

Outline

- 1 Graph Algorithms
- 2 Graph Representations
- 3 Breadth First Search (BFS)
- 4 Depth First Search (DFS)
- 5 Topological Sort
- 6 Strong Connectivity
- 7 DFS for Undirected Graphs
- 8 Biconnectivity Problem

Graph Algorithms

- Many problems in CS can be modeled as **graph problems**.

Graph Algorithms

- Many problems in CS can be modeled as **graph problems**.
- Algorithms for solving graph problems are fundamental to the field of algorithm design.

- Many problems in CS can be modeled as **graph problems**.
- Algorithms for solving graph problems are fundamental to the field of algorithm design.

Definition

- A graph $G = (V, E)$ consists of a **vertex set** V and an **edge set** E . $|V| = n$ and $|E| = m$.

- Many problems in CS can be modeled as **graph problems**.
- Algorithms for solving graph problems are fundamental to the field of algorithm design.

Definition

- A graph $G = (V, E)$ consists of a **vertex set** V and an **edge set** E . $|V| = n$ and $|E| = m$.
- Each edge $e = (x, y) \in E$ is an **unordered pair of vertices**.

Graph Algorithms

- Many problems in CS can be modeled as **graph problems**.
- Algorithms for solving graph problems are fundamental to the field of algorithm design.

Definition

- A graph $G = (V, E)$ consists of a **vertex set** V and an **edge set** E . $|V| = n$ and $|E| = m$.
- Each edge $e = (x, y) \in E$ is an **unordered pair of vertices**.
- If $(u, v) \in E$, we say v is a **neighbor of u** .

Graph Algorithms

- Many problems in CS can be modeled as **graph problems**.
- Algorithms for solving graph problems are fundamental to the field of algorithm design.

Definition

- A graph $G = (V, E)$ consists of a **vertex set** V and an **edge set** E . $|V| = n$ and $|E| = m$.
- Each edge $e = (x, y) \in E$ is an **unordered pair of vertices**.
- If $(u, v) \in E$, we say v is a **neighbor of u** .
- The **degree** $\deg(u)$ of a vertex u is the number of edges **incident to u** .

Fact

$$\sum_{v \in V} \deg(v) = 2m$$

This is because, for each $e = (u, v)$, e is counted twice in the sum, once for $\deg(v)$ and once for $\deg(u)$.

Definition

- If the two end vertices of e are ordered, the edge is directed, and we write $e = xy$.

Definition

- If the two end vertices of e are ordered, the edge is directed, and we write $e = xy$.
- If all edges are directed, then G is a directed graph.

Directed Graphs

Definition

- If the two end vertices of e are ordered, the edge is directed, and we write $e = xy$.
- If all edges are directed, then G is a directed graph.
- The in-degree $\deg_{in}(u)$ of a vertex u is the number of edges that are directed into u .

Definition

- If the two **end vertices of e** are ordered, the edge is **directed**, and we write $e = xy$.
- If all edges are directed, then G is a **directed graph**.
- The **in-degree** $\deg_{in}(u)$ of a vertex u is the number of edges **that are directed into u** .
- The **out-degree** $\deg_{out}(u)$ of a vertex u is the number of edges **that are directed from u** .

Fact

$$\sum_{v \in V} \deg_{in}(v) = \sum_{v \in V} \deg_{out}(v) = m$$

This is because, for each $e = (u \rightarrow v)$, e is counted once ($\deg_{in}(v)$) in the sum of in-degrees, and once ($\deg_{out}(u)$) in the sum of out-degrees.

- The numbers $n (= |V|)$ and $m (= |E|)$ are two important parameters to describe the size of a graph.

Graph Algorithms

- The numbers n ($= |V|$) and m ($= |E|$) are two important parameters to describe the size of a graph.
- It is easy to see $0 \leq m \leq n^2$.

Graph Algorithms

- The numbers $n (= |V|)$ and $m (= |E|)$ are two important parameters to describe the size of a graph.
- It is easy to see $0 \leq m \leq n^2$.
- It is also easy to show: if G is a tree (namely undirected, connected graph with no cycles), then $m = n - 1$.

Graph Algorithms

- The numbers n ($= |V|$) and m ($= |E|$) are two important parameters to describe the size of a graph.
- It is easy to see $0 \leq m \leq n^2$.
- It is also easy to show: if G is a tree (namely undirected, connected graph with no cycles), then $m = n - 1$.
- If m is close to n , we say G is **sparse**. If m is close to n^2 , we say G is **dense**.

Graph Algorithms

- The numbers n ($= |V|$) and m ($= |E|$) are two important parameters to describe the size of a graph.
- It is easy to see $0 \leq m \leq n^2$.
- It is also easy to show: if G is a tree (namely undirected, connected graph with no cycles), then $m = n - 1$.
- If m is close to n , we say G is **sparse**. If m is close to n^2 , we say G is **dense**.
- Because n and m are rather independent to each other, we usually use both parameters to describe the runtime of a graph algorithm. Such as $O(n + m)$ or $O(n^{1/2}m)$.

Outline

- 1 Graph Algorithms
- 2 Graph Representations**
- 3 Breadth First Search (BFS)
- 4 Depth First Search (DFS)
- 5 Topological Sort
- 6 Strong Connectivity
- 7 DFS for Undirected Graphs
- 8 Biconnectivity Problem

Graph Representations

We mainly use two **graph representations**.

Adjacency Matrix Representation

We use a 2D array $A[1..n, 1..n]$ to represent $G = (V, E)$:

$$A[i,j] = \begin{cases} 1 & \text{if } (v_i, v_j) \in E \\ 0 & \text{if } (v_i, v_j) \notin E \end{cases}$$

Graph Representations

We mainly use two **graph representations**.

Adjacency Matrix Representation

We use a 2D array $A[1..n, 1..n]$ to represent $G = (V, E)$:

$$A[i, j] = \begin{cases} 1 & \text{if } (v_i, v_j) \in E \\ 0 & \text{if } (v_i, v_j) \notin E \end{cases}$$

- Sometimes, there are other information associated with the edges. For example, each edge $e = (v_i, v_j)$ may have a **weight** $w(e) = w(v_i, v_j)$ (for example, MST). In this case, we set $A[i, j] = w(v_i, v_j)$.

Graph Representations

We mainly use two **graph representations**.

Adjacency Matrix Representation

We use a 2D array $A[1..n, 1..n]$ to represent $G = (V, E)$:

$$A[i,j] = \begin{cases} 1 & \text{if } (v_i, v_j) \in E \\ 0 & \text{if } (v_i, v_j) \notin E \end{cases}$$

- Sometimes, there are other information associated with the edges. For example, each edge $e = (v_i, v_j)$ may have a **weight** $w(e) = w(v_i, v_j)$ (for example, MST). In this case, we set $A[i,j] = w(v_i, v_j)$.
- For undirected graph, A is always **symmetric**.

Graph Representations

We mainly use two **graph representations**.

Adjacency Matrix Representation

We use a 2D array $A[1..n, 1..n]$ to represent $G = (V, E)$:

$$A[i, j] = \begin{cases} 1 & \text{if } (v_i, v_j) \in E \\ 0 & \text{if } (v_i, v_j) \notin E \end{cases}$$

- Sometimes, there are other information associated with the edges. For example, each edge $e = (v_i, v_j)$ may have a **weight** $w(e) = w(v_i, v_j)$ (for example, MST). In this case, we set $A[i, j] = w(v_i, v_j)$.
- For undirected graph, A is always **symmetric**.
- The Adjacency Matrix Representation for **directed graph** is similar.
 $A[i, j] = 1$ (or $w(v_i, v_j)$ if G has edge weights) iff $v_i \rightarrow v_j \in E$.

Graph Representations

We mainly use two **graph representations**.

Adjacency Matrix Representation

We use a 2D array $A[1..n, 1..n]$ to represent $G = (V, E)$:

$$A[i, j] = \begin{cases} 1 & \text{if } (v_i, v_j) \in E \\ 0 & \text{if } (v_i, v_j) \notin E \end{cases}$$

- Sometimes, there are other information associated with the edges. For example, each edge $e = (v_i, v_j)$ may have a **weight** $w(e) = w(v_i, v_j)$ (for example, MST). In this case, we set $A[i, j] = w(v_i, v_j)$.
- For undirected graph, A is always **symmetric**.
- The Adjacency Matrix Representation for **directed graph** is similar.
 $A[i, j] = 1$ (or $w(v_i, v_j)$ if G has edge weights) iff $v_i \rightarrow v_j \in E$.
- For directed graphs, $A[*, *]$ is not necessarily symmetric.

Graph Representations

Adjacency List Representation

- For each vertex $v \in V$, there's a **linked list** $Adj[v]$. Each entry of $Adj[v]$ is a vertex w such that $(v, w) \in E$.

Adjacency List Representation

- For each vertex $v \in V$, there's a **linked list** $Adj[v]$. Each entry of $Adj[v]$ is a vertex w such that $(v, w) \in E$.
- If there are other information associated with the edges (such as edge weight), they can be stored in the entries of the adjacency list.

Adjacency List Representation

- For each vertex $v \in V$, there's a **linked list** $Adj[v]$. Each entry of $Adj[v]$ is a vertex w such that $(v, w) \in E$.
- If there are other information associated with the edges (such as edge weight), they can be stored in the entries of the adjacency list.
- For undirected graphs, each edge $e = (u, v)$ has **two entries in this representation, one in $Adj[u]$ and one in $Adj[v]$** .

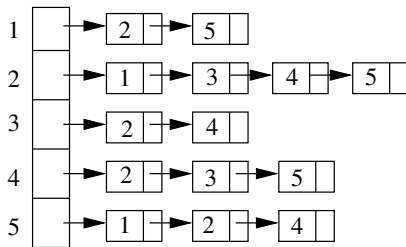
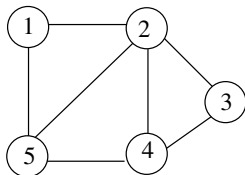
Adjacency List Representation

- For each vertex $v \in V$, there's a **linked list** $Adj[v]$. Each entry of $Adj[v]$ is a vertex w such that $(v, w) \in E$.
- If there are other information associated with the edges (such as edge weight), they can be stored in the entries of the adjacency list.
- For undirected graphs, each edge $e = (u, v)$ has **two entries in this representation, one in $Adj[u]$ and one in $Adj[v]$** .
- The Adjacency List Representation for **directed graphs** is similar. For each edge $e = u \rightarrow v$, there is an entry in $Adj[u]$.

