Rheonomic Supergravity

by

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1 Lecture 1

Mostly covered in [1], lectured by comrade Andrew Beckett.

1.1 Klein Geometry

In the lecture, let M denote a smooth manifold, G a Lie group and throughout, $G \circlearrowright M$ *transitively.* [1]

Definition 1.1.1. A homogeneous space (M, G) is a smooth manifold M together with a transitive Lie group action $G \circlearrowright M$. Right now we stick with a **left** group action, but might change in the notes depending on Andrew's taste. We study homogeneous spaces by considering subgroups $H \subset G$, and looking at the geometry of the quotient space, which can be thought as the space modelled on H. If we pick a point $o \in M$ to act as an "origin", the *stabiliser* which fixes o is the subgroup

$$\mathrm{stab}_o(G) := \{g \in G \mid g \cdot o = o\} \,. \tag{1.1}$$

We denote this subgroup of symmetries leaving o invariant $\mathrm{stab}_o(G) = G_o$, or the *isotropy* subgroup at o. There exists an isomorphism

$$\begin{aligned} M &\cong G/G_o \\ g \cdot o &\mapsto g\, G_0 \end{aligned}$$

which is a G_0-invariant diffeomorphism. It turns out that this is independent of the origin chosen as stabiliser subgroups at different points are conjugate by transitivity of the group action.

Definition 1.1.2. Given a homogeneous space (M, G), and closed subgroup $H \subset G$ the space with "features" H is G/H and it is also a homogeneous space. Sometimes, this quotient is called the homogeneous space in the literature.

Definition 1.1.3. (Klein Geometry)
A (smooth connected) Klein geometry is a pair (G, H) where H closed subgroup of G such that G/H is connected. Note that

$$\begin{tikzcd} G \arrow[d, "\pi"] \\ G/H \end{tikzcd} \tag{1.2}$$

is automatically a principal H-bundle, because the fibres are left H-cosets and the action of H on the fibres is obviously an H-torsor. Consider a Klein geometry (G, H), then the Lie functor gives us a pair of Lie algebras $(G, H) \xrightarrow{Lie} (\mathfrak{g}, \mathfrak{h})$, where $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$. Notably, this induces a short exact sequence of H-modules. [2]

$$\begin{tikzcd} 0 \arrow[r] & \mathfrak{h} \arrow[r] & \mathfrak{g} \arrow[r] & \mathfrak{g}/\mathfrak{h} \arrow[r] \arrow[l, bend left] & 0 \end{tikzcd} \tag{1.3}$$

Definition 1.1.4. If a short exact sequence of H-modules as above splits, then we call $(\mathfrak{g}, \mathfrak{h})$ **reductive**. Equivalently[3], there exists an H-invariant subspace $\mathfrak{m} \subseteq \mathfrak{g}$ such that

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}. \tag{1.4}$$

[1] That is $\forall x, y \in M$, $\exists g \in G$ such that $g \cdot x = y$.

[2] A vector space on which H acts linearly; a representation of H. Here, H acts via the adjoint representation on $\mathfrak{g}$ and $\mathfrak{h}$ and via the quotient representation on $\mathfrak{g}/\mathfrak{h}$.

[3] $\mathfrak{m}$ is the image of the splitting map.

Invariance of $\mathfrak{m}$ under the action of H is equivalent to

$$[\mathfrak{h}, \mathfrak{m}] \subseteq \mathfrak{m}. \tag{1.5}$$

As H-modules, we have $\mathfrak{m} \cong \mathfrak{g}/\mathfrak{h}$. The reductive case will always be denoted by green throughout the rest of this lecture.

Definition 1.1.5. A *reductive* pair of Lie algebras $(\mathfrak{g}, \mathfrak{h})$ is called **symmetric** if it further follows the condition

$$[\mathfrak{m}, \mathfrak{m}] \subseteq \mathfrak{h}. \tag{1.6}$$

We will follow the convention of denoting in blue the symmetric case from now on. The terminology is summarised in the table below, where $\mathfrak{m}$ is vector space complement to $\mathfrak{h}$ in $\mathfrak{g}$ (it is not H-invariant in general):

$\subseteq$	*General*	*Reductive*	*Symmetric*
$[\mathfrak{h}, \mathfrak{h}]$	$\mathfrak{h}$	$\mathfrak{h}$	$\mathfrak{h}$
$[\mathfrak{h}, \mathfrak{m}]$	$\mathfrak{h} \oplus \mathfrak{m}$	$\mathfrak{m}$	$\mathfrak{m}$
$[\mathfrak{m}, \mathfrak{m}]$	$\mathfrak{h} \oplus \mathfrak{m}$	$\mathfrak{h} \oplus \mathfrak{m}$	$\mathfrak{h}$

