Lie algebroids, Poisson manifolds and Jacobi structures

Based on mini-course by Carlos Zapata-Carratalá

by

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Abstract: Notes based on a course given by Carlos Zapata-Carratala at the EMPG in 2020. A lot of the material stems from his thesis [1] for the first half of the course. Material was added over time by myself. Mistakes almost certainly mine. **WIP**

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Contents

1 Lecture 1: Poisson and Presymplectic geometry

The first lecture is mostly based on section 2.4 of [1].

1.1 Poisson Algebra

Definition 1.1.1. A **Poisson Algebra** is a triple $(A, \cdot, \{,\})$ such that

1. $(A, \cdot)$ is a commuttative, associative and unital $\mathbb{R}$-algebra (or $\mathbb{C}$ algebra maybe?)
2. $(A, \{,\})$ is a Lie $\mathbb{R}$-algebra, which means that the bracket follows the Jacobi identity:
$$\{\{a,b\},c\} + \{\{b,c\},a\} + \{\{c,a\},b\} = 0 \tag{1.1}$$
3. The Poisson bracket follows the Libeniz identity in the sense that for $a, b, c \in A$,
$$\begin{aligned} \{a, b \cdot c\} &= \{a,b\} \cdot c + b \cdot \{a,c\} && (1.2) \\ &:= \mathrm{ad}_a\,(b \cdot c) && (1.3) \end{aligned}$$
where we have defined the adjoint map of the Lie algebra.
4. Equivalently to 3, the $\mathrm{ad}_{\{,\}} : A \to \mathrm{Der}(A, \cdot)$, which takes an element of the algebra to a derivation on the commuttative algebra $(A, \cdot)$. We also see that the $\mathrm{ad}_{\{\}}$ induces a derivation on $(A, \{,\})$ using the Jacobi identity.

Therefore the adjoint map of a Lie Algebra is a morphism from A to $\mathrm{Der}(A, \cdot) \cap \mathrm{Der}(A, \{,\})$, the derivations of both bilinear structures of a Poisson algebra.

Definition 1.1.2. A **Poisson derivation** is a derivation on both bilinear forms of a Poisson algebra, that is $X \in \mathrm{Der}(A, \cdot) \cap \mathrm{Der}(A, \{,\}) \subset \mathrm{End}_{\mathbb{R}}(A)$. If a Poisson derivation is generated by the adjoint map, $X_a = \{a, \}$, we say that it is a **Hamiltonian derivation**.

Definition 1.1.3. A Poisson Algebra morphism is a linear map $\psi : A \to B$ such that $\psi : (A, \cdot) \to (B, \cdot)$ is an algebra morphism and $\psi : (A, \{,\}) \to (B, \{,\})$ is a Lie algebra morphism.

Definition 1.1.4. A subalgebra $I \subset A$ is **coisotrope** if

- $I \subset (A, \cdot)$ is a multiplicative ideal
- $I \subset (A, \{,\})$ is a Lie subalgebra

Proposition 1.1.1. *Reduction of Poisson algebra*
Suppose $I \subset A$ coisotrope and consider the Lie normaliser (or in ring theory the idealiser)

$$N(I) = \{a \in A | \{a, I\} \subset I\}, \tag{1.4}$$

which is the largest subalgebra of A that contains I as an ideal. We claim that $A' := {}^{N(I)}\!/_{I}$ inherits a Poisson algebra structure.

Proof. Condition 1 is automatically satisfied as A' is a subalgebra of A, with a Lie algebra structure given by the same bracket. For $a', b', c' \in A'$, consider the adjoint action of a' on $b' \cdot c'$ and look at coset representatirve a, b, c of $N(I)$. Using the fact that I is coisotrope, we see that

$$\begin{aligned}\{a + I, (b + I) \cdot (c + I)\} &= \{a + I, b \cdot c + I\} \\ &= \{a, b \cdot c\} + I\end{aligned}$$

by linearity of the bracket and closure of elements in $N(I)$ w.r.t I. The jacobi identity is checked by similar arguments. □

Definition 1.1.5. The *reduced Poisson structure* is characterised by the projection map $p : (N(I), \cdot, \{,\}) \to (A', \cdot', \{,\}')$, and by the above proposition, this is a Poisson Algebra morphism.

1.2 Poisson Manifolds

Definition 1.2.1. A **Poisson manifold** is a smooth manifold P whose commutative algebra of smooth functions has the structure of a Poisson algebra $(\mathrm{C}^\infty(P), \cdot, \{,\})$. Alternatively, a Poisson manifold can be defined with the help of the **Poisson Bivector** $\Pi \in \Gamma(\bigwedge^2 TP)$.

Definition 1.2.2. A map $\phi : P_1 \to P_2$ is a **Poisson map** if $\phi^* : \mathrm{C}^\infty(P_2) \to \mathrm{C}^\infty(P_1)$ is a Poisson morphism of algebras.

Definition 1.2.3. So following through definition definition 1.1.2, the Poisson derivations on a Poisson manifolds are called **Poisson vector fields**. And Hamiltonian derivations on Poisson manifolds are called **Hamiltonian vector fields**. Hamiltonian vector fields are generated by the adjoint map

$$\begin{aligned}\mathrm{ad} : \mathrm{C}^\infty(P) &\to \Gamma(TP) \\ f &\mapsto X_f := \{f, \cdot\}\end{aligned}$$

Proposition 1.2.1. A manifold P, with a commutative algebra of smooth functions $(\mathrm{C}^\infty(P), \cdot, \{,\})$, and a Poisson bivector

$$\Pi \in \Gamma(\bigwedge^2 TP) \tag{1.5}$$

$$\Pi(df, dg) = \{f, g\}; \tag{1.6}$$

is a Poisson manifold if and only if Π has vanishing Schouten bracket

$$[\![\Pi, \Pi]\!] = 0. \tag{1.7}$$

Before proving this statement, we recall facts about the Schouten-Nijenhius which forms a special case of a *Gerstenhaber algebra*. We will define Gerstenhaber algebras in definition 3.3.1.

Proof. The Schouten-Nijenhius bracket is defined as a degree -1 bracket on the differential graded algebra of alternating multivector fields, so

$$[\![\cdot, \cdot]\!] : \Gamma(\bigwedge^2 TP) \times \Gamma(\bigwedge^2 TP) \to \Gamma(\bigwedge^3 TP). \tag{1.8}$$

By considering $f, g, h \in C^\infty(P)$ with corresponding forms $\alpha, \beta, \gamma \in \Omega^1(P)$, then

$$\begin{aligned} [\![\Pi, \Pi]\!](\alpha, \beta, \gamma) &= \Pi(\Pi(\alpha, \beta), \gamma) + \text{cyclic} \\ \Leftrightarrow \qquad &= \{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} \\ \Leftrightarrow \qquad &= 0 \qquad \text{(Jacobi)} \end{aligned}$$

□

Definition 1.2.4. Given a Poisson bivector Π, the sharp *musical map* $\cdot^\sharp$ is defined as

$$\Pi^\sharp : T^*P \to TP \tag{1.9}$$

Such a map can be used to construct the set of Hamiltonian vector field on P by

$$df \mapsto \Pi(df, \cdot) := \{f, \cdot\}. \tag{1.10}$$

Furthermore, on sections we have the **Hamiltonian distribution**

$$X_\cdot = \Pi^\sharp \circ d : C^\infty(P) \to \Gamma(TP)$$

is an *Hamiltonian map.* Note that the space of Hamiltonian distribution $\Pi^\sharp(T^*P)$ is involutive as it is a Lie algebra morphism.

Maybe expend on isotropic, coisotropic and so on here.

Definition 1.2.5. A submanifold $C \subset (P, \Pi)$ is **coisotropic** if $TC \subset (TP, \Pi)$ is a coisotropic subspace, that is $TC \supset (TC)^0$ an isotropic (sometimes denoted $TC \supset TC^\perp$) subspace of TC with respect to the bivector:

$$\Pi(\alpha, \beta) = 0 \quad \forall \alpha \in (T^*C)^0, \forall \beta \in T^*C \tag{1.11}$$

Consequently, the short exact sequence:

$$0 \to (T^*C)^0 \xrightarrow{\Pi^\sharp} TC \to C^\infty(C) \to 0^1 \tag{1.12}$$

It is useful to make the connection between geometry and algebra, relating manifolds to ideals of the algebra of functions on said manifold. In [2], a nice treatise is presented in the following.

Remark 1.2.1. It is useful to study manifolds M in a 'dual' way by considering the commutative algebra of smooth functions $C^\infty(M)$. There exits as isomorphism between M and the set $\mathcal{M}$ of all maximal ideal $I(p)$ consisting of functions $f \in C^\infty(M)$ such that $f|_p = 0$.

Proposition 1.2.2. Let $\iota : C \hookrightarrow P$ be a closed submanifold of Poisson manifold (P, Π) of codimension k. A manifold given by the zero locus of $\Phi : P \to \mathbb{R}^k$. Then the following are equivalent,

- C is coisotropic
- The vanishing ideal $I_C = \ker(\iota^*) := \{g \in C^\infty(P) \,\big|\, g|_C = 0\}$ is a coisotrope of the Poisson algebra $(C^\infty(P), \cdot, \{\cdot, \cdot\})$.
- Hamiltonian vector fields X_g generated by $g \in I_C$ are tangent to C: $X_g|_C \in \Gamma(TC)$

[1] i think this is right, but not sure

We note as well that this is equivalent to saying that the normal bundle $NC = T_{P/C}$ is trivial in the short exact sequence

$$0 \to TC \to TP \to T_{P/C} \to 0 \tag{1.13}$$

Proof.

- (1) $\Rightarrow$ (2): First $(I_C, \cdot)$ is a multiplicative ideal of $(C^\infty(P), \cdot)$ by construction. Further, if $f, g \in I_C$ then $df, dg \in (T^*C)^\perp$ and the associated Poisson bracket vanishes, making I_C into a Lie subalgebra. So I_C is coisotropic to the Poisson Algebra on P.
- (2) $\Rightarrow$ (3): for a basis $g \in I_C$, the Hamiltonian vector fields $X_g = \{g, \cdot\}$ span the $\mathrm{Der}(C^\infty(C))$ which is the space of tangent vector to C.
- (3) $\Rightarrow$ (1): $\iota^*\{g, f\} = 0$ for $g \in I_C$, $\forall f \in C^\infty(P)$ [2]

$\square$

Definition 1.2.6. Consider 2 Poisson manifold (P_1, Π_1) and (P_2, Π_2), the *product Poisson manifold* is $(P_1 \times P_2, \Pi_1 + \Pi_2)$, where the canonical isomorphism $T(P_1 \times P_2) \cong \mathrm{pr}_1^* TP_1 \oplus \mathrm{pr}_2^* TP_2$.
The Whitney sum of vector bundle A_M, B_M over manifold P is defined as above by

$$A_M \boxplus B_M = \mathrm{pr}_1^* A_M \oplus \mathrm{pr}_2^* B_M \tag{1.14}$$

Also it's easy to see that pulling back onto either P_1, P_2 commutes with the bracket structure, with "cross-pulling" bracket vanishing

Definition 1.2.7. Given a Poisson manifold (P, Π), **opposite Poisson manifold** is $\overline{P} = (P, -\Pi)$.

Proposition 1.2.3. Let two Poisson manifold $(P_1, \Pi_1), (P_2, \Pi_2)$ and a smooth map $\phi : P_1 \to P_2$, then ϕ is a Poisson map if and only if

$$\mathrm{grph}(\phi) := \{(p, \phi(p)) \mid \forall p \in P_1\} \subset P_1 \times \overline{P}_2$$

is a coisotropic submanifold.

Proof. Consider the tangent bundle of the graph submanifold

$$T\mathrm{grph}(\phi) = \{(X, Y) \mid \text{if } \exists Y \in TP_2 \text{ such that } X, Y \text{are } \phi\text{-related: } \phi^* Y = \phi_* X\}.$$

Full proof in [3] but they have a weird definition of $\Pi^\sharp$ there. [3] $\square$

We now consider the important notion of coisotropic reduction. A full treatment is given in [4], but we focus on the simpler case where the involutive distribution on $C \hookrightarrow P$ is given by the sets of its Hamiltonian vector fields.

Proposition 1.2.4. (Coisotropic Reduction of Poisson manifold)
Let (P, Π) a Poisson manifold, and $\iota : C \hookrightarrow P$ a closed coisotropic submanifold. Let $\{X_{I_C}\}$ be the set of Hamiltonian vector field tangent to C generated by the ideal $I_C = \ker(\iota^*)$. This integrates to a regular foliation χ_C on C because of the involution

$$[X_{I_C}, X_{I_C}] \subset X_{\{I_C, I_C\}} \subset X_{I_C}. \tag{1.15}$$

[2] continue later
[3] continue one day

Further assume that the leaf space is smooth $P' := {}^{C}\!/\!_{\chi_C}$ such that there is a submersion[4](quotient map) q fitting in the reductive diagram

$$\begin{tikzcd} C \arrow[r, hook, "\iota"] \arrow[d, two heads, "q"] & (P,\Pi) \\ (P',\Pi') & \end{tikzcd} \tag{1.16}$$

Then P' inherits a Poisson structure on functions $(C^\infty(P'), \{\cdot,\cdot\}')$ that is uniquely determined by the condition

$$\iota^*\{F,G\} = q^*\{f,g\}' \tag{1.17}$$

for all $f,g \in C^\infty(P')$ and $F,G \in C^\infty(P)$ such that F,G are the leaf-wise constant extensions of f,g, i.e

$$\begin{aligned} q^*f &= \iota^*F \\ q^*g &= \iota^*g \end{aligned}$$

Proof. Following [4], theorem 2.2. As opposed to the theorem in the paper, we consider the trivial case where the involutive distribution is given by the set of Hamiltonian vector fields X_{I_C}. It's clear from proposition 1.2.2 that on C, the distribution of Hamiltonian vector field vanishes: $X_{I_C} = 0 \subset TC$. $\square$

The upshot is, we have identified a Poisson submanifold (P',Π') with a reduced Poisson structure, all this is because of the coisotropic datum given.