Adjacency List Representation

- For each vertex $v \in V$, there's a **linked list** $Adj[v]$. Each entry of $Adj[v]$ is a vertex w such that $(v, w) \in E$.
- If there are other information associated with the edges (such as edge weight), they can be stored in the entries of the adjacency list.
- For undirected graphs, each edge $e = (u, v)$ has **two entries in this representation, one in $Adj[u]$ and one in $Adj[v]$** .
- The Adjacency List Representation for **directed graphs** is similar. For each edge $e = u \rightarrow v$, there is an entry in $Adj[u]$.
- **For directed graphs, each edge has only one entry in the representation.**

Example



	1	2	3	4	5
1	0	1	0	0	1
2	1	0	1	1	1
3	0	1	0	1	0
4	0	1	1	0	1
5	1	1	0	1	0

Comparisons of Representations

Graph algorithms often need the representation to support two operations.

Comparisons of Representations

Graph algorithms often need the representation to support two operations.

Neighbor Testing

Given two vertices u and v , is $(u, v) \in E$?

Comparisons of Representations

Graph algorithms often need the representation to support two operations.

Neighbor Testing

Given two vertices u and v , is $(u, v) \in E$?

Neighbor Listing

Given a vertex u , list all neighbors of u .

Comparisons of Representations

Graph algorithms often need the representation to support two operations.

Neighbor Testing

Given two vertices u and v , is $(u, v) \in E$?

Neighbor Listing

Given a vertex u , list all neighbors of u .

When deciding which representation to use, we need to consider:

Comparisons of Representations

Graph algorithms often need the representation to support two operations.

Neighbor Testing

Given two vertices u and v , is $(u, v) \in E$?

Neighbor Listing

Given a vertex u , list all neighbors of u .

When deciding which representation to use, we need to consider:

- The space needed for the representation.

Comparisons of Representations

Graph algorithms often need the representation to support two operations.

Neighbor Testing

Given two vertices u and v , is $(u, v) \in E$?

Neighbor Listing

Given a vertex u , list all neighbors of u .

When deciding which representation to use, we need to consider:

- The space needed for the representation.
- How well the representation supports the two basic operations.

Comparisons of Representations

Graph algorithms often need the representation to support two operations.

Neighbor Testing

Given two vertices u and v , is $(u, v) \in E$?

Neighbor Listing

Given a vertex u , list all neighbors of u .

When deciding which representation to use, we need to consider:

- The space needed for the representation.
- How well the representation supports the two basic operations.
- How easy to implement.

Adjacency List

- Space:
 - Each entry in the list needs $O(1)$ space.

Adjacency List

- Space:
 - Each entry in the list needs $O(1)$ space.
 - Each edge has two entries in the representation. So there are totally $2m$ entries in the representation.

Adjacency List

- Space:
 - Each entry in the list needs $O(1)$ space.
 - Each edge has two entries in the representation. So there are totally $2m$ entries in the representation.
 - We also need $O(n)$ space for the headers of the lists.

Adjacency List

- Space:
 - Each entry in the list needs $O(1)$ space.
 - Each edge has two entries in the representation. So there are totally $2m$ entries in the representation.
 - We also need $O(n)$ space for the headers of the lists.
 - Total Space: $\Theta(m + n)$.

Adjacency List

- Space:
 - Each entry in the list needs $O(1)$ space.
 - Each edge has two entries in the representation. So there are totally $2m$ entries in the representation.
 - We also need $O(n)$ space for the headers of the lists.
 - Total Space: $\Theta(m + n)$.
- Neighbor Testing: $O(\deg(v))$ time. (We need to go through $Adj(v)$ to see if another vertex u is in there.)

Adjacency List

- Space:
 - Each entry in the list needs $O(1)$ space.
 - Each edge has two entries in the representation. So there are totally $2m$ entries in the representation.
 - We also need $O(n)$ space for the headers of the lists.
 - Total Space: $\Theta(m + n)$.
- Neighbor Testing: $O(deg(v))$ time. (We need to go through $Adj(v)$ to see if another vertex u is in there.)
- Neighbor Listing: $O(deg(v))$. (We need to go through $Adj(v)$ to list all neighbors of v .)

Adjacency List

- Space:
 - Each entry in the list needs $O(1)$ space.
 - Each edge has two entries in the representation. So there are totally $2m$ entries in the representation.
 - We also need $O(n)$ space for the headers of the lists.
 - Total Space: $\Theta(m + n)$.
- Neighbor Testing: $O(deg(v))$ time. (We need to go through $Adj(v)$ to see if another vertex u is in there.)
- Neighbor Listing: $O(deg(v))$. (We need to go through $Adj(v)$ to list all neighbors of v .)
- More complex.

Comparisons of Representations

Adjacency Matrix

- Space: $\Theta(n^2)$, independent from the number of edges.

Comparisons of Representations

Adjacency Matrix

- Space: $\Theta(n^2)$, independent from the number of edges.
- Neighbor Testing: $O(1)$ time. (Just look at $A[i,j]$.)

Comparisons of Representations

Adjacency Matrix

- Space: $\Theta(n^2)$, independent from the number of edges.
- Neighbor Testing: $O(1)$ time. (Just look at $A[i,j]$.)
- Neighbor Listing: $\Theta(n)$. (We have to look the entire row i in A to list the neighbors of the vertex i .)

Comparisons of Representations

Adjacency Matrix

- Space: $\Theta(n^2)$, independent from the number of edges.
- Neighbor Testing: $O(1)$ time. (Just look at $A[i,j]$.)
- Neighbor Listing: $\Theta(n)$. (We have to look the entire row i in A to list the neighbors of the vertex i .)
- Easy to implement.

Comparisons of Representations

Adjacency Matrix

- Space: $\Theta(n^2)$, independent from the number of edges.
 - Neighbor Testing: $O(1)$ time. (Just look at $A[i,j]$.)
 - Neighbor Listing: $\Theta(n)$. (We have to look the entire row i in A to list the neighbors of the vertex i .)
 - Easy to implement.
-
- If an algorithm needs neighbor testing more often than the neighbor listing, we should use Adj Matrix.

Comparisons of Representations

Adjacency Matrix

- Space: $\Theta(n^2)$, independent from the number of edges.
 - Neighbor Testing: $O(1)$ time. (Just look at $A[i,j]$.)
 - Neighbor Listing: $\Theta(n)$. (We have to look the entire row i in A to list the neighbors of the vertex i .)
 - Easy to implement.
-
- If an algorithm needs neighbor testing more often than the neighbor listing, we should use Adj Matrix.
 - If an algorithm needs neighbor testing less often than the neighbor listing, we should use Adj List.

Comparisons of Representations

Adjacency Matrix

- Space: $\Theta(n^2)$, independent from the number of edges.
 - Neighbor Testing: $O(1)$ time. (Just look at $A[i,j]$.)
 - Neighbor Listing: $\Theta(n)$. (We have to look the entire row i in A to list the neighbors of the vertex i .)
 - Easy to implement.
-
- If an algorithm needs neighbor testing more often than the neighbor listing, we should use Adj Matrix.
 - If an algorithm needs neighbor testing less often than the neighbor listing, we should use Adj List.
 - If we use Adj Matrix, the algorithm takes at least $\Omega(n^2)$ time since even set up the representation data structure requires this much time.

Comparisons of Representations

Adjacency Matrix

- Space: $\Theta(n^2)$, independent from the number of edges.
 - Neighbor Testing: $O(1)$ time. (Just look at $A[i,j]$.)
 - Neighbor Listing: $\Theta(n)$. (We have to look the entire row i in A to list the neighbors of the vertex i .)
 - Easy to implement.
-
- If an algorithm needs neighbor testing more often than the neighbor listing, we should use Adj Matrix.
 - If an algorithm needs neighbor testing less often than the neighbor listing, we should use Adj List.
 - If we use Adj Matrix, the algorithm takes at least $\Omega(n^2)$ time since even set up the representation data structure requires this much time.
 - If we use Adj List, it is possible the algorithm can run in linear $\Theta(m + n)$ time.

Outline

- 1 Graph Algorithms
- 2 Graph Representations
- 3 Breadth First Search (BFS)**
- 4 Depth First Search (DFS)
- 5 Topological Sort
- 6 Strong Connectivity
- 7 DFS for Undirected Graphs
- 8 Biconnectivity Problem

Breadth First Search (BFS)

BFS is a simple algorithm that **travels the vertices of a given graph in a systematic way**. Roughly speaking, it works like this:

Breadth First Search (BFS)

BFS is a simple algorithm that **travels the vertices of a given graph in a systematic way**. Roughly speaking, it works like this:

- It starts at a given **starting vertex s** .

Breadth First Search (BFS)

BFS is a simple algorithm that **travels the vertices of a given graph in a systematic way**. Roughly speaking, it works like this:

- It starts at a given **starting vertex s** .
- From s , we visits **all neighbors of s** .

Breadth First Search (BFS)

BFS is a simple algorithm that **travels the vertices of a given graph in a systematic way**. Roughly speaking, it works like this:

- It starts at a given **starting vertex s** .
- From s , we visit **all neighbors of s** .
- These neighbors are placed in a **queue Q** .

Breadth First Search (BFS)

BFS is a simple algorithm that **travels the vertices of a given graph in a systematic way**. Roughly speaking, it works like this:

- It starts at a given **starting vertex s** .
- From s , we visits **all neighbors of s** .
- These neighbors are placed in a **queue Q** .
- Then the first vertex u in Q is considered. All neighbors of u **that have not been visited yet** are visited, and are placed in Q ...

Breadth First Search (BFS)

BFS is a simple algorithm that **travels the vertices of a given graph in a systematic way**. Roughly speaking, it works like this:

- It starts at a given **starting vertex s** .
- From s , we visits **all neighbors of s** .
- These neighbors are placed in a **queue Q** .
- Then the first vertex u in Q is considered. All neighbors of u **that have not been visited yet** are visited, and are placed in Q ...
- When finished, it builds a **spanning tree (called BFS tree)**.

