Assignment 3

Evaluation of G/G/1 SYSTEM

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1 Introduction

1.1 Objective

The *objective* of this work is to compare results obtained by simulation for a given G/G/1 system with the Allen-Cuneen's approximation formula and to analyze the performance of a single server queuing system by simulation when service times correspond to a long-tailed distribution of probability.

1.2 Process

The *process* will be following:
For each loading factor ρ:

1. Find parameter b so that it corresponds to loading factor. ρ
2. Evaluate average waiting time in queue W_q using Allen Cuneen'approximation formula.
3. Simulate G/G/1 system using recurrent relationship given in section 1.3
4. Analyze simulated G/G/1 system, obtain confidence intervals, and check if stable state has achieved.
5. Compare simulated average waiting time at queue with one obtained using Allen-Cuneen's Approximation Formula.

1.3 Recurrent Relationship for Generating G/G/1 System

- x_i - service time for client i following Lognormal distribution with $\sigma = 1.6333$ (INPUT)
- τ_i - interarrival time for client i following Weibull Distribution with parameters a and b (INPUT)
- θ_i - exit time instant from W.S. for client i
- t_i^S - arrival time instant to the service system
- t_i - entrance time instant to the W.S.
- $w_i = \theta_i - t_i$ - sojourn time of client i in W.S.
- $w_{q,i} = t_i^S - t_i$ - sojourn time of client i in queue
- $\mathbf{L}_i = w_i$ - contribution of client i to the occupancy

Before starting the recurrence we need to randomly generate interarrival (τ) and service times (x) for N clients, following Weibull Distribution with parameters **a** and b. In addition to that we need to initialize average occupancy, average waiting time in W.S., average occupancy in queue, and average waiting time in queue as following: $\mathbf{L} = 0; W = 0; L_q = 0; W_q = 0; \theta_0 = -\infty; t_1 = 0$. After initialization we might begin with recurrent relationship:
For $i = 1, 2, 3, ..., N$, where N is number of clients

1. $t_i^S = max\{\theta_{i-1}, t_i\}$
2. $\omega_i = t_i^S + x_i$
3. $t_{i+1} = t_i + \tau_i$
4. $\mathbf{L}_i = w_i = \omega_i - t_i; \mathbf{L} = \mathbf{L} + \mathbf{L}_i; \mathbf{L}_{T_i} = \frac{\mathbf{L}}{t_i - t_1}; W = W + w_i$
5. $\mathbf{L}_{q,i} = w_{q,i} = t_i^S - t_i; \mathbf{L}_q = \mathbf{L}_q + \mathbf{L}_{q,i}; W = W + w_{q,i}$

And finally, after client N report we adjust average waiting time, waiting time in queue, average occupancy, and average occupancy in queue:

$W = W/N; W_q = W_q/N; \mathbf{L} = \frac{\mathbf{L}}{t_N - t_1}; \mathbf{L}_q = \frac{\mathbf{L}_q}{t_N - t_1}$

1.4 Parameters and Distributions

We are given following parameters and distributions from PDF file:

- Interarrival times: *Weibull* with parameters a and b
- Service times: *Lognormal* with parameters σ and m
- $a = 2; b = 72; \sigma = 1.6333$ and m - determined for specific ρ
- $\rho = [0.4, 0.7, 0.85, 0.925]$
- $E[\tau] = 64$
- $N = 100000$

2 Analysis of Service Time

Before starting analysis we need to deduct formula for finding free parameter m for Lognormal distribution of service time given just σ. We know that $\omega = e^{\sigma^2}$ and

