Deriving the Euler-Bernoulli Simple Beam Theory

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MATH-080 - Differential Equations

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Problem 1: Symmetric/simple bending. $\frac{1}{\rho} = \frac{M(x)}{EI} = \frac{d^2y}{dx^2}$ where ρ is the radius of curvature, $M(x)$ is the internal bending moment, I the moment of inertia, and E is Young's modulus.

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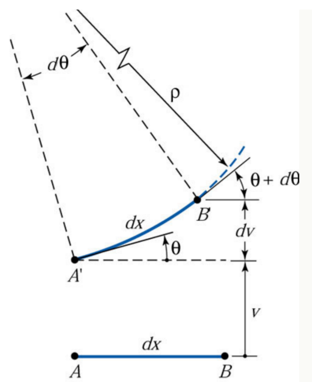
1. Deriving the differential equation:



From the above diagram, we can approximate the angle θ with dv and dx. We approximate (applying the small-angle approximation):

$$\frac{dv}{dx} = sin\theta \approx \theta$$

(where ρ is the radius of curvature)
Furthermore from the diagram, we can derive that:

$$dx = \rho d\theta$$

We can rewrite the previous as:

$$\frac{1}{\rho} = \frac{d\theta}{dx}$$

And substituting for θ results in the differential equation:

$$\frac{1}{\rho} = \frac{d^2y}{dx^2}$$

We know from the moment-curvature relationship that :

$$\frac{1}{\rho} = \frac{M(x)}{EI}$$

thus:

$$\frac{M(x)}{EI} = \frac{d^2y}{dx^2}$$

This is the differential equation of the elastic curve for a beam undergoing bending in the plane of symmetry. Its solution $y(x)$ represents the shape of the deflection curve for a given beam.

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2. Solving the differential equation:

We can solve this (separable) differential equation through double integration. Considering EI as a constant throughout the beam (EI is the "flexural rigidity", and varies per material), integration returns:

$$\begin{aligned}
\int EI\frac{d^2y}{dx^2}\mathrm{d}x &= \int M(x)\mathrm{d}x \\
EI\frac{dy}{dx} &= \int M(x)\mathrm{d}x + C1 \\
y &= \frac{\int[\int M(x)\mathrm{d}x]\mathrm{d}x}{EI} + C1x + C2
\end{aligned}$$

Solving the differential equation shows the appearance of two constant of integration **C1** and **C2**. We can use boundary conditions relative to the deflection at length $x = 0$ and $x = L$ to solve this.

NOTE: If the flexural rigidity EI is not constant over the beam, new differential equations must be calculated for each segment of the beam.

A worked-out examples of this equation can be found in appendix A.

Problem 2: Static beam bending under applied load. $\frac{d}{dx^2}[EI\frac{d^2w}{dx^2}] = p$ The previous derivation is valid for beams undergoing bending in the plane of symmetry. Beams undergoing distributed loading (here represented by p) require the derivation of another differential equation.

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1. Deriving the differential equation:

a) Assumption:

- The product EI (flexural rigidity) does not vary along the length of the beam.

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