Production Theory

Shan Gui

Shanghai University of Finance and Economics

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Production Theory

Recall: Consumer Theory

- Assumes the consumer chooses the best bundle (x_1, x_2) they can afford.
  $\Rightarrow$ Individual demand functions: $x_1^*(p_1, p_2, m)$, $x_2^*(p_1, p_2, m)$
- Aggregate individual demands to obtain the market demand for each product.

Key Questions in Production Theory

- Assume a firm specializes in producing a single product.
- What production plans are feasible? (Technology constraints, Ch19)
- What input combinations minimize cost for a given output? (Cost minimization, Ch21)
- What is the optimal output level? (Profit maximization, Ch20)

The Profit Maximization Problem

Basic Setting

- A firm specializes in producing a single product, denoted by y.
- It purchases inputs x_1 and x_2 to produce output y, using a production technology given by $y = f(x_1, x_2)$.
- Factors of Production include general inputs such as labor (L), capital (K), land, and raw materials.
- p is the price of the product, w_1 and w_2 are the prices of inputs.
- The firm's objective is to maximize profit

Sellers Under Different Market Institutions

- In a competitive market, sellers take w_1, and w_2, and p as given (price takers).
- A monopolist takes w_1 and w_2 as given but can set p (price maker).

Outline

Part I: Technology Constraint ($y = f(x_1, x_2)$)

- Relate the firm's production process to a consumer's utility production.
- Illustrate the technology constraint.

Part II: Cost Minimization

- Given input prices w_1, w_2 and a target output level y, determine the optimal input bundle (x_1^*, x_2^*) that minimizes total cost: $w_1 x_1 + w_2 x_2$.
- Derive the cost function: $c(y) = w_1 x_1^* + w_2 x_2^*$.

Part III. Profit Maximization

- Determine the output level y^* that maximizes profit.
- Profit-maximizing output may differ across market structures (e.g., competitive firms vs. monopolists).

Part I: Technology Constraint ($y = f(x_1, x_2)$, Ch19)

- Relate the firm's production process to a consumer's utility production.
- Illustrate the technology constraint that limits feasible combinations of inputs and outputs.

Technology: the Production Process

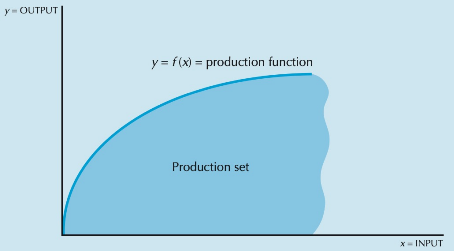
Firm(Consumer) produces Products (Utility)

	Consumer	**Firm**
(x_1, x_2)	Consumption bundle	Input bundle
$f(x_1, x_2)$	Utility function	Production function
y	Utility level	Output quantity
$l_i : \{(x_1, x_2)\} \Rightarrow y_i$	Indifference curve	Isoquant
$\frac{\Delta x_2}{\Delta x_1}$ along l_i	MRS	TRS (Technical Rate of Substitution)
$\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}$	Marginal utility	Marginal product
	Subjective Process	Observable, fixed $f(x_1, x_2)$

Comparison between Consumer and Firm Concepts

To Illustrate the Technological Constraints

One Input Case: $y = f(x)$



A production set. Here is a possible shape for a production set.

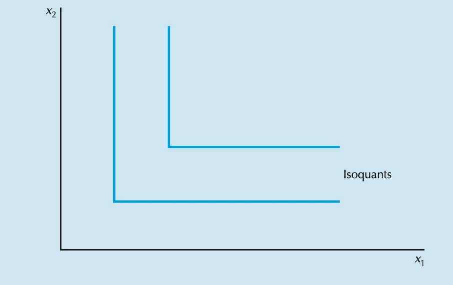
- The production set: all possible technological choices facing a firm.
- Production function: measures the maximum possible output that you can get from a given amount of input, and is the boundary of the production set.

- Marginal product of Input $MP := \frac{\Delta y}{\Delta x}$
- Law of diminishing marginal product.

To Illustrate the Technological Constraints

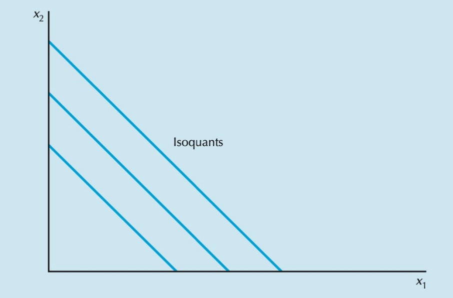
Two-input case: $y = f(x_1, x_2)$

- An isoquant: is the set of all possible combinations of inputs 1 and 2 that are just sufficient to produce a given amount of output



Fixed proportions. Isoquants for the case of fixed proportions.