Remark 1.2.2. Let $\tilde{P} \subset P$ a Poisson submanifold, then $\Pi^\sharp(T^*\tilde{P}) = X_{I_{\tilde{P}}} = 0$ is equivalent to $\tilde{P} \hookrightarrow P$ is a Poisson morphism.

An important concept to grasp is the notion of **reduction**, in this case Poisson reduction. Consider a Lie group action $G \circlearrowright (P,\Pi)$ via the Poisson map $G \times P \to P$ that is also a morphism of Poisson algebras. Infinitesimally, this is the action of $\mathfrak{g} = \mathrm{Lie}(G)$ on P by the Lie algebra-valued Hamiltonian vector fields given by the map ψ and the comoment map $\overline{\mu}$ defined as

$$\begin{aligned} \psi &: \mathfrak{g} \to \Gamma(TP) \\ \overline{\mu} &: \mathfrak{g} \to C^\infty(P) \end{aligned}$$

such that, for all $\xi, \zeta \in \mathfrak{g}$

$$\psi(\xi) = X_{\overline{\mu}(\xi)} := \{\overline{\mu}(\xi), \cdot\} \tag{1.20}$$

$$\overline{\mu}([\xi,\zeta]) = \{\overline{\mu}(\xi), \overline{\mu}(\zeta)\} \tag{1.21}$$

[4] Recall that a *submersion* is a differential map $\phi : M \to N$ such that

$$D\phi_p : T_pM \twoheadrightarrow T_{\phi_p}N \tag{1.18}$$

for all $p \in M$. Dually, an *immersion* is a differential map $\phi : M \to N$ such that

$$D\phi_p : T_pM \hookrightarrow T_{\phi_p}M \tag{1.19}$$

for all $p \in M$.

The map $\overline{\mu}$ is called the **comoment map** and it is a Lie algebra morphism between the Lie algebra $\mathfrak{g}$ and the Poisson algebra. Dually, we define the **moment map** such that the following diagram commutes

$$\begin{tikzcd} g \arrow[r, "\overline{\mu}"] & C^\infty(P) \\ g^* \arrow[u, "\mathrm{Hom}"] & P \arrow[l, "\mu"] \arrow[u] \end{tikzcd} \tag{1.22}$$

or by the relation

$$\langle \mu(p), \xi \rangle = X_{\overline{\mu}(\xi)}(p) \tag{1.23}$$

for $p \in P$ and $\xi \in \mathfrak{g}$.
If $0 \in \mathfrak{g}^*$ is a regular value, then $C := \ker(\mu) \subset P$ is a coisotropic submanifold because the equation above defines a tangent distribution $X_{\overline{\mu}(\xi)} = 0$ for all $\xi \in \mathfrak{g}$ (see proposition 1.2.2). We say that C is the zero locus of an *equivariant moment map*. An important consequence is that if the Poisson action is free and proper, then $\ker(\mu)/G$ is the coisotropic reduction of (P, Π), and in this case this is an *Hamiltonian reduction*.

Add example on 2d chern simons from geometry of gauge fields here

Question: If the quotient space is not a manifold, can we still construct a groupoid action and reduction?

1.3 Presymplectic manifold

Definition 1.3.1. A **presymplectic manifold** (S, ω) is a smooth manifold M and a closed $2-$form $\omega \in \Omega^2(S) \cong \Gamma(\bigwedge^2 T^*S)$ with $d\omega = 0$. We also say that S carries a symplectic structure ω.
If the symplectic structure ω is exact, we say that S has an exact presymplectic structure.

Definition 1.3.2. A smooth map $\phi : S_1 \to S_2$ between presymplectic manifold is a *presymplectic map* if

$$\phi^*\omega_2 = \omega_1.$$

Definition 1.3.3. A diffeomorphism $\phi : S_1 \to S_2$ that is also a presymplectic is called a **symplectomorphism**. Infinitesimally, this is generated by vector fields $X \in \Gamma(TS_1)$ that respect the symplectic form on S_1,

$$\mathcal{L}_X \omega_1 = 0. \tag{1.24}$$

In this case, we say that $X \in \Gamma(TS_1)$ is a *symplectic vector field*. Notice that a symplectic vector field $X \in \Gamma(TS)$ has by Cartan's magic formula

$$d\omega^\flat(X) = 0 \tag{1.25}$$

Definition 1.3.4. Given a presymplectic manifold (S, ω), we define the flat *musical map* $\cdot^\flat$ as

$$\begin{aligned} \omega^\flat &: TS \to T^*S \\ X &\mapsto \omega(X, \cdot) := \iota_X \omega. \end{aligned}$$

The kernel of this map is called its characteristic distribution $\ker(\omega^\flat) \subset TS$. A standard calculation in (pre)symplectic geometry is to show that vector field in the characteristic distribution are in involution.

For $X, Y \in \ker(\omega^\flat)$, $Z \in TS$,

$$\begin{aligned} d\omega(X, Y, Z) &= 0 \\ X(\omega(Y, Z)) + Y(\omega(Z, X)) + Z(\omega(X, Y)) & \\ -\omega([X, Y], Z) - \omega([Y, Z], X) - \omega([Z, X], Y) &= 0, \end{aligned}$$

which implies $\omega([X, Y], Z) = 0$ for all $Z \in TS$, or in other words $[X, Y] \in \ker(\omega^\flat)$ is involutive.

Unlike Poisson manifold, we can't always cook up Hamiltonian distributions on presymplectic manifolds. We can however do so on a subset of $C^\infty(S)$.

Definition 1.3.5. A vector field $\zeta \in \Gamma(TS)$ on a presymplectic manifold (S, ω) is said to be **Hamiltonian** if there exists some $f \in C^\infty(S)$ such that

$$\omega^\flat(\zeta) = -df. \tag{1.26}$$

The notation χ_f for associated Hamiltonian vector field is sometimes used. The set of *Admissible functions* on (S, ω) is the subring of $C^\infty(S)$ such that there exists associated Hamiltonian vector fields, i.e:

$$C^\infty(S)_\omega = \{f \in C^\infty(S) |\, \exists\, \chi_f \in \Gamma(TS) \text{ is Hamiltonian}\} \tag{1.27}$$

We sign convention chosen will be made clear in the following proposition.

Lemma 1.3.1. The Lie bracket of 2 Symplectic vector field is Hamiltonian, for $\zeta, \xi \in \Gamma(TS)$

$$[\zeta, \xi] = \chi_{\omega(\zeta,\xi)}. \tag{1.28}$$

To prove this, we use Cartan's second fomula $\iota_{[X,Y]} = [\mathcal{L}_X, \iota_Y]$ for $X, Y \in \Gamma(TS)$, then

$$\begin{aligned} \iota_{[\zeta,\xi]}\omega &= [\mathcal{L}_\zeta, \iota_\xi]\omega \\ &= -d\left(\omega(\zeta, \xi)\right) \\ &= \iota_{\chi_{\omega(\zeta,\xi)}}. \end{aligned}$$

We identify the Hamiltonian function tabove o be $\omega(\zeta, \xi)$ and so eq. (1.28) is verified.

Proposition 1.3.1. The subring of admissible function $C^\infty(S)$ on a presymplectic manifold forms a Poisson Algebra (definition 1.1.1) with

$$\{f, g\} = \omega(\chi_f, \chi_g). \tag{1.29}$$

The sign convention is such that $\omega(\chi_f, \chi_g) = dg(\chi_f) = -df(\chi_g)$

Proof. We check that this defines a derivation on admissible functions,

$$\omega(\chi_f, \chi_{gh}) = \omega(\chi_f, \chi_g) \cdot h + g \cdot \omega(\chi_f, \chi_h). \tag{1.30}$$

To check the Jacobi identity, we consider Hamiltonian vector field (which are symplectic) and using vector fields lemma 1.3.1, denote

$$[\chi_f, \chi_g] = \chi_{\omega(\chi_f,\chi_g)} = \chi_{\{f,g\}}. \tag{1.31}$$

Since this forms a closed subalgebra, the Jacobi identity is satisfied and the proposition is proven. Can also show directly the Jacobi identity by considering $d\omega(\chi_f, \chi_g, \chi_h) = 0$. ☐

Definition 1.3.6. A submanifold $C \hookrightarrow S$ of a presymplectic manifold is **isotropic** the presymplectic form is zero on C, ie:

$$i^*\omega = 0 \tag{1.32}$$

Let's now have a look at the reduction of presymplectic manifolds. Consider a Lie group action on presymplectic manifold (S, ω) that preserves the form.

$$\phi : G \times S \to S \tag{1.33}$$

$$\phi_g^*\omega = \omega \quad \forall g \in G \tag{1.34}$$

1.4 Symplectic manifold

Definition 1.4.1. A **Symplectic manifold** (S, ω) is a smooth manifold M with a closed *non-degenerate* 2-form $\omega \in \Omega^2(S)$.

Definition 1.4.2. Symplectic vf and short exact sequence..

$$0 \to \mathfrak{X}_{\mathrm{Ham}}(S) \to \mathfrak{X}_{\mathrm{Symp}}(S) \to \mathrm{H}^1_{\mathrm{dR}}(S, \mathbb{R}) \to 0 \tag{1.35}$$

Definition 1.4.3.

Definition 1.4.4.

Definition 1.4.5.

2 Lie Groupoids

2.1 Definition and structure maps

Some notation will use some category theory, so we recall here that a small category is a category where objects and morphisms are *small*. This means that they they are small enough to fit in the category of **Set**.

Definition 2.1.1. A Groupoid $\mathcal{G} :- (G_1 \rightrightarrows G_0)$ is a *small category* with morphisms that are all invertible. Let's unpack this definition (or parts of it for now):

- G_0 is the set of objects
- G_1 is the set of morphisms
- An element $g \in G_1$ is denoted, given a pair of functions called the *source* and *target* $s, t : G_1 \rightrightarrows G_0$ such that $s(g) \to t(g)$. Technically, this defines the set of composable arrows $G_1 \,{}_s\!\times_t G_1$.

2.2 Bisections

2.3 Lie Groupoids

2.4 Examples of Lie Groupoids

2.5 Morphisms of Lie Groupoids

2.6 Vector fields on Lie Groupoids

2.7 Action Lie Groupoid

We follow [5] for a brief but useful description of the action of Lie groupoids on manifolds. We recall that the orbit space M/G of a group action $\rho : G \times M \to M$ by a Lie group G can be badly behaved (eg: not free, so singular quotient space). One way out of this problem is to consider the action of a Lie groupoid.

Definition 2.7.1. The action Lie groupoid of a group G on manifold M is the category $G \ltimes M \rightrightarrows M$ (sometimes denote $M//G$) or $M//_\rho G$ for map ρ an automorphism on the manifold). For $g \in G$ and $x \in M$, the map

$$x \mapsto g \triangleright x$$

There is a good review about this topic on blog entry. Talk about functor $M//G \rightrightarrows G$ that is faithful.

3 Lie Algebroids

A good reference to add to the mix is [6].

3.1 Vector Bundles

Let's first review facts about vector bundles, but with a more 'categorical' mindset.

Definition 3.1.1. A **vector bundle** $\pi : E \to M$ over a smooth manifold M is a fibre bundle whose fibre E_x is a vector space $V \in \textbf{Vect}\ \forall x \in M$. The dimension of the typical fibre E_M is called the rank and $\dim(E_M) := \dim(E_x)$ for all x.

Definition 3.1.2. A *local trivialisation* is a map φ such that on $U \subset M$ open, $\varphi : \pi^{-1}(U) \to U \times V$ is a diffeomorphism.
On overlaps $U_1 \cap U_2 \subset M$, local trivialisations define $\mathrm{GL}(V)$-valued *transition functions.*

Definition 3.1.3. A smooth map F between 2 vector bundles $F : E_1 \to E_2$ is a bundle **vector bundle morphism** if there exists smooth map $\phi \in C^\infty(M_1, M_2)$ between the bases such that

$$\begin{tikzcd} E_1 \arrow[r, "F"] \arrow[d, "\pi_1"] & E_2 \arrow[d, "\pi_2"] \\ M_1 \arrow[r, "\phi"] & M_2 \end{tikzcd} \tag{3.1}$$

commutes. We say that F is a *covering* for ϕ. Equivalently, F restricts to a linear map on the fibre $F_x : (E_1)_x \to (E_2)_{\phi(x)}$.

Definition 3.1.4. Vector bundles over smooth manifolds with vector bundle morphism forms the **category of vector bundles** denoted $\textbf{Vect}_{\mathrm{Man}}$.

Remark 3.1.1 (Categorification)**.** Fixing a base manifold M, the point-wise construction of fibre bundles over M restricts us to the subcategory of vector bundles over M denoted $\textbf{Vect}_M$. So the point-wise construction of vector bundle over base manifold M forms a **abelian symmetric monoidal category**. [5]

Definition 3.1.5. Given vector bundle E_1, E_2 over manifold M,

- the vector bundle direct sum called the *Whitney sum* is the fiberwise direct sum $E_1 \boxplus E_2$ (as seen in definition 1.2.6),
- the vector bundle tensor product is the fiberwise tensor product $(E_1)_x \otimes (E_2)_x$ for all $x \in M$,
- and the dual vector bundle is given by $E^* = \mathrm{Hom}(E, \mathbb{R})$ (which is a functor by the way).

Definition 3.1.6. Given a vector bundle $E \xrightarrow{\pi} M$, and a smooth map $\phi : N \to M$, **pullback vector bundle** by ϕ is defined as the categorical pull-back $\phi^* E = N \ {}_\phi\!\times_\pi \ E$ such that the diagram commutes:

$$\begin{tikzcd} N \ {}_\phi\!\times_\pi \ E \arrow[r] \arrow[d] & E \arrow[d, "\pi"] \\ N \arrow[r, "\phi"] & M \end{tikzcd} \tag{3.2}$$

which is simply $\phi^* E = \{(p, e) \in N \times E \mid \phi(p) = \pi(e)\}$.

