Lie algebroids, Poisson manifolds and Jacobi structures

Based on mini-course by Carlos Zapata-Carratalá

by

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Abstract: Mistakes almost certainly mine, thanks for course etc... A lot of material is taken from [1] in the first half of the course. Some things are missing but most everything written here is material I somewhat understand right now. Some proofs are missing and I will hopefully get around to it eventually.

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Contents

1 Lecture 1: Poisson and Presymplectic geometry

The first lecture is mostly based on section 2.4 of [1].

1.1 Poisson Algebra

Definition 1.1.1. A **Poisson Algebra** is a triple $(A, \cdot, \{,\})$ such that

1. $(A, \cdot)$ is a commuttative, associative and unital $\mathbb{R}$-algebra (or $\mathbb{C}$ algebra maybe?)
2. $(A, \{,\})$ is a Lie $\mathbb{R}$-algebra, which means that the bracket follows the Jacobi identity:
$$\{\{a, b\}, c\} + \{\{b, c\}, a\} + \{\{c, a\}, b\} = 0 \tag{1.1}$$
3. The Poisson bracket follows the Libeniz identity in the sense that for $a, b, c \in A$,
$$\{a, b \cdot c\} = \{a, b\} \cdot c + b \cdot \{a, c\} \tag{1.2}$$
$$:= \mathrm{ad}_a\, (b \cdot c) \tag{1.3}$$
where we have defined the adjoint map of the Lie algebra.
4. Equivalently to 3, the $\mathrm{ad}_{\{,\}} : A \to \mathrm{Der}(A, \cdot)$, which takes an element of the algebra to a derivation on the commuttative algebra $(A, \cdot)$. We also see that the $\mathrm{ad}_{\{\}}$ induces a derivation on $(A, \{,\})$ using the Jacobi identity.

Therefore the adjoint map of a Lie Algebra is a morphism from A to $\mathrm{Der}(A, \cdot) \cap \mathrm{Der}(A, \{,\})$, the derivations of both bilinear structures of a Poisson algebra.

Definition 1.1.2. A **Poisson derivation** is a derivation on both bilinear forms of a Poisson algebra, that is $X \in \mathrm{Der}(A, \cdot) \cap \mathrm{Der}(A, \{,\}) \subset \mathrm{End}_{\mathbb{R}}(A)$. If a Poisson derivation is generated by the adjoint map, $X_a = \{a, \}$, we say that it is a **Hamiltonian derivation**.

Definition 1.1.3. A Poisson Algebra morphism is a linear map $\psi : A \to B$ such that $\psi : (A, \cdot) \to (B, \cdot)$ is an algebra morphism and $\psi : (A, \{,\}) \to (B, \{,\})$ is a Lie algebra morphism.

Definition 1.1.4. A subalgebra $I \subset A$ is **coisotrope** if

- $I \subset (A, \cdot)$ is a multiplicative ideal
- $I \subset (A, \{,\})$ is a Lie subalgebra

Proposition 1.1.1. *Reduction of Poisson algebra*
Suppose $I \subset A$ coisotrope and consider the Lie normaliser (or in ring theory the idealiser)
$$N(I) = \{a \in A | \{a, I\} \subset I\}, \tag{1.4}$$
which is the largest subalgebra of A that contains I as an ideal. We claim that $A' := {}^{N(I)}\!/_{I}$ inherits a Poisson algebra structure.

Proof. Condition 1 is automatically satisfied as A' is a subalgebra of A, with a Lie algebra structure given by the same bracket. For $a', b', c' \in A'$, consider the adjoint action of a' on $b' \cdot c'$ and look at coset representatirve a, b, c of $N(I)$. Using the fact that I is coisotrope, we see that

$$\begin{aligned}\{a + I, (b + I) \cdot (c + I)\} &= \{a + I, b \cdot c + I\} \\ &= \{a, b \cdot c\} + I\end{aligned}$$

by linearity of the bracket and closure of elements in $N(I)$ w.r.t I. The jacobi identity is checked by similar arguments. □

Definition 1.1.5. The *reduced Poisson structure* is characterised by the projection map $p : (N(I), \cdot, \{,\}) \to (A', \cdot', \{,\}')$, and by the above proposition, this is a Poisson Algebra morphism.

1.2 Poisson Manifolds

Definition 1.2.1. A **Poisson manifold** is a smooth manifold P whose commutative algebra of smooth functions has the structure of a Poisson algebra $(\mathrm{C}^\infty(P), \cdot, \{,\})$.

Definition 1.2.2. A map $\phi : P_1 \to P_2$ is a **Poisson map** if $\phi^* : \mathrm{C}^\infty(P_2) \to \mathrm{C}^\infty(P_1)$ is a Poisson morphism of algebras.

Recall that derivations on smooth functions are isomorphic to vector fields:

$$\mathrm{Der}(\mathrm{C}^\infty(P)) \simeq \Gamma(TP), \tag{1.5}$$

where the isomorphism is due to

$$\{f, g\} \mapsto X_{\{f,g\}} = [X_f, X_g] \tag{1.6}$$

Definition 1.2.3. So following through definition definition 1.1.2, the Poisson derivations on a Poisson manifolds are called **Poisson vector fields**. And Hamiltonian derivations on Poisson manifolds are called **Hamiltonian vector fields**. Hamiltonian vector fields are generated by the adjoint map

$$\begin{aligned}\mathrm{ad} : \mathrm{C}^\infty(P) &\to \Gamma(TP) \\ f &\mapsto X_f := \{f, \cdot\}\end{aligned}$$

Proposition 1.2.1. A manifold P; with a commutative algebra of smooth functions $(\mathrm{C}^\infty(P), \cdot, \{,\})$, and a bivector $\Pi \in \Gamma(\bigwedge T^2 P)$ defined as

$$\Pi(df, dg) = \{f, g\}; \tag{1.7}$$

is a Poisson manifold if and only if Π has vanishing Schouten bracket

$$[\![\Pi, \Pi]\!] = 0. \tag{1.8}$$

Before proving this statement, we recall facts about the Schouten-Nijenhius which forms a special case of a *Gerstenhaber algebra.* [1]

Definition 1.2.4. Given a Poisson bivector Π, the musical map (sharp)

$$\Pi^\sharp : T^*P \to TP \tag{1.9}$$

$$df \mapsto \Pi(df, \cdot) := \{f, \cdot\} \tag{1.10}$$

[1] CHECK THIS, will prove this later, after defining the Gerstenhaber algebra stuff.

defines an **Hamiltonian distribution**. Equivalently, $X_\cdot = \Pi^\sharp \circ d : C^\infty(P) \to \Gamma(TP)$ is an *Hamiltonian map*. Note that the space of Hamiltonian distribution $\Pi^\sharp(T^*P)$ is involutive as it is a Lie algebra morphism.

