# RHEONOMIC SUPERGRAVITY

by

# Guillaume Trojani

Supervisor : Pr Richard Szabo

 $\label{eq:ABSTRACT: Based on course at the EMPG by Andrew Beckett and al.}$ 

### 1 Lecture 1

Mostly covered in [1]

## 1.1 Klein Geometry

**Definition 1.1.1.** Homogenous spaces, group action on LEFT (M,G) manifold, G Lie group action that is transitive. That is  $\forall x, y \in M \exists g \in G | gx = y$ . For element  $o \in G$ , the stab  $G_o = \{g \in G | go = o\}$ , it turns out that

$$G/G_o \cong M$$
 (1.1)

a G-invariant diffeomorphism.

If G is a group and H a closed subgroup, then G/H is a homogeneous space.

#### **Definition 1.1.2.** (Klein Geometry)

A Klein geometry is a pair (G, H) where H closed subgroup of G and G/H is connected. The last condition is not necessary and not always there but is convenient.

**Remark 1.1.1.** Given a Klein geometry (G, H), we can get a pair of Lie algebra  $(G, H) \xrightarrow{Lie} (\mathfrak{g}, \mathfrak{h})$ . (See Lie functor??) gives a short exact sequence of H-modules

$$0 \to \mathfrak{h} \to \mathfrak{g} \to \frac{\mathfrak{g}}{\mathfrak{h}} \to 0 \tag{1.2}$$

If the sequence splits then get

$$\mathfrak{g}=\mathfrak{h}\oplus\mathfrak{m}$$
 $\mathfrak{m}\cong\mathfrak{g}/\mathfrak{h}$ 

as H-modules. m is a submodule, we call it reductive? Put in the table here and explain.

Given  $G \odot M$  such that (M,G) homogeneous then the

$$(r_g)_*: T_x M \to T_{gx} M \tag{1.3}$$

Taking an origin  $o \in M$  and let  $H = G_o$  then for  $h \in H$   $(l_h)_* \in GL(T_oM)$  that sends  $h \to (l_h)_*$  is a rep of H on  $T_oM$  called the linear isotropy rep of (M, G, o).

If  $M \cong G/H$ ,

$$T_o M \xrightarrow{\sim} T_H(G/H) \xrightarrow{\sim} \mathfrak{g}/\mathfrak{h} \cong m$$
 (1.4)

$$0 \to T_e H \to T_e G \to T_H(G/H) \to 0 \tag{1.5}$$

the upshot is we get

$$T_o M \cong \mathfrak{g}/\mathfrak{h} \cong \mathfrak{m}$$
 (1.6)

and linear structures on  $\mathfrak{g}/\mathfrak{h}$  induce a geometric structures on M.

The correspondences between H-invariant tensors of  $T_oM$  and G-equivariant tensor fields on M.

# Example 1.1.1.

$$\tau \in T_o M \to t_o = \tau \tag{1.7}$$

such that  $t_{g \cdot o} = (l_g)_* \tau$  this is well defined because for g'o = go we have  $g'g^{-1} \in H$  by H-invariance of  $\tau$ , meaning t is well defined. The other way is by evaluating at  $o \in M$ .

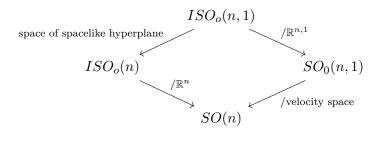
In particular, a pseudo inner product on  $\mathfrak{g}/\mathfrak{h}$  gives rise to a pseudo-riemanian metric on M.

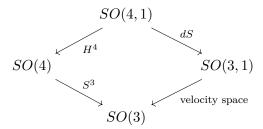
**Definition 1.1.3.** (Metric Klein geometry)  $(G, H, \eta)$ 

- (G, H) klein geo
- $\eta$  is a pseudo inner product on g/h which is H-invariant

**Example 1.1.2.** •  $G = ISO_o(\mathbb{R}^{d-1,1}) \cong SO_o(d-1,1) \ltimes \mathbb{R}^{d-1,1}$ , with H the SO part, then  $G/H \cong \mathbb{R}^{d-1,1}$  has the structure steming from the lie algebra structure of  $g/h \cong \mathbb{R}^{d-1,1}$ 

•  $G = ISO_o(\mathbb{R}^d)$  with H = SO(d) then  $G/H \cong \mathbb{R}^d$ 





ISO means inhomogeneous something

#### 1.2 Cartan geometry

# **Definition 1.2.1.** (Cartan geom)

 $(\pi: P \to M, A)$  modelled on a Klein geometry (G, H) is a principle right H-bundle  $P \to M$  with a Cartan connection  $A \in \Omega^1(P, \mathfrak{g})$ . Noting that the cartan connection takes values in the larger Lie algebra. this satisfies the conditions

- $A_p: T_pP \to \mathfrak{g}$  is a linear isomorphism
- $(R_h)^*A = \operatorname{Ad}_{h^{-1}} \circ A \text{ for all } h \in H$
- $A(\xi_X) = X$  for  $X \in \mathfrak{h}$  for fundamental vector field  $\xi$  of  $H \circlearrowleft P$ , an Ehresmannn-like connection