[5] One day, try to understand category stuff

Definition 3.1.7. A **section** s of vector bundle $E \xrightarrow{\pi} M$ is a map $s : M \to E$ such that $\pi \circ s = \mathrm{id}_M$. The set of all sections of E forms a $C^\infty(M)$-module and is denoted

$$\Gamma(E) = \{s : M \to E \mid \pi \circ s = \mathrm{id}_M\}. \tag{3.3}$$

On an open set $U \subset M$, a basis of sections $\{e_i : U \to \pi^{-1}(U) | \pi \circ e_i = \mathrm{id}_U\}_{i=1}^{\mathrm{rk}(E)}$ defines a local trivialisation (definition 3.1.2). If such sections are globally defined, the bundle is trivial (or trivialisable).

Remark 3.1.2. We would naturally guess that the functor C^∞ : **Man** $\to$ **Ring** would similarly extend to sections on manifold. But the assignment $\Gamma : \mathbf{Vect}_{\mathrm{Man}} \to R\text{-}\mathbf{Mod}$ fails to be a functor. To see this, consider the vector bundle morphism $F : E_1 \to E_2$ covering $\phi : M \to N$, and the pullback bundle along ϕ:

$$\begin{tikzcd}
E_1 \arrow[r, "F"] \arrow[d, "\pi_1"'] & \phi^* E_2 \arrow[r, "\mathrm{id}_{E_2}"] \arrow[d, "\mathrm{pr}_2"'] & E_2 \arrow[d, "\pi_2"'] \\
M \arrow[r, "\mathrm{id}_M"] \arrow[u, dashed, bend left, "s_1"] & M \arrow[r, "\phi"] \arrow[u, bend right, "\phi^* s_2"', color=red] & N \arrow[u, dashed, bend right, "s_2"']
\end{tikzcd} \tag{3.4}$$

For sections on the vector bundle, we have the maps

$$\begin{gathered}
\Gamma(E_1) \xrightarrow{F} \Gamma(\phi^* E_2) \xleftarrow{\phi^*} \Gamma(E_2) \\
s_1 \to F \circ s_1 \;;\; s_2 \circ \phi \leftarrow s_2
\end{gathered}$$

If ϕ is **not** a diffeomorphism, then one *cannot* in general construct such maps. However, when maps agree

$$F \circ s_1 = s_2 \circ \phi \qquad \Leftrightarrow \qquad s_2 \sim_F s_1 \tag{3.5}$$

we say that the sections are **F-related**.

Definition 3.1.8. If $\phi : M \to N$ **is** a diffeomorphism, and $F : E_1 \to E_2$ a covering of vector bundle then the **pushforward** of sections

$$F_* : \Gamma(E_1) \to \Gamma(E_2) \tag{3.6}$$

$$s_1 \mapsto F \circ s_1 \circ \phi^{-1} \tag{3.7}$$

is well-defined. The pushforward satisfies

$$F_*(s + r) = F_*(s) + F_*(r) \tag{3.8}$$

$$F_*(f \cdot s) = (\phi^{-1})^* f \cdot F_*(s) \tag{3.9}$$

for $s, r \in \Gamma(E_1)$ and $f \in C^\infty(M)$.

Remark 3.1.3. Note that you can *always* pushforward tangent vectors via the differential map $\phi_* : TM \to TN$, but to pushforward vector fields, you need φ to be a diffeomorphism.

Definition 3.1.9. In contrast, ϕ need not be a diffeomorphism for the **pullback** of dual sections (aka: forms) to be well-defined.

$$\begin{aligned}
F : \Gamma(E_2^*) &\to \Gamma(E_1^*) \\
\alpha &\mapsto F^* \alpha = \alpha \circ \phi
\end{aligned}$$

So for any sections $s_1 \in \Gamma(E_1)$,

$$F^* \alpha(s_1) = (\alpha \circ \phi)\,(F \circ s_1) \tag{3.10}$$

since $F(s_1) \in \Gamma(\phi^* E_2)$. [6]

Definition 3.1.10. The pullback naturally extends to all tensor powers of the dual bundles $\bigotimes^k E_2^*$, such that for $\eta, \omega \in \Gamma(\bigotimes^\bullet E_2^*)$, and function $f \in \Gamma(\bigotimes^0 E_2) \cong C^\infty(N)$, we have

$$\begin{aligned} F^*(\eta + \omega) &= F^*\eta + F^*\omega \\ F^*(\eta \otimes \omega) &= F^*\eta \otimes F^*\omega \\ F^*(f) &= f \circ \phi = \phi^* f. \end{aligned}$$

Definition 3.1.11. Given vector bundle morphisms $F : E_1 \to E_2$, $G : E_2 \to E_3$, we see that on the tensor powers of dual bundles we have the contravariant identity:

$$(G \circ F)^* = F^* \circ G^*. \tag{3.11}$$

This motivates the definition of the *contravariant* **tensor functor** to the category of associative unital algebras

$$\begin{aligned} \mathcal{T} : \mathbf{Vect}_{Man} &\to \mathbf{AssAlg} \\ E &\mapsto \Gamma\left(\bigotimes{}^\bullet E^*\right) \\ F &\mapsto F^* \end{aligned}$$

Look into spanning functions and restrictions of C^∞ functor.

Definition 3.1.12. Let $E \to M$ be a smooth vector bundle, a **connection** is a vector bundle morphism $\nabla : E \to DE \subset \mathrm{End}(E)$ to its derivation bundle covering the identity. [7] A **Koszul connection** is a $\mathbb{R}$-linear map

$$\begin{aligned} \nabla : \mathfrak{X}(M) \times \Gamma(E) &\to \Gamma(E) \\ (X, s) &\mapsto \nabla_X s \end{aligned}$$

such that the Leibniz identity is satisfied.

$$\nabla(f \cdot s) = f \nabla s + df \cdot s \tag{3.12}$$

Definition 3.1.13. Given a vector bundle $E \to M$, we construct the **Koszul complex** on sections of the vector bundle. We define a boundary map $\delta : \Gamma(\bigwedge^q E) \to \Gamma(\bigwedge^{q-1} E)$ such that

$$\delta(e_1 \wedge \ldots \wedge e_q) = \sum_{i=1}^{q} (-)^{i+1} \rho(e_i)(e_1 \wedge \ldots \hat{e_i} \wedge \ldots \wedge e_q) \tag{3.13}$$

for $e_i \in \Gamma(E)$ and $\rho \in \Gamma(E^*)$. This map enjoys the property $\delta^2 = 0$, from which we can take the *homology* of this complex.

Definition 3.1.14. Let $E \to M$ be a vector bundle, the space of E**-valued** p**-forms** is denoted

$$\Omega^p(M; E) = \Gamma(E \otimes \bigwedge^p T^* M). \tag{3.14}$$

[6] In the case of cotangeant bundle, I should expand on the fact that ϕ needs to be diffeomorphism to define pullbacks. See notes on int systems

[7] Derivation bundle will be studied more in 4.1

Given a Koszul connection ∇ on E, there is a unique way to extend ∇ to an **exterior covariant derivative** d^∇, mapping

$$d^\nabla : \Omega^p(M;E) \to \Omega^{p+1}(M;E). \tag{3.15}$$

If the connection d^∇ is flat, then this forms the *Koszul complex* with $(d^\nabla)^2 = 0$.
We extend linearly from

$$\begin{aligned} d^\nabla(\omega \otimes s) &= d\omega \otimes s + (-)^{|\omega|}\omega \otimes d^\nabla s \\ (d^\nabla s)X &= \iota_X ds \end{aligned}$$

for $s \in \Gamma(E), \omega \in \Omega^p(M;E), X \in TM$.

3.2 Definition and Examples

Definition 3.2.1. A **Derivative Lie algebra** is a vector bundle $\alpha : A \to M$ whose $\mathbb{R}$-linear module of sections $(\Gamma(A), [\cdot,\cdot])$ acts as a derivation on each of its arguments. That is

$$\mathrm{ad}_{[\cdot,\cdot]} : \Gamma(A) \to \mathrm{Der}(A)$$

is well-defined, which in this case means that the derivation property respects the $C^\infty(M)$-module structure.

Example 3.2.1. A Poisson algebra is a derivative Lie algebra.

Definition 3.2.2. A **Lie Algebroid** $\{A \to M; \rho : A \to TM; [\cdot,\cdot]\}$ over smooth manifold M is a vector bundle $A \to M$ together with a derivative Lie algebra structure on $(\Gamma(A), [\cdot,\cdot])$ and a vector bundle morphism $\rho : A \to TM$ called the *anchor map*.
The anchor map induces a Lie algebra homomorphism between modules of sections $\rho_* : \Gamma(A) \to \Gamma(TM)$ by the *symbol-squiggle theorem* which we will see in 6.1.1 with

$$\rho_*[s_1, s_2]_A = [\rho(s_1), \rho(s_2)]_{TM} \tag{3.16}$$

Being a *derivative Lie algebra* means that for $s_1, s_2, s_3 \in \Gamma(A)$ and $f \in C^\infty(M)$, the following equations are satified:

$$[s_1, f \cdot s_2]_A = f \cdot [s_1, s_2]_A + \rho_*(s_1)[f] \cdot s_2 \tag{3.17}$$

$$[s_1, [s_2, s_3]_A]_A = [[s_1, s_2]_A, s_3]_A + [s_2, [s_1, s_3]_A]_A \tag{3.18}$$

Locally, if the algebroid structure is given by structure functions $C_{ab}{}^c(x) \in \Gamma(\Lambda^2 A^* \otimes A)$ in a chart U_α with coordinates x^i, we deduce from the above a Jacobi type identity:

$$C_{[ab}{}^e C_{c]e}{}^d + \frac{\partial}{\partial x^i} C_{[ab}{}^d \rho^i{}_{c]} = 0, \tag{3.19}$$

where we have put $\rho^i_\star(\xi_a) = \rho^i{}_a$ is the components of vector field associated to a local trivialisation when anchored down to the tangent bundle. Unlike Leibniz algebras (definition 9.2.1), the bracket over sections is skew-symmetric.

Remark 3.2.1. In general, vector bundle morphism **DO NOT** induce well-defined maps between modules of sections. This is discussed further in [6] by introducing the notion of double pullback algebroid and probably needs to be understood further here.

> could this be related to remark 3.1.2 ?

Remark 3.2.2. The anchor map naturally defines two distributions $\ker(\rho) \subset A$ and $\mathrm{im}(\rho) \subset TM$. Now fiberwise, the vector spaces $\mathfrak{g}_x := \ker(\rho_x) \subset A_x$ form a Lie algebra with bracket extending on the fibres A_x. We call $(\mathfrak{g}_x, [\cdot,\cdot]_{\Gamma(A)}|_x)$ the **isotropy Lie algebras** of Lie algebroid A.
SInce ρ is a Lie algebra morphism, the image distribution $\rho(A)$ is involutive and it will be integrable by a singular foliation on M. We call the image distribution the **characteristic distribution** of the Lie algebroid A.

Example 3.2.2.

- A natural first example is the vector bundle $\mathfrak{g} \to \star$, with $\mathfrak{g}$ having a Lie algebra structure. This is a trivial Lie algebroid.
- The tangent bundle $TM \to M$ is a Lie algebroid with projection anchor map and the vector field forming a derivative algebra.
- Likewise, involutive distributions $D \hookrightarrow TM$ with trivial anchor form a sub Lie algebroid.
- The vector bundle $\mathbb{R} \to M$, with derivative Lie algebra being the commutative algebra of smooth functions over M. Then for $h \in C^\infty(M)$ and associated vector field $X_h \in \Gamma(TM)$, the anchor map is given by
$$\rho_X : f \mapsto f \cdot X$$
and the bracket is
$$[f, g]_X = fX(g) - gX(f). \tag{3.20}$$
We've just reinterpreted the regular Lie algebra of functions over a manifold as a Lie algebroid.
- The Atiyah algebroids A_P of a principal G-bundle $\pi : P \to M$, as a vector bundle appears in the sequence
$$0 \to \mathrm{ad}(P) \to A_P \to TM \to 0,$$
where the adjoint bundle $\mathrm{ad}(P) \cong P \times_G \mathfrak{g}$ of Lie algebra $\mathfrak{g}$ is the associated bundle of P on $\mathfrak{g}$ by G. This is constructed from the short exact sequence
$$0 \to VP \to TP \xrightarrow{\pi_*} TM \to 0, \tag{3.21}$$
where VP is the *vertical bundle*, the kernel of the differential surjective map. Now the vertical bundle is isomorphic to the trivial bundle $VP \cong P \otimes \mathfrak{g}$. Now since P is a principal bundle, G acts on this short exact sequence yielding the Atiyah sequence as G acts as the adjoint map on the vertical bundle.
- The bundle of derivations $DE \to M$ for a vector bundle $E \to M$ is a Lie algebroid.

3.3 Algebraic structures associated with Lie algebroid

Let's first discuss a notion that combines the structure of $\mathbb{Z}$-graded Lie superalgebras and supercommutative rings. Attention not to confuse with regular supersymmetry and $\mathbb{Z}_2$-grading for Poisson superalgebras for example.

Definition 3.3.1. A **Gerstenhaber algebra** $A^\bullet$ is a graded commutative algebra with a Lie bracket $[\![\cdot,\cdot]\!]$ of degree -1 that satisfies the *Poisson identity* and a multiplication $\cdot$ in the supercommutative associative ring. For $a, b, c \in A^\bullet$, we denote $|a|$ the degree of a and we have the following

- $|ab| = |a| + |b|$
- $|[\![a, b]\!]| = |a| + |b| - 1$
- $ab = (-1)^{|a||b|} ba$
- $[\![a, bc]\!] = [\![a, b]\!]c + (-1)^{(|a|-1)|b|} b[\![a, c]\!]$
- $[\![a, b]\!] = -(-1)^{(|a|-1)(|b|-1)} [\![b, a]\!]$
- $[\![a, [\![b, c]\!]]\!] = [\![[\![a, b]\!], c]\!] + (-1)^{(|a|-1)(|b|-1)} [\![b, [\![a, c]\!]]\!]$

Given a Lie algebroid $(A, \rho, [,])$ over M, we can *uniquely* extend the bracket to a Gerstenhaber bracket on the graded algebra of multisections.