Definition 1.2.5. A submanifold $C \subset (P, \Pi)$ is **coisotropic submanifold** if $TC \subset (TP, \Pi)$ is a coisotropic subspace, that is $TC \supset (TC)^0$ an isotropic (i.e: normal) subspace of TC with respect to the bivector:

$$\Pi(\alpha, \beta) = 0 \quad \forall \alpha \in (T^*C)^0, \beta \in T^*C \tag{1.11}$$

Consequently, the short exact sequence:

$$0 \to (T^*C)^0 \xrightarrow{\Pi^\sharp} TC \to C^\infty(C) \to 0^2 \tag{1.12}$$

Proposition 1.2.2. Let $\iota : C \hookrightarrow P$ be a closed submanifold of Poisson manifold (P, Π), then the following are equivalent:

- C is coisotropic
- The vanishing ideal $I_C = \ker(\iota^*) := \{g \in C^\infty(P) \,\big|\, g|_C = 0\}$ is a coisotrope of the Poisson algebra $(C^\infty(P), \cdot, \{\cdot, \cdot\})$.
- Hamiltonian vector fields X_g generated by $g \in I_C$ are tangent to C: $X_g|_C \in \Gamma(TC)$

Proof.
- (1) $\Rightarrow$ (2): First $(I_C, \cdot)$ is a multiplicative ideal of $(C^\infty(P), \cdot)$ by construction. And $(I_C, \{\cdot, \cdot\})$ is a Lie subalgebra because the Poisson bracket vanishes on $C \hookrightarrow P$.
- (2) $\Rightarrow$ (3): for a basis $g \in I_C$, the Hamiltonian vector fields $X_g = \{g, \cdot\}$ span the $\mathrm{Der}(C^\infty(C))$ which is the space of tangent vector to C.
- (3) $\Rightarrow$ (1): $\iota^*\{g, f\} = 0$ for $g \in I_C$, $\forall f \in C^\infty(P)$ [3]

□

Definition 1.2.6. Consider 2 Poisson manifold (P_1, Π_1) and (P_2, Π_2), the *product Poisson manifold* is $(P_1 \times P_2, \Pi_1 + \Pi_2)$, where the canonical isomorphism $T(P_1 \times P_2) \cong \mathrm{pr}_1^* TP_1 \oplus \mathrm{pr}_2^* TP_2$.
The Whitney sum of vector bundle A_M, B_M over manifold P is defined as above by

$$A_M \boxplus B_M = \mathrm{pr}_1^* A_M \oplus \mathrm{pr}_2^* B_M \tag{1.13}$$

Also it's easy to see that pulling back onto either P_1, P_2 commutes with the bracket structure, with "cross-pulling" bracket vanishing

Definition 1.2.7. Given a Poisson manifold (P, Π), **opposite Poisson manifold** is $\overline{P} = (P, -\Pi)$.

Proposition 1.2.3. Let two Poisson manifold $(P_1, \Pi_1), (P_2, \Pi_2)$ and a smooth map $\phi : P_1 \to P_2$, then ϕ is a Poisson map if and only if

$$\mathrm{grph}(\phi) := \{(p, \phi(p)) \mid \forall p \in P_1\} \subset P_1 \times \overline{P}_2$$

is a coisotropic submanifold.

[2] i think this is right, but not sure
[3] continue later

Proof. Consider the tangent bundle of the graph submanifold

$$T\mathrm{grph}(\phi) = \{(X, Y) \mid \text{if } \exists Y \in TP_2 \text{ such that } X, Y \text{are } \phi\text{-related: } \phi^* Y = \phi_* X\}.$$

Full proof in [2] but they have a weird definition of $\Pi^\sharp$ there. [4] □

Recall that a *submersion* is a differential map $\phi : M \to N$ such that

$$D\phi_p : T_pM \twoheadrightarrow T_{\phi_p}N \tag{1.14}$$

for all $p \in M$. Dually, an *immersion* is a differential map $\phi : M \to N$ such that

$$D\phi_p : T_pM \hookrightarrow T_{\phi_p}M \tag{1.15}$$

for all $p \in M$.

Proposition 1.2.4. Let (P, Π) a Poisson manifold, and $\iota : C \hookrightarrow P$ a closed coisotropic submanifold. Suppose X_{I_C}, the Hamiltonian vector field tangent to C, or equivalently generated by the ideal $I_C = \ker(\iota^*)$ as in proposition 1.2.2; integrates to a regular foliation on C. Further assume that the leaf space is smooth $P' := {}^{C}\!/_{\chi_C}$ such that there is a surjective submersion (quotient map) q fitting in the diagram

$$\begin{tikzcd} C \arrow[r, hook, "\iota"] \arrow[d, two heads, "q"] & (P, \Pi) \\ (P', \Pi') & \end{tikzcd} \tag{1.16}$$

Then inherits a Poisson bracket on functions $(C^\infty(P'), \{\cdot, \cdot\}')$ uniquely determined by the condition

$$\iota^*\{F, G\} = q^*\{f, g\}' \tag{1.17}$$

for all $f, g \in C^\infty(P')$ and $F, G \in C^\infty(P)$ such that F, G are the leaf-wise constant extensions of f, g, i.e

$$\begin{aligned} q^* f &= \iota^* F \\ q^* g &= \iota^* g \end{aligned}$$

2 Lie Groupoids

3 Lie Algebroids

3.1 Vector Bundles

Let's first review facts about vector bundles, but with a more 'categorical' mindset.

Definition 3.1.1. A **vector bundle** $\pi : E \to M$ over a smooth manifold M is a fibre bundle whose fibre E_x is a vector space $V \in \mathbf{Vect}\ \forall x \in M$. The dimension of the typical fibre E_M is called the rank and $\dim(E_M) := \dim(E_x)$ for all x. A *local trivialisation* is a map φ such that on $U \subset M$ open, $\varphi : \pi^{-1}(U) \to U \times V$ is a diffeomorphism.

[4] continue one day

On overlaps $U_1 \cap U_2$, local trivialisations define $\mathrm{GL}(V)$-valued *transition functions*. A basis of sections $\{e_i : U \to \pi^{-1}(U) | \pi \circ e_i = \mathrm{id}_U\}_{i=1}^{\mathrm{rk}(E)}$ defines a local trivialisations as well; if such sections are globally defined, the bundle is trivial (or trivialisable).

Definition 3.1.2. A smooth map F between 2 vector bundles $F : E_1 \to E_2$ is a bundle **vector bundle morphism** if there exists smooth map $\phi \in C^\infty(M_1, M_2)$ between the bases such that

$$\begin{tikzcd} E_1 \arrow[r, "F"] \arrow[d, "\pi_1"] & E_2 \arrow[d, "\pi_2"] \\ M_1 \arrow[r, "\phi"] & M_2 \end{tikzcd} \tag{3.1}$$

commutes. We say that F is a *covering* for ϕ. Equivalently, F restricts to a linear map on the fibre $F_x : (E_1)_x \to (E_2)_{\phi(x)}$.

