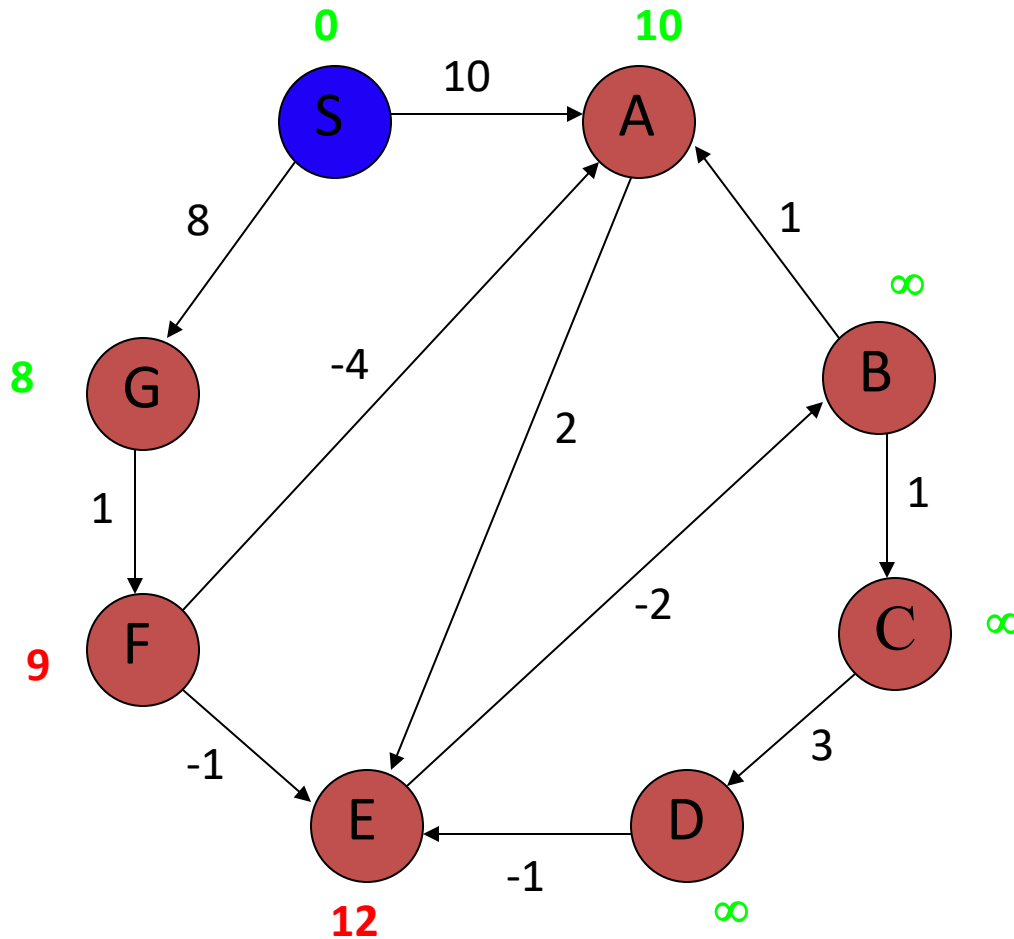
# Bellman-Ford Algorithm Example 1



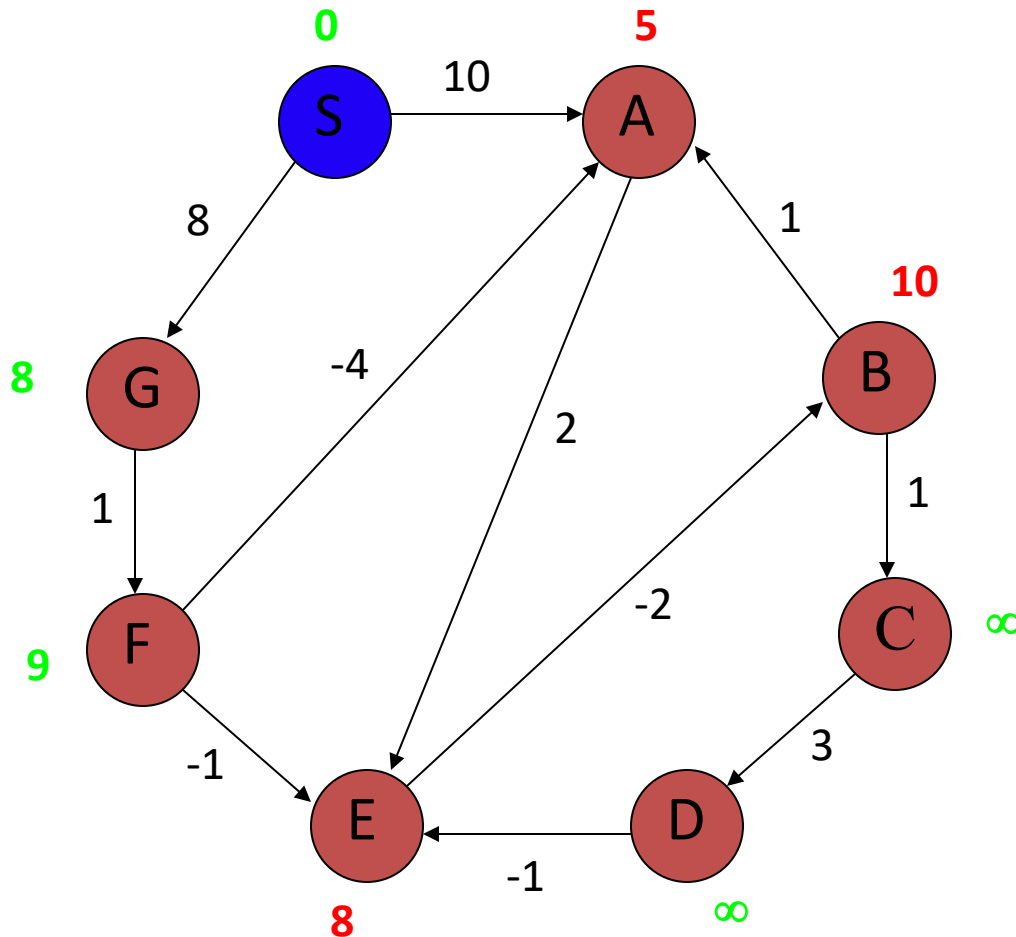
Iteration: 1

# Bellman-Ford Algorithm Example 1

Iteration: 2



# Bellman-Ford Algorithm Example 1

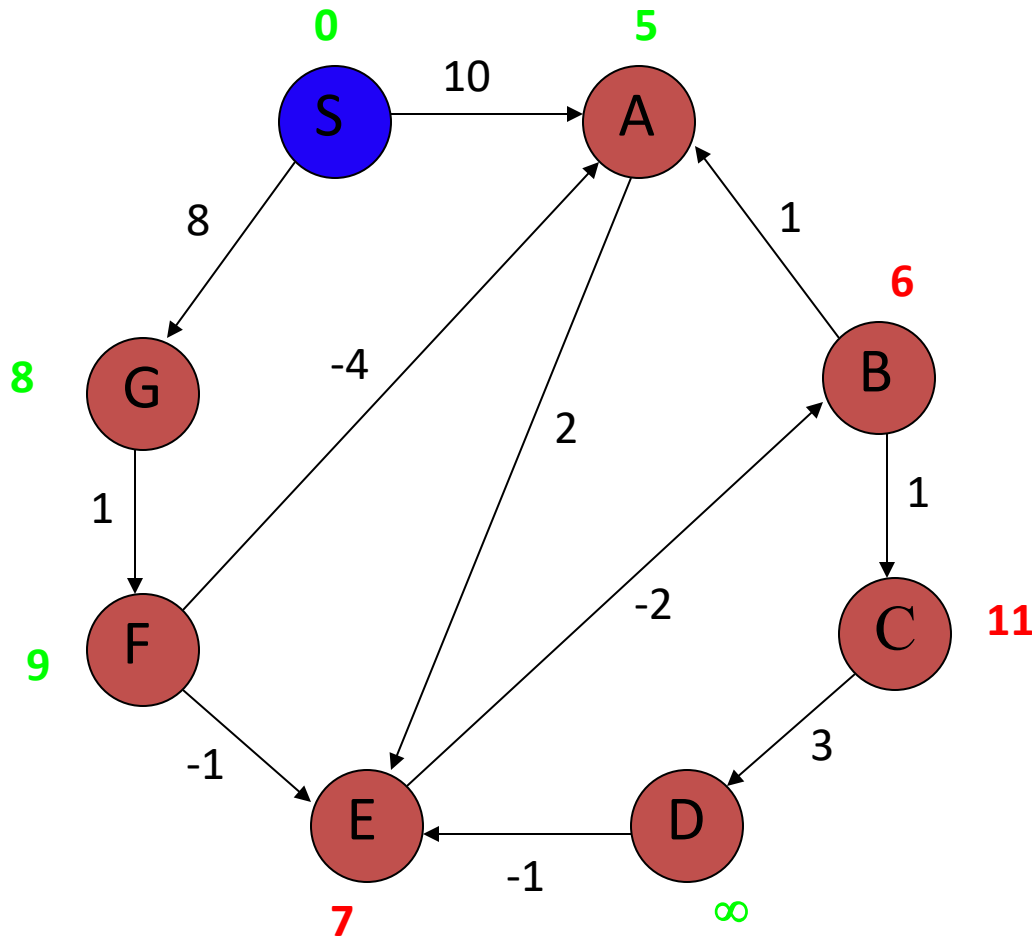


Iteration: 3

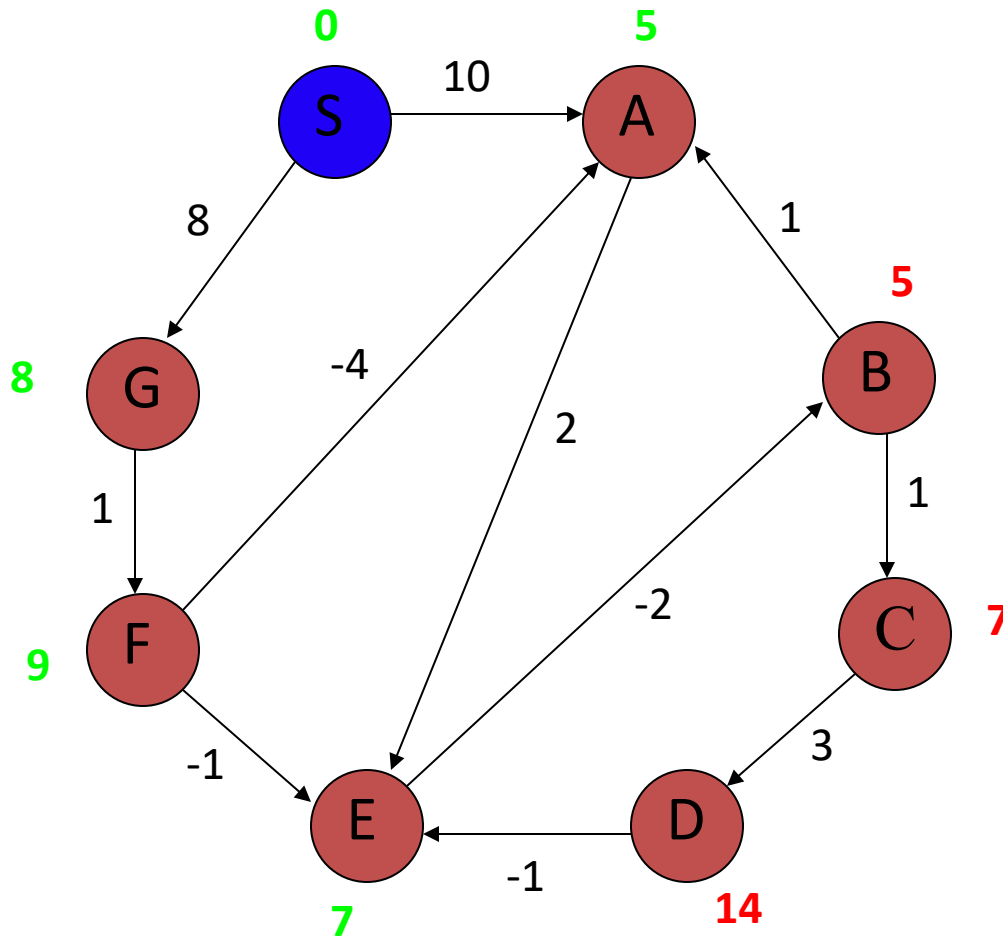
A has the correct  
