# Project 2021 lmeca2660
# Part 1

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In this project, it is proposed to study numerically the 2D flow over a hill. The computational domain $\Omega$ is a rectangular box of size $L \times H$. The flow is modeled as a boundary layer which has a height $H = 4\, h_{hill}$ and evolves over a distance $L = 20\, h_{hill}$. The hill geometry is represented by a Gaussian function which has a width $\sigma = \frac{3}{2} h_{hill}$ and which is centered at $d_{hill} = 8\, h_{hill}$. The sketch of the situation is represented on Figure 1.

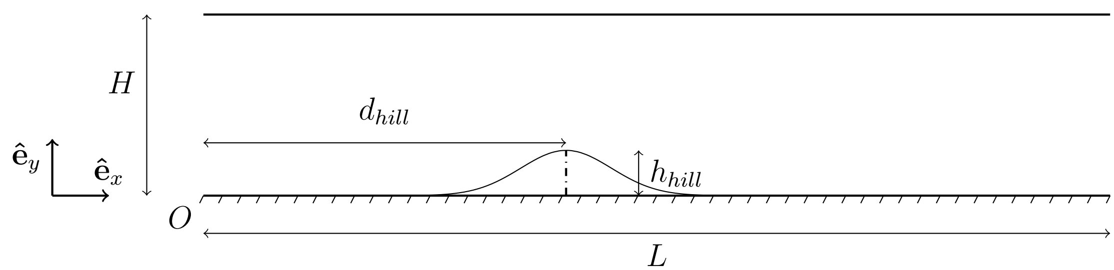


Figure 1: Description of the case study

The considered Reynolds number for this flow is too high ($\mathcal{O}(10^8)$ !) to solve all the scales. To model the impact of the unresolved subgrid scales on the resolved ones, we therefore add in the momentum equation an effective "subgrid scale viscosity" ($\nu_{SGS}$) based on the local strain rate of the flow. The hill will be taken into account using a penalization method. Inside the hill, an extra source term is added to the momentum equation in order to force the flow velocity $\mathbf{v}$ to match the prescribed velocity $\mathbf{v}_s$ (which is zero here). The area $\Omega_s \subset \Omega$, where the source term is added (represented on Figure 2), is specified using a mask function $\chi$. Thus, the equations to be solved are:

$$\nabla \cdot \mathbf{v} = 0 \tag{1}$$

$$\frac{D\mathbf{v}}{Dt} = -\nabla P + \nabla \cdot (2\, \nu_{SGS}\, \mathbf{d}) - \chi \, \frac{(\mathbf{v} - \mathbf{v_s})}{\boldsymbol{\Delta\tau}} \,, \tag{2}$$

with $P = \frac{(p-p_e)}{\rho_f}$ the kinematic pressure (where $p_e$ is a reference pressure, which we will choose to be the pressure at the outflow), $\nu_{SGS}$ the SGS kinematic viscosity, $\mathbf{d}$ is the strain rate tensor, $\Delta\tau$ a parameter (the smaller $\Delta\tau$, the better the presence of the body is accounted for, but the stiffer the problem becomes) which has the dimension of time (thus $\Delta\tau$ must be chosen as a function of $\Delta t$), and $\chi$ the mask function ($\chi = 1$ for $\mathbf{x} \in \Omega_s$ and $\chi = 0$ elsewhere).
The (local!) SGS viscosity is computed via the following expression $\nu_{SGS} = C\, (\Delta x)^2 \sqrt{2\, \mathbf{d} : \mathbf{d}}$.

The parameter C will be chosen such as $1/C =\simeq 2$.
Due to the large Reynolds number, the simulations performed with a discontinuous definition of $\chi$ would introduce large oscillations into the flow solution. To prevent this, we mollify the mask function in order to make smoother the transition from $\chi = 1$ (inside the hill) to $\chi = 0$ (in the flow) using a Heaviside function :

$$\chi(x, y) = \frac{1}{2}\left[1 - \operatorname{erf}\left(\frac{(y - y_{hill}(x))}{\sigma_\chi(x)}\right)\right] \tag{3}$$

where the x-variation of the mollified Heaviside function width, $\sigma_{\sigma_\chi}$, is here taken as Gaussian:

$$\sigma_\chi(x) = (2\Delta x)\ e^{-\left(\frac{y-y_{hill}(x)}{\sigma}\right)^2} \tag{4}$$

Thus, $\sigma_\chi = 2\Delta x$ at the top of the hill and it progressively decreases to zero. This reduction to zero was necessary in order to not penalize the inflow and outflow boundaries. Note that the use of such a mollification elevates artificially the hill (by about $5\Delta x$); it is nevertheless consistent as this slight elevation is converging to zero when $\Delta x \to 0$. Use this correction or adapt it when plotting your results !

