Simple v/s non-simple repair policies for systems under simultaneous failures of its components

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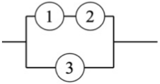

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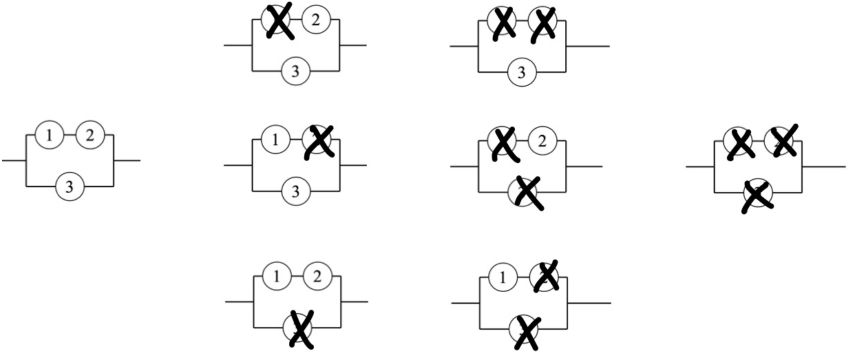

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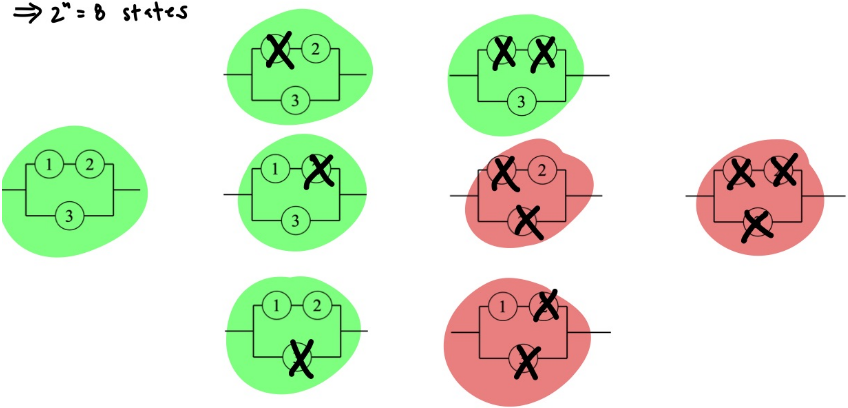

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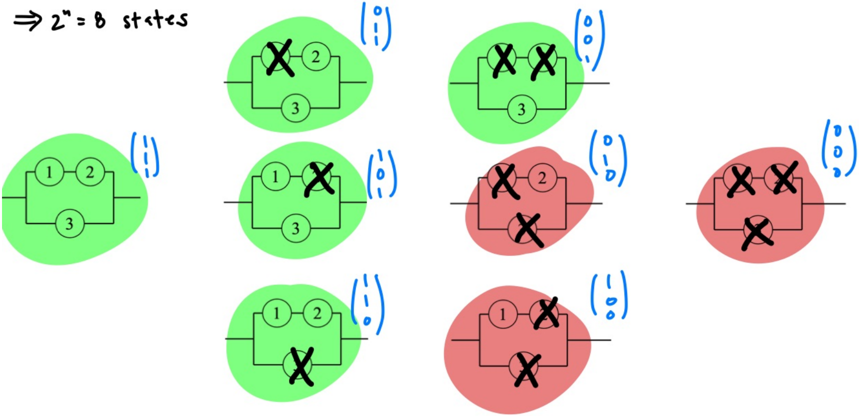

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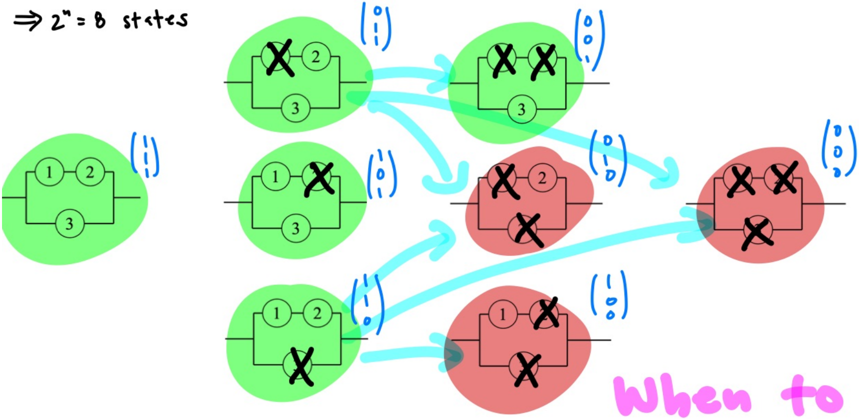

When to repair?

