Modeling the Number of Insureds' Cars Using Queuing Theory

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June 23, 2016

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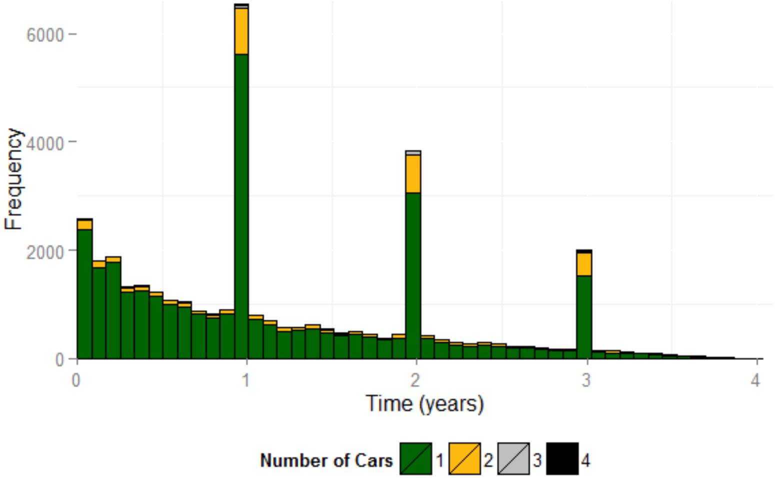

FIGURE – Life of an insurance policy

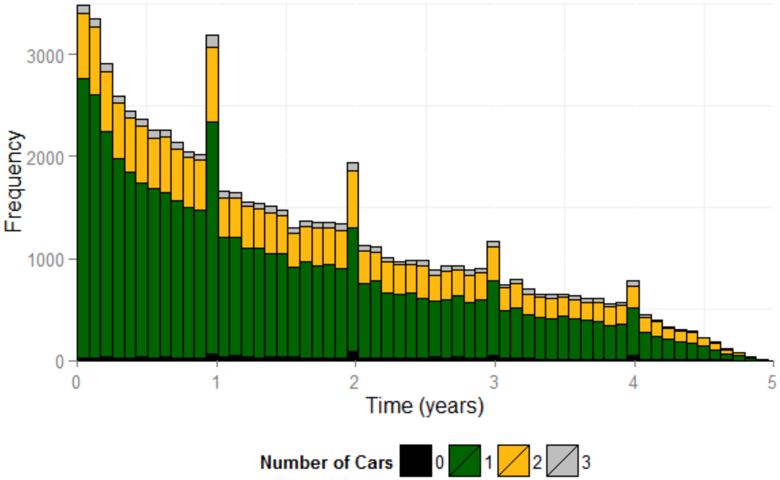

FIGURE – Time prior the addition of a car

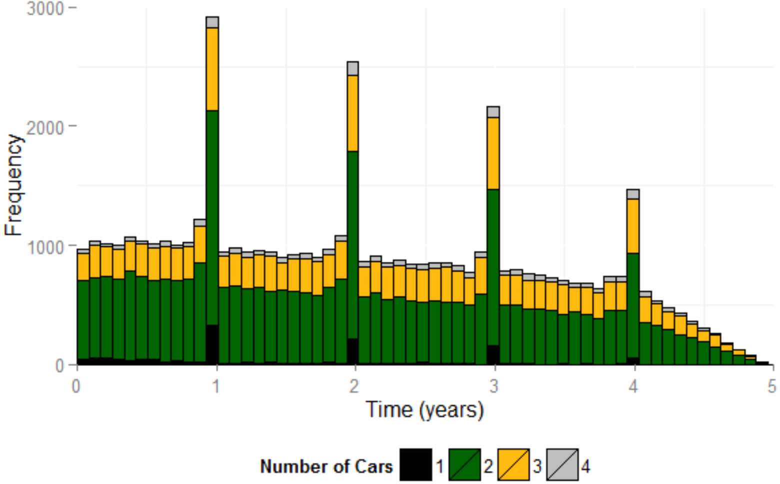

FIGURE – Time prior the removal of a car

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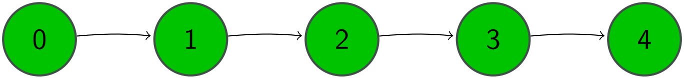

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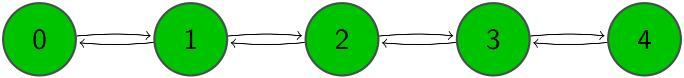