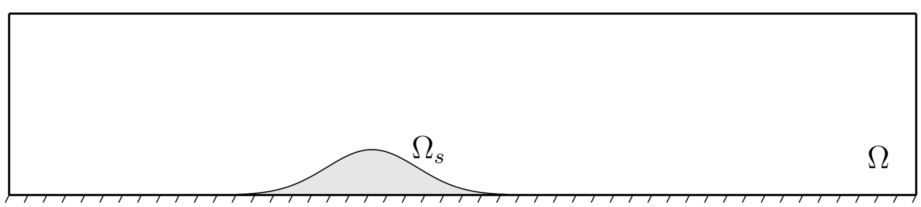


Figure 2: Computational domain

At the inflow, the velocity profile flow is imposed using a log-law: $u(y) = \frac{u_\tau}{\kappa}\log(\frac{y+y_0}{y_0})$, where $y_0$ correspond to the roughness of the ground above which the boundary layer is evolving, $\kappa$ is the *Von Karman* constant and $u_\tau$ represents the friction velocity. This friction velocity can easily be computed knowing the velocity at a certain altitude.

At the outflow, we use "natural" outflow condition:

$$\frac{\partial v}{\partial x} \simeq \frac{(v_{N-1,j} - v_{N-2,j})}{\Delta x} = 0, \tag{5}$$

$$\frac{\partial u}{\partial t} + u_p(y)\frac{\partial u}{\partial x} \simeq \frac{\left(u^*_{N-1,j} - u^n_{N-1,j}\right)}{\Delta t} + u_p(y)\frac{\left(u^n_{N-1,j} - u^n_{N-2,j}\right)}{\Delta x} = 0. \tag{6}$$

The convective outflow velocity $u_p(y)$ is chosen to be the velocity profile at the inflow. Note that the continuity demands the mass flow rate at the outflow be equal to the mass flow rate at the inflow (See *"PETSc installation notes"* for the demonstration). You must thus correct the outflow velocity after applying Eq. (6) so as to ensure that the numerically evaluated mass flow rate is conserved

We assume a slip wall for the upper boundary. We thus impose the no-through flow condition ($v = 0$) and no gradient on the horizontal velocity component: $\frac{\partial u}{\partial y} \simeq \frac{\left(u_{i,\,N-1} - u_{i,\,N-2}\right)}{\Delta y} = 0$

## Flow Solving

For the integration of Eqs. (1) et (2), we use a staggered MAC mesh. The convective term is integrated using the second order Adams-Bashforth scheme (and using the Euler scheme for the first time step). The Euler scheme is used for the integration of the SGS diffusion term. A "pseudo-implicit" Euler scheme is used for the integration of the penalization term. The overall integration scheme thus reads:

$$
\begin{aligned}
\frac{(\mathbf{v}^* - \mathbf{v}^n)}{\Delta\, t} &= -\frac{1}{2}\left(3\,\mathbf{H}^n - \mathbf{H}^{n-1}\right) - \nabla_h P^n \\
&\quad + \nabla_h \cdot (2\nu_{SGS}^n \mathbf{d^n}) - \chi\, \frac{(\mathbf{v}^* - \mathbf{v_s^{n+1}})}{\boldsymbol{\Delta}\tau} && (7) \\
\nabla_h^2 \Phi &= \frac{1}{\Delta t} \nabla_h \cdot \mathbf{v}^* && (8) \\
\frac{(\mathbf{v}^{n+1} - \mathbf{v}^*)}{\Delta\, t} &= -\nabla_h \Phi && (9) \\
P^{n+1} &= P^n + \Phi, && (10)
\end{aligned}
$$

with $\mathbf{H}$ the convective term.

The Poisson equation (8) will be solved using the PETSc library. To install this library and get more precisions for the solving of the equation, refer to the "*PETSc installation notes*" document.

We will here use a mesh with $\Delta x = \Delta y = h$, where $h$ fulfills a constraint on the precision of the solution, that is to say that $Re_h = \frac{(|u|+|v|)h}{\nu_{SGS}}$ must be "sufficiently moderate". The parameter $\Delta\tau$ has also to be chosen so that the presence of the hill is "sufficiently well captured" (meaning that $\mathbf{v}$ remains "sufficiently close" to $\mathbf{v}_s$ in $\Omega_s$). Finally, the time step $\Delta t$ is chosen so that the stability of the problem is ensured (in which the factors $r = \frac{\nu_{SGS}\,\Delta t}{h^2}$ and CFL $= \beta = \frac{(|u|+|v|)\Delta t}{h} = r\, Re_h$ are involved). The penalization is unconditionally unstable (see class notes). In any case, the classical bounds having been determined using linear stability theory on linear and periodic problems (i.e., without boundary condition), we strongly advice you to use a safety factor for $\Delta t$.

## First part of the project

As first part of this project, you are asked to simulate the flow over the hill. To this end, we ask you to initialize the velocity to the log profile everywhere in the computational box, and then to activate the presence of the hill via the penalization term and let the flow evolves until a statistically converged regime is attained.

To summarize, you are asked to:

1. Produce a numerical code to study the present situation.
2. Obtain a statistically converged solution for the following parameters:
   (a) $h_{hill} = 500\ m$
   (b) $y_0 = 0.1\ m$ (which represents the surface roughness of a vineyard),
   (c) $u_{hill} \triangleq u(y = h_{hill}) = 50\ km/h$, the inflow velocity at the altitude $h_{hill}$.
3. Plot contours of the dimensionless vorticity field $\frac{\omega\, h_{hill}}{u_{hill}}$ at different dimensionless times $\frac{t\, u_{hill}}{h_{hill}}$.
4. Comment your results.