Outline

Approximation Algorithms

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- In many applications, a nearly optimal solution might be good enough.
- This is the subject of Approximation Algorithms: Try to find solutions not too far from optimal.

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- So we now switch back to optimization problems.

Approximation Algorithms for Minimization Problems

Approximation Algorithm

- Q: a minimization problem.
- A: an algorithm for solving Q.
- I: an instance of Q.
- $Opt(I)$: the optimal solution of I.
- $|Opt(I)|$: the value of $Opt(I)$.
- $A(I)$: the solution found by A on input I.
- $|A(I)|$: the value of $A(I)$.
- If $\frac{|A(I)|}{|Opt(I)|} \leq r$ for some constant r and for ALL input instances I, then we say "A is an approximation algorithm for Q with performance ratio r."

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- I: an instance of TSP: Given a complete graph $G = (V, E)$ and weight function $w(*)$, find a HC C of G with minimum total length $w(C)$.

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- If $\frac{|A(I)|}{|Opt(I)|} \leq 1.5$ for ALL input instances I, then we say "A is an approximation algorithm for TSP with performance ratio 1.5."
- So, for any input G, w, A will always find a HC of G within 50% of the optimal length.

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- The goals of approximation algorithm design:
  - Reduce the performance ratio r.
  - Reduce the run time.

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- Depending on applications, one goal might be more important than the other goal.
- For most approximation algorithm research, the primary goal is to reduce the performance ratio as long as we stay within polynomial time.

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- Our goal is still to reduce r.

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- The worst drawback: You never know how far is your solution from the optimal.
- In some cases, the solutions produced by heuristic algorithms can be very bad. And you don't know it!

Approximation Algorithms vs Heuristic Algorithms

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In contrast, the approximation algorithm:

- Must prove the performance ratio.
- Harder to design. (How do you compare the solution constructed by the algorithm with the optimal solution which is UNKNOWN?)
- Sometimes, they are counter-intuitive. (They must consider the worst cases.)
- However, because of the existence of the performance ratio, we know how the solution constructed by the algorithm compares with the optimal solution. (If $r = 1.5$, our solution is at most within 50% of optimal.)

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Approximation Algorithms: Minimum Vertex Cover (MVC)

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Input: An undirected graph $G = (V, E)$.
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- If $deg(v) = k$, then the vertex v will cover k edges.
- Thus, the heuristic algorithm should include in C the vertices with high degrees.

MVC: Heuristic Algorithm

Heuristic-MVC(G)

1. $C \leftarrow \emptyset$
2. **while** $E \neq \emptyset$ **do**
3. pick a vertex v with the highest $deg(v)$ (break ties arbitrarily)
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- It can be infinitely bad!

Heuristic Algorithm: Minimum Vertex Cover (MVC)

Fact

For any $r > 0$, there exists graphs G so that $\frac{|\text{Heuristic-MVC}(G)|}{|\text{Opt}(G)|} \geq r$.

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- $|X_3| = n/3$. Each vertex in X_3 is adjacent to 3 vertices in Y.

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- $\ldots$
- $|X_k| = n/k$. Each vertex in X_k is adjacent to k vertices in Y.

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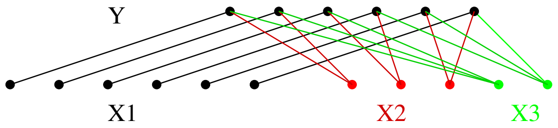

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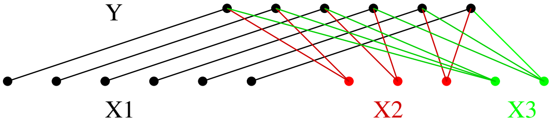

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- Every vertex in X_1 has degree 1.
- Every vertex in X_2 has degree 2.
- $\ldots$
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- Every vertex in Y has degree k

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- However, Opt$(G) = Y$.

