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# RHEONOMIC SUPERGRAVITY

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by

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ABSTRACT: Based on course at the EMPG by Andrew Beckett and al.

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# 1 Lecture 1

Mostly covered in [1], lectured by comrade Andrew Beckett.

## 1.1 Klein Geometry

In the lecture, let  $M$  denote a smooth manifold,  $G$  a Lie group and throughout,  $G \curvearrowright M$  *transitively*.<sup>1</sup>

**Definition 1.1.1.** A homogeneous space  $(M, G)$  is a smooth manifold  $M$  together with a transitive Lie group action  $G \curvearrowright M$ . Right now we stick with a **left** group action, but might change in the notes depending on Andrew's taste. We study homogeneous spaces by considering subgroups  $H \subset G$ , and looking at the geometry of the quotient space, which can be thought as the space modelled on  $H$ . If we pick a point  $o \in M$  to act as an "origin", the *stabiliser* which fixes  $o$  is the subgroup

$$\text{stab}_o(G) := \{g \in G \mid g \cdot o = o\}. \quad (1.1)$$

We denote this subgroup of symmetries leaving  $o$  invariant  $\text{stab}_o(G) = G_o$ , or the *isotropy* subgroup at  $o$ . There exists an isomorphism

$$\begin{aligned} M &\cong G/G_o \\ g \cdot o &\mapsto g G_o \end{aligned}$$

which is a  $G_o$ -invariant diffeomorphism. It turns out that this is independent of the origin chosen as stabiliser subgroups at different points are conjugate by transitivity of the group action.


**Definition 1.1.2.** Given a homogeneous space  $(M, G)$ , and closed subgroup  $H \subset G$  the space with "features"  $H$  is  $G/H$  and it is also a homogeneous space. Sometimes, this quotient is called the homogeneous space in the literature.

**Definition 1.1.3.** (Klein Geometry)

A (smooth connected) Klein geometry is a pair  $(G, H)$  where  $H$  closed subgroup of  $G$  such that  $G/H$  is connected. Note that

$$\begin{array}{c} G \\ \downarrow \pi \\ G/H \end{array} \quad (1.2)$$

is automatically a principal  $H$ -bundle, because the fibres are left  $H$ -cosets and the action of  $H$  on the fibres is obviously an  $H$ -torsor. Consider a Klein geometry  $(G, H)$ , then the Lie functor gives us a pair of Lie algebras  $(G, H) \xrightarrow{\text{Lie}} (\mathfrak{g}, \mathfrak{h})$ , where  $\mathfrak{h}$  is a Lie subalgebra of  $\mathfrak{g}$ . Notably, this induces a short exact sequence of  $H$ -modules<sup>2</sup>

$$0 \longrightarrow \mathfrak{h} \longrightarrow \mathfrak{g} \longrightarrow \mathfrak{g}/\mathfrak{h} \longrightarrow 0 \quad (1.3)$$


**Definition 1.1.4.** If a short exact sequence of  $H$ -modules as above splits, then we call  $(\mathfrak{g}, \mathfrak{h})$  **reductive** and define the orthogonal complement  $\mathfrak{m} \subseteq \mathfrak{g}$  as the  $H$ -module such that

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} \quad (1.4)$$

A consequence of a splitting is

$$[\mathfrak{h}, \mathfrak{m}] \subseteq \mathfrak{m}. \quad (1.5)$$

<sup>1</sup>That is  $\forall x, y \in M, \exists g \in G$  such that  $g \cdot x = y$ .

<sup>2</sup>A abelian group on which  $H$  acts compatibly. Here it is the vector space structure that is the abelian group.

The reductive case will always be denoted by green throughout the rest of this lecture.

