

A Relationship Between Nonphysical Quasi-probabilities and Nonlocality Objectivity

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Abstract

Density matrices are the most general descriptions of quantum states, covering both pure and mixed states. Positive semidefiniteness is a physical requirement of density matrices, imposing nonnegative probabilities of measuring physical values. Separately, nonlocality is a property shared by some bipartite quantum systems, indicating a correlation of the component parts that cannot be described by local classical variables. In this work, we show that breaking the positive-semidefinite requirement and allowing states with a negative minimal eigenvalue arbitrarily close to zero, allows for the construction of states that are nonlocal under one component labelling but local when the labelling is interchanged. This is an observer-dependent nonlocality, showing the connection between nonlocal objectivism and negative quasi-probabilities.

1 Introduction

A 2-qubit density matrix ρ is a 4×4 matrix describing a (generally mixed) quantum system composed of two qubits. They can be written as a linear combination of 2-qubit spin operators including the identity:

$$\rho = \frac{1}{4} \left(I_4 + \sum_{i,j} c_{ij} \sigma_i \otimes \sigma_j \right) \quad (1)$$

where $c_{ij} \in \mathbb{R}$, σ_i refer to 1-qubit Pauli spin operators hereafter labelled X, Y, Z, I , and the tensor products of two identities I_4 is singled out with coefficient $\frac{1}{4}$. Density states describing physical systems obey the following two conditions:

$$\text{Tr}(\rho) = 1, \quad \lambda(\rho) \geq 0 \quad (2)$$

where one also has $\text{Tr}(\rho^2) \leq 1$, with the equality saturated for pure states. The second condition above is the statement that ρ is positive semidefinite, in that all of its eigenvalues $\lambda(\rho)$ are nonnegative. As 2-qubit operators are hermitian, ρ is hermitian by construction.

Nonlocality is a quantum-specific property that some bipartite systems have, indicating correlations that cannot be described by classical “hidden” variables. It was first described by Bell in 1964[1], with an experimentally testable inequality discovered in 1966[2]. Then, in 1995 the Horodecki brothers gave an equivalent mathematical statement [3] for nonlocality for 2-qubit density matrices. The statement is as follows:

Theorem 1.1 (Horodecki). *A 2-qubit density matrix ρ describes a nonlocal state iff the matrix β of coefficients of maximally mixed operators obeys the following:*

$$\mathcal{M} > 1 \quad (3)$$

where $\mathcal{M} = u + \bar{u}$, the two largest eigenvalues of $\beta^T \beta$.

The matrix β of maximally mixed coefficients is the matrix whose entries are c_{ij} from (1) where neither σ_i nor σ_j are the identity operator.

In this work, we examine nonlocal conditions on a certain subset of states. We introduce the following definition:

Definition 1.2 (Qubit-swapping). For a given density matrix $\rho = \frac{1}{4} \left(I + \sum_{ij} c_{ij} \sigma_i \otimes \sigma_j \right)$, we define the *qubit-swapped* state ρ' by interchanging 1-qubit Pauli matrices without changing the associated coefficients:

$$\rho' = \frac{1}{4} \left(I + \sum_{ij} c_{ij} \sigma_j \otimes \sigma_i \right) \quad (4)$$

Definition 1.2 amounts to relabelling the qubits $A \otimes B \mapsto B \otimes A$. One would expect that a simple relabelling would not affect the physical properties of a system described by ρ . However one can define states where nonlocality is dependent on the operator-order labelling:

Definition 1.3. A *swap-intolerant nonlocal* (SIN) state is a state ρ that is nonlocal, but whose qubit-swapped state ρ' is local.

Now we can move on to the main theorem of this article:

Theorem 1.4. For a given $\lambda_0 < 0$ arbitrarily close to zero, one can construct a non-positive-semidefinite density state ρ whose smallest eigenvalue is λ_0 , such that it is swap-intolerant nonlocal.

Theorem 1.4 shows that even arbitrary breaking of positive-semidefiniteness allows for unphysical interpretations of nonlocality. It will be proven by construction later in this work. First we must build up a particular type of state called *perp states*.