$$E[x] = m\omega^{\frac{1}{2}} \tag{1}$$

Dividing both sides of (1) by $E[\tau]$ we get

$$\rho = \frac{E[x]}{E[\tau]} = \frac{m\omega^{\frac{1}{2}}}{E[\tau]} \tag{2}$$

From (2) we deduct the formula for m parameter of Lognormal distribution

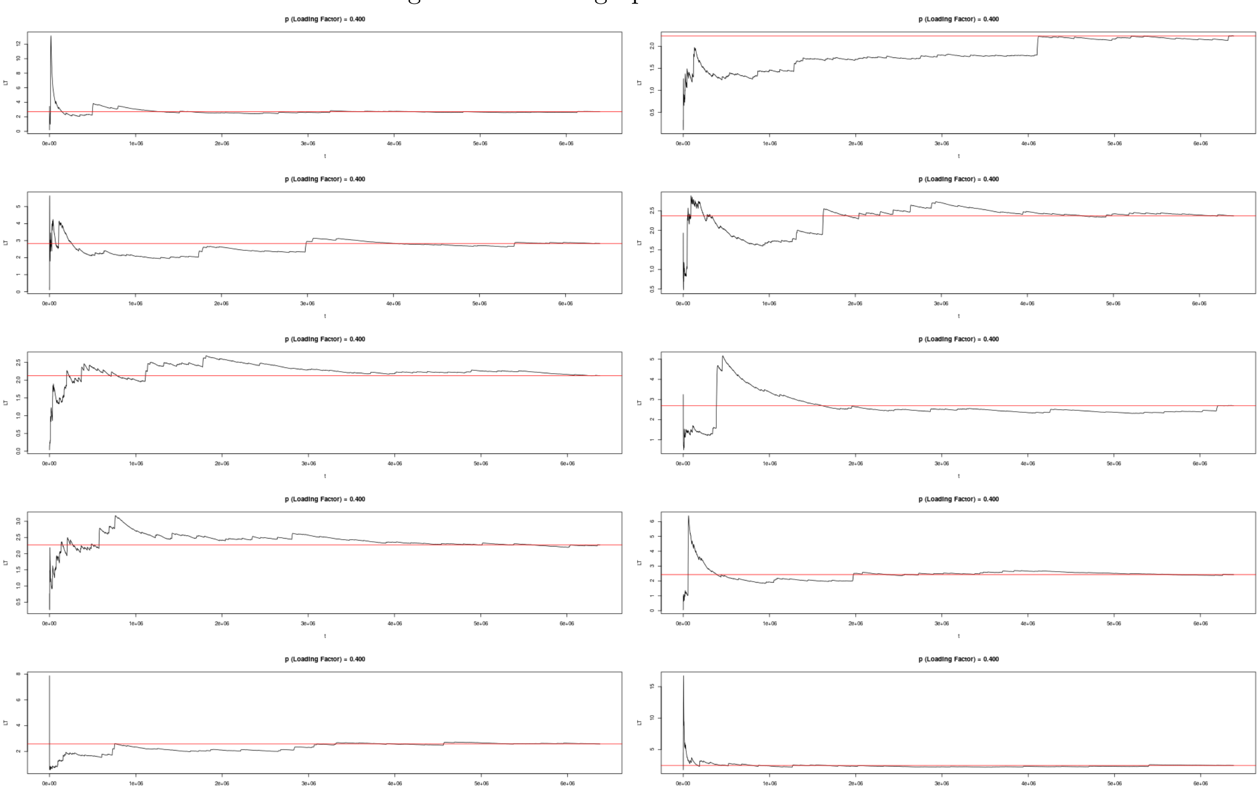
$$m = \frac{\rho E[\tau]}{\omega^{\frac{1}{2}}} \tag{3}$$

Using constants from PDF $E[\tau] = 64$ and $\omega = e^{1.6333^2}$ and formula (3) we determine m. After we can generate random number r following Lognormal distribution with $\sigma = 1.6333$ and $\mu = \log m$, such that r satisfies loading factor ρ.

2.1 Analysis for $\rho = 0.4$

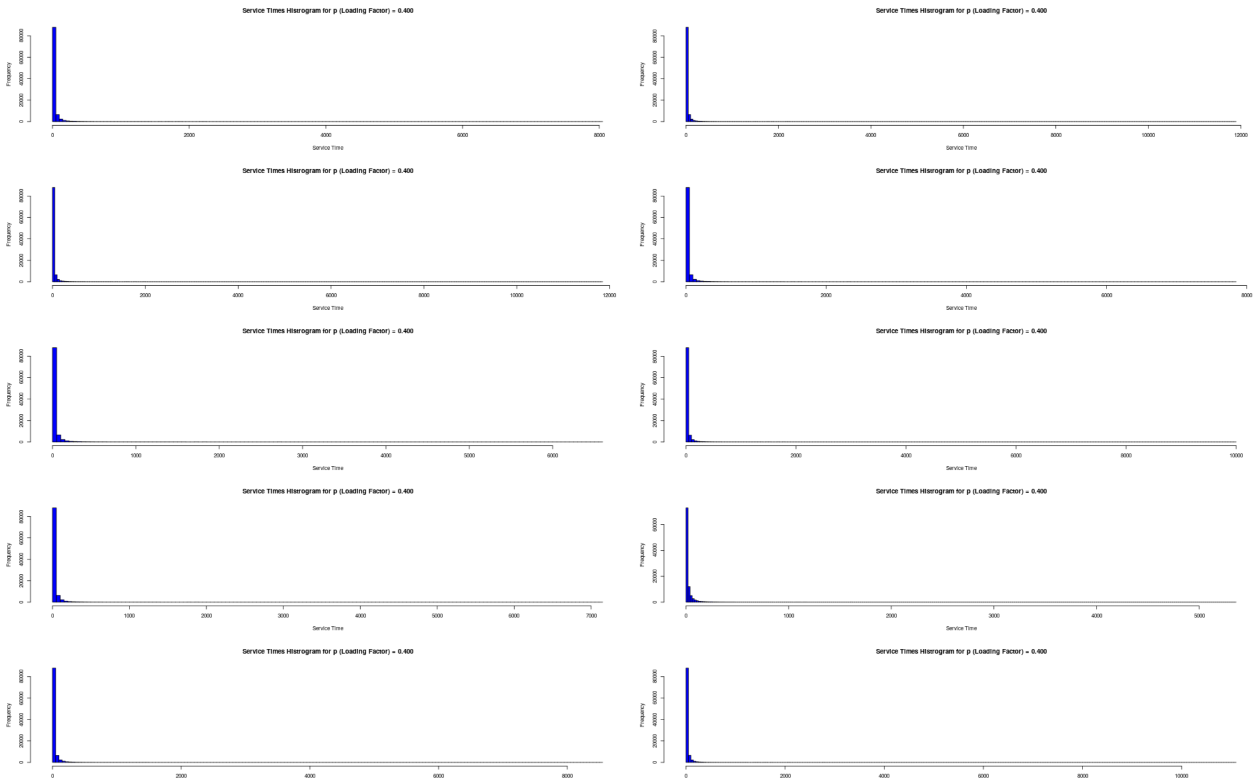
From formula (3) we find that $m = 6.744706$. Now we can proceed with performing 10 simulations and generating 100000 instances of service following Lognormal distribution with $\mu = \log m = 1.908758$ and $\sigma = 1.6333$. Before doing any analysis we must check if stable state has been achieved with $N = 100000$ instances. From Figure 1. we observe that stable values are reached in

