The Calabi Conjectures

Based on mini-course by Vamsi Pritham Pingali

by

Guillaume Trojani

Supervisor : Pr Richard Szabo

Abstract: "It's not rocket science, well in fact it's more fundamental than rocket science"

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1 Lecture 1

1.1 Motivation

Complex manifolds provide an interface between algebra and analysis. The Calabi conjecture has many applications in algebraic geometry, for example the proof that $\mathbb{C}P^2$ is rigid (holomorphic implies biholomorphic), or the fact that Fano manifolds are simply connected. Applications also to moduli spaces, for example the sides of triangles have a structure in their own rights. The Calabi conjecture has application to finding Riemannian manifolds with vanishing Ricci curvature but not vanishing Riemannian curvature tensor.

How many objects in $\mathbb{R}^n$ satisfy certain conditions? Classify the zeroes of degree d polynomials. This is a hard question to answer. And the answers to such questions are very dependant on the category on which the analysis is done.
We will be working in the complex category and study *Complex Algebraic Geometry.* So for example the question of finding all rational points on $x^2 + y^2 = 1$ makes sense also on $\mathbb{C}^2$, so one can try to use differential geometry and hope to get something like number theory. This is called Arakelov geometry.

1.2 Complex manifolds

We study the loci of polynomials or more generally, analytic function on $\mathbb{C}^n$.

Definition 1.2.1. A complex manifold M is a smooth manifold that is diffeomorphic to an open subset to $\mathbb{C}^n$, such that transition function are biholomorphic.

Definition 1.2.2. An holomorphic function $f : U \subset \mathbb{C}^n \to \mathbb{C}$, is a function f is C^1 and $\frac{\partial f}{\partial \bar{z}^i} = 0$ for $i \in \{1, \ldots, n\}$.
But equivalently using Hartogs' theorem, f is C^1 and $\frac{\partial f}{\partial \bar{z}^i} = 0$ (and by extension *separately analytic*), then it is holomorphic. Another way to say this is f locally it admits a power series expansion. **All** of these are false for real smooth function but true for complex function.

Remark 1.1. Holomorphicity in more than **2** variables is very different from the one dimension counterpart. An holomorphic function on $\mathbb{C}^n \setminus \{0\}$ can be holomorphic extended to $\mathbb{C}^n$!! This means that singularities of holomorphic function in more than 2 variable **cannot** be isolated. This is called *Hartogs'* phenomenon. Moreover, $\mathbb{B}^1 \times \mathbb{B}^1$ is **NOT** biholomorphic to $\mathbb{B}^2$ unlike the real analogue.

Example 1.2.1. A first example of a complex manifold is given by the trivial inclusion of open $U \hookrightarrow \mathbb{C}^n$. But it is difficult to come up with **compact** complex manifold.

Definition 1.2.3. The holomorphic tangent space $\mathrm{T}^{1,0}_p\mathbb{C}^n$ is the vector space generated by $\frac{\partial}{\partial \bar{z}^i}$, is spanned by derivative of complex functions. Similarly, we define anti-holomorphic vector space and their respective duals.

Remark 1.2. The complexification of a real tangent space $\mathrm{T}_pM \otimes \mathbb{C} = \mathrm{T}^{1,0}_pM \oplus \mathrm{T}^{0,1}_pM$ for real even-dimension manifold. But here, the complexified vector bundle $T_{\mathbb{C}}M$ is **NOT** a holomorphic vector bundle, but it is a complex vector bundle over M. More on this below.

Definition 1.2.4. We define holomorphic/anti-holomorphic (p, q)-form as the space $\Omega^{(p,q)}(M)$ generated by $dz^i \wedge d\bar{z}^j$. The exterior derivative $d : \Omega^k \to \Omega^{k+1}$ splits into holomorphic and anti-holomorphic ∂ and $\bar{\partial}$ in the obvious way, with $d = \partial + \bar{\partial}$.
e.g: $\bar{\partial}(\overline{z^1} d\bar{z}^2) = d\bar{z}^1 \wedge d\bar{z}^2$.

