Accelerated Combinatorial Search for Outlier Detection with Provable Bound on Sub-Optimality

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Abstract

Outliers negatively affect the accuracy of data analysis. In this paper we are concerned with their influence on the accuracy of principal component analysis (PCA). Algorithms that attempt to detect the outliers and remove them from the data prior to applying PCA are sometimes called Robust PCA, or Robust Subspace Recovery algorithms. The current state of the art includes algorithms that are either inaccurate or slow. We propose a new algorithm for outlier detection that combines two ideas. The first is "chunk recursive elimination" that was used effectively to accelerate feature selection, and the second is combinatorial search, in a setting similar to the classical A* search algorithm. Our main result is showing how to combine these two ideas. One variant is guaranteed to compute the optimal result according to some natural criteria, but its running time makes it impractical for large datasets. Another variant is much faster and comes with provable bound on sub-optimality. Experimental results show the effectiveness of the proposed approach.

1 Introduction

An important challenge in the analysis of data is the design of a model that fits the data. In practical situations this requires handling data outliers. Typically, the data model can be made significantly more accurate if it is allowed to ignore a small fraction of the data items, considered to be outliers. See, e.g., (Aggarwal 2016; Hodge and Austin 2004; Pincus 1995).

The particular data model that we discuss in this paper is the one produced by Principal Component Analysis (PCA). Consider for example the data shown in Figure 1, consisting of 9 points. In this case, rank-1 PCA gives a bad model for all 9 points, but a good model for the 8 non-outliers. Observe the huge effect that a single outlier can have on the model.

PCA is arguably the most widely used dimension reduction technique, with a variety of applications. See, e.g., (Jolliffe 2002; Burges 2010; Cabral et al. 2013; Vidal, Ma, and Sastry 2016; Boutsidis, Woodruff, and Zhong 2016; Gray 2017). For recent work that combines PCA with deep learning techniques see, e.g., (Wang et al. 2015; Rehman and Asad 2017; Ye et al. 2019; Ma and Yuan 2019). Until recently the task of computing PCA on large data was considered a challenge, and specialized techniques were designed for datasets consisting of images and video. See, e.g., (Ross et al. 2008; Levy and Lindenbaum 2000; Brand 2002; Vidal, Ma, and Sastry 2016). Recent algorithmic approaches that use randomization give algorithms that can easily handle large datasets e.g., (Halko, Martinsson, and Tropp 2011; Halko et al. 2011; Musco and Musco 2015; Li et al. 2017). However, computing the robust variants that ignore outliers is still a big challenge.

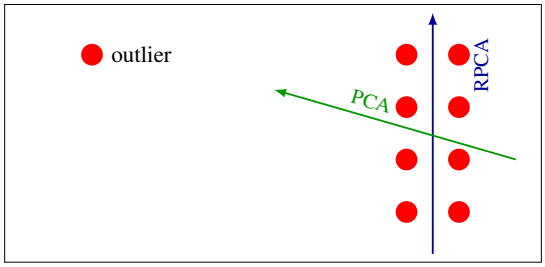

Figure 1: The direction of the dominant principal component computed from 9 points with one outlier. The direction labeled PCA (green arrow) was computed from the entire data (9 points). The direction labeled RPCA (blue arrow) was computed from the 8 non-outliers.

There is a large number of studies on outlier detection and removal specifically for computing robust PCA. The robust models are sometimes called "Robust Principal Component Analysis" (RPCA), or, alternatively, "Robust Subspace Recovery" (RSR). A recent review paper is (Lerman and Maunu 2018b).

One way to solve the RPCA problem is to first filter outliers and then fit a subspace to the remaining data by using classical PCA. Many studies view the data as coming from a fixed distribution, and attempt to detect outliers as data points at the margins of the distribution e.g., (Roberts 1999; Scheirer et al. 2011; Hubert and Engelen 2004; Xu, Caramanis, and Sanghavi 2010; Zhang et al. 2015). Another typical recipe to detect outliers is to rank the data points based on some measures. The outliers are identified as points with low scores. See, e.g., (Chen and Lerman 2009; Soltanolkotabi, Candes et al. 2012; Rahmani and Atia 2017; You, Robinson, and Vidal 2017; Rahmani and Li 2019). The "Cop" algo-

Algorithm	time complexity	memory complexity
Cop (Rahmani and Atia 2017)	$O(n^2m)$	$O(n^2+mn)$
R-graph (You, Robinson, and Vidal 2017)	$O(n^2m+n^3)$	$O(n^2+mn)$
iSearch (Rahmani and Li 2019)	$O(n^2m)$	$O(mn)$
FMS (Lerman and Maunu 2018a)	$O(T\cdot nmr)$	$O(mn)$
Shah (Shah et al. 2018)	$O(T\cdot nmk)$	$O(Tn+mn)$
Chunk-A^*(our)	$O(T\cdot nmk)$	$O(Tn+mn)$

