Lie algebroids, Poisson manifolds and Jacobi structures

Based on mini-course by Carlos Zapata-Carratalá

by

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Abstract: Mistakes almost certainly mine, thanks for course etc... main refs is [1]

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Contents

1 Lecture 1: Poisson and Presymplectic geometry

1.1 Poisson Algebra

Definition 1.1.1. A **Poisson Algebra** is a triple $(A, \cdot, \{,\})$ such that

1. $(A, \cdot)$ is a commuttative, associative and unital $\mathbb{R}$-algebra (or $\mathbb{C}$ algebra maybe?)
2. $(A, \{,\})$ is a Lie $\mathbb{R}$-algebra, which means that the bracket follows the Jacobi identity:
$$\{\{a,b\},c\} + \{\{b,c\},a\} + \{\{c,a\},b\} = 0 \tag{1}$$
3. The Poisson bracket follows the Libeniz identity in the sense that for $a, b, c \in A$,
$$\begin{aligned}\{a, b \cdot c\} &= \{a,b\} \cdot c + b \cdot \{a,c\} &(2)\\ &:= \mathrm{ad}_a\,(b \cdot c) &(3)\end{aligned}$$
where we have defined the adjoint map of the Lie algebra.
4. Equivalently to 3, the $\mathrm{ad}_{\{,\}} : A \to \mathrm{Der}(A, \cdot)$, which takes an element of the algebra to a derivation on the commuttative algebra $(A, \cdot)$. We also see that the $\mathrm{ad}_{\{\}}$ induces a derivation on $(A, \{,\})$ using the Jacobi identity.

Therefore the adjoint map of a Lie Algebra is a morphism from A to $\mathrm{Der}(A, \cdot) \cap \mathrm{Der}(A, \{,\})$, the derivations of both bilinear structures of a Poisson algebra.

Definition 1.1.2. A **Poisson derivation** is a derivation on both bilinear forms of a Poisson algebra, that is $X \in \mathrm{Der}(A, \cdot) \cap \mathrm{Der}(A, \{,\}) \subset \mathrm{End}_{\mathbb{R}}(A)$. If a Poisson derivation is generated by the adjoint map, $X_a = \{a, \}$, we say that it is a **Hamiltonian derivation**.

Definition 1.1.3. A Poisson Algebra morphism is a linear map $\psi : A \to B$ such that $\psi : (A, \cdot) \to (B, \cdot)$ is an algebra morphism and $\psi : (A, \{,\}) \to (B, \{,\})$ is a Lie algebra morphism.

Definition 1.1.4. A subalgebra $I \subset A$ is **coisotrope** if

- $I \subset (A, \cdot)$ is a multiplicative ideal
- $I \subset (A, \{,\})$ is a Lie subalgebra

Proposition 1.1.1. *Reduction of Poisson algebra*
Suppose $I \subset A$ coisotrope and consider the Lie normaliser (or in ring theory the idealiser)
$$N(I) = \{a \in A | \{a, I\} \subset I\}, \tag{4}$$
which is the largest subalgebra of A that contains I as an ideal. We claim that $A' := {}^{N(I)}\!/_{I}$ inherits a Poisson algebra structure.

Proof. Condition 1 is automatically satisfied as A' is a subalgebra of A, with a Lie algebra structure given by the same bracket. For $a', b', c' \in A'$, consider the adjoint action of a' on $b' \cdot c'$ and look at coset representatirve a, b, c of $N(I)$. Using the fact that I is coisotrope, we see that
$$\begin{aligned}\{a + I, (b + I) \cdot (c + I)\} &= \{a + I, b \cdot c + I\}\\ &= \{a, b \cdot c\} + I\end{aligned}$$

by linearity of the bracket and closure of elements in $N(I)$ w.r.t I. The jacobi identity is checked by similar arguments. □

Definition 1.1.5. The *reduced Poisson structure* is characterised by the projection map $p : (N(I), \cdot, \{,\}) \to (A', \cdot', \{,\}')$, and by the above proposition, this is a Poisson Algebra morphism.

1.2 Poisson Manifolds

Definition 1.2.1. A **Poisson manifold** is a smooth manifold P whose commutative algebra of smooth functions has the structure of a Poisson algebra $(\mathrm{C}^\infty(P), \cdot, \{,\})$.

Definition 1.2.2. A map $\phi : P_1 \to P_2$ is a *Poisson map* if $\phi^* : \mathrm{C}^\infty(P_2) \to \mathrm{C}^\infty(P_1)$ is a Poisson morphism of algebras.

