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Article *in* Pacific Journal of Mathematics · July 2022

DOI: 10.2140/pjm.2022.319.55

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Pacific Journal of Mathematics

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Volume 319 No. 1 July 2022

PACIFIC JOURNAL OF MATHEMATICS
Vol. 319, No. 1, 2022
https://doi.org/10.2140/pjm.2022.319.55

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This paper redevelops Nevanlinna theory for meromorphic functions on $\mathbb{C}$ in the viewpoint of holomorphic forms. According to our observation, Nevanlinna's functions can be formulated by a holomorphic form. Applying this thought to Riemann surfaces, one then extends the definition of Nevanlinna's functions using a holomorphic form $\mathscr{S}$. With the new settings, an analogue of Nevanlinna theory for the *$\mathscr{S}$-exhausted Riemann surfaces* is obtained, which is viewed as a generalization of the classical Nevanlinna theory for $\mathbb{C}$ and $\mathbb{D}$.

1. Introduction

Motivation. The paper is motivated by the early work of the first named author [4] (see also Atsuji [1]), who considered Nevanlinna theory for complete Kähler manifolds with nonpositive sectional curvature. To make it simpler, instead of Kähler manifolds, we restrict ourselves to the Riemann surfaces. In what follows, we introduce it without going into the details.

Let S be a noncompact Riemann surface equipped with a complete Hermitian metric h of nonpositive Gauss curvature K. Fix a reference point $o \in S$. Set

$$\kappa(r) = \min\{K(x) : x \in \overline{D(r)}\},$$

where $D(r)$ is a geodesic ball of radius r centered at o. Let f be a nonconstant meromorphic function on S. One can well define Nevanlinna's functions $T_f(r)$, $m_f(r, a)$ and $N_f(r, a)$ (see Section 3.1 in [4]). The first named author showed that (see Theorem 1.2 in [4]) for any $\delta > 0$

$$(q-2)T_f(r) + T(r, \mathscr{R}) \le \sum_{j=1}^{q} \overline{N}_f(r, a_j) + O\big(\log T_f(r) + \log C(o, r, \delta)\big)$$

holds for all $r \in (0, \infty)$ outside a set E_δ of finite Lebesgue measure, where $C(o, r, \delta)$ is a positive function with the estimate (see (19) in [4])

$$\log C(r, o, \delta) \le O(r\sqrt{-\kappa(r)} + \delta \log r),$$

MSC2020: 30D35.

Keywords: Nevanlinna theory, value distribution, second main theorem, defect relation, Riemann surface.

and the curvature term $T(r, \mathscr{R})$ is bounded by

$$r^2\kappa(r) \le T(r, \mathscr{R}) \le 0.$$

As showed above, $\log C(o, r, \delta)$ and $T(r, \mathscr{R})$ are estimated, however, these two estimates are rough and hard to improve by using the previous methods in [4]. For example, we consider $S = \mathbb{D}$, where $\mathbb{D}$ is the unit disc. According to the conditions for metrics, one needs to equip $\mathbb{D}$ with Poincaré metric (of curvature -1). In this situation, we have $\kappa(r) \equiv -1$. This gives that

$$-r^2 \le T(r, \mathscr{R}) \le 0.$$

However, the best lower bound of $T(r, \mathscr{R})$ is $O(-r)$.

The main drawback of the method in [4] is that the selectivity of metrics is restricted, i.e., the metrics have to be complete and nonpositively curved, but this will result in a rough estimate.

A viewpoint of holomorphic forms. Let f be a nonconstant meromorphic function on $\mathbb{C}$, namely, the complex plane with standard Euclidean metric. Nevanlinna's *characteristic function* $T(r, f)$ of f is well known [10] as

$$T(r, f) := m(r, f) + N(r, f),$$

where

$$m(r, f) = \int_{|z|=r} \log^+ |f| \frac{d\theta}{2\pi}, \quad N(r, f) = \int_0^r \frac{n(t, f) - n(0, f)}{t} dt + n(0, f) \log r$$

are called the *proximity function* and *counting function* of f respectively, in which, $n(r, f)$ denotes the number of poles of f on the disc $D(r) := \{|z| < r\}$. Characteristic function is an important notion, it characterizes the growth of meromorphic functions, using which Nevanlinna [10] in 1925 established two fundamental theorems: *First Main Theorem* and *Second Main Theorem*, named Nevanlinna theory. Nevanlinna theory plays a central role in complex analysis and hyperbolic geometry. Roughly speaking, this theory studies the size of images of meromorphic functions or mappings in target spaces; it is a great generalization of the Little Picard Theorem saying that a meromorphic function must be a constant if it omits three distinct values in $\overline{\mathbb{C}}$.

In this paper, we shall investigate Nevanlinna theory from the viewpoint of holomorphic forms $\mathscr{S}$. With this idea, one generalizes the Nevanlinna theory to one class of noncompact Riemann surfaces which we call the *$\mathscr{S}$-exhausted Riemann surfaces*. To begin with, let us describe how Nevanlinna's functions can be formulated by a holomorphic form $\mathscr{S}$. To see it more clearly, we shall use the Ahlfords' characteristic function

(1) $$T_f(r) = \int_1^r \frac{dt}{t} \int_{D(t)} f^*\omega_{FS},$$

which is equivalent to $T(r, f)$ up to a bounded term, here ω_{FS} is the Fubini–Study form on $\mathbb{P}^1(\mathbb{C})$. We need the proximity function

$$m_f(r, a) = \int_{|z|=r} \log \frac{1}{\|f, a\|} \frac{d\theta}{2\pi}, \tag{2}$$

where $\|\cdot, \cdot\|$ is the spherical distance on $\mathbb{P}^1(\mathbb{C})$. Taking the holomorphic form $\mathscr{S} = dz$, we will see that $\mathbb{C}$ is $\mathscr{S}$-exhausted (see Definition 2.6). Define

$$\hat{z} = \int_0^z \mathscr{S},$$

which is a holomorphic function in z with a unique zero $z = 0$. The $\mathscr{S}$-disc and $\mathscr{S}$-circle of radius r centered at 0 are defined respectively by

$$D^{\mathscr{S}}(r) = \{z : |\hat{z}| < r\} \quad \text{and} \quad C^{\mathscr{S}}(r) = \{z : |\hat{z}| = r\}.$$

Let $g_r(0, z)$ be the Green function of $\frac{\Delta}{2}$ for $D^{\mathscr{S}}(r)$ with a pole at 0 and Dirichlet boundary condition and let $d\pi_0^r$ be the harmonic measure on $C^{\mathscr{S}}(r)$ with respect to 0. Notice that $g_r(0, z) = (1/\pi)\log(r/|\hat{z}|)$, $d\pi_0^r = d\theta/2\pi$, then by integration by part, (1) can be rewritten as

$$T_f(r) = \int_1^r \frac{dt}{t} \int_{D^{\mathscr{S}}(t)} f^*\omega_{FS} = \pi \int_{D^{\mathscr{S}}(r)} g_r(0, z) f^*\omega_{FS} - \pi \int_{D^{\mathscr{S}}(1)} g_1(0, z) f^*\omega_{FS}$$

and (2) can be replaced by

$$m_f(r, a) = \int_{C^{\mathscr{S}}(r)} \log \frac{1}{\|f, a\|} d\pi_0^r.$$

Similarly, the counting function is that

$$N_f(r, a) = \int_1^r \frac{n_f^{\mathscr{S}}(t, a)}{t} dt,$$

where $n_f^{\mathscr{S}}(r, a)$ denotes the number of zeros of $f - a$ on $D^{\mathscr{S}}(r)$.

