CANONICAL REPRESENTATION OF THE YAGER'S CLASSES OF FUZZY IMPLICATIONS

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Abstract. The aim of this work is to study an interval extension of the Yager's classes of implications based on the canonical constructor. Focused on the Yager implication, such construction preserves similar and extra properties of fuzzy implications, also aggregating the correctness and optimality criteria.

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1. Introduction

The narrow sense of Interval Valued Fuzzy Logic (IVFL) [28], receives independent contributions from various research groups, aiming at the treatment of uncertainty not only about the membership functions but also about the membership degree of relevance. Thus, the notion of "precise number", representing a membership degree, is extended to an interval value carrying its uncertainty in the unity interval. According to R. Moore and W. Lodwick [15], basis on interval valued fuzzy set theory, IVFL may be thought as arising from the need of a more complete and inclusive logical model of uncertainty, expressing computations with real numbers and their interrelations in Scott and Moore topologies. Such approach emphasizes the synergism between interval analysis (as developed by R. E. Moore, see [18, 19, 14]) and fuzzy set theory (as conceived by [27]).

See, e.g., fuzzy arithmetic, interval arithmetic on alpha-cuts, the extension principle and interval representability of fuzzy connectives. Focused on the latter, we have considered the canonical representation to interpret the truth degree of a fuzzy implication related to conditional rule in inference systems based on fuzzy logic, modeling all the (lack of) knowledge about the value of a variable. So, by the best interval representation of a fuzzy implication, the correct result of such variable also aggregates the optimality criteria[13].

This paper study interval Yager's classes implications, introducing the concepts of interval f and g-generator implications. Focusing on the study of the canonical representation of the Yager implication, a characterization based on interval f- and g-generator is obtained. We also prove that an interval Yager implication satisfies similar and extra properties of fuzzy implications. For that, the paper is organized as follows: The main concepts of interval representations of real functions and related fuzzy connectives are discussed in Sects. 2 and 3, respectively. Fuzzy implications, their several properties and main classes are considered in Sect. 4. Section 5 introduces the canonical representation of the Yager's classes implications presenting a discussion about the interval extension of the related properties and results, followed by the Conclusion.

2. Interval representations

Consider $\mathbb{I}_{[a,b]} = \{[x,y] \,|\, a \leq x \leq y \leq b \text{ and } a,b \in \Re\}$ as the family of intervals of the extended real number set. The projections $l, r : \mathbb{I}_{[a,b]} \rightarrow [a,b]$ are defined by $l([x_1,x_2]) = x_1$ and $r([x_1,x_2]) = x_2$, respectively. For $X \in \mathbb{I}_{[a,b]}$, $l(X)$ and $r(X)$ are also denoted by $\underline{X}$ and $\overline{X}$, respectively. In addition, the binary operations $\vee, \wedge : {\mathbb{I}_{[a,b]}}^2 \rightarrow \mathbb{I}_{[a,b]}$ are given by $X_1 \wedge X_2 = [\min\{\underline{X_1}, \underline{X_2}\}, \min\{\overline{X_1}, \overline{X_2}\}]$ and $X \vee X_2 = [\max\{\underline{X_1}, \underline{X_2}\}, \max\{\overline{X_1}, \overline{X_2}\}]$, respectively.

For n-uplas $(\vec{X}) = (X_1, \ldots, X_n) \in {\mathbb{I}_{[a,b]}}^n$, it holds that the subsets are in $[a,b]^n$: (i) $l(\vec{X}) = (\underline{X_1}, \ldots, \underline{X_n})$ and $r(\vec{X}) = (\overline{X_1}, \ldots, \overline{X_n})$; (ii) $*(\vec{X}) = [*(\underline{X_1}, \ldots, \underline{X_n}), *(\overline{\overline{X_1}}, \ldots, \overline{\overline{X_n}})]$, when $* \in \{\vee, \wedge\}$.

When $F{:}{\mathbb{I}_{[a,b]}}^n \to \mathbb{I}_{[a,b]}$, we define the functions (i) $\underline{F}, \overline{F} : [a,b]^n \to [a,b]$, respectively by $\underline{F}(x_1,\ldots,x_n) = l(F([x_1,x_1],\ldots,[x_n,x_n]))$ and $\overline{F}(x_1,\ldots,x_n) = r(F([x_1,x_1],\ldots,[x_n,x_n]))$;

Consider the partial orders on the unit interval $U = [0,1]$:

O1 *Inclusion order*: $X \subseteq Y$ iff $\underline{X} \geq \underline{Y}$ and $\overline{X} \leq \overline{Y}$;

O2 *Product order*: $X \leq Y$ iff $\underline{X} \leq \underline{Y}$ and $\overline{X} \leq \overline{Y}$.

In addition, in the sense of domain theory, we also consider the induced-product relation: $X \lll Y$ iff $\underline{X} < \underline{Y} \wedge \overline{X} < \overline{Y}, \forall X, Y \in \mathbb{I}_U$. So, by [10, Def. 1], an interval function $F : {\mathbb{I}_U}^2 \to \mathbb{I}_U$ is $\lll$-increasing ($\lll$-decreasing) function when it is an increasing (decreasing) function w.r.t both the product order and the induced-product order.

