Stats exam 2014

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Please submit your answers by email to `guillaume.filion@gmail.com` before the deadline. Each question is worth 1 point. Every fraction of 24 hours passed the deadline will be penalized by 2 points. You are encouraged to work in group on these exercises. You can submit a joint answer sheet and will receive the same grade as your team members. In this case, indicate the name of all the students participating to the work.

The answers will be posted online at the address below.
`www.genomearchitecture.com/static/misc/statsexam_answers_2014.pdf`

1 Course questions

Exercise 1

What are the main steps of a test?

Exercise 2

What is the p-value of a test?

Exercise 3

What is the power of a test?

Exercise 4

How can you increase the power of a test?

Exercise 5

What is the statistic of the t-test?

2 Problem

An insecure implementation of a password-protected acess is to compare letter by letter the text enterred by the user to the real password, and to deny access as soon as a different letter is found. This is insecure because it opens a possibility for *a timing attack*, which is an attempt to hack a password by measuring the time it takes for the computer to respond.

`http://en.wikipedia.org/wiki/Timing_attack`

Suppose that my real password is `kotiki125`. If the hacker enters `abcdefg`, the insecure method will deny access immediately after comparing the first letter because `a` is different from `k`. But if the hacker enters `kotiki123`, this method will deny access upon comparing the 9-th letter, which will take roughly 9 times as long. With this information, the hacker could deduce that the first 8 letters of `kotiki123` are correct.

The idea of a timing attack is to try all the letters one by one and measure the time it takes for the computer to respond. Every time a letter matches the password, the response time will be slightly slower. By repeating the process, the password can be decrypted completely.

Exercise 6

Assume that there are 64 valid password letters (small letters, capital letters, numbers, _ and ~, and that you want to decrypt a password generated at random. You enter a text. What is the probability that the first letter matches the password? What is the probability that the first 3 letters match the password?

Based on previous measurements, you know that the time to compare two letters is 10 ns. In addition, for every letter comparison, there is a random overhead due to other processes running on the server. This overhead has an exponential distribution with rate 0.125 ns^{-1}.

Exercise 7

The function `rexp(8, rate=.125)` in `R` generates a random sample of size 8 drawn from an exponential distribution with rate 0.125 ns^{-1}. How to generate a random sample of size 8 representing the response time of the server when the first letter of the text does not match the password?

You have entered the text `aaaaaaaa` 8 times and have observed the following response times (in ns) 23.0, 15.7, 21.4, 12.8, 13.3, 21.3, 38.2, 15.4.

Exercise 8

Do you think that the first letter of the password is `a`? Quantify your certainty. Before you set up a complicated statistical test, look at the numbers carefully.

You know that the password starts with `wy6qU`. You now enter `wy6qUaaaaa` 8 times and obtain the following response times (in ns) 122.0, 83.8, 71.4, 136.6, 93.6, 88.1, 84.1, 123.0

In `R` you can compute the sum of 6 exponential random variables with rate 0.125 ns^{-1} by using `sum(rexp(6, rate=.125))`. Similarly, you can generate a sample of size 8 representing the response time of the server if the 6-th letter does not match by using `replicate(8,60+sum(rexp(6,rate=.125)))`.

Exercise 9

Using this information, design a statistical test to know whether the 6-th letter of the password is `a`. Quantify your certainty.

Exercise 10

What is the power of your test?

Answer to Exercise 1

1. State the null hypothesis.
2. Choose a statistic.
3. Compute the null distribution.
4. Compare the observed value of the stastic to the null.

Answer to Exercise 2

The probability that the statistic is more extreme, given that the null hypothesis is true.

Answer to Exercise 3

The probability of rejecting the null hypothesis.

Answer to Exercise 4

By increasing the level or by increasing the sample size.

Answer to Exercise 5

The effect size.

Answer to Exercise 6

$\left(\frac{1}{64}\right)^3 \approx 3.8 \cdot 10^{-6}$

Answer to Exercise 7

```
10+rexp(8, rate=.125)
```

Answer to Exercise 8

If the first letter is `a`, there will be at least 2 letter comparisons, which takes at least 20 ns. Since at least one of the numbers is lower than 20 ns, this cannot be the case.

Answer to Exercise 9

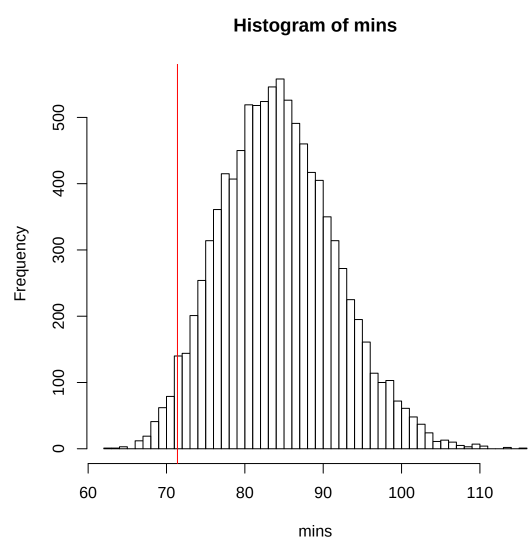
1. Ho: the 6-th letter is not `a`
2. The statistic is the minimum response time.
3. To identify the null distribution we do the following in `R`.

```
mins <- rep(NA, 10000)
for (i in 1:10000) {
  mins[i] <- min(replicate(8,60+sum(rexp(6,rate=.125))))
}
```

4. To compare the sample to the null distribution we do the following.

```
hist(mins, breaks=50)
abline(v=71.4, col=2)
mean(mins < 71.4) # 0.0264
```

We get the graphical results shown below. In only 2.6% of the cases we get a lower minimum value out of 8. This means that the minimum value for our sample is *low* and that we should *not* reject the null hypothesis, because if the null hypothesis is true, we expect the minimum value to be low. This is a special case where it makes sense to use a one-sided test and put all the rejection on the right side of the distribution, which corresponds to our intuition in the case that the null hypothesis is false.



This exercise is interesting because several choices are possible for the statistic. We could as well take the average instead of the minimum, but Exercise 8 suggests that taking the minimum contains more information. If we take the average, the null distribution is different, and also the power. Also, it is one of a few cases where the rejection *should* be one-sided because the alternative hypothesis is simple (*i.e.* not composite), if the null hypothesis is false, the response time will be longer (and not either longer or shorter). Also note that the null hypothesis cannot be that the 6-th letter is `a` because this does not allow to compute the null distribution (the distribution then depends on whether the 7-th and further letters are guessed correctly).

Answer to Exercise 10

In this case, it makes sense to compute the power when the 6-th letter is `a` (the null hypothesis is false) and the 7-th letter is wrong. In this case, the distribution of the sample will be `replicate(8,70+sum(rexp(7,rate=.125)))`.

```
mins2 <- rep(NA, 10000)
for (i in 1:10000) {
   mins2[i] <- min(replicate(8,70+sum(rexp(7,rate=.125))))
}
threshold <- quantile(mins, 0.95) # 97.00
mean(mins2 > threshold) # 0.63
```

So when the null hypothesis is false, we reject it 63% of the time, which is the power of the test. Note that if the 7-th letter is correct, the response time will be longer and the rejection will happen more than 63% of the time, so this number represents the minimum power.

If the queries are free (in terms of time or risk for the hacker), it makes sense to do more than 8 per letter in order to raise the power close to 1.0. Otherwise, it is a matter of computing the cost/benefit ratio for the hacker keeping in mind that the hacker would have only 63% chance of identifying the right letter with 8 queries.