Vanilla MDP: 2^{2^n} policies

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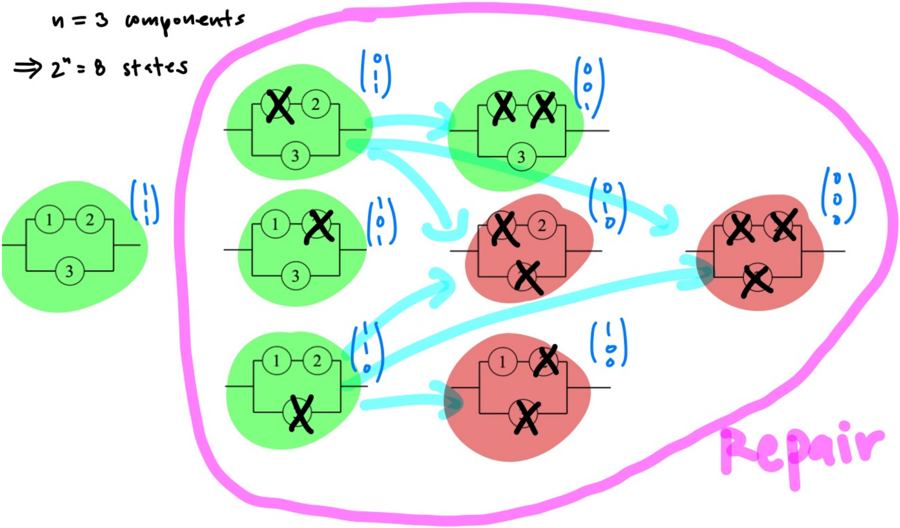

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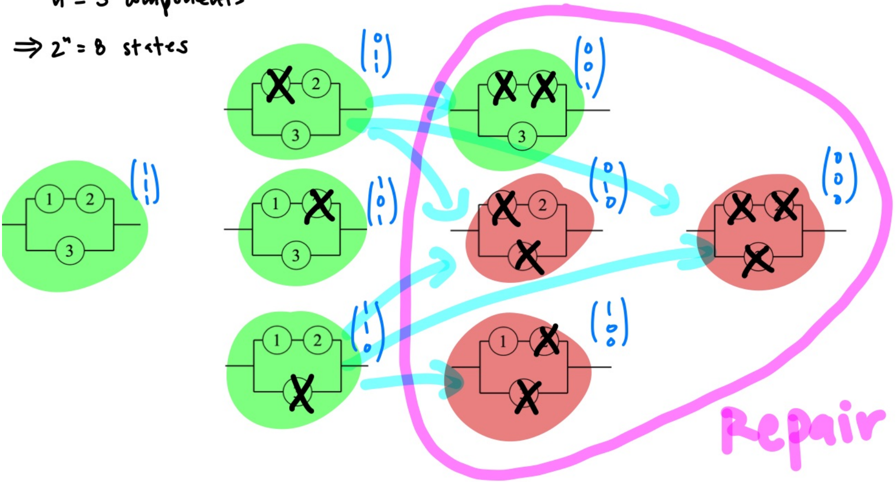

