Lie algebroids, Poisson manifolds and Jacobi structures

Based on mini-course by Carlos Zapata-Carratalá

by

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Abstract: Mistakes almost certainly mine, thanks for course etc... A lot of material is taken from [1] in the first half of the course. Some things are missing but most everything written here is material I somewhat understand right now. Some proofs are missing and I will hopefully get around to it eventually.

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Contents

1 Lecture 1: Poisson and Presymplectic geometry

The first lecture is mostly based on section 2.4 of [1].

1.1 Poisson Algebra

Definition 1.1.1. A **Poisson Algebra** is a triple $(A, \cdot, \{,\})$ such that

1. $(A, \cdot)$ is a commuttative, associative and unital $\mathbb{R}$-algebra (or $\mathbb{C}$ algebra maybe?)
2. $(A, \{,\})$ is a Lie $\mathbb{R}$-algebra, which means that the bracket follows the Jacobi identity:
$$\{\{a, b\}, c\} + \{\{b, c\}, a\} + \{\{c, a\}, b\} = 0 \tag{1}$$
3. The Poisson bracket follows the Libeniz identity in the sense that for $a, b, c \in A$,
$$\{a, b \cdot c\} = \{a, b\} \cdot c + b \cdot \{a, c\} \tag{2}$$
$$:= \mathrm{ad}_a (b \cdot c) \tag{3}$$
where we have defined the adjoint map of the Lie algebra.
4. Equivalently to 3, the $\mathrm{ad}_{\{,\}} : A \to \mathrm{Der}(A, \cdot)$, which takes an element of the algebra to a derivation on the commuttative algebra $(A, \cdot)$. We also see that the $\mathrm{ad}_{\{\}}$ induces a derivation on $(A, \{,\})$ using the Jacobi identity.

Therefore the adjoint map of a Lie Algebra is a morphism from A to $\mathrm{Der}(A, \cdot) \cap \mathrm{Der}(A, \{,\})$, the derivations of both bilinear structures of a Poisson algebra.

Definition 1.1.2. A **Poisson derivation** is a derivation on both bilinear forms of a Poisson algebra, that is $X \in \mathrm{Der}(A, \cdot) \cap \mathrm{Der}(A, \{,\}) \subset \mathrm{End}_{\mathbb{R}}(A)$. If a Poisson derivation is generated by the adjoint map, $X_a = \{a, \}$, we say that it is a **Hamiltonian derivation**.

Definition 1.1.3. A Poisson Algebra morphism is a linear map $\psi : A \to B$ such that $\psi : (A, \cdot) \to (B, \cdot)$ is an algebra morphism and $\psi : (A, \{,\}) \to (B, \{,\})$ is a Lie algebra morphism.

Definition 1.1.4. A subalgebra $I \subset A$ is **coisotrope** if

- $I \subset (A, \cdot)$ is a multiplicative ideal
- $I \subset (A, \{,\})$ is a Lie subalgebra

Proposition 1.1.1. *Reduction of Poisson algebra*
Suppose $I \subset A$ coisotrope and consider the Lie normaliser (or in ring theory the idealiser)
$$N(I) = \{a \in A | \{a, I\} \subset I\}, \tag{4}$$
which is the largest subalgebra of A that contains I as an ideal. We claim that $A' := {}^{N(I)}\!/\!{}_{I}$ inherits a Poisson algebra structure.

Proof. Condition 1 is automatically satisfied as A' is a subalgebra of A, with a Lie algebra structure given by the same bracket. For $a', b', c' \in A'$, consider the adjoint action of a' on $b' \cdot c'$ and look at coset representatirve a, b, c of $N(I)$. Using the fact that I is coisotrope, we see that

$$\begin{aligned} \{a + I, (b + I) \cdot (c + I)\} &= \{a + I, b \cdot c + I\} \\ &= \{a, b \cdot c\} + I \end{aligned}$$

by linearity of the bracket and closure of elements in $N(I)$ w.r.t I. The jacobi identity is checked by similar arguments. □

Definition 1.1.5. The *reduced Poisson structure* is characterised by the projection map $p : (N(I), \cdot, \{,\}) \to (A', \cdot', \{,\}')$, and by the above proposition, this is a Poisson Algebra morphism.

