A form of refined Roth's theorem and its application to the *abc*-conjecture

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Abstract

In this paper, we give a form of refined Roth's theorem. As an application, we prove a special case of the *abc*-conjecture.

1 Introduction

We begin by recalling some notations and notions. Let κ be a number field and $\bar{\kappa}$ an algebraic closure of κ. Let $\mathcal{O}_\kappa$ denote the ring of integers of κ. Let M_κ be the canonical set of distinct inequivalent valuations of κ satisfying the product formula

$$\prod_{v\in M_\kappa}\|x\|_v=\prod_{v\in M_\kappa}|x|_v^{n_v}=1,\ x\in\kappa_*=\kappa-\{0\}.$$

Here $\|\cdot\|_v$ is normalized by its absolute value $|\cdot|_v$ as follows. A non-Archimedean place $v\in M_\kappa$ corresponds to a nonzero prime ideal $\mathfrak{p}=\mathfrak{p}_v\subseteq\mathcal{O}_\kappa$, and we set

$$\|x\|_v=|x|_v^{n_v}=\mathcal{N}(\mathfrak{p})^{-v_{\mathfrak{p}}(x)},$$

where $v_{\mathfrak{p}}=\mathrm{ord}_v$ is the valuation defined by $\mathfrak{p}$, $\mathcal{N}(\mathfrak{p})$ is the absolute norm of $\mathfrak{p}$, i.e., the number of residue classes mod $\mathfrak{p}$, and the multiplicity $n_v=[\kappa_v:\mathbb{Q}_p]$ is the local degree of v if $v|p$ for some $p\in M_{\mathbb{Q}}$ such that $\sum_{v|p}n_v=[\kappa:\mathbb{Q}]$. If v is Archimedean, then v corresponds to a real embedding $\sigma:\kappa\longrightarrow\mathbb{R}$ or a complex conjugate pair of complex embeddings $\sigma,\overline{\sigma}:\kappa\longrightarrow\mathbb{C}$, and we set $\|x\|_v=|x|_v=|\sigma(x)|$ or $\|x\|_v=|x|_v^2=|\sigma(x)|^2$. Let M_κ^0 be the set of non-Archimedean places in M_κ and $M_\kappa^\infty=M_\kappa-M_\kappa^0$ be the set of all Archimedean places. Let S be a finite subset of M_κ containing the subset of all Archimedean valuations in M_κ.

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If v is non-Archimedean, set

$$\chi_v(x,a) = \begin{cases} \frac{|x-a|_v}{|x|_v^\vee |a|_v^\vee} & : \quad x, a \in \kappa \\ \frac{1}{|x|_v^\vee} & : \quad a = \infty, \end{cases} \tag{1}$$

where, by definition,

$$r^\vee = \max\{1, r\} \ (r \in \mathbb{R}).$$

If v is Archimedean, set

$$\chi_v(x,a) = \begin{cases} \frac{|x-a|_v}{(1+|x|_v^2)^{1/2}(1+|a|_v^2)^{1/2}} & : \quad x, a \in \kappa \\ \frac{1}{\sqrt{1+|x|_v^2}} & : \quad a = \infty. \end{cases} \tag{2}$$

The proximity function for a is defined by

$$m(x,a) = \frac{1}{[\kappa : \mathbb{Q}]} \sum_{v \in S} \log \frac{1}{\chi_v(x,a)^{n_v}}$$

and similarly the valence function for a

$$N(x,a) = \frac{1}{[\kappa : \mathbb{Q}]} \sum_{v \in M_\kappa - S} \log \frac{1}{\chi_v(x,a)^{n_v}}.$$

Fix $a, x \in \kappa$. Obviously, there exists a constant C depending only on $|a|_v$ such that

$$\max\left\{1, \frac{1}{|x-a|_v}\right\} \leq \frac{1}{\chi_v(x,a)} \leq C \max\left\{1, \frac{1}{|x-a|_v}\right\}.$$