Example 1.1.1. Consider the homogeneous space $G \circlearrowright M$. The left group action induces a morphism of tangent bundles

$$(L_g)_* : T_x M \to T_{g \cdot x} M. \tag{1.7}$$

Considering the isotropy subgroup $H := G_o$ for a chosen origin $o \in M$, then for $h \in H$, the pushforward $(L_h)_* \in \mathrm{GL}(T_o M)$ is an automorphism of the tangent bundle at the origin. Further, the map

$$\begin{aligned} \rho : H &\to \mathrm{GL}(T_o M) \\ h &\mapsto (L_h)_* \end{aligned}$$

is a representation of H on $T_o M$. We call this the *linear isotropy representation* of (M, G, o).

Remark 1.1.1. For a Klein geometry $M \cong G/H$, we have the sequence of isomorphisms of H-representations

$$T_o M \xrightarrow{\sim} T_H(G/H) \xrightarrow{\sim} \mathfrak{g}/\mathfrak{h} \quad \cong m \tag{1.8}$$

which can be constructed from the transitivity of the group action. However, looking at the tangent bundles on the group manifolds we find the following short exact sequence

$$0 \to T_e H \xrightarrow{(L_e)_*} T_e G \xrightarrow{(\pi_e)_*} T_H(G/H) \to 0, \tag{1.9}$$

which is really the same as eq. (1.3). But noticing the isomorphisms of H-representations in eq. (1.8), the tangent bundle of the manifold M at the origin is modelled on

$$T_o M \cong \mathfrak{g}/\mathfrak{h} \quad \cong \mathfrak{m} \tag{1.10}$$

This facts leads us to notice a broader correspondence between geometry and algebra for Klein geometries.

Proposition 1.1.1. (Correspondence)
Let (G, H) be a Klein geometry, then linear structures on $\mathfrak{g}/\mathfrak{h}$ correspond to geometric structures on the manifold $M \cong G/H$ in the following sense:

$$\{H\text{-invariant tensors of } \mathfrak{g}/\mathfrak{h} \cong T_o M\} \qquad \longleftrightarrow \qquad \{G\text{-invariant tensor fields on } M\} \tag{1.11}$$

Proof. $(\Rightarrow)$
Consider $\tau \in \bigotimes T_oM$ an H-invariant tensor, such that $(L_h)_*\tau = \tau$ for $h \in H$. Consider the map $\tau \mapsto T_o$ for a fixed origin, we then left translate by $g \in G$ to span the manifold. So define

$$T_{g\cdot o} = (L_g)_* T_o \ \in \Gamma(\bigotimes T_{g\cdot o}M).$$

for all $g \in G$. This map is well defined since if $g \cdot o = g' \cdot o$ are two left translation yielding the same tensor field at $T_{g\cdot o} = T_{g'\cdot o}$, then $g^{-1}g' \in H$ an isotropy. But $\tau = T_o$ is H-invariant implying that $(L_{g^{-1}g'})_* T_o = T_o$ is G-invariant.
$(\Leftarrow)$
Given a G-invariant $T \in \Gamma(TM)$, the evaluation map $\mathrm{ev}_o : \Gamma(TM) \to T_oM$ sends $T \to T_o$ which has isotropy group H. ☐

In particular, a pseudo inner product on $\mathfrak{g}/\mathfrak{h}$ gives rise to a pseudo-riemannian metric on M.

Definition 1.1.6. A Metric Klein geometry (G, H, η) is

- (G, H) a Klein geometry
- η is a pseudo inner product on $\mathfrak{g}/\mathfrak{h}$ which is H-invariant in the sense describe in the proposition above.

Let's just recall the isometries of flat spacetime for completeness.

Definition 1.1.7. The Poincaré group $\mathrm{ISO}(d-1, 1)$ of d-dimensional Minkowski spacetime is the isotropy group that leaves the split quadratic form invariant. We also call it the *inhomogeneous special orthonormal* group as it consists of disconnected components. It consists of the semidirect product of the Lorentz group (not **necessarily** orthochronous) and the group of spacetime translations,

$$\mathrm{ISO}(d-1, 1) \cong \mathrm{SO}(d-1, 1) \ltimes \mathbb{R}^{d-1,1} \tag{1.12}$$