distance and path

# Bellman-Ford Algorithm Example 1

Iteration: 4



# Bellman-Ford Algorithm Example 1

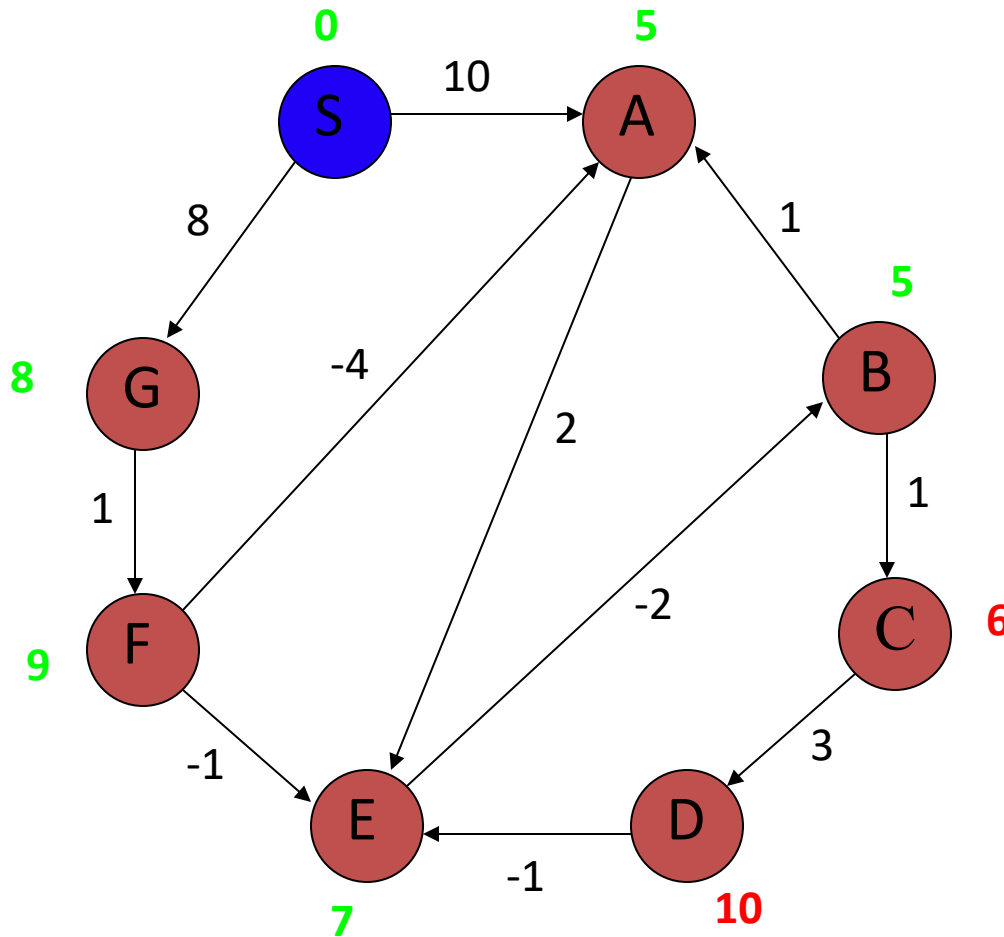


Iteration: 5

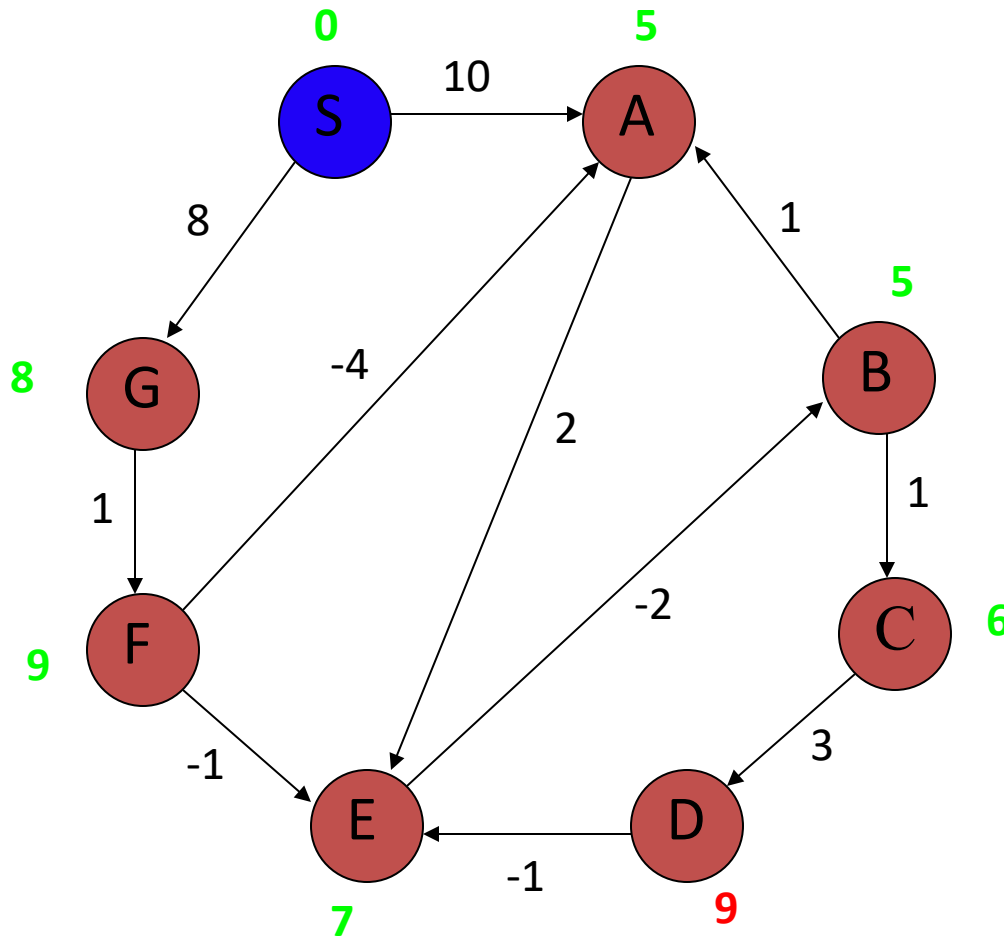
B has the correct  
distance and path

# Bellman-Ford Algorithm Example 1

Iteration: 6



# Bellman-Ford Algorithm Example 1



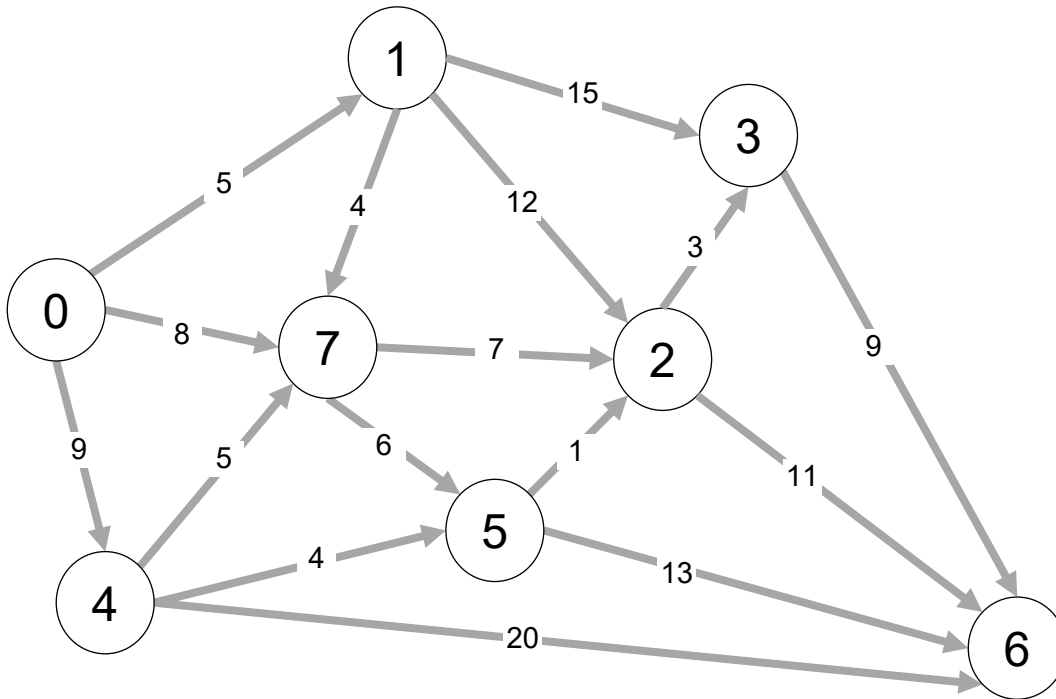