1.2 Poisson Manifolds

Definition 1.2.1. A **Poisson manifold** is a smooth manifold P whose commutative algebra of smooth functions has the structure of a Poisson algebra $(\mathrm{C}^\infty(P), \cdot, \{,\})$.

Definition 1.2.2. A map $\phi : P_1 \to P_2$ is a **Poisson map** if $\phi^* : \mathrm{C}^\infty(P_2) \to \mathrm{C}^\infty(P_1)$ is a Poisson morphism of algebras.

Recall that derivations on smooth functions are isomorphic to vector fields:

$$\mathrm{Der}(\mathrm{C}^\infty(P)) \simeq \Gamma(TP), \tag{5}$$

where the isomorphism is due to

$$\{f, g\} \mapsto X_{\{f,g\}} = [X_f, X_g] \tag{6}$$

Definition 1.2.3. So following through definition definition 1.1.2, the Poisson derivations on a Poisson manifolds are called **Poisson vector fields**. And Hamiltonian derivations on Poisson manifolds are called **Hamiltonian vector fields**. Hamiltonian vector fields are generated by the adjoint map

$$\begin{aligned} \mathrm{ad} : \mathrm{C}^\infty(P) &\to \Gamma(TP) \\ f &\mapsto X_f := \{f, \cdot\} \end{aligned}$$

Proposition 1.2.1. A manifold P; with a commutative algebra of smooth functions $(\mathrm{C}^\infty(P), \cdot, \{,\})$, and a bivector $\Pi \in \Gamma(\bigwedge T^2 P)$ defined as

$$\Pi(df, dg) = \{f, g\}; \tag{7}$$

is a Poisson manifold if and only if Π has vanishing Schouten bracket

$$[\![\Pi, \Pi]\!] = 0. \tag{8}$$

Before proving this statement, we recall facts about the Schouten-Nijenhius which forms a special case of a *Gerstenhaber algebra.* [1]

Definition 1.2.4. Given a Poisson bivector Π, the musical map (sharp)

$$\Pi^\sharp : T^*P \to TP \tag{9}$$
$$df \mapsto \Pi(df, \cdot) := \{f, \cdot\} \tag{10}$$

[1] CHECK THIS, will prove this later, after defining the Gerstenhaber algebra stuff.

defines an **Hamiltonian distribution**. Equivalently, $X_{\cdot} = \Pi^\sharp \circ d : C^\infty(P) \to \Gamma(TP)$ is an *Hamiltonian map*. Note that the space of Hamiltonian distribution $\Pi^\sharp(T^*P)$ is involutive as it is a Lie algebra morphism.

Definition 1.2.5. A submanifold $C \subset (P, \Pi)$ is **coisotropic submanifold** if $TC \subset (TP, \Pi)$ is a coisotropic subspace, that is $TC \supset (TC)^0$ an isotropic (i.e: normal) subspace of TC with respect to the bivector:

$$\Pi(\alpha, \beta) = 0 \quad \forall \alpha \in (T^*C)^0, \beta \in T^*C \tag{11}$$

Consequently, the short exact sequence:

$$0 \to (T^*C)^0 \xrightarrow{\Pi^\sharp} TC \to C^\infty(C) \to 0^2 \tag{12}$$

Proposition 1.2.2. Let $\iota : C \hookrightarrow P$ be a closed submanifold of Poisson manifold (P, Π), then the following are equivalent:

- C is coisotropic
- The vanishing ideal $I_C = \ker(\iota^*) := \{g \in C^\infty(P) \,\big|\, g|_C = 0\}$ is a coisotrope of the Poisson algebra $(C^\infty(P), \cdot, \{\cdot, \cdot\})$.
- Hamiltonian vector fields X_g generated by $g \in I_C$ are tangent to C: $X_g|_C \in \Gamma(TC)$

Proof.
- (1) $\Rightarrow$ (2): First $(I_C, \cdot)$ is a multiplicative ideal of $(C^\infty(P), \cdot)$ by construction. And $(I_C, \{\cdot, \cdot\})$ is a Lie subalgebra because the Poisson bracket vanishes on $C \hookrightarrow P$.
- (2) $\Rightarrow$ (3): for a basis $g \in I_C$, the Hamiltonian vector fields $X_g = \{g, \cdot\}$ span the $\mathrm{Der}(C^\infty(C))$ which is the space of tangent vector to C.
- (3) $\Rightarrow$ (1): $\iota^*\{g, f\} = 0$ for $g \in I_C$, $\forall f \in C^\infty(P)$ [3]

□

Definition 1.2.6. Consider 2 Poisson manifold (P_1, Π_1) and (P_2, Π_2), the *product Poisson manifold* is $(P_1 \times P_2, \Pi_1 + \Pi_2)$, where the canonical isomorphism $T(P_1 \times P_2) \cong \mathrm{pr}_1^* TP_1 \oplus \mathrm{pr}_2^* TP_2$. [4] Also it's easy to see that pulling back onto either P_1, P_2 commutes with the bracket structure, with "cross-pulling" bracket vanishing

Definition 1.2.7. Given a Poisson manifold (P, Π), **opposite Poisson manifold** is $\overline{P} = (P, -\Pi)$.

Proposition 1.2.3. Let two Poisson manifold $(P_1, \Pi_1), (P_2, \Pi_2)$ and a smooth map $\phi : P_1 \to P_2$, then ϕ is a Poisson map if and only if

$$\mathrm{grph}(\phi) := \{(p, \phi(p)) \mid \forall p \in P_1\} \subset P_1 \times \overline{P}_2$$

is a coisotropic submanifold.

[2] i think this is right, but not sure
[3] continue later
[4] This is the Whitney sum by the way

Proof. Consider the tangent bundle of the graph submanifold

$$T\text{grph}(\phi) = \{(X, Y) \mid \text{if } \exists Y \in TP_2 \text{ such that } X, Y \text{are } \phi\text{-related: } \phi^* Y = \phi_* X\}.$$

Full proof in [2] but they have a weird definition of $\Pi^\sharp$ there. [5] □

Recall that a *submersion* is a differential map $\phi : M \to N$ such that

$$D\phi_p : T_p M \twoheadrightarrow T_{\phi_p} N \tag{13}$$

for all $p \in M$. Dually, an *immersion* is a differential map $\phi : M \to N$ such that

$$D\phi_p : T_p M \hookrightarrow T_{\phi_p} M \tag{14}$$

for all $p \in M$.

Proposition 1.2.4. Let (P, Π) a Poisson manifold, and $\iota : C \hookrightarrow P$ a closed coisotropic submanifold. Suppose X_{I_C}, the Hamiltonian vector field tangent to C, or equivalently generated by the ideal $I_C = \ker(\iota^*)$ as in proposition 1.2.2; integrates to a regular foliation on C. Further assume that the leaf space is smooth $P' := {}^{C}\!/_{\chi_C}$ such that there is a surjective submersion (quotient map) q fitting in the diagram

$$\begin{tikzcd} C \arrow[r, hook, "\iota"] \arrow[d, two heads, "q"] & (P, \Pi) \\ (P', \Pi') & \end{tikzcd} \tag{15}$$

Then inherits a Poisson bracket on functions $(C^\infty(P'), \{\cdot, \cdot\}')$ uniquely determined by the condition

$$\iota^*\{F, G\} = q^*\{f, g\}' \tag{16}$$

for all $f, g \in C^\infty(P')$ and $F, G \in C^\infty(P)$ such that F, G are the leaf-wise constant extensions of f, g, i.e

$$q^* f = \iota^* F$$
$$q^* g = \iota^* g$$

2 Lie Groupoids

3 Lie Algebroids

3.1 Vector Bundles

Let's first review facts about vector bundles, but with a more 'categorical' mindset.