Definition 3.1.3. Vector bundles over smooth manifolds with vector bundle morphism forms the **category of vector bundles** denoted $\mathbf{Vect}_{\mathrm{Man}}$.

Remark 3.1.1 (Categorification)**.** Fixing a base manifold M, the point-wise construction of fibre bundles over M restricts us to the subcategory of vector bundles over M denoted $\mathbf{Vect}_M$. So the point-wise construction of vector bundle over base manifold M forms a **abelian symmetric monoidal category**. [5]

Definition 3.1.4. Given vector bundle E_1, E_2 over manifold M,

- the vector bundle direct sum called the *Whitney sum* is the fiberwise direct sum $E_1 \boxplus E_2$ (as seen in definition 1.2.6),
- the vector bundle tensor product is the fiberwise tensor product $(E_1)_x \otimes (E_2)_x$ for all $x \in M$,
- and the dual vector bundle is given by $E^* = \mathrm{Hom}(E, \mathbb{R})$ (which is a functor by the way).

Definition 3.1.5. Given a vector bundle $E \xrightarrow{\pi} M$, and a smooth map $\phi : N \to M$, **pullback vector bundle** by ϕ is defined as the categorical pull-back $\phi^* E = N \ {}_\phi\!\times_\pi \ E$ such that the diagram commutes:

$$\begin{tikzcd} N \ {}_\phi\!\times_\pi \ E \arrow[r] \arrow[d] & E \arrow[d, "\pi"] \\ N \arrow[r, "\phi"] & M \end{tikzcd} \tag{3.2}$$

which is simply $\phi^* E = \{(p, e) \in N \times E \mid \phi(p) = \pi(e)\}$.

Definition 3.1.6. A **section** s of vector bundle $E \xrightarrow{\pi} M$ is a map $s : M \to E$ such that $\pi \circ s = \mathrm{id}_M$. The set of all sections of E is denoted

$$\Gamma(E) = \{s : M \to E \mid \pi \circ s = \mathrm{id}_M\}. \tag{3.3}$$

Remark 3.1.2. We would naturally guess that the functor $C^\infty : \mathbf{Man} \to \mathbf{Ring}$ would similarly extend to sections on manifold. But the assignment $\Gamma : \mathbf{Vect}_{\mathrm{Man}} \to R\text{-}\mathbf{Mod}$ fails to be a functor. To see this,

[5] One day, try to understand category stuff

consider the vector bundle morphism $F : E_1 \to E_2$ covering $\phi : M \to N$, and the pullback bundle along ϕ:

$$\begin{tikzcd}
E_1 \arrow[r, "F"] \arrow[d, "\pi_1"] & \phi^* E_2 \arrow[r, "\mathrm{id}_{E_2}"] \arrow[d, "\mathrm{pr}_2"'] & E_2 \arrow[d, "\pi_2"'] \\
M \arrow[r, "\mathrm{id}_M"] \arrow[u, dashed, bend left, "s_1"] & M \arrow[r, "\phi"] \arrow[u, bend right, red, "\phi^* s_2"'] & N \arrow[u, dashed, bend right, "s_2"']
\end{tikzcd} \tag{3.4}$$

For sections on the vector bundle, we have the maps

$$\Gamma(E_1) \xrightarrow{F} \Gamma(\phi^* E_2) \xleftarrow{\phi^*} \Gamma(E_2)$$
$$s_1 \to F \circ s_1 \; ; \; s_2 \circ \phi \leftarrow s_2$$

If ϕ is **not** a diffeomorphism, then one *cannot* in general construct such maps. However, when maps agree

$$F \circ s_1 = s_2 \circ \phi \qquad \Leftrightarrow \qquad s_2 \sim_F s_1 \tag{3.5}$$

we say that the sections are **F-related**.

Definition 3.1.7. If $\phi : M \to N$ **is** a diffeomorphism, and $F : E_1 \to E_2$ a covering of vector bundle then the **pushforward** of sections

$$F_* : \Gamma(E_1) \to \Gamma(E_2) \tag{3.6}$$
$$s_1 \mapsto F \circ s_1 \circ \phi^{-1} \tag{3.7}$$

is well-defined. The pushforward satisfies

$$F_*(s + r) = F_*(s) + F_*(r) \tag{3.8}$$
$$F_*(f \cdot s) = (\phi^{-1})^* f \cdot F_*(s) \tag{3.9}$$

for $s, r \in \Gamma(E_1)$ and $f \in C^\infty(M)$. Note that you can *always* pushforward vector fields via $\phi_* : \mathcal{X}(M) \to \mathcal{X}(N)$.

Definition 3.1.8. In contrast, ϕ need not be a diffeomorphism for the **pullback** of dual sections (aka: forms) to be well-defined.

$$F : \Gamma(E_2^*) \to \Gamma(E_1^*)$$
$$\alpha \mapsto F^* \alpha = \alpha \circ \phi$$

So for any sections $s_1 \in \Gamma(E_1)$,

$$F^* \alpha(s_1) = (\alpha \circ \phi) \, (F \circ s_1) \tag{3.10}$$

since $F(s_1) \in \Gamma(\phi^* E_2)$. [6]

Definition 3.1.9. The pullback naturally extends to all tensor powers of the dual bundles $\bigotimes^k E_2^*$, such that for $\eta, \omega \in \Gamma(\bigotimes^\bullet E_2^*)$, and function $f \in \Gamma(\bigotimes^0 E_2) \cong C^\infty(N)$, we have

$$F^*(\eta + \omega) = F^*\eta + F^*\omega$$
$$F^*(\eta \otimes \omega) = F^*\eta \otimes F^*\omega$$
$$F^*(f) = f \circ \phi = \phi^* f.$$

[6] In the case of cotangeant bundle, I should expand on the fact that ϕ needs to be diffeomorphism to define pullbacks. See notes on int systems

Definition 3.1.10. Given vector bundle morphisms $F : E_1 \to E_2$, $G : E_2 \to E_3$, we see that on the tensor powers of dual bundles we have the contravariant identity:

$$(G \circ F)^* = F^* \circ G^*. \tag{3.11}$$

This motivates the definition of the *contravariant* **tensor functor** to the category of associative unital algebras

$$\begin{aligned} \mathcal{T} : \mathbf{Vect}_{Man} &\to \mathbf{AssAlg} \\ E &\mapsto \Gamma\left(\bigotimes{}^\bullet E^*\right) \\ F &\mapsto F^* \end{aligned}$$

Look into spanning functions and restrictions of C^∞ functor.

Definition 3.1.11. Let $E \to M$ be a smooth vector bundle, a **connection** is a vector bundle morphism $\nabla : E \to DE \subset \mathrm{End}(E)$ covering the identity. (We go into some details of derivation bundle in 4.1).
A **Kozul connection** is a $\mathbb{R}$-linear map

$$\begin{aligned} \nabla : \mathcal{X}(M) \times \Gamma(E) &\to \Gamma(E) \\ (X, s) &\mapsto \nabla_X s \end{aligned}$$

such that the Liebniz identity is satisfied.

