

Token Jumping in Planar Graphs has Linear Sized Kernels

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Abstract

Let G be a planar graph and I_s and I_t be two independent sets in G , each of size k . We begin with a “token” on each vertex of I_s and seek to move all tokens to I_t , by repeated “token jumping”, removing a single token from one vertex and placing it on another vertex. We require that each intermediate arrangement of tokens again specifies an independent set of size k . Given G , I_s , and I_t , we ask whether there exists a sequence of token jumps that transforms I_s to I_t . When k is part of the input, this problem is known to be PSPACE-complete. However, it was shown by Ito, Kamiński, and Ono [4] to be fixed-parameter tractable. That is, when k is fixed, the problem can be solved in time polynomial in the order of G . Here we strengthen the upper bound on the running time in terms of k by showing that the problem has a kernel of size linear in k . More precisely, we transform an arbitrary input problem on a planar graph into an equivalent problem on a (planar) graph with order $O(k)$.

1 Introduction

Given a graph G , a subset of $V(G)$ is *independent* if it induces no edges. In this paper we study the problem of transforming one independent set into another by a sequence of small changes. We model this as follows.

Let G be a planar graph and I_s and I_t be two independent sets in G , each of size k . We begin with a “token” on each vertex of I_s and seek to move all tokens to I_t , by repeated “token jumping”, removing a single token from one vertex and placing it on another vertex. We require that each intermediate arrangement of tokens again specifies an independent set of size k . Given G , I_s , and I_t , we ask whether there exists a sequence of token jumps that transforms I_s to I_t . We call this problem *independent set reconfiguration via token jumping* and denote it by $\text{ISR-TJ}(G, I_s, I_t)$. When k is part of the input, this problem is known to be PSPACE-complete [2]. However, it was shown by Ito, Kamiński, and Ono [4]

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to be fixed-parameter tractable. That is, when k is fixed, the problem can be solved in time polynomial in the order of G .

In an excellent recent survey paper, Bousquet, Mouawad, Nishimura, and Siebertz [1] asked whether the problem admits a linear kernel. That is, given an arbitrary planar instance $\text{ISR-TJ}(G, I_s, I_t)$, does there exist an equivalent instance on a (planar) graph G' where $|V(G')| = O(k)$? We answer their question affirmatively. (For a thorough history of independent set reconfiguration, and other related problems, we recommend [1].)

2 Main Result

Theorem 1. *On the class of planar graphs, ISR-TJ parametrized by k is fixed-parameter tractable with a kernel that has size linear in k .*

Proof. Fix an input graph G , along with source and target independent sets, I_s and I_t , each of size k . We will show that either $\text{ISR-TJ}(G, I_s, I_t)$ is trivially answered YES, or else $\text{ISR-TJ}(G, I_s, I_t)$ is equivalent to a problem $\text{ISR-TJ}(G', I_s, I_t)$, where G' is a subgraph of G and $|V(G')| = O(k)$. Let $X := I_s \cup I_t$ and note that $|X| \leq 2k$. For each $Y \subseteq X$, the X -projection class \mathcal{C}_Y is defined by $\mathcal{C}_Y := \{v \in V(G) \text{ s.t. } N(v) \cap X = Y\}$. Let

I_s, I_t, k
 X
 X -projection class
 \mathcal{C}_Y

$$\mathcal{C}_1 := \bigcup_{\substack{Y \subseteq X \\ |Y| \leq 1}} \mathcal{C}_Y \quad \text{and} \quad \mathcal{C}_2 := \bigcup_{\substack{Y \subseteq X \\ |Y| = 2}} \mathcal{C}_Y \quad \text{and} \quad \mathcal{C}_3 := \bigcup_{\substack{Y \subseteq X \\ |Y| \geq 3}} \mathcal{C}_Y. \quad \mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$$

It is easy to show by planarity, as we do below, that $|\mathcal{C}_3| = O(k)$. And it is also easy to show that either $|\mathcal{C}_1| = O(k)$ or else the answer to $\text{ISR-TJ}(G, I_s, I_t)$ is trivially YES. So we assume the former. Thus, in forming G' from G we will delete some (possibly empty) subset of \mathcal{C}_2 to reach \mathcal{C}'_2 such that $|\mathcal{C}'_2| = O(k)$, and also $\text{ISR-TJ}(G', I_s, I_t)$ is equivalent to $\text{ISR-TJ}(G, I_s, I_t)$.