2 Perp States

Perp states are a subset of *hyperplane* states introduced in [4]. Hyperplane states are formed by allowing nonzero coefficients $c_{ij} \neq 0$ in (1) only for 2-qubit operators existing in a hyperplane of $\mathcal{W}(3,2)$, the symplectic polar space of 2-qubit operators. This space contains all 2-qubit operators up to a phase factor, and commutation relations between subsets of operators of size 3. It can be represented visually by a finite geometry known as the doily, see Fig. 1, and [5, 6, 7, 8] for further reading on its relationship with quantum information.

A *hyperplane* H of $\mathcal{W}(3,2)$ is a subset of points of $\mathcal{W}(3,2)$ such that any line of $\mathcal{W}(3,2)$ is either fully contained in H or intersects H at exactly one point. A perp-set $P(p)$ is a hyperplane constructed by taking one point p of $\mathcal{W}(3,2)$, with associated operator \mathcal{O}_p , and all points that commute with \mathcal{O}_p :

$$P(p) := \{\mathcal{O}_i \in \mathcal{W}(3,2), [\mathcal{O}_i, \mathcal{O}_p] = 0\} \quad (5)$$

Perp-sets are hyperplanes of $\mathcal{W}(3,2)$ (see Fig. 1, right). Another type of hyperplane are *grids* (or *hyperbolic quadrics* in the mathematical literature), formed by 9 points lying on 6 intersecting lines with each point lying on two lines. An example of a grid is the set of maximally mixed operators G_2 , i.e. those without identities in their qubit components. The grid G_2 contains all negative lines of the doily, see Fig. 2.

We define a *perp state* ρ as a state of the form (1) whose only nonzero coefficients c_{ij} are those associated to the operators in $P(p)$, the perp set of the point p . For example, taking $p = ZZ$ as in Fig. 1, one has that the only nontrivial operators furnishing ρ_{ZZ} are $\{ZZ, XX, YY, XY, YX, IZ, ZI\}$.

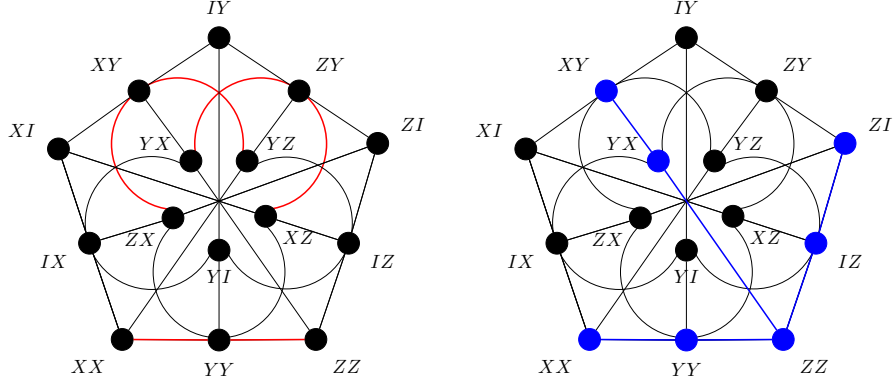


Figure 1: The doily geometry describing $\mathcal{W}(3,2)$, the symplectic polar space of 2-qubit operators. Points are labelled by 2-qubit operators with tensor symbol suppressed. Lines contain 3 points of operators, known as contexts. Operators along a context mutually commute and multiply to $\pm I_4$, with red lines indicating lines whose product is $-I_4$ (left). Right shows a doily containing a perp-set hyperplane in blue, with chosen point $p = ZZ$.

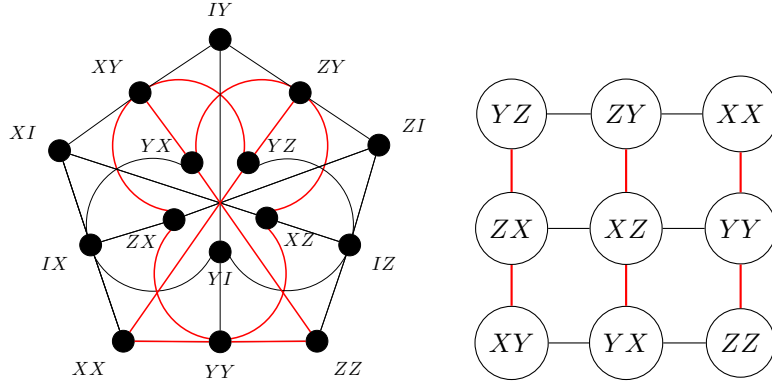


Figure 2: The grid G_2 of all nontrivial operators (right), and as a subgeometry of the doily (left, red).