Remark 1.2.1. Because of the first condition,

$$dimP = dimG$$

$$dimM = dim(G/H)$$

$$(A_p)^{-1} : \mathfrak{g} \to T_pP$$

$$(A_0)^{-1} : \mathfrak{g} \to \mathfrak{X}(P)$$

$$(A_0)^{-1} : \mathfrak{h} \to \mathfrak{X}_{\text{vert}}(P)$$

$$X \mapsto \xi_X$$

If G/H is metrci klein, M inherits a vetric of save sign with

$$T_x M \cong T_p P / \ker(\pi_*) \cong \mathfrak{g}/\mathfrak{h} \cong \mathfrak{m}$$
 (1.8)

$$TM \cong P \times_H \mathfrak{g}/\mathfrak{h} \tag{1.9}$$

**Definition 1.2.2.** The curvature of a cartan geometry

$$F(A) = dA + \frac{1}{2}[A, A] \in \Omega^2(P, \mathfrak{g})$$
 (1.10)

Diagrams diagrams

Take curvature of cartan connection

$$F[A] = dA + \frac{1}{2}[A, A]$$

$$= d\omega + de + \frac{1}{2}[\omega + e, \omega + e]$$

$$= \left(d\omega + \frac{1}{2}[\omega, \omega]_h + [\omega, e]_h + [e, e]_h + de + \frac{1}{2}[\omega, \omega]_m + [\omega, e]_m + \frac{1}{2}[e, e, ]_h\right)$$

colors:

$$\hat{F} = \Omega(\omega) + \frac{1}{2}[e, e]_h, T = d_\omega e + \frac{1}{2}[e, e]_m \text{ If } \hat{F} \text{ is flat, then } \Omega(\omega) = -\frac{1}{2}[e, e]_h \text{ and } T = 0 \text{ then } d_\omega e = \frac{1}{2}[e, e]_m \text{ and } T = 0 \text{ then } d_\omega e = \frac{1}{2}[e, e]_m \text{ and } T = 0 \text{ then } d_\omega e = \frac{1}{2}[e, e]_m \text{ and } T = 0 \text{ then } d_\omega e = \frac{1}{2}[e, e]_m \text{ and } T = 0 \text{ then } d_\omega e = \frac{1}{2}[e, e]_m \text{ and } T = 0 \text{ then } d_\omega e = \frac{1}{2}[e, e]_m \text{ and } T = 0 \text{ then } d_\omega e = \frac{1}{2}[e, e]_m \text{ and } T = 0 \text{ then } d_\omega e = \frac{1}{2}[e, e]_m \text{ and } T = 0 \text{ then } d_\omega e = \frac{1}{2}[e, e]_m \text{ and } T = 0 \text{ then } d_\omega e = \frac{1}{2}[e, e]_m \text{ and } T = 0 \text{ then } d_\omega e = \frac{1}{2}[e, e]_m \text{ and } T = 0 \text{ then } d_\omega e = \frac{1}{2}[e, e]_m \text{ and } T = 0 \text{ then } d_\omega e = \frac{1}{2}[e, e]_m \text{ and } T = 0 \text{ then } d_\omega e = \frac{1}{2}[e, e]_m \text{ and } T = 0 \text{ then } d_\omega e = \frac{1}{2}[e, e]_m \text{ and } T = 0 \text{ then } d_\omega e = \frac{1}{2}[e, e]_m \text{ and } T = 0 \text{ then } d_\omega e = \frac{1}{2}[e, e]_m \text{ and } T = 0 \text{ then } d_\omega e = \frac{1}{2}[e, e]_m \text{ and } T = 0 \text{ then } d_\omega e = \frac{1}{2}[e, e]_m \text{ and } T = 0 \text{ then } d_\omega e = \frac{1}{2}[e, e]_m \text{ and } T = 0 \text{ then } d_\omega e = \frac{1}{2}[e, e]_m \text{ and } T = 0 \text{ then } d_\omega e = \frac{1}{2}[e, e]_m \text{ and } T = 0 \text{ then } d_\omega e = \frac{1}{2}[e, e]_m \text{ and } T = 0 \text{ then } d_\omega e = \frac{1}{2}[e, e]_m \text{ and } T = 0 \text{ then } d_\omega e = \frac{1}{2}[e, e]_m \text{ and } T = 0 \text{ then } d_\omega e = \frac{1}{2}[e, e]_m \text{ and } T = 0 \text{ then } d_\omega e = \frac{1}{2}[e, e]_m \text{ and } T = 0 \text{ then } d_\omega e = \frac{1}{2}[e, e]_m \text{ and } T = 0 \text{ then } d_\omega e = \frac{1}{2}[e, e]_m \text{ and } T = 0 \text{ then } d_\omega e = \frac{1}{2}[e, e]_m \text{ and } T = 0 \text{ then } d_\omega e = \frac{1}{2}[e, e]_m \text{ and } T = 0 \text{ then } d_\omega e = \frac{1}{2}[e, e]_m \text{ and } T = 0 \text{ then } d_\omega e = \frac{1}{2}[e, e]_m \text{ and } T = 0 \text{ then } d_\omega e = \frac{1}{2}[e, e]_m \text{ and } T = 0 \text{ then } d_\omega e = \frac{1}{2}[e, e]_m \text{ and } T = 0 \text{ then } d_\omega e = \frac{1}{2}[e, e]_m \text{ and } T = 0 \text{ then } d_\omega e = \frac{1}{2}[e, e]_m \text{ and } T = 0 \text{ then } d_\omega e = \frac{1}{2}[e, e]_m \text{ and } T = 0 \text{ then } d_\omega e = \frac{1}{2}[e, e]_m \text{ and } T = 0 \text{$$

# **Bibliography**

[1] Derek K. Wise. MacDowell-Mansouri gravity and Cartan geometry. Classical and Quantum Gravity, 27(15), nov 2010. ISSN 02649381. doi: 10.1088/0264-9381/27/15/155010. URL http://arxiv.org/abs/gr-qc/0611154.