UNIVERSIDADE FEDERAL DO MARANHÃO
DEPARTAMENTO DE MATEMÁTICA

DISCIPLINA: EQUAÇÕES DIFERENCIAIS I PROF: GREICIANE

3ª AVALIAÇÃO DE EQUAÇÕES DIFERENCIAIS I

1. Calcule as transformadas de Laplace usando a definição.

   a) $\mathcal{L}\{t^2 \cosh(kt)\}$

   b) $\mathcal{L}\{t^2 \sinh(kt)\}$

   c) $\mathcal{L}\{te^{at}\}$

2. Calcule as transformadas de Laplace abaixo.

   a) $\mathcal{L}\{2e^{-3t}t^4 + e^{2t-7}t^2 \sin^2(t)\}$

   b) $\mathcal{L}\{(t - 2t + e^t)^3\}$

3. Calcule as transformadas inversas.

   a) $\mathcal{L}^{-1}\left\{\dfrac{8s - 4s + 12}{s(s^2 + 4)}\right\}$

   b) $\mathcal{L}^{-1}\left\{e^{-23s}\dfrac{3}{(s+2)^6}\right\}$

4. Calcule os problemas de valores iniciais dados.

   a) $y'' - y' - 2y = x^2$, $y(0) = 1$ e $y'(0) = 3$

   b) $y'' - y' = \sin t$, $y(0) = 1$ e $y'(0) = 1$

   c) $y'' + 4y' + 23y = 2$, $y(0) = 0$ e $y'(0) = 2$

Reposição - EDO - P3

① a) $\int_0^\infty t^2 \cosh(Kt) \cdot e^{-st}\, dt = \lim_{a\to\infty} \int_0^a t^2 \left(\frac{e^{Kt}+e^{-Kt}}{2}\right) \cdot e^{-st}\, dt$

$= \lim_{a\to\infty} \frac{1}{2}\int_0^a t^2 \left(e^{t(K-s)} + e^{t(-K-s)}\right) dt = \lim_{a\to\infty} \frac{1}{2}\underbrace{\int_0^a t^2 e^{t(K-s)} dt}\, + \frac{1}{2}\int_0^a t^2 e^{t(-K-s)} dt$

$\begin{cases} u = t^2 \to du = 2t\,dt \\ dv = e^{t(K-s)} dt \to v = \dfrac{e^{t(K-s)}}{K-s} \end{cases}$

$= \dfrac{t^2 e^{t(K-s)}}{K-s} - \underbrace{\int \dfrac{2t e^{t(K-s)}}{K-s} dt}$

$= \dfrac{2}{K-s} \underbrace{\int t e^{t(K-s)} dt}$

$\begin{cases} u = t \to du = dt \\ dv = e^{t(K-s)} dt \to v = \dfrac{e^{t(K-s)}}{K-s} \end{cases}$

$\dfrac{t e^{t(K-s)}}{K-s} - \underbrace{\int \dfrac{e^{t(K-s)}}{K-s} dt}$

$u = t(K-s) \to du = (K-s)dt$

$= \int \dfrac{e^u}{K-s} \cdot \dfrac{du}{K-s} = \dfrac{1}{(K-s)^2}\int e^u du$

$= \dfrac{e^{t(K-s)}}{(K-s)^2}$

$$\frac{t e^{t(K-s)}}{K-s} - \frac{e^{t(K-s)}}{(K-s)^2} = \frac{(K-s) t e^{t(K-s)} - e^{t(K-s)}}{(K-s)^2}$$

$$\hookrightarrow \frac{2}{K-s}\left[\frac{(K-s) t e^{t(K-s)} - e^{t(K-s)}}{(K-s)^2}\right] = \frac{2(K-s) t e^{t(K-s)} - 2 e^{t(K-s)}}{(K-s)^3}$$

$$\hookrightarrow \underbrace{\frac{t^2 e^{t(K-s)}}{K-s} - \frac{2(K-s) t e^{t(K-s)} - 2 e^{t(K-s)}}{(K-s)^3}}_{\hookrightarrow 1^{a} \text{ integral}}$$