Claim 1. *If $|\mathcal{C}_1| \geq 4k$, then the answer to $\text{ISR-TJ}(G, I_s, I_t)$ is YES.*

Proof. Assume $|\mathcal{C}_1| \geq 4k$. Since $G[\mathcal{C}_1]$ is planar, it is 4-colorable. By Pigeonhole, \mathcal{C}_1 contains an independent set I_m (for middle) of size $4k/4 = k$. Starting with tokens on I_s , for each $v \in I_s$ with a neighbor $w_v \in I_m$, move the token on v to some such w_v . Now move all remaining tokens (in an arbitrary order) to the unoccupied vertices of I_m . By symmetry, we can also move all tokens from I_t to I_m . Thus, the answer to $\text{ISR-TJ}(G, I_s, I_t)$ is YES, as claimed. \diamond

Henceforth we assume $|\mathcal{C}_1| < 4k$.

Claim 2. $|\mathcal{C}_3| \leq 8k$.

Proof. First note that the number of sets Y with $|Y| \geq 3$ and $\mathcal{C}_Y \neq \emptyset$ is less than $2|X|$. To see this, we draw a plane graph G_X with vertex set X where each such set Y corresponds to a face of G_X . (Think of restricting G to X and one vertex v_Y in \mathcal{C}_Y for each such Y . For each pair of vertices, y_i and y_j ,

that appear successively around v_Y , add edge $y_i y_j$, if it is not already present, following the path $y_i v_Y y_j$. Finally, delete each v_Y ; and “assign” the resulting newly created face to Y .) By Euler’s Formula, the resulting plane graph has at most $3|X| - 6$ edges, so at most $2|X| - 4$ faces. Since G is planar, it is $K_{3,3}$ -free, so $|\mathcal{C}_Y| \leq 2$ for all such Y . Thus, $|\mathcal{C}_3| \leq 2(2|X|) \leq 4(2k)$. \diamond

An X -pair Y is a set $\{y', y''\}$ such that $y', y'' \in X$ and $\mathcal{C}_{\{y', y''\}} \neq \emptyset$. Let \mathcal{P} denote the set of all X -pairs. As in the proof of Claim 2, we can show that $|\mathcal{P}| \leq 3|X|$. (Each X -pair corresponds to an edge of a plane graph with vertex set X .) We assume we are given a plane embedding¹ of G . For each X -pair Y , this embedding induces a linear order on \mathcal{C}_Y ; intuitively, this order is “left-to-right”, but there are subtleties, which we highlight below in Figure 1.

Suppose $Y = \{y', y''\}$. Let $H_Y := G[Y \cup \mathcal{C}_Y] - y' y''$. Consider the outer face f_0 . If f_0 contained at least three vertices of \mathcal{C}_Y , then we could add a new vertex y''' to the interior of f_0 , making it adjacent to all vertices on f_0 . This would give a plane embedding of $K_{3,3}$ (with one part equal to $\{y', y'', y'''\}$ and the other part contained in \mathcal{C}_Y), a contradiction. So f_0 contains at most two vertices of \mathcal{C}_Y . Further, since $|f_0| \geq 3$ and $y' y'' \notin E(H_Y)$, in fact f_0 contains exactly two vertices in \mathcal{C}_Y . Arbitrarily denote one of these z^ℓ and the other z^r (for left and right). We say z^ℓ is the leftmost vertex in \mathcal{C}_Y . Deleting z^ℓ , we can repeat the argument on $H_Y - z^\ell$. The new vertex on the outer face is now considered the “next-leftmost”. Recursively, we can extend this ordering to all of \mathcal{C}_Y . At some step in this process, we may transition from having one vertex of Y on the outer face to having two. However, at each step we will add exactly one new vertex of \mathcal{C}_Y , and that new vertex will be the next in the linear order. Let $N^{\text{Int}}(Y) := \mathcal{C}_Y \setminus \{z^\ell, z^r\}$; this denotes the “interior” vertices in \mathcal{C}_Y .

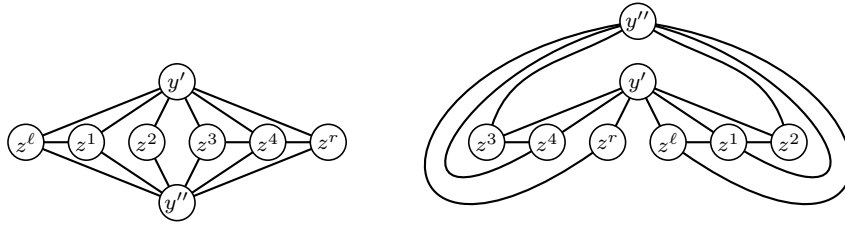


Figure 1: Distinct plane embeddings of a graph give rise to distinct linear orders on \mathcal{C}_Y .