Breadth First Search (BFS)

BFS is a simple algorithm that **travels the vertices of a given graph in a systematic way**. Roughly speaking, it works like this:

- It starts at a given **starting vertex s** .
- From s , we visits **all neighbors of s** .
- These neighbors are placed in a **queue Q** .
- Then the first vertex u in Q is considered. All neighbors of u **that have not been visited yet** are visited, and are placed in Q ...
- When finished, it builds a **spanning tree (called BFS tree)**.

Before describing details, we need to pick a graph representation. Because we need to **visit all neighbors of a vertex**, it seems we need the **neighbor listing operation**. So we use **Adj list** representation.

Input: An undirected graph $G = (V, E)$ given by Adj List.
 s : the starting vertex.

Input: An undirected graph $G = (V, E)$ given by Adj List.
 s : the starting vertex.

Basic Data Structures: For each vertex $u \in V$, we have

- $\text{Adj}[u]$: the Adj list for u .

Input: An undirected graph $G = (V, E)$ given by Adj List.
 s : the starting vertex.

Basic Data Structures: For each vertex $u \in V$, we have

- $\text{Adj}[u]$: the Adj list for u .
- $\text{color}[u]$: It can be one of the following;
 - white, (u has not been visited yet.)

Input: An undirected graph $G = (V, E)$ given by Adj List.
 s : the starting vertex.

Basic Data Structures: For each vertex $u \in V$, we have

- $\text{Adj}[u]$: the Adj list for u .
- $\text{color}[u]$: It can be one of the following;
 - white, (u has not been visited yet.)
 - grey, (u has been visited, but some neighbors of u have not been visited yet.)

Input: An undirected graph $G = (V, E)$ given by Adj List.
 s : the starting vertex.

Basic Data Structures: For each vertex $u \in V$, we have

- $\text{Adj}[u]$: the Adj list for u .
- $\text{color}[u]$: It can be one of the following;
 - white, (u has not been visited yet.)
 - grey, (u has been visited, but some neighbors of u have not been visited yet.)
 - black, (u and all neighbors of u have been visited.)

Input: An undirected graph $G = (V, E)$ given by Adj List.
 s : the starting vertex.

Basic Data Structures: For each vertex $u \in V$, we have

- $\text{Adj}[u]$: the Adj list for u .
- $\text{color}[u]$: It can be one of the following;
 - white, (u has not been visited yet.)
 - grey, (u has been visited, but some neighbors of u have not been visited yet.)
 - black, (u and all neighbors of u have been visited.)
- $\pi[u]$: the parent of u in the BFS tree.

Input: An undirected graph $G = (V, E)$ given by Adj List.
 s : the starting vertex.

Basic Data Structures: For each vertex $u \in V$, we have

- $\text{Adj}[u]$: the Adj list for u .
- $\text{color}[u]$: It can be one of the following;
 - white, (u has not been visited yet.)
 - grey, (u has been visited, but some neighbors of u have not been visited yet.)
 - black, (u and all neighbors of u have been visited.)
- $\pi[u]$: the parent of u in the BFS tree.
- $d[u]$: the distance from u to the starting vertex s .

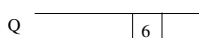
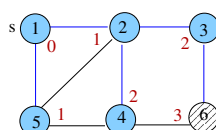
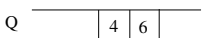
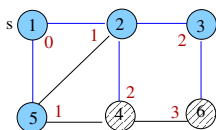
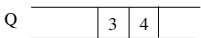
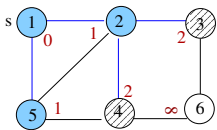
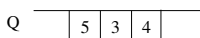
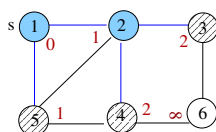
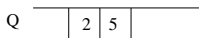
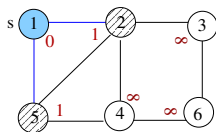
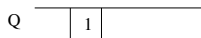
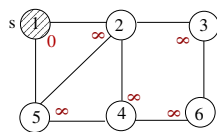
In addition, we also use a queue Q as mentioned earlier.

BFS: Algorithm

BFS(G, s)

- 1 $Q \leftarrow \emptyset$
- 2 **for** each $u \in V - \{s\}$ **do**
- 3 $\pi[u] = \text{NIL}; \quad d[u] = \infty; \quad \text{color}[u] = \text{white}$
- 4 $d[s] = 0; \quad \text{color}[s] = \text{grey}; \quad \pi[s] = \text{NIL}$
- 5 **Enqueue**(Q, s)
- 6 **while** $Q \neq \emptyset$ **do**
- 7 $u \leftarrow \text{Dequeue}(Q)$
- 8 **for** each $v \in \text{Adj}[u]$ **do**
- 9 **if** $\text{color}[v] = \text{white}$
- 10 **then** $\text{color}[v] = \text{grey}; \quad d[v] \leftarrow d[u] + 1; \quad \pi[v] \leftarrow u; \quad \text{Enqueue}(Q, v)$
- 11 $\text{color}[u] = \text{black}$

BFS: Example



2:
d[u] value

edges in BFS tree

white

grey

black

BFS: Algorithm

- BFS is not unique.

BFS: Algorithm

- BFS is not unique.
- The execution depends on the order in which the neighbors of a vertex i appear in $\text{Adj}(i)$.

BFS: Algorithm

- BFS is not unique.
- The execution depends on the order in which the neighbors of a vertex i appear in $\text{Adj}(i)$.
- In the example above, the neighbors of i appear in $\text{Adj}(i)$ in increasing order.

BFS: Algorithm

- BFS is not unique.
- The execution depends on the order in which the neighbors of a vertex i appear in $\text{Adj}(i)$.
- In the example above, the neighbors of i appear in $\text{Adj}(i)$ in increasing order.
- If the order is different, then the progress of the BFS algorithm would be different. And the BFS tree T constructed by the algorithm would be different.

BFS: Algorithm

- BFS is not unique.
- The execution depends on the order in which the neighbors of a vertex i appear in $\text{Adj}(i)$.
- In the example above, the neighbors of i appear in $\text{Adj}(i)$ in increasing order.
- If the order is different, then the progress of the BFS algorithm would be different. And the BFS tree T constructed by the algorithm would be different.
- However, regardless of which order we use, the properties of the BFS algorithm and BFS tree are always true.

BFS: Analysis

- Lines 1 and 5: The queue operations take $O(1)$ time.

BFS: Analysis

- Lines 1 and 5: The queue operations take $O(1)$ time.
- Line 2-3: Loop takes $\Theta(n)$ time.

BFS: Analysis

- Lines 1 and 5: The queue operations take $O(1)$ time.
- Line 2-3: Loop takes $\Theta(n)$ time.
- Lines 4: $O(1)$ time.

BFS: Analysis

- Lines 1 and 5: The queue operations take $O(1)$ time.
- Line 2-3: Loop takes $\Theta(n)$ time.
- Lines 4: $O(1)$ time.
- Lines 6-11:
 - Each vertex is **enqueued and dequeued exactly once**.

BFS: Analysis

- Lines 1 and 5: The queue operations take $O(1)$ time.
- Line 2-3: Loop takes $\Theta(n)$ time.
- Lines 4: $O(1)$ time.
- Lines 6-11:
 - Each vertex is **enqueued and dequeued exactly once**.
 - Since each queue operation takes $O(1)$ time, **the total time needed for all queue operations is $\Theta(n)$** .

BFS: Analysis

- Lines 1 and 5: The queue operations take $O(1)$ time.
- Line 2-3: Loop takes $\Theta(n)$ time.
- Lines 4: $O(1)$ time.
- Lines 6-11:
 - Each vertex is **enqueued and dequeued exactly once**.
 - Since each queue operation takes $O(1)$ time, **the total time needed for all queue operations is $\Theta(n)$** .
 - Lines 8-10: **Each item in $Adj[u]$ is processed once**.
 - **When an item is processed, $O(1)$ operations are needed.**

BFS: Analysis

- Lines 1 and 5: The queue operations take $O(1)$ time.
- Line 2-3: Loop takes $\Theta(n)$ time.
- Lines 4: $O(1)$ time.
- Lines 6-11:
 - Each vertex is **enqueued and dequeued exactly once**.
 - Since each queue operation takes $O(1)$ time, **the total time needed for all queue operations is $\Theta(n)$** .
 - Lines 8-10: **Each item in $Adj[u]$ is processed once**.
 - **When an item is processed, $O(1)$ operations are needed**.
 - So the total time needed is $\Theta(m) \cdot \Theta(1) = \Theta(m)$.

BFS: Analysis

- Lines 1 and 5: The queue operations take $O(1)$ time.
- Line 2-3: Loop takes $\Theta(n)$ time.
- Lines 4: $O(1)$ time.
- Lines 6-11:
 - Each vertex is **enqueued and dequeued exactly once**.
 - Since each queue operation takes $O(1)$ time, **the total time needed for all queue operations is $\Theta(n)$** .
 - Lines 8-10: **Each item in $Adj[u]$ is processed once**.
 - **When an item is processed, $O(1)$ operations are needed**.
 - So the total time needed is $\Theta(m) \cdot \Theta(1) = \Theta(m)$.
- Add everything together:

BFS: Analysis

- Lines 1 and 5: The queue operations take $O(1)$ time.
- Line 2-3: Loop takes $\Theta(n)$ time.
- Lines 4: $O(1)$ time.
- Lines 6-11:
 - Each vertex is **enqueued and dequeued exactly once**.
 - Since each queue operation takes $O(1)$ time, **the total time needed for all queue operations is $\Theta(n)$** .
 - Lines 8-10: **Each item in $Adj[u]$ is processed once**.
 - **When an item is processed, $O(1)$ operations are needed**.
 - So the total time needed is $\Theta(m) \cdot \Theta(1) = \Theta(m)$.
- Add everything together:

BFS algorithm takes $\Theta(n + m)$ time.

BFS: Main Property

Theorem

Let $G = (V, E)$ be a graph. Let $d[u]$ be the value computed by BFS algorithm. Then for any $(u, v) \in E$, $|d[u] - d[v]| \leq 1$.

BFS: Main Property

Theorem

Let $G = (V, E)$ be a graph. Let $d[u]$ be the value computed by BFS algorithm. Then for any $(u, v) \in E$, $|d[u] - d[v]| \leq 1$.