An interval $X \in \mathbb{I}_{[a,b]}$ is said to be an interval representation of a real number α if $\alpha \in X$. Considering two interval representations X and Y of a real number α, X is a better representation of α than Y if X is narrower than Y, that is, if $X \subseteq Y$.

Definition 2.1. [22, Section 1] A function $F : {\mathbb{I}_{[a,b]}}^n \to \mathbb{I}_{[c,d]}$ is an **interval representation** of a function $f : [a,b]^n \to [c,d]$ if, for each $\vec{X} = (X_1,\ldots,X_n) \in {\mathbb{I}_{a,b}}^n$ and $\vec{x} \in \vec{X}$, $f(\vec{x}) \in F(\vec{X})$.

So, an interval function $F : {\mathbb{I}_{[a,b]}}^n \to \mathbb{I}_{[a,b]}$ is a *better interval representation* of $f : [a,b]^n \to [a,b]$ than $G : {\mathbb{I}_{[a,b]}}^n \to \mathbb{I}_{[a,b]}$, denoted by $G \sqsubseteq F$, if for each $\vec{X} \in {\mathbb{I}_{[a,b]}}^n$, the inclusion $F(\vec{X}) \subseteq G(\vec{X})$ holds.

Definition 2.2. [22, Section 2] The **best interval representation** (**canonical representation**) of a real function $f : [a,b]^n \to [a,b]$, is the interval function $\widehat{f} : {\mathbb{I}_{[a,b]}}^n \to \mathbb{I}_{[a,b]}$ defined by

$$\widehat{f}(\vec{X}) = [\inf\{f(\vec{x}) | \vec{x} \in \vec{X}\}, \sup\{f(\vec{x}) | \vec{x} \in \vec{X}\}]. \tag{1}$$

The interval function $\widehat{f}$ is well defined and for any other interval representation F of f, $F \sqsubseteq \widehat{f}$, providing a narrower interval than any other interval representation of f. Thus, $\widehat{f}$ has the *optimality property* of interval algorithms mentioned by Hickey et al. [13], when it is seen as an algorithm to compute a real function f.

By [22, Sect. 2.2], for an interval function $f : [a,b]^n \to [a,b]$, the following statements are equivalent: (i) f is continuous; (ii) $\widehat{f}$ is Scott continuous; (iii) $\widehat{f}$ is Moore continuous [22]. So, when f is continuous in usual sense, $\widehat{f}(\vec{X}) = \{f(\vec{x}) | \vec{x} \in \vec{X}\} = f(\vec{X})$, $\forall \vec{X} \in {\mathbb{I}_{[a,b]}}^n$.

3. Interval fuzzy connectives

An interval fuzzy connective may be considered as an interval representation of a fuzzy connective [7]. This generalization fits the fuzzy principle, which means that the interval degree of membership may be thought as an approximation of the exact degree.

3.1. **Interval t-norms and interval t-conorms.** Notice that a t-conorm (t-norm) is a function $S : U^2 \to U$ which is commutative, associative, monotonic and has 0 (1) as neutral element. In the following, denoting $\mathbb{I}_U = \mathbb{U}$, an extension of the t-conorm (t-norm) notion is considered, following the same approach introduced in [7].

Definition 3.1. [7, Definition5.1] A function $\mathbb{T}(\mathbb{S}) : \mathbb{U}^2 \to \mathbb{U}$ is an **interval t-norm** (**t-conorm**) if it is commutative, associative, monotonic w.r.t. both, the product and inclusion orders[1], and has $\mathbf{1} = [1, 1]$ ($\mathbf{0} = [0, 0]$) as the neutral element.

Proposition 3.2. [7, Theorem 5.1] *If S (T) is a t-conorm then $\widehat{S}(\widehat{T}) : \mathbb{U}^2 \to \mathbb{U}$ is an interval t-conorm (t-norm). And, a characterization of $\widehat{S}$ ($\widehat{T}$) can be expressed by:*

$$\widehat{S}(X, Y) = [S(\underline{X}, \underline{Y}), S(\overline{X}, \overline{Y})], \ \widehat{T}(X, Y) = [T(\underline{X}, \underline{Y}), T(\overline{X}, \overline{Y})] \tag{2}$$

3.2. **Interval fuzzy negation.** A *fuzzy negation* $N : U \to U$ verifies two properties: ($N1$) the boundary conditions, i.e., $N(0) = 1$ and $N(1) = 0$; and ($N2$) it is a non-increasing function, i.e., if $x \geq y$ then $N(x) \leq N(y)$, $\forall x, y \in I$. In addition, fuzzy negations satisfying the involutive property are called *strong fuzzy negations* (shorten by SFN) : ($N3$) $N(N(x)) = x$, $\forall x \in U$. See [8], for more details.

