## RHEONOMIC SUPERGRAVITY

by

# Guillaume Trojani

Supervisor : Pr Richard Szabo

 $\label{eq:ABSTRACT: Based on course at the EMPG by Andrew Beckett and al.}$ 

## 1 Lecture 1

Mostly covered in [1], lectured by comrade Andrew Beckett.

### 1.1 Klein Geometry

In the lecture, let M denote a smooth manifold, G a Lie group and throughout,  $G \circlearrowright M$  transitively. <sup>1</sup>

**Definition 1.1.1.** A homogeneous space (M,G) is a smooth manifold M together with a transitive Lie group action  $G \supset M$ . Right now we stick with a **left** group action, but might change in the notes depending on Andrew's taste. We study homogeneous spaces by considering subgroups  $H \subset G$ , and looking at the geometry of the quotient space, which can be thought as the space modelled on H. If we pick a point  $o \in M$  to act as an "origin", the *stabiliser* which fixes o is the subgroup

$$\operatorname{stab}_{o}(G) := \{ g \in G \mid g \cdot o = o \}. \tag{1.1}$$

We denote this subgroup of symmetries leaving o invariant  $\operatorname{stab}_o(G) = G_o$ , or the *isotropy* subgroup at o. There exists an isomorphism

$$M \cong G/G_o$$
$$g \cdot o \mapsto g G_0$$

which is a  $G_0$ -invariant diffeomorphism. It turns out that this is independent of the origin chosen as stabiliser subgroups at different points are conjugate by transitivity of the group action.

**Definition 1.1.2.** Given a homogeneous space (M, G), and closed subgroup  $H \subset G$  the space with "features" H is G/H and it is also a homogeneous space. Sometimes, this quotient is called the homogeneous space in the literature.

#### **Definition 1.1.3.** (Klein Geometry)

A (smooth connected) Klein geometry is a pair (G, H) where H closed subgroup of G such that G/H is connected. Note that

$$G \downarrow_{\pi}$$

$$G/H$$

$$(1.2)$$

is automatically a principal H-bundle, because the fibres are left H-cosets and the action of H on the fibres is obviously an H-torsor. Consider a Klein geometry (G, H), then the Lie functor gives us a pair of Lie algebras  $(G, H) \xrightarrow{Lie} (\mathfrak{g}, \mathfrak{h})$ , where  $\mathfrak{h}$  is a Lie subalgebra of  $\mathfrak{g}$ . Notably, this induces a short exact sequence of H-modules  $^2$ 

$$0 \longrightarrow \mathfrak{h} \longrightarrow \mathfrak{g} \longrightarrow \mathfrak{g}/\mathfrak{h} \longrightarrow 0 \tag{1.3}$$

**Definition 1.1.4.** If a short exact sequence of H-modules as above splits, then we call  $(\mathfrak{g}, \mathfrak{h})$  reductive and define the orthogonal complement  $\mathfrak{m} \subseteq \mathfrak{g}$  as the H-module such that

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} \tag{1.4}$$

A consequence of a splitting is

$$[\mathfrak{h},\mathfrak{m}]\subseteq\mathfrak{m}.$$
 (1.5)

<sup>&</sup>lt;sup>1</sup>That is  $\forall x, y \in M$ ,  $\exists g \in G$  such that  $g \cdot x = y$ .

 $<sup>^{2}</sup>$ A abelian group on which H acts compatibly. Here it is the vector space structure that is the abelian group.

The reductive case will always be denoted by green throughout the rest of this lecture.

**Definition 1.1.5.** A pair of *reductive* Lie algebras  $(\mathfrak{g}, \mathfrak{h})$  is called **symmetric** if it further follows the condition

$$[\mathfrak{m},\mathfrak{m}] \subseteq \mathfrak{h}.$$
 (1.6)

We will follow the convention of denoting in blue the symmetric case from now on. The terminology is summarised in the table below for  $\mathfrak{m} = \mathfrak{g}/\mathfrak{h}$  (in general):

$\subseteq$	General	Reductive	Symmetric
$[\mathfrak{h},\mathfrak{h}]$	h	ħ	ħ
$[\mathfrak{h},\mathfrak{m}]$	$\mathfrak{h}\oplus\mathfrak{m}$	m	m
$[\mathfrak{m},\mathfrak{m}]$	$\mathfrak{h}\oplus\mathfrak{m}$	$\mathfrak{h}\oplus\mathfrak{m}$	h