Theorem 1.1 (Poincaré Lemma)**.** Let $U \subset M$ open, contractible subset of real manifold M, then for $\alpha \in \Omega^p(U)$, we have $d\alpha = 0 \Rightarrow \alpha = d\beta$. In the complex case, the analogous result for (p, q)-form holds for open balls in $\mathbb{C}^n$. To build submanifold of $\mathbb{C}^n$, it is useful to prove the complex equivalent of the implicit function theorem (IFT). This is a good way to generate manifold as the zero locus of some function.

Theorem 1.2 (Complex inverse function theorem)**.** If $f : \mathbb{C}^n \to \mathbb{C}^n$ is holomorphic and $\mathrm{D}f(0)$ is invertible, then f is a local biholomorphism near 0.
This leads to analogous implicit function theorem and constant rank theorem for complex functions.

Definition 1.2.5. A complex submanifold $S \subset M$ is a complex manifold such that the inclusion map is a *holomorphic* embedding. We show that it is a complex manifold by using the implicit function theorem. In a neighbourhood of $s \in S$, there is a coordinate chart such that S is the zero locus of the first few coordinate function.

Definition 1.2.6. A regular value $p \in N$ for function $f : M \to N$ is such that $df : T_{f^{-1}(p)}M \to T_pN$ is surjective in a neighbourhood of p. It can be shown that the set of regular values for holomorphic maps is *dense* in $\mathbb{C}^n$.

The IFT implies that if $f : M \to N$ a holomorphic map between complex manifold, and if $p \in N$ a regular value for f, then $f^{-1}(p)$ is a complex sub-manifold of M of dimension $m - n$.

Example 1.2.2. Non-trivial example of complex manifold

- $x^2 + y^2 = 1$ is an example because complex $\nabla f = (2x, 2y)$ non zero, so this is a non-compact complex sub-manifold of $\mathbb{C}^2$. It is a *fact* that there are no compact complex submanifold of $\mathbb{C}^n$ (unless a point).
- More generally, we will prove in prop. 1.1 that a holomorphic function on a compact complex manifold is **constant**. This statement implies the above since if there was a compact submanifold of $\mathbb{C}^n$, then the embedding is a constant function. So by the implicit function theorem, the coordinates on the submanifold are constants, giving a point.

Proposition 1.1. A holomorphic function on a compact complex manifold is *constant.*

Proof. $f : M \to N$ holomorphic on complex manifold with $f = u + iv$ then if u attains its maximum at p, then in any coordinates chart of a complex manifold. But we know that harmonic function satisfy the mean value property (the value at point p is given by the average over a ball centred at that point). So if the maximum is achieved at the centre of the ball then it is constant on that ball. Then using connectedness, we prove the claim. □

Example 1.2.3 (Examples of compact complex manifolds)**.**

- Use a quotient construction : ${}^{\mathbb{C}^n}\!/_{\Lambda}$, the complex torus for lattice Λ.
- Hopf surface ${}^{\mathbb{C}^2 \setminus \{0\}}\!/_{z \sim 2^n z} \cong S^1 \times S^3$
- Chart for $\mathbb{C}P^n$: the homogeneous chart given by $z^i = \frac{X^i}{X^0}$ for $X^0 \neq 0$ with transition map given by ratio of 2 holomorphic functions.

1.3 Projective Varieties

Let's study other kinds of compact submanifolds of $\mathbb{C}P^n$, which must be a quotient construction by above arguments.

Example 1.3.1. Consider the set $\sum a_i X^i = 0$ in $\mathbb{C}^{n+1}$ for $a_i \not\equiv 0$ and taking the quotient map to homogeneous coordinates, we get a submanifold of $\mathbb{C}P^n$ ad in fact it is biholomorphic to $\mathbb{C}P^{n-1}$.

More generally, taking a homogeneous polynomial $F : \mathbb{C}P^n \to \mathbb{C}$, then $\ker(F)$ is a well-defined subset of $\mathbb{C}P^n$ but **not** necessarily a submanifold.
For example $Y^2X = Z^3$ in $\mathbb{C}P^2$ is not a submanifold, as the chart where $X \neq 0$ defines a 'complex' cusp by analogy with $\mathbb{R}^2$ (therefore derivative not invertible at the cusp). However, using the inverse function theorem, if $\nabla F \neq 0$ on the zeros locus, then it defines a submanifold of $\mathbb{C}P^n$. This is not obvious as we start with coordinates on $\mathbb{C}^{n+1}$, meaning that we have a submanifold of $\mathbb{C}^{n+1}$ and not $\mathbb{C}P^n$; some care is required.

