期中考试

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一.

1.

记 A 发生的次数为 X，则 $X \sim B(n,p)$

$$\begin{aligned}
P(\text{至少发生一次}) &= P(X \geq 1) \\
&= 1 - P(X = 0) \\
&= 1 - C_n^0 p^0 (1-p)^n \\
&= 1 - (1-p)^n
\end{aligned}$$

$$\begin{aligned}
P(\text{至多发生一次}) &= P(X = 0) + P(X = 1) \\
&= C_n^0 p^0 (1-p)^n + C_n^1 p (1-p)^{n-1} \\
&= (1-p)^n + np(1-p)^{n-1}
\end{aligned}$$

答案：$1-(1-p)^n$；$(1-p)^n + np(1-p)^{n-1}$

2.

$$\begin{aligned}
P(A \cup B) &= P(A) + P(B) - P(AB) \\
&= P(A) + P(B) - P(B|A)P(A) \\
&= 0.5 + 0.6 - 0.8 \times 0.5 \\
&= 0.7
\end{aligned}$$

答案：0.7

3.

由于 X 与 Y 的联合分布未知，故不相关未必独立

答案：C

4.

由分布函数的规范性，有

$$\lim_{x\to+\infty} F(x) = a - b = 1$$

由分布函数的非负性，有

$$aF_1(x) - bF_2(x) \geq 0,\ \forall x \in R$$

即对二元函数 $f(u,v) = au - bv$，有 $\min\limits_{(u,v)\in[0,1]^2} f(u,v) \geq 0$，从而

$$\begin{cases} a \geq 0 \\ b \leq 0 \\ a - b \geq 0 \end{cases}$$

综上，有

$$\begin{cases} a \geq 0 \\ b \leq 0 \\ a - b = 1 \end{cases}$$

符合条件的仅有 A

答案：A

5.

由正态分布的对称性，有

$$\alpha = P(|X| \le x) = 1 - 2P(X > x)$$

即 $P(X > x) = \frac{1-\alpha}{2}$，则 $x = u_{\frac{1-\alpha}{2}}$

答案：B

6.

二次方程无实根的概率 p 为

$$\begin{aligned} p &= P(16 - 4X < 0) \\ &= P(X > 4) \\ &= P(X - \mu > 4 - \mu) = \frac{1}{2} \end{aligned}$$

则 $4 - \mu = 0$，即 $\mu = 4$

答案：4

7.

记 $A_i =$" 第 i 个零件合格 "，则

$$\begin{aligned} P(X = 2) &= P(\bar{A}_1 A_2 A_3) + P(A_1 \bar{A}_2 A_3) + P(A_1 A_2 \bar{A}_3) \\ &= (1 - p_1)p_2 p_3 + p_1(1 - p_2)p_3 + p_1 p_2 (1 - p_3) \\ &= \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} + \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{3}{4} + \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{1}{4} \\ &= \frac{11}{24} \end{aligned}$$

答案：$\frac{11}{24}$

8.

$F_X(t) = F_Y(t) = (1 - e^{-\lambda t})I(t > 0)$ 则 $Z := \min\{X, Y\}$ 的分布函数为

$$\begin{aligned} F_Z(z) &= P(\min\{X,Y\} \le z) \\ &= 1 - P(X > z, Y > z) \\ &= 1 - (1 - F_X(z))(1 - F_Y(z)) \\ &= (1 - e^{-2\lambda z})I(z > 0) \end{aligned}$$

即 $Z \sim Exp(2\lambda)$

答案：2λ；指数

9.

要求 $\rho_{X,Y+Z} = \frac{Cov(X,Y+Z)}{\sqrt{Var(X)Var(Y+Z)}}$

$$\begin{aligned} Cov(X, Y+Z) &= Cov(X,Y) + Cov(X,Z) \\ &= \rho_{X,Y}\sqrt{Var(X)Var(Y)} + \rho_{X,Z}\sqrt{Var(X)Var(Z)} \\ &= -2Var(X) + \frac{1}{2}Var(X) \\ &= -\frac{3}{2}Var(X) \end{aligned}$$

由于 $\rho_{X,Y} = -1$，则存在 $a < 0, b \in R$，使得 $Y = aX + b$，又 $Var(Y) = 4Var(X)$，则 $a = -2$，从而

$$\begin{aligned} Var(Y+Z) &= Var(Y) + Var(Z) + 2Cov(Y,Z) \\ &= 5Var(X) - 4Cov(X,Z) \\ &= 3Var(X) \end{aligned}$$

