SMDE Lab3 - G/G/1 Queues

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Contents

Abstract

The work is divided into three main parts: in the first we describe the analytical or theoretical part of the work, in the second we do an analysis over the service time, and in the third we describe the results. We summarize the conclusions at the end.

Our random variable modeling the inter-arrival times (τ) is an Erlang with $k = 3$ and $E(\tau) = 66$. The service time x corresponds to a Weibull distribution with $a = 0.6521$. Since we only have one server, our queue is a E_3 $/G/1$ case.

1 Theoretical considerations

1.1 Computing b

We have to adapt b in order to obtain the desired loading factor. The mean value of the Weibull distribution is:

$$E[x] = b\Gamma\big(\frac{a+1}{a}\big) \tag{1}$$

The loading factor is computed as:

$$\rho = \frac{\lambda}{s\mu} \tag{2}$$

where s is the number of servers. In our case, there is only one server, so $\rho = \lambda/\mu$.

The parameters λ and μ correspond to the average input rate and the service rate. So, $\lambda = E(\tau)^{-1}$ and $\mu = E(x)^{-1}$. In our case, since the queue has not length restriction, all the clients are accepted, and the average input rate is the frequency associated to the inter-arrival time (in queues of finite length, it is not). The parameter μ is the inverse of the service time:

$$\rho = \frac{E(\tau)}{\Gamma(\frac{a+1}{a})} \tag{3}$$

Combining equations 1, 2 and 3, we can find the expression for b:

$$b = \rho\frac{E(\tau)}{\Gamma(\frac{a+1}{a})} \tag{4}$$

we can find the value for b such that the loading factor ρ takes the desired value. For $\rho = 0.4, 0.7, 0.85, 0.925$ we obtain $b = 19.39173, 33.93552, 41.20742, 44.84337$, respectively.

1.2 Allen-Cuneen's Formula

The Allen-Cuneen's approximation formula is given by the expression:

$$W_q = \frac{C(s,\theta)(\lambda^2\sigma_\tau^2 + \mu^2\sigma_x^2)}{2s\mu(1-\rho)} \tag{5}$$

where λ is the input rate, μ the service rate, and σ_τ^2 and σ_x^2 their variances. The parameter s represents the number of servers, which in our case is 1, and $\theta = \lambda/\mu$. The coefficient C is:

$$C(s,\theta) = \frac{\frac{\theta^2}{s!(1-\rho)}}{\sum_{l=0}^{s-1} \frac{\theta^l}{l!} + \frac{\theta^s}{s!(1-\rho)}} \tag{6}$$

1.3 Intervals of Confidence

The interval of confidence can be computed using the Student's t, as described by Bose.

Let us consider that we have n observations of the parameter W and we are studying (W_q in our case). Then, we compute

2 Analysis of Service Time

In analyzing the service time we used the following **Weibull** formulas:

Probability Distribution Function:

$$F_x(x) = 1 - exp(-(x/b)^a) \tag{7}$$

Mean:

$$E[x] = b\Gamma(\frac{a+1}{a}) \tag{8}$$

Variance:

$$Var[x] = b^2(\Gamma(\frac{a+2}{a}) - \Gamma^2(\frac{a+1}{a})) \tag{9}$$

Coefficient of Variance (Cv):

$$\frac{\sqrt{Var[x]}}{E[x]} \tag{10}$$

Euler Function:

$$\Gamma(x+1) = x\Gamma(x), \Gamma(1/2) = \sqrt{\pi} \tag{11}$$

The function **analysisServTime** is used to simulate service times of a weibull distribution. We generated 10,000 clients to capture sample service time values of a small simulation using $\rho = 0.4$ and a calculated $b = 19.39173$ from formula (4). Our basic statistics obtained were:

Statistic	**10K Sample**	**Theoretical**
$E[x]$	25.95162	26.4
$Var[x]$	1853.307	1758.977
Cv	1.658858	1.588643

Table 1: Basic statistics Weibull 10K Sampling vs Theoretical

Our result of the small simulation show that our sample mean is fairly close to the theoretical, while the variance is higher among the samples.