Definition 1.3.1. For a collections of homogeneous polynomials. The simultaneous zero locus is called an *algebraic set*. If it irreducible and connected, it is called a *projective variety*. But note that not all projective variety are manifold.

So we have a way to construct compact complex submanifold of $\mathbb{C}P^n$ by taking $r < n$ homogeneous polynomials, then given a simultaneous null regular value (i.e: $\nabla f|_p \neq 0$ with $p = 0$) defines a complex an $(n - r)$-compact complex submanifold. Even though we have generated lot of compact complex manifold, they are **not** all projective.

- Q: Which compact complex manifold can you embed into $\mathbb{C}P^n$?
- Q: Which of them are projective varieties?

Theorem 1.3. The Kodaira embedding theorem gives an if and only if criterion for Q1. basically you are in projective space if you admit a nice holomorphic line bundle.

2 Lecture 2

2.1 Line Bundles

In the real case, we find examples of smooth compact manifold by taking the locus of functions. But as we have seen, an interesting way to construct compact complex submanifold is to take $\mathbb{C}P^n$ as the domain of said functions. But holomorphic functions on $\mathbb{C}^n$ are not necessarily holomorphic on $\mathbb{C}P^n$.

Example 2.1.1. Consider the functions coordinate functions X^i, from which we have homogeneous coordinates $[X^i] \in \mathbb{C}P^n$ (i.e: $X^i \sim \lambda X^i$ by the usual relation). These functions are *NOT* holomorphic on $\mathbb{C}P^n$. One must choose an appropriate atlas and have holomorphic transition functions which we introduce now.

Definition 2.1.1. We define *inhomogeneous coordinates* on chart $U_i \subset \mathbb{C}P^n$ by

$$Z_i^a = \frac{X^a}{X^i} \quad \text{on} \quad U_i = \{X^i \neq 0\} \tag{1}$$

We readily see that on $U_i \cap U_j$,

$$Z_i^a = \frac{Z_j^a}{Z_i^j} \tag{2}$$

Therefore, the holomorphic transition functions are given by

$$\begin{aligned} \varphi : U_i \cap U_j &\to U_i \cap U_j \\ \varphi(Z_j^a) &\mapsto Z_i^a = \frac{Z_j^a}{Z_i^j}. \end{aligned}$$

Coming back to homogeneous coordinates, X^i are **not** holomorphic maps from $\mathbb{C}P^n$ to $\mathbb{C}$ but they are holomorphic maps of another manifold called $\mathcal{O}(1)$. We first introduce some line bundles.

Definition 2.1.2. We define the holomorphic line bundle $\mathcal{O}(1)$ as the rank 1 vector bundle over $\mathbb{C}P^n$ such that on patch $U_i \cap U_J \subset \mathbb{C}P^n$ we have:

$$\mathcal{O}(1) = \frac{\sqcup_i U_i \times \mathbb{C}}{[(p, v^i) \sim (p, g_{ij} v^j)]}. \tag{3}$$

The transition functions are given by $g_{ij} = \frac{X_j}{X_i}$, which a holomorphic matrix-valued function. Here $X^i \in \mathrm{Hom}(\mathbb{C}P^n, \mathcal{O}(1))$, or in another words it is a section of $\mathcal{O}(1)$. This bundle can be thought of as a line bundle $\mathbb{C}$ with inhomogeneous transition function over the disjoin union of charts $X^i = 0$. Can also be thought of the vector space spanned by the differential 1-forms on $\mathbb{C}P^n$ (right?) (this is abuse of notation actually, sheaves)

Definition 2.1.3. Dually, the *tautological line bundle* $\mathcal{O}(-1)$ (sometimes called *universal line bundle*) is defined as the subset of $\mathbb{C}P^n \times \mathbb{C}^{n+1}$ consisting of $([X^0 : \ldots], v_0, \ldots, v_n)$, such that $\mathbf{v} = \lambda \mathbf{X}$. Meaning, that the fibre at point $\mathbf{X} \in \mathbb{C}P^n$ represents the line in $\mathbb{C}^{n+1}$. It is called tautological because it is the most obvious holomorphic vector bundle representing $\mathbb{C}P^n$ and its lines.