3.2 Definition and Examples

Definition 3.2.1. A **Derivative Lie algebra** is a vector bundle $\alpha : A \to M$ whose module of sections with an $\mathbb{R}$-linear structure $(\Gamma(A), [\cdot, \cdot])$ acts as derivations on each of its arguments. That is

$$\mathrm{ad}_{[\cdot,\cdot]} : \Gamma(A) \to \mathrm{Der}(A)$$

is well-defined.

Example 3.2.1. A Poisson algebra is a derivative Lie algebra.

Definition 3.2.2. A **Lie Algebroid** $\{A \to M; \rho : A \to TM; [\cdot, \cdot]\}$ over smooth manifold M is a vector bundle $A \to M$ together with a derivative Lie algebra structure on $(\Gamma(A), [\cdot, \cdot])$ and a vector bundle morphism $\rho : A \to TM$ called the *anchor map*.
The anchor map induces a Lie algebra homomorphism on modules of sections $\rho_* : \Gamma(A) \to \Gamma(TM)$ by the *symbol-squiggle theorem* which we will see in 6.0.1 [7].
with

$$\rho_*[s, s']_E = [\rho(s), \rho(s')] \tag{3.12}$$

This means that the derivative algebra structure is respected for $s, s' \in \Gamma(A)$ and $f \in C^\infty(M)$,

$$[s, f \cdot s'] = f \cdot [s, s'] + \rho_*(s) f \cdot s' \tag{3.13}$$

In general, vector bundle morphism **DO NOT** induce well-defined maps between modules of sections. This again will be highlighted in a future section.

Remark 3.2.1. The anchor map naturally defines two distributions $\ker(\rho) \subset A$ and $\mathrm{im}(\rho) \subset TM$. Now fiberwise, the vector spaces $\mathfrak{g}_x := \ker(\rho_x) \subset A_x$ form a Lie algebra with bracket extending on the fibres

[7] see later absolutely

A_x. We call $(\mathfrak{g}_x, [\cdot,\cdot]_{\Gamma(A)}|_x)$ the **isotropy Lie algebras** of Lie algebroid A.
SInce ρ is a Lie algebra morphism, the image distribution $\rho(A)$ is involutive and it will be integrable by a singular foliation on M. We call the image distribution the **characteristic distribution** of the Lie algebroid A.

Example 3.2.2.
- A natural first example is the vector bundle $\mathfrak{g} \to \star$, with $\mathfrak{g}$ having a Lie algebra structure. This is a trivial Lie algebroid.
- The tangent bundle $TM \to M$ is a Lie algebroid with projection anchor map and the vector field forming a derivative algebra.
- Likewise, involutive distributions $D \hookrightarrow TM$ with trivial anchor form a sub Lie algebroid.
- The vector bundle $\mathbb{R} \to M$, with derivative Lie algebra being the commutative algebra of smooth functions over M. Then for $h \in C^\infty(M)$ and associated vector field $X_h \in \Gamma(TM)$, the anchor map is given by
$$\rho_X : f \mapsto f \cdot X$$
and the bracket is
$$[f,g]_X = fX(g) - gX(f). \tag{3.14}$$
We've just reinterpreted the regular Lie algebra of functions over a manifold as a Lie algebroid.
- The Atiyah algebroids A_P of a principal G-bundle $\pi : P \to M$, as a vector bundle appears in the sequence
$$0 \to \mathrm{ad}(P) \to A_P \to TM \to 0,$$
where the adjoint bundle $\mathrm{ad}(P) \cong P \times_G \mathfrak{g}$ of Lie algebra $\mathfrak{g}$ is given via the adjoint action of G on its Lie algebra. (A more general statement can be made using the associated bundle). This is constructed from the short exact sequence
$$0 \to VP \to TP \xrightarrow{\pi_*} TM \to 0, \tag{3.15}$$
where VP is the *vertical bundle*, the kernel of the differential surjective map. Now the vertical bundle is isomorphic to the trivial bundle $VP \cong \mathfrak{g} \times P$. Now since P is a principal bundle, G acts on this short exact sequence yielding the Atiyah sequence as G acts as the adjoint map on the vertical bundle.
- The bundle of derivations $DE \to M$ for a vector bundle $E \to M$ is a Lie algebroid.

3.3 Algebraic structures associated with Lie algebroid

Let's first discuss a notion that combines the structure of $\mathbb{Z}$-graded Lie superalgebras and supercommutative rings. Attention not to confuse with regular supersymmetry and $\mathbb{Z}_2$-grading for Poisson superalgebras for example.

Definition 3.3.1. A **Gerstenhaber algebra** $A^\bullet$ is a graded commutative algebra with a Lie bracket $[\![\cdot,\cdot]\!]$ of degree -1 that satisfies the *Poisson identity* and a multiplication $\cdot$ in the supercommutative associative ring. For $a, b, c \in A^\bullet$, we denote $|a|$ the degree of a and we have the following

- $|ab| = |a| + |b|$
- $|[\![a,b]\!]| = |a| + |b| - 1$
- $ab = (-1)^{|a||b|}ba$
- $[\![a,bc]\!] = [\![a,b]\!]c + (-1)^{(|a|-1)|b|}b[\![a,c]\!]$
- $[\![a,b]\!] = -(-1)^{(|a|-1)(|b|-1)}[\![b,a]\!]$
- $[\![a,[\![b,c]\!]]\!] = [\![[\![a,b]\!],c]\!] + (-1)^{(|a|-1)(|b|-1)}[\![b,[\![a,c]\!]]\!]$

Given a Lie algebroid $(A, \rho, [,])$ over M, we can *uniquely* extend the bracket to a Gerstenhaber bracket on the graded algebra of multisections.

Definition 3.3.2. There exists a Gerstenhaber algebra $(\Gamma(\bigwedge^\bullet A), \wedge, [\![\cdot,\cdot]\!])$ such that, for $a, b \in \Gamma(A)$ and $f, g \in C^\infty(M)$

- $[\![a,b]\!] = [a,b]$
- $[\![a,f]\!] = \rho_* a[f]$
- $[\![f,g]\!] = 0$

Definition 3.3.3. We can construct a Lie algebroid counterpart to exterior calculus of the differential geometry of tangent bundles. This is a dual construction to the Gerstenhaber algebra seen above. Given a Lie algebroid $A \to M$, we construct a differential graded algebra $\Omega(A) = \bigoplus \Omega^k(A)$ of the sections $\Gamma(\bigwedge^\bullet A^*)$. The map

$$d_A : \Omega^k(A) \to \Omega^{k+1}$$

is called the *differential* and explicitly, acting on homogeneous element $\omega \in \Gamma(\bigwedge^k A^*)$ we have:

$$d_A\omega(a_0, \dots, a_k) = \sum_{i<j}(-1)^{i+j-1}\omega([a_i, a_j], a_1, \dots, \hat{a}_i, \dots \hat{a}_j \dots, a_k) + \sum_{i=0}^{k}(-1)^{i-1}\rho_* a_i[\omega(a_0, \dots \hat{a}_i, \dots a_k)], \tag{3.16}$$

for $a_i \in \Gamma(A)$. This differential graded algebra $\Omega^\bullet(A) = (\Gamma(\bigwedge^\bullet A^*), \wedge, d_A)$ is called the **exterior algebra** of the Lie algebroid A.