Recall that derivations on smooth functions are isomorphic to vector fields:

$$\mathrm{Der}(\mathrm{C}^\infty(P)) \simeq \Gamma(TP), \tag{5}$$

where the isomorphism is due to

$$\{f, g\} \mapsto X_{\{f,g\}} = [X_f, X_g] \tag{6}$$

Definition 1.2.3. So following through definition definition 1.1.2, the Poisson derivations on a Poisson manifolds are called **Poisson vector fields**. And Hamiltonian derivations on Poisson manifolds are called **Hamiltonian vector fields**. Hamiltonian vector fields are generated by the adjoint map

$$\begin{aligned} \mathrm{ad} : \mathrm{C}^\infty(P) &\to \Gamma(TP) \\ f &\mapsto X_f := \{f, \cdot\} \end{aligned}$$

Proposition 1.2.1. A manifold P; with a commutative algebra of smooth functions $(\mathrm{C}^\infty(P), \cdot, \{,\})$, and a bivector $\Pi \in \Gamma(\bigwedge T^2 P)$ defined as

$$\Pi(df, dg) = \{f, g\}; \tag{7}$$

is a Poisson manifold if and only if Π has vanishing Schouten bracket

$$[\![\Pi, \Pi]\!] = 0. \tag{8}$$

Before proving this statement, we recall facts about the Schouten-Nijenhius which forms a special case of a *Gerstenhaber algebra*. CHECK THIS!!

Definition 1.2.4. Let P be an n-dimensional manifold and let $A^k(P) = \Gamma(\bigwedge^{k+1} TP)$. There exists a unique bracket $[\cdot, \cdot] : A^k(P) \times A^l(P) \to A^{k+l}(P)$ such that

- $\forall X \in A^0(P) = \mathcal{X}(P)$, the bracket of vector fields (degree 0) is the Lie derivative $[X, \cdot] = \mathcal{L}_X$,
- $\forall X \in A^k(P)\, \forall Y \in A^l(P)$, the graded antisymmetry: $[X, Y] = -(-1)^{kl}[Y, X]$,
- $\forall X \in A^k(P)$, $[X, \cdot]$ is a derivation of degree k. [1]

[1] recall that a derivation D of degree k has $D(ab) = D(a)b + (-1)^{k|b|} aD(b)$.

The **Schouten-Nijenhius** bracket is the unique extension of the Lie bracket to a $\mathbb{Z}$-graded bracket on the space of forms.

Proof of proposition 1.2.1. One needs only prove that the Poisson bracket $\{,\}$ satisfies the Jacobi identity if and only if Π has vanishing Schouten bracket to complete the proof that (P, Π) defines a Poisson manifold. $\square$

COMPLETE LECTURE LATER

2 Lecture 4: Differential Operators

2.1 Derivations

Let A be a associative commutative unital $\mathbb{C}$-algebra, a vector space over $\mathbb{C}$ such that for any pair $a, b \in A$, the product $ab \in A$ is bilinear and associative.

Definition 2.1.1. A **derivation** $\partial \in \mathrm{Der}_{\mathbb{C}}(A)$ is a $\mathbb{C}$-linear map $\partial : A \to A$ such that the Leibniz identity is satisfied,

$$\partial(ab) = \partial(a)b + a\partial(b) \tag{9}$$

for $a, b \in A$. Clearly, $\mathrm{Der}_{\mathbb{C}}(A) \subseteq \mathrm{End}_{\mathbb{C}}(A)$

Definition 2.1.2. More generally, if B is a commutative ring, A is a B-algebra and M an A-bimodule then $\mathrm{Der}_B(A, M) = \{\partial \in \mathrm{Hom}_B(A, M) | \forall a, b \in A, \partial(ab) = a\partial(b) + \partial(a)b\}$.

Proposition 2.1.1. If $\partial \in \mathrm{End}_{\mathbb{C}}(A)$ is a derivation $\Leftrightarrow \partial(\mathbb{C}) = 0$ and for all $a \in A$, $\partial a - a\partial \in A$.

Proof. Let $b \in A$, then the Leibniz identity is equivalent to

$$\begin{aligned}(\partial a - a\partial)(b) &= \partial(ab) - a\partial(b) \\ &= \partial(a)b.\end{aligned}$$

- $\Rightarrow$ Assuming ∂ is a derivation, then the argument above shows that $\partial a - a\partial \in A \subseteq \mathrm{End}_{\mathbb{C}}(A)$, where left multiplication by this operator is the endomorphism map induced. Furthermore, since ∂ is $\mathbb{C}$-linear, and considering $\mathbb{C}$ as a $\mathbb{C}$-vector space over itself, the Leibniz identity implies

$$\begin{aligned}\partial(1z) &= \partial(1)z + 1\partial(z) \\ &\Rightarrow \partial(1) = 0.\end{aligned}$$

  Therefore $\partial(\mathbb{C}) = 0$.