It turns out that are no global holomorphic sections of $\mathcal{O}(-1)$??

Definition 2.1.4. The $\mathcal{O}(k)$ line bundle by $\mathcal{O}(k) = \bigotimes^k \mathcal{O}(1)$, with transition function given by product of transition functions. We define similarly $\mathcal{O}(-k)$. Note that these are also line bundles as $\dim(A \otimes B) = \dim(A) \times \dim(B)$.

Fact. Given U, V finite dimensional vector spaces over a field K, we have

$$U^* \otimes V \cong \mathrm{Hom}(U, V), \tag{4}$$

and tensor product is a left-adjoint functor of Hom.

Remark 2.1. Given a holomorphic section S of $\mathcal{O}(1)$, then S is necessarily a degree-1 function in $[\mathbf{X}]$. Consider the bilinear paring between sections of $\mathcal{O}(1)$ and fibre coordinates $v \in \mathbb{C}^{n+1}$ from the tautological bundle $\mathcal{O}(-1)$.

$$F_{\mathbf{X}} : \mathcal{O}(1) \times \mathcal{O}(-1) \to \mathbb{C}^{n+1} \tag{5}$$
$$(S, \mathbf{v}) \mapsto \langle S[X], \mathbf{v} \rangle \tag{6}$$

This function has homogeneous degree 1 for the homogeneous fibre coordinate $\mathbf{v} \sim \lambda \mathbf{X}$ for $\lambda \in \mathbb{C} \setminus \{0\}$. Given S and $\mathbf{v}$, F defines a homogeneous degree-1 holomorphic function on $\mathbb{C}^{n+1} \setminus \{0\}$. But by Hartog, it extends to **all** of $\mathbb{C}^{n+1}$ and its quotient is defined on all of $\mathbb{C}P^n$.
We note that $\frac{\partial F}{\partial X^i}$ are degree 0 holomorphic homogeneous function on a compact space, so they are constant. Therefore F is a degree 1 holomorphic homogeneous **polynomial** to $\mathbb{C}^{n+1}$.

So we have shown that sections of $\mathcal{O}(1)$ are spanned by homogeneous degree-1 polynomials on $\mathbb{C}^{n+1}$. But what about higher line bundles?

Remark 2.2. Let $k \geq 0$ and consider the section $S : \mathbb{C}P^n \to \mathcal{O}(k)$. Since the transition function of $\mathcal{O}(k)$ are a product of $\mathcal{O}(1)$ transition function, sections must have homogeneous degree-k, and $\mathcal{O}(k)$ is the line bundle of homogeneous degree-k holomorphic functions. By analogy with remark 2.1, we construct a function

$$\begin{aligned} F_{\mathbf{X}} :& \mathcal{O}(k) \times \mathcal{O}(-k) \to \mathbb{C}^{n+1} \\ & (S, \mathbf{v}) \mapsto \prod_{i=1}^{k} \langle s_i[X], \mathbf{v}^i \rangle. \end{aligned}$$

But the maps $\frac{\partial^k F}{\partial(X^0)^{i_0} \ldots \partial(X^n)^{i_n}}$ with $\sum_n i_n = k$ are degree-0 homogeneous holomorphic functions on compact space $\mathbb{C}P^n$, so they are constant. Therefore by the same argument as before, sections of $\mathcal{O}(k)$ are spanned by homogeneous degree-k polynomials on $\mathbb{C}P^n$.

It can be shown that these are the **only** sections of $\mathcal{O}(k)$, meaning that the space of sections is finite dimensional. This is completely **untrue** for smooth manifold. The space of sections of smooth line bundles over smooth manifold has infinite dimensions. (hard to prove, needs PDE theory). This highlights the importance of Hartogs phenomenon in holomorphic functions of several variables.

Finally, we state without proof that the tangent bundle to $\mathbb{C}P^n$ is related to the 'top' line bundle and forms the short exact sequence (see[1]):

$$0 \to \mathbb{C} \to \mathcal{O}(n+1) \to T\mathbb{C}P^n \to 0. \tag{7}$$

2.2 Almost Complex Structures

How do we identify if an even dimensional real manifold is a complex manifold?