Follow the idea as shown above, we consider a noncompact Hermitian Riemann surface $(\mathcal{S}, h)$. Choosing $\mathscr{S}$, a nowhere-vanishing holomorphic form on $\mathcal{S}$ such that $\hat{x} = \int_o^x \mathscr{S}$ defines a holomorphic function, where o is a fixed point in $\mathcal{S}$. Following [6], such a form always exists. If $\mathcal{S}$ is *$\mathscr{S}$-exhausted*, i.e., any sequence of $\mathscr{S}$-discs exhausts $\mathcal{S}$ when radius approaches increasingly to $R^{\mathscr{S}}$, where $R^{\mathscr{S}}$ is the $\mathscr{S}$-radius of $\mathcal{S}$ with respect to o, see Definition 2.7. To a $\mathscr{S}$-exhausted surface $\mathcal{S}$, $T_f^{\mathscr{S}}(r)$, $m_f^{\mathscr{S}}(r, a)$ and $N_f^{\mathscr{S}}(r, a)$ of a meromorphic function f on $\mathcal{S}$ can be similarly defined and they make sense, see Notations on page 63 for the definition. By computing Green functions and harmonic measures, we shall establish an analogue of Nevanlinna theory, which turns out to be an extension of the classical Nevanlinna theory for $\mathbb{C}$ and $\mathbb{D}$ (unit disc); see, e.g., [10; 11; 14; 15].

Main results. In what follows, we state the main results of the paper.

Theorem I (Theorem 3.4). Let $(\mathcal{S}, h; \mathscr{S})$ be a $\mathscr{S}$-exhausted Hermitian Riemann surface of $\mathscr{S}$-radius $R^{\mathscr{S}}$ with respect to o. Let γ be an integrable function on $(0, R^{\mathscr{S}})$ with $\int_0^{R^{\mathscr{S}}} \gamma(r)\,dr = \infty$. Let f be a nonconstant meromorphic function on $\mathcal{S}$ and $a_1, \dots, a_q$ be distinct values in $\overline{\mathbb{C}}$. Then for any $\delta > 0$

$$(q-2)\,T_f^{\mathscr{S}}(r) + T^{\mathscr{S}}(r, \mathscr{R}) \\ \leq \sum_{j=1}^{q} \overline{N}_f^{\mathscr{S}}(r, a_j) + O\big(\log T_f^{\mathscr{S}}(r) + \log \|\mathscr{S}\|_{r,\sup} + \log \gamma(r) + \delta \log r\big)$$

holds for all $r \in (0, R^{\mathscr{S}})$ outside a set E_δ with $\int_{E_\delta} \gamma(r)\,dr < \infty$, where

$$\|\mathscr{S}\|_{r,\sup} = \sup\{\|\mathscr{S}_x\|_h : |\hat{x}| < r\}.$$

In Theorem I, the term $T^{\mathscr{S}}(r, \mathscr{R})$ is called the *characteristic* of the Ricci form $\mathscr{R} := -dd^c \log h$ of $\mathcal{S}$, which depends on the Gauss curvature of $\mathcal{S}$, see (8). Notice that $\mathbb{C}$ is dz-exhausted and $\|dz\| = 1$ under standard Euclidean metric. In this case, the Nevanlinna's functions agree with the classical ones. Hence, Theorem I yields (by letting $\gamma = 1$) a classical consequence for $\mathbb{C}$ that:

Corollary I. Let f be a nonconstant meromorphic function on $\mathbb{C}$, and let $a_1, \dots, a_q$ be distinct points in $\overline{\mathbb{C}}$. Then for any $\delta > 0$

$$(q-2)\,T_f(r) \leq \sum_{j=1}^{q} \overline{N}_f(r, a_j) + O\big(\log T_f(r) + \delta \log r\big)$$

holds for all $r \in (0, \infty)$ outside a set E_δ of finite Lebesgue measure.

Equipping $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ with standard Euclidean metric, then $\mathbb{D}$ is a dz-exhausted Riemann surface. Take $\gamma = (1-r)^{-1}$, it concludes another classical consequence for $\mathbb{D}$ that:

Corollary II. Let f be a nonconstant meromorphic function on $\mathbb{D}$, and let $a_1, \dots, a_q$ be distinct points in $\overline{\mathbb{C}}$. Then for any $\delta > 0$

$$(q-2)\,T_f(r) \leq \sum_{j=1}^{q} \overline{N}_f(r, a_j) + O\Big(\log T_f(r) + \log \frac{1}{1-r}\Big)$$

holds for all $r \in (0, 1)$ outside a set E_δ with $\int_{E_\delta} (1-r)^{-1} dr < \infty$.

Associating a holomorphic mapping $f : \mathcal{S} \to \mathcal{R}$, where $(\mathcal{R}, \omega)$ is a compact Hermitian Riemann surface. Following a theorem of Chern [3], we can similarly define Nevanlinna's functions via $\mathscr{S}$. We prove a more generalized theorem of Theorem I:

Theorem II (Theorem 4.1). Let $(\mathcal{S}, h; \mathscr{S})$ be a $\mathscr{S}$-exhausted Hermitian Riemann surface of $\mathscr{S}$-radius $R^{\mathscr{S}}$ with respect to o, and let $\mathcal{R}$ be a compact Riemann surface of genus g. Fix a positive $(1,1)$-form ω on $\mathcal{R}$. Let γ be an integrable function on $(0, R^{\mathscr{S}})$ with $\int_0^{R^{\mathscr{S}}} \gamma(r)\,dr = \infty$. Let f be a nonconstant holomorphic mapping from $\mathcal{S}$ into $\mathcal{R}$ and $a_1, \ldots, a_q$ be distinct points in $\mathcal{R}$. Then for any $\delta > 0$

$$\begin{aligned}(q-2+2g)\,&T^{\mathscr{S}}_{f,\omega}(r) + T^{\mathscr{S}}(r, \mathscr{R}) \\ &\le \sum_{j=1}^{q} \overline{N}^{\mathscr{S}}_f(r, a_j) + O\big(\log T^{\mathscr{S}}_{f,\omega}(r) + \log \|\mathscr{S}\|_{r,\sup} + \log \gamma(r) + \delta \log r\big)\end{aligned}$$

holds for all $r \in (0, R^{\mathscr{S}})$ outside a set E_δ with $\int_{E_\delta} \gamma(r)\,dr < \infty$, where

$$\|\mathscr{S}\|_{r,\sup} = \sup\{\|\mathscr{S}_x\|_h : |\hat{x}| < r\}.$$

Remark 1.1. In order to better estimate $\|\mathscr{S}\|$, we can put Hermitian metric $\alpha = \frac{\sqrt{-1}}{2}\mathscr{S} \wedge \overline{\mathscr{S}}$ induced by $\mathscr{S}$ on $\mathcal{S}$. Under this metric, we have $\|\mathscr{S}\| \equiv 1$. Then the result in Theorem II becomes

$$\begin{aligned}(q-2+2g)\,&T^{\mathscr{S}}_{f,\omega}(r) + T^{\mathscr{S}}(r, \mathscr{R}) \\ &\le \sum_{j=1}^{q} \overline{N}^{\mathscr{S}}_f(r, a_j) + O\big(\log T^{\mathscr{S}}_{f,\omega}(r) + \log \gamma(r) + \delta \log r\big).\end{aligned}$$

We consider a defect relation of f in Nevanlinna theory. The *simple defect* $\bar{\delta}_f(a)$ of f with respect to a is defined by

$$\bar{\delta}_f(a) = 1 - \limsup_{r \to R^{\mathscr{S}}} \frac{\overline{N}^{\mathscr{S}}_f(r, a)}{T^{\mathscr{S}}_{f,\omega}(r)}. \tag{3}$$

If f is nonconstant, then we can check $T^{\mathscr{S}}_{f,\omega}(r) \ge O(\log r)$. By using the First Main Theorem (see page 65), we see that $0 \le \bar{\delta}_f(a) \le 1$.

By estimating the lower bound of $T^{\mathscr{S}}(r, \mathscr{R})$, we obtain a defect relation:

Theorem III. Assume the same conditions as in Theorem II. Suppose, in addition, that $-C \le K \le 0$ for a nonnegative constant C. If f satisfies

$$\limsup_{r \to R^{\mathscr{S}}} \frac{Cr^2 \|\mathscr{S}\|^{-1}_{r,\inf} + \log(\gamma(r)\|\mathscr{S}\|_{r,\sup})}{T^{\mathscr{S}}_{f,\omega}(r)} = 0,$$

where

$$\|\mathscr{S}\|_{r,\inf} = \inf\{\|\mathscr{S}_x\|_h : |\hat{x}| < r\},$$

then we have the defect relation

$$\sum_{j=1}^{q} \bar{\delta}_f(a_j) \le 2 - 2g.$$

The earlier study of Nevanlinna theory for noncompact Riemann surfaces dates back to the work of H. Wu [17] (also refer to Shabat [16]). More details on this aspect, the readers may refer to the recent very nice papers of He and Ru [7] and Păun and Sibony [13]. We have to indicate that the paper presents a new method of studying Nevanlinna theory by putting a metric h on $\mathcal{S}$. However, we should also point out that our results can be derived by Wu's method (as well as He and Ru's arguments) since $\mathscr{S}$ produces an exhaustion function

$$\sigma(x) := \left|\int_o^x \mathscr{S}\right|$$

on $\mathcal{S}$, so that $\log\sigma^2$ is harmonic outside a compact set. We refer the readers to the standard arguments of He and Ru [7] without going into any details.