**Definition 1.1.5.** A pair of *reductive* Lie algebras  $(\mathfrak{g}, \mathfrak{h})$  is called **symmetric** if it further follows the condition

$$[\mathfrak{m}, \mathfrak{m}] \subseteq \mathfrak{h}. \quad (1.6)$$

We will follow the convention of denoting in blue the symmetric case from now on. The terminology is summarised in the table below for  $\mathfrak{m} = \mathfrak{g}/\mathfrak{h}$  (in general):

| $\subseteq$                    | <i>General</i>                     | <i>Reductive</i>                   | <i>Symmetric</i> |
|--------------------------------|------------------------------------|------------------------------------|------------------|
| $[\mathfrak{h}, \mathfrak{h}]$ | $\mathfrak{h}$                     | $\mathfrak{h}$                     | $\mathfrak{h}$   |
| $[\mathfrak{h}, \mathfrak{m}]$ | $\mathfrak{h} \oplus \mathfrak{m}$ | $\mathfrak{m}$                     | $\mathfrak{m}$   |
| $[\mathfrak{m}, \mathfrak{m}]$ | $\mathfrak{h} \oplus \mathfrak{m}$ | $\mathfrak{h} \oplus \mathfrak{m}$ | $\mathfrak{h}$   |

**Example 1.1.1.** Consider the homogeneous space  $G \curvearrowright M$ . The left group action induces a morphism of tangent bundles

$$(L_g)_* : T_x M \rightarrow T_{g \cdot x} M. \quad (1.7)$$

Considering the isotropy subgroup  $H := G_o$  for a chosen origin  $o \in M$ , then for  $h \in H$ , the pushforward  $(L_h)_* \in \text{GL}(T_o M)$  is an automorphism of the tangent bundle at the origin. Further, the map

$$\begin{aligned} \rho : H &\rightarrow \text{GL}(T_o M) \\ h &\mapsto (L_h)_* \end{aligned}$$

is a representation of  $H$  on  $T_o M$ . We call this the *linear isotropy representation* of  $(M, G, o)$ .

**Remark 1.1.1.** For a Klein geometry  $M \cong G/H$ , we have the sequence of isomorphisms of  $H$ -representations

$$T_o M \xrightarrow{\sim} T_H(G/H) \xrightarrow{\sim} \mathfrak{g}/\mathfrak{h} \cong \mathfrak{m} \quad (1.8)$$

which can be constructed from the transitivity of the group action. However, looking at the tangent bundles on the group manifolds we find the following short exact sequence

$$0 \rightarrow T_e H \xrightarrow{(L_e)_*} T_e G \xrightarrow{(\pi_e)_*} T_H(G/H) \rightarrow 0, \quad (1.9)$$

which is really the same as eq. (1.3). But noticing the isomorphisms of  $H$ -representations in eq. (1.8), the tangent bundle of the manifold  $M$  at the origin is modelled on

$$T_o M \cong \mathfrak{g}/\mathfrak{h} \cong \mathfrak{m} \quad (1.10)$$

This facts leads us to notice a broader correspondence between geometry and algebra for Klein geometries.

**Proposition 1.1.1.** (Correspondence)

Let  $(G, H)$  be a Klein geometry, then linear structures on  $\mathfrak{g}/\mathfrak{h}$  correspond to geometric structures on the manifold  $M \cong G/H$  in the following sense:

$$\{H\text{-invariant tensors of } \mathfrak{g}/\mathfrak{h} \cong T_o M\} \longleftrightarrow \{G\text{-invariant tensor fields on } M\} \quad (1.11)$$

*Proof.* ( $\Rightarrow$ )

Consider  $\tau \in \bigotimes T_o M$  an  $H$ -invariant tensor, such that  $(L_h)_* \tau = \tau$  for  $h \in H$ . Consider the map  $\tau \mapsto T_o$  for a fixed origin, we then left translate by  $g \in G$  to span the manifold. So define

$$T_{g \cdot o} = (L_g)_* T_o \in \Gamma(\bigotimes T_{g \cdot o} M).$$

for all  $g \in G$ . This map is well defined since if  $g \cdot o = g' \cdot o$  are two left translation yielding the same tensor field at  $T_{g \cdot o} = T_{g' \cdot o}$ , then  $g^{-1}g' \in H$  an isotropy. But  $\tau = T_o$  is  $H$ -invariant implying that  $(L_{g^{-1}g'})_* T_o = T_o$  is  $G$ -invariant.