A segunda é semelhante, alterando apenas os sinais:

$$\frac{t^2 e^{t(-K-s)}}{-K-s} - \frac{2(-K-s) t e^{t(-K-s)} - 2 e^{t(-K-s)}}{(-K-s)^3}$$

↳ Unindo tudo:

$$\lim_{a \to \infty} \frac{1}{2}\left\{\left[\frac{t^2 e^{t(K-s)}}{K-s} - \frac{2(K-s) t e^{t(K-s)} - 2 e^{t(K-s)}}{(K-s)^3}\right]_0^a + \left[\frac{t^2 e^{t(-K-s)}}{-K-s} - \frac{2(-K-s) t e^{t(-K-s)} - 2 e^{t(-K-s)}}{(-K-s)^3}\right]_0^a\right.$$

↳ Com o limite tendendo a infinito, $K-s$ e $-K-s$ precisam ser < 0 para que o limite seja zerado

$$\left.\begin{array}{l} \hookrightarrow K-s<0 \to K<s \\ \hookrightarrow -K-s<0 \to -K<s \end{array}\right\} s>|K| \to$$ Condição satisfeita, quando $t=a$, podemos considerá-lo como zero, restando apenas:

$$\frac{1}{2}\left[-\left(\frac{-2e^0}{(K-s)^3}\right) - \left(\frac{-2e^0}{(-K-s)^3}\right)\right] = \frac{1}{\cancel{2}}\left[\frac{\cancel{2}}{(K-s)^3} + \frac{\cancel{2}}{(-K-s)^3}\right] = \frac{1}{(K-s)^3} + \frac{1}{(-K-s)^3}$$

b) $\int_0^{\infty} t^2 \cdot \left(\frac{e^{kt} - e^{-kt}}{2}\right) \cdot e^{-st}\, dt = \lim_{a \to \infty} \frac{1}{2} \int_0^a t^2 \left(e^{t(K-s)} - e^{t(-K-s)}\right) dt$

$$= \lim_{a \to \infty} \underbrace{\frac{1}{2} \int_0^a t^2 e^{t(K-s)} dt}_{} - \frac{1}{2} \int_0^a t^2 e^{t(-K-s)} dt$$

$\hookrightarrow = \frac{t^2 e^{t(K-s)}}{K-s} - \frac{2(K-s)\, t e^{t(K-s)} - 2 e^{t(K-s)}}{(K-s)^3}$

$\hookrightarrow$ 2ª integral: $\frac{t^2 e^{t(-K-s)}}{-K-s} - \frac{2(-K-s)\, t e^{t(-K-s)} - 2 e^{t(-K-s)}}{(-K-s)^3}$

$\hookrightarrow$ Unindo:

$$\lim_{a \to \infty} \frac{1}{2} \left\{ \left[\frac{t^2 e^{t(K-s)}}{K-s} - \frac{2(K-s)\, t e^{t(K-s)} - 2 e^{t(K-s)}}{(K-s)^3} \right]_0^a - \left[\frac{t^2 e^{t(-K-s)}}{-K-s} - \frac{2(-K-s)\, t e^{t(-K-s)} - 2 e^{t(-K-s)}}{(-K-s)^3} \right]_0^a \right.$$

$\hookrightarrow$ Condição para o limite igual à 1a) $(s > |K|)$

$\hookrightarrow \frac{1}{2} \left[-\left(\frac{-2e^0}{(K-s)^3} \right) + \left(\frac{-2e^0}{(-K-s)^3} \right) \right] = \frac{1}{\cancel{2}} \left[\frac{\cancel{2}}{(K-s)^3} - \frac{\cancel{2}}{(-K-s)^3} \right] = \frac{1}{(K-s)^3} - \frac{1}{(-K-s)^3}$

c) $\int_0^{\infty} t\, e^{at} \cdot e^{-st}\, dt = \lim_{b \to \infty} \int_0^b t\, e^{t(a-s)}\, dt \to u = t \to du = dt$