Figure 1: LT vs t graphs for 10 simulations



all 10 simulations. So we can proceed with analysis itself. The mean service time

Figure 2: Histograms of Service Times for 10 simulations



from generated sample is $E[x] = 25.77618$, while theoretical mean service time is $E[x] = m\omega^{\frac{1}{2}} = 6.744706 * (e^{1.6333^2})^{\frac{1}{2}} = 25.6$. The variance of service time from generated sample is $Var[x] = 9041.654$, while theoretical variance of service time is $Var[x] = m^2\omega(\omega - 1) = 8785.984$. The coefficient of variation of service time from generated sample is $CV = 3.688973$, while theoretical coefficient of variation of service time is $\frac{\sigma}{mean} = \frac{8785.984}{25.6} = 3.661468$. While the histrograms of 10 simulated service times is depicted in Fibure 2.

2.1.1 Allen-Cunneen Approximation Formula for G/G/1

Before approximating $E[w_q] = W_q$, it is a good idea to simplify formulas, given some parameters constant. In our case $a = 2, b = 72, s = 1$ for *Weibull distribution* of interarrival times. So Allen-Cuneen Approximation Formula is given in (4)

$$E[w_q] = W_q \approx \frac{C(s,\omega)(\lambda^2\sigma_\tau^2 + \mu^2\sigma_x^2)}{2s\mu(1-\rho)}, \tag{4}$$

Table 1: Means and Confidence Intervals

Mean	Confidence Interval
83.59346	(81.75870 85.42822)
133.12588	(128.9750 137.2767)
127.14892	(123.5953 130.7026)
80.40804	(78.36801 82.44808)
94.04288	(91.91292 96.17284)
80.65092	(78.91879 82.38305)
86.77651	(84.54530 89.00773)
93.25557	(90.98146 95.52967)
117.4381	(75.97365 79.36727)

where $\theta = \frac{\lambda}{\mu}, \omega = E[x]/E[\tau], \lambda = 1/E[\tau], \mu = 1/E[x], \rho = E[x]/E[\tau], E[x] = m\omega^{\frac{1}{2}}, \sigma_x^2 = Var[x] = m^2\omega(\omega - 1), E[\tau] = b\Gamma(\frac{a+1}{a}), \sigma_\tau^2 = Var[\tau] = b^2(\Gamma(\frac{a+2}{a}) - \Gamma^2(\frac{a+1}{a}))$ and

$$C(s,\theta) = \frac{\frac{\theta^s}{s!(1-\rho)}}{\sum_{l=0}^{s-1} \frac{\theta^l}{l} + \frac{\theta^s}{s!(1-\rho)}} \tag{5}$$

However since some of our parameters are constant we can simplify above given equations and get following:

$$C(s,\theta) = \frac{\frac{\theta}{1-\rho}}{1 + \frac{\theta}{1-\rho}} = \frac{\theta}{1-\rho+\theta}$$