2. $\mathscr{S}$-exhausted Riemann surfaces

$\mathscr{S}$-exhausted Riemann surfaces. Let $\mathcal{S}$ be a noncompact Riemann surface. Gunning proved that:

Proposition 2.1 (Gunning and Narasimhan [6]). Let $\mathcal{S}$ be a noncompact Riemann surface. Then there is a holomorphic form $\mathscr{S}$ on $\mathcal{S}$ satisfying the conditions:

(a) $\mathscr{S}$ has no zeros on $\mathcal{S}$;

(b) $\int_\gamma \mathscr{S} = 0$ for each smooth simple closed curve γ in $\mathcal{S}$.

Remark 2.2. Condition (a) defines a holomorphic field X without zeros on $\mathcal{S}$ in the following manner: write $\mathscr{S} = \phi\, dz$ in a local holomorphic coordinate z, it is trivial to check that

$$X = \phi^{-1}\frac{\partial}{\partial z}$$

is well defined on $\mathcal{S}$. Hence, X is a nowhere-vanishing holomorphic field that is dual to $\mathscr{S}$. Condition (b) defines a holomorphic function $\hat{x} : \mathcal{S} \to \mathbb{C}$ by

$$L(x) = \int_o^x \mathscr{S} := \hat{x}, \tag{4}$$

where o is a fixed point in S. By (a), we see that o is a simple zero of L.

In this paper, besides conditions (a) and (b), one assumes that $\mathscr{S}$ satisfies an additional condition: $L(x) = 0$ if and only if $x = o$. This condition forces L to be a univalent function.

Definition 2.3. Let $\mathscr{S}$ be a holomorphic form on $\mathcal{S}$ satisfying the conditions (a) and (b) in Proposition 2.1. If, in addition, that $\mathscr{S}$ satisfies the condition: $L(x) = 0$ if and only if $x = o$, where L is defined by (4), then we say that $\mathcal{S}$ is a $\mathscr{S}$-Riemann surface.

We define $\mathscr{S}$-discs and $\mathscr{S}$-circles on a $\mathscr{S}$-Riemann surface $\mathcal{S}$.

Definition 2.4. Let $(\mathcal{S};\mathscr{S})$ be a $\mathscr{S}$-Riemann surface. The $\mathscr{S}$-disc $D^{\mathscr{S}}(r)$ is defined by

$$D^{\mathscr{S}}(r)=\{x\in\mathcal{S}:|\hat{x}|<r\},$$

and the $\mathscr{S}$-circle $C^{\mathscr{S}}(r)$ is defined by

$$C^{\mathscr{S}}(r)=\{x\in\mathcal{S}:|\hat{x}|=r\}.$$

Since L is univalent, then $D^{\mathscr{S}}(r)$ is simply connected for all $r>0$. Notice that the case $C^{\mathscr{S}}(r)=\varnothing$ may happen if r is sufficiently large, since $D^{\mathscr{S}}(r)$ could cover the whole surface $\mathcal{S}$ when r is large enough in some situations. For example, we consider the case where $\mathcal{S}=\mathbb{D}$ and $\mathscr{S}=dz$, then $C^{\mathscr{S}}(r)=\{z\in\mathbb{D}:|z|=r\}$. If $r>1$, then we have $C^{\mathscr{S}}(r)=\varnothing$.

Definition 2.5. We say that $\mathscr{S}$ is an exhaustion form, if $\{D^{\mathscr{S}}(r_n)\}_{n=1}^{\infty}$ exhausts $\mathcal{S}$ whenever $0<r_1<r_2<\cdots$ and $r_n\to+\infty$ as $n\to+\infty$. Namely, $\mathcal{S}=\bigcup_{n=1}^{\infty}D^{\mathscr{S}}(r_n)$ and one of the following holds:

(i) $D^{\mathscr{S}}(r_1)\subset\overline{D^{\mathscr{S}}(r_1)}\subset D^{\mathscr{S}}(r_2)\subset\overline{D^{\mathscr{S}}(r_2)}\subset\cdots\subset\mathcal{S}$.

(ii) There exists an integer $k\geq 1$ such that

$$D^{\mathscr{S}}(r_1)\subset\overline{D^{\mathscr{S}}(r_1)}\subset\cdots\subset D^{\mathscr{S}}(r_k)=D^{\mathscr{S}}(r_{k+1})=\cdots=\mathcal{S}.$$

Precisely, we say that $\mathscr{S}$ is a parabolic exhaustion form if condition (i) is satisfied, and a hyperbolic exhaustion form if condition (ii) is satisfied.

Definition 2.6. Let $(\mathcal{S};\mathscr{S})$ be a $\mathscr{S}$-Riemann surface. We say that $\mathcal{S}$ is a $\mathscr{S}$-exhausted Riemann surface if $\mathscr{S}$ is an exhaustion form on $\mathcal{S}$.

If $\mathcal{S}$ is a $\mathscr{S}$-exhausted Riemann surface, then $C^{\mathscr{S}}(r)$ (for $r>0$) is a closed curve contained properly in $\mathcal{S}$ whenever $C^{\mathscr{S}}(r)\cap\mathcal{S}\neq\varnothing$.

Definition 2.7. Let $(\mathcal{S};\mathscr{S})$ be a $\mathscr{S}$-exhausted Riemann surface. Define $R^{\mathscr{S}}$ by

$$R^{\mathscr{S}}=\sup\{r>0:C^{\mathscr{S}}(r)\cap\mathcal{S}\neq\varnothing\},$$

which is called the $\mathscr{S}$-radius of $\mathcal{S}$ with respect to o.

From the definition as above, we have that $0<R^{\mathscr{S}}\leq\infty$ and $C^{\mathscr{S}}(r)$ is a simple closed curve in $\mathcal{S}$ for $0<r<R^{\mathscr{S}}$. Note that the following corollary is a direct consequence of Definitions 2.6 and 2.7.

Corollary 2.8. Let $(\mathcal{S};\mathscr{S})$ be a $\mathscr{S}$-exhausted Riemann surface of $\mathscr{S}$-radius $R^{\mathscr{S}}$ with respect to o. Let $\{r_n\}_{n=1}^{\infty}$ be a sequence of positive integers r_n with $r_1<r_2<\cdots$ and $r_n\to R^{\mathscr{S}}$ as $n\to+\infty$. Then $\{D^{\mathscr{S}}(r_n)\}_{n=1}^{\infty}$ exhausts $\mathcal{S}$.

Example 2.9. Let $\mathbb{C}$ be the standard complex Euclidean plane, then $\mathbb{C}$ is a dz-exhausted Riemann surface of dz-radius ∞ with respect to 0. We obtain $\hat{z}=z$, $D^{dz}(r)=\{z\in\mathbb{C}:|z|<r\}$ and $C^{dz}(r)=\{z\in\mathbb{C}:|z|=r\}$.

Example 2.10. Equipping $\mathbb{D}$ with standard Euclidean metric, then $\mathbb{D}$ is a dz-exhausted Riemann surface of dz-radius 1 with respect to 0. We have $\hat{z} = z$, $D^{dz}(r) = \{z \in \mathbb{D} : |z| < r\}$ and $C^{dz}(r) = \{z \in \mathbb{D} : |z| = r\}$.