Thus we have

$$m(x,a) = \frac{1}{[\kappa : \mathbb{Q}]} \sum_{v \in S} \log^+ \frac{1}{\|x-a\|_v} + O(1)$$

and similarly,

$$N(x,a) = \frac{1}{[\kappa : \mathbb{Q}]} \sum_{v \in M_\kappa - S} \log^+ \frac{1}{\|x-a\|_v} + O(1),$$

where by definition,

$$\log^+ r = \log r^\vee = \max\{0, \log r\} \ (r \in \mathbb{R}_+).$$

The *relative multiplicative heights* of an element x of the number field κ is defined by

$$H_\kappa(x) = \left(\prod_{v \in M_\kappa^\infty} \left(\sqrt{1+|x|_v^2} \right)^{n_v} \right) \left(\prod_{v \in M_\kappa - M_\kappa^\infty} \max\{1, |x|_v^{n_v}\} \right),$$

$$H_{*,\kappa}(x) = \prod_{v \in M_\kappa} \max\{1, |x|_v^{n_v}\},$$

and the *absolute* (*logarithmic*) *height* $h(x)$ by

$$h(x) = \frac{1}{[\kappa : \mathbb{Q}]} \log H_\kappa(x).$$

S. Lang [4] observed that there is no reason not to let x approach infinity; for example, Roth's Theorem can be changed to the following form (see [3], Theorem 6.14):

Theorem 1.1. *Let κ be a number field and $S \subset M_\kappa$ a finite subset of absolute values on κ. Assume that each absolute value in S has been extended in some way to $\bar{\kappa}$. Let a_1, ..., a_q be distinct elements in $\bar{\kappa}$ and ε a positive constant. Then there are only finitely many $x \in \kappa$ such that*

$$\prod_{v \in S} \left(\min\left\{1, \frac{1}{\|x\|_v}\right\} \prod_{j=1}^{q} \min\{1, \|x - a_j\|_v\} \right) < \frac{1}{H_{*,\kappa}(x)^{2+\varepsilon}}. \tag{3}$$

Next, assume $M_\kappa^\infty \subseteq S$. Without loss of generality, we may assume $a_j \in \kappa$ for $j \geq 1$. The inequality (3) can be rewritten into the following form:

$$(q-1)h(x) \leq N(x, \infty) + \sum_{j=1}^{q} N(x, a_j) + \varepsilon h(x) + O(1), \tag{4}$$

where

$$N(x, a_j) = \frac{1}{[\kappa : \mathbb{Q}]} \sum_{v \in S^c(x - a_j)} \mathrm{ord}_v(x - a_j) \log \mathcal{N}(\mathfrak{p}_v) + O(1) \tag{5}$$

for each $j = 1, ..., q$, and

$$N(x, \infty) = \frac{1}{[\kappa : \mathbb{Q}]} \sum_{v \in S^c(x^{-1})} \mathrm{ord}_v(x^{-1}) \log \mathcal{N}(\mathfrak{p}_v) + O(1), \tag{6}$$

where

$$S^c(y) = \{v \in M_\kappa - S \mid \mathrm{ord}_v(y) > 0\}.$$

Define

$$\overline{N}(x, a_j) = \frac{1}{[\kappa : \mathbb{Q}]} \sum_{v \in S^c(x - a_j)} \log \mathcal{N}(\mathfrak{p}_v) \tag{7}$$

and

$$\overline{N}(x, \infty) = \frac{1}{[\kappa : \mathbb{Q}]} \sum_{v \in S^c(x^{-1})} \log \mathcal{N}(\mathfrak{p}_v). \tag{8}$$

It is a simple fact (see [3], p.358) that the following Conjecture 1.2, which strengthens the inequality (4), implies the following Conjecture 1.3, the *abc*-conjecture (see [7], [9]).

Conjecture 1.2. *Let a_1,..., a_q be distinct elements in $\bar{\kappa}$ and ε a positive constant. All but finitely many $x \in \kappa$ satisfy the inequality*

$$(q-1)h(x) \leq \overline{N}(x, \infty) + \sum_{j=1}^{q} \overline{N}(x, a_j) + \varepsilon h(x) + O(1). \tag{9}$$

Conjecture 1.3. *Given $\varepsilon > 0$, there exists a number $C(\varepsilon)$ having the following property. For any nonzero relatively prime integers a, b, c such that $a + b = c$, we have*

$$\max\{|a|, |b|, |c|\} \leq C(\varepsilon) r\,(abc)^{1+\varepsilon}, \tag{10}$$

where $r\,(abc)$ is the radical of abc defined by $r(abc) = \prod_{p|abc} p$, i.e. the product of distinct primes dividing abc.