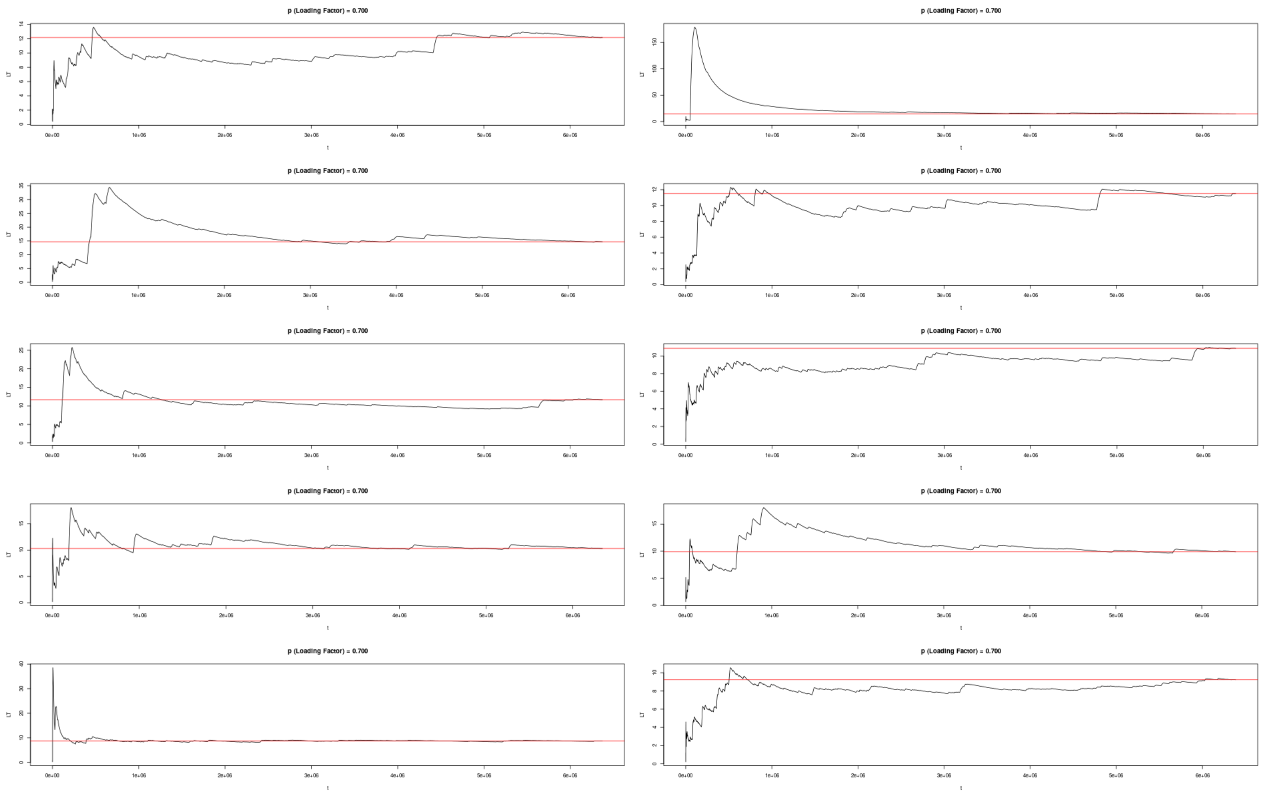
$$E[\tau] = \frac{b\sqrt{\pi}}{2}, \sigma_\tau^2 = Var[\tau] = b^2(1 - \pi/4))$$

Using above given formulas, constants parameters and variables $\rho = 0.4, m = 7.377022$, we obtain $E[w_q] = W_q \approx 134.8319$, while W_q from simulated service times is 144.4573. We notice that W_q of simulated service times doesn't differ from approximated theoretical value. We might construct 95% confidence interval for W_q from 10 simulations to understand the range of ρ can get on 95% significance level.

2.1.2 Confidence Intervals for W_q

In order to construct 95% confidence intervals for W_q, i.e. sojourn time in queue, we can apply t-statistics from R package for each of 10 simulations. The Table 1 shows Means and Confidence Intervals for each simulation. From Table 1, we observe that 2^{nd} and 3^{rd} simulations provide the closest mean value to approximation of $E[w_q]$ with Allen Cuneen's fomula. Which means we can use values of those simulations for doing other statistical analysis, which is out of our scope of investigation.