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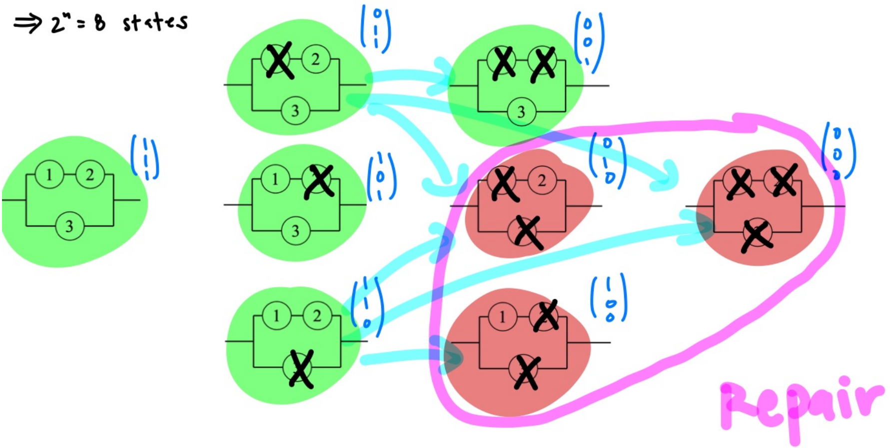

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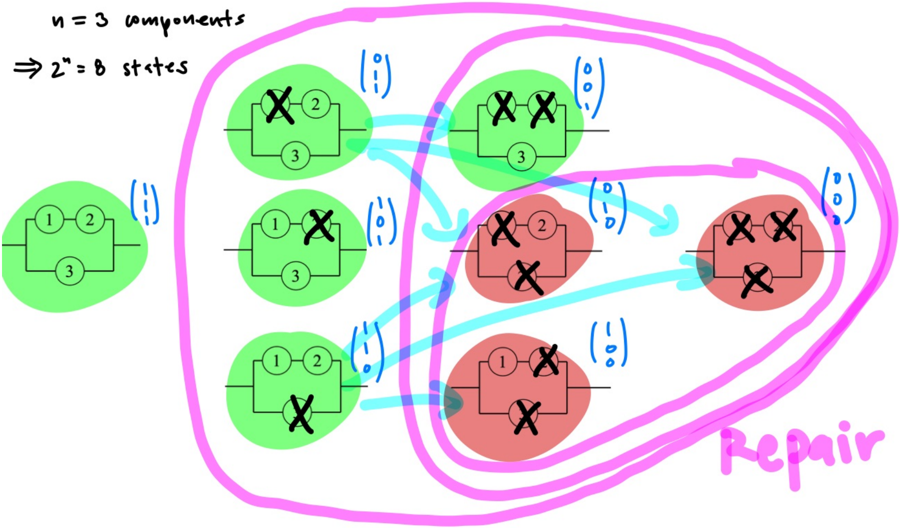

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Simple "r-out-of-n:R" repair : n policies

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First result for simultaneous failures and/or general monotone systems!

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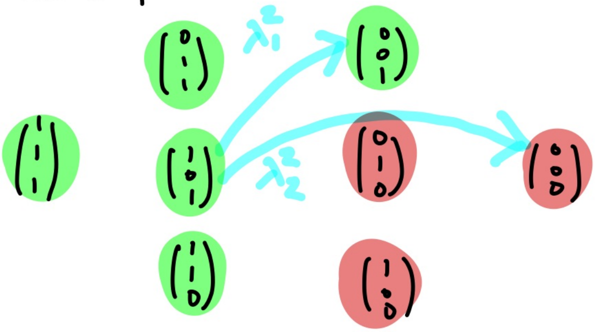

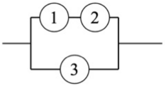

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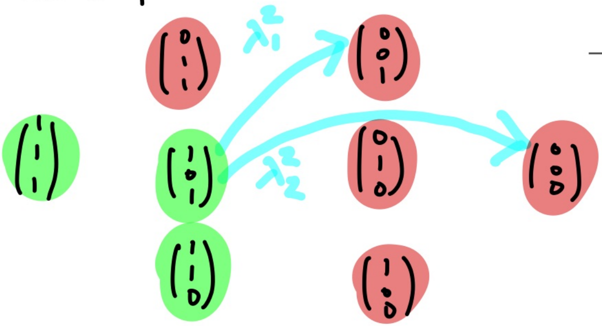

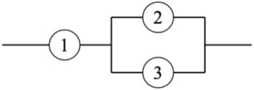

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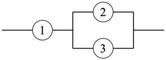

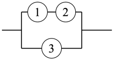

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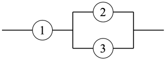