In this paper, we will prove Conjecture 1.2 for algebraic integers, that is, we obtain the following result:

Theorem 1.4. *Let a_1,..., a_q be distinct elements in $\bar{\kappa}$. All but finitely many algebraic integers $x \in \kappa$ satisfy the inequality*

$$(q-1)h(x) \leq \sum_{j=1}^{q} \overline{N}(x, a_j) + O(1). \tag{11}$$

Theorem 1.5. *Let a_1,a_2 be distinct elements in $\bar{\kappa}$. All but finitely many simple numbers $x \in \kappa$ satisfy the inequality*

$$h(x) \leq \overline{N}(x, \infty) + \overline{N}(x, a_1) + \overline{N}(x, a_2) + O(1). \tag{12}$$

In Theorem 1.5, a number $x \in \kappa$ is called simple if negative powers of ideals occurred in x are all -1. Theorem 1.5 immediately yields a special case of the *abc*-conjecture as follows:

Theorem 1.6. *For any nonzero relatively prime integers a, b, c such that $a + b = c$ and that one of a, b, c only has prime factors of power 1, we have*

$$\max\{|a|, |b|, |c|\} \leq Cr\,(abc)\,, \tag{13}$$

where C is an absolute constant.

Some results that would follow from the *abc*-conjecture can be found in [1], [8], pp. 185-188, [14]; see also [2], [5], [6], [15]. In [10], C. L. Stewart and R. Tijdeman proved that

$$\max\{|a|, |b|, |c|\} < \exp\left\{Cr(abc)^{15}\right\},$$

where C is an absolute constant. In [11], C. L. Stewart and K. Yu obtained that

$$\max\{|a|, |b|, |c|\} < \exp\left\{C(\varepsilon)r(abc)^{2/3+\varepsilon}\right\}.$$

In [12], C. L. Stewart and K. Yu further proved that

$$\max\{|a|, |b|, |c|\} < \exp\left\{C(\varepsilon)r(abc)^{1/3+\varepsilon}\right\}.$$

2 Proofs of Theorem 1.4 and Theorem 1.5

We consider the following rational function

$$Q(X) = \frac{1}{(X - a_1)\cdots(X - a_q)} = \sum_{j=1}^{q} \frac{A_j}{X - a_j},$$

where

$$A_j = \frac{1}{(a_j - a_1)\cdots(a_j - a_{j-1})\cdot(a_j - a_{j+1})\cdots(a_j - a_q)}.$$

Theorem 1.4 and Theorem 1.5 will follow from the following result:

Theorem 2.1. *Let a_1,..., a_q be distinct elements in $\bar{\kappa}$ and ε a positive constant. There exist an extension field K of κ and an element $x' \in K$ such that all but finitely many $x \in \kappa$ satisfy the inequality*

$$(q-1)h(x) \leq \overline{N}(x, \infty) + \sum_{j=1}^{q} \overline{N}(x, a_j) + 2m\left(x'Q(x), \infty\right) + O(1). \tag{14}$$

Proof. Set

$$\rho_v = \min_{1 \leq j \leq q} |A_j|_v, \ \sigma_v = \max_{1 \leq j \leq q} |A_j|_v, \ \delta_v = \min_{1 \leq i < j \leq q} |a_i - a_j|_v,$$

$$E_{vj} = \left\{ x \in \kappa \ \middle| \ |x - a_j|_v < \frac{\delta_v}{2\varrho_v} \right\},$$

where

$$\varrho_v = \frac{1}{2} + (q-1)\frac{\sigma_v}{\rho_v}.$$

When $i \neq j$, $x \in E_{vj}$, we have

$$|x - a_i|_v \geq |a_i - a_j|_v - |x - a_j|_v \geq \delta_v \left(1 - \frac{1}{2\varrho_v}\right) \geq \frac{\delta_v}{2\varrho_v}.$$