For distinct $Y_1, Y_2 \in \mathcal{P}$, we would like to move tokens in \mathcal{C}_{Y_1} independently of those in \mathcal{C}_{Y_2} . This is made possible by restricting ourselves to only moving tokens in $N^{\text{Int}}(Y_1)$ and $N^{\text{Int}}(Y_2)$, as confirmed by Claim 3.

Claim 3. *If $Y_1, Y_2 \in \mathcal{P}$, with $Y_1 \neq Y_2$, and $z_i \in N^{\text{Int}}(Y_i)$ for each $i \in \{1, 2\}$, then $z_1 z_2 \notin E(G)$.*

¹If not, we can find such an embedding [3, 5] in time $O(|V(G)|)$.

Proof. Since $N^{\text{Int}}(Y_i) \neq \emptyset$, the leftmost and rightmost vertex in \mathcal{C}_{Y_i} are distinct; call them z_i^ℓ and z_i^r . Denote Y_i by $\{y_i', y_i''\}$, for each $i \in \{1, 2\}$. Let $C_i := y_i' z_i^\ell y_i'' z_i^r$ for each $i \in \{1, 2\}$. Note that C_1 and C_2 might not be disjoint, but they intersect in at most one vertex; that is, $|Y_1 \cap Y_2| \leq 1$. Consider the regions with C_1 and C_2 as their boundaries; call them \mathcal{R}_1 and \mathcal{R}_2 . Because C_1 and C_2 intersect at no more than one point, either \mathcal{R}_1 and \mathcal{R}_2 are disjoint or else one lies completely inside (topologically) the other; say \mathcal{R}_1 lies inside \mathcal{R}_2 . In both cases, z_1 lies inside C_1 , and z_2 lies outside it. Thus, $z_1 z_2 \notin E(G)$. \diamond

It is helpful to note, for each $Y \in \mathcal{P}$, that $G[\mathcal{C}_Y]$ is a linear forest, since each $z \in \mathcal{C}_Y$ has edges only to (possibly) its predecessor and/or successor in the linear order.

Let $\mathcal{P}_{\text{Big}} := \{\{y', y''\} \in \mathcal{P} \text{ s.t. } N^{\text{Int}}(y', y'') \text{ has two non-adjacent vertices.}\}$. \mathcal{P}_{Big}

Claim 4. *Let $G_{\mathcal{P}_{\text{Big}}}$ be the subgraph of G induced by $\cup_{Y \in \mathcal{P}_{\text{Big}}} N^{\text{Int}}(Y)$. If $G_{\mathcal{P}_{\text{Big}}}$ has no independent set of size k , then $|V(G)| \leq 40k$, so G is a linear kernel.* $G_{\mathcal{P}_{\text{Big}}}$

Proof. Assume $G_{\mathcal{P}_{\text{Big}}}$ has no independent set of size k . Now

$$\begin{aligned}
|\cup_{Y \in \mathcal{P}} \mathcal{C}_Y| &\leq |\cup_{Y \in \mathcal{P}} (2 + |N^{\text{Int}}(Y)|)| \\
&= 2|\mathcal{P}| + |\cup_{Y \in \mathcal{P} \setminus \mathcal{P}_{\text{Big}}} N^{\text{Int}}(Y)| + |\cup_{Y \in \mathcal{P}_{\text{Big}}} N^{\text{Int}}(Y)| \\
&\leq 2|\mathcal{P}| + (\sum_{Y \in \mathcal{P} \setminus \mathcal{P}_{\text{Big}}} 2) + |\cup_{Y \in \mathcal{P}_{\text{Big}}} N^{\text{Int}}(Y)| \\
&\leq 2|\mathcal{P}| + 2|\mathcal{P}| + 2(k-1) \\
&\leq 4|\mathcal{P}| + 2(k-1) \\
&\leq 4(3(|X|)) + 2k \leq 12(2k) + 2k = 26k.
\end{aligned} \tag{1}$$

Here (1) uses that $N^{\text{Int}}(Y)$ is a linear forest (disjoint union of paths) for every $Y \in \mathcal{P}$, that $2\alpha(H) \geq |V(H)|$ when H is a linear forest, and that $\alpha(H_1 + H_2 + \dots) = \sum_i \alpha(H_i)$, when $H_1 + H_2 + \dots$ is a disjoint union of linear forests. Claims 1 and 2 give $|V(G)| = |X| + |\mathcal{C}_1| + |\mathcal{C}_2| + |\mathcal{C}_3| \leq 2k + 4k + 26k + 8k = 40k$. \diamond