Definition 3.3. [7, Def. 4.1] An interval function $\mathbb{N} : \mathbb{U} \to \mathbb{U}$ is an **interval fuzzy negation** if, for all $X, Y \in \mathbb{U}$, it holds that: (i) $\mathbb{N}1$: $\mathbb{N}(\mathbf{0}) = \mathbf{1}$ and $\mathbb{N}(\mathbf{1}) = \mathbf{0}$; ($ii$) $\mathbb{N}2$: If $X \geq Y$ then $\mathbb{N}(X) \leq \mathbb{N}(Y)$; and ($iii$) $\mathbb{N}3$: If $X \subseteq Y$ then $\mathbb{N}(X) \subseteq \mathbb{N}(Y)$. Moreover, if $\mathbb{N}$ also meets the involutive property, it is a **strong interval fuzzy negation** (shorten by IV-SFN): $\mathbb{N}4$: $\mathbb{N}(\mathbb{N}(X)) = X$, $\forall X \in \mathbb{U}$.

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[1]Such notion of monotonicity of t-(co)norms w.r.t an inclusion order is equivalent to the notion of t-representable t-(co)norms, as presented in [9].

By [7, Theorem4.1.1], if $N : U \to U$ be a fuzzy negation, $\widehat{N}$ is an interval fuzzy negation. In addition, if N is a strong fuzzy negation then $\widehat{N}$ is a strong interval fuzzy negation. And, a characterization of $\widehat{N}$ is given by $\widehat{N}(X) = [N(\overline{X}), N(\underline{X})]$.

4. Fuzzy implications

Fuzzy logic is a powerful methodology for formalizing our incomplete and imprecise knowledge of complex systems, whose construction frequently relies on the expert's statements combining the aim to model human reasoning in a more natural way with the necessity to get "if-then" statements modeled by fuzzy implications. Thus, in the fuzzy logic framework, the study of fuzzy logic in the narrow sense enlarges the classes of fuzzy implications. A fuzzy implication should present the behavior of the classical implication when the crisp case is considered. So, A binary function $I : U^2 \to U$ is a **fuzzy implication** if it satisfies the minimal boundary conditions:

I1: : $I(1,1) = I(0,1) = I(0,0) = 1$ and $I(1,0) = 0$.

Several reasonable properties may be required for fuzzy implications [1]. Let T be a t-norm and $x, y, z \in U$, the properties considered in this paper are described in the following:

I2: If $x \leq z$ then $I(x,y) \geq I(z,y)$ (first place antitonicity);
I3: If $y \leq z$ then $I(x,y) \leq I(x,z)$ (second place isotonicity);
I4: $I(0,y) = I(x,1) = 1$ (absorption principle);
I5: $I(1,y) = y$ (left neutrality principle);
I6: $I(x, I(y,z)) = I(y, I(x,z))$ (exchange principle);
I7: $I(x,y) \geq y$; (consequence boudary)
I8: $I(x,0) = N_I(x)$ is strong fuzzy negation;
I9: $I(x,y)=I(N(y),N(x))$, if N is a SFN (contraposition law);
I10: $I(x,x) = 1$ (identity principle);
I11: $x \leq y$ iff $I(x,y) = 1$ (ordering principle);

4.1. Main classes of fuzzy implications.

Among other classes of fuzzy implications (D-implications [21], E-implications and Xor-implications [5], force implications [11]), H-implications [17], etc.) associated to an **explicit representation** obtained from aggregation functions and negations [16], the classes of **S-implications** [4] and **QL-implications** [20, 23] will be reported in the following. In addition, the **implicit representation** of implications is another approach including the class of **R-implications** [1], arising from the notion of residuum of T in Intuitionistic Logic.

Definition 4.1. Let S be a t-conorm, T be a t-norm and N be a strong fuzzy negation. Thus, for all $(x, y) \in [0, 1]$, it holds that:
(i) an S-implication is given by $I_{S,N}(x, y) = S(N(x), y)$;
(i) a QL-implication is given by $I_{S,N,T}(x, y) = S(N(x), T(x, y))$;
(iii) an R-implication is given by $I_T(x, y) = \sup\{z : T(x, z) \leq y\}$[2].

Proposition 4.2. [12, Theorem 1.13] *An implication $I : U^2 \to U$ is an S-implication iff the properties* **I2** *(or* **I3***),* **I5***,* **I6** *and* **I9** *are met.*

Proposition 4.3. [1, Theorem 2.6.19] *Let $I : U^2 \to U$ be a QL-implication. I satisfies* **I5** *if and only if I is an S-implication.*
Proposition 4.4. [12, Theorem 1.14] *A function $I : U^2 \to U$ is an R-implication based on a left-continuous t-norm T iff I verifies* **I3***,* **I6***,* **I11** *and it is right-continuous w.r.t. its first argument.*

However, there are reasonable fuzzy implication functions that can not be easily represented in any of these three classes, see e.g.[25]. In order to describe such implications, a new class of **axiomatic representation** of fuzzy implications, named A-implications, is described in [24] in terms of non-commutativity property related to t-norms. The A-implications are based on a subset of the axioms listed in [12]. In this paper, we focus on Yager implication.