**Example 1.1.1.** Consider the homogeneous space  $G \supset M$ . The left group action induces a morphism of tangent bundles

$$(L_q)_*: T_x M \to T_{q \cdot x} M. \tag{1.7}$$

Considering the isotropy subgroup  $H := G_o$  for a chosen origin  $o \in M$ , then for  $h \in H$ , the pushforward  $(L_h)_* \in GL(T_oM)$  is an automorphism of the tangent bundle at the origin. Further, the map

$$\rho: H \to \mathrm{GL}(T_o M)$$
$$h \mapsto (L_h)_*$$

is a representation of H on  $T_oM$ . We call this the linear isotropy representation of (M, G, o).

**Remark 1.1.1.** For a Klein geometry  $M \cong G/H$ , we have the sequence of isomorphisms of H-representations

$$T_oM \xrightarrow{\sim} T_H(G/H) \xrightarrow{\sim} \mathfrak{g}/\mathfrak{h} \cong m$$
 (1.8)

which can be constructed from the transitivity of the group action. However, looking at the tangent bundles on the group manifolds we find the following short exact sequence

$$0 \to T_e H \xrightarrow{(L_e)_*} T_e G \xrightarrow{(\pi_e)_*} T_H(G/H) \to 0, \tag{1.9}$$

which is really the same as eq. (1.3). But noticing the isomorphisms of H-representations in eq. (1.8), the tangent bundle of the manifold M at the origin is modelled on

$$T_o M \cong \mathfrak{g}/\mathfrak{h} \cong \mathfrak{m}$$
 (1.10)

This facts leads us to notice a broader correspondence between geometry and algebra for Klein geometries.

#### **Proposition 1.1.1.** (Correspondence)

Let (G, H) be a Klein geometry, then linear structures on  $\mathfrak{g}/\mathfrak{h}$  correspond to geometric structures on the manifold  $M \cong G/H$  in the following sense:

$$\{H\text{-invariant tensors of }\mathfrak{g}/\mathfrak{h} \cong T_oM\} \longleftrightarrow \{G\text{-invariant tensor fields on }M\}$$
 (1.11)

Proof.  $(\Rightarrow)$ 

Consider  $\tau \in \bigotimes T_o M$  an H-invariant tensor, such that  $(L_h)_*\tau = \tau$  for  $h \in H$ . Consider the map  $\tau \mapsto T_o$  for a fixed origin, we then left translate by  $g \in G$  to span the manifold. So define

$$T_{q \cdot o} = (L_q)_* T_o \in \Gamma(\bigotimes T_{q \cdot o} M).$$

for all  $g \in G$ . This map is well defined since if  $g \cdot o = g' \cdot o$  are two left translation yielding the same tensor field at  $T_{g \cdot o} = T_{g' \cdot o}$ , then  $g^{-1}g' \in H$  an isotropy. But  $\tau = T_o$  is H-invariant implying that  $(L_{g^{-1}g'})_*T_o = T_o$  is G-invariant.  $(\Leftarrow)$ 

Given a G-invariant  $T \in \Gamma(TM)$ , the evaluation map  $\operatorname{ev}_o : \Gamma(TM) \to T_oM$  sends  $T \to T_o$  which has isotropy group H.

In particular, a pseudo inner product on  $\mathfrak{g}/\mathfrak{h}$  gives rise to a pseudo-riemannian metric on M.

**Definition 1.1.6.** A Metric Klein geometry  $(G, H, \eta)$  is

- (G, H) a Klein geometry
- $\eta$  is a pseudo inner product on  $\mathfrak{g}/\mathfrak{h}$  which is H-invariant in the sense describe in the proposition above.

Let's just recall the isometries of flat spacetime for completeness.

**Definition 1.1.7.** The Poincaré group ISO(d-1,1) of d-dimensional Minkowski spacetime is the isotropy group that leaves the split quadratic form invariant. We also call it the *inhomogeneous special* orthonormal group as it consists of disconnected components. It consists of the semidirect product of the Lorentz group (not **necessarily** orthochronous) and the group of spacetime translations,