Definition 3.1.1. A **vector bundle** $\pi : E \to M$ over a smooth manifold M is a fibre bundle whose fibre E_x is a vector space $V \in \mathbf{Vect}\ \forall x \in M$. The dimension of the typical fibre E_M is called the rank and $\dim(E_M) := \dim(E_x)$ for all x. A *local trivialisation* is a map φ such that on $U \subset M$ open, $\varphi : \pi^{-1}(U) \to U \times V$ is a diffeomorphism.

[5] continue one day

On overlaps $U_1 \cap U_2$, local trivialisations define $\mathrm{GL}(V)$-valued *transition functions.* A basis of sections $\{e_i : U \to \pi^{-1}(U) | \pi \circ e_i = \mathrm{id}_U\}_{i=1}^{\mathrm{rk}(E)}$ defines a local trivialisations as well; if such sections are globally defined, the bundle is trivial (or trivialisable).

Definition 3.1.2. A smooth map F between 2 vector bundles $F : E_1 \to E_2$ is a bundle **vector bundle morphism** if there exists smooth map $\phi \in C^\infty(M_1, M_2)$ between the bases such that

$$\begin{array}{ccc} E_1 & \xrightarrow{F} & E_2 \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_2} \\ M_1 & \xrightarrow{\phi} & M_2 \end{array} \tag{17}$$

commutes. We say that F is a covering for ϕ. Equivalently, F restricts to a linear map on the fibre $F_x : (E_1)_x \to (E_2)_{\phi(x)}$.

Definition 3.1.3. Vector bundles over smooth manifolds with vector bundle morphism forms the **category of vector bundles** denoted $\mathbf{Vect}_{\mathrm{Man}}$.

Remark 3.1.1 (Categorification)**.** Fixing a base manifold M, the point-wise construction of fibre bundles over M restricts us to the subcategory of vector bundles over M denoted $\mathbf{Vect}_M$. This subcategory has objects the vector bundles over M and morphisms F covering the identity morphism. Further, point-wise constructions are symmetric abelian products $\bigotimes : \mathbf{Vect}_M \times \mathbf{Vect}_M \to \mathbf{Vect}_M$ such that $\bigotimes$ is a **binary functor** (or bifunctor). By that we mean for morphisms $F, G, H \in \mathrm{Mor}(\mathbf{Vect}_M)$, we have

$$(F \circ G) \otimes H = (F \otimes H) \circ (G \otimes H) \tag{18}$$

and so on. So the point-wise construction of vector bundle over base manifold M forms a **abelian symmetric monoidal category**. [6]

4 Lecture 4: Differential Operators

4.1 Derivations

Let A be a associative commutative unital $\mathbb{C}$-algebra, a vector space over $\mathbb{C}$ such that for any pair $a, b \in A$, the product $ab \in A$ is bilinear and associative.

Definition 4.1.1. A **derivation** $\partial \in \mathrm{Der}_{\mathbb{C}}(A)$ is a $\mathbb{C}$-linear map $\partial : A \to A$ such that the Leibniz identity is satisfied,

$$\partial(ab) = \partial(a)b + a\partial(b) \tag{19}$$

for $a, b \in A$. Clearly, $\mathrm{Der}_{\mathbb{C}}(A) \subseteq \mathrm{End}_{\mathbb{C}}(A)$

Definition 4.1.2. More generally, if B is a commutative ring, A is a B-algebra and M an A-bimodule then $\mathrm{Der}_B(A, M) = \{\partial \in \mathrm{Hom}_B(A, M) | \forall a, b \in A, \partial(ab) = a\partial(b) + \partial(a)b\}$.