Proof: First, we make the following observations:

- Each vertex $v \in V$ is **enqueued** and **dequeued** exactly once.

BFS: Main Property

Theorem

Let $G = (V, E)$ be a graph. Let $d[u]$ be the value computed by BFS algorithm. Then for any $(u, v) \in E$, $|d[u] - d[v]| \leq 1$.

Proof: First, we make the following observations:

- Each vertex $v \in V$ is **enqueued** and **dequeued** exactly once.
- Initially $\text{color}[v] = \text{white}$. When it is **enqueued**, $\text{color}[v]$ becomes grey. When it is **dequeued**, $\text{color}[v]$ becomes black. The color remains black until the end.

BFS: Main Property

Theorem

Let $G = (V, E)$ be a graph. Let $d[u]$ be the value computed by BFS algorithm. Then for any $(u, v) \in E$, $|d[u] - d[v]| \leq 1$.

Proof: First, we make the following observations:

- Each vertex $v \in V$ is **enqueued** and **dequeued** exactly once.
- Initially $\text{color}[v] = \text{white}$. When it is **enqueued**, $\text{color}[v]$ becomes grey. When it is **dequeued**, $\text{color}[v]$ becomes black. The color remains black until the end.
- The $d[v]$ value is set when v is **enqueued**. It is never changed again.

BFS: Main Property

Theorem

Let $G = (V, E)$ be a graph. Let $d[u]$ be the value computed by BFS algorithm. Then for any $(u, v) \in E$, $|d[u] - d[v]| \leq 1$.

Proof: First, we make the following observations:

- Each vertex $v \in V$ is **enqueued** and **dequeued** exactly once.
- Initially $\text{color}[v] = \text{white}$. When it is **enqueued**, $\text{color}[v]$ becomes grey. When it is **dequeued**, $\text{color}[v]$ becomes black. The color remains black until the end.
- The $d[v]$ value is set when v is **enqueued**. It is never changed again.
- At any moment during the execution, the vertices in Q consist of two parts, Q_1 followed by Q_2 (either of them can be empty).
 - For all $w \in Q_1$, $d[w] = k$ for some k .

BFS: Main Property

Theorem

Let $G = (V, E)$ be a graph. Let $d[u]$ be the value computed by BFS algorithm. Then for any $(u, v) \in E$, $|d[u] - d[v]| \leq 1$.

Proof: First, we make the following observations:

- Each vertex $v \in V$ is **enqueued** and **dequeued** exactly once.
- Initially $\text{color}[v] = \text{white}$. When it is **enqueued**, $\text{color}[v]$ becomes grey. When it is **dequeued**, $\text{color}[v]$ becomes black. The color remains black until the end.
- The $d[v]$ value is set when v is **enqueued**. It is never changed again.
- At any moment during the execution, the vertices in Q consist of two parts, Q_1 followed by Q_2 (either of them can be empty).
 - For all $w \in Q_1$, $d[w] = k$ for some k .
 - For all $x \in Q_2$, $d[x] = k + 1$.

BFS: Main Property

Without loss of generality, suppose that u is visited by the algorithm before v . Consider the while loop in BFS algorithm, when u is at the front of Q .

BFS: Main Property

Without loss of generality, suppose that u is visited by the algorithm before v . Consider the while loop in BFS algorithm, when u is at the front of Q . There are two cases.

BFS: Main Property

Without loss of generality, suppose that u is visited by the algorithm before v . Consider the while loop in BFS algorithm, when u is at the front of Q . There are two cases.

Case 1: $\text{color}[v] = \text{white}$ at that moment.

BFS: Main Property

Without loss of generality, suppose that u is visited by the algorithm before v . Consider the while loop in BFS algorithm, when u is at the front of Q . There are two cases.

Case 1: $\text{color}[v] = \text{white}$ at that moment.

- Since $v \in \text{Adj}[u]$, the algorithm set $d[v] = d[u] + 1$, and $\text{color}[v] = \text{grey}$.

BFS: Main Property

Without loss of generality, suppose that u is visited by the algorithm before v . Consider the while loop in BFS algorithm, when u is at the front of Q . There are two cases.

Case 1: $\text{color}[v] = \text{white}$ at that moment.

- Since $v \in \text{Adj}[u]$, the algorithm set $d[v] = d[u] + 1$, and $\text{color}[v] = \text{grey}$.
- $d[v]$ is never changed again. Thus $d[v] - d[u] = 1$.

BFS: Main Property

Without loss of generality, suppose that u is visited by the algorithm before v . Consider the while loop in BFS algorithm, when u is at the front of Q .

There are two cases.

Case 1: $\text{color}[v] = \text{white}$ at that moment.

- Since $v \in \text{Adj}[u]$, the algorithm set $d[v] = d[u] + 1$, and $\text{color}[v] = \text{grey}$.
- $d[v]$ is never changed again. Thus $d[v] - d[u] = 1$.

Case 2: $\text{color}[v] = \text{grey}$ at that moment.

BFS: Main Property

Without loss of generality, suppose that u is visited by the algorithm before v . Consider the while loop in BFS algorithm, when u is at the front of Q . There are two cases.

Case 1: $\text{color}[v] = \text{white}$ at that moment.

- Since $v \in \text{Adj}[u]$, the algorithm set $d[v] = d[u] + 1$, and $\text{color}[v] = \text{grey}$.
- $d[v]$ is never changed again. Thus $d[v] - d[u] = 1$.

Case 2: $\text{color}[v] = \text{grey}$ at that moment.

- Then v is in Q at that moment.

BFS: Main Property

Without loss of generality, suppose that u is visited by the algorithm before v . Consider the while loop in BFS algorithm, when u is at the front of Q .

There are two cases.

Case 1: $\text{color}[v] = \text{white}$ at that moment.

- Since $v \in \text{Adj}[u]$, the algorithm set $d[v] = d[u] + 1$, and $\text{color}[v] = \text{grey}$.
- $d[v]$ is never changed again. Thus $d[v] - d[u] = 1$.

Case 2: $\text{color}[v] = \text{grey}$ at that moment.

- Then v is in Q at that moment.
- By the previous observation, $d[u] = k$ for some k , and $d[v] = k$ or $k + 1$. Thus $d[v] - d[u] \leq 1$.

BFS: Main Property

Definition

Let $G = (V, E)$ be a graph and T a spanning tree of G rooted at the vertex s . Let x and y be two vertices. Let (u, v) be an edge of G .

- If x is on the path from y to s , we say x is an ancestor of y , and y is a descendent of x

BFS: Main Property

Definition

Let $G = (V, E)$ be a graph and T a spanning tree of G rooted at the vertex s . Let x and y be two vertices. Let (u, v) be an edge of G .

- If x is on the path from y to s , we say x is an ancestor of y , and y is a descendent of x
- If $(u, v) \in T$, we say (u, v) is a tree edge.

BFS: Main Property

Definition

Let $G = (V, E)$ be a graph and T a spanning tree of G rooted at the vertex s . Let x and y be two vertices. Let (u, v) be an edge of G .

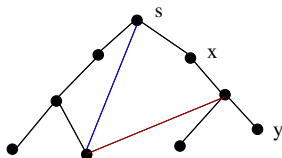
- If x is on the path from y to s , we say x is an ancestor of y , and y is a descendent of x
- If $(u, v) \in T$, we say (u, v) is a tree edge.
- If $(u, v) \notin T$ and u is an ancestor of v , we say (u, v) is a back edge.

BFS: Main Property

Definition

Let $G = (V, E)$ be a graph and T a spanning tree of G rooted at the vertex s . Let x and y be two vertices. Let (u, v) be an edge of G .

- If x is on the path from y to s , we say x is an ancestor of y , and y is a descendent of x
- If $(u, v) \in T$, we say (u, v) is a tree edge.
- If $(u, v) \notin T$ and u is an ancestor of v , we say (u, v) is a back edge.
- If neither u is an ancestor of v , nor v is an ancestor of u , we say (u, v) is a cross edge.



— tree edges
— back edges
— cross edges

BFS: Main Property

Theorem

Let T be the BFS tree constructed by the BFS algorithm. Then there are no **back edges** for T .

BFS: Main Property

Theorem

Let T be the BFS tree constructed by the BFS algorithm. Then there are no **back edges** for T .

Proof: Suppose there is an back edge (u, v) for T . Then $|d[u] - d[v]| \geq 2$. This is impossible.

BFS: Main Property

Theorem

Let T be the BFS tree constructed by the BFS algorithm. Then there are no **back edges** for T .

Proof: Suppose there is an back edge (u, v) for T . Then $|d[u] - d[v]| \geq 2$. This is impossible.

Shortest Path Problem

Let $G = (V, E)$ be a graph and s a vertex of G . For each $u \in V$, let $\delta(s, u)$ be the length of the shortest path between s and u .

Problem: For all $u \in V$, find $\delta(s, u)$ and the shortest path between s and u .

Theorem

Let $d[u]$ be the value computed by BFS algorithm and T the BFS tree constructed by BFS algorithm. Then for each vertex $u \in V$,

- $d[u] = \delta(s, u)$.
- The tree path in T from u to s is the shortest path.

Theorem

Let $d[u]$ be the value computed by BFS algorithm and T the BFS tree constructed by BFS algorithm. Then for each vertex $u \in V$,

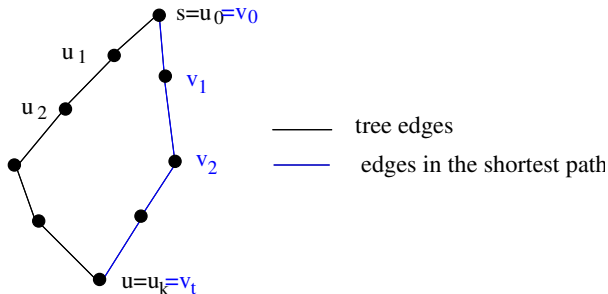
- $d[u] = \delta(s, u)$.
- The tree path in T from u to s is the shortest path.

