UNIVERSIDADE FEDERAL DO PIAUÍ

CENTRO DE CIÊNCIAS DA NATUREZA

DEPARTAMENTO DE MATEMÁTICA

PROFESSOR: ÍTALO AUGUSTO OLIVEIRA DE ALBUQUERQUE

DISCIPLINA: CÁLCULO DIFERENCIAL E INTEGRAL III

Lista de Exercícios - Integral de Superfície

1. Identifique a superfície parametrizada por $\varphi(u,v) = (v\cos u, v\sin u, 1-v^2)$ e encontre a equação da reta normal e do plano tangente a superfície em (0,1).

2. Encontre uma parametrização para as superfícies abaixo:

a. S: parte da esfera $x^2+y^2+z^2=4$ que fica acima do plano $z=\sqrt{2}$.
b. S: parte do cilindro $x^2+y^2=4$ que fica entre os planos $z=-2$ e $z=2-y$.
c. S: parte do plano $x+y+z=2$ no interior do cilindro $x^2+y^2=1$.
d. S: é o cone gerado pela semireta $z=2y, y\geqslant 0$ girada em torno do eixo z.

3. Calcule a área das superfícies abaixo:

a. S: é a parte do cilindro $x^2+y^2=4$, com $0\leqslant z\leqslant 5$, delimitada pelos semiplanos $y=2x, y=x$ e $x\geqslant 0$.
b. S: é a parte da esfera $x^2+y^2+z^2=4$ no interior do cone $3z^2=x^2+y^2, z>0$.
c. S: é a parte do cone $z^2=x^2+y^2$ que se encontra dentro do cilindro $x^2+y^2\leqslant 2y$ e fora do cilindro $x^2+y^2\leqslant 1$.
d. S: é a parte do cone $z=\sqrt{x^2+y^2}$ que está entre os planos xy e $2z-y=2$.

4. Calcula a massa de uma lâmina, que tem a forma da parte do plano $z=x$ recortada pelo cilindro $(x-1)^2+y^2=1$ cuja densidade no ponto (x,y,z) é proporcional à distância desse ponto ao plano xy.

5. Abaixo, determine o momento de inércia em relação a superfície S:

a. S: parte do cone $z^2=x^2+y^2$ entre os planos $z=1$ e $z=2$, sendo a densidade constante.
b. S: é a parte do cone $z^2=x^2+y^2, z\geqslant 0$ em relação ao eixo z, de altura h que está no primeiro octante.
c. S: é uma superfície homogênea, de massa M e equação $x^2+y^2=R^2, R>0$, com $0\leqslant z\leqslant 1$, em torno do eixo z.

5. Calcule o fluxo do campo vetorial pedidos abaixo:

a. $F=(x-y-4, y, z)$ através da semi-esfera superior $x^2+y^2+z^2=1$, com campo de vetores normais $\mathbf{n}$ tal que $\mathbf{n}\cdot\overrightarrow{k}>0$.
b. $F=(0,0,-z)$ e S é a parte da esfera $x^2+y^2+z^2=4$ fora do cilindro $x^2+y^2=1$ com $\mathbf{n}$ apontando para fora.

c. $F = (-x, -y, 3y^2z)$ sobre o cilindro $x^2 + y^2 = 16$ situado no primeiro octante entre $z = 0$ e $z = 5 - y$ com orientação normal que aponta para o eixo z.

6. Calcule $\displaystyle\iint_S F \cdot \mathbf{n} dS$, onde:

a. $F = (xze^y, -xze^y, z)$ e S é a parte do plano $x + y + z = 1$ no primeiro octante com orientação para baixo.
b. $F = (-x, 0, 2z)$ e S é a fronteira com a região limitada por $z = 1$ e $z = x^2 + y^2$, com $\mathbf{n}$ exterior a S.

7. Seja Q uma carga elétrica localizada na origem. Pela Lei de Coulomb, a força elétrica F exercida por essa carga sobre uma carga q localizada no ponto (x, y, z) com vetor posição X é $\dfrac{\varepsilon q Q}{\|x\|^3}X$, onde ε é uma constante. Considere a força por unidade de carga:

$$E(X) = \frac{1}{q}F(X) = \frac{\varepsilon Q}{\|x\|^3}X = \frac{\varepsilon Q(x, y, z)}{(x^2 + y^2 + z^2)^{3/2}}$$

chamada de campo elétrico de Q. Mostre que o fluxo elétrico de E é igual a $4\pi\varepsilon Q$, através de qualquer superfície fechada S que contenha a origem, com normal $\mathbf{n}$ apontando para fora de S. Essa é a Lei de Gauss para uma carga simples.

8.Seja $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ de classe C^2 tal que $\nabla^2 f = x^2 + y^2$ e $\dfrac{\partial f}{\partial z} = 1/3$. Calcule $\displaystyle\iint_S \frac{\partial f}{\partial \mathbf{n}} dS$, onde S é a lata cilíndrica com fundo e sem tampa dada por $x^2 + y^2 = 1, 0 \leqslant z \leqslant 1, x^2 + y^2 \leqslant 1$ e $z = 0$, com a normal apontando para fora de S.

9. Sejam $F = (\dfrac{-cy}{2} + ze^x, \dfrac{cx}{2} - ze^y, xy)$ com $c > 0$, e S uma superfície aberta união do hiperbolóide folha $x^2 + y^2 - z^2 = 1, 0 \leqslant z \leqslant \sqrt{c}$ com o disco $x^2 + y^2 \leqslant 1, z = 0$. Calcule o valor c sabendo que $\displaystyle\iint_S \operatorname{rot} F \cdot \mathbf{n} dS = -6\pi$ com $\mathbf{n}$ apontando para fora de S.

10. Calcule a circulação do campo $F = (y, xz, z^2)$ ao redor da curva C fronteira do triângulo cortado do plano $x + y + z = 1$ pelo primeiro octante, no sentido horário quando vista da origem.

11. Calcule o trabalho realizado pelo campo $F = (x^x + z^2, y^y + x^2, z^z y^2)$ quando uma partícula se move sob sua influência ao redor da borda da esfera $x^2 + y^2 + z^2 = 4$ que está no primeiro octante, na direção anti-horário quando vista por cima.

12. Calcule $\displaystyle\int_C F dr$ onde $F = (-2y + e^{\sin x}, -z + y, x^2 + e^{\sin x})$ e C é a curva interseção da superfície $z = y^2$ com o plano $x + y = 1$, orientada no sentido de crescimento de y.

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cálculo 3 - prova 3 - professor: Ítalo

1) a. identificando a superfície

seja S: $\begin{cases} x = v\cos u \\ y = v\,\text{sen}\,u \\ z = 1 - v^2 \end{cases}$, $0 \leq u \leq 2\pi$, $v \geq 0$, eliminando u e v, temos $x^2+y^2 = v^2 = 1-z$.

então, S é a paraboloide circular : $z = 1-x^2-y^2$ //

b. reta normal e eq plano tangente a S em $\varphi(0,1)$.