4.2. **Yager's classes of fuzzy implications.** In [25], Yager proposed two new classes of fuzzy implications, called f-implications and g-implications, which can not be fulfilled in the above presented classes. An application of contrapositivisation technics on such functions is presented in [2]. A study focusing on the general form of the

[2]An R-implications is well-defined only if a t-norm T is left-continuous [7]

law of importation for such is considered in [3]. We concentrate our discussion on the two extra properties:

I12: $I(x, T(y,z)) = T(I(x,y), I(x,z))$ (distributivity);
I13: $T(I(x,y), I(N(x),y)) = y$;

Definition 4.5. [25, Sect.3] Let $f : [0,1] \to [0,\infty]$ be f-generator, which means, a strictly decreasing and continuous function such that $f(0) = 1$ and its pseudo-inversa $f^{(-1)} : [0,\infty] \to [0,1]$ is defined by:

$$f^{(-1)}(x) = f^{-1}(x), \text{if } x \leq f(0); \text{ and } 0, \text{otherwise.} \tag{3}$$

When $0{\times}\infty{=}0$, an ***f*-generated implication** $I_f{:}U^2{\to}U$ is given by

$$I_f(x,y) = f^{(-1)}\left(x \cdot f(y)\right). \tag{4}$$

Definition 4.6. [25, Sect.4] Let $g : [0,1] \to [0,\infty]$ be a g-generator, which means, it is a strictly increasing and continuous function such that $g(0) = 0$ and its pseudo-inversa $g^{(-1)}$ is defined by:

$$g^{(-1)}(x) = g^{-1}(x), \text{if } x \leq g(1); \text{ and } 1, \text{otherwise.} \tag{5}$$

When $\infty{\times}0{=}\infty$, a ***g*-generated implication** $I{:}U^2 \to U$ is given by

$$I(x,y) = g^{(-1)}\left(\frac{1}{x} \cdot g(y)\right). \tag{6}$$

An (g-) f-generated fuzzy implication I_f verifies the property **I1**:

Proposition 4.7. [1, Props.3.1.2 and 3.2.2] *An (g-) f-generated implication $(I_g)I_f : U^2 \to U$, generated as in (Def. 4.6) Def. 4.5, is a fuzzy implication.*

Proposition 4.8. [25, Sect.3, p.197 and Sect.4, p.202] *An (g-) f-generated implication $(I_g)I_f : U^2 \to U$, generated as in (Def. 4.6) Def. 4.5 verifies the properties* **Ik**, *for each $2 \leq k \leq 7$.*

Proposition 4.9. [25, Sect.3, p.194 and Sect.4, p.197][1, Theorems 3,1.7 and 3.2.8] *An (g-) f-generated implication $(I_g)I_f : U^2 \to U$, generated as in (Def. 4.6) Def. 4.5 does not verify* **I8**, **I10** *and* **I11**.

From Props. 4.4 and 4.9 (see [1, Theorem 4.6.1]) it follows that:

Proposition 4.10. *An f-generated implication $I_f : U^2 \to U$, generated by an f-generator as in Def. 4.6, is not an R-implication.*

Proposition 4.11. [1, Theorem 3.2.8] *A g-generated implication $I_g : U^2 \to U$, generated as in Def. 4.6, does not verify* **I9**.

From Props. 4.2 and 4.11, it is immediate that:

Corollary 4.12. *A g-generated implication $I_g : U^2 \to U$, generated as in Def. 4.6, is not an S-implication.*

4.2.1. *Yager implication.* The study of Yager implication is considered according with [25, 2, 1, 26] and [24].

Proposition 4.13. *The binary function proposed by Yager* [25]*:*

(7) $I_Y(x, y) = 1$, *if* $x = y = 0$*; and* $I_Y(x, y) = y^x$, *otherwise.*

is an f- and a g-generated implication.

Proof. Consider $f, g : [0, 1] \to [0, \infty]$, such that $f(x) = -\log x$ and $g(x) = -\frac{1}{\ln(x)}$. Thus, f and g are an f- and a g-generator of the I_Y, generated as in Defs. 4.5 and 4.6, respectively. □

Corollary 4.14. *I_Y verifies the properties* **Ik***, for each $2 \leq k \leq 7$.*

Proof. Straightforward from Propositions 4.13 and 4.8. □

Proposition 4.15. *I_Y also satisfies* **I8***,* **I12** *and* **I13**.

Proof. When $x = y = 0$, it is trivial. Otherwise, it holds that: **I**8: Firstly, I_Y is a non-decreasing function on U and $I_Y(1, 0) = 0$ and $I_Y(1, 1) = 1$. By Corollary. 4.14, $I_Y(I_Y(x, 0), 0) = I_Y(x, I_Y(0, 0)) = I_Y(x, 1) = x$. So, N_{I_Y} is involutive. Now, let T be a left continuous t-norm. **I**12: $I_Y(x, T(y, z)) = I_Y(x, y \cdot z) = (y \cdot z)^x = y^x \cdot z^x = T(I_Y(x, y), I_Y(x, z))$. **I**13:$T(I_Y(x, y), I_Y(N(x), y)) = y^x \cdot y^{1-x} = y$. □