Figure 3: LT vs t graphs for 10 simulations



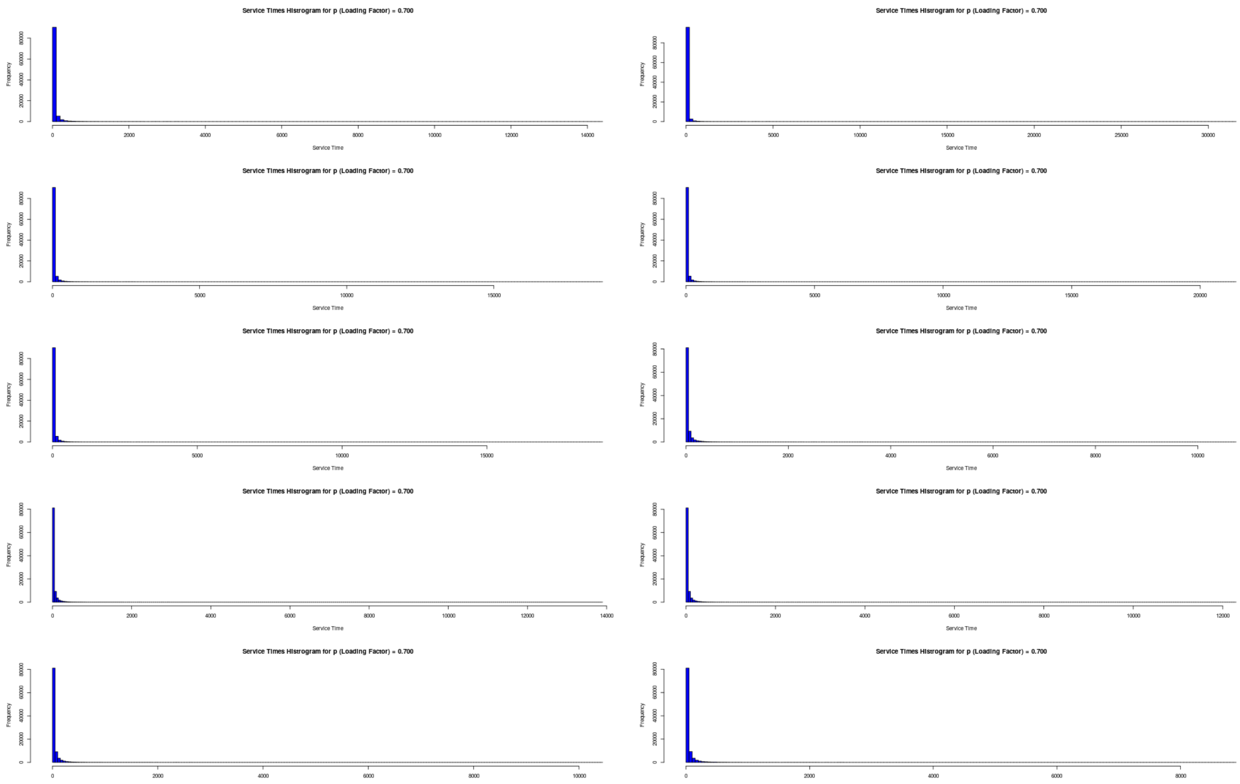
2.2 Analysis for $\rho = 0.7$

From formula (3) we find that $m = 11.80324$. Now we can proceed with performing 10 simulations and generating 100000 instances of service following Lognormal distribution with $\mu = \log m = 2.468374$ and $\sigma = 1.6333$. Before doing any analysis we must check if stable state has been achieved with $N = 100000$ instances.

From Figure 3. we observe that stable values are reached in all 10 simulations. So we can proceed with analysis itself. The mean service time from generated sample is $E[x] = 45.43346$, while theoretical mean service time is $E[x] = m\omega^{\frac{1}{2}} = 11.80324 * (e^{1.6333^2})^{\frac{1}{2}} = 44.8$. The variance of service time from generated sample is $Var[x] = 26025.52$, while theoretical variance of service time is $Var[x] = m^2\omega(\omega - 1) = 26907.08$. The coefficient of variation of service time from generated sample is $CV = 3.550781$, while theoretical coefficient of variation of service time is $\frac{\sigma}{mean} = \frac{164.0338}{44.8} = 3.661468$. While the histrograms of 10 simulated service times is depicted in Fibure 4.

In order to find approximated value of $E[w_q]$ we might use Allen Cuneen's

Figure 4: Histograms of Service Times for 10 simulations



approximation formula and simplified formulas from section 2.1.1, since values of constants have not changed.

Using formulas from 2.1.1, constant parameters and variables $\rho = 0.7, m = 11.80324$, we obtain $E[w_q] = W_q \approx 715.6293$, while W_q from simulated service times is 730.0697. We notice that W_q of simulated service times doesn't differ from approximated theoretical value. We might construct 95% confidence interval for W_q from 10 simulations to understand the range of ρ can get on 95% significance level.

In order to construct 95% confidence intervals for W_q, i.e. sojourn time in queue, we can apply t-statistics from R package for each of 10 simulations. The Table 2 shows Means and Confidence Intervals for each simulation. From Table 2, we observe that 1^{st} simulation produces closest mean value to approximation of $E[w_q]$ with Allen Cuneen's fomula. Which means we can use values of that simulation for doing other statistical analysis, which is out of our scope of investigation.

Table 2: Means and Confidence Intervals

Mean	Confidence Interval
730.0697	(719.5029 740.6365)
886.47240	(870.7601 902.1847)
893.60797	(880.7896 906.4264)
689.64981	(678.7907 700.5089)
698.64304	(688.3243 708.9618)
647.8496	(639.4242 656.2750)
608.72399	(600.4552 616.9928)
589.4436	(580.9568 597.9304)
508.97454	(502.7689 515.1802)

2.3 Analysis for $\rho = 0.85$

From formula (3) we find that $m = 14.3325$. Now we can proceed with performing 10 simulations and generating 100000 instances of service following Lognormal distribution with $\mu = \log m = 2.66253$ and $\sigma = 1.6333$. Before doing any analysis we must check if stable state has been achieved with $N = 100000$ instances.