Definition 3.3.2. There exists a Gerstenhaber algebra $(\Gamma(\bigwedge^\bullet A), \wedge, [\![\cdot, \cdot]\!])$ such that, for $a, b \in \Gamma(A)$ and $f, g \in C^\infty(M)$

- $[\![a, b]\!] = [a, b]$
- $[\![a, f]\!] = \rho_* a[f]$
- $[\![f, g]\!] = 0$

Definition 3.3.3. We can construct a Lie algebroid counterpart to exterior calculus of the differential geometry of tangent bundles. This is a dual construction to the Gerstenhaber algebra seen above. Given a Lie algebroid $A \to M$, we construct a differential graded algebra $\Omega(A) = \bigoplus \Omega^k(A)$ of the sections $\Gamma(\bigwedge^\bullet A^*)$. The map

$$d_A : \Omega^k(A) \to \Omega^{k+1}(A)$$

is called the *differential* and explicitly, acting on homogeneous element $\omega \in \Gamma(\bigwedge^k A^*)$ we have:

$$d_A\omega(a_0, \dots, a_k) = \sum_{i<j} (-1)^{i+j-1} \omega([a_i, a_j], a_1, \dots, \hat{a}_i, \dots \hat{a}_j \dots, a_k) + \sum_{i=0}^{k} (-1)^{i-1} \rho_* a_i[\omega(a_0, \dots \hat{a}_i, \dots a_k)], \tag{3.22}$$

for $a_i \in \Gamma(A)$. This differential graded algebra $\Omega^\bullet(A) = (\Gamma(\bigwedge^\bullet A^*), \wedge, d_A)$ is called the **exterior algebra** of the Lie algebroid A.

As usual we find $d_A^2 = 0$ which forms a *de Rham* complex and its cohomology is the **Lie algebroid cohomology with trivial coefficients**.

Definition 3.3.4. Just like a regular exterior algebra, there are natural notions of *interior product* and *Lie derivatives*, defined for $w \in \Gamma(\bigwedge^k A^*)$ and $a, a_0, \dots, a_{k-1} \in \Gamma(A)$ as

$$\iota_a\omega(a_0, \dots, a_{k-1}) = \omega(a, a_0, \dots, a_{k-1})$$
$$\mathcal{L}_a\omega(a_0, \dots, a_{k_1}) = \sum_{i=0}^{k} \omega(\dots, a_{i-1}, [a, a_i], \dots, a_{k-1}) - \rho_* a\omega(a_0, \dots, a_{k-1}),$$

with extension by linearity.

Fact 3.3.1. These follow the usual Cartan calculus identity:

$$\mathcal{L}_a = \iota_a \circ d_A + d_A \circ \iota_a \tag{3.23}$$
$$[\mathcal{L}_a, \mathcal{L}_b] = \mathcal{L}_{[a,b]} \tag{3.24}$$
$$[\mathcal{L}_a, \iota_b] = \iota_{[a,b]} \tag{3.25}$$
$$[\iota_a, \iota_b] = 0 \tag{3.26}$$
$$[\mathcal{L}_a, d] = 0, \tag{3.27}$$

where we consider the commuttators on the graded algebra, so be carefull. So the operators $\iota, \mathcal{L}, d$ have respectively degrees $-1, 0, 1$.

Proposition 3.3.1. There is a 1-1 correspondence between Lie algebroids and *Linear Poisson structures*.

Maybe talk about splitting and covariant derivatives?

3.4 Morphisms of Lie algebroids

We will talk about the notion of local Lie algebras later, but in the case of Lie algebroid, the extra structure of a vector bundle constrains morphisms between these algebraic structures. We start by considering a general vector bundle morphism $F : A \to B$, and demand that this map induces a Lie algebra morphism of the modules of sections, which is not always guaranteed.

3.5 Connections on Lie algebroids

A good ref is [7]. Can view the concept of a Lie algebroid as a tool to transfer the differential geometry of tangent bundles to abstract vector bundles.

Definition 3.5.1. A *A-connection* ${}^A\nabla$ on a Lie algebroid $A \to M$ is an anchor compatible *Kozul connection* on the vector bundle $A \to M$. [8] So it is a bundle morphism $\nabla : A \to DA$ covering the identity[8]. Simply put, this is a $C^\infty(M)$-module with values in regular differential operators of order 1 on M

$$\begin{aligned} {}^A\nabla : \Gamma(A) \times \Gamma(TM) &\to \Gamma(TM) \\ (s, X) &\mapsto {}^A\nabla_s X \end{aligned}$$

satisfying the Leibniz identity

$${}^A\nabla_s(f \cdot X) = \rho_*(s)(f)X + f\ {}^A\nabla_s X \tag{3.28}$$

The compatibility with the anchor is integrated in the Leibniz identity, this is really the only equation we can write down. Below we explore other connections that can be induced on the algebroid itself and the tangent bundle. In some sense the A-connection intertwines between connections on an abstract vector bundle and a regular tangent bundle.

Definition 3.5.2. The **curvature** $R_\nabla \in \Omega^2(A)$ of A-connection ∇ of a Lie algebroid $A \to M$

$$\begin{aligned} R_\nabla : \Gamma(A) \times \Gamma(A) &\to \mathrm{End}(\Gamma(TM)) \\ R_\nabla(a, b) &= [\nabla_a, \nabla_b] - \nabla_{[a,b]} \end{aligned}$$

[8] Some subtleties I don't get here yet.

for $a, b \in \Gamma(A)$. Alternatively, given $X \in \Gamma(TM)$, the curvature is a 2-form $R_\nabla \in \Omega(A)$. If the curvature vanishes, we say that the connection is *flat* and we call (A, ∇) a **representation** of the Lie algebroid over itself. This is because this definition can be extended where $A \to M$ a Lie algebroid and $E \to M$ a vector bundle and looking at morphisms between them. Fortunately we only need $A = E$.

You can play the same game with $\Omega^\bullet_E(A) = (\Gamma(\bigwedge^\bullet A^* \otimes E), \wedge, d_A^E)$ which will lead to *Lie algebroid cohomology with coefficient in E.* This maybe what I need for my problem.

3.6 Lie integration of Algebroids

4 Lecture 4: Differential Operators I

4.1 Derivations

Let A be a associative commutative unital $\mathbb{C}$-algebra, a vector space over $\mathbb{C}$ such that for any pair $a, b \in A$, the product $ab \in A$ is bilinear and associative.

Definition 4.1.1. A **derivation** $\partial \in \mathrm{Der}_{\mathbb{C}}(A)$ is a $\mathbb{C}$-linear map $\partial : A \to A$ such that the Leibniz identity is satisfied,

$$\partial(ab) = \partial(a)b + a\partial(b) \tag{4.1}$$

for $a, b \in A$. Clearly, $\mathrm{Der}_{\mathbb{C}}(A) \subseteq \mathrm{End}_{\mathbb{C}}(A)$

Definition 4.1.2. More generally, if B is a commutative ring, A is a B-algebra and M an A-bimodule then $\mathrm{Der}_B(A, M) = \{\partial \in \mathrm{Hom}_B(A, M) | \forall a, b \in A, \partial(ab) = a\partial(b) + \partial(a)b\}$.

Proposition 4.1.1. If $\partial \in \mathrm{End}_{\mathbb{C}}(A)$ is a derivation $\Leftrightarrow$ $\partial(\mathbb{C}) = 0$ and for all $a \in A$, $\partial a - a\partial \in A$.

Proof. Let $b \in A$, then the Leibniz identity is equivalent to

$$\begin{aligned}(\partial a - a\partial)(b) &= \partial(ab) - a\partial(b) \\ &= \partial(a)b.\end{aligned}$$

- $\Rightarrow$ Assuming ∂ is a derivation, then the argument above shows that $\partial a - a\partial \in A \subseteq \mathrm{End}_{\mathbb{C}}(A)$, where left multiplication by this operator is the endomorphism map induced. Furthermore, since ∂ is $\mathbb{C}$-linear, and considering $\mathbb{C}$ as a $\mathbb{C}$-vector space over itself, the Leibniz identity implies

  $$\begin{aligned}\partial(1z) &= \partial(1)z + 1\partial(z) \\ &\Rightarrow \partial(1) = 0.\end{aligned}$$

  Therefore $\partial(\mathbb{C}) = 0$.

- $\Leftarrow$ Suppose $a\partial - a\partial = c$ for some $c \in A$ and $\partial(\mathbb{C}) = 0$, then

  $$\begin{aligned}(\partial a - a\partial)(1) &= c(1) \\ \partial(a) &= c\end{aligned}$$

  therefore ∂ follows Leibniz identity.

□

Example 4.1.1. On polynomial rings, we have $\mathrm{Der}_{\mathbb{C}}(\mathbb{C}[x]) = \mathbb{C}(x)\frac{d}{dx}$. Clearly, the inclusion $\mathbb{C}[x]\frac{d}{dx} \subseteq \mathrm{Der}_{\mathbb{C}}(\mathbb{C}[x])$ is trivial by just checking that it satisfies Leibniz identity. However, for the reverse inclusion, consider a derivation $\partial \in \mathrm{Der}_{\mathbb{C}}(\mathbb{C}[x])$, then we claim that a basis is given by

$$\partial := \partial(x)\frac{d}{dx}. \tag{4.2}$$

Easy to check that acting on the unit $1 \in \mathbb{C}$ and x, these definitions agree. Therefore, by $\mathbb{C}$-linearity and Leibniz property, they agree on $\mathbb{C}[x]$.

More generally,

$$\mathrm{Der}_{\mathbb{C}}(\mathbb{C}[x_1, \ldots, c_n]) = \bigoplus_{i=1}^{n} \mathbb{C}[x_1, \ldots, x_n]\frac{\partial}{\partial x^i} \tag{4.3}$$

Example 4.1.2. If $A = C^\infty(M)$, the algebra of smooth functions on M, then

$$\mathrm{Der}_{\mathbb{R}}(C^\infty(M)) = \mathcal{X}(M) \tag{4.4}$$

4.2 Differential operators

In this section we define the more general concept of a differential operator, which are **not** necessarily derivations. There are two different ways to define them.

Definition 4.2.1 (First definition)**.** The ring $D(A)$ of $\mathbb{C}$-linear **differential operators** on A is the subalgebra of $\mathrm{End}_{\mathbb{C}}(A)$ generated by A and $\mathrm{Der}_{\mathbb{C}}(A)$. Let $\theta \in D(A)$, it has *order p* if it is the sum of products on at most p derivations.
e.g: $\frac{d^2}{dx^2} + 1 = \left(\frac{d}{dx}\right)^2 + 1$ has order 2.

We can generalise this definition a little.

Definition 4.2.2 (Second definition)**.** A **regular** differential operator of order p is an element of

$$D^p(A) = \{\theta \in \mathrm{End}_{\mathbb{C}}(A) \quad | \quad \theta a - a\theta = \theta(a) \in D^{p-1}(A) \quad \forall a \in A\}, \tag{4.5}$$

with $D^0(A) = A$. The ring of **regular differential operators** is $D(A) = \bigcup D^p(A)$ and it is easy to see that

$$D^p(A) D^r(A) \subseteq D^{p+r}(A). \tag{4.6}$$

and $D^{p+1}(A) \supseteq D^p(A)$ so this defines a filtration.

We relate the two definitions in the following sense. Suppose $\theta \in D^1(A)$, then

$$\theta = (\theta - \theta(1)) + \theta(1) \tag{4.7}$$

implying that $D^1(A) \cong \mathrm{Der}_{\mathbb{C}}(A) \oplus A$. So we can generate the ring of differential operators on A and clearly def1 $\subset$ def2.

Theorem 4.2.1 (Grothendieck)**.** The two definitions are equivalent if and only if $X = \mathrm{Spec}_A$ is non-singular. In this case, the ring of differential has the simple expression

$$D(A) = {}^{T_A(\mathrm{Der}_{\mathbb{C}}(A))}\big/_{\theta \otimes \theta' - \theta' \otimes \theta - [\theta, \theta']} \tag{4.8}$$

where T_A is the tensor algebra. Recall that the *spectrum* of a ring $\mathrm{Spec}(R)$ is the set of all prime ideals of R with the Zariski topology. [9]

Example 4.2.1. Consider the ring $A = \mathbb{C}[x]$ of rational functions over $\mathbb{C}$, then the algebra of derivations over this ring

$$\mathrm{Der}_{\mathbb{C}}(A) = \mathbb{C}[x]\frac{d}{dx} := W \tag{4.9}$$

is called the *Witt* algebra. However, the ring $D(A)$ of differential operators on A can also be viewed as the polynomial ring constructed by quotienting the free $\mathbb{C}$-algebra on x, ∂ by the ideal

$$D(A) = {}^{\mathbb{C}\langle x, \partial = \frac{d}{dx}\rangle}\big/_{x\partial - \partial x - 1}. \tag{4.10}$$

This is called a *Weyl* algebra.

As noted earlier, the second definition is more general. Here is an example where the equality fails.

Example 4.2.2. Consider $A = \mathbb{C}[t^2, t^3]$. Then $\mathrm{Spec}(A)$ is the space of proper prime ideals

$$\mathrm{Spec}(A) = \left\{ \langle t^2 - a, t^3 - b \rangle, (a, b) \in \mathbb{C}^2 \right\} \bigcup \left\{ \langle f(t^2, t^3) \rangle, \text{f is irreducible} \right\} \cup \{ \langle 0 \rangle \} \tag{4.11}$$

This space has a singular point and somehow this implies that there exists differential operators at that point that are not generated by sum-products of derivations. EXPAND ON THIS

Lemma 4.2.1. Let $\theta \in D^p(A)$ and $\theta' \in D^r(A)$ then

$$[\theta, \theta'] := \theta \cdot \theta' - \theta' \cdot \theta \in D^{p+r-1}(A) \tag{4.12}$$

In particular, $D^1(A)$ and $\mathrm{Der}_{\mathbb{C}}(A)$ are *Lie algebras*. Not true for higher order as it doesn't close. But below we will see a way to make it into a Lie algebra.