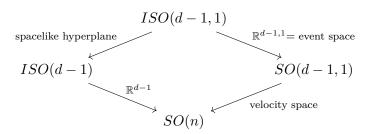
$$ISO(d-1,1) \cong SO(d-1,1) \ltimes \mathbb{R}^{d-1,1}$$
 (1.12)

with multiplication

$$(g,\alpha)\cdot(f,\beta) = (gf,\alpha + f\cdot\beta) \tag{1.13}$$

**Example 1.1.2.** Using our new-found understanding of the Poincaré group, we understand it as a homogeneous space. For example, taking G = ISO(d-1,1) and considering the isotropy group at an origin o to be the Lorentz group H = SO(d-1,1). The manifold  $G/H \cong \mathbb{R}^{d-1,1}$  is connected and at the level of algebras  $\mathfrak{g}/\mathfrak{h} \cong \mathbb{R}^{d-1,1}$ . Further, the bilinear form on  $\mathfrak{g}/\mathfrak{h}$  is invariant under rotations (H-invariant). Therefore, the Poincaré group with its Lorentz subgroups is a Klein geometry. A similar argument can be made for Euclidean Poincaré groups. It is interesting to note however, that using the correspondence established in proposition 1.1.1, the Lorentz invariant bilinear form on  $\mathfrak{g}/\mathfrak{h}$  corresponds to Poincaré invariant tensor fields on the full Minkowski spacetime.

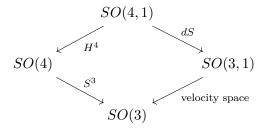
**Example 1.1.3.** We can summarise the ideas presented above in the following reductive diagrams, where arrows represent the subgroup that is quotiented out:



By "event space", we mean that we quotient by all possible translations of spacetime, while the "spacelike hyperplane" is quotienting by the Lorentz boosts. <sup>3</sup> In 4 spacetime dimensions, the more familiar

<sup>&</sup>lt;sup>3</sup>not sure here

diagram:



I don't know much about dS, so maybe someone can complete here.

#### 1.2 Cartan geometry

Assuming familiarity with Maurer-Cartan form  $\omega_G \in \Omega^1(G, \mathfrak{g})$  for a Lie group G, which are the G-invariant one forms on G, we review the more general notion of connections on a principal G-bundle. For some reason, the literature often prefers right action on principal bundles. idk

**Definition 1.2.1.** An Ehresmann connection on a principal right G-bundle  $(P \to M)$  is a  $\mathfrak{g}$ -valued one-form A on P, that is a map

$$\omega: TP \to \mathfrak{g}$$
 (1.14)

such that

$$(R_g)^*\omega = \operatorname{Ad}_{g^{-1}}\omega$$
 for all  $g \in G$   
 $\omega(\xi_X) = X$  for  $X \in V_p$ .

Equivalently an Ehresmann connection is a *choice* of Horizontal distribution such that  $T_pP = V_p \oplus H_p$  where  $H_p$  is G-equivariant. This is better explained in the EKC notes from last year. We now turn our attention to the case of a principal bundle modelled on a reduced geometry.

#### **Definition 1.2.2.** (Cartan Geometry)

A Cartan Geometry  $(\pi: P \to M, A)$  modelled on a Klein geometry (G, H) is a principle right H-bundle  $P \to M$  with a **Cartan connection**  $A \in \Omega^1(P, \mathfrak{g})$  satisfying the conditions:

- $A_p: T_pP \to \mathfrak{g}$  is a linear **isomorphism** (or simply put a  $\mathfrak{g}$ -valued one-form for on P).
- $(R_h)^*A = \operatorname{Ad}_{h^{-1}} \circ A \text{ for all } h \in H$
- $A(\xi_X) = X$  for  $X \in \mathfrak{h}$  and fundamental vector field  $\xi_X$  of  $H \circlearrowright P$ .

Importantly, the Cartan connection one-form takes values in the larger Lie algebra  $\mathfrak{g}$ , because the tangent space of this principal bundle is isomorphic to the larger Lie algebra  $\mathfrak{g}$ . Morally speaking, the algebra  $\mathfrak{g} \cong T_p P$ , while  $\mathfrak{h} \cong V_p$  and  $\mathfrak{m} \cong H_P \cong T_{\pi(p)} M$ .

