Deriving the Euler-Bernoulli Beam Theory

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MATH-080 - Differential Equations

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Problem 1: Symmetric/simple bending. $\frac{1}{\rho} = \frac{M(x)}{EI} = \frac{d^2w}{dx^2}$ where ρ is the radius of curvature, $M(x)$ is the internal bending moment, I the moment of inertia, and E is Young's modulus.

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1. Deriving the differential equation:

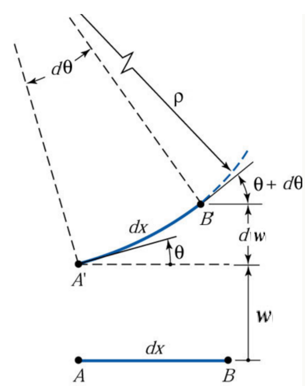

Figure 1: Symmetric bending - diagram

From the above diagram, we can approximate the angle θ with dv and dx. We approximate (applying the small-angle approximation):

$$\frac{dw}{dx} = sin\theta \approx \theta$$

(where ρ is the radius of curvature)

Furthermore from the diagram, we can derive that:

$$dx = \rho d\theta$$

We can rewrite the previous as:

$$\frac{1}{\rho} = \frac{d\theta}{dx}$$

And substituting for θ results in the differential equation:

$$\frac{1}{\rho} = \frac{d^2w}{dx^2}$$

We know from the moment-curvature relationship that :

$$\frac{1}{\rho} = \frac{M(x)}{EI}$$

thus:

$$\frac{M(x)}{EI} = \frac{d^2w}{dx^2}$$

This is the differential equation of the elastic curve for a beam undergoing bending in the plane of symmetry. Its solution $w(x)$ represents the shape of the deflection curve for a given beam.

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2. Solving the differential equation:

We can solve this (separable) differential equation through double integration. Considering EI as a constant throughout the beam (EI is the "flexural rigidity", and varies per material), integration returns:

$$\begin{aligned} \int EI\frac{d^2w}{dx^2}\mathrm{d}x &= \int M(x)\mathrm{d}x \\ EI\frac{dw}{dx} &= \int M(x)\mathrm{d}x + C_1 \\ w &= \frac{\int[\int M(x)\mathrm{d}x]\mathrm{d}x}{EI} + C_1x + C_2 \end{aligned}$$

Solving the differential equation shows the appearance of two constant of integration C_1 and C_2. We can use boundary conditions relative to the deflection at length $x = 0$ and $x = L$ to solve this.

NOTE: If the flexural rigidity EI is not constant over the beam, new differential equations must be calculated for each segment of the beam.

A worked-out example putting this equation to use can be found under appendix A.

Problem 2: Static beam bending under applied load. $\frac{d}{dx^2}[EI\frac{d^2w}{dx^2}] = q(x)$ The previous derivation is valid for beams undergoing bending in the plane of symmetry. Beams undergoing distributed loading (here represented by q) require the derivation of another differential equation.

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1. Deriving the differential equation:

a) Assumption:

- The product EI (flexural rigidity) does not vary along the length of the beam.

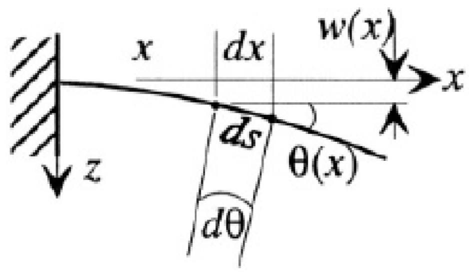

Figure 2: Static beam bending under load - diagram

We will derive our differential equation according to the above diagram. To avoid confusion, $w(x)$ again represents the deflection of the beam along the z-axis.

The Euler-Bernoulli static beam theory arises from four subsets: *the kinematic, constitutive, force resultant, and equilibrium equations.* Originating from four subsets of equations signifies that its derivation requires much substitution.
For the sake of concision in this paper, the origin of these four equations is not included but can be found in the reference list.

We first combine the kinematic equations [3] from which we get:

$$\frac{d^2M}{dx^2} = -q$$

We can then replace M, the moment resultant by its definition [3]:

$$\frac{d^2}{dx^2}(\iint y \cdot \sigma \cdot dydz) = -q$$

The constitutive relation further eliminates σ (stress), replacing it by the strain ϵ. ϵ is further replaced by the displacement w (kinematics):

$$\frac{d^2}{dx^2}(E\frac{d^2w}{dx^2}\iint y^2 \cdot dydz) = q$$

The double integral of y^2 over $dydz$ is the definition of the beam's moment of inertia.

$$\therefore \frac{d^2}{dx^2}[EI\frac{d^2w}{dx^2}] = q = EI\frac{d^4w}{dx^4}$$

(for a distributed load q).

Solving this differential equation for w returns an equation that can be used to find the deflection of a beam (under load) given boundary conditions.

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2. Solving the differential equation:

This differential equation is of fourth order, meaning that its solution requires a bit more than in Problem 1. We again consider EI as constant throughout the entirety of the studied beam:

$$\begin{aligned}
EI\frac{d^4w}{dx^4} &= q(x) \\
\int EI\frac{d^4w}{dx^4}\mathrm{d}x &= \int q(x)\mathrm{d}x \\
EI\frac{d^3w}{dx^3} &= \int q(x)\mathrm{d}x + C_1 \\
\int EI\frac{d^3w}{dx^3}\mathrm{d}x &= \int [\int q(x)\mathrm{d}x]\mathrm{d}x + C_1 \\
EI\frac{d^2w}{dx^2} &= \int [\int q(x)\mathrm{d}x]\mathrm{d}x + C_1x + C_2 \\
\int EI\frac{d^2w}{dx^2}\mathrm{d}x &= \int [\int [\int q(x)\mathrm{d}x]\mathrm{d}x]\mathrm{d}x + C_1x + C_2 \\
EI\frac{dw}{dx} &= \int [\int [\int q(x)\mathrm{d}x]\mathrm{d}x]\mathrm{d}x + \frac{C_1x^2}{2} + C_2x + C_3 \\
\int EI\frac{dw}{dx}\mathrm{d}x &= \int [\int [\int [\int q(x)\mathrm{d}x]\mathrm{d}x]\mathrm{d}x]\mathrm{d}x + \frac{C_1x^2}{2} + C_2x + C_3 \\
EIw(x) &= \int [\int [\int [\int q(x)\mathrm{d}x]\mathrm{d}x]\mathrm{d}x]\mathrm{d}x + \frac{C_1x^3}{6} + \frac{C_2x^2}{2} + C_3x + C_4
\end{aligned}$$

The solution of the differential equation shows the manifestation of four constants of integration, C_1, C_2, C_3, and C_4. Boundary conditions specific to an engineering problem can be used to solve for these constants. We also see the dependence of the function on $q(x)$, the load (that needs to be integrated four times).