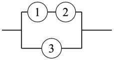

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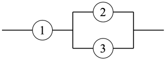

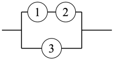

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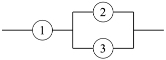

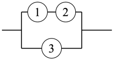

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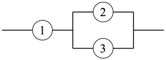

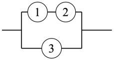

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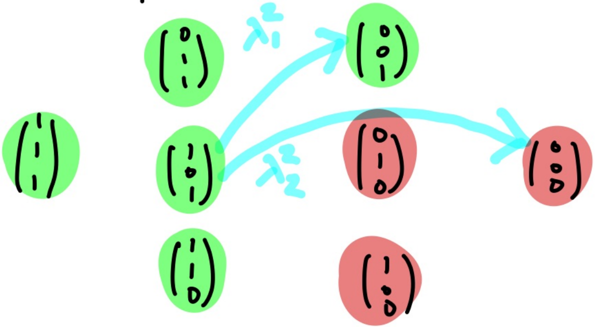

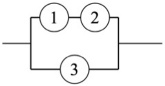

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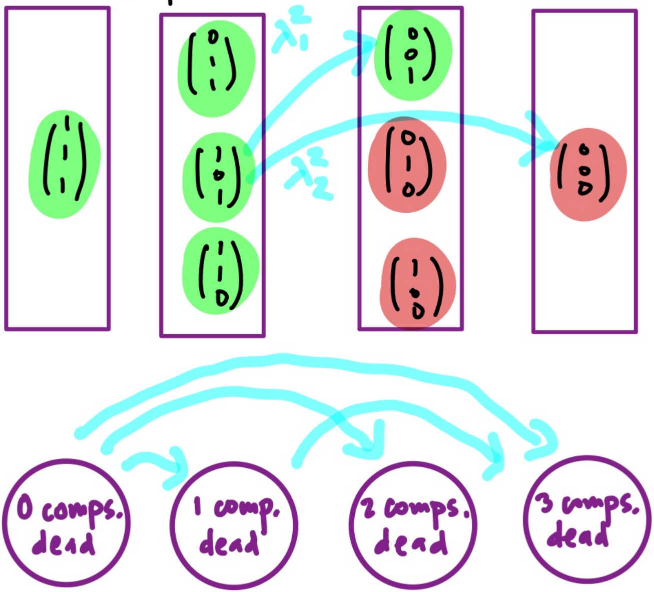

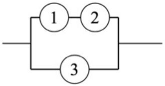

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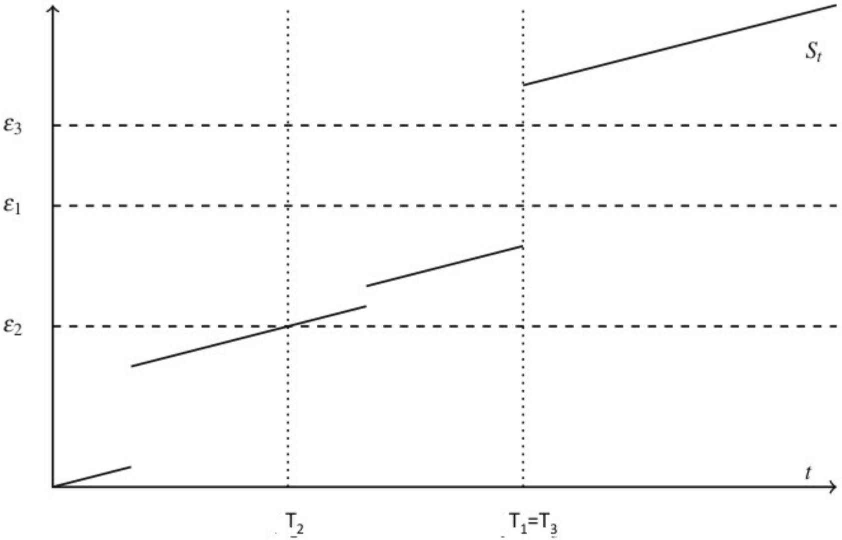