Table 1: Complexity of various algorithms. n is the number of data items. m is the dimension of each item. r is the number of principal components. k is the number of outliers. T is the number of iterations. The T of Chunk-A^*can be much smaller than the T of Shah algorithm.

rithm (Rahmani and Atia 2017) computes the scores from the Gram matrix of the normalized data matrix. The score for a data point is the Frobenius norm of corresponding row of the Gram matrix. It runs very fast, but requires an unacceptably large amount of memory. The "R-graph" algorithm (You, Robinson, and Vidal 2017) measures the points based on the self-expressiveness property. The scores are computed by solving an elastic net minimization problem. The "Innovation" of each point is used to rank the points in (Rahmani and Li 2019). The authors refer the algorithm as "iSearch".

A different approach was taken in (Shah et al. 2018). The authors view outlier detection as a search problem, and apply the weighted A^* algorithm, a classical combinatorial search technique from AI, to solve it. The search is motivated by the PCA model error. As reported in (Shah et al. 2018) the algorithm comes with an accuracy parameter called ϵ. With ϵ=0 the algorithm is guaranteed to terminate with an optimal outlier selection. With larger ϵ values the algorithm terminates with outliers that are typically less accurate than the optimal, but the running time is much faster. Unfortunately, as we show in Section 7 even with the choice ϵ=∞ the algorithm is slow, and cannot be applied to large datasets.

There are other variants of robust PCA that do not entirely discard outliers. A well known approach allows outlier values to be wrong in some coordinates but right in other coordinates. Computing the PCA in this case can be formulated as partitioning the data matrix into the sum of two matrices, one low rank and the other sparse. See, e.g., (Candes et al. 2011; Bouwmans et al. 2017) and the recent survey (Vaswani and Narayanamurthy 2018).

As we show experimentally in Section 7 current state-of-the-art algorithms that compute outliers encounter difficulties when applied to large datasets with hundreds of thousands / millions of data points. Most algorithms run too slow or require an unacceptably large amount of memory. Table 1 shows the complexity of some recent algorithms. The time complexity for fastest variant of Chunk-$A^*$is $O(k/c\cdot nmr)$, where c is the chunk size (introduced later). The memory complexity is $O(mn)$.

The approach we take is outlined below.

1.1 Our approach

Our goal is to design a scalable algorithm capable of detecting outliers in large datasets. A useful algorithm for the closely related feature selection problem (Guyon and Elisseeff 2003) is "backward elimination", that requires a measure of the importance of each individual feature. Such a measure is typically called a "filter". Using filters a typical implementation of backward elimination for selecting k features to be eliminated from among n features is as follows:

> *Run k iterations. In the jth iteration rank all remaining features (there are $n-j+1$ features to be considered) according to their filter values and eliminate the least important feature.*

Observe that this requires roughly nk filter evaluations which may be impractical. Instead, classical implementations of this approach (for example the well known SVM-RFE algorithm (Guyon et al. 2002)) eliminate chunks of multiple points in each iteration. A chunk consists of a fraction of the features that are ranked as the least important features. For sufficiently big chunks this reduces the number of filter evaluations to $O(n)$. We refer to this approach as the "Chunk-RFE", for "Chunk Recursive Feature Elimination".

Our main result is showing how to wrap a combinatorial search framework around the Chunk-RFE. This combines the speed of Chunk-RFE with the accuracy of the combinatorial search approach. The resulting algorithm that we call "Chunk-A^*" has two parameters that control the balance between speed and accuracy: the chunk size of the Chunk-RFE, and the ϵ parameter of the weighted A^*.

1.2 Our results

We describe an outlier detection algorithm for detecting k outliers. (k is assumed to be known.) The algorithm error is defined as the PCA error of modeling the data without the outliers (see Section 2). The algorithm accuracy and running time are controlled by the two parameters c, ϵ, where $1 \le c \le k$ is the chunk size, and $0 \le \epsilon \le \infty$ is the optimality parameter. In general, larger values of c and ϵ result in a faster running time, and smaller values give results that are more accurate.

In addition to identifying the outliers the algorithm computes sub-optimality bound on how close the result is to the optimum. We are not aware of any other outlier detection algorithm that computes such bound on sub-optimality.

The following are special cases that we prove:

- If $\epsilon = 0$ the algorithm terminates with the optimal outlier selection regardless of the value of chunk size c.
- If $c = 1$ the algorithm is identical to (Shah et al. 2018). Still, it does not produce sub-optimality bound as ours.

1.3 Paper organization

The paper is organized as follows. The PCA error of modeling the data is defined in Section 2. The proposed algorithm is described in Section 3. The three variants of the algorithm along with the proof are given in Section 4. An intuitive algorithm is introduced to improve the results in Section 5. The sub-optimality bound is discussed in Section 6. Experimental results are shown in Section 7.

Algorithm 1: The Chunk-A^* algorithm.

