Fisher's test

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1 The problem

In 1935, Muriel Bristol, a colleague of the legendary statistician R.A Fisher claimed to be able to distinguish by taste alone whether milk or tea was poured first in the cup. Fisher challenged her to test this claim in a way that she found agreeable. They agreed that Fisher would secretly pour tea first in 4 cups, milk first in 4 other cups and then present the cups to Muriel in a random order.

Here is the outcome of the test:

		She guesses	
		Milk first	Tea first
He pours	Milk first	3	1
	Tea first	1	3

This kind of data is called a (2×2) contingency table. This problem got Fisher so enthusiastic that he invented a test for that purpose.

Exercise 1

What is the null hypothesis of the test? What are the expected values of the table?

Exercise 2

How many cases are possible, how many different tables can be observed with this setup?

Exercise 3

Can you suggest a statistic and an alternative hypothesis for the test?

2 The distribution of the statistic

With the null hypothesis at hand, we need to study the distribution of the statistic. Not every possible table are equally likely, so we must be careful about that. Under the null hypothesis, Muriel puts labels at random, so she simply orders them at random. This is what will guide the resampling.

Exercise 4

Create at `factor` called `catego`, of length 8 consisting of 4 elements called `"Tea first"` and 4 elements called `"Milk first"`.
Hint: Use `factor`.

Exercise 5

Reorder `catego` at random.
Hint: Use `sample`.

Exercise 6

Write a function `S` of two factors of same length that returns the number of co-occurrences of their categories. Here, co-occurrence means that factors contain the same category at the same position.
Hint: Use `==`.

Exercise 7

Resample a vector of length 10,000 consisting of the S statistic under the null hypothesis. Plot the cumulative distribution with `plot(ecdf(...))` which will give more information than density or histogram in that case.
Hint: Use your imagination.

Exercise 8

Define the shape of the rejection region. Is it unilateral or bilateral? Estimate the p-value of the test. What do we conclude?
For the aces: If instead of the agreed protocol, Fisher had flipped a coin to know if he would pour tea or milk first, what would be the p-value of the test for the given data?

Exercise 9

Compare the results with the output of `fisher.test`. For this you need to create the 2×2 contingency table with the function `matrix`.

Answer of Exercise 1

In English terms, the null hypothesis is that Muriel Bristol guesses at random, as if she had 4 'Milk first' and 4 'Tea first' stickers that she distributed randomly among the cups.

In mathematical terms, the null hypothesis can be formulated as:

1. The data consists of 2 categories of 2 variables.
2. Occurrences of the categories of different variables are independent.
3. The numbers of each category are known and not random.
4. Sampling is exchangeable.

Note that sampling here is **not** independent because there are mutual influences among the observations. For example, if the first 4 guesses were 'Tea first', the next 4 have to be 'Milk first'. But sampling is assumed to be **exchangeable** which means that all orderings of the categories have the same probability, in other words, the order in which the data is collected is irrelevant.

The expected values are all equal to 2. For each guess, Muriel has a chance of 1/2 and there are 4 guesses per line.

Answer of Exercise 2

5 only because the margin have to sum up to 4. For example, if the top-left cell is 0, the bottom-right cell also has to be 0: if Muriel guessed wrong all the 'Milk first', it is because she gave them all the label 'Tea first', and therefore she *must* also have guessed wrong all the 'Tea first'.

This test is called Fisher's *exact* test because he could compute the exact distribution of the statistic, simply because the number of cases is so small that he could compute their probabilities exhaustively.

Answer of Exercise 3

Several options are possible. Because the numbers in opposite cells vary in the same fashion, the score has to be high when cells of the descending diagonal are high and low when the cells of the ascending diagonal are high. We can choose the statistic S to be the sum of the cells of the descending diagonal. In the data, $S = 3 + 3 = 6$. This is not the 'official' statistic of Fisher's test... so what? As long as we have the distribution of our statistic, we can obtain exactly the same result as `R` does.

In the alternative hypothesis, item 2 is replaced by its complementary: 'Occurrences of the categories are not independent'.

The hypotheses of the test may seem very general and cover all the cases, however, only few problems can be assimilated to a 2×2 contingency table.

Answer of Exercise 4

```
> catego <- factor(rep(c("Tea first", "Milk first"), each=4));
```

Answer of Exercise 5

```
> sample(catego);
```

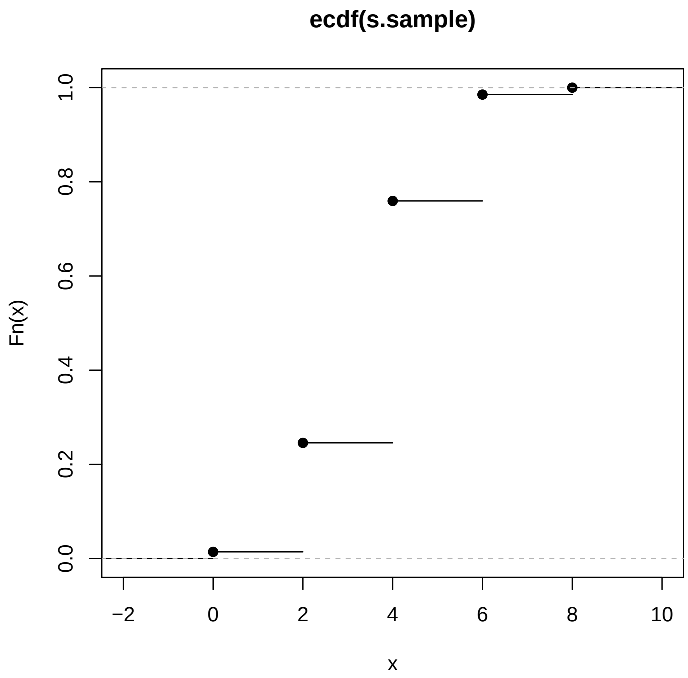
```
[1] Tea first  Milk first Milk first Tea first  Tea first  Milk first Milk first
[8] Tea first
Levels: Milk first Tea first
```

Answer of Exercise 6

```
> S <- function(x, y) { return (sum(x == y)); }
```

Answer of Exercise 7

```
> s.sample <- rep(NA, 10000);
> for (i in 1:10000) {
+    s.sample[i] <- S(sample(catego), sample(catego));
+ }
> plot(ecdf(s.sample));
```



Notice that only even values of S are possible. The jumps in the center of the plot, around 4, are the biggest, showing that those values are more likely.

Answer of Exercise 8

In that case, unilateral testing makes sense, because if Muriel guesses extremely poorly, we'd rather conclude that she is unlucky (and not that she has the special ability to not guess if tea or milk was first). Then estimating the p-value is just a matter of

```
> mean(s.sample >= 6);
```

```
[1] 0.2406
```

Answer of Exercise 9

```
> contab <- matrix(c(3,1,1,3), ncol=2);
> fisher.test(contab, alternative="greater");

        Fisher's Exact Test for Count Data

data:  contab
p-value = 0.2429
alternative hypothesis: true odds ratio is greater than 1
95 percent confidence interval:
 0.3135693       Inf
sample estimates:
odds ratio
  6.408309
```

For the anecdote, Fisher found the same result. Still, his daughter later wrote that he was convinced that Muriel's claim was true. Even legendary statisticians sometimes lose faith in statistics.