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ON THE UNIQUENESS OF THE STRICTLY CONVEX QUADRILATERAL CENTRAL CONFIGURATION WITH A FIXED ANGLE

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ABSTRACT. The conjecture of the existence and the uniqueness of the strictly convex quadrilateral central configuration for the Newtonian 4-body problem is one of the most-talked open problems in the study of the classical n -body problems in celestial mechanics. MacMillan and Bartky first gave its general existence in the 1930s and a particular case for its uniqueness. Still, the general case has yet to be solved perfectly since it was considered by Sim'ò and Yoccoz in the 1980s and was first mentioned by Albouy and Fu in 2008 in the formal publication. Using coordinates of mutual distances and Morse's critical point theory, we give the (at most) uniqueness of the planar strictly convex 4-body central configuration when the angle of one pair of the opposite sides is given.

1. INTRODUCTION

The classical n -body problem mainly concentrates on the study of n particles P_1, \dots, P_n with masses $m_i > 0$ and positions $x_i = (x_{i,1}, \dots, x_{i,d})^T \in \mathbb{R}^d$ ($i = 1, \dots, n$; $d = 2, 3$) interacting with each other and satisfying the following differential equations

$$(1) \quad m_i \ddot{x}_{i,j} = \frac{\partial U}{\partial x_{i,j}}, \quad i = 1, \dots, n, j = 1, \dots, d,$$

where

$$U = \sum_{i < j} \frac{m_i m_j}{|x_j - x_i|} = \sum_{i < j} \frac{m_i m_j}{r_{i,j}}$$

is the Newtonian potential and $r_{i,j} = |x_i - x_j|$ is the mutual distance between P_i and P_j . We denote by $x = (x_1, \dots, x_n) \in \mathbb{R}^{dn} \setminus \Delta$ a collision free configuration, where $\Delta = \{x \in \mathbb{R}^{dn} | x_i = x_j, \forall i \neq j\}$ is the *singular set*, or *the collision set*, and $r = (r_{12}, r_{13}, \dots, r_{n-1,n}) \in (\mathbb{R}^+)^{C_n^2}$ its corresponding mutual distance vector.

Key words and phrases. celestial mechanics, Dziobek central configurations, mutual distances, Newtonian 4-body problem, Morse's theory.

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Central configurations are particular arrangements of the n -body system satisfying a set of nonlinear algebraic equations

$$(2) \quad -\lambda m_j(x_j - c) = \frac{\partial U(x)}{\partial x_j} = \sum_{i \neq j, i=1}^n \frac{m_i m_j (x_i - x_j)}{r_{i,j}^3},$$

where $c = \frac{1}{m} \sum_{i=1}^n m_i x_i$ is the *center of mass* with $m = \sum_{i=1}^n m_i$ the *total mass* and λ the *configuration constant*. Furthermore, if we denote by

$$(3) \quad I(x) = \frac{1}{2} \sum_{i=1}^n m_i (x_i - c)^2 = \frac{1}{2m} \sum_{i < j} m_i m_j r_{ij}^2$$

the *moment of inertia* of the system, a more compact form of (2) is as follows

$$(4) \quad \nabla U(x) + \lambda \nabla I(x) = 0,$$

in which ∇ denotes the gradient operator with respect to x . Hence, λ can be seen as the Lagrangian multiplier of the potential $U(x)$ with the constraint $I(x) = I_0$. For a more intuitive illustration, we can simultaneously release the n particles with zero velocity from this special position, and all bodies will collide at the center of mass in a limited time. This can be seen as a special homothetic case of the *homographic solutions* of (1), i.e.,

$$x(t) - c(t) = r(t)Q(t)(x_0 - c_0),$$

where x_0 is a central configuration with c_0 its center of mass, $r(t) > 0$ is a real scaling factor, namely, a solution of Kepler's problem (i.e., 2-body problem), and $Q(t) \in \text{SO}(d)$ is a rotation. Furthermore, if we take $r(t) = 1$, only considering the rotation, we get the well-known relative equilibrium solutions. These self-similar solutions may be the only analytic solutions to the n -body problems up to the recent research. The first two kinds of homographic solutions were constructed by Euler in 1767 and Lagrange in 1772 with initial particular positions for the 3-body problem, respectively, which came to be known as the Eulerian collinear and Lagrangian equilateral triangle central configurations.