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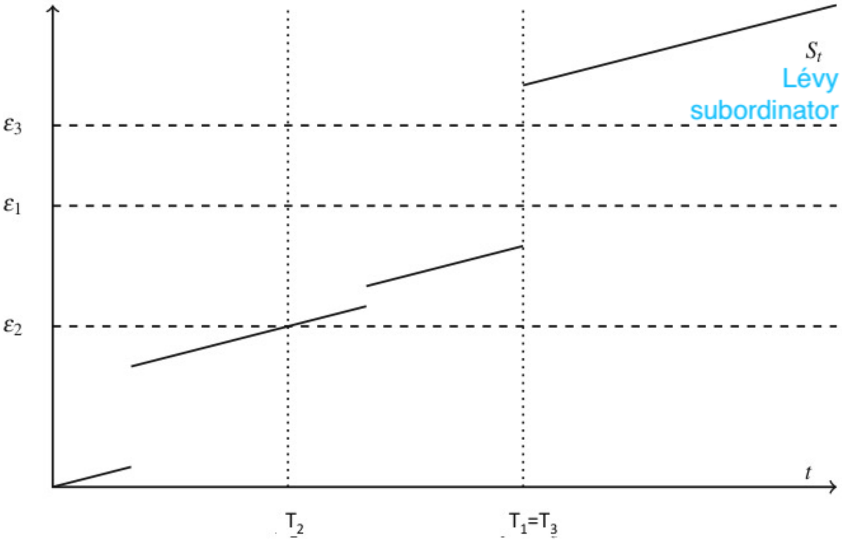

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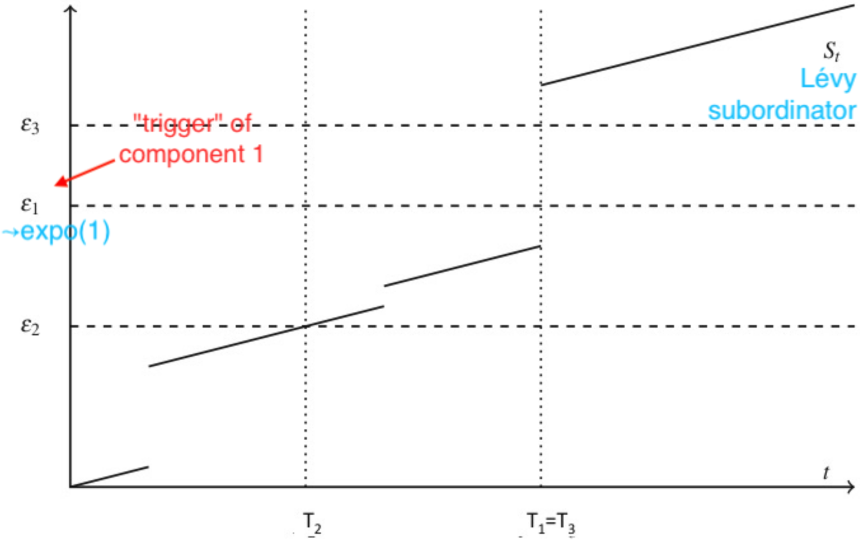

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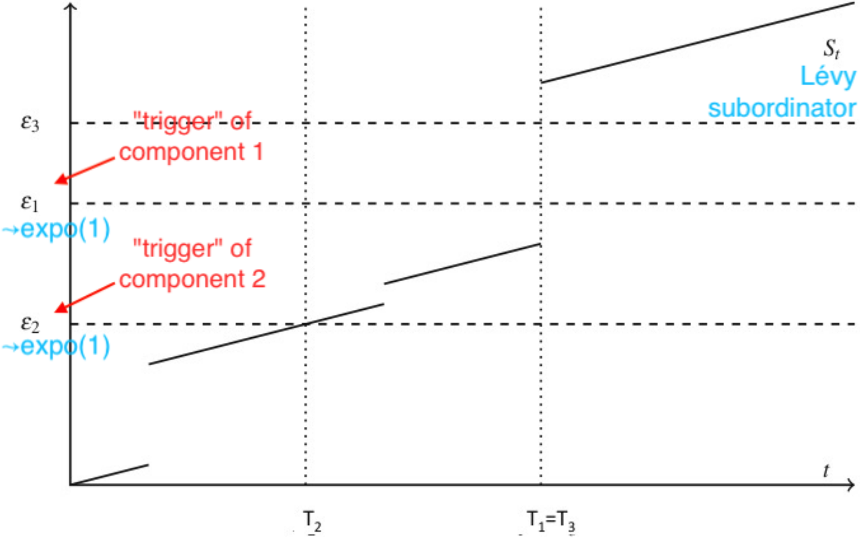