Example 2.2.1. We can turn $\mathbb{R}^2$ into $\mathbb{C}$ by appropriate definitions of $\sqrt{-1}$. This is governed by knowing *how* $\sqrt{-1}$ acts on vectors of $\mathbb{R}^2$.

Definition 2.2.1. An almost complex structure (a.c.s) on vector space V is an endomorphism $J : V \to V$ such that $J^2 = -1$. This implies that V is even-dimensional.

Example 2.2.2. In the case of $\mathbb{R}^2$, the almost complex structure *rotates* everything by 90 degrees.

Remark 2.3. One can always find a basis of V such that the almost complex structure is given by

$$\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \tag{8}$$

An almost complex manifold (M, J) is a manifold with an a.c.s on every tangent space and varies smoothly. Not every even-dimensional manifold admits an a.c.s.

Remark 2.4. On a complex manifold M, one can trivially the a.c.s locally as

$$\begin{aligned} J(\partial_{x^i}) &= \partial_{y^i} \\ J(\partial_{y^i}) &= -\partial_{x^i} \end{aligned}$$

Using holomorphicity of transition function, we can globalise this result.

Remark 2.5. Not all almost complex manifolds are complex manifold. (Newlander and Nirenberg theorem) But the real question is, given a manifold, can I find a complex structure? (not almost, relates to integrability)

2.3 Metrics on Complex manifolds

Consider the complexified tangent bundle over even-dimensional manifold $T_{\mathbb{C}}M$, this decomposes into $T^{(1,0)}M \oplus T^{(0,1)}M$ as the $\pm i$ eigenspace of J. Likewise, this has duals $\Omega^{1,0}(M)$ and $\Omega^{0,1}(M)$, with the more general $\Omega^{p,q}(M)$.

Remark 2.6. There is a natural isomorphism $(T_{\mathbb{R}}M, J)$ to $(T^{1,0}M, i)$ via

$$v \mapsto \frac{v - iJv}{2} \tag{9}$$

and similarly for the anti-holomorphic tangent bundle

$$v \mapsto \frac{v + iJv}{2} \tag{10}$$

In general, we can decompose real $2n$-dimensional real components in holomorphic/anti-holomorphic components by identifying

$$J^a{}_b = i\delta^\alpha{}_\beta - i\delta^{\overline{\alpha}}{}_{\overline{\beta}} \tag{11}$$

Example 2.3.1. On $T^{1,0}\mathbb{C}$, the standard hermitian metric is

$$g = dz \otimes d\overline{z} = g_{\mathbb{R}^2} - idx \wedge dy \tag{12}$$

So the real part is a Riemannian metric while the imaginary part is a 2-form. Further we note:

$$\omega(X, Y) = g(JX, Y) \tag{13}$$

Definition 2.3.1. A compatible Riemannian (or **hermitian**) metric $g \in \bigotimes^2 T^*M$, is a metric such that, for $X, Y \in \mathrm{T}_{\mathbb{C}}M$ we have

$$g(J(X), J(Y)) = g(X, Y). \tag{14}$$

or equivalently

$$\omega(X, Y) = \frac{1}{2}\left(g(JX, Y) + g(Y, JX)\right) \tag{15}$$

Definition 2.3.2. Given a hermitian Riemannian metric g on M, we define a compatible 2-form as

$$\omega(X, Y) = g(JX, Y) \tag{16}$$

In fact upon the decomposition of $T_{\mathbb{C}}M$ into holomorphic and anti-holomorphic bundles, we see that $\omega \in \Omega^{(1,1)}(M)$.

Example 2.3.2. Back to our example above, we see that the standard hermitian metric g on $\mathrm{T}^{1,0}\mathbb{C}$ is recovered by:

$$g = g_{\mathbb{R}^2} - i\omega. \tag{17}$$

The upshot is, given a Riemannian metric on a complex manifold that is compatible with an almost complex structure, then there is a **natural** hermitian metric on (TM, J).

Proposition 2.1. All complex manifolds are orientable with the orientation $\{\partial_{x^1}, \partial_{y^1}, \ldots\}$. This is proved considering an holomorphic map $\phi : M \to M$ and show that orientation lie in same equivalence class.