( $\Leftarrow$ )

Given a  $G$ -invariant  $T \in \Gamma(TM)$ , the evaluation map  $\text{ev}_o : \Gamma(TM) \rightarrow T_o M$  sends  $T \rightarrow T_o$  which has isotropy group  $H$ .  $\square$

In particular, a pseudo inner product on  $\mathfrak{g}/\mathfrak{h}$  gives rise to a pseudo-riemannian metric on  $M$ .

**Definition 1.1.6.** A Metric Klein geometry  $(G, H, \eta)$  is

- $(G, H)$  a Klein geometry
- $\eta$  is a pseudo inner product on  $\mathfrak{g}/\mathfrak{h}$  which is  $H$ -invariant in the sense describe in the proposition above.

Let's just recall the isometries of flat spacetime for completeness.

**Definition 1.1.7.** The Poincaré group  $\text{ISO}(d-1, 1)$  of  $d$ -dimensional Minkowski spacetime is the isotropy group that leaves the split quadratic form invariant. We also call it the *inhomogeneous special orthonormal* group as it consists of disconnected components. It consists of the semidirect product of the Lorentz group (not **necessarily** orthochronous) and the group of spacetime translations,

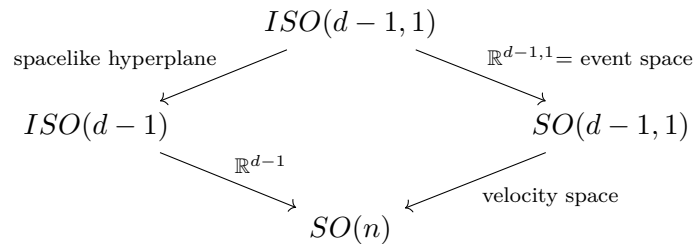
$$\text{ISO}(d-1, 1) \cong \text{SO}(d-1, 1) \ltimes \mathbb{R}^{d-1, 1} \quad (1.12)$$

with multiplication

$$(g, \alpha) \cdot (f, \beta) = (gf, \alpha + f \cdot \beta) \quad (1.13)$$

**Example 1.1.2.** Using our new-found understanding of the Poincaré group, we understand it as a homogeneous space. For example, taking  $G = \text{ISO}(d-1, 1)$  and considering the isotropy group at an origin  $o$  to be the Lorentz group  $H = \text{SO}(d-1, 1)$ . The manifold  $G/H \cong \mathbb{R}^{d-1, 1}$  is connected and at the level of algebras  $\mathfrak{g}/\mathfrak{h} \cong \mathbb{R}^{d-1, 1}$ . Further, the bilinear form on  $\mathfrak{g}/\mathfrak{h}$  is invariant under rotations ( $H$ -invariant). Therefore, the Poincaré group with its Lorentz subgroups is a Klein geometry. A similar argument can be made for Euclidean Poincaré groups. It is interesting to note however, that using the correspondence established in proposition 1.1.1, the Lorentz invariant bilinear form on  $\mathfrak{g}/\mathfrak{h}$  corresponds to Poincaré invariant tensor fields on the full Minkowski spacetime.

**Example 1.1.3.** We can summarise the ideas presented above in the following reductive diagrams, where arrows represent the subgroup that is quotiented out:

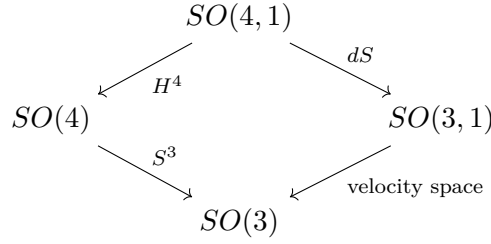


By "event space", we mean that we quotient by all possible translations of spacetime, while the "spacelike hyperplane" is quotienting by the Lorentz boosts. <sup>3</sup> In 4 spacetime dimensions, the more familiar

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<sup>3</sup>not sure here

diagram:



I don't know much about  $dS$ , so maybe someone can complete here.