Finally one defines *semi-trivial* perp states ρ as those whose defining point p contains one identity operator, i.e. $p \notin G_2$. It is these states that we will show can become swap-intolerant nonlocal under arbitrary positive-definite breaking.

It was found in [4] that the eigenvalues of perp states are given by the following:

$$\lambda(\rho) = \begin{cases} 1 + c_p \pm \sqrt{\sum_{\text{pos}}(c_{i1} + c_{i2})^2 + \sum_{\text{neg}}(c_{i1} - c_{i2})^2} \\ 1 - c_p \pm \sqrt{\sum_{\text{pos}}(c_{i1} - c_{i2})^2 + \sum_{\text{neg}}(c_{i1} + c_{i2})^2} \end{cases} \quad (6)$$

where c_p is the coefficient associated to the point p , c_{i1}, c_{i2} are colinear coefficients sharing the i th context with p , and the sums are taken over positive and negative lines as defined by the operator products.

We now split the coefficients into two types: ones associated to the fully nontrivial points in G_2 , which we label β_{ij} , and the rest, which we label τ_{ij} . It is easy to see that for a point $p \notin G_2$, its incident lines contain one operator within G_2 and one outside it. Also, all of these contexts are positive, as the only negative lines are fully contained in G_2 . Thus for a semi-trivial perp state we can label the eigenvalues as such:

$$\lambda(\rho) = \begin{cases} 1 + \tau_p \pm \sqrt{\sum_{i=1}^3 (\beta_i + \tau_i)^2} \\ 1 - \tau_p \pm \sqrt{\sum_{i=1}^3 (\beta_i - \tau_i)^2} \end{cases} \quad (7)$$

We let the τ_i , $i = 1, 2, 3$ coefficients be coordinates of a point C in 3-dimensional parameter space \mathcal{S} . We let the coordinates β_i , $i = 1, 2, 3$ define a point E which we call the state point. The distance $r_{\pm} := 1 \pm \tau_p$ define radii of spheres centred at the point C , which we call (C, r_{\pm}) . Then positive definiteness becomes a condition on the coordinates of the points:

Proposition 2.1. *The semi-trivial perp state ρ is positive-semidefinite iff the state point E and its reflected point $-E$ are contained on the surface or in the interior of the spheres (C, r_+) , (C, r_-) respectively.*

This is proven in [4], but is evident upon inspection of the eigenvalues. What's more, the eigenvalues are dependent upon the radii r_{\pm} and the distances $|\pm E, C|$, so under translations and rotations of the parameter space \mathcal{S} , any eigenvalue-dependent property such as positive-definiteness is unchanged. Therefore for any perp state ρ we are free to form a state $R(\rho)$ by rotating the parameters in \mathcal{S} by some rotation R , then ρ and $R(\rho)$ will have the same positive-definite property.

Now we introduce the nonlocality condition (3) into the parameter space \mathcal{S} .

3 SIN States in \mathcal{S}

Let the matrix β be the matrix of coefficients β_{ij} of fully nontrivial operators for a given state ρ . Then for ρ a semi-trivial perp state, it is easy to see that β will only

have one column or row vector as nonzero entries. This is because for a given $p \notin G_2$, points in G_2 commuting with it must share the same nontrivial 1-qubit operator as the one in p . Whether it is a row or column vector depends on whether the identity is in the right-hand or left-hand of p :

$$\beta^T \beta = \begin{cases} \begin{pmatrix} \beta_{ix}^2 & & \\ & \beta_{iy}^2 & \\ & & \beta_{iz}^2 \end{pmatrix}, & p = i \otimes I \\ \begin{pmatrix} \sum_j \beta_{ji}^2 & & \\ & 0 & \\ & & 0 \end{pmatrix}, & p = I \otimes i \end{cases} \quad (8)$$

where the index i matches the chosen nontrivial operator in p . This asymmetry in $\beta^T \beta$ furnishes a swap-intolerant nonlocality as \mathcal{M} is given by the sum of the largest two eigenvalues of the above matrix. Let $(\beta_1, \beta_2, \beta_3)$ be the vector entries of β for a given semi-trivial perp state ρ , and WLOG let β_3 be the smallest of the three. Then the following conditions give a SIN state:

$$\begin{aligned} \beta_1^2 + \beta_2^2 + \beta_3^2 &> 1 \\ \beta_1^2 + \beta_2^2 &\leq 1 \\ \beta_3^2 &\leq \min(\beta_1^2, \beta_2^2) \end{aligned} \quad (9)$$