- $\Leftarrow$ Suppose $a\partial - a\partial = c$ for some $c \in A$ and $\partial(\mathbb{C}) = 0$, then

$$\begin{aligned}(\partial a - a\partial)(1) &= c(1) \\ \partial(a) &= c\end{aligned}$$

  therefore ∂ follows Leibniz identity.

$\square$

Example 2.1.1. On polynomial rings, we have $\mathrm{Der}_{\mathbb{C}}(\mathbb{C}[x]) = \mathbb{C}(x)\frac{d}{dx}$. Clearly, the inclusion $\mathbb{C}[x]\frac{d}{dx} \subseteq \mathrm{Der}_{\mathbb{C}}(\mathbb{C}[x])$ is trivial by just checking that it satisfies Leibniz identity. However, for the reverse inclusion, consider a derivation $\partial \in \mathrm{Der}_{\mathbb{C}}(\mathbb{C}[x])$, then we claim that a basis is given by

$$\partial := \partial(x)\frac{d}{dx}. \tag{10}$$

Easy to check that acting on the unit $1 \in \mathbb{C}$ and x, these definitions agree. Therefore, by $\mathbb{C}$-linearity and Leibniz property, they agree on $\mathbb{C}[x]$.

More generally,

$$\mathrm{Der}_{\mathbb{C}}(\mathbb{C}[x_1, \ldots, c_n]) = \bigoplus_{i=1}^{n} \mathbb{C}[x_1, \ldots, x_n]\frac{\partial}{\partial x^i} \tag{11}$$

Example 2.1.2. If $A = C^\infty(M)$, the algebra of smooth functions on M, then

$$\mathrm{Der}_{\mathbb{R}}(C^\infty(M)) = \mathcal{X}(M) \tag{12}$$

2.2 Differential operators

In this section we define the more general concept of a differential operator, which are **not** necessarily derivations. There are two different ways to define them.

Definition 2.2.1 (First definition)**.** The ring $D(A)$ of $\mathbb{C}$-linear **differential operators** on A is the subalgebra of $\mathrm{End}_{\mathbb{C}}(A)$ generated by A and $\mathrm{Der}_{\mathbb{C}}(A)$. Let $\theta \in D(A)$, it has *order p* if it is the sum of products on at most p derivations.
e.g: $\frac{d^2}{dx^2} + 1 = \left(\frac{d}{dx}\right)^2 + 1$ has order 2.

We can generalise this definition a little.

Definition 2.2.2 (Second definition)**.** A **regular** differential operator of order p is an element of $D^p(A) = \{\theta \in \mathrm{End}_{\mathbb{C}}(A) \,|\theta a - a\theta = \theta(a) \in D^{p-1}(A) \quad \forall a \in A\}$, with $D^0(A) = A$. The ring of **regular differential operators** is $D(A) = \bigcup D^p(A)$ and it is easy to see that

$$D^p(A)D^r(A) \subseteq D^{p+r}(A). \tag{13}$$

and $D^{p+1}(A) \supseteq D^p(A)$ so this defines a filtration.

We relate the two definitions in the following sense. Suppose $\theta \in D^1(A)$, then

$$\theta = (\theta - \theta(1)) + \theta(1) \tag{14}$$

implying that $D^1(A) \cong \mathrm{Der}_{\mathbb{C}}(A) \oplus A$. So we can generate the ring of differential operators on A and clearly def1 $\subset$ def2.

Theorem 2.2.1 (Grothendieck)**.** The two definitions are equivalent if and only if $X = \mathrm{Spec}_A$ is non-singular. In this case, the ring of differential has the simple expression

$$D(A) = {T_A(\mathrm{Der}_{\mathbb{C}}(A))}\big/{\theta \otimes \theta' - \theta' \otimes \theta - [\theta, \theta']} \tag{15}$$

where T_A is the tensor algebra. Recall that the *spectrum* of a ring $\mathrm{Spec}(R)$ is the set of all prime ideals of R with the Zariski topology. [2]

Example 2.2.1. Consider the ring $A = \mathbb{C}[x]$ of rational functions over $\mathbb{C}$, then the algebra of derivations over this ring

$$\mathrm{Der}_{\mathbb{C}}(A) = \mathbb{C}[x]\frac{d}{dx} := W \tag{16}$$

is called the *Witt* algebra. However, the ring $D(A)$ of differential operators on A can also be viewed as the polynomial ring constructed by quotienting the free $\mathbb{C}$-algebra on x, ∂ by the ideal

$$D(A) = {}^{\mathbb{C}\langle x, \partial = \frac{d}{dx}\rangle}\!\big/_{x\partial - \partial x - 1}. \tag{17}$$

This is called a *Weyl* algebra.