## 1.2 Cartan geometry

Assuming familiarity with Maurer-Cartan form  $\omega_G \in \Omega^1(G, \mathfrak{g})$  for a Lie group  $G$ , which are the  $G$ -invariant one forms on  $G$ , we review the more general notion of connections on a principal  $G$ -bundle. For some reason, the literature often prefers *right* action on principal bundles. idk

**Definition 1.2.1.** An Ehresmann connection on a principal right  $G$ -bundle  $(P \rightarrow M)$  is a  $\mathfrak{g}$ -valued one-form  $A$  on  $P$ , that is a map

$$\omega : TP \rightarrow \mathfrak{g} \quad (1.14)$$

such that

$$\begin{aligned}
 (R_g)^* \omega &= \text{Ad}_{g^{-1}} \omega \quad \text{for all } g \in G \\
 \omega(\xi_X) &= X \quad \text{for } X \in V_p.
 \end{aligned}$$

Equivalently an Ehresmann connection is a *choice* of Horizontal distribution such that  $T_p P = V_p \oplus H_p$  where  $H_p$  is  $G$ -equivariant. This is better explained in the EKC notes from last year. We now turn our attention to the case of a principal bundle modelled on a reduced geometry.

**Definition 1.2.2.** (Cartan Geometry)

A Cartan Geometry  $(\pi : P \rightarrow M, A)$  modelled on a Klein geometry  $(G, H)$  is a principle right  $H$ -bundle  $P \rightarrow M$  with a **Cartan connection**  $A \in \Omega^1(P, \mathfrak{g})$  satisfying the conditions:

- $A_p : T_p P \rightarrow \mathfrak{g}$  is a linear **isomorphism** (or simply put a  $\mathfrak{g}$ -valued one-form for on  $P$ ).
- $(R_h)^* A = \text{Ad}_{h^{-1}} \circ A$  for all  $h \in H$
- $A(\xi_X) = X$  for  $X \in \mathfrak{h}$  and fundamental vector field  $\xi_X$  of  $H \curvearrowright P$ .

Importantly, the Cartan connection one-form takes values in the larger Lie algebra  $\mathfrak{g}$ , because the tangent space of this principal bundle is isomorphic to the larger Lie algebra  $\mathfrak{g}$ . Morally speaking, the algebra  $\mathfrak{g} \cong T_p P$ , while  $\mathfrak{h} \cong V_p$  and  $\mathfrak{m} \cong H_P \cong T_{\pi(p)} M$ .

**Remark 1.2.1.** Consequently, in a Cartan geometry we identify because of the first condition  $\dim(P) = \dim(G)$  and  $\dim(M) = \dim(G/H)$ . Moreover, the Cartan connection isomorphisms implies that upon restricting to the subalgebra  $\mathfrak{h}$  and choosing an origin

$$\begin{aligned}
 (A_o)^{-1} : \mathfrak{h} &\rightarrow \mathfrak{X}_{\text{vert}}(P) \\
 X &\mapsto \xi_X.
 \end{aligned}$$

**Remark 1.2.2.** For a Cartan geometry  $(P \rightarrow M : A)$  modelled on a *metric* Klein geometry  $(G, H)$ , the correspondence 1.1.1 implies that the isomorphism

$$T_x M \cong T_p P / \ker(\pi_*) \cong \mathfrak{g}/\mathfrak{h} \cong \mathfrak{m} \quad (1.15)$$

give a metric of the same sign to the manifold  $M$ . Where in the above  $x := \pi(p)$ . In general, the full tangent bundle of the manifold can be thought of as the associated bundle of the  $H$ -action

$$TM \cong P \times_H \mathfrak{g}/\mathfrak{h} \quad (1.16)$$

**Definition 1.2.3.** The curvature of a Cartan Geometry  $(\pi : P \rightarrow M, A)$  modelled on a Klein geometry  $(G, H)$  is

$$F(A) = dA + \frac{1}{2}[A, A] \in \Omega^2(P, \mathfrak{g}) \quad (1.17)$$

We summarise the different connections in a Cartan geometry, whether it is reductive or not in the following diagram. We denote  $e : TP \rightarrow \mathfrak{g}/\mathfrak{h} \cong TM$  as the veilbein of this geometry, while the Ehresmann connection  $\omega : TP \rightarrow \mathfrak{h}$  is denoted in green.