则 $\rho_{X,Y+Z} = \frac{-\frac{3}{2}Var(X)}{\sqrt{3}Var(X)} = -\frac{\sqrt{3}}{2}$

答案：$-\frac{\sqrt{3}}{2}$

10.

$$
\begin{aligned}
af(x)+bg(x)\text{为密度函数} \Longleftrightarrow &\int_{-\infty}^{\infty} af(x)+bg(x)dx = a+b=1;\\
&af(x)+bg(x)\ge 0, \forall x\in R\\
\Longleftrightarrow & a+b=1, a\ge 0, b\ge 0
\end{aligned}
$$

答案：$a+b=1, a\ge 0, b\ge 0$

二.

(1).

记 $A=$"检测呈阳性"，$B=$"该人带菌"，则所求概率即为

$$
\begin{aligned}
P(B|A) &= \frac{P(AB)}{P(A)}\\
&= \frac{P(A|B)P(B)}{P(A|B)P(B)+P(A|\overline{B})P(\overline{B})}\\
&= \frac{0.95\times 0.1}{0.95\times 0.1+0.01\times 0.9}\\
&= \frac{95}{104}\\
&\approx 0.9135
\end{aligned}
$$

(2).

记 $C=$"三次检测中 2 次阳性 1 次阴性"，则所求概率即为

$$
\begin{aligned}
P(B|C) &= \frac{P(C|B)P(B)}{P(C|B)P(B)+P(C|\overline{B})P(\overline{B})}\\
&= \frac{0.95\cdot C_2^1\cdot 0.05\cdot 0.95\times 0.1}{0.95\cdot C_2^1\cdot 0.05\cdot 0.95\times 0.1+0.01\cdot C_2^1\cdot 0.01\cdot 0.99\times 0.9}\\
&= \frac{45125}{46016}\\
&\approx 0.9806
\end{aligned}
$$

三.

(1).

由规范性，有

$$
\begin{aligned}
1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dxdy = A\int_{0}^{\infty}\int_{0}^{\infty} e^{-3x-4y}dxdy \\
= \frac{A}{12}
\end{aligned}
$$

则 $A = 12$

(2).

$$
\begin{aligned}
f_X(x) &= \int_{-\infty}^{\infty} f(x,y)dy \\
&= 3e^{-3x}I(x>0)\int_{0}^{\infty} 4e^{-4y}dy \\
&= 3e^{-3x}I(x>0)
\end{aligned}
$$

同理有 $f_Y(y) = 4e^{-4y}I(y>0)$

则 $f(x,y) = f_X(x)f_Y(y)$，X 与 Y 独立

(3).

由卷积公式，有

$$
\begin{aligned}
f_Z(z) &= \int_{-\infty}^{\infty} f(x, z-x)dx \\
&= \int_{0}^{z} 12e^{x-4z}I(z>0)dx \\
&= 12(e^{-3z} - e^{-4z})I(z>0)
\end{aligned}
$$

(4).

X 与 $Z=X+Y$ 的联合分布函数为

$$\begin{aligned}
F_{XZ}(x,z) &= P(X\le x, X+Y\le z)\\
&= \iint\limits_{\substack{u\le x\\ u+v\le z}} f(u,v)dudv\\
&= \begin{cases} \int_0^z \int_0^{z-u} 12e^{-3u-4v}dvdu, & 0<z\le x\\ \int_0^x \int_0^{z-u} 12e^{-3u-4v}dvdu, & 0<x<z\\ 0, & else \end{cases}\\
&= \begin{cases} 1-e^{-3z}+3e^{-4z}-3e^{-5z}, & 0<z\le x\\ 1-e^{-3x}+3e^{-4z}-3e^{x-4z}, & 0<x<z\\ 0, & else \end{cases}
\end{aligned}$$

则给定 $Z=X+Y=1$ 的条件下，X 的条件密度为

$$\begin{aligned}
f_{X|Z=1}(x|z=1) &= \frac{f_{XZ}(x,1)}{f_Z(1)}\\
&= \frac{e^x}{e-1} I(0<x<1)
\end{aligned}$$

从而 $E(X|X+Y=1)=\frac{1}{e-1}$，$E(X^2|X+Y=1)=\frac{e-2}{e-1}$

故

$$\begin{aligned}
Var(X|X+Y=1) &= E(X^2|X+Y=1)-(E(X|X+Y=1))^2\\
&= \frac{e^2-3e+1}{(e-1)^2}
\end{aligned}$$

四.