Proposition 4.1.1. If $\partial \in \mathrm{End}_{\mathbb{C}}(A)$ is a derivation $\Leftrightarrow$ $\partial(\mathbb{C}) = 0$ and for all $a \in A$, $\partial a - a\partial \in A$.

Proof. Let $b \in A$, then the Leibniz identity is equivalent to

$$\begin{aligned} (\partial a - a\partial)(b) &= \partial(ab) - a\partial(b) \\ &= \partial(a)b. \end{aligned}$$

[6] expand on this please

- $\Rightarrow$ Assuming ∂ is a derivation, then the argument above shows that $\partial a - a\partial \in A \subseteq \mathrm{End}_{\mathbb{C}}(A)$, where left multiplication by this operator is the endomorphism map induced. Furthermore, since ∂ is $\mathbb{C}$-linear, and considering $\mathbb{C}$ as a $\mathbb{C}$-vector space over itself, the Leibniz identity implies

$$\begin{aligned} \partial(1z) &= \partial(1)z + 1\partial(z) \\ &\Rightarrow \partial(1) = 0. \end{aligned}$$

  Therefore $\partial(\mathbb{C}) = 0$.

- $\Leftarrow$ Suppose $a\partial - a\partial = c$ for some $c \in A$ and $\partial(\mathbb{C}) = 0$, then

$$\begin{aligned} (\partial a - a\partial)(1) &= c(1) \\ \partial(a) &= c \end{aligned}$$

  therefore ∂ follows Leibniz identity.

□

Example 4.1.1. On polynomial rings, we have $\mathrm{Der}_{\mathbb{C}}(\mathbb{C}[x]) = \mathbb{C}(x)\frac{d}{dx}$. Clearly, the inclusion $\mathbb{C}[x]\frac{d}{dx} \subseteq \mathrm{Der}_{\mathbb{C}}(\mathbb{C}[x])$ is trivial by just checking that it satisfies Leibniz identity. However, for the reverse inclusion, consider a derivation $\partial \in \mathrm{Der}_{\mathbb{C}}(\mathbb{C}[x])$, then we claim that a basis is given by

$$\partial := \partial(x)\frac{d}{dx}. \tag{20}$$

Easy to check that acting on the unit $1 \in \mathbb{C}$ and x, these definitions agree. Therefore, by $\mathbb{C}$-linearity and Leibniz property, they agree on $\mathbb{C}[x]$.

More generally,

$$\mathrm{Der}_{\mathbb{C}}(\mathbb{C}[x_1, \dots, c_n]) = \bigoplus_{i=1}^{n} \mathbb{C}[x_1, \dots, x_n]\frac{\partial}{\partial x^i} \tag{21}$$

Example 4.1.2. If $A = C^\infty(M)$, the algebra of smooth functions on M, then

$$\mathrm{Der}_{\mathbb{R}}(C^\infty(M)) = \mathcal{X}(M) \tag{22}$$

4.2 Differential operators

In this section we define the more general concept of a differential operator, which are **not** necessarily derivations. There are two different ways to define them.

Definition 4.2.1 (First definition)**.** The ring $D(A)$ of $\mathbb{C}$-linear **differential operators** on A is the subalgebra of $\mathrm{End}_{\mathbb{C}}(A)$ generated by A and $\mathrm{Der}_{\mathbb{C}}(A)$. Let $\theta \in D(A)$, it has *order* p if it is the sum of products on at most p derivations.
e.g: $\frac{d^2}{dx^2} + 1 = \left(\frac{d}{dx}\right)^2 + 1$ has order 2.

We can generalise this definition a little.