Since

$$Q(x) = \frac{A_j}{x - a_j} \left\{ 1 + \sum_{i \neq j} \frac{A_i}{A_j} \cdot \frac{x - a_j}{x - a_i} \right\},$$

we deduce that

$$|Q(x)|_v > \frac{|A_j|_v}{|x - a_j|_v} \left\{ 1 - (q-1)\frac{\frac{\delta_v}{2q}}{\delta_v\left(1 - \frac{1}{2q}\right)} \right\} \geq \frac{\rho_v}{2|x - a_j|_v}$$

and thus that

$$\begin{aligned} \log^+ \|Q(x\|_v &> \log^+ \frac{1}{\|x - a_j\|_v} - n_v \log \frac{2}{\rho_v} \\ &\geq \sum_{i=1}^{q} \log^+ \frac{1}{\|x - a_i\|_v} - qn_v \log^+ \frac{2\varrho_v}{\delta_v} - n_v \log \frac{2}{\rho_v}. \end{aligned}$$

Obviously, this inequality is also true if $x \notin \cup_i E_{vi}$. Thus we obtain

$$m(Q(x), \infty) \geq \sum_{j=1}^{q} m(x, a_j) - C_S,$$

where

$$C_S = \frac{1}{[\kappa : \mathbb{Q}]} \sum_{v \in S} \left(qn_v \log^+ \frac{2\varrho_v}{\delta_v} + n_v \log \frac{2}{\rho_v} \right).$$

On the other hand, for some $x' \in K_*$, where K is an extension field of κ (see below for its construction), we have

$$
\begin{aligned}
m(Q(x),\infty) &= \frac{1}{[K:\mathbb{Q}]}\sum_{w\in S_K}\log^+\|Q(x)\|_w + O(1)\\
&= \frac{1}{[K:\mathbb{Q}]}\sum_{v\in S}\sum_{w|v}\log^+\|Q(x)\|_v^{[K_w:\kappa_v]} + O(1)\\
&= \frac{1}{[\kappa:\mathbb{Q}]}\sum_{v\in S}\log^+\|Q(x)\|_v + O(1),
\end{aligned}
$$

where S_K is the set of $w \in M_K$ such that $w|v$ for some $v \in S$ so that

$$
\begin{aligned}
m(Q(x),\infty) &\leq m\left(x'Q(x),\infty\right) + m\left(x',0\right)\\
&\leq m\left(x'Q(x),\infty\right) + h(x') - N\left(x',0\right) + O(1).
\end{aligned}
$$

Hence for each $j \in \{1,...,q\}$, we have

$$
\begin{aligned}
h(x') &= m(x',\infty) + N(x',\infty)\\
&\leq m(x-a_j,\infty) + N(x',\infty) + m\left(\frac{x'}{x-a_j},\infty\right) + O(1)\\
&\leq h(x) + N(x',\infty) - N(x,\infty) + m\left(\frac{x'}{x-a_j},\infty\right) + O(1).
\end{aligned}
$$

Therefore

$$
m(x,\infty) + \sum_{j=1}^{q} m(x,a_j) \leq 2h(x) - N_{x'}(x) + S_{j,x'}(x) + O(1), \tag{15}
$$

where

$$
N_{x'}(x) = 2N(x,\infty) - N(x',\infty) + N(x',0),
$$

$$
S_{j,x'}(x) = m\left(\frac{x'}{x-a_j},\infty\right) + m\left(x'Q(x),\infty\right),
$$

and hence

$$
(q-1)h(x) \leq N(x,\infty) + \sum_{j=1}^{q} N(x,a_j) - N_{x'}(x) + S_{x'}(x) + O(1), \tag{16}
$$

where

$$
S_{x'}(x) = \frac{1}{q}\sum_{j=1}^{q} S_{j,x'}(x) = \frac{1}{q}\sum_{j=1}^{q} m\left(\frac{x'}{x-a_j},\infty\right) + m\left(x'Q(x),\infty\right).
$$

We claim that there exists a constant c_v satisfying

$$
c_v \prod_{j=1}^{q} \left|\frac{x'}{x-a_j}\right|_{v+} \leq \left|\frac{x'}{(x-a_1)\cdots(x-a_q)}\right|_{v+}^{q}, \tag{17}
$$

where $x_+ = x^\vee = \max\{1, x\}$. When $|x - a_i|_v \le \delta_v/2$ for some i, then for $j \ne i$

$$|x - a_j|_v \ge |a_j - a_i|_v - |x - a_i|_v \ge \delta_v/2,$$

and

$$|x - a_j|_v \le |x - a_i|_v + |a_i - a_j|_v \le \delta_v/2 + \lambda_v,$$

where

$$\lambda_v = \max_{i,j} |a_i - a_j|_v.$$

Hence we have either

$$\prod_{j=1}^{q} \left| \frac{x'}{x - a_j} \right|_{v+} \le \left| \frac{x'}{x - a_i} \right|_{v+}$$

when $|x'|_v \le \delta_v/2$, or

$$\prod_{j=1}^{q} \left| \frac{x'}{x - a_j} \right|_{v+} \le \left(\frac{2|x'|_v}{\delta_v} \right)^{q-1} \left| \frac{x'}{x - a_i} \right|_v \le \left| \frac{x'}{x - a_i} \right|_v^q$$

if $|x'|_v > \delta_v/2$.