Input:
X, a data matrix of n columns.
r, the desired number of principal components.
k, the desired number of outliers.
$f(S)$, a "filter" function. Here S is an outlier subset, and $f(S)$ is an estimate to the PCA error if the outliers in S are included in the solution subset.
$c(S)$, a criterion for determining the chunk size.
r_d (optional), the rank for dimensionality reduction. Default: $r_d = \min(m, n)$.
Output: a subset S of k outliers.
Preprocessing: dimensionality reduction.
Data Structures: Two global lists of subsets: the fringe list F, and the closed list C.
Initialization: Put the empty subset into F.

1 **while** F *is nonempty* **do**
2 Pick S_i with the smallest $f(S_i)$ from F. Ties are resolved in favor of the larger size(S_i).
3 **if** S_i *contains* k *outliers* **then**
4 Stop and return S_i as the solution subset.
5 **else**
6 Create U_i as the list of all subsets that can be obtained by adding a single column to S_i.
7 Remove from U_i all the subsets that are in C.
8 Compute $f(S_j)$ for each subset $S_j \in U_i$.
9 Select chunk size $c(S_i)$ satisfying: $1 \le c(S_i) \le k-\text{size}(S_i)$.
10 Compute the subset $\overline{S}_j$ as the union of the $c(S_i)$ subsets in U_i with the smallest $f(S_j)$.
11 Add $\overline{S}_j$ to F and C.
12 Add to F and C all the subsets $S_j \in U_i$ that are not used in the creation of $\overline{S}_j$.
13 **end**
14 **end**

2 The error of modeling data by PCA

Let $X=(x_1 \dots x_n)$ be the data matrix of size $m \times n$, where n is the number of data items and m is the dimension of each item. The PCA computes a linear mapping from the m dimensional x_i to the r dimensional y_i, where $r \le m$. The inverse of this mapping gives an approximate reconstruction of x_i from y_i. The reconstruction error of x_i is denoted by e_i. As in the classical development of PCA e.g., (Jolliffe 2002) the quality of the mapping can be measured by the sum of reconstruction errors as follows:

$$e_i(r) = \|x_i - VV^T x_i\|^2, \quad E(r) = \sum_{i=1}^{n} e_i(r).$$

It is known (e.g., (Jolliffe 2002)) that the V can be computed from the r eigenvectors of the matrix $B = XX^T$ corresponding to its r largest eigenvalues. A known result about matrix engenvalues gives an explicit expression to this error in terms of the eigenvalues of B (e.g., (Golub and Van-Loan 2013)). Suppose $\lambda_1 \dots \lambda_m$ are the eigenvalues of B and V is $m \times r$ then:

$$E(r) = \sum_{j=r+1}^{m} \lambda_j = \text{trace}(B) - \sum_{j=1}^{r} \lambda_j.$$

Suppose that X is partitioned into a subset of outliers S and the reminder subset $\tilde{X} = X \setminus S$. The rank-$r$ PCA is computed from the data after the outliers in S are removed. Accordingly, the PCA error is computed from the reminder subset:

$$\begin{aligned} &e_i(S,r) = \|x_i - VV^T x_i)\|^2 \\ &E(S,r) = \sum_{i \notin S} e_i(S,r) = \sum_{j=r+1}^{m} \tilde{\lambda}_j = \text{trace}(\tilde{B}) - \sum_{j=1}^{r} \tilde{\lambda}_j \end{aligned} \quad (1)$$

where $\tilde{\lambda}_j$ are the eigenvalues of $\tilde{B} = \tilde{X}\tilde{X}^T$.

3 The Chunk-A^* algorithm

In this section we describe the algorithm in terms of a general filter function that will be defined later, as shown in Algorithm 1. It is closely related to standard combinatorial search algorithms such as the A^* (Pearl 1984; Russell and Norvig 2010; He et al. 2019). The major difference is that previous studies do not have the Chunk-RFE, as shown in lines 9-11 of the algorithm.

The goal of the algorithm is to compute a subset of k outliers, minimizing the "filter" function f. It maintains in F a list of subsets that need to be further evaluated and in C a list of visited subsets that need not to be added into F again. Employing the standard "best first" strategy the algorithm selects the subset from F with the smallest f to be expanded. However, unlike standard combinatorial search algorithms, not all of the direct expansions of the selected subset are treated equally. Instead, the algorithm "guesses" that the best c among them are part of the solution, and packages them together in a "chunk". We refer the "chunk" as "super child" and the children obtained by adding a column as "direct children".

3.1 The subsets graph

The subset graph is created with nodes corresponding to outlier subsets. When the chunk size c=1, there is an edge from subset S_i to subset S_j if adding one column to S_i creates S_j. When the chunk size c>1, the super children are created. The graphs for the matrix $X = (x_1, \dots, x_5), k = 3$ are shown in Figure 2.