Proposition 4.16. *I_Y is neither an S-implication nor a QL-implication nor an R-implication.*

Proof. By Prop. 4.13, I_Y is an f- and g-generated implication. Thus, by Prop. 4.10 and by Corollary 4.12, it is neither an S-implication nor an R-implication. But I_Y met **I5**, by Corollary. 4.14. So, it is not a QL-implication by Prop. 4.3. □

5. Interval fuzzy implication

According to the idea that values in interval mathematics are identified with degenerate intervals[3], the minimal properties of fuzzy implications can be naturally extended from interval approach. Thus, for all $X, Y, Z \in \mathbb{U}$, an interval function $\mathbb{I} : \mathbb{U}^2 \to \mathbb{U}$ is an *interval fuzzy implication* if the following conditions hold:

$\mathbb{I}$1: $\mathbb{I}(\mathbf{1},\mathbf{1}) = \mathbb{I}(\mathbf{0},\mathbf{0}) = \mathbb{I}(\mathbf{0},\mathbf{1}) = \mathbf{1}$, and $\mathbb{I}(\mathbf{1},\mathbf{0}) = \mathbf{0}$.

Some extra properties can be naturally extended.

$\mathbb{I}$2: If $X \leq Z$ then $\mathbb{I}(X,Y) \geq \mathbb{I}(Z,Y)$;
$\mathbb{I}$3: If $Y \leq Z$ then $\mathbb{I}(X,Y) \leq \mathbb{I}(X,Z)$;
$\mathbb{I}$4: $\mathbb{I}(\mathbf{0},Y) = \mathbb{I}(X,\mathbf{1}) = \mathbf{1}$;
$\mathbb{I}$5: $\mathbb{I}(\mathbf{1},Y) = Y$;
$\mathbb{I}$6: $\mathbb{I}(X,\mathbb{I}(Y,Z)) = \mathbb{I}(Y,\mathbb{I}(X,Z))$;
$\mathbb{I}$7: $\mathbb{I}(X,Y) \geq Y$;
$\mathbb{I}$8: $\mathbb{I}(X,\mathbf{0}) = \mathbb{N}_{\mathbb{I}}(X)$ is an IV-SFN;
$\mathbb{I}$9: $\mathbb{I}(X,Y) = \mathbb{I}(\mathbb{N}(Y),\mathbb{N}(X))$, if $\mathbb{N}$ is an IV-SFN;
$\mathbb{I}$10: $\mathbb{I}(X,X) = \mathbf{1}$;
$\mathbb{I}$11: $\mathbb{I}(X,Y) = \mathbf{1}$ iff $\overline{X} \leq \underline{Y}$;
$\mathbb{I}$12: $\mathbb{I}(X,\mathbb{T}(Y,Z)) = \mathbb{T}(\mathbb{I}(X,Y),\mathbb{I}(X,Z))$;
$\mathbb{I}$13: $\mathbb{T}(\mathbb{I}(X,Y),\mathbb{I}(N(X),Y)) = Y$;

It is always possible to obtain canonically an interval fuzzy implication from any fuzzy implication, which also meets the optimality property and preserves the same properties satisfied by the fuzzy implication. Now, as a particular case of Eq.(1), the best interval representation $\widehat{I}$ of a fuzzy implication I, is shown as an inclusion-monotonic function in both arguments.

Proposition 5.1. [4, Prop. 16] *If I is a fuzzy implication then $\widehat{I}$ is an interval fuzzy implication.*

Proposition 5.2. [4, Prop. 21] *An implication $I : U^2 \to U$ satisfies the Properties* **I1** *and* **I2** *iff $\widehat{I}$ can be expressed as*

$$\widehat{I}(X,Y) = [I(\overline{X},\underline{Y}), I(\underline{X},\overline{Y})]. \tag{8}$$

[3] A degenerate interval $[x,x] \in \mathbb{U}$, identified with x, is denoted by $\mathbf{x}$.

Proposition 5.3. [4, Prop. 23] [6, Theorem 11] *Let I be a fuzzy implication satisfying* **I2** *and* **I3**. *I satisfies the Property* **Ik**, *for some* $\mathbf{k} = 4, \ldots, 7$ *iff* $\widehat{I}$ *satisfies the Property* $\mathbb{I}\mathbf{k}$.

Proposition 5.4. *Let I be a fuzzy implication satisfying* **I1** *and* **I2**. *I satisfies* **I12** *and* **I13** *iff* $\widehat{I}$ *satisfies* $\mathbb{I}\mathbf{12}$ *and* $\mathbb{I}\mathbf{13}$:

Proof. Let T be an interval T norm. It holds that:
I12 : ($\Rightarrow$) When $u \in \widehat{I}(X, \widehat{T}(Y,Z))$, there exist $x \in X$ and $v \in \widehat{T}(Y,Z)$ with $u = I(x,v)$. If $v \in \widehat{T}(Y,Z)$, then there exists $y \in Y$ and $z \in Z$ such that $v = T(y,z)$. So, $u = I(x, T(y,z))$ and, by **I**12, $u = T(I(x,y), I(x,z))$. Thus, since $I(x,y) \in \widehat{I}(X,Y)$ and $I(x,z) \in \widehat{I}(X,Z)$, $u \in \widehat{T}(\widehat{I}(X,Y), \widehat{I}(X,Z))$. Therefore, we can conclude that $\widehat{I}(X, \widehat{T}(Y,Z)) \subseteq \widehat{T}(\widehat{I}(X,Y), \widehat{I}(X,Z))$.