Harmonic measures on $\mathscr{S}$-circles. Let $(\mathcal{S}, h; \mathscr{S})$ be a $\mathscr{S}$-exhausted Hermitian Riemann surface of $\mathscr{S}$-radius $R^{\mathscr{S}}$ with respect to o. We denote by Δ the Laplace–Beltrami operator on $\mathcal{S}$ defined by h. For $0 < r < R^{\mathscr{S}}$, we shall compute Green function $g_r(o, x)$ of $\frac{\Delta}{2}$ for $D^{\mathscr{S}}(r)$ with Dirichlet boundary and a pole at o, i.e.,

$$-\tfrac{1}{2}\Delta g_r(o, x) = \delta_o(x), \quad x \in D^{\mathscr{S}}(r); \qquad g_r(o, x) = 0, \quad x \in C^{\mathscr{S}}(r)$$

in the sense of distributions, as well as harmonic measure $d\pi_o^r$ on $C^{\mathscr{S}}(r)$ with respect to o, for $0 < r < R^{\mathscr{S}}$. We have the well-known formula

$$d\pi_o^r(x) = -\frac{1}{2}\frac{\partial g_r(o, x)}{\partial \vec{n}} d\sigma_r(x),$$

where $\partial/\partial\vec{n}$ is the inward normal derivative on $C^{\mathscr{S}}(r)$.

Lemma 2.11. For $0 < r < R^{\mathscr{S}}$, we have

$$g_r(o, x) = \frac{1}{\pi} \log \frac{r}{|\hat{x}|}.$$

Proof. Since L is a holomorphic function on $\mathcal{S}$ with a unique simple zero o, then it follows that

$$\tfrac{1}{2\pi}\Delta \log |L(x)| = \delta_o(x), \quad x \in D^{\mathscr{S}}(r)$$

in the sense of distribution. Indeed, it is clear that $|L(x)| = r$ for $x \in C^{\mathscr{S}}(r)$. Hence, we obtain

$$g_r(o, x) = \frac{1}{\pi} \log \frac{r}{|\hat{x}|}. \qquad \square$$

Lemma 2.12. For $0 < r < R^{\mathscr{S}}$, we have

$$d\pi_o^r = \frac{\|\mathscr{S}\|_h}{2\pi r} d\sigma_r,$$

where $d\sigma_r$ is the Riemannian line element on $C^{\mathscr{S}}(r)$ defined by h.

Proof. Fix an arbitrary point $x_0 \in C^{\mathscr{S}}(r)$, we take a holomorphic coordinate z near x_0 such that $h|_{x_0} = 1$. Note that

$$d\pi_o^r(x) = -\frac{1}{2}\frac{\partial g_r(o, x)}{\partial \vec{n}} d\sigma_r(x),$$

where $\partial/\partial\vec{n}$ is the inward normal derivative on $\partial C^{\mathscr{S}}(r)$. By $|L(x_0)| = r$, we have

$$d\pi_o^r(x_0) = \frac{1}{2\pi r}\frac{\partial |L(x_0)|}{\partial \vec{n}} d\sigma_r(x_0).$$

Set $z=\xi_1+\sqrt{-1}\xi_2$. Differentiating the equation $|L|=r$ near x_0, then

$$\frac{\partial |L|}{\partial \xi_1}d\xi_1+\frac{\partial |L|}{\partial \xi_2}d\xi_2=0,$$

which gives an inward normal vector

$$\vec{n}|_{x_0}=\left(\frac{\partial|L(x_0)|/\partial\xi_1}{\sqrt{\left(\frac{\partial|L(x_0)|}{\partial\xi_1}\right)^2+\left(\frac{\partial|L(x_0)|}{\partial\xi_2}\right)^2}},\ \frac{\partial|L(x_0)|/\partial\xi_2}{\sqrt{\left(\frac{\partial|L(x_0)|}{\partial\xi_1}\right)^2+\left(\frac{\partial|L(x_0)|}{\partial\xi_2}\right)^2}}\right).$$

Write $\mathscr{S}=\phi\, dz$ in the local coordinate z, a direct computation follows that

$$\frac{\partial|L(x_0)|}{\partial\xi_1}=\Re\left[\phi(x_0)\frac{\overline{L(x_0)}^{1/2}}{L(x_0)^{1/2}}\right],\qquad \frac{\partial|L(x_0)|}{\partial\xi_2}=-\Im\left[\phi(x_0)\frac{\overline{L(x_0)}^{1/2}}{L(x_0)^{1/2}}\right].$$

Then

$$\frac{\partial|L(x_0)|}{\partial\vec{n}}=\left(\frac{\partial|L(x_0)|}{\partial\xi_1},\frac{\partial|L(x_0)|}{\partial\xi_2}\right)\cdot\vec{n}|_{x_0}=|\phi(x_0)|=\|\mathscr{S}x_0\|_h.$$

We conclude that

$$d\pi_o^r(x_0)=\frac{\|\mathscr{S}x_0\|_h}{2\pi r}d\sigma_r(x_0).$$ □

3. Value distribution of meromorphic functions on $\mathcal{S}$

First Main Theorem.

Notations. Let $(\mathcal{S},h;\mathscr{S})$ be a $\mathscr{S}$-exhausted Hermitian Riemann surface of $\mathscr{S}$-radius $R^{\mathscr{S}}$ with respect to o. Using $\mathscr{S}$, we extend the notion of Nevanlinna's functions to $\mathcal{S}$. Let f be a meromorphic function on $\mathcal{S}$ and let r be satisfied with $0<r_0<r<R^{\mathscr{S}}$. Viewing $f=f_1/f_0=[f_0:f_1]$ as a holomorphic mapping from $\mathcal{S}$ into $\mathbb{P}^1(\mathbb{C})$. We define the *characteristic function* of f by

$$T_f^{\mathscr{S}}(r)=\int_{r_0}^{r}\frac{dt}{t}\int_{D^{\mathscr{S}}(t)}f^*\omega_{FS},$$

where $\omega_{FS}=dd^c\log(|\zeta_0|^2+|\zeta_1|^2)$ and

$$d=\partial+\bar\partial,\quad d^c=\frac{\sqrt{-1}}{4\pi}(\bar\partial-\partial),\quad dd^c=\frac{\sqrt{-1}}{2\pi}\partial\bar\partial.$$

By integration by part, it yields that

$$T_f^{\mathscr{S}}(r)=\int_{D^{\mathscr{S}}(r)}\log\frac{r}{|\hat{x}|}dd^c\log(|f_0(x)|^2+|f_1(x)|^2)\\ -\int_{D^{\mathscr{S}}(r_0)}\log\frac{r_0}{|\hat{x}|}dd^c\log(|f_0(x)|^2+|f_1(x)|^2).$$

Locally, we write

$$dV=\sqrt{-1}h\,dz\wedge d\bar{z},\quad \Delta=\frac{2}{h}\frac{\partial^2}{\partial z\partial\bar{z}},$$

where dV is the Riemannian area element of $\mathcal{S}$. We get

$$dd^c \log(|f_0|^2+|f_1|^2)=\tfrac{1}{4\pi}\Delta\log(|f_0|^2+|f_1|^2)\,dV.$$

Notice (see Lemma 2.11) that

$$g_r(o,x)=\frac{1}{\pi}\log\frac{r}{|\hat{x}|},$$

then $T_f^{\mathscr{S}}(r)$ can be written in terms of Green function as

$$(5)\quad T_f^{\mathscr{S}}(r)=\frac{1}{4}\int_{D^{\mathscr{S}}(r)} g_r(o,x)\Delta\log(|f_0(x)|^2+|f_1(x)|^2)\,dV(x) \\ -\frac{1}{4}\int_{D^{\mathscr{S}}(r_0)} g_{r_0}(o,x)\Delta\log(|f_0(x)|^2+|f_1(x)|^2)\,dV(x).$$