Question. Given algebras A, B with respective spectrum $X = \mathrm{Spec}(A)$, $Y = \mathrm{Spec}(B)$. If $D(A) \cong D(B)$, does that mean that $X \cong Y$? This turns out to be **false** if the algebraic varieties are allowed to be singular.

4.3 From differential operators to Poisson algebras

We have seen in lemma 4.2.1 that $[D^p(A), D^r(A)] \subseteq D^{p+r-1}(A)$. In particular, for Lie subalgebra $\mathrm{Der}_{\mathbb{C}}(A) \subseteq D(A)$, if $\delta, \delta' \in \mathrm{Der}_{\mathbb{C}}(A)$ then $[\delta, \delta'] \in \mathrm{Der}_{\mathbb{C}}(A)$,

$$\begin{aligned}
[\delta, \delta'](ab) &= \delta\delta'(ab) - \delta'\delta(ab) \\
&= \delta(\delta'(a)b + a\delta'(b)) - \delta'(\delta(a)b + a\delta(b)) \\
&= \delta\delta'(a)b + a\delta\delta'(b) - \delta'\delta(a)b - a\delta'\delta(b) \\
&= [\delta, \delta'](a)b + a[\delta, \delta'](b)
\end{aligned}$$

Definition 4.3.1. Given the filtration of regular differential operators $D(A)$ on algebra A, we define its grading $\mathrm{gr}\, D(A)$ as

$$\mathrm{gr}\, D(A) = \bigoplus_p D^p(A) / D^{p-1}(A) \tag{4.13}$$

Proposition 4.3.1. The grading of differential operators on A is a commutative ring under composition and a Poisson algebra with bracket generated by the commutator $[\cdot, \cdot]$.

Proof.

- Let $\pi \in D^p(A)$ and $\rho \in D^r(A)$, then $\pi\rho, \rho\pi \in D^{p+r}(A)$ while $[\pi, \rho] \in D^{p+r-1}(A)$. So

$$\pi\rho \sim \rho\pi + D^{p+r-1}(A) \tag{4.14}$$

  but as elements $\mathrm{gr}\,(\pi\rho), \mathrm{gr}\,(\rho\pi) \in \mathrm{gr}\, D(A)$, we have $\mathrm{gr}\,(\pi\rho) = \mathrm{gr}\,(\rho\pi)$

- $(\mathrm{gr}\, D(A), \{\cdot, \cdot\})$ is a Lie algebra. Taking the bracket on differential operators, we induce the Lie bracket $\{\cdot, \cdot\} : \mathrm{gr}\, D(A) \times \mathrm{gr}\, D(A) \to \mathrm{gr}\, D(A)$ by

$$\begin{aligned}
\{\mathrm{gr}\, \rho, \mathrm{gr}\, \pi\} &:= \mathrm{gr}\, [\rho, \pi] \\
&= [\rho, \pi] + D^{p+r-2}(A)
\end{aligned}$$

  for $\pi \in D^p(A)$, $\rho \in D^r(A)$. Given that $[\cdot, \cdot]$ is a Lie bracket on $D^1(A)$, we extend it to $\mathrm{gr}\, D(A)$ so that $\{\cdot, \cdot\}$ is a bracket up to an element of the quotient.

- The adjoint action is a derivation. [9]

□

In fact, if $X = \mathrm{Spec}(A)$ is non-singular,

$$\begin{aligned}\mathrm{gr}\, D(A) &= \mathrm{gr}\left(\frac{T_A(\mathrm{Der}_{\mathbb{C}}(A))}{\delta\otimes\delta' - \delta'\otimes\delta - [\delta,\delta']}\right)\\ &= \frac{T_A(\mathrm{Der}_{\mathbb{C}}(A))}{\delta\otimes\delta' - \delta'\otimes\delta}.\end{aligned}$$

So in this case, $\mathrm{gr}\, D(A) = \mathrm{Sym}_A(\mathrm{Der}_{\mathbb{C}}(A))$, and since we can identify the derivations with category of vector fields on X,

$$\mathrm{Der}_{\mathbb{C}}(A) = \mathbb{V}ect(X) = \mathbb{C}[T^*X] \tag{4.15}$$

Possible connection with L_∞-algebras. see [10]

4.4 Weyl algebras

Let $A = \mathbb{C}[x_1, \dots, x_n]$ then the ring of differential operator on A is constructed akin to example 4.2.1 as the free algebra in $\{x_i, y_i = -\partial_i\}$ variables

$$D(A) \cong \frac{\mathbb{C}\langle x_1, \dots, x_n, y_1, \dots, y_n\rangle}{[x_i, y_j] = \delta_{ij},\ \text{rest commutes}}. \tag{4.16}$$

This is the n^{th} **Weyl Algebra** $D(A)$ which is a simple ring (i.e: it does not have a proper 2-sided ideal). Its grading is the Poisson simple algebra

$$\mathrm{gr}\, D(A) = \mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_n] \tag{4.17}$$

with Poisson brackets

$$\{x_i, y_j\} = \delta_{ij} \qquad \{x_i, x_j\} = 0 = \{y_i, y_j\} \tag{4.18}$$

This is sometimes called *the first example*.

Remark 4.4.1. $D(\mathbb{C}[x])$ has no non-trivial finite dimensional modules. This is because, assuming V is a $D(\mathbb{C}[x])$-module of complex dimension d. Then $D(\mathbb{C}[x])$ acts on V as an endomorphism. Let $X, Y \in \mathrm{Mat}_{d\times d}(\mathbb{C})$ such that $[X, Y] = \mathbb{1}_d$, then $\mathrm{tr}([X, Y]) = 0 \neq d$, which is a contradiction.

Proposition 4.4.1. Let I be a right ideal of $D(A)$. Then $J = \mathrm{gr}\,(I)$ is an ideal of $\mathrm{gr}\, D(A)$ and it is **involutive/coisotrope**

$$\{J, J\} \subseteq J \tag{4.19}$$

Proof. Let $\theta, \eta \in I$ then $[\theta, \eta] \in I$ since it is a right ideal. Taking the grading, $\mathrm{gr}\,[\theta, \eta] \equiv \{\mathrm{gr}\,\theta, \mathrm{gr}\,\eta\} \subseteq J$ □

Theorem 4.4.1 (Gabber)**.** If $J = \mathrm{gr}\,(I)$ is coisotrope for some right ideal I of $D(A)$, then the radical $\sqrt{J} := \{\theta | \exists\ \text{k, s.t}\ \theta^k \in J\}$ is also coisotrope.

[9] do this SOMEDAY

Corollary 4.4.1 (Bernstein's inequality)**.** Using Gabber's theorem and Hilbert Nullstellensatz $\sqrt{J} = I(V(J))$, we see that

$$\dim(V(J) \subseteq \mathbb{C}^{2n}) \geq n \tag{4.20}$$

Example 4.4.1. Let $A = \mathbb{C}[x, y]$ with $\{x, y\} = 1$. Consider the coisotrope subring $J = \langle x^2, xy, y^2 \rangle$. It has radical $\sqrt{J} = \langle x, y \rangle$, but the radical is *not* coisotrope. Therefore J is **not** the grading of some right ideal of $D(\mathbb{C}[x, y])$.

5 Lecture 5: Differential Operators II

5.1 Differential operators on Manifolds

In this section, we will build up a correspondence between the algebraic theory and the geometric theory of differential operators on manifolds. Throughout this lecture, we assume the algebra $\mathcal{A}$ to be unital, associative and commutative over the real, and we'll restrict very soon to the algebra of smooth functions over a manifold.

Definition 5.1.1. A *point* x is an algebra morphism $x : \mathcal{A} \to \mathbb{R}$. The dual $|\mathcal{A}|$ of the algebra $\mathcal{A}$ is the set of all such algebra morphisms.

Definition 5.1.2. The set of $\mathbb{R}$-valued functions on $|\mathcal{A}|$ is denoted

$$\tilde{\mathcal{A}} = \{f_a : |\mathcal{A}| \to \mathbb{R} \quad | \quad f_a(x) = x(a)\ \forall a \in \mathcal{A}\} \tag{5.1}$$

Definition 5.1.3. An algebra $\mathcal{A}$ is said to be *geometric* if

$$\bigcap_{x\in|\mathcal{A}|} \ker(x) = 0. \tag{5.2}$$

The named geometric comes from the canonical isomorphism

$$\mathcal{A} \cong \tilde{\mathcal{A}}, \tag{5.3}$$

therefore putting an algebra structure on the $\mathbb{R}$-valued functions on the set $|\mathcal{A}|$. This is an identification between geometric algebras and sets. It is possible to put a topology on such spaces, or to find a smooth structure in order to define the notion of continuous/smooth algebras.

Let's consider a smooth manifold M and its ring of smooth functions $C^\infty(M)$, then the ring of regular differential operators $\mathrm{Diff}(M)$ has the 2 equivalent definitions from theorem 4.2.1 [10]:

$$\mathrm{Diff}(M) = \bigcup_p D^p(C^\infty(M)) \tag{5.4}$$

We sometimes denote $\mathrm{Diff}_p(M) = D^p(C^\infty(M))$ for simplicity. [11] A useful equivalence between elements of the algebra of smooth functions on a manifold, relating to differential operators is the following local relation. For $f, g \in C^\infty(M)$

$$\begin{aligned} & f \sim_x^k g \\ \Leftrightarrow \qquad & f(x) = g(x), \quad \partial^{|k|} f(x) = \partial^{|k|} g(x), \end{aligned}$$

where $|k|$ is the order of the differential operator (potentially the sum of integers). This equivalence relation can be globalised into a vector bundle.

Definition 5.1.4. The **k-jet** bundle of a manifold M is

$$J^k M := \bigcup_{x\in M} J_x^k M, \qquad J_x^k := \left\{[f]_x^k, f \in C^\infty(M)\right\} \tag{5.5}$$

under the equivalence relation defined above. There is a natural projective resolution

$$\dots \xrightarrow{\pi^{k+1}} J^k M \xrightarrow{\pi^k} J^{k-1} M \to \dots \xrightarrow{\pi^1} C^\infty(M) \to 0, \tag{5.6}$$

[10] I'm assuming here that the spectrum of the ring of smooth functions is non-singular...

[11] Or even $\mathrm{Diff}(M, \mathbb{R})$ more precisely.

where the projective maps π^k sends classes of functions in the k-jet to the class of function that agree on a lower order derivative, i.e: $\pi^{n-1} \circ \pi^n = 0$. There exists also a map

$$\begin{aligned} j^k : C^\infty(M) &\to \Gamma(J^k M) \\ f &\mapsto j^k f(x) := [f]^k_x \end{aligned}$$

sending functions to their $k-$jet.

Proposition 5.1.1.

$$\mathrm{Diff}_k(M) \cong \Gamma(J^k M^*) \tag{5.7}$$

Proof. (Sketch)
Let $\Delta \in \mathrm{Diff}_k(M) \subset \mathrm{End}(C^\infty(M))$ an $\mathbb{R}$-linear endomorphism then, by the universal property, there exists a $C^\infty(M)$-linear map $\delta : J^k M \to \mathbb{R}$ (in other words, the dual jet bundle) fitting the diagram

$$\begin{tikzcd} C^\infty(M) \arrow[r, "j^k_x"] \arrow[d, "\Delta"] & J^k_x M \arrow[d, "\delta"] \\ C^\infty(M) & \mathbb{R} \arrow[l, "\delta_*"] \end{tikzcd} \tag{5.8}$$

Meaning that at the level of sections, we have

$$\Delta = \delta_* \circ j^k \tag{5.9}$$

where the pushforward $\delta_* : \Gamma(J^k M^*) \to C^\infty(M)$ makes the diagram commute. □

5.2 Differential operators on Vector Bundles

Definition 5.2.1. Let $\mathcal{A}$ be a $\mathbb{K}$-algebra and $\mathcal{P}, \mathcal{Q}$ be $\mathcal{A}$-modules with

$$\mu_\mathcal{P} : \mathcal{A} \times \mathcal{P} \to \mathcal{P}$$
$$\mu_\mathcal{Q} : \mathcal{A} \times \mathcal{Q} \to \mathcal{Q}$$

their respective operation. For $\phi \in \mathrm{Hom}_\mathbb{K}(\mathcal{P}, \mathcal{Q})$ and $a \in \mathcal{A}$, we define the **commutator** of $\mathbb{K}$-linear morphisms

$$\begin{aligned} c_a(\phi) &\in \mathrm{Hom}_\mathbb{K}(\mathcal{P}, \mathcal{Q}) \\ c_a(\phi) &:= [\phi, a] = \phi \circ \mu_\mathcal{P}(a) - \mu_\mathcal{Q}(a) \circ \phi \end{aligned} \tag{5.10}$$

Definition 5.2.2. The **Differential Operators** of order less than n between the $\mathcal{A}$-modules $\mathcal{P}, \mathcal{Q}$ is

$$D^n(\mathcal{P}, \mathcal{Q}) := \{\Delta \in \mathrm{Hom}_\mathbb{K}(\mathcal{P}, \mathcal{Q}) \mid c_{a_0} \circ \ldots \circ c_{a_n}(\Delta) = 0 \quad \forall a_i \in \mathcal{A}\} \tag{5.11}$$

Making the connection with regular differential operators on an algebra A over the complex numbers ineq. (4.5), the definition above is equivalent to the previous statement using $\theta a - a\theta = \theta(a) \equiv c_a(\theta)$.

We previously identified geometric/continuous/smooth algebras with sets/topological spaces/manifolds. Similarly, there is an equivalence between vector bundles and finitely generated projective modules over smooth algebras.