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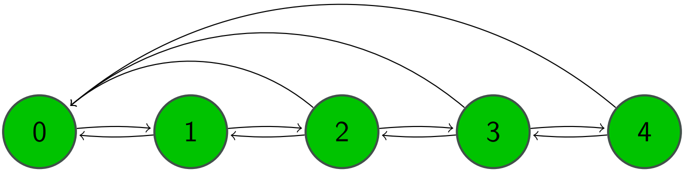

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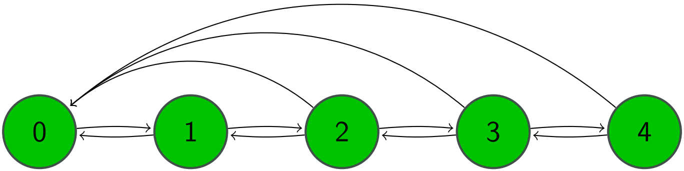

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- a shock at contract renewal.

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- p : probability of insurance renewal.

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$$\mathcal{Z}(t) \sim \text{Poisson}(\lambda t),$$

with the distribution :

$$\Pr(\mathcal{Z}(t) = j) = e^{-\lambda t} \frac{(\lambda t)^j}{j!}.$$

Kendall's notation

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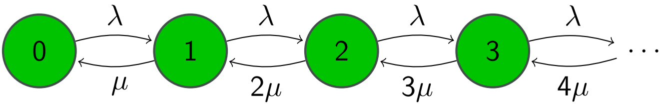
- D : queuing **D**iscipline (ex : first come, first served).

$M/M/\infty$ Model



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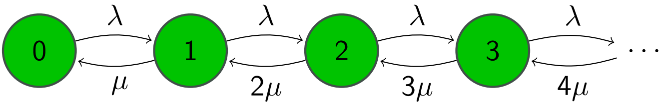

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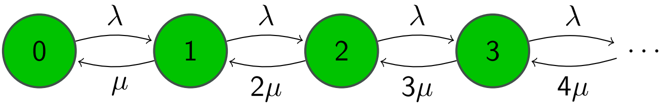

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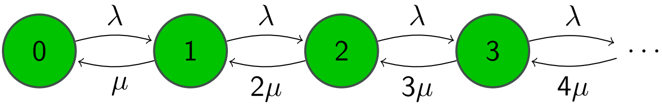

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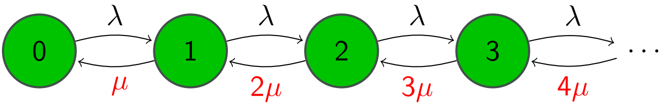

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From ∞, there is no limit to the number of cars insured.

$M/M/\infty$ Model distribution

Using the law of total probability :

$$\Pr(\mathcal{N}(t) = i) = \sum_{j=i} \Pr(\mathcal{N}(t) = i | \mathcal{Z}(t) = j) \Pr(\mathcal{Z}(t) = j)$$

For the next step, define :

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Pour $i = j = 1$ et $\mathcal{N}(0) = 0$, nous avons :

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A PGF is defined as :

$$P_X(z) = \sum \Pr(X = x) \times z^x,$$

and allows to easily find moments (expectation, variance) as well as probabilities by mathematical transformation.

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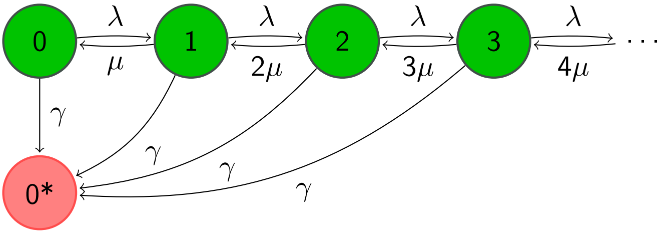

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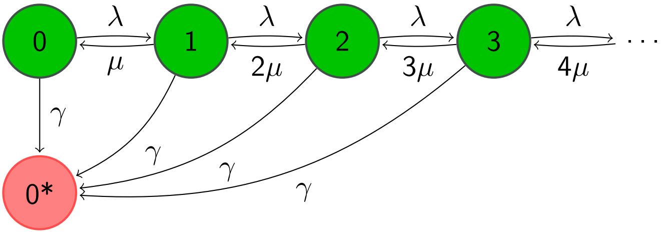