$\varphi(0,1) = (1,0,0)$

$\vec{N}(0,1) = \frac{\partial\varphi}{\partial u}(0,1) \times \frac{\partial\varphi}{\partial v}(0,1) = (-v\,\text{sen}\,u, v\cos u, 0) \times (\cos u, \text{sen}\,u, -2v)\Big|_{(0,1)}$

$= (0,1,0) \times (1,0,-2) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & 1 & 0 \\ 1 & 0 & -2 \end{vmatrix} = (-2,0,-1)$

plano tg a S em $\varphi(0,1) = (1,0,0) \rightarrow [(x,y,z) - \varphi(0,1)] \cdot \vec{N}(0,1) = 0$.

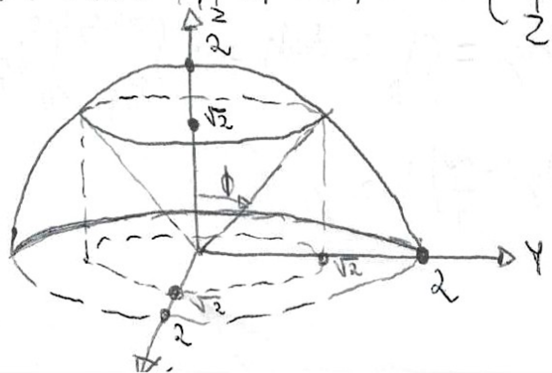
$[(x,y,z)-(1,0,0)] \cdot (-2,0,-1) = 0 \rightarrow (x-1,y,z)\cdot(-2,0,-1) = 0 \rightarrow -2(x-1)-z = 0$

$2x+z-2 = 0$. // equação plano tangente

$[(x,y,z)-\varphi(0,1)] = \lambda\vec{N}(0,1)$, $\lambda \in \mathbb{R} \rightarrow [(x,y,z)-(1,0,0)] = \lambda(-2,0,-1)$

$\begin{cases} x = 1-2\lambda \\ y = 0 \\ z = -\lambda \end{cases}$

2) a. se $(x,y,z) \in S$, então $\begin{cases} x = 2\,\text{sen}\,\phi\cos\theta \\ y = 2\,\text{sen}\,\phi\,\text{sen}\,\theta \\ z = 2\cos\phi \end{cases}$; $\begin{cases} 0 \leq \theta \leq 2\pi \\ \cos\phi = \frac{\sqrt{2}}{2} \rightarrow \phi = \frac{\pi}{4} \end{cases}$



então, uma parametrização de S é dada por: $\varphi(\phi,\theta) = (2\,\text{sen}\,\phi\cos\theta, 2\,\text{sen}\,\phi\,\text{sen}\,\theta, 2\cos\phi)$

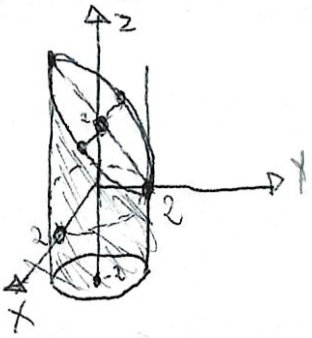
com $(\phi,\theta) \in D$: $\begin{cases} 0 \leq \theta \leq 2\pi \\ 0 \leq \phi \leq \pi/4 \end{cases}$

b. $\begin{cases} x = 2\cos t \\ y = 2\,\text{sen}\, t \\ z = z \end{cases}$, $0 \leq t \leq 2\pi$
$-2 \leq z \leq 2 - y = 2 - 2\,\text{sen}\, t$

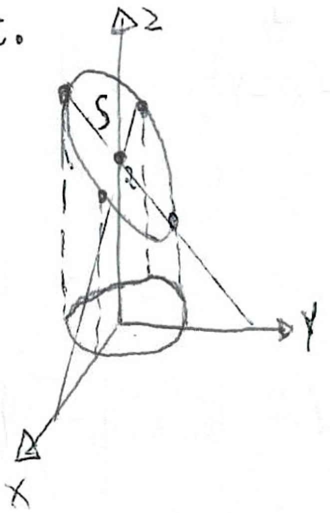
; Uma parametrização de S é

$\varphi(t,z) = (2\cos t, 2\,\text{sen}\, t, z)$

$(t,z) \in D: \begin{cases} 0 \leq t \leq 2\pi \\ -2 \leq z \leq 2 - 2\,\text{sen}\, t \end{cases}$



c.

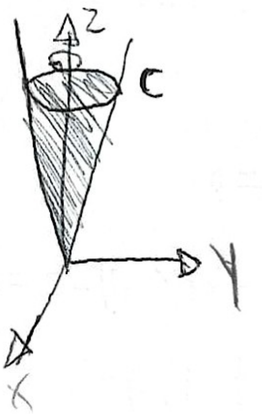


$z = 2 - x - y$, $(x,y) \in D: x^2 + y^2 \leq 1$; então a parametrização de S é dada por $\phi(x,y) = (x, y, 2-x-y)$. Outra é

$\psi(r,\theta) = (r\cos\theta, r\,\text{sen}\,\theta, 2 - r\cos\theta - r\,\text{sen}\,\theta)$

$(r,\theta) \in D: \begin{cases} 0 \leq r \leq 1 \\ 0 \leq \theta \leq 2\pi \end{cases}$

d.

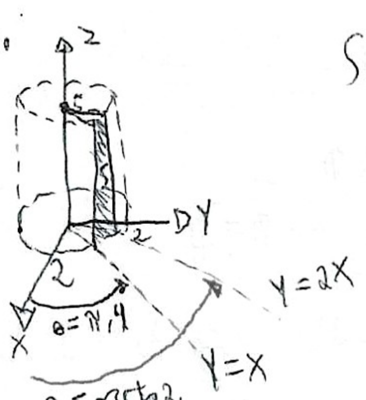


$C: \begin{cases} x(t) = 0 \\ y(t) = t \\ z(t) = 2t \end{cases}$; $t \geq 0$; $(x,y,z) \in$ circunferência de raio $y(t) = t$ e centro $(0,0,z(t)) = (0,0,2t)$. Então:

$\begin{cases} x = t\cos\theta \\ y = t\,\text{sen}\,\theta \\ z = 2t \end{cases}$, $t > 0$ e $\theta \in [0, 2\pi]$

Uma parametrização de S é dada por: $\varphi(t,\theta) = (t\cos\theta, t\,\text{sen}\,\theta, 2t)$
$(t,\theta) \in D: t \geq 0, \theta \in [0, 2\pi]$

3) a.