Note that

$$\left| \frac{x'}{(x - a_1) \cdots (x - a_q)} \right|_v \ge \left(\frac{1}{\delta_v/2 + \lambda_v} \right)^{q-1} \left| \frac{x'}{x - a_i} \right|_v.$$

Thus when $|x'|_v \le \delta_v/2$, there exists a constant c_v such that

$$c_v \prod_{j=1}^{q} \left| \frac{x'}{x - a_j} \right|_{v+} \le \left| \frac{x'}{(x - a_1) \cdots (x - a_q)} \right|_{v+},$$

so that (17) follows. If $|x'|_v > \delta_v/2$, then

$$\left| \frac{x'}{(x - a_1) \cdots (x - a_q)} \right|_v^q \ge \left(\frac{1}{\delta_v/2 + \lambda_v} \right)^{(q-1)q} \prod_{j=1}^{q} \left| \frac{x'}{x - a_j} \right|_{v+}$$

which also yields the estimate (17) .

If $|x - a_i|_v > \delta_v/2$ for all i, then

$$\prod_{j=1}^{q} \left| \frac{x'}{x - a_j} \right|_{v+} \le \left(\frac{2}{\delta_v} \right)_+^q \le \left(\frac{2}{\delta_v} \right)_+^q \left| \frac{x'}{(x - a_1) \cdots (x - a_q)} \right|_{v+},$$

and hence (17) follows again.

Hence, we obtain the estimate

$$S_{x'}(x) \le 2m\left(x'Q(x), \infty\right) + O(1), \tag{18}$$

which implies by (16) that

$$(q-1)h(x) \le \overline{N}(x, \infty) + \sum_{j=1}^{q} \overline{N}(x, a_j) + 2m\left(x'Q(x), \infty\right) + O(1). \tag{19}$$

It is easy to show that if (11) is true for all algebraic integers a_j, then it is true for all algebraic numbers a_j so that we may assume that all a_j are algebraic integers. We may construct the above x' so that it is in κ but it would be hard to compute the term $m\left(x'Q(x),\infty\right)$ (see Remark below). We turn to use the following result in [3], Theorem 2.32 (see also [13]): For a number field κ, there is a number field $K \supseteq \kappa$ such that for each ideal $\mathfrak{a}$ in the ring of integers $\mathcal{O}_\kappa$ of κ, it holds that

(I) $\mathcal{O}_K\mathfrak{a}$ is a principal ideal;

(II) $(\mathcal{O}_K\mathfrak{a}) \cap \mathcal{O}_\kappa = \mathfrak{a}$.

Using this result we obtain an extension field $K \supseteq \kappa$ such that for each ideal $\mathfrak{a}$ in the ring of integers $\mathcal{O}_\kappa$ of κ, (I) and (II) hold. Thus there exist x_0, $x_\infty \in \mathcal{O}_K$ such that

$$(x_0) = \mathcal{O}_K\mathfrak{a}_0,\ (x_\infty) = \mathcal{O}_K\mathfrak{a}_\infty,$$

and

$$(\mathcal{O}_K\mathfrak{a}_0) \cap \mathcal{O}_\kappa = \mathfrak{a}_0,\ (\mathcal{O}_K\mathfrak{a}_\infty) \cap \mathcal{O}_\kappa = \mathfrak{a}_\infty.$$