($\Leftarrow$) On the other hand, if $u \in \widehat{T}(\widehat{I}(X,Y), \widehat{I}(X,Z))$ then there exist $w \in \widehat{I}(X,Y)$ and $v \in \widehat{I}(X,Z)$ such that $u = T(w,v)$. But, when $w \in \widehat{I}(X,Y)$, there exist $x \in X$ and $y \in Y$, and $w = I(x,y)$. In addition, if $v \in \widehat{I}(X,Z)$ then there exist $x \in Z$ and $z \in Z$ and $v = I(x,z)$. So, $u = T(I(x,y), I(x,z))$ and therefore, by Property **I**8, $u = I(x, T(y,z))$. Thus, since $x \in X$ and $T(y,z) \in \widehat{T}(Y,Z)$, it holds that $u \in \widehat{I}(X, \widehat{T}(Y,Z))$. Concluding, we proved that $\widehat{T}(\widehat{I}(X,Y), \widehat{I}(X,Z)) = \widehat{I}(X, \widehat{T}(Y,Z))$, so $\widehat{I}$ satisfies $\mathbb{I}\mathbf{12}$.
I13 : ($\Rightarrow$) If $u \in \widehat{T}(\widehat{I}(X,Y), \widehat{I}(N(X),Y))$ then there exist $w \in \widehat{I}(X,Y)$ and $v \in \widehat{I}(N(X),Y)$ such that $u = T(w,v)$. Analogously, when $w \in \widehat{I}(X,Y)$, there exist $x \in X$ and $y \in Y$ such that $w = I(x,y)$. And, $v \in \widehat{I}(N(X),Y)$ implies that, for $N(x) \in N(X)$ and $y \in Y$, $v = I(N(x),y)$. So, $u = T(I(x,y), I(N(x),y))$ and, by Property **I**13, it holds that $u = y$. Thus, since $y \in Y$, $u \in Y$. Therefore, $\widehat{T}(\widehat{I}(X,Y), \widehat{I}(N(X),Y)) \subseteq Y$.

($\Leftarrow$) When $y \in Y$, by Property **I**13, $T(I(x,y), I(N(x),y)) \in Y$. Since $x \in X$, $y \in Y$ and $z \in Z$, $I(x,y) \in \widehat{I}(X,Y)$ and $I(x,z) \in \widehat{I}(X,Z)$. So, $y = T(I(x,y), I(N(x),y)) \in \widehat{T}(\widehat{I}(X,Y), \widehat{I}(X,Z))$, which means $Y \subseteq \widehat{T}(\widehat{I}(X,Y), \widehat{I}(X,Z))$. Thus, $\widehat{T}(\widehat{I}(X,Y), \widehat{I}(X,Z)) = Y$. ☐

5.1. **Main classes of interval implications.**

This section extends the classes of implications presented in Sect. 4.1.

5.1.1. *Explicit and implicit interval implications.* Let $\mathbb{S}$ be an interval t-conorm, $\mathbb{T}$ be an interval t-norm and $\mathbb{N}$ be a strong interval fuzzy negation. Thus, expressions of the main classes of interval implications are in the following:
(i)**interval R-implication**, as $\mathbb{I}_{\mathbb{T}}(X,Y)=\sup\{Z\in\mathbb{U}|\mathbb{T}(X,Z)\leq Y\}$;
(ii) **interval QL-implication**, as $\mathbb{I}_{\mathbb{S},\mathbb{N},\mathbb{T}}(X,Y)=\mathbb{S}(\mathbb{N}(X),\mathbb{T}(X,Y))$;
(iii) **interval S-implication**, as $\mathbb{I}_{\mathbb{S},\mathbb{N}}(X,Y)=\mathbb{S}(\mathbb{N}(X),Y)$.

Theorem 5.5. [4, Theorem 24] *Let S be a t-conorm and N be a strong fuzzy negation. Then $\mathbb{I}_{\widehat{S},\widehat{N}} = \widehat{I_{S,N}} = [I_{S,N}(\overline{X},\underline{Y}), I_{S,N}(\underline{X},\overline{Y})]$.*

Proposition 5.6. [4, Theorem 29] *Let $\mathbb{I}$ be an interval fuzzy implication. $\mathbb{I}$ is an interval S-implication iff $\mathbb{I}2$, $\mathbb{I}3$,$\mathbb{I}4$, $\mathbb{I}5$ and $\mathbb{I}9$ hold.*

Theorem 5.7. [20, Theorema 4] *Let S be a t-conorm, T be a t-norm and N be a strong fuzzy negation. If S and T are continuous, $\mathbb{I}_{\widehat{S},\widehat{N},\widehat{T}}=\widehat{I_{S,N,T}}=[I_{S,N,T}(S(N(\overline{X}),T(\underline{X},\underline{Y})),I_{S,N,T}(S(N(\underline{X}),T(\overline{X},\overline{Y}))]$.*