Definition 5.2.3. A *Differential operator* between two vector bundles over the same manifold $A, B \to M$ are defined as

$$\mathrm{Diff}(A, B) := \bigcup_k \mathrm{Diff}_k(A, B), \tag{5.12}$$

where $\mathrm{Diff}_k(A, B) := D^k(\Gamma(A), \Gamma(B))$ the set of k differential operators on sections of vector bundles.

Remark 5.2.1. As we have seen, differential operators form a filtered structure and we now show that this gives the *symbol short exact sequence*

$$0 \to \mathrm{Diff}_{k-1}(A, B) \to \mathrm{Diff}_k(A, B) \xrightarrow{\sigma} \Gamma(\bigodot^k TM \otimes A^* \otimes B) \to 0 \tag{5.13}$$

Note that since vector bundles are commutative algebras that respect the Jacobi identity, we have

$$c_f \circ c_g = c_g \circ c_f \tag{5.14}$$

for $f, g \in C^\infty(M)$. The map σ is called the **symbol** and completes this sequence in the following way. For $f_i \in C^\infty(M)$,

$$\sigma : \Delta \mapsto \sigma_\Delta$$
$$\sigma_\Delta(df_1, \ldots, df_k) := c_{f_1} \circ \ldots \circ c_{f_k}(\Delta)$$

We prove this is a multiderivation by considering the case $k = 1$ and extending linearly. Acting on $a \in \Gamma(A)$, for $df \in T^*M$

$$\begin{aligned}\sigma_\Delta(df)(a) &= [\Delta, f](a)\\ &= \Delta(f \cdot a) - f \cdot \Delta(a)\end{aligned}$$

It is also easy to show that for any $a \in \Gamma(A)$, this acts as a derivation on functions on the manifold M, therefore $[\Delta, \cdot](a) \in \Gamma(TM)$.

Definition 5.2.4. The **k-jet bundle** of a vector bundle $A \to M$ is

$$J^k A := \bigcup J^k_x A, \tag{5.15}$$

where the equivalence class is on sections of the vector bundle i.e $J^k_x A = \left\{ [s]^k_x := j^k s(x) \,|\, s \in \Gamma(A) \right\}$. The induced jet maps $j^k : \Gamma(A) \to \Gamma(J^k A)$ are defined analogously. As in definition 5.1.4, we have the long exact sequence

$$\ldots \xrightarrow{\pi^{k+1}} J^k A \xrightarrow{\pi^k} J^{k-1} A \to \ldots \xrightarrow{\pi^1} \Gamma(A) \to 0, \tag{5.16}$$

Proposition 5.2.1.

$$\mathrm{Diff}_k(A, B) \cong \Gamma((J^k A)^* \otimes B) \tag{5.17}$$

5.3 Derivations on Vector Bundles

finish this section

6 Lecture 6: Local Lie Algebras

We will now explore the important notion of **locality**, which we formalise from the physical intuition that things should only depend on local variables, or that an open neighbourhood around a point should be sufficient to reconstruct sections (onto whatever) at that point.

6.1 General Local Lie Algebras

Definition 6.1.1. The **support** of a map $f : X \to Y$ is

$$\text{supp}(f) = \{x \in X | x \notin \ker(f)\} \tag{6.1}$$

Definition 6.1.2. A structure is **local**, if for any two maps $f, g : X \to Y$

$$\text{supp}(fg) \subset \text{supp}(f) \cap \text{supp}(g) \tag{6.2}$$

and the support is compatible with the structure. Some examples to illumintes what we mean:

- The algebra of smooth functions on manifold M has,

  $$\text{supp}(fg) \subset \text{supp}(f) \cap \text{supp}(g)$$

  for $f, g \in C^\infty(M)$.

- Similarly for a vector bundle $A \to M$,

  $$\text{supp}(f \cdot s) \subset \text{supp}(f) \cap \text{supp}(s)$$

  for $f \in C^\infty(M), s \in \Gamma(A)$.

- The Poisson bracket in a Poisson algebra as seen in definition 1.1.1 has manifestly the same property since it is an algebra over smooth functions.

Definition 6.1.3. A vector bundle $A \to M$ is a **local Lie Algebra** if the $\mathbb{R}$-linear bracket on the smooth sections is local, i.e

$$\text{supp}([a, b]) \subset \text{supp}(a) \cap \text{supp}(b) \tag{6.3}$$

for $a, b \in \Gamma(A)$. Regarding the Lie bracket as a differential operator on smooth sections above prompts us to extend our the historical definition of local Lie algebra to vector bundles with a local differential structure.

Definition 6.1.4. A vector bundle $A \to M$ is said to carry a local lie algebra *structure* on its sections $\Gamma(A)$ if the lie bracket on the space of sections is $\mathbb{R}$-linear and the adjoint map is a differential operator of degree 1.

$$\text{ad.} : \Gamma(A) \to \text{Diff}_1(A)$$

Or in other words, ad. $\in \text{Diff}(A, J^1 A^* \otimes A)$, where $J^k A^* \otimes A$ is algebroid-valued k-jet on its dual using proposition 5.2.1.

Theorem 6.1.1. test

7 Jacobi Geometry I

7.1 Local Lie algebra

We recall that according to, a local Lie algbra is

8 Jacobi Geometry II

9 Lecture 9: Dirac Geometry

Aim of the lecture is to recover Dirac geometry. A lot of the ideas can be traced back to Courant's thesis [11]. As usual Gualtieri's thesis [12] serves us well. In the following the underlying field is $\mathbb{R}$.

9.1 Courant Spaces

Definition 9.1.1. Let V be a vector space over $\mathbb{R}$. A **Linear Courant space** is a triple $(C, \langle\cdot,\cdot\rangle, \rho)$ where $(C, \langle\cdot,\cdot\rangle)$ is a inner product space (meaning the inner product is bilinear and non-degenerate but **not** positive-definite) and $\rho : C \to V$ is a linear homomorphism called the *anchor*.

Definition 9.1.2. With respect to the *bilinear form*, a subspace $N \subset C$ and its orthogonal complement $N^\perp := \{x \in C \mid \langle x, y\rangle = 0 \,\forall\, y \in N\}$, need not be disjoint as the inner product is not in general positive definite. Therefore a subspace $N \subset C$ is called the following ways if the orthogonality conditions hold:

- $N \subset N^\perp$ is isotropic,
- $N \supset N^\perp$ is coisotropic,
- $N = N^\perp$ is Lagrangian.

Definition 9.1.3. Since $(C, \langle\cdot,\cdot\rangle)$ is an inner product space, we have the usual *musical isomorphisms*:

$$C \xrightarrow{\flat} C^*$$
$$C^* \xrightarrow{\sharp} C$$

This data of a Courant space implies that we can construct a map $k : V^* \to V$ such that the following diagram commutes:

$$\begin{tikzcd} C^* \arrow[d, "\sharp"] & V^* \arrow[l, "\rho^*"'] \arrow[d, dashed, "k"] \\ C \arrow[r, "\rho"] & V \end{tikzcd} \tag{9.1}$$

Definition 9.1.4. A Courant space is **exact** when we have the short exact sequence

$$0 \to V^* \xrightarrow{\sharp\rho^*} C \xrightarrow{\rho} V \to 0. \tag{9.2}$$

Define from now on $j = \sharp\rho^*$. In such a case, $\mathrm{im}(j) = \ker(\rho) \subset C$ is maximally isotropic (i.e: Lagrangian) in C. Further, the injective map j is also isotropic in the sense that

$$\langle j(V^*), j(V^*)\rangle = 0 \tag{9.3}$$

Definition 9.1.5. Given a Courant space C, C', we define

- the *opposite Courant space* $(\overline{C}, -\langle\cdot,\cdot\rangle, \rho)$,
- the *direct sum* of two Courant spaces $(C \oplus C, \langle\cdot,\cdot\rangle \oplus \langle\cdot,\cdot\rangle', \rho \oplus \rho')$

Definition 9.1.6. For $V \in \mathbf{Vect}_{\mathbb{R}}$ we define the **Standard Courant space** as $\mathbb{V} = (V \oplus V', \langle \cdot, \cdot \rangle, \mathrm{pr}_1)$, with bilinear pairing

$$\langle v \oplus \alpha, w \oplus \beta \rangle = \frac{1}{2} \left(\alpha(w) + \beta(v) \right) \tag{9.4}$$

Note that we can also define a skew-symmetric bilinear form as well, but we will only call the *inner* product the symmetric one. Further note that, the symmetry group preserving orientation is $\mathrm{SO}(d, d)$, the non-compact special orthogonal group. [12]

Definition 9.1.7. A **Dirac space** D is a Lagrangian subspace of Courant space $(C, \langle \cdot, \cdot \rangle, \rho)$ for which there exists $W \subset V$ and $\overline{W} \subset V^*$ such that the following sequence is exact

$$0 \to \overline{W} \xrightarrow{\sharp \rho^*} D \xrightarrow{\rho} W \to 0 \tag{9.5}$$

The space $W = \rho(D) = {D}/{\overline{W}}$ is generally called the *range* of D. We remark that the Lagrangian condition imposed on the space implies that the bilinear form is 0 along this subspace.

Proposition 9.1.1. A Dirac space $D \subset (C, \langle \cdot, \cdot \rangle, V)$ specifies a 2-form on its range $\rho(D)$,

$$\omega_D \in \bigwedge^2 W^* \tag{9.6}$$

Such that for $w_i = \rho(d_i) + j(\epsilon_i) := \rho(a_i)$, with $d_i \in D$, $\epsilon_i \in V \setminus W$ and some $a_i \in C$ (that restricts to ω_i in D); the 2-form is:

$$\begin{aligned} \omega_D(w_1, w_2) &= \langle a_1, \sharp \rho^*(\epsilon_2) \rangle \\ &= -\langle \sharp \rho^*(\epsilon_1), a_2 \rangle \end{aligned} \tag{9.7}$$

Proof. D is maximally isotropic so $\forall d_1, d_2 \in D \subset C$,

$$\langle d_1, d_2 \rangle = 0. \tag{9.8}$$

Since ρ is surjective there exists $w_1, w_2 \in W$ such that $\rho(d_i) = w_i$. Consider the extension of elements $w_1, w_2 \in W \subset V$ by ρ, that is $a_1, a_2 \in C$ such that

$$w_i = \rho(a_i). \tag{9.9}$$

where $a_i = d_i + \sharp \rho^*(\epsilon_i)$ for some $\epsilon_1, \epsilon_2 \in V^* \setminus W^*$.

$$\langle a_1 - \sharp \rho^*(\epsilon_1) \ , \ a_2 - \sharp \rho^*(\epsilon_2) \rangle = 0 \tag{9.10}$$

The cross terms must vanish while the non-cross terms carry the non-zero part of the inner product, therefore

$$\langle a_1, \sharp \rho^*(\epsilon_2) \rangle = -\langle \sharp \rho^*(\epsilon_1), a_2 \rangle. \tag{9.11}$$

sketchy af proof. coset construction? do this ☐

Definition 9.1.8. An *isotropic* relation (or Lagrangian relation) $\Lambda : A \dashrightarrow B$ between two exact Courant spaces $(A, \langle , \rangle_A, \alpha : A \to V)$, $(B, \langle , \rangle_B, \beta : B \to W)$ is a morphism such that $\Lambda \subset A \oplus \overline{B}$ is a Lagrangian subspace with the relations

$$\begin{aligned} a_1 \sim_\Lambda b_1, &\quad a_2 \sim_\Lambda b_2 \\ \Rightarrow &\quad \langle a_1, a_2 \rangle_A = \langle b_1, b_2 \rangle_B \end{aligned}$$

[12] probably add some example such as B transform here, for posterity

In this case, we say that elements $a, b \in \Lambda$ are Λ-related.

Definition 9.1.9. An isotropic relation $\Gamma : A \dashrightarrow B$ is a **Courant morphism** if there exists a map $\gamma : V \to W$ such that elements of $\mathrm{graph}(\Gamma) \subset B \oplus \overline{A}$ have

$$\begin{aligned} & b \oplus a \in \Gamma \\ & \Rightarrow \beta(b) = (\gamma \circ \alpha)(a) \end{aligned}$$

Remark 9.1.1. A Courant morphism induces a Dirac space $\Gamma_\gamma \subset B \oplus \overline{A}$ and enters the following short exact sequence,

$$0 \to \mathrm{graph}(\gamma^*) \to \Gamma_\gamma \xrightarrow{\beta \oplus \alpha} \mathrm{graph}(\gamma) \to 0 \tag{9.12}$$

where $\mathrm{graph}(\gamma) \subset W \oplus V^*$ and similarly for γ^*. This exact sequence is easily read as both piece of the graphs are exact Courant morphisms, on which we take the direct sum. Courant morphisms are morphism in the category of Courant algebroids, and we have seen that elements are mapped appropriately. A lesser constraint would be to consider maps such that the inner product is preserved.

Definition 9.1.10. A **Courant map** $\Psi : A \to B$ of Courant algebroids is a linear map together with $\psi : V \to W$ such that the inner product is preserved,

$$\Psi^* \langle \cdot, \cdot \rangle_B = \langle \cdot, \cdot \rangle_A \tag{9.13}$$

and the following diagram commutes

$$\begin{tikzcd} A \arrow[r, "\Psi"] \arrow[d, "\alpha"] & B \arrow[d, "\beta"] \\ V \arrow[r, "\psi"] & W. \end{tikzcd} \tag{9.14}$$

Proposition 9.1.2. A map $\Psi : A \to B$ is Courant if and only if $\mathrm{graph}(\Psi) \in B \oplus \overline{A}$ is a Courant morphism.

Proof. do later, but similar to proposition 1.2.3, done in p47 of [1]. □

Definition 9.1.11. The category of exact Courant spaces **Crnt** is the category whose objects are exact Courant spaces $C \to V$ and morphisms are Courant morphisms defined in definition 9.1.9 that extend linear maps between vector spaces.

Remark 9.1.2. It's easy to see that a linear map $\Psi : C \to D$ is Courant for some linear map $\psi : V \to W$ (definition 9.1.10) if and only if $\mathrm{grph}(\psi) \subset D \oplus \overline{C}$.