(1).

对 $n = 10000, p = 0.017$，记

$$X_i = \begin{cases} 1, & i\text{老人在年内死亡} \\ 0, & \text{否则} \end{cases}$$

则 $X_1, \ldots, X_n \overset{i.i.d.}{\sim} B(1, p)$

由中心极限定理，亏本概率 p 为

$$\begin{aligned}
p &= P\left(10000\sum_{i=1}^{n} X_i > 2000000\right) \\
&= P\left(\sum_{i=1}^{n} X_i > 200\right) \\
&= P\left(\frac{\sum\limits_{i=1}^{n} X_i - np}{\sqrt{np(1-p)}} > \frac{200 - np}{\sqrt{np(1-p)}}\right) \\
&\approx 1 - \Phi(\frac{200 - np}{\sqrt{np(1-p)}}) \\
&\approx 1 - \Phi(2.32) \\
&\approx 0.01
\end{aligned}$$

(2).

利用中心极限定理，所求概率 p' 为

$$
\begin{aligned}
p' &= P\left(100000 \le 2000000 - 10000\sum_{i=1}^{n} X_i \le 200000\right) \\
&= P\left(180 \le \sum_{i=1}^{n} X_i \le 190\right) \\
&= P\left(\frac{180-np}{\sqrt{np(1-p)}} \le \frac{\sum_{i=1}^{n} X_i - np}{\sqrt{np(1-p)}} \le \frac{190-np}{\sqrt{np(1-p)}}\right) \\
&\approx \Phi(1.55) - \Phi(0.77) \\
&\approx 0.16
\end{aligned}
$$

五.

(1)

由于 $Z := X + Y - 3\mu \sim N(0, 4\sigma^2)$

则

$$
\begin{aligned}
E[(X+Y-3\mu)_+] &= EZ_+ \\
&= \int_0^{\infty} z f_Z(z) dz \\
&= \int_0^{\infty} \frac{z}{\sqrt{8\pi\sigma^2}} e^{-\frac{z^2}{8\sigma^2}} dz \\
&= \sqrt{\frac{2}{\pi}}\sigma
\end{aligned}
$$

(2)

$$\begin{aligned}
E[(X+Y-3\mu)_+^2] &= EZ_+^2 \\
&= \int_0^\infty z^2 f_Z(z)dz \\
&= \int_0^\infty \frac{z^2}{\sqrt{8\pi\sigma^2}} e^{-\frac{z^2}{8\sigma^2}} dz \\
&= 2\sigma^2
\end{aligned}$$

则 $Var[(X+Y-3\mu)_+] = EZ_+^2 - (EZ_+)^2 = (2-\frac{2}{\pi})\sigma^2$

六.

(1)

由于 $0 \le x \le 1$，则有

$$\begin{aligned}
EF_n(x) &= \frac{1}{n}\sum_{k=1}^{n} EI(U_k \le x) \\
&= P(U_1 \le x) \\
&= x
\end{aligned}$$

$$\begin{aligned}
Var(F_n(x)) &= \frac{1}{n} Var(I(U_1 \le x)) \\
&= \frac{1}{n}\left(EI^2(U_1 \le x) - [EI(U_1 \le x)]^2\right) \\
&= \frac{1}{n}\left(EI(U_1 \le x) - [EI(U_1 \le x)]^2\right) \\
&= \frac{x-x^2}{n}
\end{aligned}$$

(2)

由于 $0 < x, y < 1$，则

$$
\begin{aligned}
Cov(F_n(x), F_n(y)) &= Cov\left(\frac{1}{n}\sum_{k=1}^{n} I(U_k \le x), \frac{1}{n}\sum_{l=1}^{n} I(U_l \le y)\right) \\
&= \frac{1}{n^2}\sum_{k=1}^{n} Cov\left(I(U_k \le x), \sum_{l=1}^{n} I(U_l \le y)\right) \\
&= \frac{1}{n^2}\sum_{k=1}^{n} Cov\left(I(U_k \le x), I(U_k \le y)\right) \\
&= \frac{1}{n} Cov(I(U_1 \le x), I(U_1 \le y)) \\
&= \frac{1}{n}\left(E[I(U_1 \le x)I(U_1 \le y)] - EI(U_1 \le x)EI(U_1 \le y)\right) \\
&= \frac{\min\{x, y\} - xy}{n}
\end{aligned}
$$