Iteration: 7

D (and all other nodes) have the correct distance and path



# Bellman-Ford Algorithm Example 2

Repeat  $V - 1$  times: relax all  $E$  edges.



v	distTo[]		
0	<del>∞</del>	0	
1	<del>∞</del>	5	
2	<del>∞</del>	<del>17</del>	14
3	<del>∞</del>	<del>20</del>	17
4	<del>∞</del>	9	
5	<del>∞</del>	13	
6	<del>∞</del>	<del>28</del>	<del>26</del> 25
7	<del>∞</del>	8	

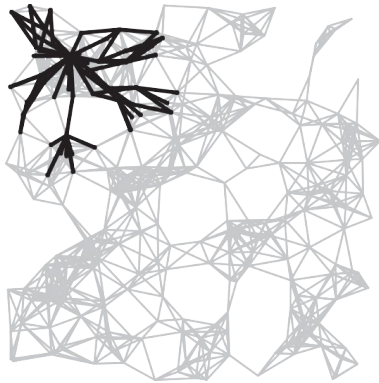
v	edgeTo[]		
0	-		
1	<del>-</del>	0	
2	<del>-</del>	<del>1</del>	5
3	<del>-</del>	<del>1</del>	2
4	<del>-</del>	0	
5	<del>-</del>	4	
6	<del>-</del>	<del>2</del>	<del>5</del> 2
7	<del>-</del>	0	

pass 1 pass 2 pass 3 (no further changes) pass 4-7 (no further changes)

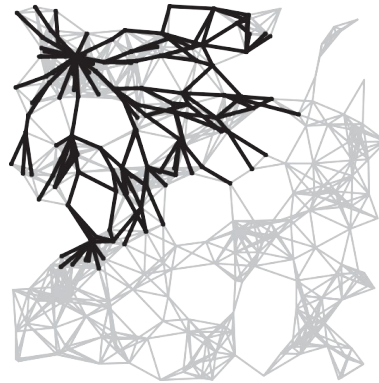
0→1 0→4 0→7 1→2 1→3 1→7 2→3 2→6 3→6 4→5 4→6 4→7 5→2 5→6 7→2 7→5

# Bellman-Ford Algorithm Visualization

passes 4



7



10



13



SPT



# Quiz

- Given a directed graph where weight of every edge is same, we can most efficiently find shortest path from a given source to destination using?
  - A. Breadth First Traversal
  - B. Dijkstra's Shortest Path Algorithm
  - C. Neither Breadth First Traversal nor Dijkstra's algorithm can be used
  - D. Depth First Search
- ANS: A
- With BFS, we first explore vertices at one edge distance, then all vertices at 2 edge distance, and so on.