with multiplication

$$(g, \alpha) \cdot (f, \beta) = (gf, \alpha + f \cdot \beta) \tag{1.13}$$

Example 1.1.2. Using our new-found understanding of the Poincaré group, we understand it as a homogeneous space. For example, taking $G = ISO(d-1, 1)$ and considering the isotropy group at an origin o to be the Lorentz group $H = \mathrm{SO}(d-1, 1)$. The manifold $G/H \cong \mathbb{R}^{d-1,1}$ is connected and at the level of algebras $\mathfrak{g}/\mathfrak{h} \cong \mathbb{R}^{d-1,1}$. Further, the bilinear form on $\mathfrak{g}/\mathfrak{h}$ is invariant under rotations (H-invariant). Therefore, the Poincaré group with its Lorentz subgroups is a Klein geometry. A similar argument can be made for Euclidean Poincaré groups. It is interesting to note however, that using the correspondence established in proposition 1.1.1, the Lorentz invariant bilinear form on $\mathfrak{g}/\mathfrak{h}$ corresponds to Poincaré invariant tensor fields on the full Minkowski spacetime.

Example 1.1.3. We can summarise the ideas presented above in the following reductive diagrams, where arrows represent the subgroup that is quotiented out:

$$\begin{tikzcd}
& ISO(d-1,1) \arrow[dl, "\text{spacelike hyperplane}"'] \arrow[dr, "\mathbb{R}^{d-1,1}= \text{event space}"] & \\
ISO(d-1) \arrow[dr, "\mathbb{R}^{d-1}"] & & SO(d-1,1) \arrow[dl, "\text{velocity space}"] \\
& SO(n) &
\end{tikzcd}$$

By "event space", we mean that we quotient by all possible translations of spacetime, while the "spacelike hyperplane" is quotienting by the Lorentz boosts. [4] In 4 spacetime dimensions, the more familiar diagram:

$$\begin{tikzcd}
 & SO(4,1) \arrow[dl, "H^4"'] \arrow[dr, "dS"] & \\
SO(4) \arrow[dr, "S^3"] & & SO(3,1) \arrow[dl, "\text{velocity space}"] \\
 & SO(3) &
\end{tikzcd}$$

I don't know much about dS, so maybe someone can complete here.

1.2 Cartan geometry

Assuming familiarity with Maurer-Cartan form $\omega_G \in \Omega^1(G, \mathfrak{g})$ for a Lie group G, which are the G-invariant one forms on G, we review the more general notion of connections on a principal G-bundle. For some reason, the literature often prefers *right* action on principal bundles. idk

Definition 1.2.1. An Ehresmann connection on a principal right G-bundle $(P \to M)$ is a $\mathfrak{g}$-valued one-form A on P, that is a map

$$\omega : TP \to \mathfrak{g} \tag{1.14}$$

such that

$$(R_g)^*\omega = \mathrm{Ad}_{g^{-1}}\omega \;\text{ for all } g \in G$$
$$\omega(\xi_X) = X \;\text{ for } X \in V_p.$$

Equivalently an Ehresmann connection is a *choice* of Horizontal distribution such that $T_pP = V_p \oplus H_p$ where H_p is G-equivariant. This is better explained in the EKC notes from last year. We now turn our attention to the case of a principal bundle modelled on a reduced geometry.

Definition 1.2.2. (Cartan Geometry)
A Cartan Geometry $(\pi : P \to M, A)$ modelled on a Klein geometry (G, H) is a principle right H-bundle $P \to M$ with a **Cartan connection** $A \in \Omega^1(P, \mathfrak{g})$ satisfying the conditions:

- $A_p : T_pP \to \mathfrak{g}$ is a linear **isomorphism** (or simply put a $\mathfrak{g}$-valued one-form for on P).
- $(R_h)^*A = \mathrm{Ad}_{h^{-1}} \circ A$ for all $h \in H$
- $A(\xi_X) = X$ for $X \in \mathfrak{h}$ and fundamental vector field ξ_X of $H \circlearrowright P$.

Importantly, the Cartan connection one-form takes values in the larger Lie algebra $\mathfrak{g}$, because the tangent space of this principal bundle is isomorphic to the larger Lie algebra $\mathfrak{g}$. Morally speaking, the algebra $\mathfrak{g} \cong T_pP$, while $\mathfrak{h} \cong V_p$ and $\mathfrak{m} \cong H_P \cong T_{\pi(p)}M$.