$S: \varphi(\theta,z) = (2\cos\theta, 2\,\text{sen}\,\theta, z)$

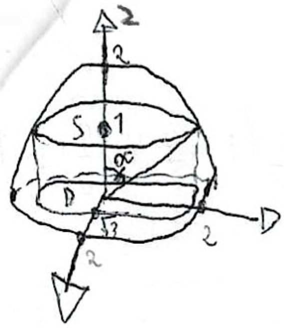
$(\theta,z) \in D: \begin{cases} 0 \leq z \leq 5 \\ \pi/4 \leq \theta \leq \text{arctg}\, 2 \end{cases}$

$\varphi_\theta \times \varphi_z = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2\,\text{sen}\,\theta & 2\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = (2\cos\theta, 2\,\text{sen}\,\theta, 0)$

$\|\varphi_\theta \times \varphi_z\| = \sqrt{4\cos^2\theta + 4\,\text{sen}^2\theta} = \sqrt{4} = 2$

$A(S) = \iint_D \|\varphi_\theta \times \varphi_z\|\, d\theta dz = \iint_D 2\, d\theta dz = 2\int_{\pi/4}^{\text{arctg}\,2}\int_0^5 dz\, d\theta$

$= 10\int_{\pi/4}^{\text{arctg}\,2} dz = 10\left(\text{arctg}\, 2 - \frac{\pi}{4}\right)$ u.a. //



$$x^2+y^2+\frac{x^2+y^2}{3}=4 \to x^2+y^2=3$$
$$z=1$$

S: $\varphi(x,y)=(x,y,\sqrt{4-x^2-y^2})$

$(x,y)\in D: x^2+y^2\le 3$

as coordenadas esféricas são: ρ, ϕ e θ, logo, $\rho=2$, $x=2\sin\phi\cos\theta$

$tg\,\alpha=\frac{\sqrt3}{1}$, $\alpha=\frac{\pi}{3}$, então: $y=2\sin\phi\sin\theta$

$z=2\cos\phi$

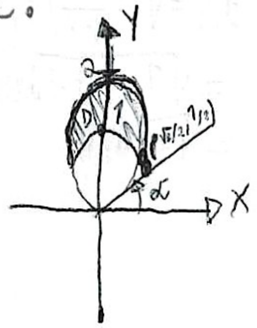
$A(S)=\iint_S dS=\iint_D 4\sin\phi\, d\phi\, d\theta$ S: $\varphi(\phi,\theta)=(2\sin\phi\cos\theta, 2\sin\phi\sin\theta, 2\cos\phi)$ //

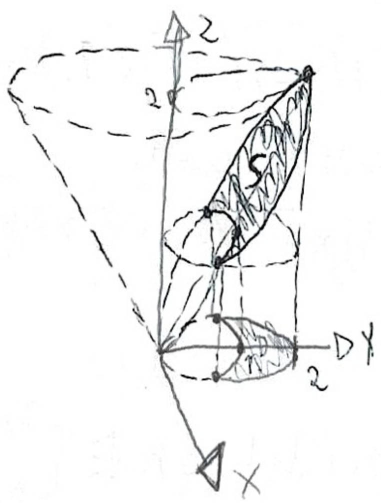
$(\phi,\theta)\in D: \begin{cases} 0\le\phi\le\pi/3 \\ 0\le\theta\le 2\pi\end{cases}$

$=4\int_0^{\pi/3}\int_0^{2\pi}\sin\phi\, d\theta\, d\phi$

$=8\pi\int_0^{\pi/3}\sin\phi\, d\phi=8\pi\left[-\cos\phi\right]_0^{\pi/3}=8\pi\left(1-\frac{1}{2}\right)=4\pi$ u.a //

c.





S: $\varphi(x,y)=(x,y,\sqrt{x^2+y^2})$; $(x,y)\in D$

S: $z=\sqrt{x^2+y^2}$, $(x,y)\in D$, então:

$A(S)=\iint_D\sqrt{1+(z_x)^2+(z_y)^2}\,dx\,dy=$

$=\iint_D\sqrt{1+\left(\frac{x}{\sqrt{x^2+y^2}}\right)^2+\left(\frac{y}{\sqrt{x^2+y^2}}\right)^2}\,dx\,dy$

$=\iint_D\sqrt{1+\frac{x^2+y^2}{x^2+y^2}}\,dx\,dy=\sqrt2\iint_D dx\,dy=$

$\sqrt2\int_{\pi/6}^{5\pi/6}\int_1^{2\sin\theta} r\,dr\,d\theta=\sqrt2\int_{\pi/6}^{5\pi/6}\left[\frac{r^2}{2}\right]_1^{2\sin\theta}d\theta$

$=\frac{\sqrt2}{2}\int_{\pi/6}^{5\pi/6}(4\sin^2\theta-1)\,d\theta=\frac{\sqrt2}{2}\left[\frac{4}{2}\left(\theta-\frac{\sin2\theta}{2}\right)-\ldots\right.$

$=\frac{\sqrt2}{2}\left[\theta-\sin2\theta\right]_{\pi/6}^{5\pi/6}=\frac{\sqrt2}{2}\left[\left(\frac{5\pi}{6}-\sin\frac{5\pi}{3}\right)-\left(\frac{\pi}{6}-\ldots\right.\right.$

$=\frac{\sqrt2}{2}\left(\frac{2\pi}{3}+2\sin\frac{\pi}{3}\right)=\frac{\sqrt2}{2}\left(\frac{2\pi}{3}+\sqrt3\right)$ u.a //

$\begin{cases} x=r\cos\theta \\ y=r\sin\theta \\ z=\sqrt{x^2+y^2}=r\end{cases}$ $tg\,\alpha=\frac{\frac12}{\frac{\sqrt3}{2}}\to\alpha=\frac{\pi}{6}$

$\left\{\frac{\pi}{6}\le\theta\le\frac{5\pi}{6}\right.$

$x^2+y^2=2y\to r^2=2r\sin\theta$, $\{1\le r\le 2\sin\theta$

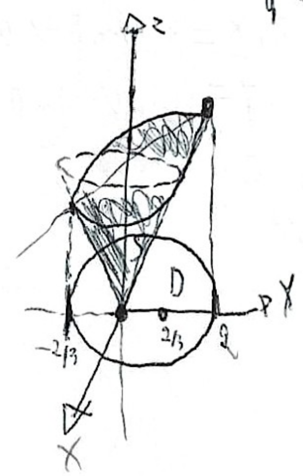
$r=2\sin\theta$

S: $\varphi(r,\theta)=(r\cos\theta, r\sin\theta, r)$,

d. seja $z=\sqrt{x^2+y^2}$ e $z-\frac{y}{2}=1$;

$$x^2+y^2=\left(1+\frac{y}{2}\right)^2 \to x^2+y^2=1+y+\frac{y^2}{4} \to x^2+\frac{3}{4}y^2-y=1 \to x^2+\frac{3}{4}\left(y^2-\frac{4}{3}y+\frac{4}{9}\right)=1+\frac{1}{3}$$

$$\to x^2+\frac{3}{4}\left(y-\frac{2}{3}\right)^2=\frac{4}{3} \to \frac{x^2}{\frac{4}{3}}+\frac{\left(y-\frac{2}{3}\right)^2}{\frac{16}{9}}=1$$