One can easily see that the equations (2) and (4) are invariant under translation, dilation, and rotation, which allows us to consider the equivalent classes of central configurations naturally. In addition, classical Morse's theory requires the non-degeneracy of central configurations as critical points at least, which are not isolated under the continuous action of the rotation group; hence, they are degenerate. Two central configurations are equivalent if they can be transformed from one to another by combining the above three transformations. There are several open problems concerning central configurations, one of which is the finiteness conjecture, namely, the Chazy-Wintner-Smale conjecture [6, 33, 35]: *Is the number of equivalent classes of central configurations (or relative equilibria for $d = 2$) finite for any given n positive masses?* For $n = 3$, Wintner [35] showed that there are

totally three Eulerian collinear central configurations and two Lagrangian equilateral triangle central configurations. Hampton and Moeckel [15] gave a positive answer for the case $n = 4$, and it is still open for $n \geq 5$ with several generic results on $n = 5$, see [5, 14].

If we consider the particular shapes of the configurations, which are related to the classification of central configurations, some excellent results also come out. For example, Moulton [24] studied the n collinear case in 1910, concluding that there are just $n!/2$ collinear central configurations for any given n positive masses along a straight line. One can also refer to [23, 32] for different proofs of this collinear case. In other words, if the order of the n masses along a line is fixed, the central configuration is unique. Similarly, the uniqueness of some non-collinear central configurations can also be obtained, provided the corresponding order of the masses is fixed. Related to this idea, one widely focused issue is the existence and uniqueness conjecture of the planar strictly convex 4-body central configuration. The existence property dates back to the work of MacMillan and Bartky [21] in 1932, and Xia [36] provided a concise proof in 2004 using perturbation techniques. One can also refer to Moeckel [23] for more details. For the uniqueness part, one can refer to the clear narrative by Santoprete in his recent papers [28, 29] that it is reasonable to be named after Simó and Yoccoz since, according to Alain Albouy, one can trace this uniqueness problem back to several conversations between Simó and Yoccoz, as well as their related works [31, 38].

Conjecture 1 (Simó-Yoccoz). *Given four positive masses m_1, m_2, m_3, m_4 , is there only one convex central configuration for each cyclic order?*

Albouy and Fu first mentioned this conjecture in a formally published paper [1], and then Albouy, Cabral, and Santos collected it in [3] as Problem 10 on the classical n -body problems. There have been several excellent results around this issue, with different kinds of constraints on masses [2, 4, 7, 11, 18, 20, 26], shapes [19, 27–29, 37], potentials, and their combinations [8, 12, 13], and so on, using different analytical or topological approaches. Recently, Sun, Xie, and You [34] gave a computer-assisted proof in some generic cases using interval analysis based on the Krawczyk operator and the implicit function theorem, except for the ones with masses very close to 0.

In the current paper, we use a rigorous topological method inspired by the idea of Santoprete in [28, 29] to get the uniqueness result of the strictly convex planar 4-body central configuration with a given angle of one pair of opposite sides. We summarize the main steps as follows:

- (i) Use mutual distances as variables instead of positions to eliminate the rotation symmetry naturally;
- (ii) Replace the Cayley-Menger determinants constraint with a simple new one such that the gradients of both are paralleled;
- (iii) Determine the type, i.e., the Morse index of the critical point of the reduced Lagrangian function;

- (iv) Study the topology of a proper space formed by the corresponding constraints;
- (v) Get the uniqueness of the non-degenerate critical point via Morse inequality.

For (ii), replacing the Cayley-Menger determinant constraints with more simple ones can significantly simplify the calculations in (iii), (iv) and (v), and was first noticed and raised by Cors and Roberts [9] in the study of co-circular central configurations of the 4-body problem in which they used the Ptolemy's theorem instead, supported by an important property, i.e., the gradients of both constraints with respect to r are paralleled. Santoprete in [27] succeeded in replacing with a trapezoidal geometrical constraint. In this current paper, we use a necessary condition for a planar convex quadrilateral given by Josefsson in [16] (Theorem 10), which plays a crucial role in (ii), and we will see in the following.

Finally, we have

Theorem 1. *Given four positive masses m_1, m_2, m_3, m_4 , there is at most only one convex central configuration for each cyclic order with a given angle of one pair of opposite sides.*

2. THE STRICTLY CONVEX PLANAR 4-BODY CENTRAL CONFIGURATION WITH A GIVEN ANGLE OF ONE PAIR OF OPPOSITE SIDES

We need to point out that according to the famous Perpendicular Bisector Theorem in [22], a convex planar 4-body central configuration must be strictly convex, i.e., any one of the four bodies is located outside the convex hull of the left three. In other words, there is no 3-body collinearity in a planar 4-body case and no $(n-1)$ -body collinearity in a planar n -body central configuration. Before going into the main steps, some notations are necessary.