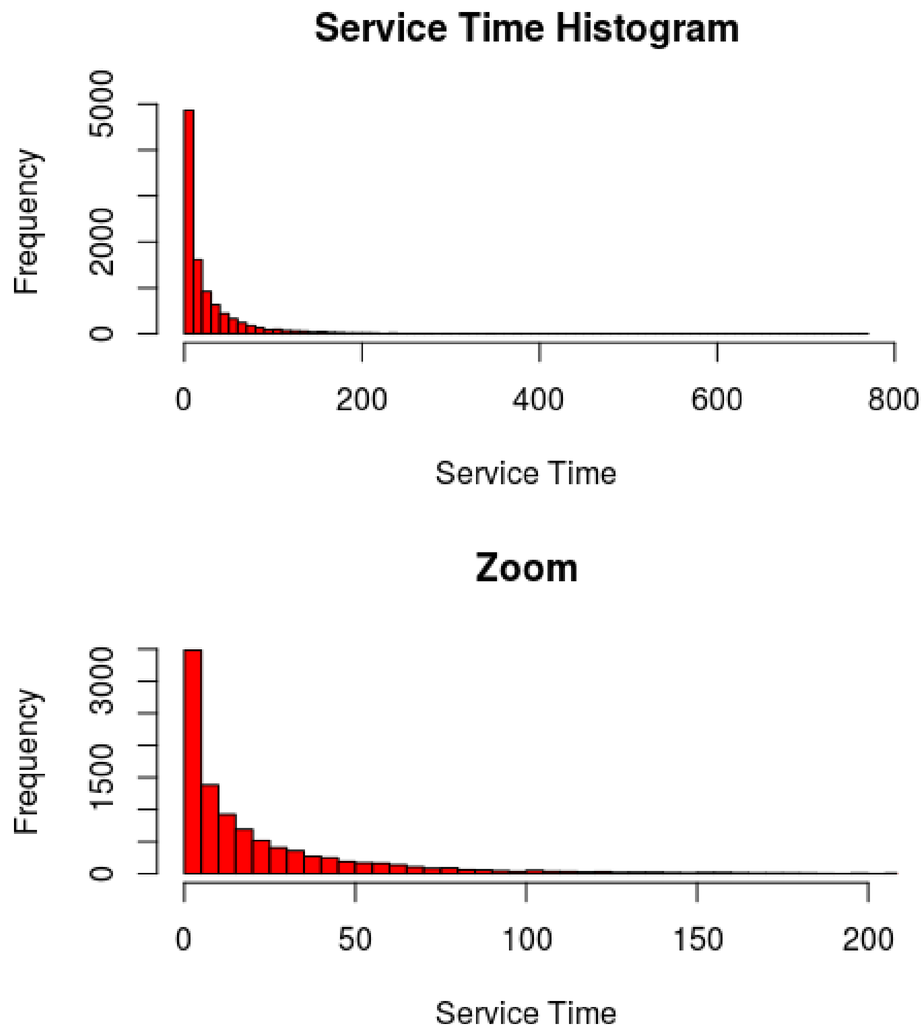

Figure 1: Histograms for sample of 10,000 service times

3 Results

The Results part is divided in three sections. In the first section, we describe the behavior of the random values we use as input, in the second, we perform a simulation for each one of the ρ values, and in the third section, we use ten simulations for each ρ, in order to find intervals of confidence.

3.1 Input

We use as input data the inter-arrival time for client $i(\tau_i)$ and the service time for client $i(x_i)$.

It has been necessary to check that the input data correspond to the given distribution.

As we have previously mentioned, the b parameter is computed depending on the loading factor. Thus, the mean and variance of the service time change for th different values of ρ, while the parameters of the inter-arrival time remain constant. The following table shows the theoretical and the experimental statistical parameters for the different cases (100,000 clients).

ρ	$E(x)$	$\overline{x}$	σ_x^2 (Theor.)	σ_x^2 (Exp.)
0.4	26.4	26.11787	1758.977	1749.347
0.7	46.2	46.93617	5386.868	5648.105
0.85	56.1	54.92763	7942.882	7367.936
0.925	61.05	62.02113	9406.406	9693.216

Table 2: Service experimental and theoretical parameters (100,000 clients)

The service time does not depend on ρ:

$E(x)$	$\overline{x}$	σ_x^2 (Theor.)	σ_x^2 (Exp.)
66	66.04972	1452	1452.592

Table 3:

These values have been obtained with only 100,000 clients. That explains the differences between the theoretical and the experimental values. In experiments involving a larger number of clients, the values are closer.

The following histograms show the experimental distribution of these random parameters (10^6 samples, $\rho = 0.925$). The plot converges to the corresponding probability density function.