$S: z=\sqrt{x^2+y^2}$, $(x,y)\in D: \frac{x^2}{4/3}+\frac{(y-2/3)^2}{16/9}\le 1$

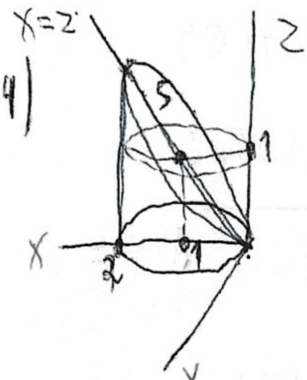
$$A(S)=\iint_D \sqrt{1+(z_x)^2+(z_y)^2}\,dxdy$$

$$A(S)=\iint_D\sqrt{1+\frac{x^2}{x^2+y^2}+\frac{y^2}{x^2+y^2}}\,dxdy=\iint_D\sqrt{1+1}\,dxdy$$

$$=\sqrt{2}\cdot A(D)=\sqrt{2}\,\pi ab=\sqrt{2}\,\pi\,\frac{8}{3\sqrt{3}}=\frac{8\pi\sqrt{6}}{9} \text{ u.a.}$$

$a=\frac{2}{\sqrt{3}}$, $b=\frac{4}{3}$

4)



$S: z=z(x,y)=x$, $(x,y)\in D: (x-1)^2+y^2\le 1$, temos $dS=\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}$

$=\sqrt{1+1^2+0^2}\,dxdy$

$=\sqrt{2}\,dxdy$

a densidade $f(x,y,z)=kz$, onde k é constante.

$M=\iint_S f(x,y,z)\,dS \to M=k\iint_D x\sqrt{2}\,dxdy=k\sqrt{2}\,x\,dxdy$

Em coord polar, temos:

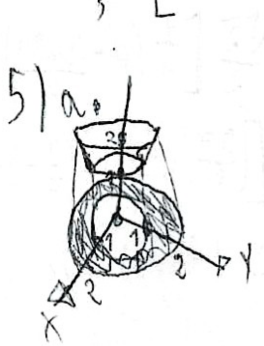
$$M=k\sqrt{2}\int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta} r\cos\theta\, r\,dr\,d\theta=k\sqrt{2}\int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta} r^2\cos\theta\,d$$

$$=k\sqrt{2}\int_{-\pi/2}^{\pi/2}\cos\theta\left[\frac{r^3}{3}\right]_0^{2\cos\theta}d\theta=\frac{8k\sqrt{2}}{3}\int_{-\pi/2}^{\pi/2}(1+2\cos 2\theta+\cos^2 2\theta)\,d\theta=\frac{2k\sqrt{2}}{3.2}\int_{-\pi/2}^{\pi/2}(1+2\cos 2\theta+\cos$$

$d(2$

$$=\frac{k\sqrt{2}}{3}\left[2\theta+2\sin 2\theta+\frac{1}{2}\left(2\theta+\frac{\sin 4\theta}{2}\right)\right]_{-\pi/2}^{\pi/2}=\frac{k\sqrt{2}}{3}(2\pi+\pi)=k\cdot\sqrt{2}\,\pi \text{ u.m.}$$

5) a.



$S=O_z$, então $I_z=\iint_S (x^2+y^2)\,\rho(x,y,z)\,dS$, $\rho(x,y,z)=\rho$

$$I_z=\rho\iint_S(x^2+y^2)\,dS$$

$S: z=\sqrt{x^2+y^2}=f(x,y)$, $(x,y)\in D: 1\le x^2+y^2\le 4$; $z_x=f_x=\frac{x}{\sqrt{x^2+y^2}}$

$1+(z_x)^2+(z_y)^2=1+\frac{x^2}{x^2+y^2}+\frac{y^2}{x^2+y^2}=1+\frac{x^2+y^2}{x^2+y^2}=2 \to dS=\sqrt{1+(z_x)^2+(z_y)^2}\,dxdy$; $dS=\sqrt{2}\,dxdy$

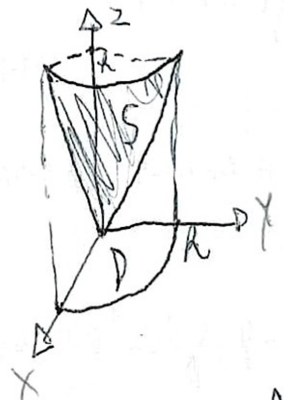
$I_z=\rho\iint_S(x^2+y^2)\,dS=\rho\iint_D(x+y^2)\sqrt{2}\,dxdy=\sqrt{2}\rho\iint_D(x^2+y^2)\,dxdy \to \begin{cases} x=r\cos\theta \\ y=r\sin\theta \\ dxdy=r\,dr\,d\theta \\ x^2+y^2=r^2 \end{cases}$ e $D_{r\theta}: \begin{cases} 0\le\theta \\ 1\le r \end{cases}$

$= \sqrt{2}\rho \iint_{D_{r\theta}} r^2 \cdot r\, dr d\theta = \sqrt{2}\rho \iint_{D_{r\theta}} r^3 dr d\theta = \sqrt{2}\rho \int_1^2 r^3 \int_0^{2\pi} d\theta dr = 2\sqrt{2}\rho\pi \int_1^2 r^3 dr$

$= 2\sqrt{2}\rho \left[\frac{r^4}{4}\right]_1^2 = \frac{\sqrt{2}\rho\pi}{2}(16-1) = \underline{\frac{15\sqrt{2}\rho\pi}{2}}$ //

N. $S: z = \sqrt{x^2+y^2}$, $(x,y) \in D: x^2+y^2 \leq R^2$, $x \geq 0$ e $y \geq 0$, então $dS = \sqrt{2}\, dxdy$

$I_z = \iint_S (x^2+y^2) k\, dS = k \iint_D (x^2+y^2)\sqrt{2}\, dxdy = k\sqrt{2} \iint_D (x^2+y^2)\, dxdy.$



Em coordenadas polares temos:

$I_z = k\sqrt{2} \int_0^{\pi/2} \int_0^R r^2 \cdot r\, dr d\theta = k\sqrt{2} \int_0^{\pi/2} \int_0^R r^3 \ldots$