Definition 2.3.3. Given a compatible Riemannian metric g, there is a *volume form*

$$\mathrm{vol}_g = \sqrt{\det(g)}dx^1 \wedge dy^1 \wedge dx^2 \wedge dy^2 \dots \tag{18}$$

We can always go to Darboux chart around point p such that g, ω are standard (done by diagonalising g). So locally,

$$w = \sum_i dx^i \wedge dy^i \tag{19}$$

By antisymmetry, the volume form of a complex manifold is

$$\mathrm{vol}_g = \frac{\omega^n}{n!}. \tag{20}$$

Given a complex submanifold $S \subset M$ of dimension k, the induced 2-form is simply the pullback

$$\omega_S = \iota^* \omega. \tag{21}$$

So the volume form of S is given by $\mathrm{vol}_{g_S} = \frac{\omega_S^k}{k!}$, meaning that the volume form of a complex submanifold is given by the restriction of a globally defined bulk volume form. (this is definitely true for symplectic manifold but not sure when ω not closed) expand on this a bit

3 Lecture 3

Let's explore how to construct Kähler metrics on compact spaces. In the following lecture, we are being a bit loose with $J \leftrightarrow i$. When using the complex structure, we think of real components while using the i prescription is thinking in terms of complex components.

3.1 Kähler metrics

Example 3.1.1. Consider $\mathbb{C}^n$, as an even-dimensional real manifold with complex structure i. The usual flat euclidean metric can be written in terms of potential

$$g = \partial\overline{\partial}|\mathbf{z}|^2 \tag{22}$$

$$= \sum dz^i \vee d\overline{z}^i \tag{23}$$

By taking quotient with lattice Λ, we induce a metric on compact complex tori ${\mathbb{C}^m}/{\Lambda}$.

But in general, how do we find Riemannian metrics on $\mathbb{C}^n$? Actually how do we do in $\mathbb{R}^n$? A metric on $\mathbb{R}^n$ is positive definite $(0,2)$-tensor, so it can be locally given by taking the Hessian of a convex function at point p.

Example 3.1.2. Given $\phi : \mathbb{C}^n \to \mathbb{R}$ a convex function, we can construct a Hermitian metric on $\mathbb{C}^n$ similarly,

$$g_{i\bar{j}} = \frac{\partial^2 \phi}{\partial z^i \partial \overline{z}^j}. \tag{24}$$

Such smooth metrics are called *plurisubharmonic* (i.e: $\mathrm{tr}(g_{i\bar{j}}) \propto \Delta\phi > 0$ is subharmonic). However, not all hermitian metrics on $\mathbb{C}^n$ of this form as these are highly over-determined constraints.

Definition 3.1.1. Given a hermitian metric g of the Kähler form, the induced 2-form as defined in definition 2.3.2 is simply $\omega = \mathrm{Im}(g)$.

Proposition 3.1. The induced 2-form ω is d-closed if and only if it is $\partial\overline{\partial}$-closed a a $(1,1)$-form.

Proof. This is easily seen under the decomposition $\mathrm{T}_{\mathbb{C}}M \cong T^{(1,0)}M \oplus T^{(0,1)}M$, giving

$$d = \partial + \overline{\partial}. \tag{25}$$

But using $d^2 = 0$ and $\partial^2 = \overline{\partial}^2 = 0$, we find

$$\partial\overline{\partial} + \overline{\partial}\partial = 0, \tag{26}$$

thus proving our claim. $\square$

There is an equivalent Poincaré lemma for holomorphic manifold in the following.

Theorem 3.1 (The $\partial\overline{\partial}$-lemma)**.** For any d-closed real $(1,1)$-form ω on an open ball $B \subset \mathbb{C}^n$, there exists a function $\phi : B \to \mathbb{R}$ such that

$$\omega = i\partial\overline{\partial}\phi. \tag{27}$$

Definition 3.1.2. A Riemannian manifold (M, g) is *Kähler* if g is a metric compatible with complex structure J and the induced $(1,1)$-form ω is d-closed.

We sometimes interchange the names ω and g as the Kähler metric in the literature, but they are related by the complex structure so it doesn't matter.

Remark 3.1. Not every complex manifold admits a Kähler metric! As a counterexample, the Hopf surface!

We recall that in Riemannian geometry manifold are locally flat, that is there exists a local coordinates chart such that the metric takes the form $g \sim g_{\mathbb{R}^n} + O(g'')$. Can we have a similar setup in complex geometry?