As noted earlier, the second definition is more general. Here is an example where the equality fails.

Example 2.2.2. Consider $A = \mathbb{C}[t^2, t^3]$. Then $\mathrm{Spec}(A)$ is the space of proper prime ideals

$$\mathrm{Spec}(A) = \Big\{\langle t^2 - a, t^3 - b\rangle, (a, b) \in \mathbb{C}^2\Big\} \bigcup \Big\{\langle f(t^2, t^3)\rangle, \text{f is irreducible}\Big\} \cup \{\langle 0\rangle\} \tag{18}$$

This space has a singular point and somehow this implies that there exists differential operators at that point that are not generated by sum-products of derivations. EXPAND ON THIS

Lemma 2.2.1. Let $\theta \in D^p(A)$ and $\theta' \in D^r(A)$ then

$$[\theta, \theta'] := \theta \cdot \theta' - \theta' \cdot \theta \in D^{p+r-1}(A) \tag{19}$$

In particular, $D^1(A)$ and $\mathrm{Der}_{\mathbb{C}}(A)$ are *Lie algebras*. Not true for higher order as it doesn't close. But below we will see a way to make it into a Lie algebra.

Question. Given algebras A, B with respective spectrum $X = \mathrm{Spec}(A)$, $Y = \mathrm{Spec}(B)$. If $D(A) \cong D(B)$, does that mean that $X \cong Y$? This turns out to be **false** if the algebraic varieties are allowed to be singular.

2.3 From differential operators to Poisson algebras

We have seen in lemma 2.2.1 that $[D^p(A), D^r(A)] \subseteq D^{p+r-1}(A)$. In particular, for Lie subalgebra $\mathrm{Der}_{\mathbb{C}}(A) \subseteq D(A)$, if $\delta, \delta' \in \mathrm{Der}_{\mathbb{C}}(A)$ then $[\delta, \delta'] \in \mathrm{Der}_{\mathbb{C}}(A)$,

$$\begin{aligned}
[\delta, \delta'](ab) &= \delta\delta'(ab) - \delta'\delta(ab) \\
&= \delta(\delta'(a)b + a\delta'(b)) - \delta'(\delta(a)b + a\delta(b)) \\
&= \delta\delta'(a)b + a\delta\delta'(b) - \delta'\delta(a)b - a\delta'\delta(b) \\
&= [\delta, \delta'](a)b + a[\delta, \delta'](b)
\end{aligned}$$

Definition 2.3.1. Given the filtration of regular differential operators $D(A)$ on algebra A, we define its grading $\mathrm{gr}\, D(A)$ as

$$\mathrm{gr}\, D(A) = \bigoplus_p {}^{D^p(A)}\!\big/_{D^{p-1}(A)} \tag{20}$$

Proposition 2.3.1. The grading of differential operators on A is a commutative ring under composition and a Poisson algebra with bracket generated by the commutator $[\cdot, \cdot]$.

Proof.

- Let $\pi \in D^p(A)$ and $\rho \in D^r(A)$, then $\pi\rho, \rho\pi \in D^{p+r}(A)$ while $[\pi, \rho] \in D^{p+r-1}(A)$. So

$$\pi\rho \sim \rho\pi + D^{p+r-1}(A) \tag{21}$$

but as elements $\mathrm{gr}\,(\pi\rho), \mathrm{gr}\,(\rho\pi) \in \mathrm{gr}\, D(A)$, we have $\mathrm{gr}\,(\pi\rho) = \mathrm{gr}\,(\rho\pi)$

- $(\mathrm{gr}\, D(A), \{\cdot,\cdot\})$ is a Lie algebra. Taking the bracket on differential operators, we induce the Lie bracket $\{\cdot,\cdot\} : \mathrm{gr}\, D(A) \times \mathrm{gr}\, D(A) \to \mathrm{gr}\, D(A)$ by

$$\begin{aligned}\{\mathrm{gr}\,\rho, \mathrm{gr}\,\pi\} &:= \mathrm{gr}\,[\rho,\pi]\\ &= [\rho,\pi] + D^{p+r-2}(A)\end{aligned}$$

  for $\pi \in D^p(A)$, $\rho \in D^r(A)$. Given that $[\cdot,\cdot]$ is a Lie bracket on $D^1(A)$, we extend it to $\mathrm{gr}\, D(A)$ so that $\{\cdot,\cdot\}$ is a bracket up to an element of the quotient.