Remark 1.2.1. Consequently, in a Cartan geometry we identify because of the first condition $\dim(P) = \dim(G)$ and $\dim(M) = \dim(G/H)$. Moreover, the Cartan connection isomorphisms implies that upon restricting to the subalgebra $\mathfrak{h}$ and choosing an origin

$$(A_o)^{-1} : \mathfrak{h} \to \mathfrak{X}_{\mathrm{vert}}(P)$$
$$X \mapsto \xi_X.$$

[4] not sure here

Remark 1.2.2. For a Cartan geometry $(P \to M : A)$ modelled on a *metric* Klein geometry (G, H), the correspondence 1.1.1 implies that the isomorphism

$$T_x M \cong T_p P/\ker(\pi_*) \cong \mathfrak{g}/\mathfrak{h} \quad \cong \mathfrak{m} \tag{1.15}$$

give a metric of the same sign to the manifold M. Where in the above $x := \pi(p)$. In general, the full tangent bundle of the manifold can be thought of as the associated bundle of the H-action

$$TM \cong P \times_H \mathfrak{g}/\mathfrak{h} \tag{1.16}$$

Definition 1.2.3. The curvature of a Cartan Geometry $(\pi : P \to M, A)$ modelled on a Klein geometry (G, H) is

$$F(A) = dA + \frac{1}{2}[A, A] \in \Omega^2(P, \mathfrak{g}) \tag{1.17}$$

We summarise the different connections in a Cartan geometry, whether it is reductive or not in the following diagram. We denote $e : TP \to \mathfrak{g}/\mathfrak{h} \cong TM$ as the veilbein of this geometry, while the Ehresmann connection $\omega : TP \to \mathfrak{h}$ is denoted in green.

$$\begin{tikzcd} & & \mathfrak{h} \\ TP \arrow[r, "A"] \arrow[urr, bend left, "\omega"] \arrow[drr, bend right, "e"'] & \mathfrak{g} \arrow[ur] \arrow[dr, two heads] & \\ & & \mathfrak{g}/\mathfrak{h} \cong \mathfrak{m} \end{tikzcd} \tag{1.18}$$

So in the reductive case

$$A = \omega + e. \tag{1.19}$$

Considering the curvature of these connections, we have the diagram

$$\begin{tikzcd} & & \mathfrak{h} \\ \bigwedge^2(TP) \arrow[r, "F"] \arrow[urr, bend left, "\hat{F}"] \arrow[drr, bend right, "T"'] & \mathfrak{g} \arrow[ur] \arrow[dr, two heads] & \\ & & \mathfrak{g}/\mathfrak{h} \cong \mathfrak{m} \end{tikzcd} \tag{1.20}$$

with the understanding that in the reductive case,

$$F = \hat{F} + T \tag{1.21}$$

where T is called the *torsion* and $\hat{F}$ is the Ehresmann part of the curvature.

Remark 1.2.3. A short calculation shows that if the geometry is reductive then the Cartan curvature

$$\begin{aligned} F[A] &= dA + \frac{1}{2}[A, A]_{\mathfrak{g}} \\ &= d\omega + de + \frac{1}{2}[\omega + e, \omega + e]_{\mathfrak{g}} \\ &= \left(d\omega + \frac{1}{2}[\omega, \omega]_{\mathfrak{h}} + [\omega, e]_{\mathfrak{h}} + \frac{1}{2}[e, e]_{\mathfrak{h}} + de + \frac{1}{2}[\omega, \omega]_{\mathfrak{m}} + [\omega, e]_{\mathfrak{m}} + \frac{1}{2}[e, e,]_{\mathfrak{m}} \right) \end{aligned}$$

Now because ω is an Ehresmann connection and it reduces to a Maurer-Cartan form on the fibres we its curvature 2-form

$$d\omega + \frac{1}{2}[\omega, \omega]_{\mathfrak{h}} = \Omega(\omega).$$

If we are in the reductive case, we recall that $[\mathfrak{h}, \mathfrak{m}] \subseteq \mathfrak{m}$, so the bracket $[\omega, e]_{\mathfrak{h}} = 0$. With further restriction in the symmetric case, $[\mathfrak{m}, \mathfrak{m}] \subseteq \mathfrak{h}$, the bracket $[e, e]_{\mathfrak{h}} = 0$, but we won't assume it in general so instead define a covariant derivative on $\mathfrak{m}$

$$d^{\omega} = d + \omega. \tag{1.22}$$

So in conclusion, the Ehresmann part of the curvature in a Cartan geometry is

$$\hat{F} = \Omega(\omega) + \frac{1}{2}[e, e]_{\mathfrak{h}}, \tag{1.23}$$

while the torsion is

$$T = d^{\omega} e + \frac{1}{2}[e, e]_{m} \tag{1.24}$$

Some interpretation of flatness and torsion freedom needed here.

Bibliography

[1] Derek K. Wise. MacDowell-Mansouri gravity and Cartan geometry. *Classical and Quantum Gravity*, 27(15), nov 2010. ISSN 02649381. doi: 10.1088/0264-9381/27/15/155010. URL http://arxiv.org/abs/gr-qc/0611154.