Proposition 5.8. *Let $\mathbb{I}$ be an interval fuzzy implication. If $\mathbb{I}$ is an interval QL-implication then $\mathbb{I}$ satisfies the properties $\mathbb{I}3$, $\mathbb{I}5$ and $\mathbb{I}8$.*
Proposition 5.9. [6, Theorem 24] *Let T be a left-continuous t-norm. If $\mathbb{I}_{\widehat{T}}$ is $\subseteq$-monotonic then $\mathbb{I}_{\widehat{T}} = \widehat{I}_T$.*

Proposition 5.10. [6, Theorem 14] *Let $\mathbb{T}$ be a (Moore, Scott) left-continuous interval t-norm. If $\mathbb{I}$ is an interval R-implication then $\mathbb{I}$ satisfies $\mathbb{I}3$, $\mathbb{I}6$ and $\mathbb{I}11$.*

5.1.2. *Interval Yager's classes of implications.*

Definition 5.11. Consider $F:\mathbb{I}_U\rightarrow\mathbb{I}_{[0,\infty]}$ as a $\lll$-decreasing and (Scott and Moore) continuous interval function with $F(\mathbf{1})=\mathbf{0}$ and a pseudo-inverse $F^{(-1)}:\mathbb{I}_{[0,\infty]}\rightarrow\mathbb{I}_U$ given by:

$$F^{(-1)}(Y) = F^{-1}(Y), \text{if } Y \lll F(\mathbf{0});\ \text{and } \mathbf{0}, \text{otherwise.} \tag{9}$$

An **interval $\boldsymbol{F}$-generated implication** $\mathbb{I}_F:\mathbb{U}^2\rightarrow\mathbb{U}$ is defined by

$$\mathbb{I}_F(X,Y) = F^{(-1)}(X\cdot F(Y)). \tag{10}$$

Proposition 5.12. *An interval F-generated operator $\mathbb{I}_F : \mathbb{U}^2 \to \mathbb{U}$, defined by Eq. (10) and with the understanding that $\mathbf{0} \cdot [\infty, \infty] = \mathbf{0}$, is an interval fuzzy implication.*

Proof. For all $X, Y \in \mathbb{U}$, by Definition 5.11, it holds that $\mathbb{I}_F(\mathbf{0}, Y) = F^{(-1)}(\mathbf{0} \cdot F(Y)) = F^{-1}(\mathbf{0}) = \mathbf{1}$ and $\mathbb{I}_F(X, \mathbf{1}) = F^{(-1)}(X \cdot F(\mathbf{1})) = F^{-1}(\mathbf{0}) = \mathbf{1}$. In addition, in both cases, $F(\mathbf{0}) = \infty$ or $F(\mathbf{0}) < \infty$, it holds that $\mathbb{I}_F(\mathbf{1}, \mathbf{0}) = F^{(-1)}(\mathbf{1} \cdot F(\mathbf{0})) = F^{-1}(F(\mathbf{0})) = \mathbf{0}$ Therefore, by $\mathbb{I}1$, it follows that $\mathbb{I}_F$ is an interval fuzzy implication. □

Definition 5.13. Consider $G : \mathbb{I}_U \to \mathbb{I}_{[0,\infty]}$ as a $\lll$-increasing and (Scott and Moore) continuous interval function with $G(\mathbf{0}) = \mathbf{0}$ and a pseudo-inverse $G^{(-1)} : \mathbb{I}_{[0,\infty]} \to \mathbb{U}$ given by:

$$G^{(-1)}(Y) = G^{-1}(Y), \text{if } Y \lll G(\mathbf{1}); \text{ and } \mathbf{1}, \text{otherwise.} \tag{11}$$

An **interval G-generated implication** $\mathbb{I}_G : \mathbb{U}^2 \to \mathbb{U}$ is defined by

$$\mathbb{I}_G(X, Y) = G^{(-1)}\left(\frac{\mathbf{1}}{X} \cdot G(Y)\right). \tag{12}$$

Proposition 5.14. *An interval G-generated operator $\mathbb{I}_G : \mathbb{U}^2 \to \mathbb{U}$ defined by Eq. (12), with the understanding that $\frac{\mathbf{1}}{\mathbf{0}} = [\infty, \infty]$ and $[\infty, \infty] \cdot \mathbf{0} = [\infty, \infty]$, is an interval fuzzy implication.*

Proof. By Def. 5.13, it holds that: (i) $\mathbb{I}_G(\mathbf{0}, \mathbf{0}) = G^{(-1)}(\frac{\mathbf{1}}{\mathbf{0}} \cdot G(\mathbf{0})) = G^{(-1)}([\infty, \infty]) = \mathbf{1}$; (ii) $\mathbb{I}_G(\mathbf{1}, \mathbf{0}) = G^{(-1)}(\mathbf{1} \cdot G(\mathbf{0})) = G^{-1}(\mathbf{0}) = \mathbf{0}$; (iii) $\mathbb{I}_G(\mathbf{1}, \mathbf{1}) = G^{(-1)}(G(\mathbf{1})) = \mathbf{1}$; and (iv) $\mathbb{I}_G([\infty, \infty], G(\mathbf{1})) = \mathbf{1}$. Therefore, by $\mathbb{I}1$, it follows that $\mathbb{I}_G$ is an interval fuzzy implication. □