Proposition 3.2. A complex manifold (M, g), with compatible complex structure J admits a locally flat metric g if M is Kähler.

Proof. We follow [2] here. If (M, g) is Kähler, then $(\partial + \overline{\partial})\omega = 0$, but it is easy to show that in terms of holomorphic/anti-holomorphic components,

$$\omega_{ab} = ig_{\alpha\overline{\beta}} - ig_{\overline{\alpha}\beta}. \tag{28}$$

but the holomorphic/anti-holomorphic derivative acts trivially on the second and first terms respectively, so we have

$$\partial_{[\mu}g_{\alpha]\overline{\beta}} = 0 \tag{29}$$

$$\overline{\partial}_{[\overline{\mu}}g_{\overline{\alpha}]\beta} = 0 \tag{30}$$

It turns out this is sufficient (and necessary but not proved here), to find a local chart where g is flat. $\square$

3.2 Cohomology

For compact manifold, we recall the de Rham cohomology groups $\mathrm{H}^k_{\mathrm{dR}}(M) = {}^{\ker(d)}\!/_{\mathrm{im}(d)}$ which are finite-dimensional with *betti numbers*

$$b^k(M) = \dim_{\mathbb{R}}(\mathrm{H}^k_{\mathrm{dR}}(M)) \tag{31}$$

Likewise, the Dolbeault cohomology groups $\mathcal{H}^{p,q}(M)$ for complex manifold defined using ∂ and $\overline{\partial}$, with associated hodge numbers:

$$h^{p,q} = \dim_{\mathbb{C}}\mathcal{H}^{p,q}(M) \tag{32}$$

We can use Hodge theory and harmonic analysis to prove the following theorem. [3]

Theorem 3.2. If (M, g) is a compact Kähler manifold, then the Dolbeault cohomology and de Rham cohomology have the isomorphism:

$$\mathrm{H}^k(M) \cong \bigoplus_{p+q=k} \mathrm{H}^{p,q}(M) \tag{33}$$

Remark 3.2. If ω Kähler form on a compact Kähler manifold, then its de Rham class $[\omega]$ cannot be trivial. THis is because the volume form ω^n would otherwise integrate to 0 on the boundary of the compact manifold.

Example 3.2.1. Let's consider some examples of Kähler manifold:

- submanifold of Kähler manifold are Kähler by pulling back the Kähler metric.
- complex torus with euclidean metric.
- $\mathbb{C}P^n$ has a nice Kähler metric, called the Fubini-Study metric. This is found by using the metric on ${}^{S^{2n+1}}\!/_{S^1}$ which is isomorphic to $\mathbb{C}P^n$.
  On the chart $X^0 \neq 0$, we will show that the FS metric is given by:

  $$\omega_{FS} = i\partial\overline{\partial}\ln(1 + \sum_{i=1}^{n}|z^i|^2). \tag{34}$$

  Implicitly, we have normalised $X^0 \to 1$.
- Any smooth projective variety is Kähler.

3.3 Chern connection and class

A methodology to derive metrics for Kähler manifold is to take line bundles and compute the curvature of the connections.

Definition 3.3.1. Let (E, h) be a hermitian holomorphic vector bundle over M (not necessarily flat). There exists a unique connection $\nabla : \Omega^{0,0}(M, E) \to \Omega^{0,1}(M, E)$, called the *Chern connection*, that is compatible with the metric on the bundle (metric preserving) and compatible with product structure,

$$d\langle s, t\rangle = \langle \nabla s, t\rangle + \langle s, \nabla t\rangle \tag{35}$$

$$\nabla^{0,1} = \overline{\partial} \tag{36}$$

for sections $s, t \in \Gamma(E)$. The second equation defines compatibility with the product structure, because given a holomorphic local trivialisation of the bundle, the connection acts on sections to given an E-valued $(0,1)$-form. Locally, the takes the form

$$\nabla = d + A, \tag{37}$$

for $A \in \Omega^{1,0}(M, E)$ an E-valued *connection form.*

Remark 3.3. Locally, we can express A using the following argument. Consider a holomorphic frame $\{e_i, \ldots, e_n\}$ of E, this defines a metric on E as $h_{ij} = \langle e_i, e_j \rangle$.