(a) Fixed Proportions

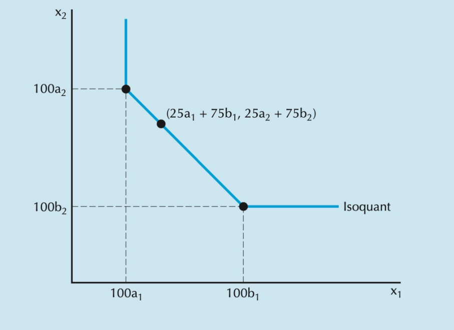


Perfect substitutes. Isoquants for the case of perfect substitutes.

(b) Perfect Substitutes

Well-behaved Technology

Monotonicity and Convexity

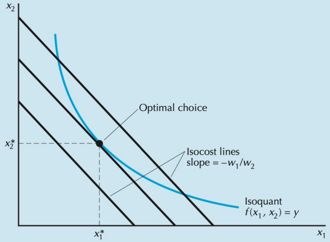


Convexity. If you can operate production activities independently, then weighted averages of production plans will also be feasible. Thus the isoquants will have a convex shape.

- **Monotonic tech:** More of both inputs increases output, $f(x_1 + \Delta_{x_1}, x_2 + \Delta_{x_2}) > f(x_1, x_2)$
- Strict Monotonic: More of any input increases output, $f(x_1 + \Delta_{x_1}, x_2) > f(x_1, x_2)$
- **Convex tech:** weighted average of two input bundles does not decrease output
- Bundle $A(x_1^A, x_2^A)$, Bundle $B(x_2^B, y_2^B)$
- Weighted averaged: $C(tx_1^A + (1-t)x_1^B, tx_2^A + (1-t)x_2^B), t \in (0,1)$
- **Convexity**: $f(C) \geq f(A), f(C) \geq f(B)$
- Strict Convex: $f(C) > f(A), f(C) > f(B)$

Slope of Isoquant

Technical Rate of Substitution (TRS)



Cost minimization. The choice of factors that minimize production costs can be determined by finding the point on the isoquant that has the lowest associated isocost curve.

- Slope:$= \frac{\Delta x_2}{\Delta x_1}$
- Technical Rate of Substitution (MRS) **of input 1** (for input 2):$= \frac{\Delta x_2}{\Delta x_1}$
- Marginal product of Input 1 $:= MP_1$
- $MP_1 := \frac{\partial f}{\partial x_1} = \frac{f(x_1+\Delta_{x_1}, x_2) - f(x_1, x_2)}{\Delta x_1}$
- Same output, $MP_1 \Delta x_1 + MP_2 \Delta x_1 = 0$
- TRS:$= \frac{\Delta x_2}{\Delta x_1} = -\frac{MP_1}{MP_2}$
- Law of diminishing technical rate of substitution.

Part II. Cost Minimization (Ch20-21)

- Given input prices p_1, p_2, a target output level y, and the technology $y = f(x_1, x_2)$.
- Determine the optimal input bundle (x_1^*, x_2^*) that minimizes the total cost (c).
  Conditional factor demand functions: $x_1^*(p_1, p_2, y)$, $x_2^*(p_1, p_2, y)$.
- Derive the **Cost function**: $c(w_1, w_2, y) = w_1 x_1^* + w_2 x_2^*$. mininum cost to produce y units of output given (w_1, w_2), simplifed as $c(y)$.

Cost Minimization in Long Run and in Short Run

Fixed and Variable Factors (Costs)

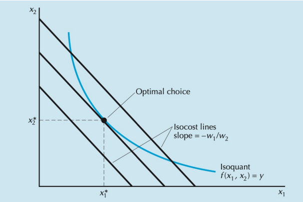
- **Fixed Factor/Costs**: a factor (costs) that is (are) in a fixed amount whatever the output level, $x_1 = a$ for all $y \geq 0$; $FC = b$ for all $y \geq 0$
  - **Sunk cost**: a type of fixed cost that cannot be recovered
- **Variable Factor/Costs**: a factor (costs) that can be in different amounts, $x_1 = x_1(y)$ and $x_1(0) = 0$; $VC = VC(y)$ and $VC(0) = 0$
  - Quasi-fixed Factor/Costs: a factor (costs) that is (are) in a fixed amount when producing a positive amount of output, $x_1 = a$ if $y > 0$; $QFC = b$ if $y > 0$

Long Run and Short Run

- In the **short run**: there are some fixed factors.
- In the **long run**: only variable factors. $c(0) = 0$.
- Fixed costs only exist in the short run. Quasi-fixed costs (which belong to variable costs) may exist in the long run.