2.1. Notations and the change of coordinates. Let

$$x = (x_1, \dots, x_4), x_i = (x_{i1}, x_{i2}) \in \mathbb{R}^2, i = 1, \dots, 4$$

be the position vector and $r = (r_{12}, r_{13}, \dots, r_{34}) \in (\mathbb{R}^+)^6$ its distance vector, where

$$P_i P_j = r_{ij} = |x_j - x_i| = \sqrt{(x_{j1} - x_{i1})^2 + (x_{j2} - x_{i2})^2}, 1 \leq i < j \leq 4.$$

The Cayley-Menger determinant is

$$(5) \quad \begin{aligned} \mathcal{F}(1, 2, \dots, n) &= F_{12 \dots n}(r) = F_n(1, 2, \dots, n) \\ &= \begin{vmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & r_{12}^2 & \cdots & r_{1n}^2 \\ 1 & r_{12}^2 & 0 & \cdots & r_{2n}^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & r_{1n}^2 & r_{2n}^2 & \cdots & 0 \end{vmatrix} = (-1)^n \cdot 2^{n-1} \cdot ((n-1)!)^2 \cdot \mathcal{V}_n^2, \end{aligned}$$

where \mathcal{V}_n is the volume of the simplex formed by n points. Dziobek used it in [10] in 1900 as a planar constraint $F_4(r) = 0$ to characterize the equations of central configurations of the 4-body problem and gave the equivalence of the equations under both coordinates. These equations are generalized to the n -body central configurations with $(n - 2)$ -dimension afterward, and one can refer to Chapter 2 in [23] for more details. Especially for $n = 3$, we have Heron's formula with $i < j < k$

$$\begin{aligned}
 \mathcal{V}_3 &= \mathcal{V}_3(i, j, k) = V_{ijk} \\
 (6) \quad &= \frac{1}{4} \sqrt{-(r_{ij} - r_{ik} - r_{jk})(r_{ij} + r_{ik} - r_{jk})(r_{ij} + r_{ik} + r_{jk})(r_{ij} - r_{ik} + r_{jk})} \\
 &= \frac{1}{4} \sqrt{-F_{ijk}(r)} = \frac{1}{4} \sqrt{-F_3(i, j, k)},
 \end{aligned}$$

where $\mathcal{V}_3 = V_{ijk}$ is the area of the triangle $\triangle P_i P_j P_k$.

We need to point out again that not each vector $r \in (\mathbb{R}^+)^{C_n^2}$ corresponds to a n -body configuration. Here, especially for four bodies, we need the concept of *geometrical realizable set* observed and raised by Cors and Roberts in [9]:

$$\begin{aligned}
 (7) \quad \mathcal{G} &= \{r \in (\mathbb{R}^+)^6 \mid r_{ij} + r_{jk} \geq r_{ik}, \forall i, j, k \in \{1, 2, 3, 4\}, i \neq j \neq k, \\
 &\quad F_4(1, 2, 3, 4) \geq 0\}.
 \end{aligned}$$

Then the *normalized configuration space* with respect to r is

$$(8) \quad \mathcal{N} = \{r \in \mathcal{G} \mid I(r) = I_0, F_4(r) = 0\}.$$

The equivalence of the equations of central configurations under both coordinates is guaranteed by the following result from Dziobek(also refer to [28, 29]):

Proposition 1. *Let x be a planar 4-body configuration, let $r \in \mathcal{N}$ be its corresponding normalized configuration, and let $U|_{\mathcal{N}} : \mathcal{N} \rightarrow \mathbb{R}$ be the restriction of the Newtonian potential U to \mathcal{N} . Then, x is a planar central configuration if and only if r is a critical point of $U|_{\mathcal{N}}$ with respect to r .*

This means that x is a planar central configuration if and only if r is a critical point of the Lagrangian function

$$(9) \quad \tilde{W}(r) = U(r) + \lambda(I(r) - I_0) + \sigma F_4(r),$$

with $\lambda > 0$ and σ its multipliers.

2.2. The reduced Lagrangian function for convex quadrilateral configurations. Without loss of generality, we fix an order of the four particles such that P_1, P_2, P_3 and P_4 lie anticlockwise along a strictly convex quadrilateral, see Figure 1. The key is finding a proper constraint for this general convex quadrilateral. Let

$$(10) \quad C(r) = 2r_{12}r_{34} \cos \theta + r_{23}^2 + r_{14}^2 - r_{24}^2 - r_{13}^2 = 0,$$