Definition 4.2.2 (Second definition)**.** A **regular** differential operator of order p is an element of $D^p(A) = \{\theta \in \mathrm{End}_{\mathbb{C}}(A) \,|\theta a - a\theta = \theta(a) \in D^{p-1}(A) \quad \forall a \in A\}$, with $D^0(A) = A$. The ring of **regular differential operators** is $D(A) = \bigcup D^p(A)$ and it is easy to see that

$$D^p(A)D^r(A) \subseteq D^{p+r}(A). \tag{23}$$

and $D^{p+1}(A) \supseteq D^p(A)$ so this defines a filtration.

We relate the two definitions in the following sense. Suppose $\theta \in D^1(A)$, then

$$\theta = (\theta - \theta(1)) + \theta(1) \tag{24}$$

implying that $D^1(A) \cong \mathrm{Der}_{\mathbb{C}}(A) \oplus A$. So we can generate the ring of differential operators on A and clearly def1 $\subset$ def2.

Theorem 4.2.1 (Grothendieck)**.** The two definitions are equivalent if and only if $X = \mathrm{Spec}_A$ is non-singular. In this case, the ring of differential has the simple expression

$$D(A) = {T_A(\mathrm{Der}_{\mathbb{C}}(A))}\big/{\theta \otimes \theta' - \theta' \otimes \theta - [\theta, \theta']} \tag{25}$$

where T_A is the tensor algebra. Recall that the *spectrum* of a ring $\mathrm{Spec}(R)$ is the set of all prime ideals of R with the Zariski topology. [3]

Example 4.2.1. Consider the ring $A = \mathbb{C}[x]$ of rational functions over $\mathbb{C}$, then the algebra of derivations over this ring

$$\mathrm{Der}_{\mathbb{C}}(A) = \mathbb{C}[x]\frac{d}{dx} := W \tag{26}$$

is called the *Witt* algebra. However, the ring $D(A)$ of differential operators on A can also be viewed as the polynomial ring constructed by quotienting the free $\mathbb{C}$-algebra on x, ∂ by the ideal

$$D(A) = {\mathbb{C}\langle x, \partial = \frac{d}{dx}\rangle}\big/{x\partial - \partial x - 1}. \tag{27}$$

This is called a *Weyl* algebra.

As noted earlier, the second definition is more general. Here is an example where the equality fails.

Example 4.2.2. Consider $A = \mathbb{C}[t^2, t^3]$. Then $\mathrm{Spec}(A)$ is the space of proper prime ideals

$$\mathrm{Spec}(A) = \left\{\langle t^2 - a, t^3 - b\rangle, (a, b) \in \mathbb{C}^2\right\} \bigcup \left\{\langle f(t^2, t^3)\rangle, \text{f is irreducible}\right\} \cup \{\langle 0\rangle\} \tag{28}$$

This space has a singular point and somehow this implies that there exists differential operators at that point that are not generated by sum-products of derivations. EXPAND ON THIS

Lemma 4.2.1. Let $\theta \in D^p(A)$ and $\theta' \in D^r(A)$ then

$$[\theta, \theta'] := \theta \cdot \theta' - \theta' \cdot \theta \in D^{p+r-1}(A) \tag{29}$$

In particular, $D^1(A)$ and $\mathrm{Der}_{\mathbb{C}}(A)$ are *Lie algebras.* Not true for higher order as it doesn't close. But below we will see a way to make it into a Lie algebra.

Question. Given algebras A, B with respective spectrum $X = \mathrm{Spec}(A)$, $Y = \mathrm{Spec}(B)$. If $D(A) \cong D(B)$, does that mean that $X \cong Y$? This turns out to be **false** if the algebraic varieties are allowed to be singular.