$$\begin{array}{ccc} TP & \xrightarrow{A} & \mathfrak{g} \\ & \searrow \omega & \nearrow \\ & & \mathfrak{h} \\ & \searrow e & \nearrow \\ & & \mathfrak{g}/\mathfrak{h} \cong \mathfrak{m} \end{array} \quad (1.18)$$

So in the reductive case

$$A = \omega + e. \quad (1.19)$$

Considering the curvature of these connections, we have the diagram

$$\begin{array}{ccc} \Lambda^2(TP) & \xrightarrow{F} & \mathfrak{g} \\ & \searrow \hat{F} & \nearrow \\ & & \mathfrak{h} \\ & \searrow T & \nearrow \\ & & \mathfrak{g}/\mathfrak{h} \cong \mathfrak{m} \end{array} \quad (1.20)$$

with the understanding that in the reductive case,

$$F = \hat{F} + T \quad (1.21)$$

where  $T$  is called the *torsion* and  $\hat{F}$  is the Ehresmann part of the curvature.

**Remark 1.2.3.** A short calculation shows that if the geometry is reductive then the Cartan curvature

$$\begin{aligned} F[A] &= dA + \frac{1}{2}[A, A]_{\mathfrak{g}} \\ &= d\omega + de + \frac{1}{2}[\omega + e, \omega + e]_{\mathfrak{g}} \\ &= \left( d\omega + \frac{1}{2}[\omega, \omega]_{\mathfrak{h}} + [\omega, e]_{\mathfrak{h}} + \frac{1}{2}[e, e]_{\mathfrak{h}} + de + \frac{1}{2}[\omega, \omega]_{\mathfrak{m}} + [\omega, e]_{\mathfrak{m}} + \frac{1}{2}[e, e]_{\mathfrak{m}} \right) \end{aligned}$$

Now because  $\omega$  is an Ehresmann connection and it reduces to a Maurer-Cartan form on the fibres we its curvature 2-form

$$d\omega + \frac{1}{2}[\omega, \omega]_{\mathfrak{h}} = \Omega(\omega).$$

If we are in the reductive case, we recall that  $[\mathfrak{h}, \mathfrak{m}] \subseteq \mathfrak{m}$ , so the bracket  $[\omega, e]_{\mathfrak{h}} = 0$ . With further restriction in the symmetric case,  $[\mathfrak{m}, \mathfrak{m}] \subseteq \mathfrak{h}$ , the bracket  $[e, e]_{\mathfrak{h}} = 0$ , but we won't assume it in general so instead define a covariant derivative on  $\mathfrak{m}$

$$d^\omega = d + \omega. \tag{1.22}$$

So in conclusion, the Ehresmann part of the curvature in a Cartan geometry is

$$\hat{F} = \Omega(\omega) + \frac{1}{2}[e, e]_{\mathfrak{h}}, \tag{1.23}$$

while the torsion is

$$T = d^\omega e + \frac{1}{2}[e, e]_{\mathfrak{m}} \tag{1.24}$$

Some interpretation of flatness and torsion freedom needed here.

# Bibliography

- [1] Derek K. Wise. MacDowell-Mansouri gravity and Cartan geometry. *Classical and Quantum Gravity*, 27(15), nov 2010. ISSN 02649381. doi: 10.1088/0264-9381/27/15/155010. URL <http://arxiv.org/abs/gr-qc/0611154>.