Proof: Let $P = \{s = u_0, u_1, \dots, u_k = u\}$ be the path from s to u in the BFS tree T . Then: $d[u] = d[u_k] = k$, $d[u_{k-1}] = k - 1$, $d[u_{k-2}] = k - 2 \dots$

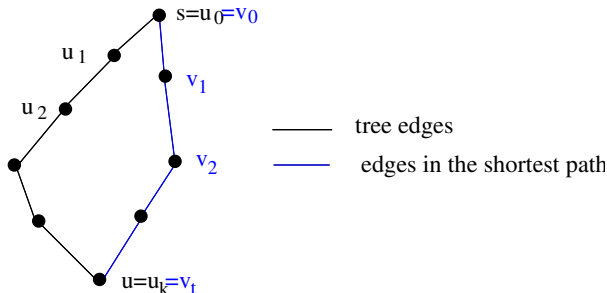
Suppose $P' = \{s = v_0, v_1, v_2, \dots, v_t\}$ is the shortest path from s to u in G . We need to show $k = t$.

Toward a contradiction, suppose $t < k$. Then there must exist $(v_i, v_{i+1}) \in P'$ such that $|d[v_i] - d[v_{i+1}]| \geq 2$. This is impossible.

Shortest Path Problem



Shortest Path Problem



BFS algorithm solves the Single Source Shortest Path problem in $\Theta(n + m)$ time.

Connectivity Problem

Definition

- A graph $G = (V, E)$ is **connected** if for any two vertices u and v in G , there exists a path in G between u and v .

Connectivity Problem

Definition

- A graph $G = (V, E)$ is **connected** if for any two vertices u and v in G , there exists a path in G between u and v .
- A **connected component** of G is a **maximal subgraph** of G that is connected.

Connectivity Problem

Definition

- A graph $G = (V, E)$ is **connected** if for any two vertices u and v in G , there exists a path in G between u and v .
- A **connected component** of G is a **maximal subgraph** of G that is connected.
- G is connected if and only if it has exactly one connected component.

Connectivity Problem

Definition

- A graph $G = (V, E)$ is **connected** if for any two vertices u and v in G , there exists a path in G between u and v .
- A **connected component** of G is a **maximal subgraph** of G that is connected.
- G is connected if and only if it has exactly one connected component.

Connectivity Problem

Given $G = (V, E)$, is G a connected graph?
If not, find the connected components of G .

We can use BFS algorithm to solve the connectivity problem.

Connectivity Problem

In the **BFS** algorithm, delete the lines 2-3 (initialization of vertex variables).

Connectivity Problem

In the **BFS** algorithm, delete the lines 2-3 (initialization of vertex variables).

Connectivity($G = (V, E)$)

- 1 **for** each $i \in V$ **do**
- 2 $\text{color}[i] = \text{white}; d[i] = \infty; \pi[i] = \text{nil};$
- 3 $\text{count} = 0;$ (count will be the number of connected components)
- 4 **for** $i = 1$ **to** n **do**
- 5 **if** $\text{color}[i] = \text{white}$ **then**
- 6 **call** **BFS**(G, i); $\text{count} = \text{count} + 1$
- 7 **output** $\text{count};$
- 8 **end**

Connectivity Problem

- This algorithm outputs **count**, the number of connected components.

Connectivity Problem

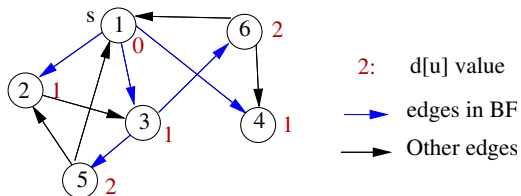
- This algorithm outputs **count**, the number of connected components.
- If $\text{count} = 1$, G is connected. The algorithm also constructs a BFS tree.

Connectivity Problem

- This algorithm outputs **count**, the number of connected components.
- If $\text{count} = 1$, G is connected. The algorithm also constructs a BFS tree.
- If $\text{count} > 1$, G is not connected. The algorithm also constructs a **BFS spanning forest F** of G . F is a collection of trees.
- Each tree corresponds to a connected component of G .

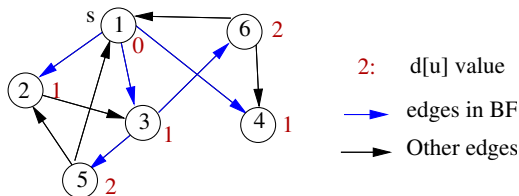
BFS for Directed Graphs

BFS algorithm can be applied to directed graphs without any change.



BFS for Directed Graphs

BFS algorithm can be applied to directed graphs without any change.



Definition

Let $G = (V, E)$ be a directed graph, T a spanning tree rooted at s . An edge $e = u \rightarrow v$ is called:

- **tree edge** if $e = u \rightarrow v \in T$.
- **backward edge** if u is a decedent of v .
- **forward edge** if u is an ancestor of v .
- **cross edge** if u and v are **unrelated**.

BFS for Directed Graphs: Property

Theorem

Let $G = (V, E)$ be a directed graph. Let T be the BFS tree constructed by BFS algorithm. Then there are no **forward edges** with respect to T .

BFS for Directed Graphs: Property

Theorem

Let $G = (V, E)$ be a directed graph. Let T be the BFS tree constructed by BFS algorithm. Then there are no **forward edges** with respect to T .

Theorem

Let $d[u]$ be the value computed by BFS algorithm and T the BFS tree constructed by BFS algorithm. Then for each vertex $u \in V$,

- The tree path in T from s to u is the shortest path.
- $d[u]$ = the length of the shortest path from s to u .

Outline

- 1 Graph Algorithms
- 2 Graph Representations
- 3 Breadth First Search (BFS)
- 4 Depth First Search (DFS)**
- 5 Topological Sort
- 6 Strong Connectivity
- 7 DFS for Undirected Graphs
- 8 Biconnectivity Problem

Depth First Search (DFS)

- Similar to BFS, **Depth First Search (DFS)** is another systematic way for visiting the vertices of a graph.

Depth First Search (DFS)

- Similar to BFS, **Depth First Search (DFS)** is another systematic way for visiting the vertices of a graph.
- It can be used on **directed** or **undirected graphs**. We discuss DFS for **directed graph** first.

Depth First Search (DFS)

- Similar to BFS, **Depth First Search (DFS)** is another systematic way for visiting the vertices of a graph.
- It can be used on **directed** or **undirected graphs**. We discuss DFS for **directed graph** first.
- DFS has special properties, making it very useful in several applications.

Depth First Search (DFS)

- Similar to BFS, **Depth First Search (DFS)** is another systematic way for visiting the vertices of a graph.
- It can be used on **directed** or **undirected graphs**. We discuss DFS for **directed graph** first.
- DFS has special properties, making it very useful in several applications.
- As a high level description, the only difference between BFS and DFS: replace the **queue Q** in BFS algorithm by a **stack S**. So it works like this:

Depth First Search (DFS)

- Similar to BFS, **Depth First Search (DFS)** is another systematic way for visiting the vertices of a graph.
- It can be used on **directed** or **undirected graphs**. We discuss DFS for **directed graph** first.
- DFS has special properties, making it very useful in several applications.
- As a high level description, the only difference between BFS and DFS: replace the **queue Q** in BFS algorithm by a **stack S** . So it works like this:

High Level Description of DFS

- Start at the starting vertex s .

Depth First Search (DFS)

- Similar to BFS, **Depth First Search (DFS)** is another systematic way for visiting the vertices of a graph.
- It can be used on **directed** or **undirected graphs**. We discuss DFS for **directed graph** first.
- DFS has special properties, making it very useful in several applications.
- As a high level description, the only difference between BFS and DFS: replace the **queue Q** in BFS algorithm by a **stack S** . So it works like this:

High Level Description of DFS

- Start at the starting vertex s .
- Visit a neighbor u of s ; visit a neighbor v of u . . .

Depth First Search (DFS)

- Similar to BFS, **Depth First Search (DFS)** is another systematic way for visiting the vertices of a graph.
- It can be used on **directed** or **undirected graphs**. We discuss DFS for **directed graph** first.
- DFS has special properties, making it very useful in several applications.
- As a high level description, the only difference between BFS and DFS: replace the **queue Q** in BFS algorithm by a **stack S** . So it works like this:

High Level Description of DFS

- Start at the starting vertex s .
- Visit a neighbor u of s ; visit a neighbor v of u . . .
- Go as far as you can go, until reaching **a dead end**.

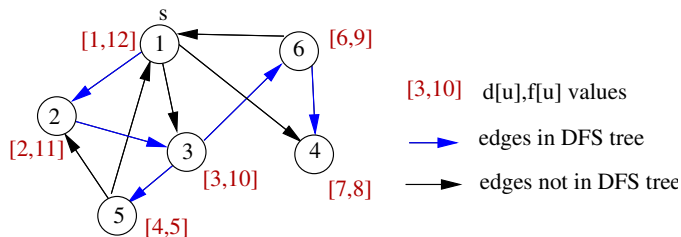
Depth First Search (DFS)

- Similar to BFS, **Depth First Search (DFS)** is another systematic way for visiting the vertices of a graph.
- It can be used on **directed** or **undirected graphs**. We discuss DFS for **directed graph** first.
- DFS has special properties, making it very useful in several applications.
- As a high level description, the only difference between BFS and DFS: replace the **queue Q** in BFS algorithm by a **stack S**. So it works like this:

High Level Description of DFS

- Start at the starting vertex s .
- Visit a neighbor u of s ; visit a neighbor v of u . . .
- Go as far as you can go, until reaching **a dead end**.
- Backtrack to a vertex that still has unvisited neighbors, and continue

DFS: Example



DFS: Recursive algorithm

- It is easier to describe the DFS by using a recursive algorithm.

DFS: Recursive algorithm

- It is easier to describe the DFS by using a recursive algorithm.
- DFS also computes two variables for each vertex $u \in V$:
 - $d[u]$: The time when u is "discovered", i.e. pushed on the stack.
 - $f[u]$: the time when u is "finished", i.e. popped from the stack.
- These variables will be used in applications.