$dv = e^{t(a-s)}\, dt \to v = \dfrac{e^{t(a-s)}}{a-s}$

$$= \frac{t\, e^{t(a-s)}}{a-s} - \underbrace{\int \frac{e^{t(a-s)}}{a-s}\, dt}$$

$u = t(a-s) \to du = (a-s)\, dt$

$$= \int \frac{e^{u}}{a-s} \cdot \frac{du}{a-s} = \frac{1}{(a-s)^2} \int e^{u}\, du$$

$$= \frac{e^{t(a-s)}}{(a-s)^2}$$

$$= \frac{t\, e^{t(a-s)}}{a-s} - \frac{e^{t(a-s)}}{(a-s)^2}$$

$\hookrightarrow \lim_{b \to \infty} \left[\dfrac{t\, e^{t(a-s)}}{a-s} - \dfrac{e^{t(a-s)}}{(a-s)^2} \right]_0^b \to a-s < 0 \to a < s \to$ Condição satisfeita;

$$-\left(\frac{-e^{0}}{(a-s)^2} \right) = \frac{1}{(a-s)^2}$$

② a) $\underbrace{\mathcal{L}\left(2e^{-3t}t^4\right)}+\underbrace{\mathcal{L}\left[e^{2t-7}t^2 \operatorname{sen}^2(t)\right]}$

$$=2\cdot\frac{4!}{(s+3)^5}=\frac{48}{(s+3)^5}$$

$$= e^{-7}\mathcal{L}\left[e^{2t}\cdot t^2\cdot\left(\frac{1-\cos(2t)}{2}\right)\right]$$

$$= \frac{e^{-7}}{2}\cdot\mathcal{L}\underbrace{\left[e^{2t}\left(t^2-t^2\cos(2t)\right)\right]}$$

$$=\underbrace{\mathcal{L}\left(e^{2t}t^2\right)}-\underbrace{\mathcal{L}\left[e^{2t}t^2\cos(2t)\right]}$$

$$=\frac{2}{(s-2)^3}$$

$$1\cdot\mathcal{L}\left[t^2\cos(2t)\right]$$

$$=(-1)^2\frac{d^2}{ds^2}\left(\frac{s}{s^2+4}\right)$$

$$=\frac{2s^3-24s}{(s^2+4)^3}$$

$$2\cdot\mathcal{L}\left[e^{2t}t^2\cos(2t)\right]=$$

$$\frac{2(s-2)^3-24(s-2)}{\left[(s-2)^2+4\right]^3}$$

$$=\frac{e^{-7}}{\cancel{2}}\left[\frac{\cancel{2}}{(s-2)^3}-\left(\frac{\cancel{2}(s-2)^3-\cancel{24}^{12}(s-2)}{\left[(s-2)^2+4\right]^3}\right)\right]$$

$$\boxed{=\frac{48}{(s+3)^5}+e^{-7}\left[\frac{1}{(s-2)^3}-\frac{(s-2)^3+12(s-2)}{\left[(s-2)^2+4\right]^3}\right]}$$

b) $\mathcal{L}_e(e^t - t)^3 = \mathcal{L}_e(e^{3t} - 3te^{2t} + 3t^2e^t - t^3)$

$\hookrightarrow = \frac{1}{s-3} - 3\cdot\frac{1}{(s-2)^2} + 3\cdot\frac{2}{(s-1)^3} - \frac{6}{s^4} = \boxed{\frac{1}{s-3} - \frac{3}{(s-2)^2} + \frac{6}{(s-1)^3} - \frac{6}{s^4}}$

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3 a) $\mathcal{L}_e^{-1}\left[\frac{4s+12}{s(s^2+4)}\right]$