As usual we find $d_A^2 = 0$ which forms a *de Rham* complex and its cohomology is the **Lie algebroid cohomology with trivial coefficients**.

Definition 3.3.4. Just like a regular exterior algebra, there are natural notions of *interior product* and *Lie derivatives*, defined for $w \in \Gamma(\bigwedge^k A^*)$ and $a, a_0, \dots, a_{k-1} \in \Gamma(A)$ as

$$\iota_a\omega(a_0, \dots, a_{k-1}) = \omega(a, a_0, \dots, a_{k-1})$$
$$\mathcal{L}_a\omega(a_0, \dots, a_{k_1}) = \sum_{i=0}^{k}\omega(\dots, a_{i-1}, [a, a_i], \dots, a_{k-1}) - \rho_* a\omega(a_0, \dots, a_{k-1}),$$

with extension by linearity. These follow the usual Cartan calculus identity:

$$\begin{aligned}\mathcal{L}_a &= \iota_a \circ d_A + d_A \circ \iota_a \\ [\mathcal{L}_a, \mathcal{L}_b] &= \mathcal{L}_{[a,b]} \\ [\mathcal{L}_a, \iota_b] &= \iota_{[a,b]} \\ [\iota_a, \iota_b] &= 0\end{aligned}$$

So the operators $\iota, \mathcal{L}, d$ have respectively degrees $-1, 0, 1$.

Proposition 3.3.1. There is a $1-1$ correspondence between Lie algebroids and *Linear Poisson structures.*

Maybe talk about splittings and covariant derivatives?

3.4 Morphisms of Lie algebroids

We will talk about the notion of local Lie algebras later, but in the case of Lie algebroid, the extra structure of a vector bundle constrains morphisms between these algebraic structures. We start by considering a general vector bundle morphism $F : A \to B$, and demand that this map induces a Lie algebra morphism of the modules of sections, which is not always guaranteed.

3.5 Connections on Lie algebroids

A good ref is [3]. Can view the concept of a Lie algebroid as a tool to transfer the differential geometry of tangent bundles to abstract vector bundles.

Definition 3.5.1. A *A-connection* ∇ on a Lie algebroid $A \to M$ is a vector bundle morphism $\nabla : A \to DA$, that is a Kozul connetion with the added property that it is compatible with the anchor map in the sense that given $s \in \Gamma(A)$, the induced connection on the tangent bundle TM is

$$\begin{aligned}\nabla : \Gamma(A) &\to \Gamma(T^*M \otimes TM) \\ s &\mapsto \nabla_{\rho_*(s)}\end{aligned}$$

In [4], the splitting is explored further.

Definition 3.5.2. The **curvature** $R_\nabla \in \Omega^2(A)$ of connection ∇ of a Lie algebroid $A \to M$ is defined as $R = \nabla^2$ or for sections of the algebroids,

$$R_\nabla(a, b) = [\nabla_a, \nabla_b] - \nabla_{[a,b]} \tag{3.17}$$

for $a, b \in \Gamma(A)$. Alternatively, the curvature can be viewed as a map

$$R_\nabla : \Gamma(TM) \to \Omega^2(A)$$

If the curvarue vanishes, we say that the connection is flat and we call (A, ∇) a **representation** of the Lie algebroid over itself. This is because this definition can be extended where $A \to M$ a Lie algebroid and $E \to M$ a vector bundle and looking at morphisms between them. Fortunately we only need $A = E$.

You can play the same game with $\Omega^\bullet_E(A) = (\Gamma(\bigwedge^\bullet A^* \otimes E), \wedge, d^E_A)$ which will lead to *Lie algebroid cohomology with coefficient in E.* This maybe what I need for my problem.

4 Lecture 4: Differential Operators I

4.1 Derivations

Let A be a associative commutative unital $\mathbb{C}$-algebra, a vector space over $\mathbb{C}$ such that for any pair $a, b \in A$, the product $ab \in A$ is bilinear and associative.

Definition 4.1.1. A **derivation** $\partial \in \mathrm{Der}_{\mathbb{C}}(A)$ is a $\mathbb{C}$-linear map $\partial : A \to A$ such that the Leibniz identity is satisfied,

$$\partial(ab) = \partial(a)b + a\partial(b) \tag{4.1}$$

for $a, b \in A$. Clearly, $\mathrm{Der}_{\mathbb{C}}(A) \subseteq \mathrm{End}_{\mathbb{C}}(A)$

Definition 4.1.2. More generally, if B is a commutative ring, A is a B-algebra and M an A-bimodule then $\mathrm{Der}_B(A, M) = \{\partial \in \mathrm{Hom}_B(A, M) | \forall a, b \in A, \partial(ab) = a\partial(b) + \partial(a)b\}$.

Proposition 4.1.1. If $\partial \in \mathrm{End}_{\mathbb{C}}(A)$ is a derivation $\Leftrightarrow$ $\partial(\mathbb{C}) = 0$ and for all $a \in A$, $\partial a - a\partial \in A$.

Proof. Let $b \in A$, then the Leibniz identity is equivalent to

$$\begin{aligned}(\partial a - a\partial)(b) &= \partial(ab) - a\partial(b) \\ &= \partial(a)b.\end{aligned}$$

- $\Rightarrow$ Assuming ∂ is a derivation, then the argument above shows that $\partial a - a\partial \in A \subseteq \mathrm{End}_{\mathbb{C}}(A)$, where left multiplication by this operator is the endomorphism map induced. Furthermore, since ∂ is $\mathbb{C}$-linear, and considering $\mathbb{C}$ as a $\mathbb{C}$-vector space over itself, the Leibniz identity implies

$$\begin{aligned}\partial(1z) &= \partial(1)z + 1\partial(z) \\ &\Rightarrow \partial(1) = 0.\end{aligned}$$

  Therefore $\partial(\mathbb{C}) = 0$.

- $\Leftarrow$ Suppose $a\partial - a\partial = c$ for some $c \in A$ and $\partial(\mathbb{C}) = 0$, then

$$\begin{aligned}(\partial a - a\partial)(1) &= c(1) \\ \partial(a) &= c\end{aligned}$$

  therefore ∂ follows Leibniz identity.