DFS: Recursive algorithm

DFS(G)

- ➊ **for** each vertex $u \in V$ **do**
- ➋ $\text{color}[u] \leftarrow \text{white}; \quad \pi[u] = \text{NIL}$
- ➌ $\text{time} \leftarrow 0$
- ➍ **for** each vertex $u \in V$ **do**
- ➎ **if** $\text{color}[u] = \text{white}$ **then DFS-Visit**(u)

DFS: Recursive algorithm

DFS(G)

- 1 **for** each vertex $u \in V$ **do**
- 2 $\text{color}[u] \leftarrow \text{white}; \quad \pi[u] = \text{NIL}$
- 3 $\text{time} \leftarrow 0$
- 4 **for** each vertex $u \in V$ **do**
- 5 **if** $\text{color}[u] = \text{white}$ **then** **DFS-Visit**(u)

DFS-Visit(u)

- 1 $\text{color}[u] \leftarrow \text{grey}; \quad \text{time} \leftarrow \text{time} + 1; \quad d[u] \leftarrow \text{time}$
- 2 **for** each vertex $v \in \text{Adj}[u]$ **do**
- 3 **if** $\text{color}[v] = \text{white}$
- 4 **then** $\pi[v] \leftarrow u; \quad \text{DFS-Visit}(v)$
- 5 $\text{color}[u] \leftarrow \text{black}$
- 6 $f[u] \leftarrow \text{time} \leftarrow \text{time} + 1$

DFS: Properties

Let T be the DFS tree of G by DFS algorithm. Let $[d[u], f[u]]$ be the **time interval** computed by DFS algorithm. Let $u \neq v$ be any two vertices of G .

- The intervals of $[d[u], f[u]]$ and $[d[v], f[v]]$ are either **disjoint** or **one is contained in another**.
- $[d[u], f[u]]$ is **contained in** $[d[v], f[v]]$ if and only if u is a descendent of v with respect to T .

Classification of Edges

Let $G = (V, E)$ be a directed graph and T a spanning tree of G . The edge $e = u \rightarrow v$ of G can be classified as:

- **tree edge** if $e = u \rightarrow v \in T$.
- **back-edge** if $e \notin T$ and v is an **ancestor** of u .
- **forward-edge** if $e \notin T$ and u is an **ancestor** of v .
- **cross-edge** if $e \notin T$, v and u are **unrelated** with respect to T .

Classification of Edges

Let $G = (V, E)$ be a directed graph and T the spanning tree of G constructed by DFS algorithm. The classification of the edges can be done as follows.

- When $e = u \rightarrow v$ is first explored by DFS, color e by the color[v].
- If color[v] is white, then e is white and is a tree edge.
- If color[v] is grey, then e is grey and is a back-edge.
- If color[v] is black, then e is black and is either a forward- or a cross-edge.

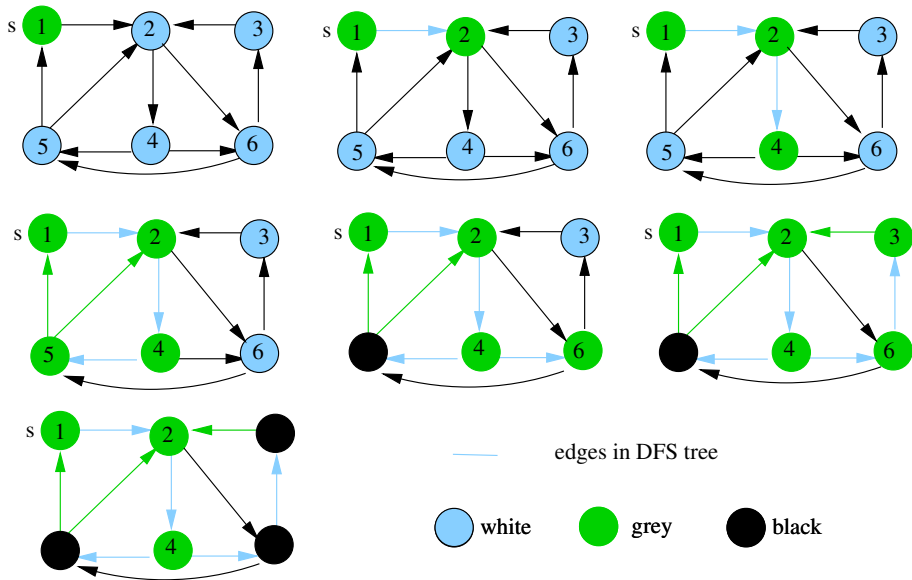
Classification of Edges

Let $G = (V, E)$ be a directed graph and T the spanning tree of G constructed by DFS algorithm. The classification of the edges can be done as follows.

- When $e = u \rightarrow v$ is first explored by DFS, color e by the color $[v]$.
- If color $[v]$ is white, then e is white and is a tree edge.
- If color $[v]$ is grey, then e is grey and is a back-edge.
- If color $[v]$ is black, then e is black and is either a forward- or a cross-edge.

For DFS tree of directed graphs, all four types of edges are possible.

DFS: Example



Definition

A directed graph $G = (V, E)$ is called a **directed acyclic graph (DAG for short)** if it contains no directed cycles.

Definition

A directed graph $G = (V, E)$ is called a **directed acyclic graph (DAG for short)** if it contains no directed cycles.

DAG Testing

Given a directed graph $G = (V, E)$, test if G is a DAG or not.

DFS: Applications

Definition

A directed graph $G = (V, E)$ is called a **directed acyclic graph (DAG for short)** if it contains no directed cycles.

DAG Testing

Given a directed graph $G = (V, E)$, test if G is a DAG or not.

Theorem

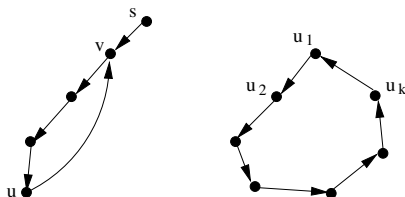
Let G be a directed graph, and T the DFS tree of G . Then G is DAG \iff there are no **back edges**.

DFS: Applications

Proof: \implies Suppose $e = u \rightarrow v$ is a back edge. Let P be the path in T from v to u . Then the directed path P followed by $e = u \rightarrow v$ is a directed cycle.

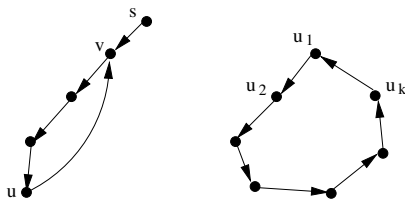
DFS: Applications

Proof: \implies Suppose $e = u \rightarrow v$ is a back edge. Let P be the path in T from v to u . Then the directed path P followed by $e = u \rightarrow v$ is a directed cycle.



DFS: Applications

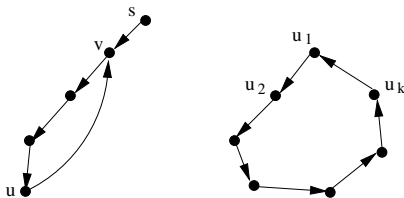
Proof: \implies Suppose $e = u \rightarrow v$ is a back edge. Let P be the path in T from v to u . Then the directed path P followed by $e = u \rightarrow v$ is a directed cycle.



\Leftarrow Suppose $C = u_1 \rightarrow u_2 \rightarrow \dots \rightarrow u_k \rightarrow u_1$ is a directed cycle. Without loss of generality, assume u_1 is the first vertex visited by DFS. Then, the algorithm visits u_2, u_3, \dots, u_k , before it backtracks to u_1 . So $u_k \rightarrow u_1$ is a back edge.

DFS: Applications

Proof: \implies Suppose $e = u \rightarrow v$ is a back edge. Let P be the path in T from v to u . Then the directed path P followed by $e = u \rightarrow v$ is a directed cycle.



\Leftarrow Suppose $C = u_1 \rightarrow u_2 \rightarrow \dots \rightarrow u_k \rightarrow u_1$ is a directed cycle. Without loss of generality, assume u_1 is the first vertex visited by DFS. Then, the algorithm visits u_2, u_3, \dots, u_k , before it backtrack to u_1 . So $u_k \rightarrow u_1$ is a back edge.

DAG Testing in $\Theta(n + m)$ time

- 1 Run DFS on G . Mark the edges “white”, “grey” or “black”,
- 2 If there is a grey edge, report “ G is not a DAG”. If not “ G is a DAG”.

Outline

- 1 Graph Algorithms
- 2 Graph Representations
- 3 Breadth First Search (BFS)
- 4 Depth First Search (DFS)
- 5 Topological Sort**
- 6 Strong Connectivity
- 7 DFS for Undirected Graphs
- 8 Biconnectivity Problem

Topological Sort

Topological Sort

Let $G = (V, E)$ be a DAG. A **topological sort** of G assigns each vertex $v \in V$ a distinct number $L(v) \in [1..n]$ such that if $u \rightarrow v$ then $L(u) < L(v)$.

Topological Sort

Topological Sort

Let $G = (V, E)$ be a DAG. A **topological sort** of G assigns each vertex $v \in V$ a distinct number $L(v) \in [1..n]$ such that if $u \rightarrow v$ then $L(u) < L(v)$.

Note: If G is not a DAG, topological sort cannot exist.

Topological Sort

Topological Sort

Let $G = (V, E)$ be a DAG. A **topological sort** of G assigns each vertex $v \in V$ a distinct number $L(v) \in [1..n]$ such that if $u \rightarrow v$ then $L(u) < L(v)$.

Note: If G is not a DAG, topological sort cannot exist.

Application

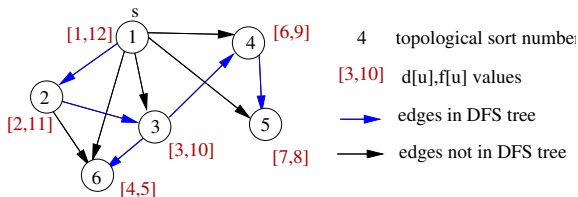
- The directed graph $G = (V, E)$ specifies a **job flow chart**.
- Each $v \in V$ is a job.
- If $u \rightarrow v$, then the job u must be done before the job v .
- A **topological sort** specifies the order to complete jobs.

Topological Sort

We can use DFS to find topological sort.

Topological-Sort-by-DFS(G)

- 1 Run DFS on G .
- 2 Number the vertices by decreasing order of $f[v]$ value. (This can be done as follows: During DFS, when a vertex v is finished, insert v in the front of a linked list.)