$= k\sqrt{2} \int_0^{\pi/2} \left[\frac{r^4}{4}\right]_0^R d\theta = \frac{R^4 k}{4}\sqrt{2} \int_0^{\pi/2} d\theta = \frac{R^4 k\sqrt{2}\pi}{8}$

$M = \iint_S k\, dS = k\sqrt{2} \iint_D dxdy = k\sqrt{2} A(D) = \frac{k\sqrt{2}\pi R^2}{4} = \frac{R^2 k\sqrt{2}\pi}{4}$

$I_z = \frac{MR^2}{2}$ //

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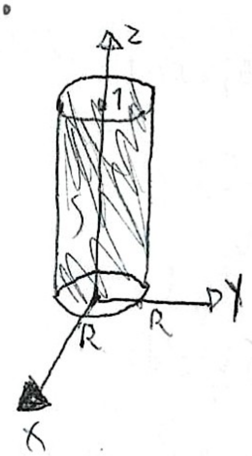


$S: \varphi(t,z) = (R\cos t, R\sin t, z)$, $(t,z) \in D: \begin{cases} 0 \leq t \leq 2\pi \\ 0 \leq z \leq 1 \end{cases}$

$\varphi_t = (-R\sin t, R\cos t, 0)$ e $\varphi_z = (0,0,1)$, portanto:

$\varphi_t \times \varphi_z = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -R\sin t & R\cos t & 0 \\ 0 & 0 & 1 \end{vmatrix} = (R\cos t, R\sin t, 0)$

$\|\varphi_t \times \varphi_z\| = R$, como $dS = \|\varphi_t \times \varphi_z\|\, dt dz$, $dS = R\, dt dz$

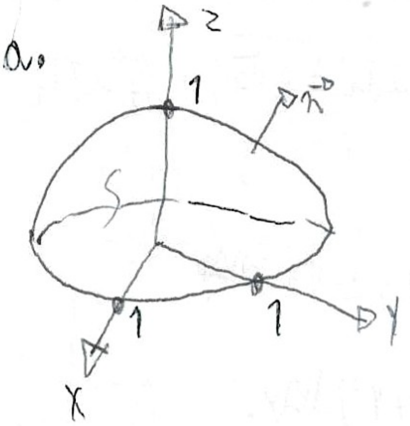
$x^2+y^2 = R^2$

$I_z = \iint_S (x^2+y^2) \underbrace{\delta(x,y,z)}_{k} dS =$

$k \iint_D (R^2\cos^2 t + R^2\sin^2 t) R\, dt dz = kR^3 \int_0^{2\pi} \int_0^1 dz dt = 2k\pi R^3.$

$M = kA(S) = k(2\pi R) \cdot 1 = 2k\pi R \rightarrow I_z = MR^2$ //

6) a.



$\vec{n}\cdot\hat{k} > 0$, então $\vec{n}$ aponta para cima e então, $\vec{n} = \frac{(x,y,z)}{a} = (x, \dots$

$a = 1$.

$\Phi = \iint_S \vec{F}\cdot\vec{n}\, dS = \iint_S (x-y-4, y, z)\cdot(x,y,z)\, dS =$

$\iint_S (\underbrace{x^2+y^2+z^2}_{1} - xy - 4x)\, dS = \iint_S (1 - xy - 4x)\, dS = \iint_S dS - \iint_S (xy \dots$

$= A(S) - \iint_S (xy - 4x)\, dS = \frac{1}{2}\cdot 4\pi\cdot 1^2 - \iint_S (xy-4x)\, dS = 2\pi - \iint_S (xy-4x) \dots$

$S: \varphi(\phi,\theta) = (\operatorname{sen}\phi\cos\theta, \operatorname{sen}\phi\operatorname{sen}\theta, \cos\phi)$

$(\phi,\theta)\in D: \begin{cases} 0 \le \phi \le \pi/2 \\ 0 \le \theta \le 2\pi \end{cases}$

$dS = a^2 \operatorname{sen}\phi\, d\phi d\theta = \operatorname{sen}\phi\, d\phi d\theta$

$\iint_S (xy - 4x)\, dS = \iint_D (\operatorname{sen}^2\phi \operatorname{sen}\theta\cos\theta - 4\operatorname{sen}\phi\cos\theta)\operatorname{sen}\phi\, d\phi d\theta$

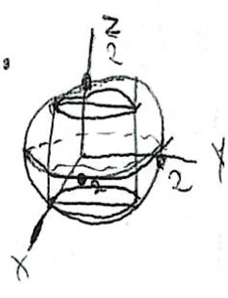
$= \iint_D \operatorname{sen}^3\phi \operatorname{sen}\theta\cos\theta\, d\phi d\theta - 4\iint_D \operatorname{sen}^2\phi\cos\theta\, d \dots$

$= \int_0^{\pi/2} \operatorname{sen}^3\phi \int_0^{2\pi} \operatorname{sen}\theta\cos\theta\, d\theta d\phi - 4\int_0^{\pi/2} \operatorname{sen}^2\phi \int_0^{2\pi} \cos\theta \dots$

$= \int_0^{\pi/2} \operatorname{sen}^3\phi \left[\frac{\operatorname{sen}^2\theta}{2}\right]_0^{2\pi} d\phi - 4\int_0^{\pi/2} \operatorname{sen}^2\phi \left[\operatorname{sen}\theta\right]_0^{2\pi} d\phi = 0$

Portanto: $\iint_S \vec{F}\cdot\vec{n}\, dS = 2\pi$ //

b.



$\operatorname{tg}\theta = \frac{1}{\sqrt{3}} \to \phi = \frac{\pi}{6}$

$S: \varphi(\phi,\theta) = (2\operatorname{sen}\phi\cos\theta, 2\operatorname{sen}\phi\operatorname{sen}\theta, 2\cos\phi)$

$(\phi,\theta)\in D = \left[\frac{\pi}{6}, \frac{5\pi}{6}\right]\times[0, 2\pi]$.