Claim 5. *In every linear forest, every independent set of size k can be reconfigured to every other (in at most $2k$ steps).*

Proof Sketch. We order the paths arbitrarily, and pick a “left end” for each. We now iteratively move each token on the first path as far left as possible, and then fill the remainder of the path with remaining tokens. We finish recursively on the remaining paths and tokens. This transforms an arbitrary independent set of size k to a canonical one, in at most k steps. \diamond

Henceforth, we assume that $G_{\mathcal{P}_{\text{Big}}}$ contains an independent set of size k ; call it J_m . We will form sets J_s and J_t (not necessarily independent, but each inducing a linear forest), each of size $O(k)$, and form G' from G by deleting $(\cup_{Y \in \mathcal{P}_{\text{Big}}} N^{\text{Int}}(Y)) \setminus (J_m \cup J_s \cup J_t)$. Clearly $|V(G')| = O(k)$. So it remains to specify J_s and J_t , and to show that $\text{ISR-TJ}(G', I_s, I_t)$ is equivalent to J_m G'

ISR-TJ(G, I_s, I_t). For the latter, we will show that if in G we can move a token to a vertex absent from G' , then in G' we can move all tokens from I_s to $J_s \cup J_m$, and we can also move all tokens from I_t to $J_t \cup J_m$. Since $G[J_s \cup J_t \cup J_m]$ is a linear forest, by Claim 4 the answer to ISR-TJ(G', I_s, I_t) is YES.

Let \mathcal{P}^1 be the set of X -pairs with at most one vertex in I_s and let \mathcal{P}^2 be the set of X -pairs with two vertices in I_s . To form J_s , start by including, for all $Y \in \mathcal{P}^1$, each vertex in C_Y such that I_s contains at most one vertex of Y , up to the point (if it occurs) where J_s contains an independent set of size k that lies entirely in $\cup_{Y \in \mathcal{P}^1} N^{\text{Int}}(Y)$. If this point occurs, we are done forming J_s ; so assume it does not. Now, we consider each $Y \in \mathcal{P}^2$. For each such Y , add to J_s the leftmost and right vertex in C_Y as well as two vertices in $N^{\text{Int}}(Y)$, if they exist, giving preference to a non-adjacent pair. Analogously, we construct J_t with I_t in the role of I_s .

Claim 6. $|V(G')| = O(k)$.

Proof. For each $Y \in \mathcal{P}^1$, its contribution to a maximum independent set contained in $\cup_{Y \in \mathcal{P}^1} N^{\text{Int}}(Y)$ is $\alpha(G[N^{\text{Int}}(Y)]) \geq |N^{\text{Int}}(Y)|/2$, because $N^{\text{Int}}(Y)$ induces a linear forest. Since the size of such a set is at most k (by construction), we have $|\cup_{Y \in \mathcal{P}^1} N^{\text{Int}}(Y)| \leq 2 \sum_{Y \in \mathcal{P}^1} \alpha(G[N^{\text{Int}}(Y)]) \leq 2k$. Thus,

$$\begin{aligned} |J_s| &\leq \sum_{Y \in \mathcal{P}^1} (2 + |N^{\text{Int}}(Y)|) + \sum_{Y \in \mathcal{P}^2} (2 + \min(2, |N^{\text{Int}}(Y)|)) \\ &\leq \sum_{Y \in \mathcal{P}^1} 2 + 2k + \sum_{Y \in \mathcal{P}^2} 4 \\ &\leq 4|\mathcal{P}| + 2k \leq 12|X| + 2k \leq 24k + 2k = 26k. \end{aligned}$$

Now $|J_s \cup J_t \cup J_m| \leq |J_s| + |J_t| + |J_m| \leq 26k + 26k + k = 53k$. Thus, $|V(G')| \leq |X| + |\mathcal{C}_1| + |J_s \cup J_t \cup J_m| + |\mathcal{C}_3| \leq 2k + 4k + 53k + 8k = 67k$. \diamond

Lastly, we show that when we restrict G to G' the problem stays equivalent.

Claim 7. ISR-TJ(G', I_s, I_t) is equivalent to ISR-TJ(G, I_s, I_t).

Proof. Since $G' \subseteq G$, if the answer to ISR-TJ(G, I_s, I_t) is NO, then clearly the answer to ISR-TJ(G', I_s, I_t) is also NO. So it suffices to show that if the answer to ISR-TJ(G, I_s, I_t) is YES, then also the answer to ISR-TJ(G', I_s, I_t) is YES.