Note that all paths leading from the root to a node can be considered equivalent, since the order of outliers in a subset does not matter. For example, if the goal node $\{x_1, x_2, x_4\}$ in the right graph is found, it is irrelevant if it is reached by the path $\{\} \to \{x_1, x_2\} \to \{x_1, x_2, x_4\}$ or by $\{\} \to \{x_4\} \to \{x_1, x_2, x_4\}$. This property makes the chunking meaningful.

3.2 The chunk size

The chunk size is used in Line 9 of the algorithm. The expectation is that using big chunks would result in fast convergence of the algorithm, while small chunks would give

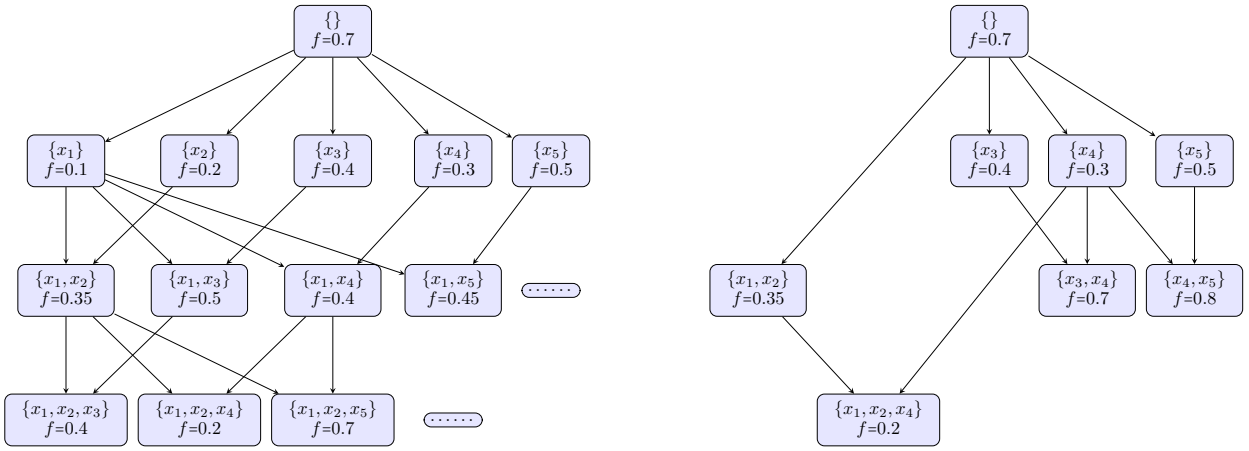

Figure 2: Examples of the subset graphs with $n = 5$. The left graph is for c=1; the right graph is for c=2. For instance, when the root node is expanded, in the left graph all direct children are added into F; in the right graph, the super child $\{x_1, x_2\}$ is created by taking the union of $\{x_1\}$ and $\{x_2\}$. This fastens the algorithm to evaluate the nodes in deeper level.

more accurate results. The chunk size should satisfy following condition:

$$1 \leq c(S_i) \leq k\text{−size}(S_i).$$

The right part k−size(S_i) is the reminding number of outliers still needed to be selected. It is the largest possible, giving the maximal acceleration. With chunk size $c = 1$, there is no acceleration at all.

3.3 The filter function

From Equation (1) it is clear that the subset with the smallest error can be found by exhaustively evaluating all the subsets S of size k, but this is clearly impractical. Instead, we use filters that allow a search algorithm to find the solution.

Suppose we are given a subset S_i of size k_i<k. We consider the error that can be obtained by completing it to size k. The best error that an outlier subset S_i can achieve is:

$$d(S_i, r, k) = \min_{\text{size}(S_i \cup T)=k} E(S_i \cup T, r) \tag{2}$$

Ideally we may wish to use $d(S_i, r, k)$ as the filter function, but computing it is inefficient since it involves going over all subsets of size $k-k_i$. Instead, we approximate it using lower and upper bounds defined as follows. Let $\lambda_1 \geq \lambda_2, \ldots, \geq \lambda_m$ be the eigenvalues of the second moments of the remaining data. Define:

$$\begin{aligned} l(S_i, r, k) &= E(S_i, r + k - k_i) = \sum_{j=r+k-k_i+1}^{m} \lambda_j \\ u(S_i, r, k) &= E(S_i, r) = \sum_{j=r+1}^{m} \lambda_j \end{aligned} \tag{3}$$

These are similar to the functions defined in (He et al. 2019; Shah et al. 2018).

Lemma 1 *For any subset S_i of size$(S_i) \leq k$:*

$$l(S_i, r, k) \leq\ d(S_i, r, k)\ \leq u(S_i, r, k)$$

when size$(S_i) = k$ both inequalities become equalities.

The proof of Lemma 1 is based on the interlacing property of eigenvalues (e.g., (Hill and Parlett 1992; Golub and Van-Loan 2013)). It is given in the supplementary material.