Let $a=[a_0:a_1]\in\mathbb{P}^1(\mathbb{C})$ such that $f\not\equiv a$. The *proximity function* of f with respect to a is defined by

$$(6)\qquad m_f^{\mathscr{S}}(r,a)=\int_{C^{\mathscr{S}}(r)}\log\frac{1}{\|f,a\|}d\pi_o^r,$$

where $\|\cdot,\cdot\|$ is the spherical distance on $\mathbb{P}^1(\mathbb{C})$, defined by

$$\|f,a\|=\frac{|\langle f;a\rangle|}{\|f\|\|a\|}=\frac{|a_0f_1-a_1f_0|}{\sqrt{|a_0|^2+|a_1|^2}\sqrt{|f_0|^2+|f_1|^2}},$$

where

$$\langle f;a\rangle:=a_0f_1-a_1f_0.$$

Using Lemma 2.11, we obtain

$$m_f^{\mathscr{S}}(r,a)=\frac{1}{2\pi r}\int_{C^{\mathscr{S}}(r)}\|\mathscr{S}\|_h\log\frac{1}{\|f,a\|}d\sigma_r,$$

where $d\sigma_r$ is the Riemannian line element on $C^{\mathscr{S}}(r)$. The *counting function* of f with respect to a is defined by

$$N_f^{\mathscr{S}}(r,a)=\int_{r_0}^r\frac{n_f^{\mathscr{S}}(t,a)}{t}dt,$$

where $n_f^{\mathscr{S}}(r,a)$ denotes the number of the zeros of $f-a$ on $D^{\mathscr{S}}(r)$ counting multiplicities. By Poincaré–Lelong formula [2], we see that

$$N_f^{\mathscr{S}}(r,a)=\int_{r_0}^r\frac{dt}{t}\int_{D^{\mathscr{S}}(t)}dd^c[\log|\langle f;a\rangle|^2],$$

where $dd^c[\log|\langle f;a\rangle|^2]$ is a current [11]. Similarly, in terms of Green function, there is an alternate expression:

$$(7)\quad N_f^{\mathscr{S}}(r,a)=\frac{1}{4}\int_{D^{\mathscr{S}}(r)}g_r(o,x)\Delta\log|\langle f(x);a\rangle|^2dV(x) \\ -\frac{1}{4}\int_{D^{\mathscr{S}}(r_0)}g_{r_0}(o,x)\Delta\log|\langle f(x);a\rangle|^2dV(x),$$

where $\Delta \log |\langle f; a\rangle|^2$ is understood as a distribution. Similarly, one can define the *simple counting function* $\overline{N}_f^{\mathscr{S}}(r, a)$ of f with respect to a, which measures the size of the set of zeros of $f - a$ without counting multiplicities.

We define several other symbols. Let $\mathscr{R}$ be the Ricci form of $(\mathcal{S}, h)$, i.e.,

$$\mathscr{R} = \mathrm{Ric}(\alpha) = -dd^c \log h,$$

where

$$\alpha = \frac{\sqrt{-1}}{\pi} h\, dz \wedge d\bar{z}$$

is the Kähler form of $\mathcal{S}$. The *characteristic* of $\mathscr{R}$ is defined by

$$T^{\mathscr{S}}(r, \mathscr{R}) = \int_{r_0}^{r} \frac{dt}{t} \int_{D^{\mathscr{S}}(t)} \mathscr{R}.$$

Since the Gauss curvature K of h is computed by

$$K = -\frac{1}{2}\Delta \log h = -\frac{1}{h}\frac{\partial^2 \log h}{\partial z \partial \bar{z}},$$

then we obtain

$$2\mathscr{R} = K\alpha.$$

By this with the above, we see that the characteristic of $\mathscr{R}$ can be expressed in terms of Gauss curvature that

$$(8)\quad T^{\mathscr{S}}(r, \mathscr{R}) = \frac{1}{2}\int_{D^{\mathscr{S}}(r)} g_r(o, x) K(x)\, dV(x) - \frac{1}{2}\int_{D^{\mathscr{S}}(r_0)} g_{r_0}(o, x) K(x)\, dV(x).$$

First Main Theorem. Let us introduce Dynkin formula which is a generalization of Green–Jensen formula [12; 14], see the probabilistic version of Dynkin formula in [4; 8; 9].

Lemma 3.1 (Dynkin formula). Let u be a function of $\mathscr{C}^2$-class except at most a polar set of singularities on a Riemannian manifold M. Let $D \subset M$ be a relatively compact domain with piecewise smooth boundary ∂D. Assume that $u(o) \neq \infty$ for a fixed point $o \in D$. Then

$$\int_{\partial D} u(x)\, d\pi_o^{\partial D}(x) - u(o) = \frac{1}{2}\int_D g_D(o, x)\Delta u(x)\, dV(x),$$

where $g_D(o, x)$ is the Green function of Laplacian $\frac{\Delta}{2}$ for D with a pole o and Dirichlet boundary condition, and $d\pi_o^{\partial D}$ is the harmonic metric on ∂D with respect to o. Here, Δu is understood as a distribution.

Proof. We are to prove the above lemma by using a probabilistic approach, i.e., the probabilistic Dynkin formula is applied to showing this lemma. Let $X_t := \{X_t\}_{t \geq 0}$ be the Brownian motion generated by $\frac{\Delta}{2}$, started at $o \in M$ (see Section 2.2 in [4]).

Denote by $\mathbb{P}_o$ the law or distribution of X_t starting from o and by $\mathbb{E}_o$ the expectation of X_t with respect to $\mathbb{P}_o$. Set the stopping time

$$\tau_D = \inf\{t > 0 : X_t \notin D\}.$$

Note that the probabilistic Dynkin formula (see Itô formula in [4], page 6) says that

$$\mathbb{E}_o[u(X_{\tau_D})] - u(o) = \frac{1}{2}\mathbb{E}_o\Big[\int_0^{\tau_D} \Delta u(X_t)\, dt\Big].$$

On the other hand, using the co-area formula (see (2) in [4]) and the relation between harmonic measures and hitting times (see (3) in [4]), we obtain

$$\mathbb{E}_o\Big[\int_0^{\tau_D} \Delta u(X_t)\, dt\Big] = \int_D g_D(o, x)\Delta u(x)\, dV(x)$$

and

$$\mathbb{E}_o[u(X_{\tau_D})] = \int_{\partial D} u(x)\, d\pi_o^{\partial D}(x).$$

Substituting the two equalities into the above probabilistic Dynkin formula, we have the lemma proved. □

Applying Dynkin formula to $\log \|f, a\|^{-1}$ and noting (5)–(7), it follows the First Main Theorem.

Theorem 3.2 (First Main Theorem). If $f \not\equiv a$, then we have

$$T_f^{\mathscr{S}}(r) = m_f^{\mathscr{S}}(r, a) + N_f^{\mathscr{S}}(r, a) + O(1).$$

Remark 3.3. Theorem 3.2 can be confirmed by using Green–Jensen formula instead of Dynkin formula, since $\mathscr{S}$ can produce a parabolic or a hyperbolic exhaustion function $\sigma(x) = |\int_o^x \mathscr{S}|$; see, e.g., [7; 13; 16; 17].

Second Main Theorem. The main purpose here is to prove the following theorem.

Theorem 3.4 (Second Main Theorem). Let $(\mathcal{S}, h; \mathscr{S})$ be a $\mathscr{S}$-exhausted Hermitian Riemann surface of $\mathscr{S}$-radius $R^{\mathscr{S}}$ with respect to o. Let γ be an integrable function on $(0, R^{\mathscr{S}})$ with $\int_0^{R^{\mathscr{S}}} \gamma(r)\, dr = \infty$. Let f be a nonconstant meromorphic function on $\mathcal{S}$ and $a_1, \dots, a_q$ be distinct values in $\overline{\mathbb{C}}$. Then for any $\delta > 0$

$$(q-2)T_f^{\mathscr{S}}(r) + T^{\mathscr{S}}(r, \mathscr{R}) \le \sum_{j=1}^{q} \overline{N}_f^{\mathscr{S}}(r, a_j) + O\big(\log T_f^{\mathscr{S}}(r) + \log \|\mathscr{S}\|_{r,\sup} + \log \gamma(r) + \delta \log r\big)$$

holds for all $r \in (0, R^{\mathscr{S}})$ outside a set E_δ with $\int_{E_\delta} \gamma(r)\, dr < \infty$, where

$$\|\mathscr{S}\|_{r,\sup} = \sup\{\|\mathscr{S}_x\|_h : |\hat{x}| < r\}.$$

The following is called the Borel's Growth Lemma.

Lemma 3.5 [15]. Let γ be an integrable function on $(0, R)$ with $\int_0^R \gamma(r)\,dr = \infty$. Let h be a nondecreasing function of $\mathscr{C}^1$-class on $(0, R)$. Let $\lim_{r\to R} h(r) = \infty$ and $h(r_0) > 0$ for some $r_0 \in (0, R)$. Then for any $\delta > 0$

$$h'(r) \leq h^{1+\delta}(r)\gamma(r)$$

holds for all $r \in (0, R)$ outside a set E_δ with $\int_{E_\delta} \gamma(r)\,dr < \infty$. In particular, when $R = \infty$, we can take $\gamma = 1$. Then for any $\delta > 0$

$$h'(r) \leq h^{1+\delta}(r)$$

holds for all $r \in (0, \infty)$ outside a set E_δ of finite Lebesgue measure.