# Quiz

- Let  $G = (V, E)$  be an undirected graph with a subgraph  $G_1 = (V_1, E_1)$ . Weights are assigned to edges of  $G$  as follows :

$$w(e) = \begin{cases} 0 & \text{if } e \in E_1 \\ 1 & \text{otherwise} \end{cases}$$

- A single-source shortest path algorithm is executed on the weighted graph  $(V, E, w)$  with an arbitrary vertex  $v_1$  of  $V_1$  as the source. Which of the following can always be inferred from the path costs computed?
- A. The number of edges in the shortest paths from  $v_1$  to all vertices of  $G$
- B.  $G_1$  is connected or not
- C.  $V_1$  forms a clique in  $G$
- D.  $G_1$  is a tree
- ANS: B
- When shortest path from  $v_1$  (one of the vertices in  $V_1$ ) is computed.  $G_1$  is connected if the distance from  $v_1$  to any other vertex in  $V_1$  is greater than 0, otherwise  $G_1$  is disconnected.

# Quiz

- Let  $G = (V, E)$  be a simple undirected graph, and  $s$  be a particular vertex in it called the source. For  $x \in V$ , let  $d(x)$  denote the shortest distance in  $G$  from  $s$  to  $x$ . A breadth first search (BFS) is performed starting at  $s$ . Let  $T$  be the resultant BFS tree. If  $(u, v)$  is an edge of  $G$  that is not in  $T$ , then which one of the following CANNOT be the value of  $d(u) - d(v)$ ?
- A. -1 B. 0 C. 1 D. 2
- ANS: D
- Note that the given graph is undirected, so an edge  $(u, v)$  also means  $(v, u)$  is also an edge. Since a shorter path can always be obtained by using edge  $(u, v)$  or  $(v, u)$ , the difference between  $d(u)$  and  $d(v)$  can not be more than 1.

# Quiz

- Which of the following algorithm can be used to efficiently calculate single source shortest paths in a Directed Acyclic Graph?
  - Dijkstra
  - Bellman-Ford
  - Topological Sort
  - Strongly Connected Component
- ANS: Topological Sort
- Using Topological Sort, we can find single source shortest paths in  $O(V+E)$  time which is the most efficient algorithm

# Quiz

- Given a graph where all edges have positive weights, the shortest paths produced by Dijkstra and Bellman Ford algorithm may be different but path weight would always be same.
- ANS: True
- Dijkstra and Bellman-Ford both work fine for a graph with all positive weights, but they are different algorithms and may pick different edges for shortest paths.

# Quiz

- Match the following
  - Group A
    - a) Dijkstra's single shortest path algo
    - b) Bellman Ford's single shortest path algo
    - c) Floyd Warshall's all pair shortest path algo
  - Group B
    - p) Dynamic Programming
    - q) Backtracking
    - r) Greedy Algorithm
- Dijkstra is a greedy algorithm where we pick the minimum distant vertex from not yet finalized vertices. Bellman Ford and Floyd Warshall both are Dynamic Programming algorithms where we build the shortest paths in bottom up manner.



# Quiz

- Let  $G$  be a directed graph whose vertex set is the set of numbers from 1 to 100. There is an edge from a vertex  $i$  to a vertex  $j$  if either  $j = i + 1$  or  $j = 3i$ . The minimum number of edges in a path in  $G$  from vertex 1 to vertex 100 is
- A. 4 B. 7 C. 23 D. 99
- ANS: 7
- The task is to find minimum number of edges in a path in  $G$  from vertex 1 to vertex 100 such that we can move to either  $i+1$  or  $3i$  from a vertex  $i$ .
- Since the task is to minimize number of edges, we would prefer to follow  $3*i$ . Let us follow multiple of 3.  $1 \Rightarrow 3 \Rightarrow 9 \Rightarrow 27 \Rightarrow 81$ , now we can't follow multiple of 3 anymore. So we will have to follow  $i+1$ . This solution gives a long path.
- What if we begin from end, and we reduce by 1 if the value is not multiple of 3, else we divide by 3.  $100 \Rightarrow 99 \Rightarrow 33 \Rightarrow 11 \Rightarrow 10 \Rightarrow 9 \Rightarrow 3 \Rightarrow 1$
- So we need total 7 edges.