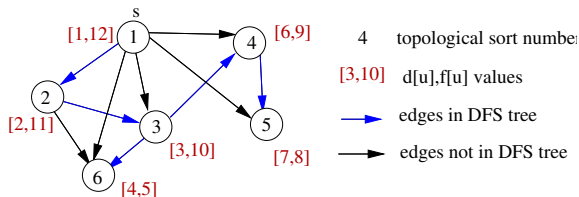


Topological Sort

We can use DFS to find topological sort.

Topological-Sort-by-DFS(G)

- 1 Run DFS on G .
- 2 Number the vertices by decreasing order of $f[v]$ value. (This can be done as follows: During DFS, when a vertex v is finished, insert v in the front of a linked list.)



Clearly, this algorithm takes $\Theta(m + n)$ time.

Outline

- 1 Graph Algorithms
- 2 Graph Representations
- 3 Breadth First Search (BFS)
- 4 Depth First Search (DFS)
- 5 Topological Sort
- 6 Strong Connectivity**
- 7 DFS for Undirected Graphs
- 8 Biconnectivity Problem

Strong Connectivity

Definition

- A directed graph $G = (V, E)$ is **strongly connected** if for any two vertices u and v in V , there exists a directed path from u to v .
- A **strongly connected component of G** is a maximal subgraph of G that is strongly connected.

Strong Connectivity

Definition

- A directed graph $G = (V, E)$ is **strongly connected** if for any two vertices u and v in V , there exists a directed path from u to v .
- A **strongly connected component** of G is a maximal subgraph of G that is strongly connected.

Strong Connectivity Problem

Given a directed graph G , find the strongly connected components of G .

Strong Connectivity

Definition

- A directed graph $G = (V, E)$ is **strongly connected** if for any two vertices u and v in V , there exists a directed path from u to v .
- A **strongly connected component** of G is a maximal subgraph of G that is strongly connected.

Strong Connectivity Problem

Given a directed graph G , find the strongly connected components of G .

Note: G is strongly connected if and only if it has exactly one strongly connected component.

Strong Connectivity

Application: Traffic Flow Map

- $G = (V, E)$ represents a street map.
- Each $v \in V$ is an intersection.
- Each edge $u \rightarrow v$ is a 1-way street from the intersection u to the intersection v .
- Can you reach from any intersection to any other intersection?
- This is so $\iff G$ is strongly connected.
- All intersections within each connected component can reach each other.

Strong Connectivity

Application: Traffic Flow Map

- $G = (V, E)$ represents a street map.
- Each $v \in V$ is an intersection.
- Each edge $u \rightarrow v$ is a 1-way street from the intersection u to the intersection v .
- Can you reach from any intersection to any other intersection?
- This is so $\iff G$ is strongly connected.
- All intersections within each connected component can reach each other.

This problem can be solved by using DFS. Without it, it would be hard to solve efficiently.

Strong Connectivity

Strong-Connectivity-by-DFS(G)

- 1 Run DFS on G , compute $f[u]$ for all $u \in V$,
- 2 Order the vertices by decreasing $f[v]$ values.
- 3 Construct the **transpose graph G^T** , which is obtained from G by reversing the direction of all edges.
- 4 Run DFS on G^T , the vertices are considered in the order of decreasing $f[v]$ values.
- 5 The vertices in each tree in the DFS forest correspond to a strongly connected component of G .

Strong Connectivity

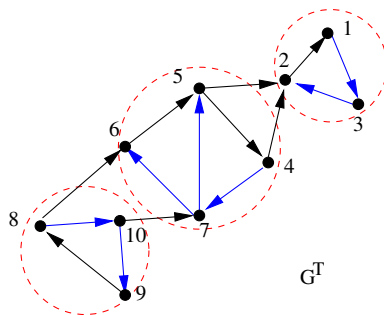
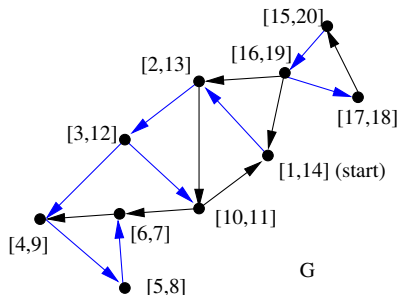
Strong-Connectivity-by-DFS(G)

- 1 Run DFS on G , compute $f[u]$ for all $u \in V$,
- 2 Order the vertices by decreasing $f[v]$ values.
- 3 Construct the **transpose graph G^T** , which is obtained from G by reversing the direction of all edges.
- 4 Run DFS on G^T , the vertices are considered in the order of decreasing $f[v]$ values.
- 5 The vertices in each tree in the DFS forest correspond to a strongly connected component of G .

Analysis:

- Steps 1 and 2: $\Theta(n + m)$ (step 2 is a part of step 1.)
- Step 3: $\Theta(n + m)$ (how?)
- Step 4 and 5: $\Theta(n + m)$ (step 5 is part of step 4.)

Strong Connectivity: Example



Outline

- 1 Graph Algorithms
- 2 Graph Representations
- 3 Breadth First Search (BFS)
- 4 Depth First Search (DFS)
- 5 Topological Sort
- 6 Strong Connectivity
- 7 DFS for Undirected Graphs**
- 8 Biconnectivity Problem

DFS for Undirected Graphs

- DFS algorithm can be used on an undirected graph $G = (V, E)$ without any change.

DFS for Undirected Graphs

- DFS algorithm can be used on an undirected graph $G = (V, E)$ without any change.
- It constructs a DFS tree T of G .

DFS for Undirected Graphs

- DFS algorithm can be used on an undirected graph $G = (V, E)$ without any change.
- It constructs a DFS tree T of G .
- Recall that: for an undirected graph $G = (V, E)$ and a spanning tree T of G , the edges of G can be classified as:

DFS for Undirected Graphs

- DFS algorithm can be used on an undirected graph $G = (V, E)$ without any change.
- It constructs a DFS tree T of G .
- Recall that: for an undirected graph $G = (V, E)$ and a spanning tree T of G , the edges of G can be classified as:
 - tree edges
 - back edges
 - cross edges

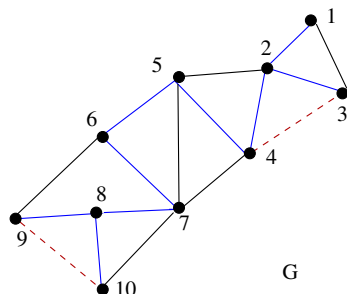
DFS for Undirected Graphs

- DFS algorithm can be used on an undirected graph $G = (V, E)$ without any change.
- It constructs a DFS tree T of G .
- Recall that: for an undirected graph $G = (V, E)$ and a spanning tree T of G , the edges of G can be classified as:
 - tree edges
 - back edges
 - cross edges

Theorem

Let G be an undirected graph, and T the DFS tree of G constructed by DFS algorithm. Then there are no cross edges.

DFS for Undirected Graphs



- back edges
- tree edges
- - - would be cross edge
(which cannot exist)

Summary: Edge Types

For Directed Graphs

	Tree	Forward	Backward	Cross
BFS	yes	no	yes	yes
DFS	yes	yes	yes	yes

Summary: Edge Types

For Directed Graphs

	Tree	Forward	Backward	Cross
BFS	yes	no	yes	yes
DFS	yes	yes	yes	yes

For Undirected Graphs

	Tree	Back-edge	Cross
BFS	yes	no	yes
DFS	yes	yes	no

Outline

- 1 Graph Algorithms
- 2 Graph Representations
- 3 Breadth First Search (BFS)
- 4 Depth First Search (DFS)
- 5 Topological Sort
- 6 Strong Connectivity
- 7 DFS for Undirected Graphs
- 8 Biconnectivity Problem**

Biconnectivity Problem

Definition

Let $G = (V, E)$ be an **undirected connected graph**.

- A vertex $v \in V$ is a **cut vertex (also called articulation point)** if deleting v and its incident edges disconnects G .
- G is **biconnected** if it is connected and has no cut vertices.
- A **biconnected component** of G is a maximal subgraph of G that is biconnected.

Biconnectivity Problem

Definition

Let $G = (V, E)$ be an **undirected connected graph**.

- A vertex $v \in V$ is a **cut vertex (also called articulation point)** if deleting v and its incident edges disconnects G .
- G is **biconnected** if it is connected and has no cut vertices.
- A **biconnected component** of G is a maximal subgraph of G that is biconnected.
- G is biconnected if and only if it has exactly one biconnected component.

Biconnectivity Problem

Definition

Let $G = (V, E)$ be an **undirected connected graph**.

- A vertex $v \in V$ is a **cut vertex (also called articulation point)** if deleting v and its incident edges disconnects G .
- G is **biconnected** if it is connected and has no cut vertices.
- A **biconnected component** of G is a maximal subgraph of G that is biconnected.
- G is biconnected if and only if it has exactly one biconnected component.

Biconnectivity Problem

Given an undirected graph $G = (V, E)$, is G biconnected?

If not, find the cut vertices and the biconnected components of G .

Biconnectivity Problem

Application

- G represents a computer network.
- Each vertex is a computer site.
- Each edge is a communication link.
- If v is a cut vertex, then the failure of v will disconnect the whole network.
- The network can survive any single site failure if and only if G is biconnected.

Biconnectivity Problem

Application

- G represents a computer network.
- Each vertex is a computer site.
- Each edge is a communication link.
- If v is a cut vertex, then the failure of v will disconnect the whole network.
- The network can survive any single site failure if and only if G is biconnected.

Simple-Biconnectivity(G)

- 1 **for** each vertex $v \in V$ **do**
- 2 delete v and its incident edges from G
- 3 test if $G - \{v\}$ is connected

This algorithm takes $\Theta(n) \times \Theta(n + m) = \Theta(n(n + m))$ time.

Biconnectivity Problem

By using DFS, the problem can be solved in $O(n + m)$ time.

Biconnectivity Problem

By using DFS, the problem can be solved in $O(n + m)$ time.

- Let T be the DFS tree of G .
- Re-number the vertices by increasing $d[v]$ values.
- For each vertex v , define:
 $\text{low}[v] = \text{the smallest vertex that can be reached from } v \text{ or a descendent of } v \text{ through a back edge.}$
- If v is a leaf of T , then $\text{low}[v] = \min \left\{ \begin{array}{c} v \\ \{w \mid (v, w) \text{ is a back-edge}\} \end{array} \right\}$

Biconnectivity Problem

By using DFS, the problem can be solved in $O(n + m)$ time.