$$\begin{aligned} dh_{ij} &= \partial h_{ij} + \overline{\partial} h_{ij} \\ &= \langle A_i^k e_k, e_j \rangle + \langle e_i, \overline{A}_j^k e_k \rangle \end{aligned}$$

Identifying the holomorphic $(1,0)$-form part, we have

$$A = h^{-1} \partial h \tag{38}$$

Definition 3.3.2. Given connection form $A \in \Omega^{1,0}(M, E)$, we define the *curvature* 2*-form* $\Theta \in \Omega^{1,1}(M, E)$ as

$$\Theta = dA + A \wedge A, \tag{39}$$

which reduces to $\Theta = \overline{\partial} A$ here. All of these are local 2-forms but it turns out that they agree on overlap and therefore are globally defined.

Remark 3.4. Now if we consider a line bundle, the metric on the line is just a scalar and the curvature $(1,1)$-form has the simple expression

$$\Theta = \overline{\partial} \partial \ln h \tag{40}$$

If we change metric on the line $h \to he^{-f}$, then curvature changes by $\partial \overline{\partial} f$. In other words, $\frac{i}{2\pi}\Theta$ is globally *real* $(1,1)$-form that is d-closed and with de Rham cohomology class independent of the metric on the line bundle! This real class is called the first Chern class $c_1(L)$ of the line bundle.

Definition 3.3.3. More generally, given $E \to M$ a holomorphic vector bundle over M with curvature $(1,1)$-form Θ. The **Total Chern class** $c(E)$ of E is

$$c(E) = \det\left(1 + \frac{i}{2\pi}\Theta\right) \tag{41}$$

Since Θ is a 2-form, the total Chern class is a direct sum of even degree forms $c_k(E) \in H^{2k}_{\mathrm{dR}}(M)$. By expanding, we recover the first Chern class

$$c_1(E) = \frac{i}{2\pi}\mathrm{Tr}(\Theta) \tag{42}$$

Remark 3.5. Since $\det(E)$ is the top exterior power of E, it is a line bundle. So for line bundle $E \to M$, $c_1(E) = c_1(\det(E))$. We now have the background to derive the Fubini-Study metric.

Example 3.3.1. Consider the tautological line bundle $\mathcal{O}(-1) \subset \mathbb{C}P^n \times \mathbb{C}^{n+1}$ with the flat metric h on the fibres. That is, for $\mathbf{v} \in \mathbb{C}^{n+1}$

$$\|\mathbf{v}\|^2 = \sum_{i=0}^{n} |v^i|^2 \tag{43}$$

Then using the flat metric, we construct the associated curvature 2-form Θ from eq. (40), which in a chart $X^0 \neq 0$ is as before

$$\Theta = \overline{\partial}\partial \ln(1 + \sum_{i=1}^{n} |z^i|^2) \tag{44}$$

$$= -i\omega_{FS}. \tag{45}$$

Note that given the curvature for $\mathcal{O}(-1)$, we extrapolate the curvature of $\mathcal{O}(1)$ by the simple fact that the dual metric of a line bundle is $\frac{1}{h}$. Inverse metric will lead simply to *negative* the curvature form, giving

$$c_1(\mathcal{O}(1)) = \frac{\omega_{FS}}{2\pi} \tag{46}$$

Definition 3.3.4. A holomorphic line bundle (E, h) is said to be positively curved is $i\Theta$ is of Kähler form.

Theorem 3.3 (Kodaira embedding theorem)**.** A compact complex manifold M is projective if and only if it admits a positively curved holomorphic line bundle (L,h)

Proposition 3.3 (Calabi volume conjecture)**.** Given a Kähler class $[\omega]$ and (n, n)-positive form η, there exists a unique Kähler metric ω' in $[\omega]$ such that $\omega'^n = \eta$.

In other words, **every** metric in $[\omega]$, there is a unique function ϕ such that

$$\omega_\phi = \omega + i\partial\overline{\partial}\phi \tag{47}$$

where $\omega_\phi^n = \eta$. Find that $\phi \sim \phi' + cte$. This requires PDE theory.

Locally, this means the determinant of ω_ϕ is given. Such equations are called complex Monge-Ampère equations in which just given the Hessian of a function. This conjecture was proved by Yau in the 80s and by now more general results about PDEs are known.

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