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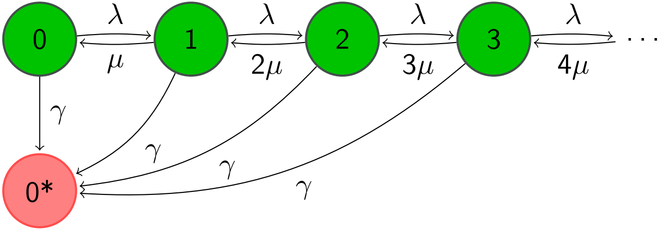

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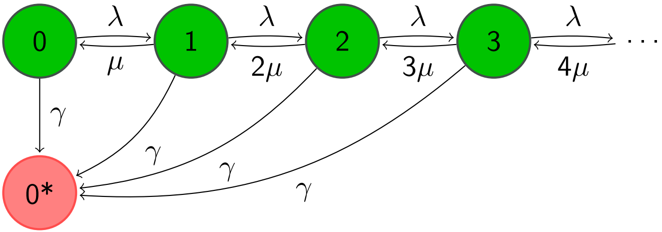

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The special case, $m = 1, n = 0$, could be, for example, an insured person without any car but is home insured or an insured who stored his car for a while.

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$$\begin{aligned} P_{\mathcal{H}_a(t)}(z,t) &= P_{\mathcal{M}(t)}(P_{\mathcal{N}_a(t)}(z,t),t) \\ &= 1 - e^{-\gamma t} p^{\lfloor t+c \rfloor} \left[1 - \left((z-1)e^{-\mu t} + 1 \right)^a e^{(z-1)(1-e^{-\mu t})\frac{\lambda}{\mu}} \right] \end{aligned}$$

Likelihood function

By independance of the various distributions, the likelihood function is simply the multiplication of exponential and Bernoulli distributions.

$$\mathcal{L}(E, A, S, Q, T, V|\lambda, \gamma, \mu, p) \propto e^{-\lambda T}\lambda^{E}e^{-\mu V}\mu^{S}e^{-\gamma T}\gamma^{A}p^{Q}(1-p)^{\lfloor T\rfloor - Q}$$

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Parameter estimates

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\hat{\mu} &= \frac{\sum^{\xi} S_i}{\sum^{\xi} V_i} \text{mean of service time,} \\
\hat{p} &= \frac{\sum^{\xi} \lfloor T_i \rfloor - Q_i}{\sum^{\xi} \lfloor T_i \rfloor} \text{proportion of renewals.}
\end{aligned}$$

Values of the general parameters

Using the data, we find the following estimates :

Car addition rate λ	Car removal rate μ	Mid-term canc. rate γ	Renewal prob. p
0,0730 (0,0003)	0,0828 (0,0003)	0,0400 (0,0003)	0,9179 (0,0003)

TABLE – Parameter estimates (standard deviation)

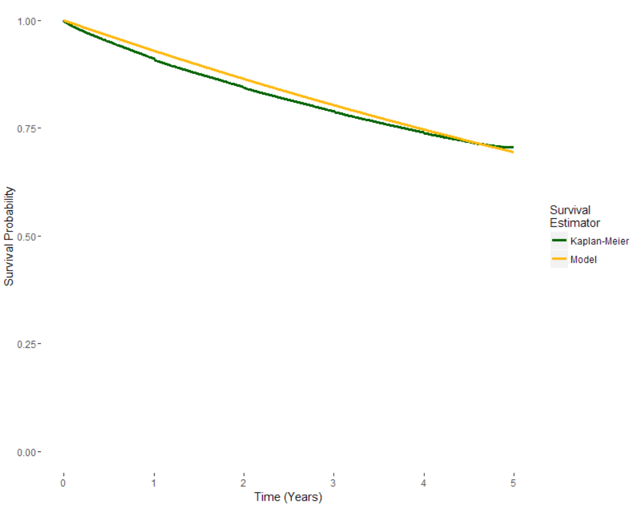

FIGURE – Arrival rate of a new car

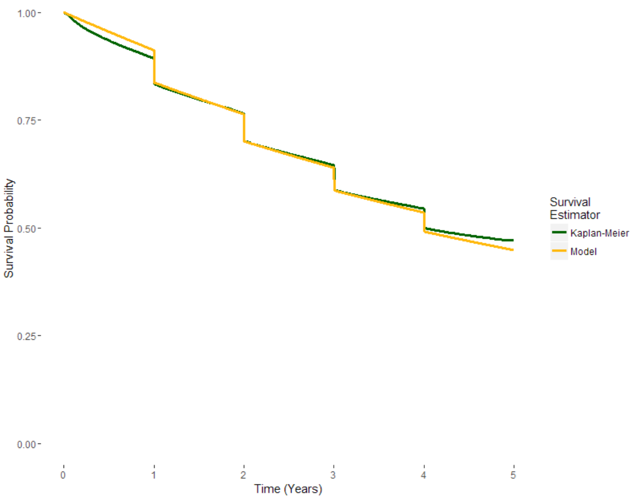

FIGURE – Departure rate of a client (model without censored data)

