Lie algebroids, Poisson manifolds and Jacobi structures

Based on mini-course by Carlos Zapata-Carratalá

by

Guillaume Trojani

Supervisor : Pr Richard Szabo

Abstract: Mistakes almost certainly mine, thanks for course etc... main refs is [1]

July 6, 2020

Contents

1 Lecture 1: Poisson and Presymplectic geometry

1.1 Poisson Algebra

Definition 1.1. A **Poisson Algebra** is a triple $(A, \cdot, \{,\})$ such that

1. $(A, \cdot)$ is a commuttative, associative and unital $\mathbb{R}$-algebra
2. $(A, \{,\})$ is a Lie $\mathbb{R}$-algebra, which means that the bracket follows the Jacobi identity:
$$\{\{a, b\}, c\} + \{\{b, c\}, a\} + \{\{c, a\}, b\} = 0 \tag{1}$$
3. The Poisson bracket follows the Libeniz identity in the sense that for $a, b, c \in A$,
$$\begin{aligned} \{a, b \cdot c\} &= \{a, b\} \cdot c + b \cdot \{a, c\} && (2) \\ &:= \mathrm{ad}_a\,(b \cdot c) && (3) \end{aligned}$$
where we have defined the adjoint map of the Lie algebra.
4. Equivalently to 3, the $\mathrm{ad}_{\{,\}} : A \to \mathrm{Der}(A, \cdot)$, which takes an element of the algebra to a derivation on the commuttative algebra $(A, \cdot)$. We also see that the $\mathrm{ad}_{\{\}}$ induces a derivation on $(A, \{,\})$ using the Jacobi identity.

Therefore the adjoint map of a Lie Algebra is a morphism from A to $\mathrm{Der}(A, \cdot) \cap \mathrm{Der}(A, \{,\})$, the derivations of both bilinear structures of a Poisson algebra.

Definition 1.2. A **Poisson derivation** is a derivation on both bilinear forms of a Poisson algebra, that is $X \in \mathrm{Der}(A, \cdot) \cap \mathrm{Der}(A, \{,\}) \subset \mathrm{End}_{\mathbb{R}}(A)$. If a Poisson derivation is generated by the adjoint map, $X_a = \{a,\}$, we say that it is a **Hamiltonian derivation**.

Definition 1.3. A Poisson Algebra morphism is a linear map $\psi : A \to B$ such that $\psi : (A, \cdot) \to (B, \cdot)$ is an algebra morphism and $\psi : (A, \{,\}) \to (B, \{,\})$ is a Lie algebra morphism.

Definition 1.4. A subalgebra $I \subset A$ is **coisotrope** if

- $I \subset (A, \cdot)$ is a multiplicative ideal
- $I \subset (A, \{,\})$ is a Lie subalgebra

Proposition 1.1. *Reduction of Poisson algebra*
Suppose $I \subset A$ coisotrope and consider the Lie normaliser (or in ring theory the idealiser)

$$N(I) = \{a \in A | \{a, I\} \subset I\}, \tag{4}$$

which is the largest subalgebra of A that contains I as an ideal. We claim that $A' := {}^{N(I)}\!/_{I}$ inherits a Poisson algebra structure.

Proof. Condition 1 is automatically satisfied as A' is a subalgebra of A, with a Lie algebra structure given by the same bracket. For $a', b', c' \in A'$, consider the adjoint action of a' on $b' \cdot c'$ and look at coset representatirve a, b, c of $N(I)$. Using the fact that I is coisotrope, we see that

$$\begin{aligned} \{a + I, (b + I) \cdot (c + I)\} &= \{a + I, b \cdot c + I\} \\ &= \{a, b \cdot c\} + I \end{aligned}$$

by linearity of the bracket and closure of elements in $N(I)$ w.r.t I. The jacobi identity is checked by similar arguments. $\square$

Definition 1.5. The *reduced Poisson structure* is characterised by the projection map $p : (N(I), \cdot, \{,\}) \to (A', \cdot', \{,\}')$, and by the above proposition, this is a Poisson Algebra morphism.

1.2 Poisson Manifolds

Definition 1.6. A **Poisson manifold** is a smooth manifold P whose commutative algebra of smooth functions has the structure of a Poisson algebra $(\mathrm{C}^\infty(P), \cdot, \{,\})$.

Definition 1.7. A map $\phi : P_1 \to P_2$ is a *Poisson map* if $\phi^* : \mathrm{C}^\infty(P_2) \to \mathrm{C}^\infty(P_1)$ is a Poisson morphism of algebras.

Recall that derivations on smooth functions are isomorphic to vector fields:

$$\mathrm{Der}(\mathrm{C}^\infty(P)) \simeq \Gamma(TP), \tag{5}$$

where the isomorphism is due to

$$\{f, g\} \mapsto X_{\{f,g\}} = [X_f, X_g] \tag{6}$$

Definition 1.8. So following through definition definition 1.2, the Poisson derivations on a Poisson manifolds are called **Poisson vector fields**. And Hamiltonian derivations on Poisson manifolds are called **Hamiltonian vector fields**. Hamiltonian vector fields are generated by the adjoint map

$$\begin{aligned} \mathrm{ad} : \mathrm{C}^\infty(P) &\to \Gamma(TP) \\ f &\mapsto X_f := \{f, \cdot\} \end{aligned}$$

Proposition 1.2. A manifold P; with a commutative algebra of smooth functions $(\mathrm{C}^\infty(P), \cdot, \{,\})$, and a bivector $\Pi \in \Gamma(\bigwedge T^2 P)$ defined as

$$\Pi(df, dg) = \{f, g\}; \tag{7}$$

is a Poisson manifold if and only if Π has vanishing Schouten bracket

$$[\![\Pi, \Pi]\!] = 0. \tag{8}$$

Before proving this statement, we recall facts about the Schouten-Nijenhius which forms a special case of a *Gerstenhaber algebra.*

Definition 1.9. Let P be an n-dimensional manifold and let $A^k(P) = \Gamma(\bigwedge^{k+1} TP)$. There exists a unique bracket $[\cdot, \cdot] : A^k(P) \times A^l(P) \to A^{k+l}(P)$ such that

- $\forall X \in A^0(P) = \mathcal{X}(P)$, the bracket of vector fields (degree 0) is the Lie derivative $[X, \cdot] = \mathcal{L}_X$,
- $\forall X \in A^k(P)\, \forall Y \in A^l(P)$, the graded antisymmetry: $[X, Y] = -(-1)^{kl}[Y, X]$,
- $\forall X \in A^k(P)$, $[X, \cdot]$ is a derivation of degree k. [1]

The **Schouten-Nijenhius** bracket is the unique extension of the Lie bracket to a $\mathbb{Z}$-graded bracket on the space of forms.

[1] recall that a derivation D of degree k has $D(ab) = D(a)b + (-1)^{|a|k} aD(b)$. not sure here though

Proof of proposition 1.2. One needs only prove that the Poisson bracket $\{,\}$ satisfies the Jacobi identity if and only if Π has vanishing Schouten bracket to complete the proof that (P, Π) defines a Poisson manifold. $\square$

Bibliography

[1] Carlos Zapata-Carratala. A Landscape of Hamiltonian Phase Spaces: on the foundations and generalizations of one of the most powerful ideas of modern science. 2019. URL http://arxiv.org/abs/1910.08469.