Proposition 5.15. *Consider $G, F : \mathbb{I}_U \to \mathbb{I}_{[0,\infty]}$ and let $\mathbb{I}_Y : \mathbb{U}^2 \to \mathbb{U}$ be the **interval Yager implication**[4] defined by*

$$\mathbb{I}_Y(X, Y) = \mathbf{1}, \textit{if } X = Y = \mathbf{0} \textit{ and } \mathbb{I}_Y(X, Y) = Y^X, \textit{otherwise.} \tag{13}$$

Then, Y^X is an interval (F-) G-generated implication.

Proof. (i) $\mathbb{I}_Y$ is an interval F-generated implication when the interval F-generated operator is given as $F(X) = -\mathbf{Ln}(X)$[5] and related

[4] $Y^X = [\underline{Y}^{\overline{X}}, \overline{Y}^{\underline{X}}]$ denotes the generalized power interval operator.

[5] $\mathbf{Ln}X = [\ln(\underline{X}), \ln(\overline{X})]$ denotes the natural logarithmic interval operator.

interval pseudo-inverse as $F^{(-1)}(Y) = \mathbf{Exp}^{-Y}$, if $Y \lll F(\mathbf{0})$ and $F^{(-1)}(Y) = \mathbf{0}$, otherwise[6]; (ii) $\mathbb{I}_Y$ is also an interval G-generated implication whose generator is given by $G(X)=-\frac{\mathbf{1}}{\mathbf{Ln}(X)}$ and $G^{(-1)}(Y)= \mathbf{Exp}^{-\frac{\mathbf{1}}{\mathbf{Y}}}$, if $Y \lll G(\mathbf{1})$ and $G^{(-1)}(Y)=\mathbf{1}$, otherwise. □

Proposition 5.16. $\widehat{I_Y}(X,Y) = \mathbb{I}_Y(X,Y) = [I_Y(\overline{X},\underline{Y}), I_Y(\underline{X},\overline{Y})]$.

Proof. It follows from Corollary 4.14 and Prop. 5.2. □

Theorem 5.17. *I_Y satisfies the properties* $\mathbf{I}k$ *iff* $\mathbb{I}_Y$ *satisfies the properties* $\mathbb{I}k$*, for* $k = 2, \ldots, 8, 12, 13$.

Proof. It follows from Corollary. 4.14 and Prop. 4.15 in Sect. 4 and Props. 5.2, 5.3 and 5.4 in the previous Subsect. 5.1.1. □

Proposition 5.18. $\mathbb{I}_Y$ *does not meet properties* $\mathbb{I}9$ *and* $\mathbb{I}10$*.*

Proof. (i) When $\mathbb{I}_Y(\mathbb{N}(X), \mathbb{N}(Y)) = \mathbb{I}_Y(X,Y)$, $I_Y(N(\underline{X}), N(\overline{Y})) = I_Y(\overline{X},\underline{Y})$ and $I_Y(N(\overline{X}), N(\underline{Y})) = I_Y(\underline{X},\overline{Y})$. So, since N is IV-SFN, X and Y are the equilibrium points. Then $\mathbb{I}_Y$ does not meet $\mathbb{I}9$.
(ii) Taking $\mathbb{I}(X,X) = \mathbf{1}$ it follows that $\underline{\mathbb{I}}(X,X) = I_Y(\overline{X},\underline{X}) = 1$, So, either $\underline{X} = 1$ or $\overline{X} = \underline{X} = 0$. Therefore, $\mathbb{I}_Y$ does not meet $\mathbb{I}10$. □

Proposition 5.19. $\mathbb{I}_Y$ *is neither an interval S-implication nor an interval R-implication and nor an interval QL-implication.*

Proof. It follows from Props. 5.6, 5.8 and 5.10 in Sect. 5.1.2. □

6. Conclusion

This work investigates the interval Yager's classes implications, in the approach of an axiomatic representation to fuzzy implications. We study the interval f- and g-generator, which is related to the type of interval additive generator used to represent interval t-norm and interval t-conorms, respectively (see [10]). Thus, the interval f- and g-generated implications derived from corresponding interval f- and g-generator are considered. Based on the canonical representation, our task has undertaken the discussion of the canonical interval

[6] $\mathbf{Exp}X = [\exp(\underline{X}), \exp(\overline{X})]$ denotes the exponential interval operator.

representation of Yager implication, emphasizing similar and extra properties of related main classes of interval fuzzy implications. Such discussion contributes to obtain of alternative approaches to generate interval fuzzy implication considering both the correctness and the optimality criteria. Further work is focused on the study the interval-valued intuitionistic Yager's classes implications.

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