Cost Minimizationn

For well-behaved technology: The Tangency Condition



Cost minimization. The choice of factors that minimize production costs can be determined by finding the point on the isoquant that has the lowest associated isocost curve.

- Isocost lines: $x_2 = \frac{C}{w_2} - \frac{w_1}{w_2} x_1$, given some level of total cost C
- **Tangency condition**: the slope of the isoquant must be equal to the slope of the isocost curve.
  $-TRS = \frac{MP_1(x_1^*, x_2^*)}{MP_1(x_1^*, x_2^*)} = \frac{w_1}{w_2}$
- Cobb-Douglas production function $y = x_1^a x_2^b$:
  $-TRS = \frac{a x_2}{b x_1} = \frac{w_1}{w_2}$
- if $w_1 = w_2 = 1$, $x_1^* = \left(\frac{a}{b}\right)^{\frac{b}{a+b}} y^{\frac{1}{a+b}}$,
  $c(y) = \left(\frac{a}{b}\right)^{\frac{b}{a+b}} y^{\frac{1}{a+b}} + \left(\frac{b}{a}\right)^{\frac{a}{a+b}} y^{\frac{1}{a+b}}$

General Rules

1. Fixed costs will not influence the best input bundle.
   - Total costs will be larger.
2. Find the lowest isocost line that reaches the isoquant.
   - *e.g.* Fixed Proportions: $y = \min\{ax_1, bx_2\}$

     $ax_1^* = y = bx_2^*$

     $c(y) = w_1x_1^* + w_2x_2^* = (\frac{w_1}{a} + \frac{w_2}{b})y$
   - *e.g.* Perfect Substitutes: $y = ax_1 + bx_2$

     $y = ax_1^*$ if $\frac{w_1}{w_2} < \frac{a}{b}$; $y = ax_2^*$ if $\frac{w_1}{w_2} > \frac{a}{b}$

     $c(y) = \min\{\frac{w_1}{a}y, \frac{w_2}{b}y\}$

General Rules

3. Don't forget the short-run input constraints.
   - *e.g.* Perfect Substitutes: $y = ax_1 + bx_2$, $\frac{w_1}{w_2} < \frac{a}{b}$, $x_1 = \bar{x}$
     Input 1 is a fixed factor, so we only need to solve x_2^*
   - if $y \leq a\bar{x}$, $x_2^* = 0$, $c(y) = w_1\bar{x}$;
   - if $y > a\bar{x}$, $x_2^* = \frac{y - a\bar{x}}{b}$, $c(y) = w_1\bar{x} + \frac{w_2}{b}y$

4. Other constraints.
   - *e.g.* Perfect Substitutes: $y = ax_1 + bx_2$, $\frac{w_1}{w_2} < \frac{a}{b}$, $x_1 < \bar{x}$
   - if $y < a\bar{x}$, $x_1^* = \frac{y}{a}$, $x_2^* = 0$, $c(y) = \frac{w_1}{a}$;
   - if $y \geq a\bar{x}$, $x_1^* = \bar{x}$, $x_2^* = \frac{y - a\bar{x}}{b}$, $c(y) = w_1 x_1^* + w_2 x_2^*$

Returns to Scales

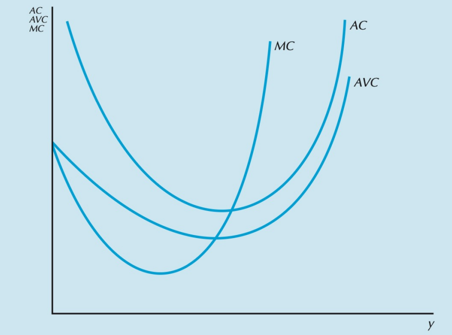
- Total costs ($c(y)$) = Variable costs ($VC(y)$) + Fixed Costs (F)
- Average cost: $AC(y) = \frac{c(y)}{y}$; Average variable cost: $AVC(y) = \frac{VC(y)}{y}$
- Marginal cost: $MC(y) = \frac{\Delta c(y)}{\Delta y}$; Marginal variable cost: $MVC(y) = \frac{\Delta VC(y)}{\Delta y}$.