4.3 From differential operators to Poisson algebras

We have seen in lemma 4.2.1 that $[D^p(A), D^r(A)] \subseteq D^{p+r-1}(A)$. In particular, for Lie subalgebra $\mathrm{Der}_{\mathbb{C}}(A) \subseteq D(A)$, if $\delta, \delta' \in \mathrm{Der}_{\mathbb{C}}(A)$ then $[\delta, \delta'] \in \mathrm{Der}_{\mathbb{C}}(A)$,

$$\begin{aligned}
[\delta, \delta'](ab) &= \delta\delta'(ab) - \delta'\delta(ab) \\
&= \delta(\delta'(a)b + a\delta'(b)) - \delta'(\delta(a)b + a\delta(b)) \\
&= \delta\delta'(a)b + a\delta\delta'(b) - \delta'\delta(a)b - a\delta'\delta(b) \\
&= [\delta, \delta'](a)b + a[\delta, \delta'](b)
\end{aligned}$$

Definition 4.3.1. Given the filtration of regular differential operators $D(A)$ on algebra A, we define its grading $\mathrm{gr}\, D(A)$ as

$$\mathrm{gr}\, D(A) = \bigoplus_p D^p(A) \big/ D^{p-1}(A) \tag{30}$$

Proposition 4.3.1. The grading of differential operators on A is a commutative ring under composition and a Poisson algebra with bracket generated by the commutator $[\cdot, \cdot]$.

Proof.

- Let $\pi \in D^p(A)$ and $\rho \in D^r(A)$, then $\pi\rho, \rho\pi \in D^{p+r}(A)$ while $[\pi, \rho] \in D^{p+r-1}(A)$. So

  $$\pi\rho \sim \rho\pi + D^{p+r-1}(A) \tag{31}$$

  but as elements $\mathrm{gr}\,(\pi\rho), \mathrm{gr}\,(\rho\pi) \in \mathrm{gr}\, D(A)$, we have $\mathrm{gr}\,(\pi\rho) = \mathrm{gr}\,(\rho\pi)$

- $(\mathrm{gr}\, D(A), \{\cdot, \cdot\})$ is a Lie algebra. Taking the bracket on differential operators, we induce the Lie bracket $\{\cdot, \cdot\} : \mathrm{gr}\, D(A) \times \mathrm{gr}\, D(A) \to \mathrm{gr}\, D(A)$ by

  $$\begin{aligned}
  \{\mathrm{gr}\,\rho, \mathrm{gr}\,\pi\} &:= \mathrm{gr}\,[\rho, \pi] \\
  &= [\rho, \pi] + D^{p+r-2}(A)
  \end{aligned}$$

  for $\pi \in D^p(A)$, $\rho \in D^r(A)$. Given that $[\cdot, \cdot]$ is a Lie bracket on $D^1(A)$, we extend it to $\mathrm{gr}\, D(A)$ so that $\{\cdot, \cdot\}$ is a bracket up to an element of the quotient.

- The adjoint action is a derivation. This is shown SOMEDAY.

□

In fact, if $X = \mathrm{Spec}(A)$ is non-singular,

$$\begin{aligned}
\mathrm{gr}\, D(A) &= \mathrm{gr}\left(\frac{T_A(\mathrm{Der}_{\mathbb{C}}(A))}{\delta \otimes \delta' - \delta' \otimes \delta - [\delta, \delta']}\right) \\
&= \frac{T_A(\mathrm{Der}_{\mathbb{C}}(A))}{\delta \otimes \delta' - \delta' \otimes \delta}.
\end{aligned}$$

So in this case, $\mathrm{gr}\, D(A) = \mathrm{Sym}_A(\mathrm{Der}_{\mathbb{C}}(A))$, and since we can identify the derivations with category of vector fields on X,

$$\mathrm{Der}_{\mathbb{C}}(A) = \mathbb{V}ect(X) = \mathbb{C}[T^*X] \tag{32}$$

Possible connection with L_∞-algebras. see [4]

4.4 Weyl algebras

Let $A = \mathbb{C}[x_1, \ldots, x_n]$ then the ring of differential operator on A is constructed akin to example 4.2.1 as the free algebra in $\{x_i, y_i = -\partial_i\}$ variables

$$D(A) \cong \frac{\mathbb{C}\langle x_1, \ldots, x_n, y_1, \ldots, y_n\rangle}{[x_i, y_j] = \delta_{ij}, \text{ rest commutes}}. \tag{33}$$

