Rheonomic Supergravity

by

Guillaume Trojani

Supervisor : Pr Richard Szabo

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1 Lecture 1

Mostly covered in [1]

1.1 Klein Geometry

Definition 1.1.1. Homogenous spaces, group action on LEFT (M, G) manifold, G Lie group action that is transitive. That is $\forall x, y \in M \exists g \in G | gx = y$. For element $o \in G$, the stab $G_o = \{g \in G | go = o\}$, it turns out that

$$G/G_o \cong M \tag{1.1}$$

a G-invariant diffeomorphism.

If G is a group and H a closed subgroup, then G/H is a homogeneous space.

Definition 1.1.2. (Klein Geometry)
A Klein geometry is a pair (G, H) where H closed subgroup of G and G/H is connected. The last condition is not necessary and not always there but is convenient.

Remark 1.1.1. Given a Klein geometry (G, H), we can get a pair of Lie algebra $(G, H) \xrightarrow{Lie} (\mathfrak{g}, \mathfrak{h})$. (See Lie functor??) gives a short exact sequence of H-modules

$$0 \to \mathfrak{h} \to \mathfrak{g} \to \frac{\mathfrak{g}}{\mathfrak{h}} \to 0 \tag{1.2}$$

If the sequence splits then get

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$$
$$\mathfrak{m} \cong \mathfrak{g}/\mathfrak{h}$$

as H-modules. m is a submodule, we call it reductive? Put in the table here and explain.

Given $G \circlearrowright M$ such that (M,G) homogeneuous then the

$$(r_g)_* : T_x M \to T_{gx} M \tag{1.3}$$

Taking an origin $o \in M$ and let $H = G_o$ then for $h \in H$ $(l_h)_* \in \mathrm{GL}(T_o M)$ that sends $h \to (l_h)_*$ is a rep of H on $T_o M$ called the linear isotropy rep of (M, G, o).

If $M \cong G/H$,

$$T_o M \xrightarrow{\sim} T_H(G/H) \xrightarrow{\sim} \mathfrak{g}/\mathfrak{h} \cong m \tag{1.4}$$

$$0 \to T_e H \to T_e G \to T_H(G/H) \to 0 \tag{1.5}$$

the upshot is we get

$$T_o M \cong \mathfrak{g}/\mathfrak{h} \cong \mathfrak{m} \tag{1.6}$$

and linear structures on $\mathfrak{g}/\mathfrak{h}$ induce a *geometric* structures on M.

The correspondances between H-invariant tensors of $T_o M$ and G-equivariant tensor fields on M.

Example 1.1.1.

$$\tau \in T_o M \to t_o = \tau \tag{1.7}$$

such that $t_{g\cdot o} = (l_g)_* \tau$ this is well defined because for $g'o = go$ we have $g'g^{-1} \in H$ by H-invariance of τ, meaning t is well defined. THe other way is by evaluating at $o \in M$.

In particular, a pseudo inner product on $\mathfrak{g}/\mathfrak{h}$ gives rise to a pseudo-riemanian metric on M.

Definition 1.1.3. (Metric Klein geometry)
(G, H, η)

- (G, H) klein geo
- η is a pseudo inner product on g/h which is H-invariant

Example 1.1.2.
- $G = ISO_o(\mathbb{R}^{d-1,1}) \cong SO_o(d-1,1) \ltimes \mathbb{R}^{d-1,1}$, with H the SO part, then $G/H \cong \mathbb{R}^{d-1,1}$ has the structure steming from the lie algebra structure of $g/h \cong \mathbb{R}^{d-1,1}$
- $G = ISO_o(\mathbb{R}^d)$ with $H = SO(d)$ then $G/H \cong \mathbb{R}^d$

$$\begin{tikzcd}
 & ISO_o(n,1) \arrow[dl, "\text{space of spacelike hyperplane}"'] \arrow[dr, "/\mathbb{R}^{n,1}"] & \\
ISO_o(n) \arrow[dr, "/\mathbb{R}^n"] & & SO_0(n,1) \arrow[dl, "/\text{velocity space}"] \\
 & SO(n) &
\end{tikzcd}$$

$$\begin{tikzcd}
 & SO(4,1) \arrow[dl, "H^4"'] \arrow[dr, "dS"] & \\
SO(4) \arrow[dr, "S^3"] & & SO(3,1) \arrow[dl, "\text{velocity space}"] \\
 & SO(3) &
\end{tikzcd}$$

ISO means inhomogeneous something

1.2 Cartan geometry

Definition 1.2.1. (Cartan geom)
$(\pi : P \to M, A)$ modelled on a Klein geometry (G, H) is a principle right H-bundle $P \to M$ with a Cartan connection $A \in \Omega^1(P, \mathfrak{g})$. Noting that the cartan connection takes values in the larger Lie algebra. this satisfies the conditions

- $A_p : T_p P \to \mathfrak{g}$ is a linear isomorphism
- $(R_h)^* A = \mathrm{Ad}_{h^{-1}} \circ A$ for all $h \in H$
- $A(\xi_X) = X$ for $X \in \mathfrak{h}$ for fundamental vector field ξ of $H \circlearrowright P$, an Ehresmannn-like connection

Remark 1.2.1. Because of the first condition,

$$\begin{aligned} dimP &= dimG \\ dimM &= dim(G/H) \\ (A_p)^{-1} &: \mathfrak{g} \to T_p P \\ (A_0)^{-1} &: \mathfrak{g} \to \mathfrak{X}(P) \\ (A_0)^{-1} &: \mathfrak{h} \to \mathfrak{X}_{\text{vert}}(P) \\ X &\mapsto \xi_X \end{aligned}$$

If G/H is metrci klein, M inherits a ùetric of saùe sign with

$$T_x M \cong T_p P/\ker(\pi_*) \cong \mathfrak{g}/\mathfrak{h} \cong \mathfrak{m} \tag{1.8}$$

$$TM \cong P \times_H \mathfrak{g}/\mathfrak{h} \tag{1.9}$$

Definition 1.2.2. The curvature of a cartan geometry

$$F(A) = dA + \frac{1}{2}[A, A] \in \Omega^2(P, \mathfrak{g}) \tag{1.10}$$

Diagrams diagrams

Take curvature of cartan connection

$$\begin{aligned} F[A] &= dA + \frac{1}{2}[A, A] \\ &= d\omega + de + \frac{1}{2}[\omega + e, \omega + e] \\ &= \left(d\omega + \frac{1}{2}[\omega, \omega]_h + [\omega, e]_h + [e, e]_h + de + \frac{1}{2}[\omega, \omega]_m + [\omega, e]_m + \frac{1}{2}[e, e,]_h \right) \end{aligned}$$

colors:

$\hat{F} = \Omega(\omega) + \frac{1}{2}[e, e]_h$, $T = d_\omega e + \frac{1}{2}[e, e]_m$ If $\hat{F}$ is flat, then $\Omega(\omega) = -\frac{1}{2}[e, e]_h$ and $T = 0$ then $d_\omega e = \frac{1}{2}[e, e]_m$

Bibliography

[1] Derek K. Wise. MacDowell-Mansouri gravity and Cartan geometry. *Classical and Quantum Gravity*, 27(15), nov 2010. ISSN 02649381. doi: 10.1088/0264-9381/27/15/155010. URL `http://arxiv.org/abs/gr-qc/0611154`.