We then take $x' = x_0/x_\infty \in K$. This completes the proof. □

Remark. As noted in the above proof, x' can be constructed so that it belongs to κ (it would however be hard to computer the term $m\left(x'Q(x),\infty\right)$). To see this, write

$$(x) = \mathfrak{P}_1^{t_1}\cdots\mathfrak{P}_l^{t_l}\mathfrak{h}^{-1},\ \mathfrak{h} = \mathfrak{Q}_1^{u_1}\cdots\mathfrak{Q}_h^{u_h}\mathfrak{Q}_{h+1}^{u_{h+1}}\cdots\mathfrak{Q}_{h+g}^{u_{h+g}},$$

where $t_i,\ u_j$ are positive integers, and

$$\mathfrak{Q}_i \in M_\kappa - S\ (i = 1, ..., h);\ \mathfrak{Q}_{h+j} \in S\ (j = 1, ..., g).$$

Similarly, we can write

$$(x - a_j) = \mathfrak{h}^{-1}\mathfrak{p}_{m_{j-1}+1}^{r_{m_{j-1}+1}}\cdots\mathfrak{p}_{m_j}^{r_{m_j}}\mathfrak{q}_{n_{j-1}+1}^{s_{n_{j-1}+1}}\cdots\mathfrak{q}_{n_j}^{s_{n_j}},$$

where r_i, s_j are positive integers, $m_0 = n_0 = 0$, and

$$\mathfrak{p}_i \in M_\kappa - S\ (i = 1, ..., m_q);\ \mathfrak{q}_j \in S\ (j = 1, ..., n_q).$$

We first assume that $\mathfrak{p}_1$, ..., $\mathfrak{p}_{m_q}$ are distinct. Write $\mathfrak{a}_0 = \prod_{i=1}^{m_q}\mathfrak{p}_i^{r_i-1}$, and further define ideals $\mathfrak{d}_i$ by

$$\mathfrak{p}_i^{r_i}\mathfrak{d}_i = \mathfrak{a}_0\mathfrak{p}_1\cdots\mathfrak{p}_{m_q},\ i = 1, 2, ..., m_q$$

so that $\mathfrak{d}_i$ is relatively prime to $\mathfrak{p}_i$. Since these $\mathfrak{d}_i$ in their totality are relatively prime, there are elements $\delta_i \in \mathfrak{d}_i$ satisfying

$$\delta_1 + \delta_2 + \cdots + \delta_{m_q} = 1.$$

Since $\mathfrak{d}_i|\delta_i$, hence $\mathfrak{p}_j|\delta_i$ $(j \neq i)$. Consequently, $\mathfrak{p}_i \nmid \delta_i$ since $\mathfrak{p}_i \nmid (1)$. We now determine elements α_i such that

$$\mathfrak{p}_i^{r_i-1}|\alpha_i,\ \mathfrak{p}_i^{r_i} \nmid \alpha_i,\ i = 1, ..., m_q,$$

which is obviously always possible since for this to happen α_i needs only to be an element from $\mathfrak{p}_i^{r_i-1}$ which does not occur in $\mathfrak{p}_i^{r_i}$. Then the element

$$x_0 = \alpha_1\delta_1 + \alpha_2\delta_2 + \cdots + \alpha_{m_q}\delta_{m_q}$$

has the property $\mathfrak{a}_0 \mid x_0$. For each of the prime ideals $\mathfrak{p}_i$ occurs in $m_q - 1$ summands at least to the power $\mathfrak{p}_i^{r_i}$; however, it occurs precisely to the power $\mathfrak{p}_i^{r_i-1}$ in the i-th summand; consequently x_0 is divisible by precisely the $(r_i - 1)$-th power of $\mathfrak{p}_i$, but no higher power. If $\mathfrak{p}_1$, ..., $\mathfrak{p}_{m_q}$ are not distinct, say $\mathfrak{p}_1 = \mathfrak{p}_2$, but $\mathfrak{p}_2$, ..., $\mathfrak{p}_{m_q}$ are distinct, replace $\mathfrak{d}_2$ by

$$\mathfrak{p}_1^{r_1}\mathfrak{p}_2^{r_2}\mathfrak{d}_2 = \mathfrak{a}_0\mathfrak{p}_1\cdots\mathfrak{p}_{m_q}$$

and determine the element α_2 such that

$$\mathfrak{p}_1^{r_1+r_2-2}|\alpha_2,\ \mathfrak{p}_1^{r_1+r_2-1} \nmid \alpha_2.$$

Then the element x_0 is replaced by

$$x_0 = \alpha_2\delta_2 + \alpha_3\delta_3 + \cdots + \alpha_{m_q}\delta_{m_q}.$$