□

Example 4.1.1. On polynomial rings, we have $\mathrm{Der}_{\mathbb{C}}(\mathbb{C}[x]) = \mathbb{C}(x)\frac{d}{dx}$. Clearly, the inclusion $\mathbb{C}[x]\frac{d}{dx} \subseteq \mathrm{Der}_{\mathbb{C}}(\mathbb{C}[x])$ is trivial by just checking that it satisfies Leibniz identity. However, for the reverse inclusion, consider a derivation $\partial \in \mathrm{Der}_{\mathbb{C}}(\mathbb{C}[x])$, then we claim that a basis is given by

$$\partial := \partial(x)\frac{d}{dx}. \tag{4.2}$$

Easy to check that acting on the unit $1 \in \mathbb{C}$ and x, these definitions agree. Therefore, by $\mathbb{C}$-linearity and Leibniz property, they agree on $\mathbb{C}[x]$.

More generally,

$$\mathrm{Der}_{\mathbb{C}}(\mathbb{C}[x_1, \ldots, c_n]) = \bigoplus_{i=1}^{n} \mathbb{C}[x_1, \ldots, x_n]\frac{\partial}{\partial x^i} \tag{4.3}$$

Example 4.1.2. If $A = C^\infty(M)$, the algebra of smooth functions on M, then

$$\mathrm{Der}_{\mathbb{R}}(C^\infty(M)) = \mathcal{X}(M) \tag{4.4}$$

4.2 Differential operators

In this section we define the more general concept of a differential operator, which are **not** necessarily derivations. There are two different ways to define them.

Definition 4.2.1 (First definition)**.** The ring $D(A)$ of $\mathbb{C}$-linear **differential operators** on A is the subalgebra of $\mathrm{End}_{\mathbb{C}}(A)$ generated by A and $\mathrm{Der}_{\mathbb{C}}(A)$. Let $\theta \in D(A)$, it has *order p* if it is the sum of products on at most p derivations.
e.g: $\frac{d^2}{dx^2} + 1 = \left(\frac{d}{dx}\right)^2 + 1$ has order 2.

We can generalise this definition a little.

Definition 4.2.2 (Second definition)**.** A **regular** differential operator of order p is an element of $D^p(A) = \{\theta \in \mathrm{End}_{\mathbb{C}}(A) \,|\theta a - a\theta = \theta(a) \in D^{p-1}(A) \quad \forall a \in A\}$, with $D^0(A) = A$. The ring of **regular differential operators** is $D(A) = \bigcup D^p(A)$ and it is easy to see that

$$D^p(A)D^r(A) \subseteq D^{p+r}(A). \tag{4.5}$$

and $D^{p+1}(A) \supseteq D^p(A)$ so this defines a filtration.

We relate the two definitions in the following sense. Suppose $\theta \in D^1(A)$, then

$$\theta = (\theta - \theta(1)) + \theta(1) \tag{4.6}$$

implying that $D^1(A) \cong \mathrm{Der}_{\mathbb{C}}(A) \oplus A$. So we can generate the ring of differential operators on A and clearly def1 $\subset$ def2.

Theorem 4.2.1 (Grothendieck)**.** The two definitions are equivalent if and only if $X = \mathrm{Spec}_A$ is non-singular. In this case, the ring of differential has the simple expression

$$D(A) = {}^{T_A(\mathrm{Der}_{\mathbb{C}}(A))}\!\big/_{\theta \otimes \theta' - \theta' \otimes \theta - [\theta, \theta']} \tag{4.7}$$

where T_A is the tensor algebra. Recall that the *spectrum* of a ring $\mathrm{Spec}(R)$ is the set of all prime ideals of R with the Zariski topology. [5]

Example 4.2.1. Consider the ring $A = \mathbb{C}[x]$ of rational functions over $\mathbb{C}$, then the algebra of derivations over this ring

$$\mathrm{Der}_{\mathbb{C}}(A) = \mathbb{C}[x]\frac{d}{dx} := W \tag{4.8}$$

is called the *Witt* algebra. However, the ring $D(A)$ of differential operators on A can also be viewed as the polynomial ring constructed by quotienting the free $\mathbb{C}$-algebra on x, ∂ by the ideal

$$D(A) = {}^{\mathbb{C}\langle x, \partial = \frac{d}{dx}\rangle}\!\big/_{x\partial - \partial x - 1}. \tag{4.9}$$

This is called a *Weyl* algebra.

As noted earlier, the second definition is more general. Here is an example where the equality fails.

Example 4.2.2. Consider $A = \mathbb{C}[t^2, t^3]$. Then $\mathrm{Spec}(A)$ is the space of proper prime ideals

$$\mathrm{Spec}(A) = \left\{ \langle t^2 - a, t^3 - b \rangle, (a, b) \in \mathbb{C}^2 \right\} \bigcup \left\{ \langle f(t^2, t^3) \rangle, \text{f is irreducible} \right\} \cup \{ \langle 0 \rangle \} \tag{4.10}$$

This space has a singular point and somehow this implies that there exists differential operators at that point that are not generated by sum-products of derivations. EXPAND ON THIS

Lemma 4.2.1. Let $\theta \in D^p(A)$ and $\theta' \in D^r(A)$ then

$$[\theta, \theta'] := \theta \cdot \theta' - \theta' \cdot \theta \in D^{p+r-1}(A) \tag{4.11}$$

In particular, $D^1(A)$ and $\mathrm{Der}_{\mathbb{C}}(A)$ are *Lie algebras*. Not true for higher order as it doesn't close. But below we will see a way to make it into a Lie algebra.

Question. Given algebras A, B with respective spectrum $X = \mathrm{Spec}(A)$, $Y = \mathrm{Spec}(B)$. If $D(A) \cong D(B)$, does that mean that $X \cong Y$? This turns out to be **false** if the algebraic varieties are allowed to be singular.

4.3 From differential operators to Poisson algebras

We have seen in lemma 4.2.1 that $[D^p(A), D^r(A)] \subseteq D^{p+r-1}(A)$. In particular, for Lie subalgebra $\mathrm{Der}_{\mathbb{C}}(A) \subseteq D(A)$, if $\delta, \delta' \in \mathrm{Der}_{\mathbb{C}}(A)$ then $[\delta, \delta'] \in \mathrm{Der}_{\mathbb{C}}(A)$,

$$\begin{aligned}
[\delta, \delta'](ab) &= \delta\delta'(ab) - \delta'\delta(ab) \\
&= \delta(\delta'(a)b + a\delta'(b)) - \delta'(\delta(a)b + a\delta(b)) \\
&= \delta\delta'(a)b + a\delta\delta'(b) - \delta'\delta(a)b - a\delta'\delta(b) \\
&= [\delta, \delta'](a)b + a[\delta, \delta'](b)
\end{aligned}$$

Definition 4.3.1. Given the filtration of regular differential operators $D(A)$ on algebra A, we define its grading $\mathrm{gr}\, D(A)$ as

$$\mathrm{gr}\, D(A) = \bigoplus_p D^p(A) \big/ D^{p-1}(A) \tag{4.12}$$

Proposition 4.3.1. The grading of differential operators on A is a commutative ring under composition and a Poisson algebra with bracket generated by the commutator $[\cdot, \cdot]$.