These conditions give $\mathcal{M} > 1$ for $p = I \otimes i$ (left-identity) but not $p = i \otimes I$ (right-identity). The first describes the region outside the unit sphere in \mathcal{S} , the second describes the interior of the unit cylinder along the β_3 axis, and the third gives the exterior of the cone along the β_3 axis with endpoint at the origin. The intersection of these regions is nonempty in \mathcal{S} . We need to place our state point E in this intersection and find suitable coordinates τ_i and τ_p to describe our state.

First we prove the following about semi-trivial perp states.

Proposition 3.1. *Semi-trivial perp states ρ cannot be both positive-semidefinite and nonlocal.*

Proof. Let $(\beta_1, \beta_2, \beta_3)$ be the coordinates of E in \mathcal{S} . Nonlocality in both left- or right-identity cases implies $|E|^2 = \sum_{i=1}^3 \beta_i^2 > 1$. Now for a given set of values for τ_p , $(\tau_i) = C$ one places C in the axis connecting E , $-E$ (see Fig. 3). This provides the maximum distance E that leaves ρ positive semidefinite. But here the distance $2|E| = 1 + \tau_p + 1 - \tau_p = 2$, in contradiction to the condition that $|E| > 1$ as required by nonlocality. \square

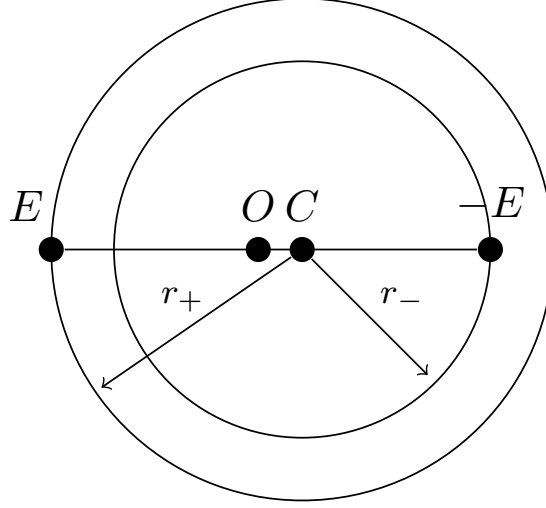


Figure 3: The points E , C , $-E$ and the origin in parameter space. C is placed along the $(E, -E)$ axis, and E , $-E$ are placed on the surfaces of the spheres with radii r_+ , r_- . For a given C , τ_p , this arrangement gives the maximal distance between E and $-E$ satisfying positive definiteness.

We now prove Theorem 1.4. For a given $\lambda_0 < 0$, we let $|E| = 1 + \frac{1}{2}|\lambda_0|$ - outside the unit sphere in \mathcal{S} thus giving a nonlocal state. By 3.1 this state is non-positive-semidefinite. Set the remaining parameters such that C is on the axis joining E and $-E$, and the point $-E$ lies on the sphere (C, r_-) . This then puts E a distance of $|\lambda_0|$ from the sphere (C, r_+) , making the smallest eigenvalue of ρ equal to λ_0 . Now perform a rotation R in \mathcal{S} such that $R(E)$ is lying in the region described in (9), making the state swap-intolerant nonlocal.

4 Conclusion

We have shown here a connection between positive semidefiniteness of density matrices and label-dependence of nonlocality. If one were to relax the positive-semidefiniteness criterion even slightly then one can construct a 2-qubit state where the nonlocality depends on the order of the qubit labels. Furthermore, the *boundary* for nonlocality in general in this case (including swap-intolerant nonlocality) is the same as the boundary for positive semidefiniteness, identifying a boundary for physicality with one separating classical from quantum mechanics. It is the boundary that defines

$|E| = 1$, with $|E|$ the point given by coordinates $(\beta_1, \beta_2, \beta_3)$, the nonzero vector in β . We have used the symplectic polar space $\mathcal{W}(3, 2)$ and perp sets of this space to construct these SIN states in the most general fashion.

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