Returns to Scale

- Increasing/Constant/Decreasing return to scale (IRS/CRS/DRS): when inputs increase, output increases more rapidly/same/less rapidly than inputs.
- (Common) CRS: $f(tx_1, tx_2) = tf(x_1, x_2)$, for all $t > 1$
  $\Rightarrow VC(y) = y \cdot VC(1) = y \cdot AVC(y)$, MVC is independent of y
- (Usually when $y < \bar{y}$) IRS: $f(tx_1, tx_2) > tf(x_1, x_2)$, for all $t > 1$
  $\Rightarrow$ when y increases, MVC decreases, AVC decreases
- (Usually when $x^{effect} \leq \overline{x}$) DRS: $f(tx_1, tx_2) < tf(x_1, x_2)$, for all $t > 1$,
  $\Rightarrow$ if inputs increase, MVC increases, AVC increases

Cost Curves

Typical Cost Curves

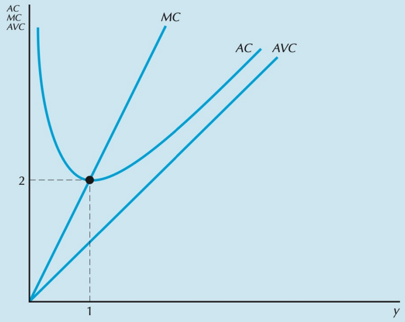


Cost curves. The average cost curve (AC), the average variable cost curve (AVC), and the marginal cost curve (MC).

- MC may initially slope down (IRS), then go up (DRS)
- True: MC=AVC at first unit of output (Δy)
- True: MC passes through the minimum point of both AVC and AC
- *e.g.*, short run: $c(y) = y^2 - 4y + 10$
- *e.g.*, long run: $c(y) = y^2 - 4y + 10$ if $y > 0$ and $c(0) = 0$ (Quasi-fixed Costs)

Cost Curves

Another Example of Cost Curves

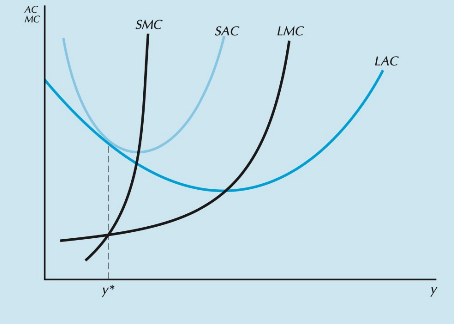


Cost curves. The cost curves for $c(y) = y^2 + 1$.

- MC may go up at a constant rate (DRS)
- True: MC=AVC at first unit of output (Δy)
- True: MC passes through the min point of both AVC and AC
- *e.g.*, short run: $c(y) = y^2 + 1$
- *e.g.*, long run: $c(y) = y^2 + 1$ if $y > 0$ and $c(0) = 0$ (Quasi-fixed Costs)

Cost Curves

Short Run Costs $\leq$ Long Run Costs



- Technology: $f(x_1, x_2)$
- Short run cost function: $c_s(y, x_2)$.
- Long run, cost function: $c(y)$
- The optimal input given y: $x_2^*(y)$
- $\Rightarrow c(y) \leq c_s(y, x_2)$ (Long-run costs no more)
- $\Rightarrow c(y) = c_s[y, x_2^*(y)]$ (Same cost if $x_2 = x_2^*$)
- True: Long-run average cost curve (LAC) is the lower envelope of the short-run average cost curves (SAC).
- True: At y^*, LMC = SMC, LAV=SAC.

Part III. Profit Maximization

- Determine the output level y^* that maximizes profit (Ch19).
- Marginal Revenue = Marginal Cost; Revenue $\geq$ Variable Cost
  - Competitive market price takers: $p = MC(y^*)$
    Individual firm supply (Ch23) and industry firm supply (Ch24)
  - Monopolists: $p(y^*) = MC(y^*)$

Purely Competitive Markets

Definition: Every Firm is a Price Taker

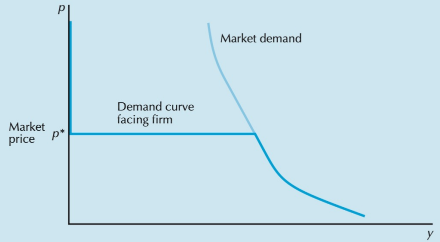
- Market price is independent of the firm's level of output.
- *e.g.*, Food delivery restaurants (not platform)

Characteristics (Not Requirements)

- Many buyers and sellers
- Identical products
- Free entry and exit of a firm
- Full information, no transaction costs

Profit Maximization for a Competitive Firm

Demand Curve Facing a Competitive Firm



The demand curve facing a competitive firm. The firm's demand is horizontal at the market price. At higher prices, the firm sells nothing, and below the market price it faces the entire market demand curve.