$$\begin{aligned}\frac{|\text{Heuristic-MVC}(G)|}{|\text{Opt}(G)|} &= \frac{n+n/2+n/3+\cdots+n/k}{n} \\ &= 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{k} \sim \ln k\end{aligned}$$

Heuristic Algorithm: Minimum Vertex Cover (MVC)

- The Heuristic-MVC algorithm include all vertices in X_k into C.
- Then all vertices in X_{k-1} into C.
- $\cdots$
- Then all vertices in X_1 into C.
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- The last sum is the Harmonic series (remember?), and it approaches to $\ln k$.
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Approximation Algorithm: MVC

Appr-MVC$(G = (V, E))$

1. $C \leftarrow \emptyset$ (C will be a VC of G)
2. $M \leftarrow \emptyset$ (M will be a matching of G. It is not really needed by the algorithm. However, it will help to prove the performance ratio.)

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2. $M \leftarrow \emptyset$ (M will be a matching of G. It is not really needed by the algorithm. However, it will help to prove the performance ratio.)
3. **while** $E \neq \emptyset$ **do:**
4. pick any edge $e = (u, v)$ in G
5. $C \leftarrow C \cup \{u, v\}$
6. $M \leftarrow M \cup \{e\}$
7. delete all edges that are incident to u or v from E
8. **output** C

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Lemma

Let $G = (V, E)$ be a graph. Let C be any vertex cover of G. Let M be any matching of G. Then $|C| \geq |M|$.

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Proof.

- Any edge $e \in M$ must be covered by a vertex in C.
- No two edges $e_1, e_2 \in M$ can be covered by the same vertex in C, because e_1 and e_2 have no common end vertex.
- Therefore $|C| \geq |M|$.

□

Approximation Algorithms: MVC

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- Nevertheless, it works well.
- This simple algorithm is the best approximation algorithm for MVC for general graphs.

Outline

Traveling Salesman Problem (TSP)

TSP

Input: A complete graph $G = (V, E)$ with edge weight function $w(*) \geq 0$.
Find: A HC C of G so that the total weight $w(C)$ is minimum.

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$$w(x, y) + w(y, z) \geq w(x, z)$$

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- Δ-TSP is still $\mathcal{NPC}$.

TSP: Heuristic Algorithm

Heuristic-TSP(G)

1. start at the beginning vertex v_1
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For any $r > 0$, there exists instance $I = \langle G, w(*)\rangle$ of Δ-TSP so that $\frac{|\text{Heuristic-TSP}(I)|}{|\text{Opt}(I)|} \geq r$.

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This heuristic algorithm can perform very badly, and you would not know it!

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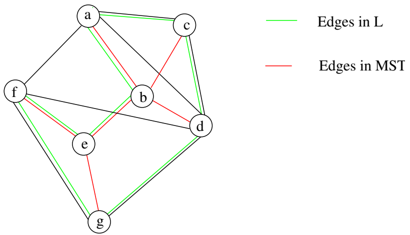

In the above example, the HC returned by the algorithm is: $abefgdca$.

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The performance ratio of Appr-TSP is $r = 2$.

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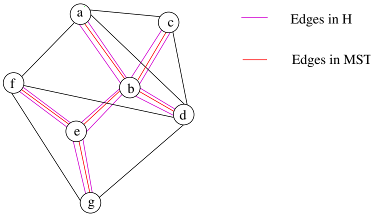
- The output L can can be viewed as constructed as follows:
- Start with the MST T.
- Travel around T, using each edge of T exactly twice. This is a tour H of G. (It is not a HC of G, since some vertices are traveled more than once).
- So $w(H) = 2 \cdot w(T)$.



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Short-Cut Operation

Suppose that a vertex v is visited by H more than once. Let $u \to v \to w$ be a section of H containing v. A short-cut at v is the operation that replaces $u \to v \to w$ by $u \to w$.

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- However, in order to prove the performance ratio, we must find a lower bound for $w(O)$.

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We can do better.