4 The three variants of the algorithm

To simplify notation we write $l_i = l(S_i, r, k)$ and $u_i = u(S_i, r, k)$. Lemma 1 shows that the best choice for a filter is "sandwiched" between l_i and u_i. Following (He et al. 2019) we express f_i as a linear combination of l_i and u_i, with three distinct options: 1. $f_i = l_i$; 2. $f_i = u_i$; 3. $f_i = l_i + \epsilon u_i$, where $\epsilon \geq 0$. In each case the Chunk-A^* has distinct characteristics as stated in the theorems below:

Theorem 1 ***(Optimal variant)*** *If $f_i = l_i$, then the Chunk-A^* terminates with an optimal solution.*

Theorem 2 ***(Greedy variant)*** *If $f_i = u_i$, then the Chunk-A^* is greedy and selects c outliers in each iteration.*

Theorem 3 ***(Suboptimal variant)*** *If $f_i = l_i + \epsilon u_i$, then Chunk-A^* guarantees a solution "close" to the optimum. Let S_* be an optimal solution subset of outliers. Let $e^* = E(S_*, r)$ be the corresponding error. Let S_{**} be the solution subset of outliers found by the algorithm. Let e^{**} be the error for S_{**}, and l_{**} be the value of l for S_{**}. Let $u_{\max}$ be the largest value of u for the subsets remaining at the Fringe list after the goal node is reached. Then:*

$$e^{**} \leq e^* + \epsilon(u_{\max} - l_{**}). \tag{4}$$

4.1 Proof of Theorems

In this section we give the technical proofs of theorems in Section 4. Similar theorems were stated and proved in (He et al. 2019) for feature selection. Their theorems cannot be extended to the outlier detection case that we study here. In particular, the acceleration of the combinatorial search requires a different proof strategy. To prove the theorems, we need following lemmas. As in the case of Lemma 1 the proofs are based on the interlacing property of eigenvalues. They are given in the supplementary material.

Lemma 2 *The value of l_i is monotonically increasing along any path.*

Lemma 3 *The value of u_i is monotonically decreasing along any path.*

Lemma 4 *Consider the choice $f_i = u_i$. The f value of the super child is smaller than the f value of any of its siblings.*

Lemma 5 *Consider the choice $f_i = u_i$. Let n_i be the node picked at Line 2 of the algorithm. Let n_j be a child of n_i. The following three properties hold:*

a. *The size of S_j associated with node n_j is larger than the size of all other nodes currently in the fringe list.*

b. *The next node to be picked is a child of n_i.*

c. *If the chunk size $c > 1$, the next node to be picked is the super child of n_i.*

Lemma 6 *Suppose Theorem 3 is false. Then for any node n_z on the path from the root to n_* the following condition holds: $f_z < f_{**}$.*

Lemma 7 *Let S^* be an optimal outlier subset of size k. If the algorithm always uses chunk size $c(S_i)$ satisfying: $1 \leq c(S_i) \leq k-size(S_i)$ then during the run of the algorithm the fringe list F always contains a subset of S^*.*

Proof of Theorem 1: From Lemma 5 when a node n_i is picked in Line 2 of the algorithm, one of its children will be examined next. If the chunk size c>1, then the super child will be examined next. This shows that the algorithm is greedy. It detects c outliers in each iteration, and terminates after $\lceil \frac{k}{c} \rceil$ iterations. ■

Proof of Theorem 2: The proof follows as a corollary of Theorem 3 with $\epsilon = 0$. ■

Proof of Theorem 3: If the theorem is false then from Lemma 6 it follows that all nodes on the path from the root to S_* have smaller f values than f_{**}. Since at any given time at least one of them is in the fringe list from Lemma 7, they should all be selected before S_{**} is selected. But this means that S_* is selected as the solution and not S_{**}. ■

4.2 Bound

Both the Greedy variant and the Suboptimal variant are not guaranteed to produce the optimal solution. We proceed to show how to obtain bound on how close their solution is to the optimum. The technique we use was originally proposed by (Thayer and Ruml 2008).

Consider a run of a non-optimal algorithm producing the non-optimal subset S_{**}. Then size$(S_{**}) = k$, and from Lemma 1 it follows that $l_{**} = u_{**} = E(S_{**}, r)$. The value of l_{**} is related to the optimal value l_* by: $l_* \leq l_{**}$. Let B be a value satisfying: $l_{**} - l_* \leq B$. We refer to B as a bound, where a smaller B indicates a better bound, and in particular $B = 0$ implies an optimal solution. An important observation is that in combinatorial search one can always compute such values. Let F be the fringe list after the algorithm terminates. Going over all the remaining nodes in F we can compute: $l_{\min} = \min_{S_i \in F} l_i$. From Lemma 2 then: $l_* \geq l_{min}$. We can take:

$$B = l_{**} - l_{\min}, \quad \text{where: } l_{\min} = \min_{S_i \in F} l_i. \tag{5}$$

5 Improvement

For the non-optimal variants of the Chunk-A^*algorithm, the chunking comes with possibly wrong outlier detection. We proceed to design an efficient algorithm for improving the result. We observe that running of the algorithm is guaranteed decrease (never increase) the PCA error, but it typically gets "stuck" in a local minimum after a small number of iterations.