From Figure 5. we observe that stable values are reached in all 10 simulations. So we can proceed with analysis itself. The mean service time from generated sample is $E[x] = 54.62658$, while theoretical mean service time is $E[x] = m\omega^{\frac{1}{2}} = 14.3325 * (e^{1.6333^2})^{\frac{1}{2}} = 54.4$. The variance of service time from generated sample is $Var[x] = 40230.74$, while theoretical variance of service time is $Var[x] = m^2\omega(\omega - 1) = 39674.21$. The coefficient of variation of service time from generated sample is $CV = 3.671766$, while theoretical coefficient of variation of service time is $\frac{\sigma}{mean} = \frac{199.1839}{54.4} = 3.661468$. While the histrograms of 10 simulated service times is depicted in Fibure 6.

In order to find approximated value of $E[w_q]$ we might use Allen Cuneen's approximation formula and simplified formulas from section 2.1.1, since values of constants have not changed.

Using formulas from 2.1.1, constant parameters and variables $\rho = 0.85, m = 14.3325$, we obtain $E[w_q] = W_q \approx 2109.428$, while W_q from simulated service times is 1968.038. We notice that W_q of simulated service times doesn't differ from approximated theoretical value. We might construct 95% confidence interval for W_q from 10 simulations to understand the range of ρ can get on 95% significance level.

In order to construct 95% confidence intervals for W_q, i.e. sojourn time in queue, we can apply t-statistics from R package for each of 10 simulations. The Table 3 shows Means and Confidence Intervals for each simulation.

From Table 3, we observe that 1^{st} and 3^{rd} simulations producs closest mean value to approximation of $E[w_q]$ with Allen Cuneen's fomula. Which means we can use values of those simulations for doing other statistical analysis.

Figure 5: LT vs t graphs for 10 simulations

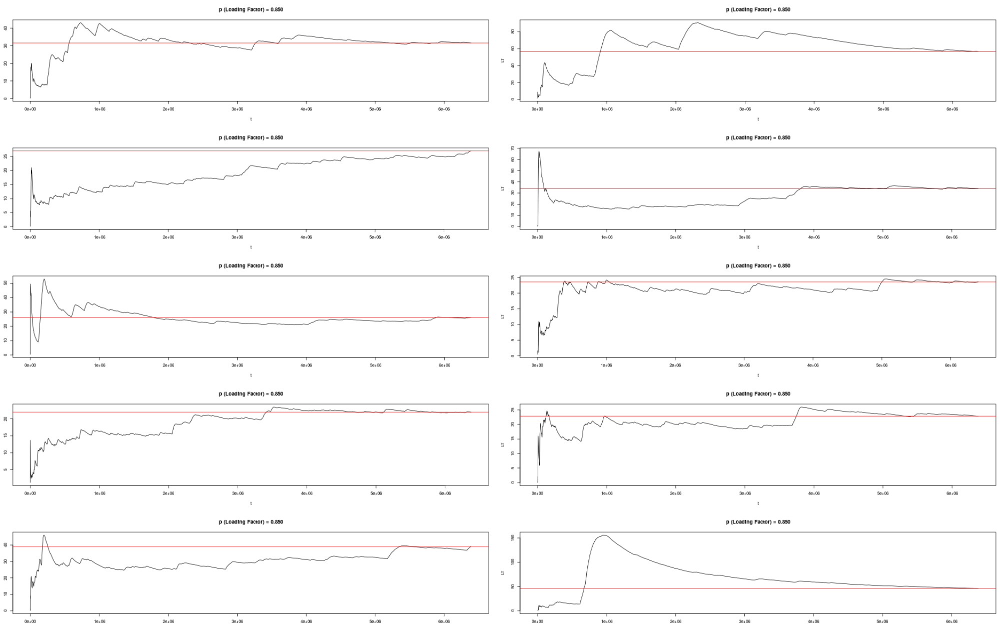
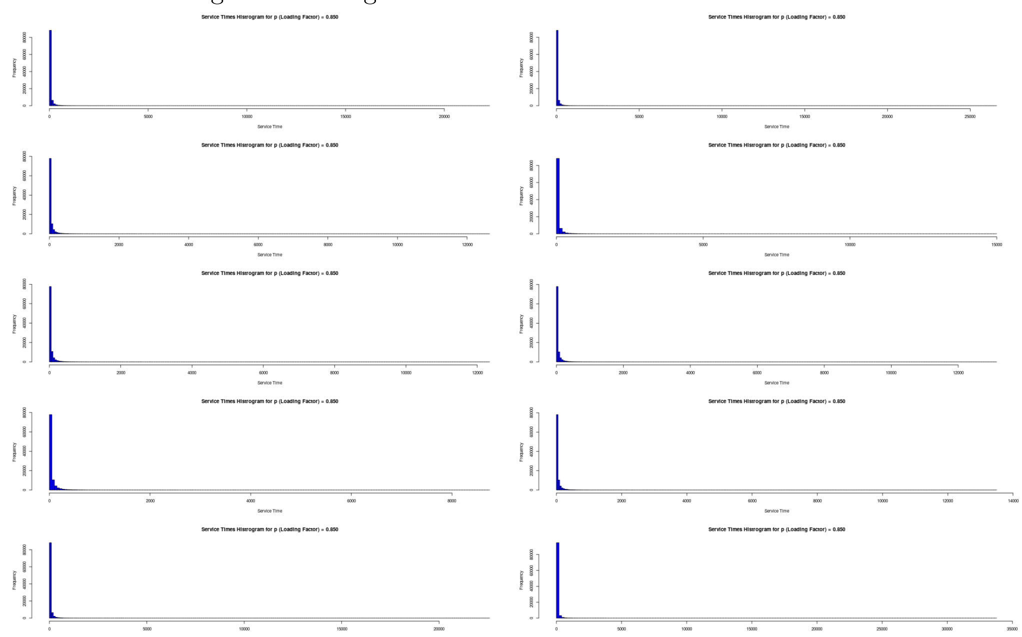