Remark 9.1.3. Since Courant morphisms $\Gamma : A \dashrightarrow B$ between exact Courant spaces $(A \to V), (B \to W)$ define a Dirac space $\Gamma_\gamma \subset B \oplus \overline{A}$ (remark 9.1.1), and Dirac spaces specify a 2-form on the range of the anchor, we have a the fibration:

$$\Lambda^2 V^* \to \mathrm{Crnt}(A, B) \to \mathrm{Vect}(V, W). \tag{9.15}$$

Furthermore, by considering Courant automorphisms, we abelianise this fibration to make the following short exact sequence exact:

$$0 \to (\Lambda^2 V^*, +) \to \mathrm{Aut}_{\mathrm{Crnt}}(A) \to \mathrm{GL}(V) \to 0. \tag{9.16}$$

It's probably a good idea to stop and think about what this means. By considering Courant algebroids over a vector space, we have extended the available symmetries by 2-forms on said vector space.

Definition 9.1.12. An **isotropic splitting** of an exact Courant space is a map ∇ such that the short exact sequence splits:

$$\begin{tikzcd} 0 \arrow[r] & V^* \arrow[r, "j"] & C \arrow[r, "\rho"] & V \arrow[l, bend left, "\nabla"] \arrow[r] & 0, \end{tikzcd}$$

with $\rho \circ \nabla = \mathrm{id}_V$, and the distribution $\nabla(V)$ is isotropic in C with respect to the bilinear form $\langle \cdot, \cdot \rangle$.

Proposition 9.1.3. Exact Courant Spaces always admit an isotropic split. Furthermore, given $\nabla, \nabla' : V \to C$ two isotropic splits, there exists a 2-form $B \in \Lambda^2 V^*$ such that

$$\nabla - \nabla' = j \circ B^\flat \tag{9.17}$$

The following proof follows [1].

Proof. Consider two not necessarily isotropic sections $\nabla, \nabla' : V \to C$ such that $\rho \circ \nabla = \mathrm{id}_V = \rho \circ \nabla'$, we can construct the map $\nabla - \nabla' = \gamma$ such that

$$\nabla - \nabla' : V \to \ker(\rho)$$

Since the sequence is exact, then $\gamma(v) \in \mathrm{im}(j)$, so there must $B \in V^* \otimes V^*$ such that $B^\flat : V \to V^*$ and

$$\gamma = j \circ B^\flat. \tag{9.18}$$

In summary we have shown that general sections differ by elements in the Lagrangian subspace $\mathrm{im}(j) = \ker(\rho)$.

$$\begin{tikzcd} 0 \arrow[r] & V^* \arrow[r, "j"] & C \arrow[r, "\rho"] & V \arrow[l, bend left, "\gamma"] \arrow[ll, bend right, "B^\flat"'] \arrow[r] & 0. \end{tikzcd}$$

Now fix a non-isotropic section ∇_0, we show that we can define an isotropic section ∇, we show that defining implicitly

$$B_{\nabla_0}(v, w) = -\frac{1}{2} \langle \nabla_0(v), \nabla_0(w) \rangle \tag{9.19}$$

allows us to make $\nabla = \nabla_0 + j \circ B^\flat$ into an isotropic map, in the sense

$$\langle \nabla(v), \nabla(w) \rangle = 0. \tag{9.20}$$

$\square$

We note that just like any split short exact sequence, a choice of isotropic section ∇ induces a Courant isomorphism with the standard Courant space,

$$(C, \langle \cdot, \cdot \rangle, V) \overset{\nabla}{\cong} (V \oplus V', \langle \cdot, \cdot \rangle, \mathrm{pr}_1). \tag{9.21}$$

EXPAND BELOW, do diagram etc... Further to remark 9.1.3, the category of exact Courant spaces is explicitly given by $\mathbf{Crnt}(\mathbb{V}, \mathbb{W}) \cong \Lambda^2 V^* \times \mathbf{Vect}(V)$, where morphisms

$$(B, \phi) : \mathbb{V} \dashrightarrow \mathbb{W}$$

are both given by an isotropic relation (as defined in definition 9.1.8) and related by a 2-form. So

$$v \oplus \alpha \sim_\psi^B w \oplus \beta \qquad \Leftrightarrow \qquad \psi(v) = w \quad \text{and} \quad \alpha + \iota_v B = \psi^* \beta. \tag{9.22}$$

Composition of morphism are given just like a semi-direct product:

$$(B', \varphi) \circ (B, \psi) = (B' + \psi^* B \ , \ \varphi \circ \psi). \tag{9.23}$$

Finally, by considering automorphisms of Courant spaces, the above argument reduces to

$$\mathrm{Aut}_{\mathbf{Crnt}}(\mathbb{V}) = \Lambda^2 V^* \rtimes \mathrm{GL}(V). \tag{9.24}$$

9.2 Dorfman Algebras

Definition 9.2.1. A **Leibniz algebra** $(\mathfrak{a}, [\cdot, \cdot])$ over $\mathbb{R}$ is a module with an $\mathbb{R}$-linear bracket that is **NOT** skew-symmetric, such that the bracket is a derivation.

$$[a, [b, c]] = [[a, b], c] + [b, [a, c]] \tag{9.25}$$

for $a, b, c \in \mathfrak{a}$. Of importance is to note that because the bracket is not antisymmetric, the right/left adjoint maps are not related, i.e:

$$\mathrm{ad}_a^L = [a, \cdot] \neq [\cdot, a] = \mathrm{ad}_a^R \tag{9.26}$$

In particular, Lie algebras are Leibniz algebras over $\mathbb{R}$ with skew-symmetric brackets.

Definition 9.2.2. A map $\Psi : \mathfrak{a} \to \mathfrak{b}$ is a Leibniz morphism if

$$\Psi([a, b]_{\mathfrak{a}}) = [\Psi(a), \Psi(b)]_{\mathfrak{b}}. \tag{9.27}$$

Definition 9.2.3. Let $\mathfrak{a}$ be a Leibniz algebra and $\mathfrak{h}$ a vector space. A **Leibniz representation** of $\mathfrak{a}$ over $\mathfrak{h}$ is a pair of maps (or equivalently right and left $\mathfrak{a}$-modules)

$$[\cdot, \cdot]_L : \mathfrak{a} \times \mathfrak{h} \to \mathfrak{h}$$
$$[\cdot, \cdot]_R : \mathfrak{h} \times \mathfrak{a} \to \mathfrak{h}$$

such that all possible combination of Leibniz identities hold.

Definition 9.2.4. Given a Leibniz algebra $\mathfrak{a}$ and an Leibniz $\mathfrak{h}$-representation, we define the Leibniz cochain complex $C^k(\mathfrak{a}, \mathfrak{h}) := \bigotimes^k \mathfrak{a}^* \otimes \mathfrak{h}$

$$\begin{aligned} \delta^k : C^k(\mathfrak{a}, \mathfrak{h}) &\to C^{k+1}(\mathfrak{a}, \mathfrak{h}) \\ h &\mapsto [\cdot, h]_L - [h, \cdot]_R \end{aligned}$$

We check explicitly that this exact forms are indeed closed in with this differential. Let $a, b \in \mathfrak{a}, h \in \mathfrak{h}$ then

$$\begin{aligned} \delta^0 h(a) &= [a, h]_L - [h, a]_R \\ \Rightarrow \qquad \delta^1 \circ \delta^0 h(a, b) &= [a, [b, h]_L]_L - [[b, h]_L, a]_R - [b, [h, a]_R]_L + [[h, a]_R, b]_R - ([[a, b], h]_L - [h, [a, b]]_R) \\ &= 0 \qquad \text{using all the Leibniz identities.} \end{aligned}$$

We extend the differential maps to higher powers and construct a cohomology.

Definition 9.2.5. Given a Leibniz algebra $\mathfrak{a}$ and an Leibniz $\mathfrak{h}$-representation, we denote $\mathrm{H}^\bullet(\mathfrak{a}, \mathfrak{h})$ the Leibniz cohomology of the cochain complex $(C^k(\mathfrak{a}, \mathfrak{h}), \delta)$, with differential maps $\delta^2 = 0$.

Definition 9.2.6. A **Dorfman Algebra** (or Courant algebra) $(\mathfrak{a}, [\cdot,\cdot])$ over a Lie algebra $\mathfrak{g}$ is a Leibniz algebra $\mathfrak{a}$, together with a morphism of Leibniz algebra

$$\rho : (\mathfrak{a}, [\cdot,\cdot]) \to (\mathfrak{g}, [\cdot,\cdot]). \tag{9.28}$$

Since ρ is a homomorphism, $\ker(\rho)$ is a 2-sided Leibniz ideal and $\rho(\mathfrak{a}) \subset \mathfrak{g}$ is a Lie subalgebra. It's probably good to underline that the bracket over the Lie algebra is antisymmetric.

Definition 9.2.7. A Dorfman algebra $(\mathfrak{a}, [\cdot,\cdot])$ over a Lie algebra $\mathfrak{g}$ is called *exact* if

$$0 \to \mathfrak{h} \to \mathfrak{a} \xrightarrow{\rho} \mathfrak{g} \to 0 \tag{9.29}$$

with $\ker(\rho) = \mathfrak{h}$ **and** $\mathfrak{h}$ is an abelian subalgebra of $\mathfrak{a}$, so

$$[\mathfrak{h}, \mathfrak{h}] = 0. \tag{9.30}$$

We can always build such a sequence but the extra data comes from the fact that $\mathfrak{h}$ is abelian.

Example 9.2.1. Consider a right $\mathfrak{g}$-module R over $\mathfrak{h}$ when $\mathfrak{g}$ is Lie

$$R : \mathfrak{g} \times \mathfrak{h} \to \mathfrak{h}. \tag{9.31}$$

Denote the *hemisemidirect product* [13]between a Lie algebra and a representation space by $\mathfrak{a} = \mathfrak{h} \oplus \mathfrak{g}$ with bracket

$$[h_1 \oplus g_1, h_2 \oplus g_2]_{\mathfrak{a}} := R(g_1, h_2) \oplus [g_1, g_2]_{\mathfrak{g}} \tag{9.32}$$

for $g_1, g_2 \in \mathfrak{g}$ and $h_1, h_2 \in \mathfrak{h}$. We claim that $\mathfrak{a}$ is a Leibniz (in particular Dorfman) algebra if and only if R is a Leibniz representation.

Proof. • Suppose $\mathfrak{a}$ is a Leibniz algebra, then the bracket over $\mathfrak{a}$ is a derivation, so by considering the bracket between $a_1, a_2, a_3 \in \mathfrak{a}$,

$$[a_1, [a_2, a_3]] = R(g_1, R(g_2, h_3)) \oplus [g_1, [g_2, g_3]].$$

We use the Leibniz identity for the Lie algebra part and we expand the LHS that also follows Leibniz and match with the $\mathfrak{h}$ part of the RHS. Therefore

$$R([g_1, g_2], h_3) + R(g_2, R(g_1, h_3)) = R(g_1, R(g_2, h_3)) \tag{9.33}$$

is a *right* Leibniz representation of $\mathfrak{g}$ on $\mathfrak{h}$.
• Conversely, if R is a Leibniz representation, we can split the brackets and set the left one to zero? □

Definition 9.2.8. A Dorfman morphism $\Psi : \mathfrak{a} \to \mathfrak{a}'$ is a Leibniz morphism covering the map of algebras $\psi : \mathfrak{g} \to \mathfrak{g}$ such that

$$\rho' \circ \Psi = \psi \circ \rho. \tag{9.34}$$

Proposition 9.2.1. Let $\mathfrak{a}$ be an exact Dorfman algebra over $\mathfrak{g}$ with anchor ρ and abelian subalgebra $\mathfrak{h} = \ker(\rho)$, then $\mathfrak{a}$ carries a Leibniz representation on $\mathfrak{h}$.

Proof. DO LATER □

If we split the sequence of an exact Dorfman algebra

[13]Hemi: because we take only the right representation and discard the left.

Definition 9.2.9. Let $\nabla : \mathfrak{g} \to \mathfrak{a}$ be a section of the following Dorfman algebra with $\mathfrak{g}$-representation

$$\begin{tikzcd} 0 \arrow[r] & \mathfrak{h} \arrow[r] & \mathfrak{a} \arrow[r, "\rho"] & \mathfrak{g} \arrow[l, bend left, "\nabla"] \arrow[r] & 0, \end{tikzcd}$$

with $[\mathfrak{h}, \mathfrak{h}] = 0$. Moreover, sections of the Dorfman algebra provide a lift of the right/left Leibniz representation as

$$\begin{aligned} [\cdot, \cdot]_L &: \mathfrak{a} \times \mathfrak{g} \to \mathfrak{g} \\ [a, g]_L &= [a, \nabla(g)]_{\mathfrak{a}} \\ [\cdot, \cdot]_R &: \mathfrak{g} \times \mathfrak{a} \to \mathfrak{g} \\ [g, a]_R &= [\nabla(g), a]_{\mathfrak{a}} \end{aligned}$$

for $a \in \mathfrak{a}, g \in \mathfrak{g}$. The **Leibniz Curvature** of ∇ is

$$\begin{aligned} C_\nabla &\in C^2(\mathfrak{g}, \mathfrak{h}) \\ C_\nabla(g, g') &:= [\nabla(g), \nabla(g')]_{\mathfrak{a}} - \nabla([g, g']_{\mathfrak{g}}). \end{aligned} \tag{9.35}$$

We note that $C_\nabla \in \ker(\rho)$ so it takes values in the representation space $\mathfrak{h}$. Important to note that $C_\nabla(g, g') \neq C_\nabla(g', g)$.