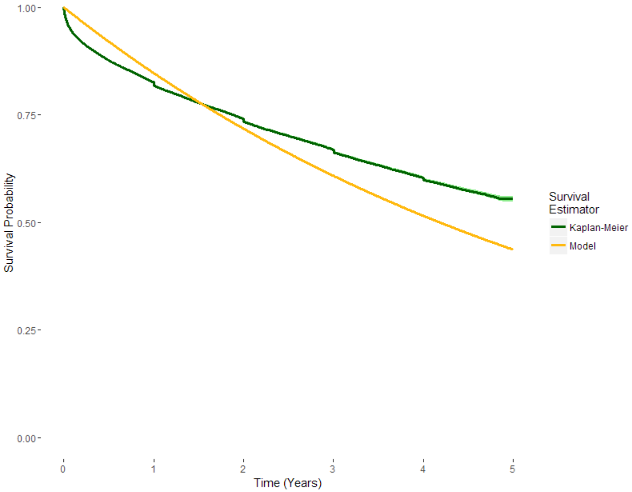

FIGURE – Removal rate with 2 cars (model without censored data)

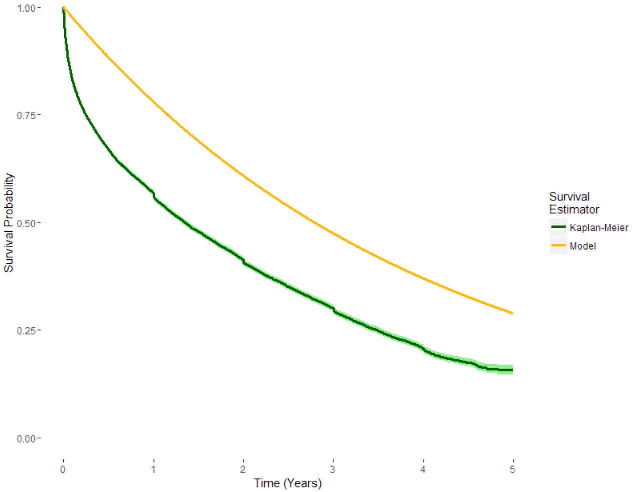

FIGURE – Removal rate with 3 cars (model without censored data)

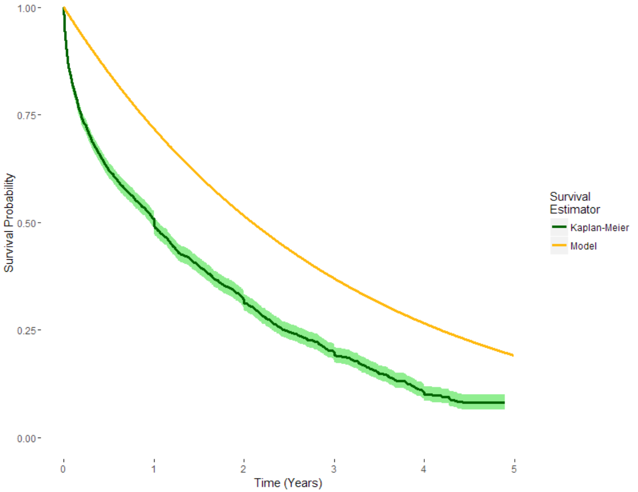

FIGURE – Removal rate with 4 cars (model without censored data)

Regression with explanatory variables

The next step is to generalize this model with explanatory variables.

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Selected explanatory variables

The characteristics chosen to define the insured are, in order, if the insured :

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Selected explanatory variables

The characteristics chosen to define the insured are, in order, if the insured :

- is from direct or affinity market,
- has at least one leased vehicle,
- is not married,
- is insured for less than 9 years (at the beginning),
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- is from direct or affinity market,
- has at least one leased vehicle,
- is not married,
- is insured for less than 9 years (at the beginning),
- has its effective contract date between the month of January and June,
- has its effective contract date in the month of July,
- has its effective contract date the first day of the month.