Similarly, if we define

$$\mathfrak{a}_\infty = \mathfrak{Q}_1^{u_1+1}\cdots\mathfrak{Q}_h^{u_h+1},$$

then we can find an element x_∞ such that when $\mathfrak{Q}_1$, ..., $\mathfrak{Q}_h$ are distinct, each of the prime ideals $\mathfrak{Q}_i$ occurs in $h-1$ summands at least to the power $\mathfrak{Q}_i^{u_i+2}$; however, it occurs precisely to the power $\mathfrak{Q}_i^{u_i+1}$ in the i-th summand; consequently x_∞ is divisible by precisely the (u_i+1)-th power of $\mathfrak{Q}_i$, but no higher power. We then take $x' \in \kappa_*$ satisfying $x' = \frac{x_0}{x_\infty}$.

Proof of Theorem 1.4: Note that the arguments in the above proof are true over K. Hence if x is an algebraic integer, then $y = \frac{1}{x'Q(x)}$ is an algebraic integer, so that

$$\begin{aligned} m\left(x'Q(x),\infty\right) &= m(y,0) = h(y) - N(y,0) + O(1) \\ &= m(y,\infty) - N(y,0) + O(1). \end{aligned}$$

By using Dirichlet's unit theorem (see Theorem 2.36 and Lemma 4.3 in [3]), there exists a constant $c(\kappa)$ such that $|y|_v \le c(\kappa)|y|_w$ for any Archimedean v, w. Now we choose $v \in M_\kappa^\infty$ such that

$$1 \le |y|_v = \max_{w\in M_\kappa^\infty} |y|_w.$$

Then we have

$$m(y,0) = \sum_{w\in S} \log^+ \frac{1}{\|y\|_w} + O(1) = O(1).$$

Thus we have

$$(q-1)h(x) \le \overline{N}(x,\infty) + \sum_{j=1}^{q} \overline{N}(x,a_j) + O(1),\ j = 1,...,q. \tag{20}$$

□

Proof of Theorem 1.5: Since x is simple, $y = \frac{1}{x'Q(x)}$ is an algebraic integer, where $q = 2$. The proof can be completed in the same way as for Theorem 1.4. □

3 Proof of Theorem 1.6

Since one of a, b, c only has prime factors of power 1, say, c does, taking an abc-point $y \in \mathbb{P}^2(\kappa)$ with a reduced representation $(a, b, c) \in \mathcal{O}_\kappa^3$ and applying (12) to $x = a/c$, we obtain

$$h\left(\frac{a}{c}\right) \leq \overline{N}\left(\frac{a}{c}, 0\right) + \overline{N}\left(\frac{a}{c}, -1\right) + \overline{N}\left(\frac{a}{c}, \infty\right) + O(1). \tag{21}$$

Since $a + b + c = 0$, and the elements a, b, c are relatively prime, we obtain

$$\begin{aligned} \overline{N}\left(\frac{a}{c}, 0\right) &= \frac{1}{[\kappa : \mathbb{Q}]} \sum_{v \in S^c(a)} \log \mathcal{N}(\mathfrak{p}_v), \\ \overline{N}\left(\frac{a}{c}, -1\right) &= \frac{1}{[\kappa : \mathbb{Q}]} \sum_{v \in S^c(b)} \log \mathcal{N}(\mathfrak{p}_v), \\ \overline{N}\left(\frac{a}{c}, \infty\right) &= \frac{1}{[\kappa : \mathbb{Q}]} \sum_{v \in S^c(c)} \log \mathcal{N}(\mathfrak{p}_v). \end{aligned}$$

Thus (21) becomes

$$h\left(\frac{a}{c}\right) \leq \overline{N}(y, E) + O(1), \tag{22}$$

where the definition of E is referred to Section 5.6.1 in [3]. Similarly, we can obtain

$$h\left(\frac{b}{c}\right) \leq \overline{N}(y, E) + O(1). \tag{23}$$

It is easy to show that

$$h(y) = \max\left\{h\left(\frac{a}{c}\right), h\left(\frac{b}{c}\right)\right\} + O(1).$$

Combining (22) and (23), we obtain

$$h\,(y) \leq \overline{N}(y, E) + O(1), \tag{24}$$

and (13) thus holds. □

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