Proposition 9.2.2. The Leibniz curvature defined above is a cocycle under the differential of Leibniz cohomology defined in definition 9.2.5,

$$\delta C_\nabla = 0. \tag{9.36}$$

Proof. For $a, b, c \in \mathfrak{g}$,

$$\begin{aligned} \delta C_\nabla(a, b, c) = {} & [a, C_\nabla(b, c)]_L - [C_\nabla(b, a), c]_R + [C_\nabla(c, a), b]_R \\ & + C_\nabla([a, b]_{\mathfrak{g}}, c) - C_\nabla([a, c]_{\mathfrak{g}}, b) + C_\nabla(a, [b, c]) \end{aligned}$$

We then use the section ∇ to lift right and left representations on the Dorfman algebra to find

$$\begin{aligned} \delta C_\nabla(a, b, c) = {} & [\nabla(a), [\nabla(b), \nabla(c)]] - [\nabla(a), \nabla([b, c])] \\ & - [[\nabla(b), \nabla(a)], \nabla(c)] + [\nabla([b, a]), \nabla(c)] \\ & + [[\nabla(c), \nabla(a)], \nabla(b)] - [\nabla([c, a]), \nabla(b)] \\ & + [\nabla([a, b]), \nabla(c)] - \nabla\left([[a, b], c]\right) \\ & - [\nabla([a, c]), \nabla(b)] + \nabla\left([[a, c], b]\right) \\ & + C_\nabla(a, [b, c]) \\ = {} & 0 \end{aligned}$$

by collecting some terms, using Leibniz everywhere and identifying terms with the last line. □

Since the Leibniz curvature is a cocycle in $C^2(\mathfrak{g}, \mathfrak{h})$, and the Leibniz differential is closed, we define:

Definition 9.2.10. The *Characteristic class* of an exact Dorfman algebra is the 2nd cohomology class $[C_\nabla] \in H^2(\mathfrak{a}, \mathfrak{g})$. This is invariant under different splits since upon choosing a different $\nabla' = \nabla + \eta$ for $\eta : \mathfrak{g} \to \mathfrak{a}$, a short computation shows

$$C_{\nabla'} = C_\nabla + \delta\eta. \tag{9.37}$$

Therefore an exact Dorfman algebra carries a Characteristic class

$$[C_\nabla] = [C_{\nabla'}] \in H^2(\mathfrak{a}, \mathfrak{g}). \tag{9.38}$$

Proposition 9.2.3. Two exact Dorfman algebras are isomorphic if and only if their Characteristic classes agree. Proof in [1], prop 2.5.7

Example 9.2.2. Construct a Dorfman algebra from the Leibniz representation of a Lie algebra $\mathfrak{g}$ over a vector space $\mathfrak{h}$ in the following way. Let $\rho : \mathfrak{g} \to \mathfrak{h}$ be a Leibniz representation with

$$\begin{aligned}
&[\cdot,\cdot]_L : \mathfrak{g} \times \mathfrak{h} \to \mathfrak{h}\\
&[\cdot,\cdot]_R : \mathfrak{h} \times \mathfrak{g} \to \mathfrak{h},
\end{aligned}$$

that can be alternatively be understood as the Dorfman algebra $\mathfrak{w} = \mathfrak{h} \oplus \mathfrak{g}$ with $\rho = \mathrm{pr}_2$. Then the semidirect product Dorfman algebra is given by the bracket on $\mathfrak{w}$ given by

$$[h \oplus g, h' \oplus g'] = \left([g, h']_L - [h, g']_R + \eta(h, h')\right) \oplus [g, g']_{\mathfrak{g}} \tag{9.39}$$

for some bilinear form $\eta : \bigotimes^2 \mathfrak{h} \to \mathfrak{h}$.

9.3 Courant Algebroids

To study of Courant algebroids is to look at a structures encompassing vector bundles (§3.1), Courant spaces (§9.1) and Dorfman algebras (§9.2). Alternatively, one can start from a Lie algebroid and build a Leibniz representation on it, giving a Dorfman structure on it using a semidirect product as in example 9.2.2.

Example 9.3.1. Consider a Lie algebroid $(A, \rho, [\cdot,\cdot])$. Sections of this Lie algebroid enjoy the usual rules of Cartan calculus with operators $\iota, \mathcal{L}, d$ of degrees $-1, 0, 1$. We show that there exists a Leibniz representation of $\Gamma(A)$ on its dual $\Gamma(A^*)$ with the brackets

$$\begin{aligned}
[\cdot,\cdot] :_L \Gamma(A) \times \Gamma(A^*) &\to \Gamma(A^*)\\
(a, \alpha) &\mapsto \mathcal{L}_a \alpha\\
[\cdot,\cdot] :_R \Gamma(A^*) \times \Gamma(A) &\to \Gamma(A^*)\\
(\alpha, a) &\mapsto \iota_a d\alpha.
\end{aligned}$$

To check that this is indeed a Leibniz representation, we need to check all the Jacobi identities. For $a, b \in \Gamma(A)$, $\alpha, \beta \in \Gamma(A^*)$, using fact 3.3.1

$$\begin{aligned}
[\alpha, [a, b]]_R &= \iota_{[a,b]} d\alpha\\
&= \mathcal{L}_{[a,b]}\alpha - d \circ \iota_{[a,b]}\alpha\\
&= \iota_b \circ d \circ \iota_a \circ d\alpha + \mathcal{L}_a(\iota_b d\alpha)\\
&= [[\alpha, a]_R, b]_R + [a, [\alpha, b]_R]_L
\end{aligned}$$

and so on... Since we have found a Leibniz representation $\Gamma(A) \to \Gamma(A^*)$, we construct the semidirect product $\Gamma(A) \oplus \Gamma(A^*)$, that is a Dorfman algebra with bracket

$$[a \oplus \alpha, b \oplus \beta] = [a, b] \oplus (\mathcal{L}_a\beta - \iota_b d\alpha)\,. \tag{9.40}$$

We note that the lack of skew-symmetry is controlled by

$$\begin{aligned}[a \oplus \alpha, a \oplus \alpha] &= \mathcal{L}_a \alpha - \iota_a d\alpha \\ &= d \circ \iota_a \alpha\end{aligned}$$

Definition 9.3.1. A **Courant algebroid** is the tuple $(E \to M, \langle\cdot,\cdot\rangle, \rho, [\cdot,\cdot])$ such that $E \to M$ is a vector bundle, $\langle\cdot,\cdot\rangle$ is a symmetric non-degenerate bilinear form on the fibres. We also ask that the anchor map $\rho : E \to TM$ is $\mathbb{R}$-bilinear on sections of the algebroid such that the pushforward of the anchor

$$\rho_* : (\Gamma(E), [\cdot,\cdot]_E) \to (\Gamma(TM), [\cdot,\cdot]) \tag{9.41}$$

forms a Dorfman algebra. Asking for a Dorfman structure means that the bracket on sections of the algebroid is non skew-symmetric but still follows Leibniz identity. So for sections $a, b \in \Gamma(E)$ and $f \in C^\infty(M)$,

$$[a, f \cdot b]_E = f \cdot [a, b]_E + \rho_*(a)(f) \tag{9.42}$$

that adjoint action of a on f descends to the pushforward. Moreover, we ask for compatibility with the bilinear form in the following way

$$\rho_*(a)(\langle b, c\rangle) = \langle [a, b]_E, c\rangle + \langle b, [a, c]_E\rangle. \tag{9.43}$$

Finally, we ask for a so called "Tame symmetry", that is failure of the bracket to close is controlled by

$$[a, a]_E = D\langle a, a\rangle, \tag{9.44}$$

where $D = \sharp \circ \rho^* \circ d : C^\infty(M) \to \Gamma(E)$.

This definition implies that Courant algebroids are bundles of linear *Courant spaces* (see definition 9.1.1)

$$\rho_x : (E_x, \langle\cdot,\cdot\rangle_x) \to T_x M, \tag{9.45}$$

whose sections carry a Dorfman algebra structure.

Definition 9.3.2. A Courant algebroid $(E \to M, \langle\cdot,\cdot\rangle, \rho, [\cdot,\cdot])$ is *exact* when it is fibrewise exact. Considering the entire bundle of sections, it is then Dorfman exact. So we have the short exact sequence of Leibniz algebras (definition 9.2.7)

$$0 \to \Omega^1(M) \xrightarrow{i} \Gamma(E) \xrightarrow{\rho_*} \mathfrak{X}(M) \to 0, \tag{9.46}$$

with

$$[i(\Omega^1(M)), i(\Omega^1(M))]_E = 0. \tag{9.47}$$

The upshot is that all structure introduced in §9.1 and §9.2 applies here. In particular we recall that exact Dorfman algebras $\mathfrak{h} \oplus \mathfrak{g} \to \mathfrak{g}$ are represented by their characteristic classes $[C_\nabla] \in H^2(\mathfrak{g}, \mathfrak{h})$.

Definition 9.3.3. Let $E \to M$ be an exact Courant algebroid, such that $(\Gamma(E), [\cdot,\cdot]_E) \to (\Gamma(TM), [\cdot,\cdot])$ is an exact Dorfman algebra, then this algebra is represented by an element

$$[\eta] \in H^2(\mathfrak{X}(M), \Omega^1(M)) \tag{9.48}$$

called the **Ševera class** of E. Since this cohomology has coefficients in the space of one-forms, it also defines a closed H-flux

$$[H] \in H^3(M). \tag{9.49}$$

The relation being $\delta\eta = 0 \Leftrightarrow dH = 0$.

Example 9.3.2. The *Standard Courant algebroid*, found when describing Supergravity is $T\mathbb{M} = TM \oplus_M T^*M$. It has inner Courant bracket and Dorfman bracket given by

$$\langle X \oplus \alpha, Y \oplus \beta \rangle = \frac{1}{2}(\alpha(Y) + \beta(X))$$
$$[X \oplus \alpha, Y \oplus \beta] = [X, Y] \oplus (\mathcal{L}_X \beta - \iota_Y d\alpha + \iota_X \iota_Y H) .$$

The standard bracket can be twisted by an H-flux. This standard result is made more clear in [1, section 2.5.2].

Definition 9.3.4. Let $(E \to M, \langle \cdot, \cdot \rangle, \rho, [\cdot, \cdot])$ be a Courant algebroid, a **Dirac Structure** supported on submanifold $Q \subset M$ is a subbundle $D \subset E|_Q$, on the restricted support, such that

- D_q is maximally isotropic with respect to $\langle \cdot, \cdot \rangle_q$ for all $q \in Q$
- $\rho : D \to TQ$ is an anchor over Q
- the space of sections on Dirac space D is involutive,

$$[\Gamma(D), \Gamma(D)] \subseteq \Gamma(D). \tag{9.50}$$

In other words, the bundle D is fiberwise a *Dirac space* (see definition 9.1.7). Since Dirac structures are fiberwise Dirac, extending proposition 9.1.1 to the entire bundle, there exists a 2-form given by eq. (9.7)

$$\omega_q^D \in \Lambda^2(\mathrm{im}(\rho_q)) \subset \Lambda^2 T_q^* M \tag{9.51}$$
$$d\omega_q^D(X_q, Y_q, Z_q) = \langle [X, Y], Z \rangle|_q \tag{9.52}$$

NEEDS more work up here

Bibliography

[1] Carlos Zapata-Carratala. A Landscape of Hamiltonian Phase Spaces: on the foundations and generalizations of one of the most powerful ideas of modern science. 2019. URL `http://arxiv.org/abs/1910.08469`.

[2] José Miguel O'Farrill. BRST Cohomology and its Applications to Two Dimensional Conformal Field Theory. 1989. URL `https://inspirehep.net/literature/289080`.

[3] Rui Loja Fernandes. Rui Loja Fernandes and Ioan M˘ arcut , Lectures on Poisson Geometry. 2015. URL `http://www.math.illinois.edu/{~}ruiloja/Math595/book.pdf`.

[4] Jerrold E. Marsden and Tudor Ratiu. Reduction of Poisson manifolds. *Letters in Mathematical Physics*, 11(2):161–169, feb 1986. ISSN 03779017. doi: 10.1007/BF00398428. URL `https://link.springer.com/article/10.1007/BF00398428`.

[5] Branislav Jurčo, Lorenzo Raspollini, Christian Sämann, and Martin Wolf. L -Algebras of Classical Field Theories and the Batalin–Vilkovisky Formalism. *Fortschritte der Physik*, 67(7), sep 2019. ISSN 15213978. doi: 10.1002/prop.201900025. URL `https://arxiv.org/abs/1809.09899`.

[6] Kyle Wright. Lie algebroid gauging of non-linear sigma models. *Journal of Geometry and Physics*, 146, 2019. ISSN 03930440. doi: 10.1016/j.geomphys.2019.103490.

[7] S. L. Lyakhovich and A. A. Sharapov. BRST quantization of quasi-symplectic manifolds and beyond. dec 2003. doi: 10.1063/1.2190775. URL `https://arxiv.org/abs/hep-th/0312075`.

[8] Alexei Kotov and Thomas Strobl. Lie algebroids, gauge theories, and compatible geometrical structures. mar 2016. doi: 10.1142/S0129055X19500156. URL `http://arxiv.org/abs/1603.04490http://dx.doi.org/10.1142/S0129055X19500156`.

[9] S. C. Coutinho. *A Primer of Algebraic D-Modules*. Cambridge University Press, may 1995. ISBN 9780521551199. doi: 10.1017/CBO9780511623653. URL `/core/books/primer-of-algebraic-dmodules/87B8F8AB3B53DBA8A8BD33A058E54473https://www.cambridge.org/core/product/identifier/9780511623653/type/book`.

[10] Christopher Braun and Andrey Lazarev. Homotopy BV algebras in Poisson geometry. apr 2013. doi: 10.1090/S0077-1554-2014-00216-8. URL `http://arxiv.org/abs/1304.6373http://dx.doi.org/10.1090/S0077-1554-2014-00216-8`.

[11] Theodore James Courant. Dirac manifolds. *Transactions of the American Mathematical Society*, 319(2):631–661, feb 1990. ISSN 0002-9947. doi: 10.1090/S0002-9947-1990-0998124-1. URL `https://www.ams.org/journal-terms-of-usehttp://www.ams.org/jourcgi/jour-getitem?pii=S0002-9947-1990-0998124-1`.

[12] Marco Gualtieri. Generalized complex geometry. jan 2004. URL `http://arxiv.org/abs/math/0401221`.