$= \frac{A}{s} + \frac{Bs+C}{s^2+4} \rightarrow 4s+12 = A(s^2+4) + Bs^2 + Cs = As^2 + 4A + Bs^2 + Cs$

$\hookrightarrow s^2(A+B) + s\cdot C + 4A$

$\hookrightarrow \begin{cases} A+B=0 \rightarrow 3+B=0 \rightarrow \underline{B=-3} \\ \underline{C=4} \\ 4A=12 \rightarrow \underline{A=3} \end{cases}$

$= \mathcal{L}_e^{-1}\left[\frac{3}{s} + \left(\frac{-3s+4}{s^2+4}\right)\right] = \boxed{3 - 3\cdot\cos(2t) + 2\cdot\mathrm{sen}(2t)}$

b) $\mathcal{L}_e(t^n e^{at}) = \frac{n!}{(s-a)^{n+1}} \rightarrow \left.\begin{matrix} a=-2 \\ n=5 \end{matrix}\right\} \frac{120}{(s+2)^6} (\div 40) = \frac{3}{(s+2)^6}$

$\hookrightarrow \mathcal{L}_e^{-1}\left[\frac{3}{(s+2)^6}\right] = \frac{1}{40}\cdot t^5 e^{-2t}$

$\hookrightarrow e^{-23s}F(s) \rightarrow$ 2º teorema da translação $(= f(t-a)\cdot U(t-a))$

$\hookrightarrow \boxed{\frac{1}{40}(t-23)^5\cdot e^{-2(t-23)}\cdot U(t-23)}$

(4) a) $s^2Y(s) - sy(0) - y'(0) - (sY(s) - y(0)) - 2Y(s) = \frac{2}{s^3}$

↳ $s^2Y(s) - s\cdot 1 - 3 - sY(s) + 1 - 2Y(s) = \frac{2}{s^3}$

↳ $Y(s)(s^2-s-2) = \frac{2}{s^3} + s + 3 - 1 \to Y(s) = \left(\frac{2}{s^3}+s+2\right)\left(\frac{1}{s^2-s-2}\right) = \left(\frac{2+s^4+2s^3}{s^3}\right)\left(\frac{1}{s^2-s-2}\right)$

$= \left(\frac{2+s^4+2s^3}{s^3}\right)\left(\frac{1}{(s-2)(s+1)}\right) = \frac{2+s^4+2s^3}{s^3(s-2)(s+1)} = \frac{A}{s}+\frac{B}{s^2}+\frac{C}{s^3}+\frac{D}{s-2}+\frac{E}{s+1}$

↳ $2+s^4+2s^3 = As^2(s-2)(s+1) + Bs(s-2)(s+1) + C(s-2)(s+1) + Ds^3(s+1) + Es^3(s-2)$

$=(As^3-2As^2)(s+1) + (Bs^2-2Bs)(s+1) + (Cs-2C)(s+1) + Ds^4+Ds^3+Es^4-2Es^3$

$=As^4+As^3-2As^3-2As^2+Bs^3+Bs^2-2Bs^2-2Bs+Cs^2+Cs-2Cs-2C+Ds^4+Ds^3+Es^4-2Es^3$

$=s^4(A+D+E)+s^3(A-2A+B+D-2E)+s^2(-2A+B-2B+C)+s(-2B+C-2C)-2C$

- $1=A+D+E \to 1 = -\cancel{3/4} + 2E + \cancel{3/4} + E \to 1=3E \to E=1/3$
- $2=-A+B+D-2E \to 3/4+1/2-2 = 2E - D = -3/4 = 2E-D \to D = 2E+3/4 \to D=\frac{17}{12}$
- $0=-2A-B+C \to 2A=-1/2-1=-3/2 \to A=-3/4$
- $0=-2B-C \to 0=-2B+1 \to 2B=1 \to B=1/2$
- $2=-2C \to C=-1$