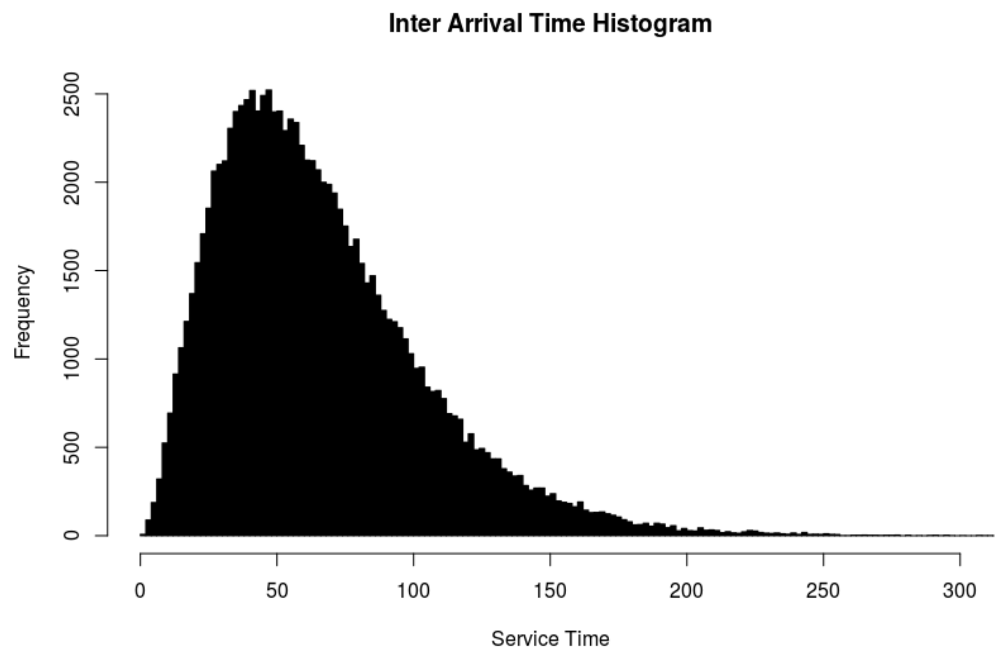

Figure 2: Histograms for sample of 10,000 service times

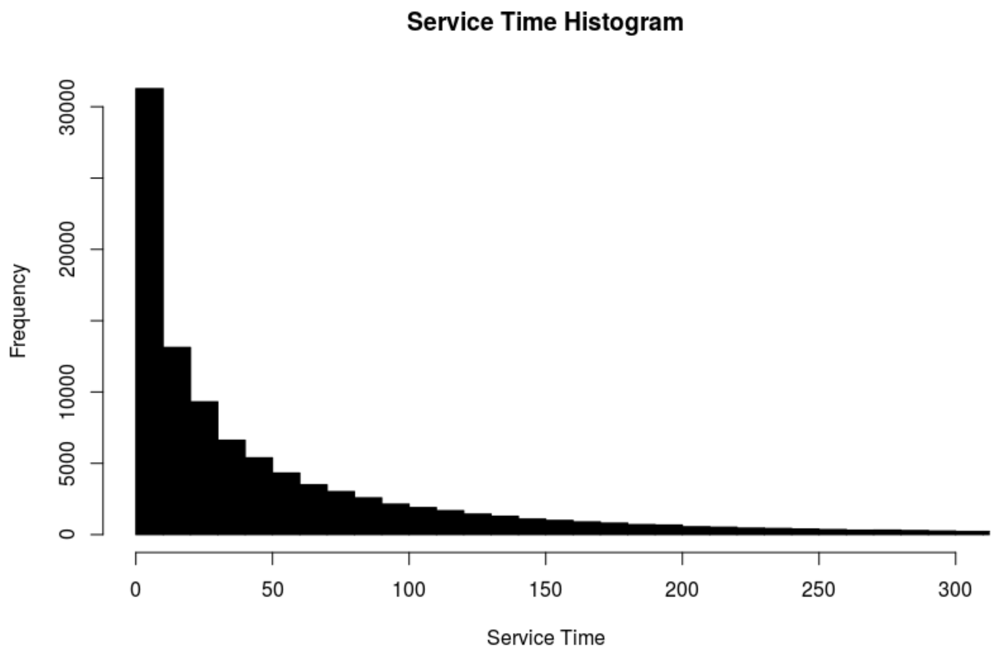

Figure 3: Histograms for sample of 10,000 service times

3.2 Experiment with a single execution

In the following section we execute a simulation for each one of the different ρ cases using 100,000 clients.