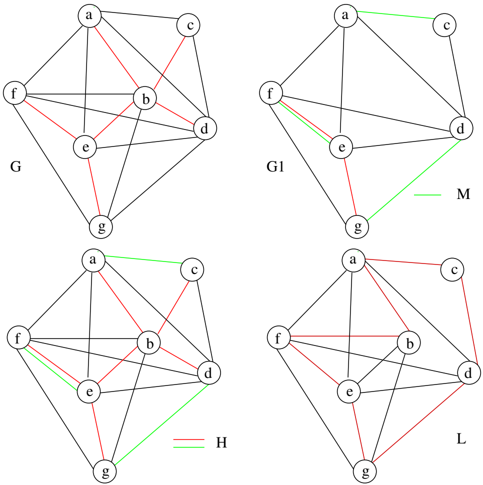
Outline

TSP: Christofides Algorithm

Christofides$(G = (V, E))$

1. construct a minimum spanning tree T of G.
2. let $V_1 \subseteq V$ be the set of vertices that have odd degrees in T.
3. let $G_1 = (V_1, E_1)$ be the subgraph of G consisting of all vertices in V_1 and all edges connecting them.
4. find a minimum weight perfect matching M in G_1. (Namely, M is a perfect matching of G_1 so that $w(M)$ is the minimum among all perfect matchings of G_1.)
5. consider the graph consisting of the edges $T \cup M$. Every vertex has even degree in this graph.
6. find an Euler tour H in $T \cup M$. (H visits each vertex of G at least once.)
7. take short-cuts on H, until it becomes a HC L of G.
8. return L

TSP: Christofides Algorithm



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- Let O be an optimal HC of G (which is unknown.)
- Same as before, we have $w(O) \geq w(T)$.
- Suppose we can show $w(M) \leq \frac{w(O)}{2}$ then:

$$\begin{aligned} w(L) &\leq w(H) && (L \text{ is obtained from } H \text{ by short-cuts.}) \\ &= w(T) + w(M) && (\text{because } H = T \cup M) \\ &\leq w(O) + w(O)/2 \\ &= 1.5w(O) \end{aligned}$$

Then we will have $\frac{w(L)}{w(O)} \leq 1.5$ (this is all we want to show.)

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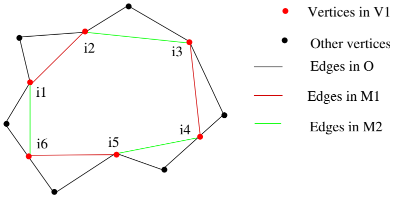
- Let $M_1 = \{(i_1, i_2), (i_3, i_4), \ldots (i_{k-1}, i_k)\}$ and $M_2 = \{(i_2, i_3), (i_4, i_6) \ldots (i_k, i_1)\}$



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$$\begin{array}{rcll} w(O) & \geq & w(O') & \text{(because } O' \text{ is obtained from } O \text{ by short-cuts)} \\ & = & w(M_1) + w(M_2) & \text{(because } O' = M_1 \cup M_2\text{.)} \\ & \geq & w(M) + w(M) & \\ & = & 2 \cdot w(M) & \end{array}$$

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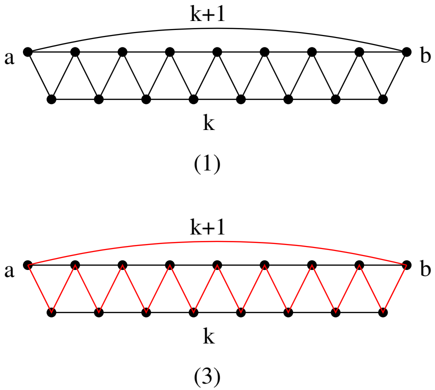
- The following is an example that shows the ratio is actually 1.5.



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  - It has $k+1$ vertices on the lower line and $k+2$ vertices on the upper line.
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- So the ratio on this graph is $\frac{3k+3}{2k+3} \rightarrow 1.5$ when $k \rightarrow \infty$.