- The adjoint action is a derivation. This is shown SOMEDAY.

□

In fact, if $X = \mathrm{Spec}(A)$ is non-singular,

$$\begin{aligned}\mathrm{gr}\, D(A) &= \mathrm{gr}\left(\frac{T_A(\mathrm{Der}_{\mathbb{C}}(A))}{\delta\otimes\delta' - \delta'\otimes\delta - [\delta,\delta']}\right)\\ &= \frac{T_A(\mathrm{Der}_{\mathbb{C}}(A))}{\delta\otimes\delta' - \delta'\otimes\delta}.\end{aligned}$$

So in this case, $\mathrm{gr}\, D(A) = \mathrm{Sym}_A(\mathrm{Der}_{\mathbb{C}}(A))$, and since we can identify the derivations with category of vector fields on X,

$$\mathrm{Der}_{\mathbb{C}}(A) = \mathbb{V}ect(X) = \mathbb{C}[T^*X] \tag{22}$$

NEED REFS

2.4 Weyl algebras

Let $A = \mathbb{C}[x_1,\ldots,x_n]$ then the ring of differential operator on A is constructed akin to example 2.2.1 as the free algebra in $\{x_i, y_i = -\partial_i\}$ variables

$$D(A) \cong \frac{\mathbb{C}\langle x_1,\ldots,x_n,y_1,\ldots,y_n\rangle}{[x_i,y_j]=\delta_{ij},\ \text{rest commutes}}. \tag{23}$$

This is the n^{th} **Weyl Algebra** $D(A)$ which is a simple ring (i.e: it does not have a proper 2-sided ideal). Its grading is the Poisson simple algebra

$$\mathrm{gr}\, D(A) = \mathbb{C}[x_1,\ldots,x_n,y_1,\ldots,y_n] \tag{24}$$

with Poisson brackets

$$\{x_i,y_j\} = \delta_{ij} \qquad \{x_i,x_j\} = 0 = \{y_i,y_j\} \tag{25}$$

This is sometimes called *the first example*.

Remark 2.4.1. $D(\mathbb{C}[x])$ has no non-trivial finite dimensional modules. This is because, assuming V is a $D(\mathbb{C}[x])$-module of complex dimension d. Then $D(\mathbb{C}[x])$ acts on V as an endomorphism. Let $X, Y \in \mathrm{Mat}_{d\times d}(\mathbb{C})$ such that $[X,Y] = \mathbb{1}_d$, then $\mathrm{tr}([X,Y]) = 0 \neq d$, which is a contradiction.

Proposition 2.4.1. Let I be a right ideal of $D(A)$. Then $J = \mathrm{gr}\,(I)$ is an ideal of $\mathrm{gr}\, D(A)$ and it is **involutive/coisotrope**

$$\{J,J\} \subseteq J \tag{26}$$

Proof. Let $\theta, \eta \in I$ then $[\theta, \eta] \in I$ since it is a right ideal. Taking the grading, $\mathrm{gr}\,[\theta, \eta] \equiv \{\mathrm{gr}\,\theta, \mathrm{gr}\,\eta\} \subseteq J$ □

Theorem 2.4.1 (Gabber)**.** If $J = \mathrm{gr}\,(I)$ is coisotrope for some right ideal I of $D(A)$, then the radical $\sqrt{J} := \{\theta | \exists$ k, s.t $\theta^k \in J\}$ is also coisotrope.

Corollary 2.4.1 (Bernstein's inequality)**.** Using Gabber's theorem and Hilbert Nullstellensatz $\sqrt{J} = I(V(J))$, we see that

$$\dim(V(J) \subseteq \mathbb{C}^{2n}) \geq n \tag{27}$$

Example 2.4.1. Let $A = \mathbb{C}[x, y]$ with $\{x, y\} = 1$. Consider the coisotrope subring $J = \langle x^2, xy, y^2 \rangle$. It has radical $\sqrt{J} = \langle x, y \rangle$, but the radical is *not* coisotrope. Therefore J is **not** the grading of some right ideal of $D(\mathbb{C}[x, y])$.

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