Algorithm 2: Improvement for outlier selection.
Input: X: a matrix of n columns. r: the PCA rank. S: a subset of current outlier selection. $T_{\max}$: the maximum number of iterations.
Output: a subset R of X satisfying: $|R| = |S|$ and $E(R, r) \leq E(S, r)$.

1 Set $k = |S|$.
2 **repeat**
3 Set old error = the current error.
4 Compute V as the rank-r PCA of $X \setminus S$.
5 Compute $e_i = \|x_i\|^2 - \|V^T x_i\|^2$ for all columns x_i of X.
6 Replace the k outliers of S by the k columns of X with the largest e_i.
7 Set new error = $\sum_{x_i \notin S} e_i$.
8 **until** *new error = old error* ***or*** *reach T_{max}*;
9 Set $R = S$ and return.

The algorithm is motivated by that the PCA error can be evaluated separately for each column of X. As in Equation (1) let V contains r orthogonal columns. Recall that the reconstruction error of x_i is given by: $e_i = \|x_i - VV^T x_i\|^2 = \|x_i\|^2 - \|V^T x_i\|^2$ and the PCA error is: $E(S, r) = \sum_{i \notin S} e_i$.

The idea of Algorithm 2 is that if we know S we can calculate V using PCA, which gives optimal reduction to the reconstruction error of the columns not in S. Once V is known the best selection of k outliers from X is the columns with the largest reconstruction error. The PCA error reduction by this process improves the outlier selection. The complexity of the algorithm is $O(T_{\max} nmr)$. Experimental results show that the number of iterations is typically smaller than 5. The correctness proof is given in the supplementary material.

6 Fractional bounds on sub-optimality

The bounds that we derive in this section measure how close the computed result is relative to the optimal result. For example, it may be interesting to know that the error of the computed outlier set is within 10% of the optimal. Let E_k be the PCA error of the outlier set S_k produced by the Chunk-A^* for input size k. Let E_k^* be the smallest possible PCA error of an outlier set of size k. The bound B_k computed from (5) satisfies: $E_k \leq E_k^* + B_k$. Let b be the fractional bound: $b = (E_k - E_k^*)/E_k^*$. (It is meaningful only when $E_k^* \neq 0$). After each run of the Chunk-A^* we can compute the upper bound on b as in (6).

The fractional bound after Chunk-A^* :

$$b = \frac{E_k - E_k^*}{E_k^*} \leq \frac{B_k}{E_k^*} \leq \frac{B_k}{E_k - B_k} = b_0 \tag{6}$$

Now consider running Algorithm 2 after the Chunk-A^* which gives a new outlier subset with the PCA the error of $E_k^1 \leq E_k$. We show that the fractional bound can be improved as in (7).

The fractional bound after Algorithm 2 :

$$b = \frac{E_k^1 - E_k^*}{E_k^*} \leq \frac{E_k^1 - (E_k - B_k)}{E_k^*} \leq \frac{B_k - (E_k - E_k^1)}{E_k - B_k} = b_1 \quad (7)$$

7 Experimental results

We evaluate the performance of the proposed algorithm on various synthetic and real datasets. The real datasets are publicly available. Comparison is conducted with various algorithms that made their code available, including: R-graph (You, Robinson, and Vidal 2017); iSearch (Rahmani and Li 2019); GGD (Maunu, Zhang, and Lerman 2019); FMS (Lerman and Maunu 2018a); Cop (Rahmani and Atia 2017); RMD (Goes et al. 2014); TME (Zhang 2016); OP (Xu, Caramanis, and Sanghavi 2012); MKF (Zhang, Szlam, and Lerman 2009); DHRPCA (Feng, Xu, and Yan 2012); TORP (Cherapanamjeri, Jain, and Netrapalli 2017); R1PCA (Ding et al. 2006).

7.1 Experiments on synthetic datasets

Following a recent survey paper (Lerman and Maunu 2018b) two synthetic data models: Haystack model and Blurryface model are used to compare the performance of various algorithms for subspace recovery. Let U_* be the ground truth subspace. The error is the squared principal angles between U_* and the recovered U.

Haystack model: With fixed parameters m=200, n=400, and r=10, inliers are generated i.i.d. $\mathbb{N}(0, U_* U_*^T/r)$ and outliers are generated i.i.d. $\mathbb{N}(0, I/D)$. all points are perturbed by adding noise from i.i.d. $\mathbb{N}(0, 10^{-2}I)$. The entire data is centered at the end.