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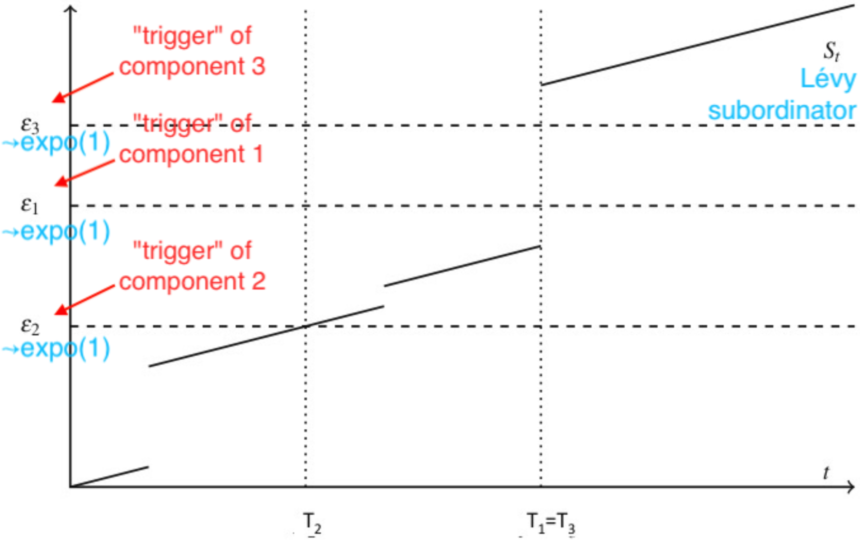

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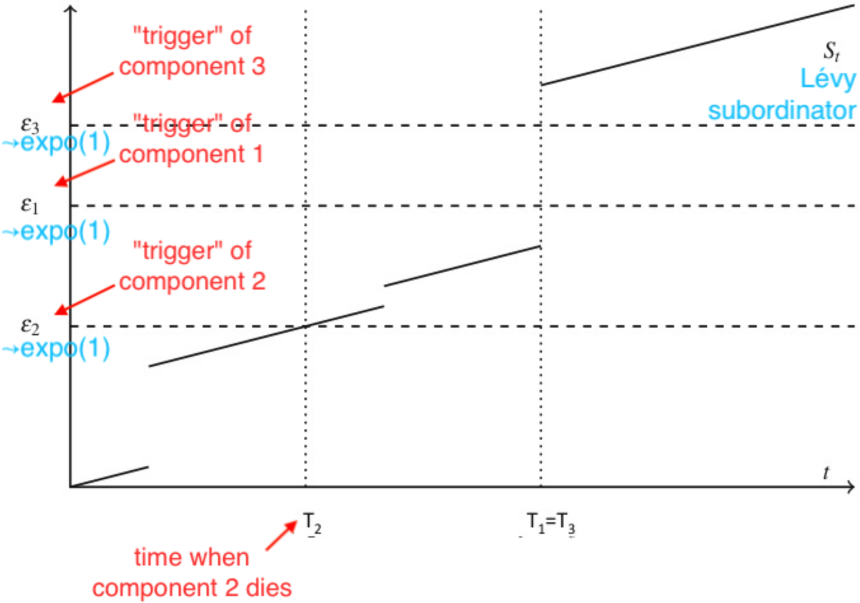