This is the n^{th} **Weyl Algebra** $D(A)$ which is a simple ring (i.e: it does not have a proper 2-sided ideal). Its grading is the Poisson simple algebra

$$\operatorname{gr} D(A) = \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n] \tag{34}$$

with Poisson brackets

$$\{x_i, y_j\} = \delta_{ij} \qquad \{x_i, x_j\} = 0 = \{y_i, y_j\} \tag{35}$$

This is sometimes called *the first example.*

Remark 4.4.1. $D(\mathbb{C}[x])$ has no non-trivial finite dimensional modules. This is because, assuming V is a $D(\mathbb{C}[x])$-module of complex dimension d. Then $D(\mathbb{C}[x])$ acts on V as an endomorphism. Let $X, Y \in \mathrm{Mat}_{d\times d}(\mathbb{C})$ such that $[X, Y] = \mathbb{1}_d$, then $\mathrm{tr}([X, Y]) = 0 \neq d$, which is a contradiction.

Proposition 4.4.1. Let I be a right ideal of $D(A)$. Then $J = \operatorname{gr}(I)$ is an ideal of $\operatorname{gr} D(A)$ and it is **involutive/coisotrope**

$$\{J, J\} \subseteq J \tag{36}$$

Proof. Let $\theta, \eta \in I$ then $[\theta, \eta] \in I$ since it is a right ideal. Taking the grading, $\operatorname{gr}[\theta, \eta] \equiv \{\operatorname{gr}\theta, \operatorname{gr}\eta\} \subseteq J$ □

Theorem 4.4.1 (Gabber)**.** If $J = \operatorname{gr}(I)$ is coisotrope for some right ideal I of $D(A)$, then the radical $\sqrt{J} := \{\theta | \exists \text{ k, s.t } \theta^k \in J\}$ is also coisotrope.

Corollary 4.4.1 (Bernstein's inequality)**.** Using Gabber's theorem and Hilbert Nullstellensatz $\sqrt{J} = I(V(J))$, we see that

$$\dim(V(J) \subseteq \mathbb{C}^{2n}) \geq n \tag{37}$$

Example 4.4.1. Let $A = \mathbb{C}[x, y]$ with $\{x, y\} = 1$. Consider the coisotrope subring $J = \langle x^2, xy, y^2\rangle$. It has radical $\sqrt{J} = \langle x, y\rangle$, but the radical is *not* coisotrope. Therefore J is **not** the grading of some right ideal of $D(\mathbb{C}[x, y])$.

Bibliography

[1] Carlos Zapata-Carratala. A Landscape of Hamiltonian Phase Spaces: on the foundations and generalizations of one of the most powerful ideas of modern science. 2019. URL `http://arxiv.org/abs/1910.08469`.

[2] Rui Loja Fernandes. Rui Loja Fernandes and Ioan M˘ arcut , Lectures on Poisson Geometry. 2015. URL `http://www.math.illinois.edu/{~}ruiloja/Math595/book.pdf`.

[3] S. C. Coutinho. *A Primer of Algebraic D-Modules*. Cambridge University Press, may 1995. ISBN 9780521551199. doi: 10.1017/CBO9780511623653. URL `/core/books/primer-of-algebraic-dmodules/87B8F8AB3B53DBA8A8BD33A058E54473https://www.cambridge.org/core/product/identifier/9780511623653/type/book`.

[4] Christopher Braun and Andrey Lazarev. Homotopy BV algebras in Poisson geometry. apr 2013. doi: 10.1090/S0077-1554-2014-00216-8. URL `http://arxiv.org/abs/1304.6373http://dx.doi.org/10.1090/S0077-1554-2014-00216-8`.