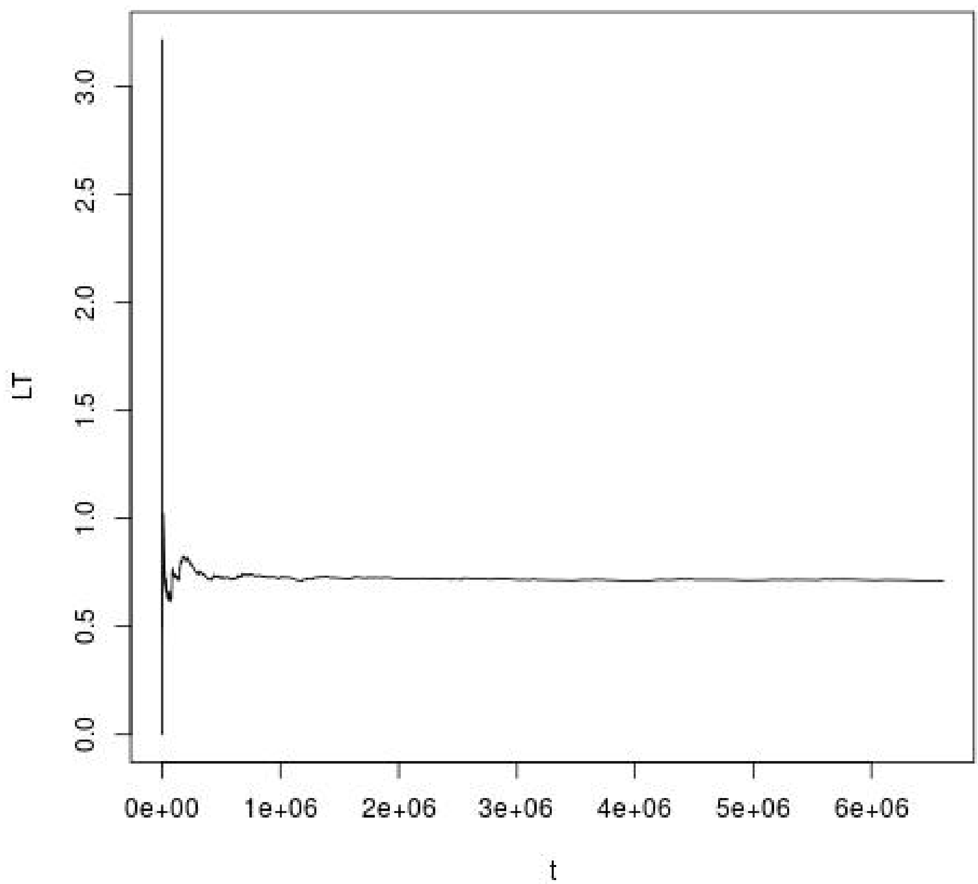

Figure 4: Histograms for $\rho = 0.4$ of 100,000 service times

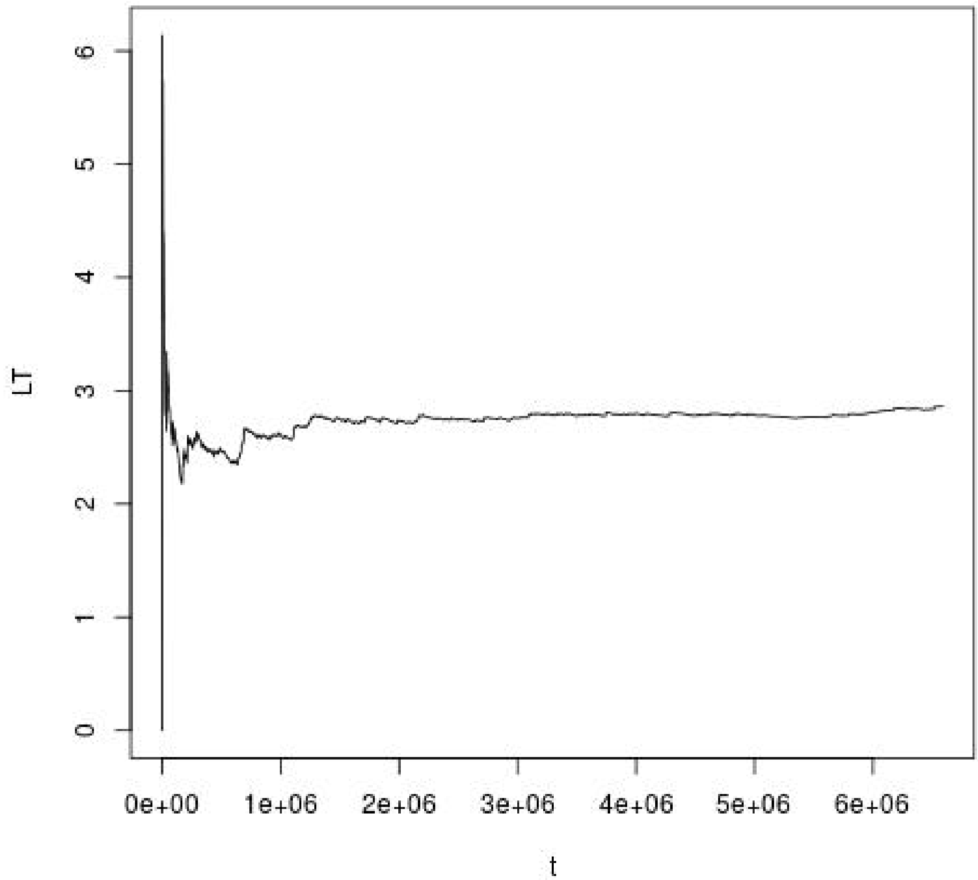

Figure 5: Histograms for $\rho = 0.7$ of 100,000 service times

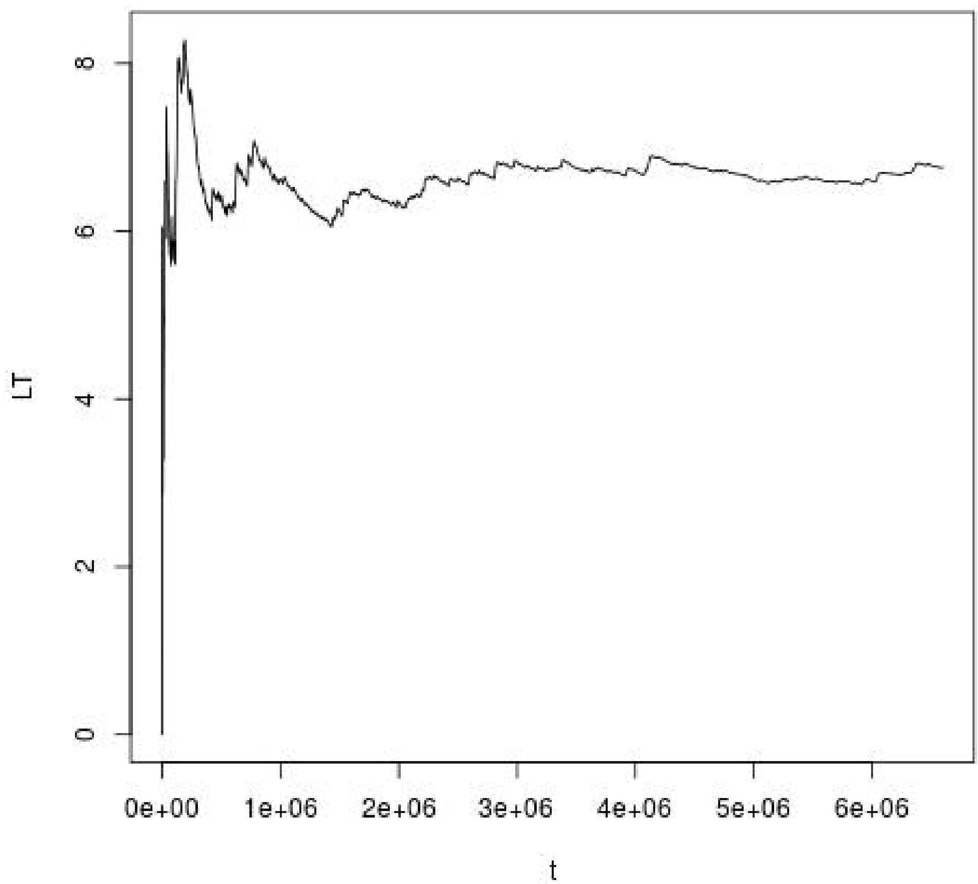

Figure 6: Histograms for $\rho = 0.85$ of 100,000 service times

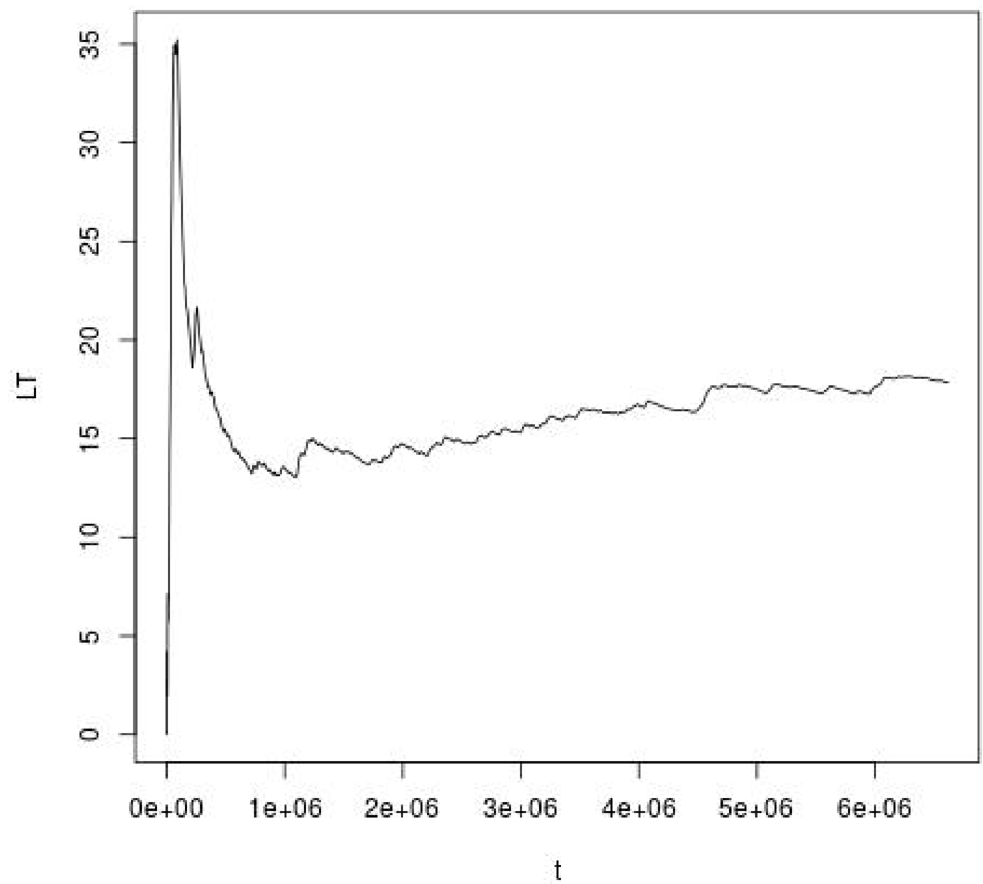

Figure 7: Histograms for $\rho = 0.925$ of 100,000 service times

3.3 10 replications

In this section we repeat each simulation ten times for each ρ value in order to obtain ten observations of W_q and L_q. Each simulation uses 10^6 clients. The table shows the Allen-Cuneen (AC) and the experimental (Exp.) values for W_q and L_q, and also the confidence intervals.