We utilize Borel's growth lemma to prove the following calculus lemma. Let k be a locally integrable function on $\mathcal{S}$. Set

$$E_k(r) = \int_{C^{\mathscr{S}}(r)} k\,d\pi_o^r, \quad A_k(r) = \int_{r_0}^r \frac{dt}{t} \int_{D^{\mathscr{S}}(t)} k\,dV.$$

Lemma 3.6. Let $(\mathcal{S}, h; \mathscr{S})$ be a $\mathscr{S}$-exhausted Hermitian Riemann surface of $\mathscr{S}$-radius $R^{\mathscr{S}}$ with respect to o. Let γ be an integrable function on $(0, R^{\mathscr{S}})$ with $\int_0^{R^{\mathscr{S}}} \gamma(r)\,dr = \infty$. Let k be a locally integrable function on $\mathcal{S}$. Then for any $\delta > 0$

$$E_k(r) \leq \frac{\|\mathscr{S}\|_{r,\sup}\, r^\delta \gamma^{2+\delta}(r)}{2\pi} A_k^{(1+\delta)^2}(r)$$

holds for all $r \in (0, R^{\mathscr{S}})$ outside a set E_δ with $\int_{E_\delta} \gamma(r)\,dr < \infty$, where

$$\|\mathscr{S}\|_{r,\sup} = \sup\{\|\mathscr{S}_x\|_h : |\hat{x}| < r\}.$$

Proof. Notice that

$$\int_{D^{\mathscr{S}}(r)} k\,dV = \int_0^r dt \int_{C^{\mathscr{S}}(t)} k\,d\sigma_r,$$

then it follows from Lemma 2.12 that

$$\frac{d}{dr}\left(r\frac{dA_k}{dr}\right) = \int_{C^{\mathscr{S}}(r)} k\,d\sigma_r \geq \frac{2\pi r}{\|\mathscr{S}\|_{r,\sup}} E_k(r).$$

Using Lemma 3.5 twice (first to rA_k' and then to A_k), we obtain

$$E_k(r) \leq \frac{\|\mathscr{S}\|_{r,\sup}\, r^\delta \gamma^{2+\delta}(r)}{2\pi} A_k^{(1+\delta)^2}(r). \qquad \square$$

Proof of Theorem 3.4. Consider a singular volume form (see [2; 5])

$$\Phi = \frac{C\omega_{FS}}{\prod_{j=1}^q \|\zeta, a_j\|^2 \log^2 \|\zeta, a_j\|^{-2}} \tag{9}$$

on $\overline{\mathbb{C}}$, where $a_1, \dots, a_q$ are distinct values in $\overline{\mathbb{C}}$ and $\omega_{FS} = dd^c \log(1 + |\zeta|^2)$. Since $\overline{\mathbb{C}}$ is compact, we can choose a positive number C such that $\int_{\overline{\mathbb{C}}} \Phi = 1$. Set

$$f^*\Phi = \xi \frac{\sqrt{-1}}{\pi} h \, dz \wedge d\bar{z}.$$

By taking the Ricci form of both sides of the above identity, it follows from $\mathrm{Ric}(\omega_{FS}) = 2\omega_{FS}$ that

$$\begin{aligned} dd^c[\log \xi] = (q-2) f^*\omega_{FS} - \sum_{j=1}^q (f - a_j)_0 + D_{f,\mathrm{ram}} \\ + \mathscr{R} - 2 \sum_{j=1}^q dd^c \log\log \|f, a_j\|^{-2} \end{aligned} \tag{10}$$

in the sense of currents, where $(f - a_j)_0$ is the zero divisor of $f - a_j$, $D_{f,\mathrm{ram}}$ is the ramification divisor of f, and $\mathscr{R} = -dd^c \log h$ is the Ricci form of $\mathcal{S}$. Applying the integral operator

$$\int_{r_0}^r \frac{dt}{t} \int_{D^{\mathscr{S}}(t)}$$

to the above identity and using Dynkin formula, we get

$$\begin{aligned} \frac{1}{2} \int_{C^{\mathscr{S}}(r)} \log \xi \, d\pi_o^r &= (q-2) T_f^{\mathscr{S}}(r) - \sum_{j=1}^q N_f^{\mathscr{S}}(r, a_j) + N^{\mathscr{S}}(r, D_{f,\mathrm{ram}}) \\ &\quad + T^{\mathscr{S}}(r, \mathscr{R}) - \sum_{j=1}^q \int_{C^{\mathscr{S}}(r)} \log\log \|f, a_j\|^{-2} d\pi_o^r + O(1) \\ &\geq (q-2) T_f^{\mathscr{S}}(r) - \sum_{j=1}^q \overline{N}_f^{\mathscr{S}}(r, a_j) + T^{\mathscr{S}}(r, \mathscr{R}) \\ &\quad - \sum_{j=1}^q \int_{C^{\mathscr{S}}(r)} \log\log \|f, a_j\|^{-2} d\pi_o^r + O(1). \end{aligned}$$

The concavity of log implies that

$$\begin{aligned} \int_{C^{\mathscr{S}}(r)} \log\log \|f, a_j\|^{-2} d\pi_o^r &\leq \log \int_{C^{\mathscr{S}}(r)} \log \|f, a_j\|^{-2} d\pi_o^r \\ &= \log m_f^{\mathscr{S}}(r, a_j) + O(1) \leq \log T_f^{\mathscr{S}}(r) + O(1). \end{aligned} \tag{11}$$

By this, we obtain

$$\begin{aligned} \frac{1}{2} \int_{C^{\mathscr{S}}(r)} \log \xi \, d\pi_o^r \geq (q-2) T_f^{\mathscr{S}}(r) - \sum_{j=1}^q \overline{N}_f^{\mathscr{S}}(r, a_j) + T^{\mathscr{S}}(r, \mathscr{R}) \\ + O(\log T_f^{\mathscr{S}}(r)) + O(1). \end{aligned} \tag{12}$$

It remains to estimate the upper bound of the term on the left-hand side of (12). By Lemma 3.6, for any $\delta > 0$

$$\begin{aligned}\int_{C^{\mathscr{S}}(r)}\log\xi\,d\pi_o^r &\le \log\int_{C^{\mathscr{S}}(r)}\xi\,d\pi_o^r\\ &\le (1+\delta)^2\log\int_{r_0}^{r}\frac{dt}{t}\int_{D^{\mathscr{S}}(t)}\xi\,dV+\log\|\mathscr{S}\|_{r,\sup}\\ &\qquad +(2+\delta)\log\gamma(r)+\delta\log r+O(1)\\ &= (1+\delta)^2\log\int_{r_0}^{r}\frac{dt}{t}\int_{D^{\mathscr{S}}(t)}f^*\Phi+\log\|\mathscr{S}\|_{r,\sup}\\ &\qquad +(2+\delta)\log\gamma(r)+\delta\log r+O(1)\end{aligned}$$

holds for all $r\in(0,R^{\mathscr{S}})$ outside a set E_δ with $\int_{E_\delta}\gamma(r)\,dr<\infty$, here γ is an integrable function on $(0,R^{\mathscr{S}})$ such that $\int_0^{R^{\mathscr{S}}}\gamma(r)\,dr=\infty$. Notice that

$$\int_{D^{\mathscr{S}}(r)}f^*\Phi=\int_{\overline{\mathbb{C}}}n_f^{\mathscr{S}}(r,\zeta)\,\Phi(\zeta)$$

due to the change of variable formula. Using Fubini theorem, then

$$\int_{r_0}^{r}\frac{dt}{t}\int_{D^{\mathscr{S}}(t)}f^*\Phi=\int_{\overline{\mathbb{C}}}N_f^{\mathscr{S}}(r,\zeta)\,\Phi(\zeta)\le c_0T_f^{\mathscr{S}}(r)$$

for some positive constant c_0. Combining the above, we conclude that

$$\begin{aligned}\int_{C^{\mathscr{S}}(r)}\log\xi\,d\pi_o^r\le(1+\delta)^2\log T_f^{\mathscr{S}}(r)+\log\|\mathscr{S}\|_{r,\sup}\\ +(2+\delta)\log\gamma(r)+\delta\log r+O(1)\end{aligned}\tag{13}$$

holds for all $r\in(0,R^{\mathscr{S}})$ outside a set E_δ with $\int_{E_\delta}\gamma(r)\,dr<\infty$. Put together (12) and (13), then we prove the theorem. □