**Remark 1.2.1.** Consequently, in a Cartan geometry we identify because of the first condition  $\dim(P) = \dim(G)$  and  $\dim(M) = \dim(G/H)$ . Moreover, the Cartan connection isomorphisms implies that upon restricting to the subalgebra  $\mathfrak{h}$  and choosing an origin

$$(A_o)^{-1} : \mathfrak{h} \to \mathfrak{X}_{\mathrm{vert}}(P)$$
  
 $X \mapsto \xi_X.$ 

**Remark 1.2.2.** For a Cartan geometry  $(P \to M : A)$  modelled on a *metric* Klein geometry (G, H), the correspondence 1.1.1 implies that the isomorphism

$$T_x M \cong T_p P / \ker(\pi_*) \cong \mathfrak{g}/\mathfrak{h} \cong \mathfrak{m}$$
 (1.15)

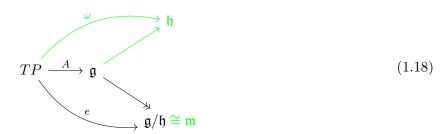
give a metric of the same sign to the manifold M. Where in the above  $x := \pi(p)$ . In general, the full tangent bundle of the manifold can be thought of as the associated bundle of the H-action

$$TM \cong P \times_H \mathfrak{g}/\mathfrak{h} \tag{1.16}$$

**Definition 1.2.3.** The curvature of a Cartan Geometry  $(\pi: P \to M, A)$  modelled on a Klein geometry (G, H) is

$$F(A) = dA + \frac{1}{2}[A, A] \in \Omega^{2}(P, \mathfrak{g})$$
 (1.17)

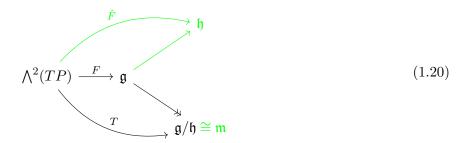
We summarise the different connections in a Cartan geometry, whether it is reductive or not in the following diagram. We denote  $e: TP \to \mathfrak{g}/\mathfrak{h} \cong TM$  as the veilbein of this geometry, while the Ehresmann connection  $\omega: TP \to \mathfrak{h}$  is denoted in green.



So in the reductive case

$$A = \omega + e. \tag{1.19}$$

Considering the curvature of these connections, we have the diagram



with the understanding that in the reductive case,

$$F = \hat{F} + T \tag{1.21}$$

where T is called the *torsion* and  $\hat{F}$  is the Ehresmann part of the curvature.

Remark 1.2.3. A short calculation shows that if the geometry is reductive then the Cartan curvature

$$\begin{split} F[A] &= dA + \frac{1}{2}[A,A]_{\mathfrak{g}} \\ &= d\omega + de + \frac{1}{2}[\omega + e,\omega + e]_{\mathfrak{g}} \\ &= \left(d\omega + \frac{1}{2}[\omega,\omega]_{\mathfrak{h}} + [\omega,e]_{\mathfrak{h}} + \frac{1}{2}[e,e]_{\mathfrak{h}} + de + \frac{1}{2}[\omega,\omega]_{\mathfrak{m}} + [\omega,e]_{\mathfrak{m}} + \frac{1}{2}[e,e,]_{\mathfrak{m}}\right) \end{split}$$

Now because  $\omega$  is an Ehresmann connection and it reduces to a Maurer-Cartan form on the fibres we its curvature 2-form

$$d\omega + \frac{1}{2}[\omega, \omega]_{\mathfrak{h}} = \Omega(\omega).$$

If we are in the reductive case, we recall that  $[\mathfrak{h},\mathfrak{m}]\subseteq\mathfrak{m}$ , so the bracket  $[\omega,e]_{\mathfrak{h}}=0$ . With further restriction in the symmetric case,  $[\mathfrak{m},\mathfrak{m}]\subseteq\mathfrak{h}$ , the bracket  $[e,e]_{\mathfrak{h}}=0$ , but we won't assume it in general so instead define a covariant derivative on  $\mathfrak{m}$ 

$$d^{\omega} = d + \omega. \tag{1.22}$$

So in conclusion, the Ehresmann part of the curvature in a Cartan geometry is

$$\hat{F} = \Omega(\omega) + \frac{1}{2}[e, e]_{\mathfrak{h}},\tag{1.23}$$

while the torsion is

$$T = d^{\omega}e + \frac{1}{2}[e, e]_m \tag{1.24}$$

Some interpretation of flatness and torsion freedom needed here.

# **Bibliography**

[1] Derek K. Wise. MacDowell-Mansouri gravity and Cartan geometry. Classical and Quantum Gravity, 27(15), nov 2010. ISSN 02649381. doi: 10.1088/0264-9381/27/15/155010. URL http://arxiv.org/abs/gr-qc/0611154.