Table 4: Experimental and theoretical statistical parameters (100,000 clients)

p	Wq (Sim)	Wq (Theory)	Interval Wq	Lq (Sim)	Lq (Theory)	Interval Lq
0.4	19.62848	25.14264	[19.29444, 19.96252]	0.2967228	0.6142825	[0.2916732, 0.3017725]
0.7	155.6886	153.9987	[154.0778, 157.2995]	2.363249	2.616647	[2.338798, 2.387701]
0.85	408.2014	454.139	[405.0268, 411.3759]	6.16383	7.189227	[6.115894, 6.211766]
0.925	988.7383	1075.634	[981.7394, 995.7372]	14.96874	16.61831	[14.86278, 15.07470]

For an accurate approximation, we could expect the approximate value to rely on the confidence interval. In our experiment, however, the Allen-Cuneen value is not within the interval.

4 Conclusion

Regarding the observations L_q and W_q we are able to reach a stationary state after a given amount of time. We expect this to happen when $\rho < 1$ which is when the service rate exceeds the request rate. The simulations confirm this behavior. We also see that there is a large difference in the period of instability as the time needed for the stationary state is very dependent on the traffic condition.

When comparing our results to the Allen-Cuneen approximations the result is somewhat inaccurate (between the CI) but the behavior is acceptable. Since the transient time is larger for high values of ρ, it follows that the simulations where $\rho = 0.925$ had more variability between trials. If we are simulating highly loaded systems it would be necessary to use longer simulations or perform an increased number of repetitions to get converge on expected results.