Proof.

- Let $\pi \in D^p(A)$ and $\rho \in D^r(A)$, then $\pi\rho, \rho\pi \in D^{p+r}(A)$ while $[\pi, \rho] \in D^{p+r-1}(A)$. So

  $$\pi\rho \sim \rho\pi + D^{p+r-1}(A) \tag{4.13}$$

  but as elements $\mathrm{gr}\,(\pi\rho), \mathrm{gr}\,(\rho\pi) \in \mathrm{gr}\, D(A)$, we have $\mathrm{gr}\,(\pi\rho) = \mathrm{gr}\,(\rho\pi)$

- $(\mathrm{gr}\, D(A), \{\cdot, \cdot\})$ is a Lie algebra. Taking the bracket on differential operators, we induce the Lie bracket $\{\cdot, \cdot\} : \mathrm{gr}\, D(A) \times \mathrm{gr}\, D(A) \to \mathrm{gr}\, D(A)$ by

  $$\begin{aligned}
  \{\mathrm{gr}\,\rho, \mathrm{gr}\,\pi\} &:= \mathrm{gr}\,[\rho, \pi] \\
  &= [\rho, \pi] + D^{p+r-2}(A)
  \end{aligned}$$

  for $\pi \in D^p(A)$, $\rho \in D^r(A)$. Given that $[\cdot, \cdot]$ is a Lie bracket on $D^1(A)$, we extend it to $\mathrm{gr}\, D(A)$ so that $\{\cdot, \cdot\}$ is a bracket up to an element of the quotient.

- The adjoint action is a derivation. [8]

□

In fact, if $X = \text{Spec}(A)$ is non-singular,

$$\begin{aligned}\text{gr}\, D(A) &= \text{gr}\left(\frac{T_A(\text{Der}_{\mathbb{C}}(A))}{\delta\otimes\delta' - \delta'\otimes\delta - [\delta,\delta']}\right)\\ &= \frac{T_A(\text{Der}_{\mathbb{C}}(A))}{\delta\otimes\delta' - \delta'\otimes\delta}.\end{aligned}$$

So in this case, $\text{gr}\, D(A) = \text{Sym}_A(\text{Der}_{\mathbb{C}}(A))$, and since we can identify the derivations with category of vector fields on X,

$$\text{Der}_{\mathbb{C}}(A) = \mathbb{V}ect(X) = \mathbb{C}[T^*X] \tag{4.14}$$

Possible connection with L_∞-algebras. see [6]

4.4 Weyl algebras

Let $A = \mathbb{C}[x_1, \ldots, x_n]$ then the ring of differential operator on A is constructed akin to example 4.2.1 as the free algebra in $\{x_i, y_i = -\partial_i\}$ variables

$$D(A) \cong \frac{\mathbb{C}\langle x_1, \ldots, x_n, y_1, \ldots, y_n\rangle}{[x_i, y_j] = \delta_{ij},\ \text{rest commutes}}. \tag{4.15}$$

This is the n^{th} **Weyl Algebra** $D(A)$ which is a simple ring (i.e: it does not have a proper 2-sided ideal). Its grading is the Poisson simple algebra

$$\text{gr}\, D(A) = \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n] \tag{4.16}$$

with Poisson brackets

$$\{x_i, y_j\} = \delta_{ij} \qquad \{x_i, x_j\} = 0 = \{y_i, y_j\} \tag{4.17}$$

This is sometimes called *the first example*.

Remark 4.4.1. $D(\mathbb{C}[x])$ has no non-trivial finite dimensional modules. This is because, assuming V is a $D(\mathbb{C}[x])$-module of complex dimension d. Then $D(\mathbb{C}[x])$ acts on V as an endomorphism. Let $X, Y \in \text{Mat}_{d\times d}(\mathbb{C})$ such that $[X, Y] = \mathbb{1}_d$, then $\text{tr}([X, Y]) = 0 \neq d$, which is a contradiction.

Proposition 4.4.1. Let I be a right ideal of $D(A)$. Then $J = \text{gr}\,(I)$ is an ideal of $\text{gr}\, D(A)$ and it is **involutive/coisotrope**

$$\{J, J\} \subseteq J \tag{4.18}$$

Proof. Let $\theta, \eta \in I$ then $[\theta, \eta] \in I$ since it is a right ideal. Taking the grading, $\text{gr}\,[\theta, \eta] \equiv \{\text{gr}\,\theta, \text{gr}\,\eta\} \subseteq J$ □

Theorem 4.4.1 (Gabber)**.** If $J = \text{gr}\,(I)$ is coisotrope for some right ideal I of $D(A)$, then the radical $\sqrt{J} := \{\theta | \exists \text{ k, s.t } \theta^k \in J\}$ is also coisotrope.

[8] do this SOMEDAY

Corollary 4.4.1 (Bernstein's inequality)**.** Using Gabber's theorem and Hilbert Nullstellensatz $\sqrt{J} = I(V(J))$, we see that

$$\dim(V(J) \subseteq \mathbb{C}^{2n}) \geq n \tag{4.19}$$

Example 4.4.1. Let $A = \mathbb{C}[x, y]$ with $\{x, y\} = 1$. Consider the coisotrope subring $J = \langle x^2, xy, y^2 \rangle$. It has radical $\sqrt{J} = \langle x, y \rangle$, but the radical is *not* coisotrope. Therefore J is **not** the grading of some right ideal of $D(\mathbb{C}[x, y])$.

5 Lecture 5: Differential Operators II

6 Lecture 6: Local Lie algebras

Theorem 6.0.1. test

7 Lecture 9: Dirac Geometry

Aim of the lecture is to recover Dirac geometry. A lot of the ideas can be traced back to Courant's thesis [7]. As usual Gualtieri's thesis [8] serves us well. In the following the underlying field is $\mathbb{R}$.

7.1 Courant Spaces

Definition 7.1.1. A **Courant space** is a triple $(C, \langle\cdot,\cdot\rangle, \rho)$ where $(C, \langle\cdot,\cdot\rangle)$ is a inner product space (meaning the inner product is bilinear and non-degenerate but **not** positive-definite) and $\rho : C \to V$ is a linear homomorphism compatible with the inner product called the *anchor*.

Definition 7.1.2. With respect to the *bilinear form*, a subspace $N \subset C$ and its orthogonal complement $N^\perp := \{x \in C \mid \langle x, y\rangle = 0 \,\forall\, y \in N\}$, need not be disjoint as the inner product is not in general positive definite. Therefore a subspace $N \subset C$ is called the following ways if the orthogonality conditions hold:

- $N \subset N^\perp$ is isotropic,
- $N \supset N^\perp$ is coisotropic,
- $N = N^\perp$ is Lagrangian.