Blurryface model: The motivation of the blurryface model is to generate data that resembles real data. The Extended Yale face database B (Kuang-Chih Lee, Ho, and Kriegman 2005) is used to generate the data. The first individual's face is used to obtain the 9-dimensional subspace U_*. The inliers are generated i.i.d. $\mathbb{N}(0, c_1 U_* U_*^T/r)$ where c_1 is to give inliers comparable magnitude. The outliers are sampled from other faces. The resulting dataset is of size m=400, n=500. See (Lerman and Maunu 2018b) for details.

To demonstrate the effect of percentage of outliers on error, for each percentage from 10% to 90%, 10 datasets are generated. The Chunk-A^* is experimented with r_d=50, ϵ=∞ and different chunk sizes. The algorithm is marked as Chunk-A^*followed by the c value. For example, Chunk-A^*-1 means the algorithm with c = 1. The number of outliers is given as input for the following algorithms: R-graph, iSearch, DHRPCA, Shah, Cop and Chunk-A^*. Parameters unspecific for other algorithms are same as the ones used in (Lerman and Maunu 2018b).

The results for the haystack model are shown in Figure 3. Our Chunk-$A^*$is fastest among those which achieve high accuracy. The results for the blurry face model are shown in Figure 4. We observed a decline of performance when c = k

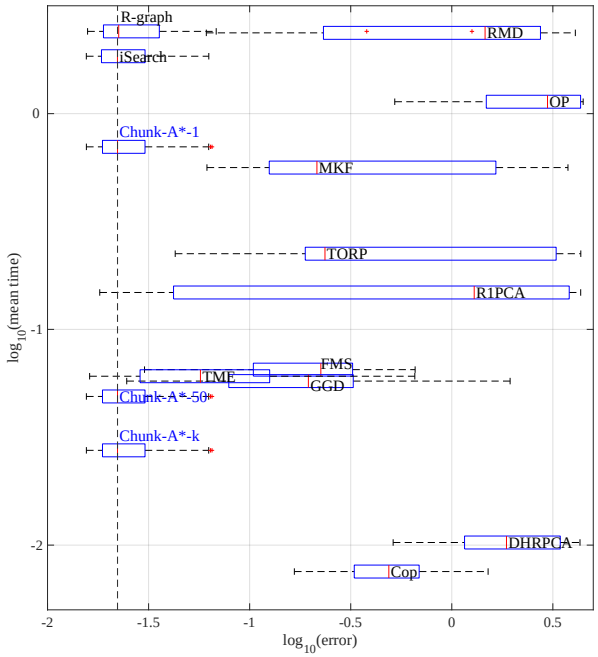

Figure 3: Comparison on Haystack model. The horizontal axis is the boxplot of errors with different outlier percentage. The red vertical line in a box corresponds to the median of errors. The vertical axis is the average runtime. The dashed line corresponds to the smallest median error among all algorithms. The closer to the left bottom, the better the algorithm is.

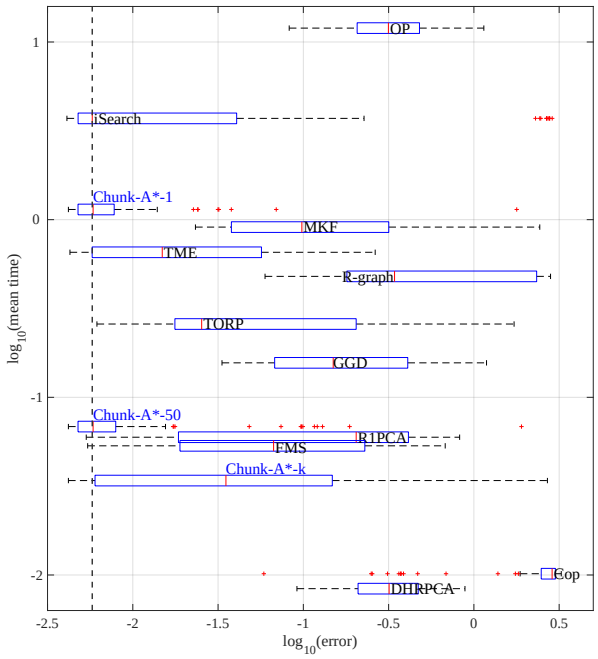

Figure 4: Comparison on Blurryface model. The c=k is the worst parameter setting.

dataset	r:k	$f = l$ (optimal)	$f = l + 0.2u$	$f = l + 0.5u$	$f = l + 1.0u$	$f = u$	$f = l + u$ bound:b_0	$f = l + 0.5u$ bound:b_0	iSearch	R-graph	TME
spectf	5:5	**290,498**	**290,498**	**290,498**	**290,498**	**290,498**	0.29	**0.09**	293,043	299,100	303,907
267×45	7:7	**203,556**	**203,556**	**203,556**	**203,556**	**203,556**	0.40	**0.19**	217,235	212,274	218,210
vehicle	5:5	**35,908**	**35,908**	**35,908**	36,211	36,211	0.39	**0.38**	52,553	54,686	42,234
846×18	10:5	**1,212**	1,242	1,242	1,242	1,580	**0.05**	**0.05**	2,122	4,140	2,354
libras	1:3	**591.43**	**591.43**	**591.43**	**591.43**	**591.43**	**0.00**	**0.00**	**591.43**	598.72	**591.43**
360×90	20:7	-	-	**1.03**	**1.03**	1.07	0.59	**0.22**	1.13	1.15	1.15