Suppose that the answer to ISR-TJ(G, I_s, I_t) is YES, and let σ be a sequence of token jumps witnessing this. If each vertex appearing in σ is a vertex of G' , then σ also witnesses that the answer to ISR-TJ(G', I_s, I_t) is YES. So assume that σ uses some vertex that is not in G' . Let z_s and z_t denote the first and last such vertex used by σ that are not in $V(G')$. We will show that (a) because $z_s \notin V(G')$, we can move all tokens from I_s to vertices of $J_s \cup J_m$. Similarly, (b) because $z_t \notin V(G')$, we can move all tokens from I_t to vertices of $J_t \cup J_m$. The arguments are essentially identical (we interchange the roles of I_s and I_t and run σ in reverse), so it suffices to prove (a).

Suppose that $z_s \in \cup_{Y \in \mathcal{P}^1} \mathcal{C}_Y$. (Recall that \mathcal{P}^1 is the set of X -pairs with at most one vertex in I_s .) What caused z_s to be absent from J_s ? It is because J_s contains an independent set, call it J_1 of size k that lies entirely in $\cup_{Y \in \mathcal{P}^1} N^{\text{Int}}(Y)$. For each token on a vertex v with a neighbor w in J_1 , we move the token to w . This cannot create any conflicts since J_1 is an independent set and is contained in $\cup_{Y \in \mathcal{P}^1} N^{\text{Int}}(Y)$. Now each vertex of J_1 has no neighbor with a token, so we can greedily move the remaining tokens to the unoccupied vertices of J_1 . Thus, we are done if J_1 exists; so we assume it does not.

Instead assume that $z_s \in \cup_{Y \in \mathcal{P}^2} \mathcal{C}_Y$. By our construction of G' , since $z_s \notin V(G')$, we know $z_s \in \cup_{Y \in \mathcal{P}_{\text{Big}}} N^{\text{Int}}(Y)$. Denote $\mathcal{P}_{\text{Big}} \cap \mathcal{P}^2$ by $\{Y_1, \dots, Y_a\}$ in an arbitrary order (for some integer a), and denote Y_i by $\{y'_i, y''_i\}$ for each $i \in \{1, \dots, a\}$. Since σ moves a token to z_s , at some (earlier) point σ moves a token off of some vertex in some Y_i . By symmetry, assume σ first moves a token off of y'_1 . We show, by induction on i , that we can move all tokens off of $\cup_{i=1}^a Y_i$. Recall, for each i , that $N^{\text{Int}}(Y_i)$ contains non-adjacent vertices that lie in J_m ; call such a pair z'_i and z''_i . For the base case, we move the token on y'_1 to z'_1 . For the induction step, we assume that no token appears on the closed neighborhood of z''_{i-1} . So we move a token from y'_i to z''_{i-1} and then we move a token from y''_i to z'_i . This finishes the induction proof.

Once all tokens are removed from $\cup_{i=1}^a Y_i$, by Claim 4 we move all tokens on $\cup_{i=1}^a \{z'_i, z''_i\}$ to J_m . Finally, we greedily move all remaining tokens to J_m . \diamond

Claim 7 finishes the proof of Theorem 1. \square

In Claim 6, we showed that $|V(G')| \leq 67k$. This upper bound can likely be strengthened, but we have not made an effort to do so. We prefer to keep the proof as simple as possible. The important point is that $|V(G')| = O(k)$.

References

- [1] N. Bousquet, A. E. Mouawad, N. Nishimura, and S. Siebertz. A survey on the parameterized complexity of the independent set and (connected) dominating set reconfiguration problems. Apr 2022, [arXiv:2204.10526](https://arxiv.org/abs/2204.10526).
- [2] R. A. Hearn and E. D. Demaine. PSPACE-completeness of sliding-block puzzles and other problems through the nondeterministic constraint logic model of computation. *Theoret. Comput. Sci.*, 343(1-2):72–96, 2005. doi:10.1016/j.tcs.2005.05.008.
- [3] J. Hopcroft and R. Tarjan. Efficient planarity testing. *J. Assoc. Comput. Mach.*, 21:549–568, 1974. doi:10.1145/321850.321852.
- [4] T. Ito, M. Kamiński, and H. Ono. Fixed-parameter tractability of token jumping on planar graphs. In *Algorithms and computation*, volume 8889 of *Lecture Notes in Comput. Sci.*, pages 208–219. Springer, Cham, 2014. doi:10.1007/978-3-319-13075-0_17.
- [5] K. Mehlhorn, P. Mutzel, and S. Näher. An implementation of the Hopcroft and Tarjan planarity test and embedding algorithm, 1993. <https://www.mpi-inf.mpg.de/~mehlhorn/ftp/planar.ps>.