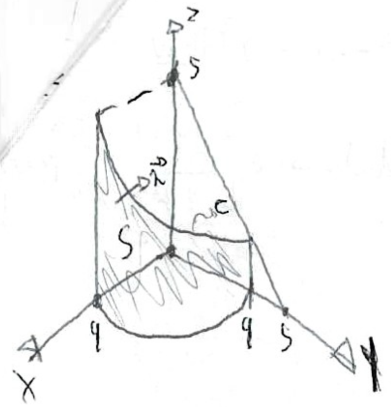
$dS = a^2 \operatorname{sen}\phi\, d\phi d\theta \overset{a=2}{=} 4\operatorname{sen}\phi\, d\phi d\theta$

$\vec{n} = \frac{(x,y,z)}{a} \overset{a=2}{=} \frac{(x,y,z)}{2} \to \iint_S \vec{F}\cdot\vec{n}\, dS = \iint_S (0,0,-z)\cdot\frac{(x,y,z)}{2}\, dS = -\iint_S z^2\, dS$

$= -\iint_D 4\cos^2\phi \cdot 4\operatorname{sen}\phi\, d\phi d\theta = -16\int_{\pi/6}^{5\pi/6}\int_0^{2\pi} \cos^2\phi \operatorname{sen}\phi\, d\theta d\phi$

$= 32\pi \int_{\pi/6}^{5\pi/6} \cos^2\phi\, d(\cos\phi) = 32\pi\left[\frac{\cos^3\phi}{3}\right]_{\pi/6}^{5\pi/6} = \frac{32\pi}{3}\left[-\left(\frac{\sqrt{3}}{2}\right)^3 - \dots\right.$

$= -\frac{32}{3}\pi \cdot \frac{3\sqrt{3}}{8} = -4\pi\sqrt{3}$ //



$S: \varphi(t,z) = (4\cos t, 4\sin t, z)$; $dS = a\,dt\,dz \overset{a=4}{=} 4\,dt\,dz$

$(t,z) \in D: \begin{cases} 0 \le t \le \pi/2 \\ 0 \le z \le 5 - 4\sin t \end{cases}$

Como $\vec{n}$ aponta para o eixo z, então:

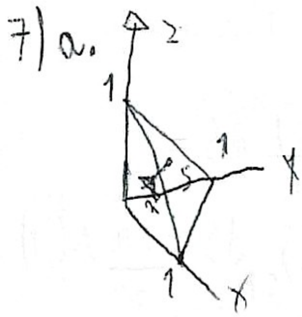
$$\vec{n} = \frac{(-x,-y,0)}{a} = \frac{(-x,-y,0)}{4}$$

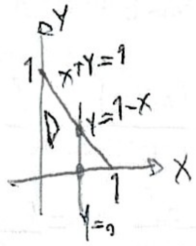
$$\iint_S \vec{F}\,\vec{n}\,dS = \iint_S (-x,-y,3y^2z)\cdot\frac{(-x,-y,0)}{4}\,dS = \frac{1}{4}\iint_S \underbrace{(x^2+y^2)}_{16}\,dS = 4\iint_S dS = 4\iint_D 4\,dt\,dz$$

$$= 16\int_0^{\pi/2}\int_0^{5-4\sin t} dz\,dt = 16\int_0^{\pi/2}(5-4\sin t)\,dt = 16\left[5t+4\cos t\right]_0^{\pi/2} = 16\left(\frac{5\pi}{2}-4\right)$$

$$= 40\pi - 64 \,//$$

7) a.





$S: z = 1-x-y = f(x,y)$; $(x,y) \in D: \begin{cases} 0 \le x \le 1 \\ 0 \le y \le 1-x \end{cases}$

Um vetor normal a S é dado por: $N = (-f_x, -f_y, 1) = (1,1,1)$

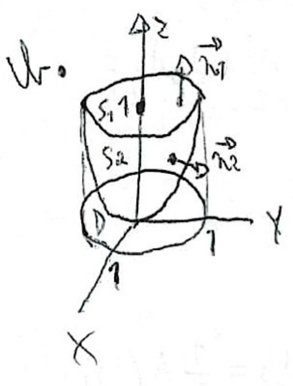
$\vec{n} = \frac{(1,1,1)}{\sqrt{3}}$; $dS = \sqrt{1+(f_x)^2+(f_y)^2}\,dxdy = \sqrt{3}\,dxdy$

Então: $\iint_S \vec{F}\,\vec{n}\,dS = \iint_S (xze^y, -xze^y, z)\cdot\frac{(-1,-1,-1)}{\sqrt{3}}\,dS = \iint_S (-xze^y + xze^y + z)\cdot\frac{(-1,-1,-1)}{\sqrt{3}}\,dS$

$$= \iint_S \frac{(xze^y - xze^y - z)}{\sqrt{3}}\,dS = \iint_S \frac{-z}{\sqrt{3}}\,dS = \iint_D d\left(\frac{-(1-x-y)}{\sqrt{3}}\right)\cdot\sqrt{3}\,dxdy = \iint_D (-1+x+y)\,dxdy$$

$$= \int_0^1\int_0^{1-x}(-1+x+y)\,dy\,dx = \int_0^1\left[-y+xy+\frac{y^2}{2}\right]_0^{1-x} dx = -\frac{1}{6}\,//$$

b.



$\iint_S \vec{F}\cdot\vec{n}\,dS = \iint_{S1} \vec{F}\,\vec{n}_1\,dS + \iint_{S2} \vec{F}\cdot\vec{n}_2\,dS$.

$S1: z = 1 = f(x,y)$; $(x,y) \in D: x^2+y^2 \le 1$; $\vec{n}_1 = \hat{k}$ e $dS = dxdy$

$\iint_{S1} \vec{F}\,\vec{n}_1\,dS = \iint_{S1} (-x,0,2.1)\cdot(0,0,1)\,dS = \iint_{S1} 2\,dS = 2A(S) = 2(\pi.1^2) = 2\pi$

$S2: z = x^2+y^2 = g(x,y)$; $(x,y) \in D: x^2+y^2 \le 1$. Um vetor normal a S é dado por

$\vec{N} = (-g_x, -g_y, 1) = (-2x,-2y,1)$; que aponta para cima. $\vec{n}_2$ para baixo.

$\vec{n}_2 = \frac{(2x,2y,1)}{\sqrt{1+4x^2+4y^2}}$. $dS = \|\vec{N}\|\,dxdy = \sqrt{1+4x^2+4y^2}\,dxdy$

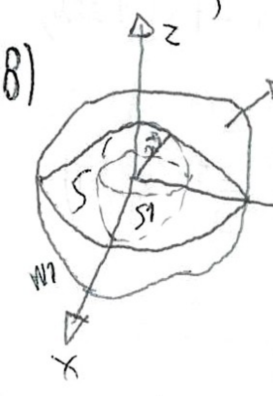
$$\iint_{S_2} \vec F\cdot\vec n_2\, dS = \iint_D (-x,0,2(x^2+y^2))\cdot(2x,2y,-1)\,dxdy = \iint_D (-2x^2-2x^2-2y^2)\,dxdy$$

$$= \iint_{D_{r\theta}} (-2r^2-2r^2\cos^2\theta)\, r\,dr\,d\theta = -2\int_0^{2\pi}\int_0^1 (1+\cos^2\theta)\, r^3\, dr\,d\theta = -2\left[\frac{r^4}{4}\right]_0^1 \int_0^{2\pi}(1+\cos^2\theta)$$

$$= -\frac{1}{2}\left[\theta+\frac{1}{2}\left(\theta+\frac{\sin 2\theta}{2}\right)\right]_0^{2\pi} = -\frac{3\pi}{2}.$$

$$\iint_S \vec F\cdot\vec n\, dS = 2\pi + \left(-\frac{3\pi}{2}\right) = \frac{\pi}{2} \,//$$