Table 2: Accuracy and bounds with different filter function. The minimum error and bounds are highlighted. "-" indicates no results after running for 5 minutes.

dataset	$r_d : r : k$		$c = 1$	$c = 0.5{*}k$	$c = k$	iSearch	R-graph	TME
Day1:sparse	100:30:100	error	> 60min	**59,976**	60,413	-	-	-
$20,000 \times 3,231,957$		runtime		884s	607s	-	-	-
Sift:dense	70:15:100	error$(+4.606e10)$	**5.95e6**	5.96e6	5.96e6	ArrayLimit	> 10min	6.08e6
$128 \times 1,000,000$		runtime	1523s	78s	65s			43s
YPMSD:dense	50:10:100	error$(+6.2e11)$	**2.79e8**	**2.79e8**	2.80e8	ArrayLimit	> 10min	3.02e8
$90 \times 515,345$		runtime	438s	22s	17s			8s
Covtype:dense	30:3:30	error$(+3.13e11)$	**8.45e7**	**8.45e7**	**8.45e7**	ArrayLimit	> 10min	9.84e7
$54 \times 581,012$		runtime	69s	7s	6s			21s

Table 3: Greedy variant on big datasets. "-" indicates no results since the implementation does not support sparse data format.

which is the worst parameter setting. However when c ≤ 50, the Chunk-$A^*$are significantly more accurate than other algorithms.

7.2 Experiments on real datasets

In this section we describe experiments with standard datasets that are commonly used for evaluating machine learning algorithms. Since the ground truth is unknown, the error is the PCA error as defined in Section 2.

Different filter function: As discussed in Section 3.3, with different selection of the filter function our algorithm produces optimal and suboptimal results. The results for various datasets with c=1 are shown in Table 2. No dimensionality reduction is conducted. Observe that: 1. With ϵ=0 (guaranteed optimality) the algorithm produces the smallest results; 2. With larger ϵ values the algorithm terminates with solutions that are typically less accurate than the optimal solution; 3. Smaller ϵ values yield tighter bounds; 4. Our algorithm is significantly more accurate than other algorithms.

Bounds on sub-optimality: The suboptimal solutions given by our algorithm come with guarantees on how close the solution is to the optimal solution. Table 4 shows the bounds and errors with different chunk size. The error_0 is the error after running Chunk-A^*. The error_1 is the error after running the Algorithm 2 for improvement. Observe that: 1. With chunking, the algorithm runs faster while the error and bound values become bigger. 2.Algorithm 2 improves the results when the chunk size is big (e.g. $c > 0.5{*}k$).

Experiments with big datasets: We describe experiments with the greedy variant applied to big datasets. The results are shown in Table 3. The common part of errors is written in brackets. Without chunking ($c = 1$), the algorithm runs much slow.

$r : k : \epsilon$		$c = 1$	$c = 0.2{*}k$	$c = 0.5{*}k$	$c = k$
5:20:0.2	b_1	**0.12**	**0.12**	**0.12**	0.14
	error_0	**563,313**	**563,313**	**563,313**	583,633
	error_1	**563,313**	**563,313**	**563,313**	577,102
	runtime	67s	21s	18s	17s
5:60:0.2	b_1	**0.26**	0.31	0.31	0.50
	error_0	**110,054**	113,716	115,004	133,020
	error_1	**110,054**	113,716	113,716	130,433
	runtime	93s	18s	16s	16s
10:70:0.1	b_1	**0.15**	0.17	0.19	0.19
	error_0	**46,461**	47,213	48,115	48,148
	error_1	**46,461**	47,213	48,115	48,148
	runtime	41s	17s	19s	16s
10:90:0.1	b_1	**0.18**	0.20	0.22	0.21
	error_0	**19,673**	19,881	20,314	20,169
	error_1	**19,673**	19,881	20,314	20,169
	runtime	417s	18s	17s	17s

Table 4: Subpotimal variant with different chunk size on TechTC01 dataset.

8 Conclusion

Identifying outliers is an important problem in data analytics and machine learning. We propose a new algorithm for outlier detection that combines the "chunk recursive elimination" and combinatorial search, similar to the classical A* search algorithm. The results are three variants of the algorithm. One variant is guaranteed to produce optimal solution. Other variants are significantly more accurate and are also much faster than most current alternatives. We also introduce bound for this problem. The bound shows how close the result is to the optimal solution. For non-optimal variants, the chunking comes with possibly wrong outlier detection. An efficient algorithm is proposed to improve both the solution and bound. Extensive experiments demonstrate the effectiveness of the proposed approach.

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