Outline

Polynomial Time Approximation Scheme

Polynomial Time Approximation Scheme (PTAS)

Let Q be a given optimization problem. A PTAS for Q is a class of algorithms so that, for any $\epsilon > 0$, we have an algorithm A_ϵ in this class with following properties:

- The runtime of A_ϵ is polynomial in n (it may depend on ϵ in any way).
- The performance ratio of A_ϵ is at most $1 + \epsilon$.

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- Let $\epsilon = 0.01$. Then we have an algorithm $A_{0.01}$ with performance ratio 1.01 (namely within 1% of optimal), with runtime $O(n^{100})$.

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- Example: The runtime of A_ϵ is $O(n^{\frac{1}{\epsilon}})$.
- Let $\epsilon = 0.1$. Then we have an algorithm $A_{0.1}$ with performance ratio 1.1 (namely within 10% of optimal), with runtime $O(n^{10})$.
- Let $\epsilon = 0.01$. Then we have an algorithm $A_{0.01}$ with performance ratio 1.01 (namely within 1% of optimal), with runtime $O(n^{100})$.
- Letting ϵ smaller and smaller, we can approximate the optimal solution with arbitrarily small error. But we pay a heavy price on run time.

Polynomial Time Approximation Scheme

Fully Polynomial Time Approximation Scheme (FPTAS)

Let Q be a given optimization problem. A FPTAS for Q is a class of algorithms so that, for any $\epsilon > 0$, we have an algorithm A_ϵ in this class with following properties:

- The runtime of A_ϵ is polynomial in both n and $1/\epsilon$.
- The performance ratio of A_ϵ is at most $1 + \epsilon$.

Polynomial Time Approximation Scheme

Fully Polynomial Time Approximation Scheme (FPTAS)

Let Q be a given optimization problem. A FPTAS for Q is a class of algorithms so that, for any $\epsilon > 0$, we have an algorithm A_ϵ in this class with following properties:

- The runtime of A_ϵ is polynomial in both n and $1/\epsilon$.
- The performance ratio of A_ϵ is at most $1 + \epsilon$.

- Example: The runtime of A_ϵ is $O(n^{\frac{1}{\epsilon}})$. It is not a FPTAS: The runtime is polynomial in n, but exp in $1/\epsilon$.
- Example: The runtime of A_ϵ is $O(n^3 \cdot (\frac{1}{\epsilon})^4)$. The runtime is polynomial in both n and $1/\epsilon$. This is a FPTAS.

Polynomial Time Approximation Scheme

- Baring the extremely unlikely event that $\mathcal{NP} = \mathcal{P}$, a FPTAS is the best we can hope for solving an $\mathcal{NPC}$ problem:
  - We can approximate the optimal solution within arbitrarily small error.
  - The runtime is polynomial
  - The runtime is polynomial in error rate $1/\epsilon$.

Polynomial Time Approximation Scheme

Euclidean TSP

This is a special case of the ΔTSP:

- The vertices are the points on the 2D plane (or high dimension space.)
- The weight function is $w(u, v)$ = the Euclidean distance between the point u and the point v.

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- The vertices are the points on the 2D plane (or high dimension space.)
- The weight function is $w(u, v)$ = the Euclidean distance between the point u and the point v.

- Euclidean TSP is still $\mathcal{NPC}$.
- Aurora showed (1992): There is a FPTAS for solving Euclidean TSP.
- In contrast, Christofides algorithm is the best known algorithm for solving the ΔTSP.

Polynomial Time Approximation Scheme

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Theorem

If there exists a polynomial time approximation algorithm for solving the Maximum Clique problem (or the Maximum Independent Set problem) for any constant performance ratio r, then $\mathcal{NP} = \mathcal{P}$.

- So unless $\mathcal{NP} = \mathcal{P}$, we cannot have an approximation algorithm for MC with performance ratio $r = 100$, $r = 10000$, any r!