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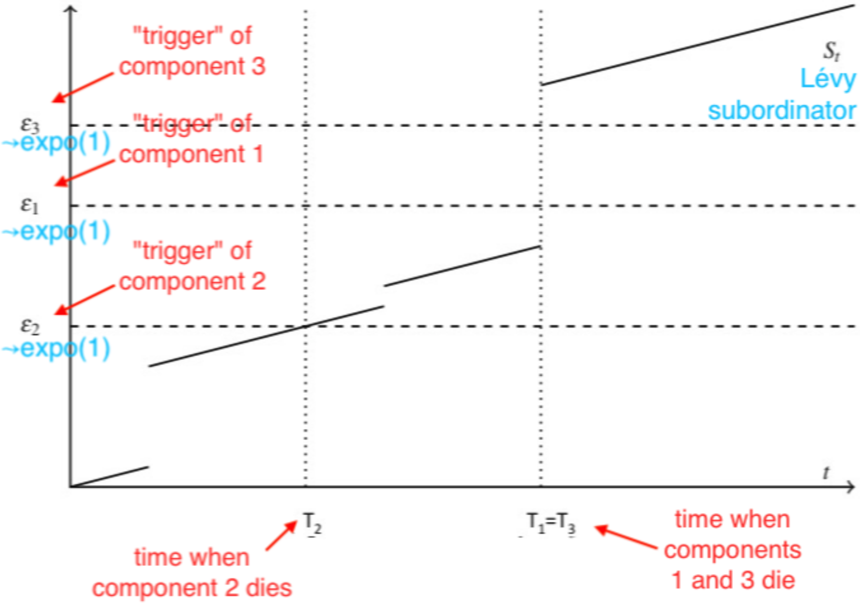

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Consider a system with n components subject to failures.

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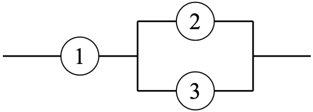
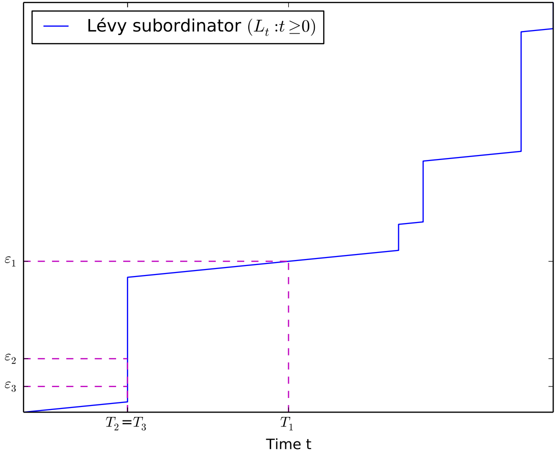
☺ Markovian structure! ☺ A Marshall-Olkin (shock) model! ☺ **Very few parameters!**

Examples of systems





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Assume that:

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Theorem (idea)

Setting:

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- Ingredients of proof: Samaniego decomposition + Markov chain analysis

Formulas of the main result

For

- $C[0, t]$ = total cost up to time t
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- $\mathbf{s}$ = structural signature vector of the system

it holds that

$$\lim_{t\to\infty} \frac{C[0,t]}{t} = \frac{\mathbb{E}\, C[0, T_{rep}^{(r)}]}{\mathbb{E}\, T_{rep}^{(r)}} \qquad \text{almost surely,}$$

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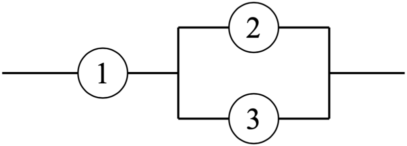
Example I

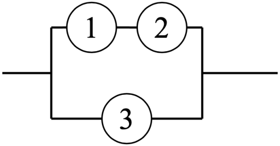
3 components, $c_{comp} = 1$, $c_{sys} = 10$



r	Probab. of failure before repair	Rate of component failures	Expected time to system failure	Long-term average cost
1	0.42	4.50	0.632	20.33
2	1.00	3.83	0.483	24.52
3	1.00	3.83	0.483	24.52

Example II

3 components, $c_{comp} = 1$, $c_{sys} = 10$



r	Probab. of failure before repair	Rate of component failures	Expected time to system failure	Long-term average cost
1	0.111	4.50	2.40	8.66
2	0.725	3.68	0.816	15.93
3	1.0	3.16	0.775	16.06