8)



$S_1: x^2+y^2+z^2=a^2$, $a>0$ tal que $S_1 \subset W$ (já que $W \not\subset \operatorname{dom}\vec E = \{\mathbb{R}^3-\{(0,0$

$W_1 \subset \operatorname{dom}\vec E \to \partial W_1 = S\cup S_1$. seja $\vec n_1$ a normal a S_1

$$\iint_{\partial W_1} \vec E\cdot\vec n\, dS = \iiint_{W_1} \operatorname{div}\vec E\, dxdydz, \quad \operatorname{div}\vec E = 0$$

$$\iint_S \vec E\cdot\vec n\, dS + \iint_{S_1} \vec E\cdot\vec n_1\, dS = \iiint_{W_1} \operatorname{div}\vec E\, dxdydz$$

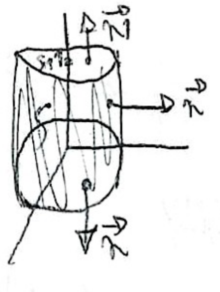
$$\iint_S \vec E\cdot\vec n\, dS = -\iint_{S_1} \vec E\cdot\vec n_1\, dS = \iint_{S_1} \vec E\cdot(-\vec n_1)\, dS = \iint_{S_1} \frac{\varepsilon Q(x,y,z)}{(x^2+y^2+z^2)^{3/2}}\cdot\frac{(x,y,z)}{a}\, dS =$$

$$-\vec n_1 = \frac{(x,y,z)}{a} \to \quad \frac{\varepsilon Q}{a}\iint_{S_1} \frac{x^2+y^2+z^2}{(x^2+y^2+z^2)^{3/2}}\, dS = \frac{\varepsilon Q}{a}\iint_{S_1}\frac{a^2}{(a^2)^{3/2}}\, dS = \frac{\varepsilon Q}{a^2}\iint_{S_1} dS = \frac{\varepsilon Q}{a^2} A(S_1)$$

$$= \frac{\varepsilon Q}{a^2}\, 4\pi a^2 = 4\pi\varepsilon Q \,//$$

9) seja $\bar S = S\cup S_1$

$S_1: z=1, (x,y)\in D: x^2+y^2\leq 1$ e $\vec n_1 = \hat k$ e $dS = dxdy$



$$\iint_{\bar S} \frac{\partial f}{\partial \vec n}\, dS = \iint_{\bar S} \nabla f\cdot\vec n\, dS = \iiint_W \nabla\cdot\nabla f\, dV = \iiint_W \nabla^2 f\, dV$$

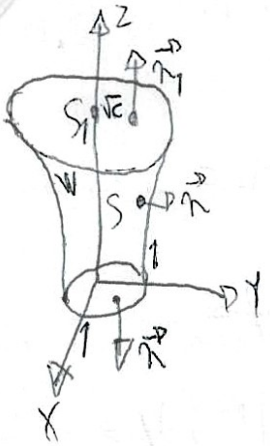
$$= \iiint_W (x^2+y^2)\, dV = \int_0^{2\pi}\int_0^1\int_0^1 r^2\, r\, dr\,dz\,d\theta = \int_0^{2\pi}\int_0^1\int_0^1 r^3\, dr\,dz\,d\theta =$$

$$\left[\frac{r^4}{4}\right]_0^1 \int_0^{2\pi}\int_0^1 dz\,d\theta = \frac{2\pi}{4} = \frac{\pi}{2}.$$

seja $\iint_{\bar S} \frac{\partial f}{\partial \vec n}\, d\bar S = \iint_S \frac{\partial f}{\partial \vec n}\, dS + \iint_{S_1} \frac{\partial f}{\partial \vec n_1}\, dS \to \iint_{S_1} \frac{\partial f}{\partial \vec n_1}\, dS = \iint_{S_1} \nabla f\cdot\vec n_1\, dS$

$$= \iint_{S_1} \left(\frac{\partial f}{\partial x}(x,y,1), \frac{\partial f}{\partial y}(x,y,1), \frac{\partial f}{\partial z}(x,y,1)\right)\cdot(0,0,1)\, dS = \iint_{S_1} \frac{\partial f}{\partial z}(x,y,1)\, dS = \iint_{S_1} \frac{1}{3}\, dS = \frac{1}{3} A(S_1)$$

$$= \frac{1}{3}\pi\cdot 1^2 = \frac{\pi}{3} \to \iint_S \frac{\partial f}{\partial \vec n}\, d\vec S = \frac{\pi}{2} - \frac{\pi}{3} = \frac{\pi}{6} \,//$$



$x^2+y^2=1+c$

$x^2+y^2=(\sqrt{1+c})^2$

$z=\sqrt{c}$

$\iint_S \text{rot}\vec F\cdot\vec n\,dS+\iint_{S_1}\text{rot}\vec F\cdot\vec n_1\,dS$

$=\iiint_W \text{div}(\text{rot}\vec F)\,dV$

$\text{div}(\text{rot}\vec F)=0;\ \iint_S \text{rot}\vec F\cdot\vec n\,dS=-6\pi$

(1) $\iint_{S_1}\text{rot}\vec F\cdot\vec n_1\,dS=6\pi$, assim: $\text{rot}\vec F=\begin{vmatrix}\hat i & \hat j & \hat k\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ -\frac{cy}{2}+ze^x & \frac{cx}{2}-ze^y & xy\end{vmatrix}=$

$(x+e^y,\ e^x-y,\ \frac{c}{2}+\frac{c}{2})=(x+e^y,\ e^x-y,\ c)$

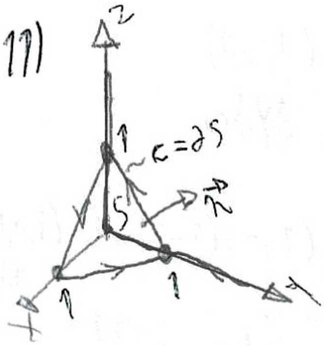
$S_1: z=\sqrt{c},\ (x,y)\in D: x^2+y^2\le(\sqrt{1+c})^2;\ \vec n_1=\hat k$

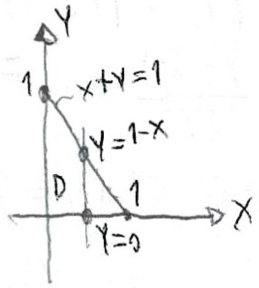
$\iint_{S_1}\text{rot}\vec F\cdot\vec n_1\,dS=\iint_D(x+e^y,e^x-y,c)\cdot(0,0,1)\,dxdy=c\iint_D dxdy=c\,A(D)=c\,[\pi(\sqrt{1+c})^2]$