Definition 7.1.3. Since $(C, \langle\cdot,\cdot\rangle)$ is an inner product space, we have the usual *musical isomorphisms*:

$$C \xrightarrow{\flat} C^*$$
$$C^* \xrightarrow{\sharp} C$$

This data of a Courant space implies that we can construct a map $j : V^* \to V$ such that the following diagram commutes:

$$\begin{tikzcd} C^* \arrow[d, "\sharp"] & V^* \arrow[l, "\rho^*"'] \arrow[d, dashed, "j"] \\ C \arrow[r, "\rho"] & V \end{tikzcd} \tag{7.1}$$

Definition 7.1.4. A Courant space is **exact** when we have the short exact sequence

$$0 \to V^* \xrightarrow{\sharp\rho^*} C \xrightarrow{\rho} V \to 0. \tag{7.2}$$

So ρ is surjective and $\ker(\rho) \subset C$ is an isotropic subspace.

Definition 7.1.5. Given a Courant space C, C', we define

- the *opposite Courant space* $(\overline{C}, -\langle\cdot,\cdot\rangle, \rho)$,
- the *direct sum* of two Courant spaces $(C \oplus C, \langle\cdot,\cdot\rangle \oplus \langle\cdot,\cdot\rangle', \rho \oplus \rho')$

Definition 7.1.6. For $V \in \mathbf{Vect}_\mathbb{R}$ we define the **Standard Courant space** as $\mathbb{V} = (V \oplus V', \langle\cdot,\cdot\rangle, \mathrm{pr}_1)$, with bilinear pairing

$$\langle v \oplus \alpha, w \oplus \beta\rangle = \frac{1}{2}\left(\alpha(w) + \beta(v)\right) \tag{7.3}$$

Note that we can also define a skew-symmetric bilinear form as well, but we will only call the *inner* product the symmetric one. Further note that, the symmetry group preserving orientation is $\mathrm{SO}(d, d)$, the non-compact special orthogonal group. [9]

[9] probably add some example such as B transform here, for posterity

Definition 7.1.7. A **Dirac space** D is a Lagrangian subspace of Courant space $(C, \langle\cdot,\cdot\rangle, \rho)$ for which there exists $W \subset V$ and $\overline{W} \subset V^*$ such that the following sequence is exact

$$0 \to \overline{W} \xrightarrow{\sharp\rho^*} D \xrightarrow{\rho} W \to 0 \tag{7.4}$$

The space $W = \rho(D) = {}^{D}\!/_{\overline{W}}$ is generally called the *range* of D. We remark that the Lagrangian condition imposed on the space implies that the bilinear form is 0 along this subspace.

Proposition 7.1.1. A Dirac space $D \subset (C, \langle\cdot,\cdot\rangle, V)$ specifies a 2-form on its range $\rho(D)$,

$$\omega_D \in \bigwedge^2 W^* \tag{7.5}$$

Such that for $w_i = \rho(d_i) + j(\epsilon_i) := \rho(a_i)$, with $d_i \in D$ and $\epsilon_i \in V \setminus W$,

$$\begin{aligned}\omega_D(w_1, w_2) &= \langle a_1, \sharp\rho^*(\epsilon_2)\rangle \\ &= -\langle \sharp\rho^*(\epsilon_1), a_2\rangle\end{aligned} \tag{7.6}$$

Proof. D is maximally isotropic so $\forall d_1, d_2 \in D \subset C$,

$$\langle d_1, d_2\rangle = 0. \tag{7.7}$$

Since ρ is surjective there exists $w_1, w_2 \in W$ such that $\rho(d_i) = w_i$. Consider the extension of elements $w_1, w_2 \in W \subset V$ by ρ, that is $a_1, a_2 \in C$ such that

$$w_i = \rho(a_i). \tag{7.8}$$

where $a_i = d_i + \sharp\rho^*(\epsilon_i)$ for some $\epsilon_1, \epsilon_2 \in V^* \setminus W^*$.

$$\langle a_1 - \sharp\rho^*(\epsilon_1)\ ,\ a_2 - \sharp\rho^*(\epsilon_2)\rangle = 0 \tag{7.9}$$

The cross terms must vanish while the non-cross terms carry the non-zero part of the inner product, therefore

$$\langle a_1, \sharp\rho^*(\epsilon_2)\rangle = -\langle \sharp\rho^*(\epsilon_1), a_2\rangle. \tag{7.10}$$

sketchy af proof. coset construction? do this □

Definition 7.1.8. An *isotropic* relation (or Lagrangian relation) $\Lambda : A \dashrightarrow B$ between two exact Courant spaces $(A, \langle,\rangle_A, \alpha : A \to V)$, $(B, \langle,\rangle_B, \beta : B \to W)$ is a morphism such that $\Lambda \subset A \oplus \overline{B}$ is a Lagrangian subspace with the relations

$$\begin{aligned} a_1 \sim_\Lambda b_1, \quad & a_2 \sim_\Lambda b_2 \\ \Rightarrow \quad & \langle a_1, a_2\rangle_A = \langle b_1, b_2\rangle_B \end{aligned}$$

In this case, we say that elements $a, b \in \Lambda$ are Λ-related.

Definition 7.1.9. An isotropic relation $\Gamma : A \dashrightarrow B$ is a **Courant morphism** if there exists a map $\gamma : V \to W$ such that elements of graph$(\Gamma) \subset B \oplus \overline{A}$ have

$$\begin{aligned} & b \oplus a \in \Gamma \\ \Rightarrow\ & \beta(b) = (\gamma \circ \alpha)(a) \end{aligned}$$

A Courant morphism becomes a Dirac space $\Gamma_\gamma \subset B \oplus \overline{A}$ and enters the following short exact sequence,

$$0 \to \mathrm{graph}(\gamma^*) \to \Gamma_\gamma \xrightarrow{\beta\oplus\alpha} \mathrm{graph}(\gamma) \to 0 \tag{7.11}$$

where graph$(\gamma) \subset W \oplus V^*$ and similarly for γ^*. This exact sequence is easily read as both piece of the graphs are exact Courant morphisms, on which we take the direct sum.

Courant morphisms are morphism in the category of Courant algebroids, and we have seen that elements are mapped appropriately. A lesser constraint would be to consider maps such that the inner product is preserved.

Definition 7.1.10. A *Courant* map $\Psi : A \to B$ of Courant algebroids is a linear map together with $\psi : V \to W$ such that the inner product is preserved,

$$\Psi\langle\cdot,\cdot\rangle_A = \langle\cdot,\cdot\rangle \tag{7.12}$$

and the following diagram commutes

$$\begin{array}{ccc} A & \xrightarrow{\Gamma} & B \\ \big\downarrow{\scriptstyle\alpha} & & \big\downarrow{\scriptstyle\beta} \\ V & \xrightarrow{\gamma} & W. \end{array} \tag{7.13}$$

Proposition 7.1.2. A map $\Psi : A \to B$ is Courant if and only if graph$(\Psi) \in B \oplus \overline{A}$ is a Courant morphism.

Proof. do later $\square$

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