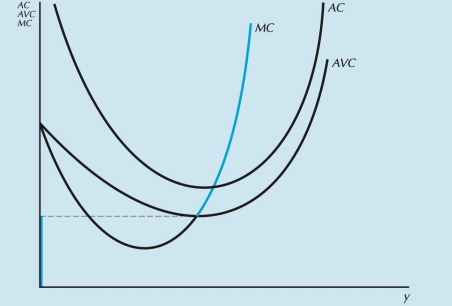
- A competitive firm is a price taker:

$$\max_y py - c(y)$$

- where $c(y) = VC(y) + F$, $VC(0) = 0$.
- F does not affect the optimal production;
- When to produce (**shutdown condition**): $p \geq AVC(y)$
- **Optimal production** (necessary condition): $MC(y^*) = p$ and $MC(y > y^*) > p$

Competitive Firm Supply

Part of the Marginal Cost Curve is its Supply Curve

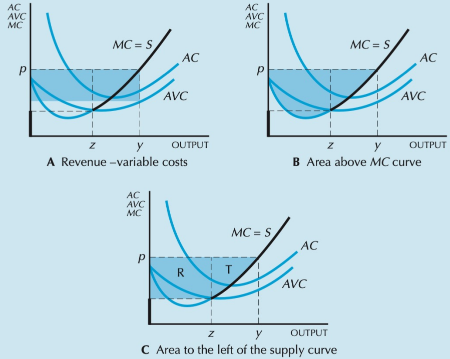


Average variable cost and supply. The supply curve is the upward-sloping part of the marginal cost curve that lies above the average variable cost curve. The firm will not operate on those points on the marginal cost curve below the average cost curve since it could have greater profits (less losses) by shutting down.

- The supply curve is the upward-sloping part of the marginal cost curve that lies above the average variable cost curve.
- Optimal production (necessary): $MC(y^*) = p$ and $MC(y > y^*) > p$
- **Inverse supply function**: $p = MC(y)$

Profit and Producer's Surplus for a Competitive Firm

Non-negative Producer's Surplus



Producer's surplus. Three equivalent ways to measure producer's surplus. Panel A depicts a box measuring revenue minus variable cost. Panel B depicts the area above the marginal cost curve. Panel C uses the box up until output z (area R) and then uses the area above the marginal cost curve (area T).

- Profit $= py - c(y) = py - VC(y) - F$
- Producer's surplus $= py - VC(y)$

- When to produce (**shutdown condition**): $p \geq AVC(y)$

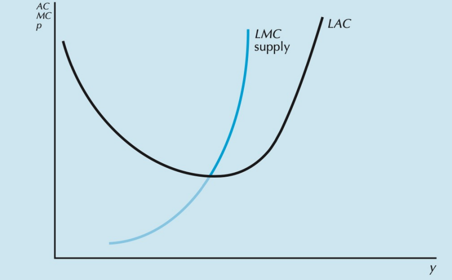
  $\Rightarrow$ Producer's surplus is non-negative

  $\Rightarrow$ Profit may be negative in the short run if $F > (p - AVC)y$

Long-run Supply Curve for a Competitive Firm

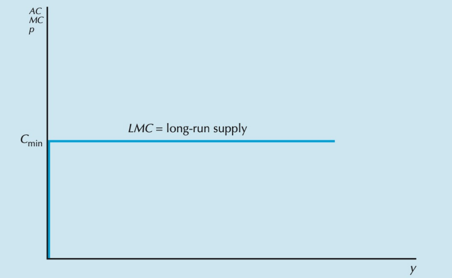
Non-negative Profit in the Long Run: $p \geq \frac{c(y)}{y}$

- In the long run, a firm can choose to remain in the business or not.
- No fixed costs: no need to pay anything if a firm chooses to leave.



The long-run supply curve. The long-run supply curve will be the upward-sloping part of the long-run marginal cost curve that lies above the average cost curve.

(a) DRS if only consider Variable Factors



Constant average costs. In the case of constant average costs, the long-run supply curve will be a horizontal line.

(b) Example of CRS

Thank You!