$=\pi c(1+c)$, em (1): $\pi c(1+c)=6\pi \Rightarrow c^2+c-6=0 \Rightarrow c=2$ ou $c=-3$

$c>0 \Rightarrow c=2$ //

11)





$S: z=1-x-y=f(x,y);\ (x,y)\in D$, D é a projeção de S sobre xy

$\vec N=(-f_x,-f_y,1)=(1,1,1)$

$\vec n=\frac{(1,1,1)}{\sqrt3}$ e $dS=\sqrt3\,dxdy$, pelo teorema de Stokes temos: $\oint_C\vec F\cdot d\vec r=\iint_S \text{rot}\vec F\cdot\vec n\,dS$

onde $\text{rot}\vec F=\begin{vmatrix}\hat i & \hat j & \hat k\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ y & xz & z^2\end{vmatrix}=(-x,0,z-1)$, logo: $\oint_C\vec F\cdot d\vec r=\iint_D(-x,0,1-x-y-1)\cdot\frac{(1,1,1)}{\sqrt3}\cdot\sqrt3$

$=\iint_D(-x-x-y)\,dxdy=-\iint_D(2x+y)\,dxdy=-\int_0^1\int_0^{1-x}(2x+y)\,dydx=-\int_0^1\left[2xy+\frac{y^2}{2}\right]_0^{1-x}dx$

$=-\int_0^1\left(2x-2x^2+\frac{1-2x+x^2}{2}\right)dx=-\frac12\int_0^1(4x-4x^2+1-2x+x^2)\,dx=-\frac12\int_0^1(2x-3x^2+1)\,dx$

$=-\frac12\left[x^2-x^3+x\right]_0^1=-\frac12$ //

12)

$\vec n=\frac{(x,y,z)}{a}=\frac{(x,y,z)}{2}$, temos $W=\int_C\vec F\cdot d\vec r=\iint_S\text{rot}\vec F\cdot\vec n\,dS$

$\text{rot}\vec F=\begin{vmatrix}\hat i & \hat j & \hat k\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ x^2+z^2 & y^2+x^2 & z^2+y^2\end{vmatrix}=(2y,2z,2x)$

$W=\iint_S(2y,2z,2x)\cdot\frac{(x,y,z)}{2}\,dS=\iint_S(xy+yz+xz)\,dS.$

parametrizando S: $\varphi(\phi,\theta) = (2\sin\phi\cos\theta, 2\sin\phi\sin\theta, 2\cos\phi)$, $(\phi,\theta)\in D: \begin{cases}0\le\phi\le\pi/2\\ 0\le\theta\le\pi/2\end{cases}$

$dS = a^2\sin\phi\, d\phi d\theta = 4\sin\phi\, d\phi d\theta$, logo:

$$W=\iint_D (4\sin^2\phi\cos\theta\sin\theta + 4\sin\phi\cos\phi\sin\theta + 4\sin\phi\cos\phi\cos\theta)\, 4\sin\phi\, d\phi d\theta =$$

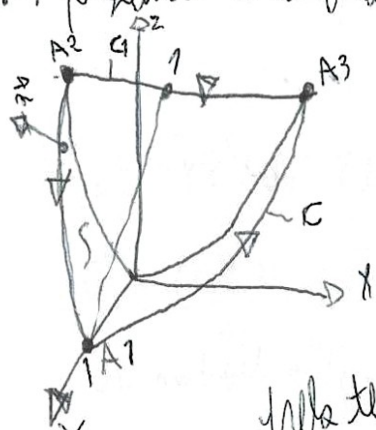
$$= 16\int_0^{\pi/2}\int_0^{\pi/2}(\sin^3\phi\cos\theta\sin\theta + \sin^2\phi\cos\phi\sin\theta + \sin^2\phi\cos\phi\cos\theta)\, d\theta d\phi =$$

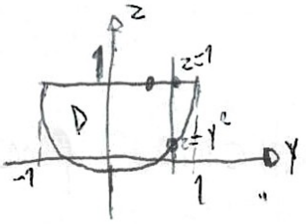
$$16\int_0^{\pi/2}\left(\sin^3\phi\cdot\frac{\sin^2\theta}{2}\Big|_0^{\pi/2} + \sin^2\phi\cos\phi(-\cos\theta)\Big|_0^{\pi/2} + \sin^2\phi\cos\phi\sin\phi\Big|_0^{\pi/2}\right)d\phi =$$

$$16\int_0^{\pi/2}\left(\frac{\sin^3\phi}{2} + 2\sin^2\phi\cos\phi\right)d\phi = 8\int_0^{\pi/2}(1-\cos^2\phi)\sin\phi\, d\phi + 32\int_0^{\pi/2}\sin^2\phi\cos\phi\, d\phi =$$

$$8\left[-\cos\phi + \frac{\cos^3\phi}{3}\right]_0^{\pi/2} + 32\left[\frac{\sin^3\phi}{3}\right]_0^{\pi/2} = 16 \,//$$

13) fechando C usando a reta C1, que liga A3 em A2, temos:





$S: x = 1 - z = f(x,y)$, $(x,y)\in D: \begin{cases}-1\le y\le 1\\ 0\le z\le y^2\end{cases}$

$\bar{C} = \partial S$

$\vec N = (1, -f_y, -f_z) = (1,0,1)$

$\vec n = \frac{(1,0,1)}{\sqrt2}$ e $dS = \sqrt2\, dy dz$.

pelo teorema de Stokes: $\int_{\bar C}\vec F\cdot d\vec r = \iint_S \text{rot}\,\vec F\cdot\vec n\, dS = \iint_D (1, -3(1-z)^2, 2)\cdot(1,0,1)\, dydz$

$$= \iint_D (1+2)\,dydz = 3\iint_D dydz = 3\int_{-1}^1\int_{y^2}^1 dz dy = 3\int_{-1}^1(1-y^2)dy = 3\left[y - \frac{y^3}{3}\right]_{-1}^1$$

$$= 6\left(1-\frac13\right) = 4.$$

$\int_C \vec F\cdot d\vec r + \int_{C1}\vec F\cdot d\vec r = 4$; $C1^-: z=1, x=0$ e $-1\le y\le 1$, $dz=dx=0$, então:

$$\int_{C1}\vec F d\vec r = -\int_{C1^-}\vec F d\vec r = -\int_{C1^-} Q(0,y,1)\,dy = -\int_{-1}^1(-1+y)\,dy = -\left[-y+\frac{y^2}{2}\right]_{-1}^1 = 2$$

logo: $\int_C \vec F\cdot d\vec r = 4 - 2 = 2 \,//$