Table 3: Means and Confidence Intervals

Mean	Confidence Interval
1968.038	(1949.299 1986.778)
3551.072	(3512.328 3589.816)
2109.691	(2089.007 2130.376)
1620.176	(1605.286 1635.065)
1455.811	(1440.952 1470.671)
1347.671	(1335.476 1359.866)
1405.688	(1390.651 1420.726)
2439.927	(2415.907 2463.948)
2868.668	(2827.561 2909.775)

Figure 6: Histograms of Service Times for 10 simulations



2.4 Analysis for $\rho = 0.925$

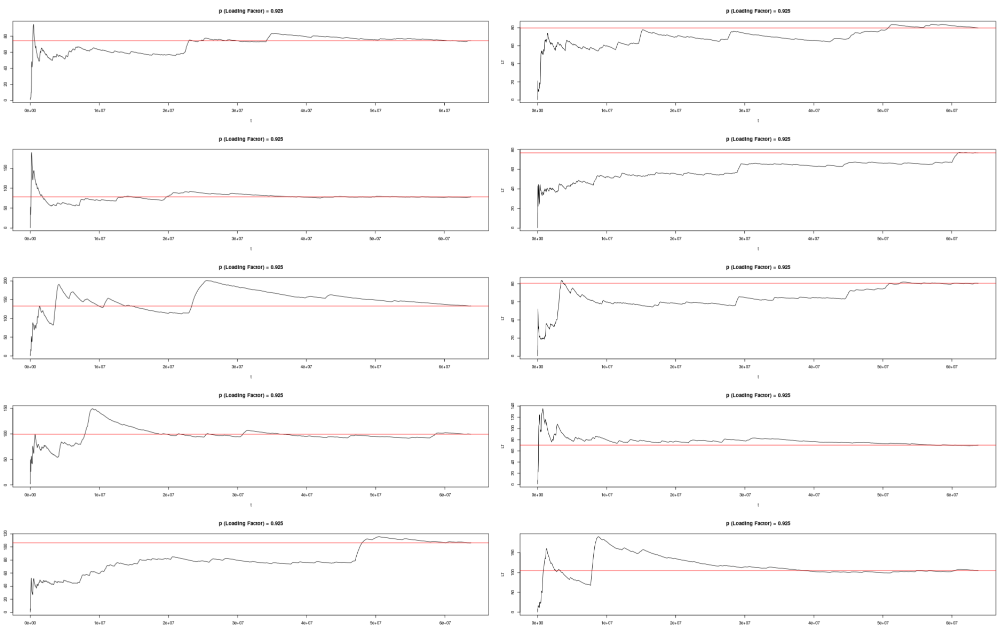
From formula (3) we find that $m = 15.59713$. Now we can proceed with performing 10 simulations and generating 100000 instances of service following Lognormal distribution with $\mu = \log m = 2.747087$ and $\sigma = 1.6333$. Before doing any analysis we must check if stable state has been achieved with $N = 100000$ instances. As a matter of fact 100000 instances is not enough to achieve stable state. So we perform 10 simulations for 1000000 instances and observe the LT vs t graphs.