By uniformization theorem, for each nonnegative constant C, there exists a metric h such that the Gauss curvature K satisfies $-C\le K\le 0$. Now, we estimate the lower bound of $T^{\mathscr{S}}(r,\mathscr{R})$. By Lemmas 2.11 and 2.12, it follows from (8) that

$$\begin{aligned}T^{\mathscr{S}}(r,\mathscr{R})&=\frac{1}{2}\int_{D^{\mathscr{S}}(r)}g_r(o,x)K(x)\,dV(x)-\frac{1}{2}\int_{D^{\mathscr{S}}(r_0)}g_{r_0}(o,x)K(x)\,dV(x)\\ &\ge-\frac{C}{2}\int_0^r dt\int_{C^{\mathscr{S}}(t)}g_r(o,x)\,d\sigma_t(x)\\ &=-C\int_0^r dt\int_{C^{\mathscr{S}}(t)}\frac{t}{\|\mathscr{S}\|_h}\log\frac{r}{t}d\pi_o^t\\ &\ge-\frac{C}{\|\mathscr{S}\|_{r,\inf}}\int_0^r t\log\frac{r}{t}dt=-\frac{Cr^2}{4\|\mathscr{S}\|_{r,\inf}},\end{aligned}$$

where

$$\|\mathscr{S}\|_{r,\inf}=\inf\{\|\mathscr{S}_x\|_h:|\hat{x}|<r\}.$$

Combining this with Theorem 3.4, we obtain:

Theorem 3.7. Assume the same conditions as in Theorem 3.4. Suppose, in addition, that $-C \le K \le 0$ for a nonnegative constant C. Then for any $\delta > 0$

$$\begin{aligned}&(q-2)\,T_f^{\mathscr{S}}(r)\\ &\le \sum_{j=1}^{q}\overline{N}_f^{\mathscr{S}}(r,a_j)+O\big(\log T_f^{\mathscr{S}}(r)+Cr^2\|\mathscr{S}\|_{r,\inf}^{-1}+\log\|\mathscr{S}\|_{r,\sup}+\log\gamma(r)+\delta\log r\big)\end{aligned}$$

holds for all $r \in (0, R^{\mathscr{S}})$ outside a set E_δ with $\int_{E_\delta}\gamma(r)\,dr < \infty$, where

$$\|\mathscr{S}\|_{r,\inf} = \inf\{\|\mathscr{S}_x\|_h : |\hat{x}| < r\}, \quad \|\mathscr{S}\|_{r,\sup} = \sup\{\|\mathscr{S}_x\|_h : |\hat{x}| < r\}.$$

We recall the definition of the *simple defect* $\bar{\delta}_f(a)$ in (3).

Corollary 3.8. Assume the same conditions as in Theorem 3.4. Suppose, in addition, that $-C \le K \le 0$ for a nonnegative constant C. If f satisfies

$$\limsup_{r\to R^{\mathscr{S}}} \frac{Cr^2\|\mathscr{S}\|_{r,\inf}^{-1}+\log(\gamma(r)\|\mathscr{S}\|_{r,\sup})}{T_f^{\mathscr{S}}(r)} = 0,$$

then we have the defect relation

$$\sum_{j=1}^{q}\bar{\delta}_f(a_j) \le 2.$$

4. Targets are compact Riemann surfaces

Let $\mathcal{R}$ be a compact Riemann surface of genus g. Fix a positive $(1,1)$-form ω on $\mathcal{R}$. In a local holomorphic coordinate ζ, we may write ω as

$$\omega = \frac{\sqrt{-1}}{2\pi} w\, d\zeta \wedge d\bar{\zeta}.$$

According to Chern's theorem [3], we have that for every $a \in \mathcal{R}$, there exists a positive function u_a on $\mathcal{R}$ such that

$$2dd^c[\log u_a] = \omega - \delta_a, \tag{14}$$

where δ_a should be understood as a current. Let $f : \mathcal{S} \to \mathcal{R}$ be a holomorphic mapping. For $a \in \mathcal{R}$ with $f \not\equiv a$, by replacing $\|f, a\|^{-1}$ by $u_a \circ f$, we define similarly the Nevanlinna's functions of f as

$$\begin{aligned} T_{f,\omega}^{\mathscr{S}}(r) &= \int_{r_0}^{r}\frac{dt}{t}\int_{D^{\mathscr{S}}(t)} f^*\omega, \\ m_{f,\omega}^{\mathscr{S}}(r,a) &= \int_{C^{\mathscr{S}}(r)} \log(u_a\circ f)\, d\pi_o^r, \\ N_f^{\mathscr{S}}(r,a) &= \int_{r_0}^{r}\frac{n_f^{\mathscr{S}}(t,a)}{t}dt. \end{aligned}$$

Since

$$n_f^{\mathscr{S}}(r,a)=\int_{D^{\mathscr{S}}(r)} f^*\delta_a,$$

then we get

$$N_f^{\mathscr{S}}(r,a)=\int_{r_0}^{r}\frac{dt}{t}\int_{D^{\mathscr{S}}(t)} f^*\delta_a.$$

Using (14) and Dynkin formula, we obtain the First Main Theorem as

$$T_{f,\omega}^{\mathscr{S}}(r)=m_{f,\omega}^{\mathscr{S}}(r,a)+N_f^{\mathscr{S}}(r,a)+O(1).$$

In what follows we derive the Second Main Theorem. A computation gives that

$$\mathrm{Ric}(\omega)=K'\omega, \tag{15}$$

where K' is the Gauss curvature of ω. If $g=0$, then $\mathcal{R}$ can be regarded as $\mathbb{P}^1(\mathbb{C})$. Since $C_1\omega_{FS}\leq\omega\leq C_2\omega_{FS}$ for two suitable positive constants, then we can confirm the theorem by using the conclusion proved in Theorem 3.4. In the following, we assume that $g\geq 1$. We need to modify the form (9) as

$$\Phi=\frac{C\omega}{\prod_{j=1}^{q}u_{a_j}^{-2}\log^2 u_{a_j}^2},$$

where C is chosen so that Φ is normalized. Set

$$f^*\Phi=\xi\frac{\sqrt{-1}}{\pi}h\,dz\wedge d\bar{z}.$$

Since (14) and (15), then in the sense of currents, (10) becomes

$$\begin{aligned} dd^c[\log\xi]=qf^*\omega-f^*(K'\omega)-\sum_{j=1}^{q}(f-a_j)_0+D_{f,\mathrm{ram}}&\\ +\mathscr{R}-2\sum_{j=1}^{q}dd^c\log\log(u_{a_j}^2\circ f),& \end{aligned} \tag{16}$$

It is similar to (11), we have

$$\int_{C^{\mathscr{S}}(r)}\log\log u_{a_j}^2\,d\pi_o^r\leq\log T_{f,\omega}^{\mathscr{S}}(r)+O(1).$$

Integrating both sides of (16) and using Dynkin formula, we get

$$\begin{aligned}\frac{1}{2}\int_{C^{\mathscr{S}}(r)}\log\xi\,d\pi_o^r\geq qT_{f,\omega}^{\mathscr{S}}(r)-\int_{r_0}^{r}\frac{dt}{t}\int_{D^{\mathscr{S}}(t)}f^*(K'\omega)-\sum_{j=1}^{q}\overline{N}_f^{\mathscr{S}}(r,a_j)&\\ +T^{\mathscr{S}}(r,\mathscr{R})+O(\log T_{f,\omega}^{\mathscr{S}}(r))+O(1).&\end{aligned}$$