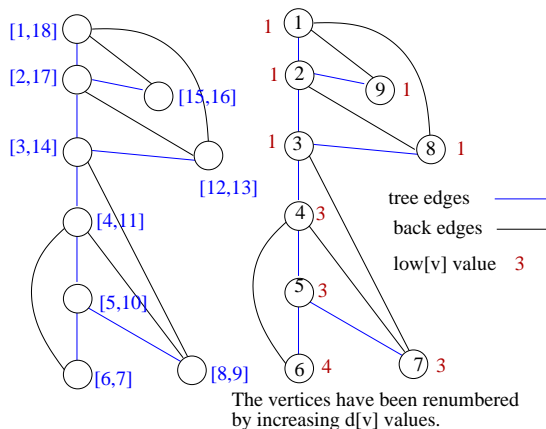
- Let T be the DFS tree of G .
- Re-number the vertices by increasing $d[v]$ values.
- For each vertex v , define:
 $\text{low}[v] = \text{the smallest vertex that can be reached from } v \text{ or a descendent of } v \text{ through a back edge.}$
- If v is a leaf of T , then $\text{low}[v] = \min \left\{ \overset{v}{\{w \mid (v, w) \text{ is a back-edge}\}} \right\}$
- If v is not a leaf of T , then $\text{low}[v] = \min \left\{ \overset{v}{\begin{array}{l} \{w \mid (v, w) \text{ is a back-edge}\} \\ \{\text{low}[t] \mid t \text{ is a son of } v\} \end{array}} \right\}$

Biconnectivity Problem

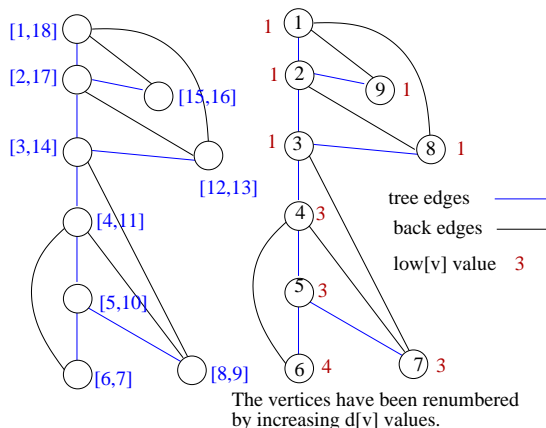
By using DFS, the problem can be solved in $O(n + m)$ time.

- Let T be the DFS tree of G .
- Re-number the vertices by increasing $d[v]$ values.
- For each vertex v , define:
 $\text{low}[v] = \text{the smallest vertex that can be reached from } v \text{ or a descendent of } v \text{ through a back edge.}$
- If v is a leaf of T , then $\text{low}[v] = \min \left\{ \overset{v}{\{w \mid (v, w) \text{ is a back-edge}\}} \right\}$
- If v is not a leaf of T , then $\text{low}[v] = \min \left\{ \overset{v}{\begin{array}{l} \{w \mid (v, w) \text{ is a back-edge}\} \\ \{\text{low}[t] \mid t \text{ is a son of } v\} \end{array}} \right\}$

Biconnectivity Problem



Biconnectivity Problem



In this figure, **low** means closer to the root. So the root is the lowest vertex. $low[v]$ is the lowest vertex that can be reached from v or a descendent of v thru a single back edge.

Biconnectivity Problem

Theorem

Let T be the DFS of $G = (V, E)$ rooted at the vertex s .

- ① s is a cut vertex $\iff s$ has at least two sons in T .
- ② A vertex $a \neq s$ is a cut vertex $\iff a$ has a son b such that $low[b] \geq a$.

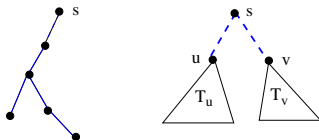
Biconnectivity Problem

Theorem

Let T be the DFS of $G = (V, E)$ rooted at the vertex s .

- ① s is a cut vertex $\iff s$ has at least two sons in T .
- ② A vertex $a \neq s$ is a cut vertex $\iff a$ has a son b such that $low[b] \geq a$.

Proof of (1): Suppose s has only one son. After deleting s , all other vertices are still connected by the remaining edges of T . So s is not a cut vertex.



Suppose s has at least two sons u and v (there may be more). Let T_u be the subtree of T rooted at u and T_v be the subtree of T rooted at v . Because there are no cross edges, no edges connect T_u with T_v . So after s is deleted, T_u and T_v become disconnected. Hence s is a cut vertex.

Biconnectivity Problem

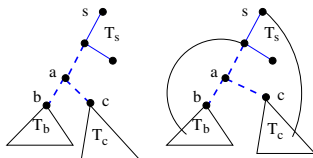
Proof of (2): Let T_s be the subtree of T above the vertex a . Let $b, c \dots$ be the sons of a . Let T_b, T_c, \dots be the subtree of T rooted at $b, c \dots$

- Suppose a has a son b with $\text{low}[b] \geq a$. Because $\text{low}[b] \geq a$, no vertex in T_b is connected to T_s . Because there are no cross edges, no edges connect vertices in T_b and T_c . So after a is deleted, T_b is disconnected from the rest of G . So a is a cut vertex.

Biconnectivity Problem

Proof of (2): Let T_s be the subtree of T above the vertex a . Let $b, c \dots$ be the sons of a . Let T_b, T_c, \dots be the subtree of T rooted at $b, c \dots$

- Suppose a has a son b with $\text{low}[b] \geq a$. Because $\text{low}[b] \geq a$, no vertex in T_b is connected to T_s . Because there are no cross edges, no edges connect vertices in T_b and T_c . So after a is deleted, T_b is disconnected from the rest of G . So a is a cut vertex.



- Suppose for every son b of a we have $\text{low}[b] < a$. This means that there is a back edge connecting a vertex in T_b to a vertex in T_s . So after a is deleted, all subtrees T_b, T_c, \dots are still connected to T_s , and G remains connected. So a is not a cut vertex.

Biconnectivity Problem

We can now describe the algorithm. For conceptual clarity, the algorithm is divided into several steps. Actually, all steps can be and should be incorporated into a single DFS run.

Biconnectivity Problem

We can now describe the algorithm. For conceptual clarity, the algorithm is divided into several steps. Actually, all steps can be and should be incorporated into a single DFS run.

Biconnectivity-by-DFS(G)

- 1 Run DFS on G
- 2 Renumber the vertices by increasing $d[*]$ values.
- 3 For all $u \in V$, compute $\text{low}[u]$ as described before.
- 4 Identify the cut vertices according to the conditions in the theorem.

Analysis

- Steps 1 and 2: takes $\Theta(m + n)$ time.

Analysis

- Steps 1 and 2: takes $\Theta(m + n)$ time.
- Step 3: $\text{low}[u]$ is the minimum of k values:
 - the $\text{low}[*]$ values for all sons of u .
 - the values for each back-edge from u .
 - 1 for u itself, we charge this to the edge between u and its parent.
 - So $k = \text{deg}(u)$.

Analysis

- Steps 1 and 2: takes $\Theta(m + n)$ time.
- Step 3: $\text{low}[u]$ is the minimum of k values:
 - the $\text{low}[*]$ values for all sons of u .
 - the values for each back-edge from u .
 - 1 for u itself, we charge this to the edge between u and its parent.
 - So $k = \text{deg}(u)$.
 - We compute $\text{low}[*]$ in post order. When computing $\text{low}[u]$, all values needed have been computed already. So it takes $\Theta(\text{deg}(u))$ time to compute $\text{low}[u]$.

Analysis

- Steps 1 and 2: takes $\Theta(m + n)$ time.
- Step 3: $\text{low}[u]$ is the minimum of k values:
 - the $\text{low}[*]$ values for all sons of u .
 - the values for each back-edge from u .
 - 1 for u itself, we charge this to the edge between u and its parent.
 - So $k = \text{deg}(u)$.
 - We compute $\text{low}[*]$ in post order. When computing $\text{low}[u]$, all values needed have been computed already. So it takes $\Theta(\text{deg}(u))$ time to compute $\text{low}[u]$.
 - So the total time needed to compute $\text{low}[u]$ for all vertices is Θ of the number of edges of G . This is $\Theta(m)$.

Analysis

- Steps 1 and 2: takes $\Theta(m + n)$ time.
- Step 3: $\text{low}[u]$ is the minimum of k values:
 - the $\text{low}[*]$ values for all sons of u .
 - the values for each back-edge from u .
 - 1 for u itself, we charge this to the edge between u and its parent.
 - So $k = \text{deg}(u)$.
 - We compute $\text{low}[*]$ in post order. When computing $\text{low}[u]$, all values needed have been computed already. So it takes $\Theta(\text{deg}(u))$ time to compute $\text{low}[u]$.
 - So the total time needed to compute $\text{low}[u]$ for all vertices is Θ of the number of edges of G . This is $\Theta(m)$.
- Step 4: The total time needed for checking these conditions for all vertices is $\Theta(n)$.

Analysis

- Steps 1 and 2: takes $\Theta(m + n)$ time.
- Step 3: $\text{low}[u]$ is the minimum of k values:
 - the $\text{low}[*]$ values for all sons of u .
 - the values for each back-edge from u .
 - 1 for u itself, we charge this to the edge between u and its parent.
 - So $k = \text{deg}(u)$.
 - We compute $\text{low}[*]$ in post order. When computing $\text{low}[u]$, all values needed have been computed already. So it takes $\Theta(\text{deg}(u))$ time to compute $\text{low}[u]$.
 - So the total time needed to compute $\text{low}[u]$ for all vertices is Θ of the number of edges of G . This is $\Theta(m)$.
- Step 4: The total time needed for checking these conditions for all vertices is $\Theta(n)$.

The Biconnectivity problem can be solved in $\Theta(n + m)$ time

- The DFS based Biconnectivity algorithm was discovered by Tarjan and Hopcroft in 1972. (See Problem 22-2, Page 558).
- They advocated the use of adjacent list representation over the adjacent matrix representation for solving complex graph problems in linear (i.e. $O(n + m)$) time.
- This DFS algorithm is a good example. Without using adjacent list representation, the problem would take at least $\Theta(n^2)$ time to solve.