Example II — simulations

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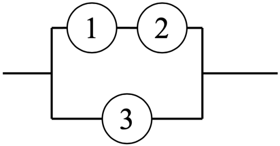

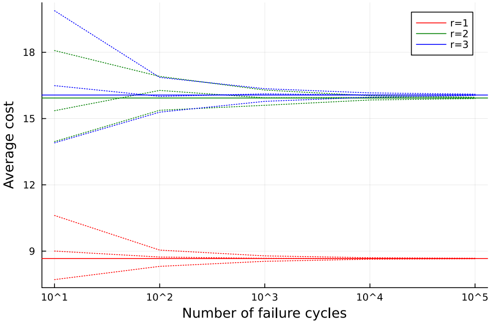

Example II — comparison with MDP solutions

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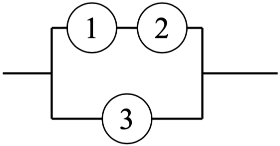
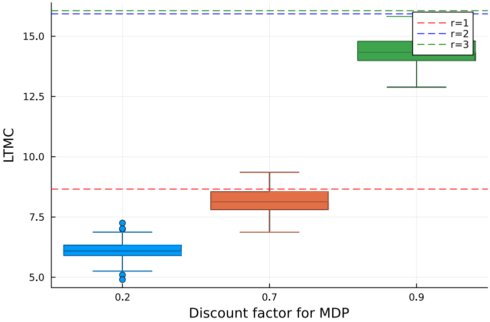




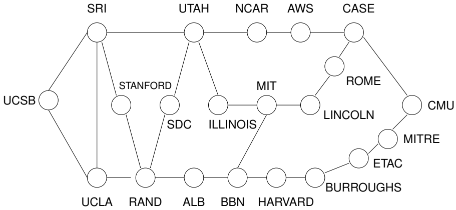
Example III

A random mixed monotone system with 10 components.

r	Probab. of failure before repair	Rate of component failures	Expected time to system failure	Long-term average cost
1	0.183	15.0	17.5	34.9
2	0.254	14.3	19.7	29.1
3	0.387	13.6	18.8	29.1
4	0.492	12.9	18.4	28.3
5	0.614	12.2	17.7	28.1
6	0.734	11.6	17.0	28.2
7	0.776	11.1	16.9	26.9
8	0.835	10.5	16.8	25.9
9	0.897	9.8	16.5	24.9
10	1.0	8.9	15.9	24.0

Example IV

RAND network: arcs fail and system works when there is path UCSB→CMU:



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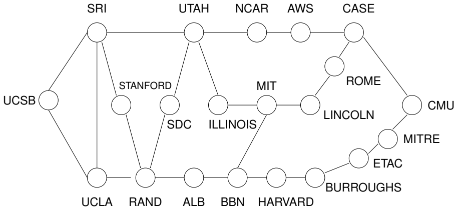

- $n = 26$ arcs

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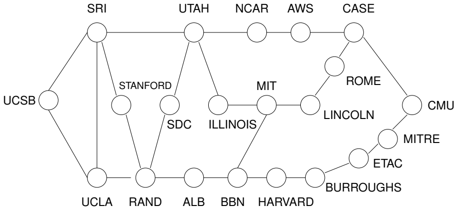

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  First state (“all components working”) has $2^n - 1$ successors

  Total: $3^n - 2^n = 2\,541\,798\,719\,465$ transitions

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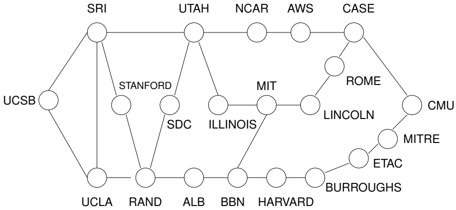

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- Trivial computation for best r-out-of-n policy

Summary & main takeaways

- Setting considered:
  - Failure times: components fail according to *Lévy-frailty Marshall-Olkin* (LFMO) model for **simultaneous failures**
  - System structure: **monotone system**
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- Comparison with MDP-obtained policies:
  - Competitive, depending on choice of discount factor
  - Bypasses exponential number of states and transitions

Thanks! — keep in touch! guidolagos.github.io / guido.lagos@uai.cl