For the second term on the right-hand side of the above identity, we use the change of variable formula and Gauss–Bonnet formula to yield that

$$\begin{aligned}\int_{r_0}^{r}\frac{dt}{t}\int_{D^{\mathscr{S}}(r)}f^*(K'\omega) &= \int_{r_0}^{r}\frac{dt}{t}\int_{\mathcal{R}}n_f^{\mathscr{S}}(t,\zeta)K'(\zeta)\,\omega(\zeta)\\ &= \int_{\mathcal{R}}\Big[\int_{r_0}^{r}\frac{n_f^{\mathscr{S}}(t,\zeta)}{t}dt\Big]K'(\zeta)\,\omega(\zeta)\\ &= \int_{\mathcal{R}}N_f^{\mathscr{S}}(r,\zeta)K'(\zeta)\,\omega(\zeta)\\ &\geq \int_{\mathcal{R}}(T_{f,\omega}^{\mathscr{S}}(r)+O(1))K'(\zeta)\,\omega(\zeta)\\ &= (2-2g)\,T_{f,\omega}^{\mathscr{S}}(r)+O(1).\end{aligned}$$

Thus

$$\begin{aligned}\frac{1}{2}\int_{C^{\mathscr{S}}(r)}\log\xi\,d\pi_o^r \geq (q-2+2g)\,T_{f,\omega}^{\mathscr{S}}(r)-\sum_{j=1}^{q}\overline{N}_f^{\mathscr{S}}(r,a_j)\\ +T^{\mathscr{S}}(r,\mathscr{R})+O(\log T_{f,\omega}^{\mathscr{S}}(r))+O(1).\end{aligned}\tag{17}$$

On the other hand, it follows by using the similar arguments as in derivation of (13) that

$$\begin{aligned}\int_{C^{\mathscr{S}}(r)}\log\xi\,d\pi_o^r \leq (1+\delta)^2\log T_{f,\omega}^{\mathscr{S}}(r)+\log\|\mathscr{S}\|_{r,\sup}\\ +(2+\delta)\log\gamma(r)+\delta\log r+O(1)\end{aligned}\tag{18}$$

holds for all $r\in(0,R^{\mathscr{S}})$ outside a set E_δ with $\int_{E_\delta}\gamma(r)\,dr<\infty$. Put together (17) and (18), then we conclude the following Second Main Theorem.

Theorem 4.1. Let $(\mathcal{S},h;\mathscr{S})$ be a $\mathscr{S}$-exhausted Hermitian Riemann surface of $\mathscr{S}$-radius $R^{\mathscr{S}}$ with respect to o, and $\mathcal{R}$ be a compact Riemann surface of genus g. Fix a positive $(1,1)$-form ω on $\mathcal{R}$. Let γ be an integrable function on $(0,R^{\mathscr{S}})$ with $\int_0^{R^{\mathscr{S}}}\gamma(r)\,dr=\infty$. Let $f:\mathcal{S}\to\mathcal{R}$ be a nonconstant holomorphic mapping and $a_1,\ldots,a_q$ be distinct points in $\mathcal{R}$. Then for any $\delta>0$

$$\begin{aligned}&(q-2+2g)\,T_{f,\omega}^{\mathscr{S}}(r)+T^{\mathscr{S}}(r,\mathscr{R})\\ &\qquad\leq\sum_{j=1}^{q}\overline{N}_f^{\mathscr{S}}(r,a_j)+O\big(\log T_{f,\omega}^{\mathscr{S}}(r)+\log\|\mathscr{S}\|_{r,\sup}+\log\gamma(r)+\delta\log r\big)\end{aligned}$$

holds for all $r\in(0,R^{\mathscr{S}})$ outside a set E_δ with $\int_{E_\delta}\gamma(r)\,dr<\infty$, where

$$\|\mathscr{S}\|_{r,\sup}=\sup\{\|\mathscr{S}_x\|_h:|\hat{x}|<r\}.$$

Similar to Theorem 3.7, we have:

Theorem 4.2. Assume the same conditions as in Theorem 4.1. Suppose, in addition, that $-C \le K \le 0$ for a nonnegative constant C. Then for any $\delta > 0$

$$(q-2+2g)\,T^{\mathscr{S}}_{f,\omega}(r)$$
$$\le \sum_{j=1}^{q} \overline{N}^{\mathscr{S}}_{f}(r, a_j) + O\Big(\log T^{\mathscr{S}}_{f,\omega}(r) + Cr^2 \|\mathscr{S}\|^{-1}_{r,\inf} + \log \|\mathscr{S}\|_{r,\sup} + \log \gamma(r) + \delta \log r\Big)$$

holds for all $r \in (0, R^{\mathscr{S}})$ outside a set E_δ with $\int_{E_\delta} \gamma(r)\,dr < \infty$, where

$$\|\mathscr{S}\|_{r,\inf} = \inf\{\|\mathscr{S}_x\|_h : |\hat{x}| < r\}, \qquad \|\mathscr{S}\|_{r,\sup} = \sup\{\|\mathscr{S}_x\|_h : |\hat{x}| < r\}.$$

Theorem 4.2 derives a defect relation, i.e., Theorem III in Introduction.

Acknowledgement

The authors are grateful to the referee for valuable comments on this paper.

References

[1] A. Atsuji, “Nevanlinna-type theorems for meromorphic functions on non-positively curved Kähler manifolds”, *Forum Math.* **30**:1 (2018), 171–189. MR Zbl

[2] J. Carlson and P. Griffiths, “A defect relation for equidimensional holomorphic mappings between algebraic varieties”, *Ann. of Math.* (2) **95** (1972), 557–584. MR Zbl

[3] S.-S. Chern, “Complex analytic mappings of Riemann surfaces, I”, *Amer. J. Math.* **82** (1960), 323–337. MR Zbl

[4] X. Dong, “Carlson–Griffiths theory for complete Kähler manifolds”, *J. Inst. Math. Jussieu* (2022), 1–29.

[5] P. Griffiths and J. King, “Nevanlinna theory and holomorphic mappings between algebraic varieties”, *Acta Math.* **130** (1973), 145–220. MR Zbl

[6] R. C. Gunning and R. Narasimhan, “Immersion of open Riemann surfaces”, *Math. Ann.* **174** (1967), 103–108. MR Zbl

[7] Y. He and M. Ru, “Nevanlinna and algebraic hyperbolicity”, *Internat. J. Math.* **32**:12 (2021), art. id. 2140015. MR Zbl

[8] N. Ikeda and S. Watanabe, *Stochastic differential equations and diffusion processes*, 2nd ed., North-Holland Mathematical Library **24**, North-Holland Publishing Co., Amsterdam, 1989. MR Zbl

[9] K. Itô and H. P. McKean, Jr., *Diffusion processes and their sample paths*, Grundl. Math. Wissen. **125**, Academic Press, New York, 1965. MR Zbl

[10] R. Nevanlinna, “Zur Theorie der Meromorphen Funktionen”, *Acta Math.* **46**:1-2 (1925), 1–99. MR Zbl

[11] J. Noguchi and T. Ochiai, *Geometric function theory in several complex variables*, Translations of Mathematical Monographs **80**, American Mathematical Society, Providence, RI, 1990. MR Zbl

[12] J. Noguchi and J. Winkelmann, *Nevanlinna theory in several complex variables and diophantine approximation*, Grundl. Math. Wissen. **350**, Springer, Tokyo, 2014. MR

[13] M. Păun and N. Sibony, "Value distribution theory for parabolic Riemann surfaces", 2014. arXiv 1403.6596

[14] M. Ru, *Nevanlinna theory and its relation to diophantine approximation*, World Scientific Publishing Co., River Edge, NJ, 2001. MR Zbl

[15] M. Ru and N. Sibony, "The second main theorem in the hyperbolic case", *Math. Ann.* **377**:1-2 (2020), 759–795. MR Zbl

[16] B. V. Shabat, *Distribution of values of holomorphic mappings*, Translations of Mathematical Monographs **61**, American Mathematical Society, Providence, RI, 1985. MR Zbl

[17] H.-H. Wu, *The equidistribution theory of holomorphic curves*, Annals of Mathematics Studies **64**, Princeton University Press, Princeton, N.J., 1970. MR Zbl

Received January 4, 2022. Revised April 26, 2022.

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The Pacific Journal of Mathematics (ISSN 1945-5844 electronic, 0030-8730 printed) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

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PACIFIC JOURNAL OF MATHEMATICS

Volume 319 No. 1 July 2022