Insured examples

For purposes of simplification, I will show only the results of 5 examples of insureds from the many possible combinations.

Parameter	Direct Market	Leased	Not married	0-9 yrs	January-June	July	1st of month
Household A	1	0	1	1	0	0	0
Household B	1	1	1	0	0	0	0
Household C	0	0	1	0	0	0	1
Household D	1	1	0	0	0	0	0
Household E	0	1	0	0	0	1	1

TABLE – Features of the insured examples selected

Insureds parameters

Car addition rate	Car removal rate λ	Mid-term canc. rate γ	Renewal p μ
Household A	0,0492	0,0557	0,1270
Household B	0,0349	0,0283	0,1063
Household C	0,0460	0,0159	0,0868
Household D	0,0550	0,0228	0,0705
Household E	0,0570	0,0107	0,0530

TABLE – Values of the parameters for each profile

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\begin{aligned}
E[\mathcal{H}_a(t)] &= \frac{\partial P_{H_a(t)}(z=1,t)}{\partial z} \\
&= e^{-\gamma t} p^{\lfloor t+c \rfloor} \left(a e^{-\mu t} + \left(1 - e^{-\mu t}\right) \frac{\lambda}{\mu} \right).
\end{aligned}
$$

Number of cars expected, 5 years

For $t = 5$ years, after the renewal shock, we have the following results :

Number of initial vehicles	1	2	3	4
Household A	0,281	0,490	0,700	0,909
Household B	0,417	0,756	1,096	1,435
Household C	0,541	0,961	1,381	1,801
Household D	0,604	1,058	1,513	1,967
Household E	0,812	1,424	2,036	2,648

TABLE – Expected number of cars after 5 years

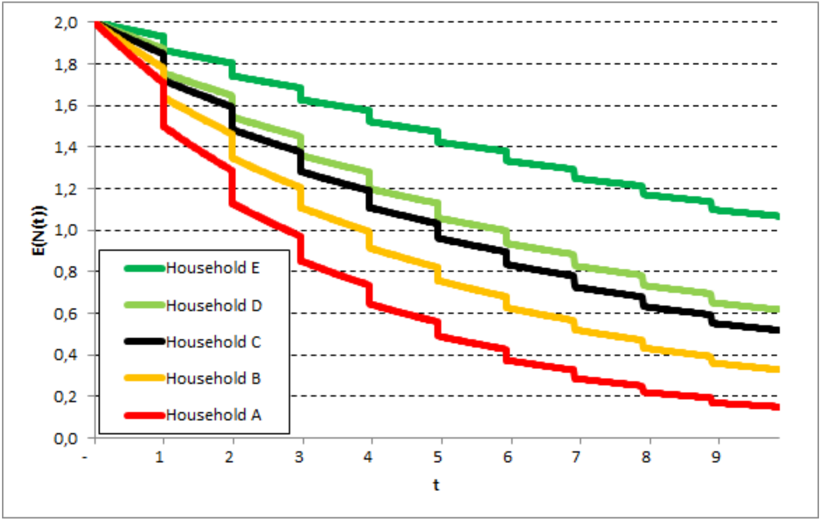

FIGURE – Expected number of cars at time t

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$$\begin{aligned}
\text{Profits} &= \int_0^\infty \mathbb{E}(\mathcal{H}_a(t))e^{-\delta t}dt \\
&= \frac{\left(1-e^{-(\gamma+\delta+\mu)}\right)\left(a-\frac{\lambda}{\mu}\right)}{(\gamma+\delta+\mu)\left(1-e^{-(\gamma+\delta+\mu)}p\right)} \\
&\quad + \frac{(1-e^{-(\gamma+\delta)})\lambda}{(\gamma+\delta)\mu(1-e^{-(\gamma+\delta)}p)},
\end{aligned}$$

Customer lifetime value with $\delta = 0.02$

Number of initial cars	1	2	3	4
Household A	3,964	7,161	10,358	13,555
Household B	5,588	9,992	14,395	18,799
Household C	7,661	13,011	18,361	23,711
Household D	8,782	14,586	20,390	26,194
Household E	16,130	24,735	33,340	41,945

Conclusion

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- model with seasonality and growth of the company ;
- effect of claims in customer retention ;
- generalization of constant parameters.

Thank you everyone