Indeed from Figure 7. we observe that stable values are reached in all 10 simulations. So we can proceed with analysis itself. The mean service time from generated sample is $E[x] = 59.08197$, while theoretical mean service time is $E[x] = m\omega^{\frac{1}{2}} = 15.59713 * (e^{1.6333^2})^{\frac{1}{2}} = 59.2$. The variance of service time from generated sample is $Var[x] = 46882.25$, while theoretical variance of service time is $Var[x] = m^2\omega(\omega - 1) = 46984.42$. The coefficient of variation of service time from generated sample is $CV = 3.664791$, while theoretical coefficient of variation of service time is $\frac{\sigma}{mean} = \frac{216.7589}{59.2} = 3.661468$. While the histrograms of 10 simulated service times is depicted in Fibure 8.

In order to find approximated value of $E[w_q]$ we might use Allen Cuneen's approximation formula and simplified formulas from section 2.1.1, since values of constants have not changed.

Using formulas from 2.1.1, constant parameters and variables $\rho = 0.925, m =$

Figure 7: LT vs t graphs for 10 simulations



15.59713, we obtain $E[w_q] = W_q \approx 4995.083$, while W_q from simulated service times is 4687.221. We notice that W_q of simulated service times doesn't differ from approximated theoretical value. We might construct 95% confidence interval for W_q from 10 simulations to understand the range of ρ can get on 95% significance level.

In order to construct 95% confidence intervals for W_q, i.e. sojourn time in queue, we can apply t-statistics from R package for each of 10 simulations. The Table 4 shows Means and Confidence Intervals for each simulation.

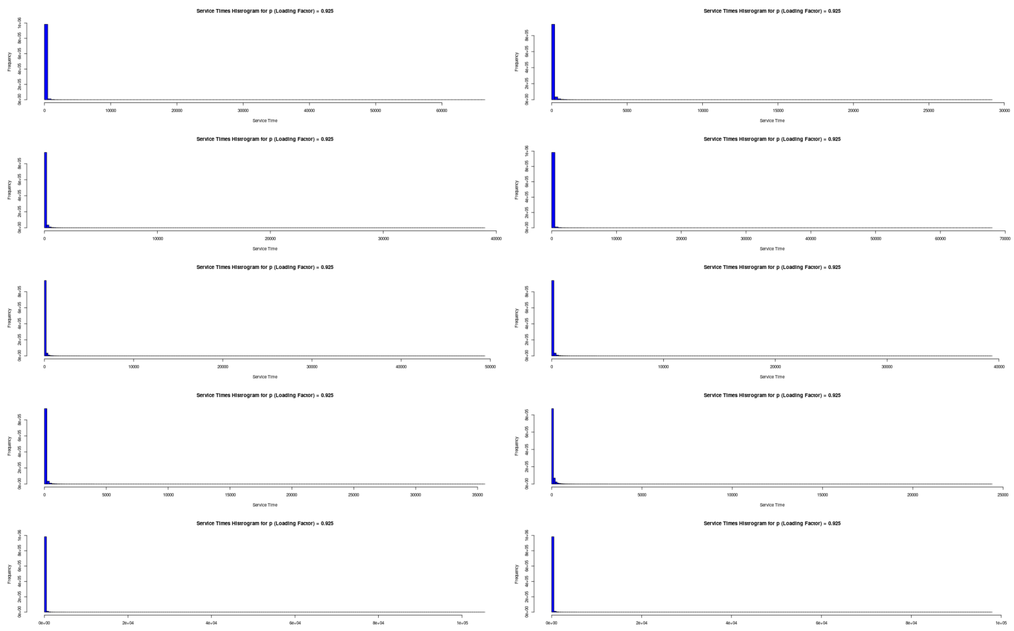
From Table 4, we observe that 1^{st}, 3^{rd}, and 4^{th} simulations producs closest mean value to approximation of $E[w_q]$ with Allen Cuneen's fomula. Which means we can use values of those simulations for doing other statistical analysis,which is out of our scope of investigation.

3 Conclusions

After performing 10 simulations for each ρ, we see that the magnitudes of interarrival and service times are increasing with ρ, because ρ is a loading factor of serving system. The higher the factor the longer it takes to process clients. That's the reason why we get increase in magnitute of Waiting System variables.

Another interesting fact that should be underlined is that theoretical approximation of $E[w_q]$ given by Allen Cuneen's Approximation formula is very close to $E[w_q]$ of generated data. Which means the simulation seems pretty

Figure 8: Histograms of Service Times for 10 simulations



accurate and can be used for other kind of analysis if needed.

Table 4: Means and Confidence Intervals

Mean	Confidence Interval
4687.221	(4673.687 4700.755)
5029.918	(5015.508 5044.328)
4942.788	(4929.991 4955.586)
4838.988	(4824.082 4853.893)
8445.695	(8416.488 8474.902)
5077.94	(5062.765 5093.115)
6289.404	(6271.295 6307.514)
4426.818	(4415.892 4437.743)
6722.485	(6697.156 6747.814)
6658.021	(6635.772 6680.270)