↳ $Y(s) = -\frac{3}{4s}+\frac{1}{2s^2}-\frac{1}{s^3}+\frac{17}{12(s-2)}+\frac{1}{3(s+1)}$

↳ $\mathcal{L}^{-1}[Y(s)] = \boxed{-\frac{3}{4}+\frac{x}{2}-\frac{x^2}{2}+\frac{17}{12}\cdot e^{2x}+\frac{e^{-x}}{3}}$

b) $s^2Y(s) - sy(0) - y'(0) - (sY(s) - y(0)) = \frac{1}{s^2+1}$

↳ $Y(s)(s^2-s) = \frac{1}{s^2+1} + s + \cancel{1-1} \to Y(s) = \left(\frac{1}{s^2+1} + s\right)\left(\frac{1}{s^2-s}\right) = \left(\frac{1+s(s^2+1)}{s^2+1}\right)\left(\frac{1}{s^2-s}\right)$

$= \frac{1+s^3+s}{(s^2+1)s(s-1)} = \frac{As+B}{s^2+1} + \frac{C}{s} + \frac{D}{s-1} \to 1+s^3+s = (As+B)(s^2-s) + (Cs^2+C)(s-1) + (Ds^2+D)s$

$= As^3 - As^2 + Bs^2 - Bs + Cs^3 - Cs^2 + Cs - C + Ds^3 + Ds = s^3(A+C+D) + s^2(-A+B-C) + s(-B+C+D) - C$

- $1 = A+C+D \to 2 = A+D \to 2 = 1/2 + D \to D = 3/2$
- $0 = -A+B-C \to A = -B \to A = 1/2$
- $1 = -B+C+D \to 2 = -B+D \to -1 = 2B \to B = -1/2$
- $1 = -C \to C = -1$

↳ $Y(s) = \left(\frac{s}{2} - \frac{1}{2}\right)\left(\frac{1}{s^2+1}\right) - \frac{1}{s} + \left(\frac{3}{2}\right)\left(\frac{1}{s-1}\right) = \frac{s-1}{2s^2+2} - \frac{1}{s} + \frac{3}{2s-2}$

↳ $\mathcal{L}^{-1}[Y(s)] = \boxed{\frac{1}{2}(\cos(t) - \text{sen}(t)) - 1 + \frac{3}{2} \cdot e^t}$

c) $s^2Y(s)-sy(0)-y'(0)+4sY(s)-4y(0)+23Y(s)=\frac{2}{s}$

$\hookrightarrow Y(s)(s^2+4s+23)=\frac{2}{s}+2 \rightarrow Y(s)=\left(\frac{2+2s}{s}\right)\left(\frac{1}{s^2+4s+23}\right)=\frac{2+2s}{s(s^2+4s+23)}$

$=\frac{A}{s}+\frac{Bs+C}{s^2+4s+23} \rightarrow 2+2s=As^2+4As+23A+Bs^2+Cs=s^2(A+B)+s(4A+C)+23A$

$\left.\begin{array}{l} \cdot 0=A+B \rightarrow B=-\frac{2}{23} \\ \cdot 2=4A+C \longrightarrow \frac{38}{23}=C \\ \cdot 2=23A \rightarrow A=\frac{2}{23} \end{array}\right\} Y(s)=\frac{2}{23}\cdot\frac{1}{s}\ \frac{-2s+38}{23}\cdot\frac{1}{s^2+4s+23}$

$\hookrightarrow \mathcal{L}^{-1}[Y(s)]=\frac{2}{23}-\frac{2}{23}\mathcal{L}^{-1}\left(\frac{s-19}{(s+2)^2+19}\right)=\boxed{\frac{2}{23}-\frac{2e^{-2t}}{23}\cdot\cos(t\sqrt{19})+\frac{42e^{-2t}}{23\sqrt{19}}\cdot\operatorname{sen}(t\sqrt{19})}$