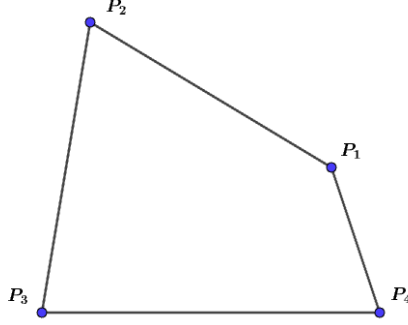


FIGURE 1. A planar convex 4-body configuration.

where θ denotes the angle between the extensions of the sides r_{12} and r_{34} . Since we can define that $\theta = 0$ if the line P_1P_2 and P_3P_4 are parallel, i.e., a trapezoid shape, hence $\theta \in [0, \pi/2]$, see Figure 2. This constraint comes from Josefsson's paper [16], in which, in Theorem 10, he gave (10) as a necessary condition of a convex quadrilateral. If we denote by

$$\tilde{\mathcal{C}} = \{r \in \mathcal{G} | C(r) = 0\},$$

we can easily check that the collinear and the strictly concave configurations are not in this set. However, the 3-body collinear case does, i.e., the non-strictly convex one with a triangular convex hull. Still, fortunately, it cannot be a central configuration according to the Perpendicular Bisector Theorem.

Proposition 2. *For a strictly convex quadrilateral configuration $r \in \mathcal{G}$,*

$$(11) \quad F_4(r) = C(r) \cdot A(r),$$

where $A(r) = 32V_{124}V_{234} \neq 0$.

Proof. Firstly, since $r \in \mathcal{G}$ is a strictly convex quadrilateral configuration, we have $C(r) = 0$. Substituting

$$(12) \quad r_{13}^2 = 2r_{12}r_{34} \cos \theta + r_{23}^2 + r_{14}^2 - r_{24}^2$$

into $F_4(r)$ with some simplification and using the notations in (6), we have

$$(13) \quad F_4(r) = C(r) \cdot A(r) - 2 \left(\left(\frac{A(r)}{4r_{24}} \right)^2 - B(r)^2 \right),$$

where

$$(14) \quad \begin{cases} A(r) = 2(r_{24}^2(r_{13}^2 - r_{34}^2) + r_{12}^2(r_{23}^2 - r_{24}^2 - r_{34}^2) + r_{14}^2(-r_{23}^2 + r_{34}^2) \\ \quad + 2r_{12}r_{24}^2r_{34} \cos \theta), \\ \quad = 8r_{12}r_{24}^2r_{34} \left(\cos \theta - \frac{(r_{12}^2 - r_{14}^2 + r_{24}^2)(-r_{23}^2 + r_{24}^2 + r_{34}^2)}{4r_{12}r_{24}^2r_{34}} \right), \\ B(r) = \frac{\sqrt{F_3(1, 2, 4)F_3(2, 3, 4)}}{2r_{24}} = \frac{8V_{124}V_{234}}{r_{24}}, \end{cases}$$

where the second equality of $A(r)$ is obtained by eliminating r_{13}^2 with (12). In general, we can eliminate any of the four square items in $C(r)$ to get different versions of the decomposition of F_4 . Secondly, we claim that

$$A(r) = 4r_{24}B(r)$$

for $\theta \in [0, \pi/2]$. It suffices to show that

$$(15) \quad \cos \theta = \frac{(r_{12}^2 - r_{14}^2 + r_{24}^2)(-r_{23}^2 + r_{24}^2 + r_{34}^2)}{4r_{12}r_{24}^2r_{34}} + \frac{16V_{124}V_{234}}{4r_{12}r_{24}^2r_{34}}.$$

If we denote by $\alpha = \angle P_1P_2P_4$ and $\beta = \angle P_2P_4P_3$, then $\alpha, \beta \in (0, \pi)$ in a strictly convex quadrilateral. One can see in Figure 2 that with the law of cosines and the area formula of triangles in sines, we have

$$(16) \quad \begin{cases} r_{12}^2 - r_{14}^2 + r_{24}^2 = 2r_{12}r_{24} \cos \alpha, \\ -r_{23}^2 + r_{24}^2 + r_{34}^2 = 2r_{24}r_{34} \cos \beta, \\ V_{124} = \frac{r_{12}r_{24} \sin \alpha}{2}, \\ V_{234} = \frac{r_{24}r_{34} \sin \beta}{2}. \end{cases}$$

Substituting (16) into the right side of (15) we have

$$(17) \quad \frac{(r_{12}^2 - r_{14}^2 + r_{24}^2)(-r_{23}^2 + r_{24}^2 + r_{34}^2)}{4r_{12}r_{24}^2r_{34}} + \frac{16V_{124}V_{234}}{4r_{12}r_{24}^2r_{34}} = \cos(\beta - \alpha).$$

We draw a parallel line of P_1P_2 crossing P_4 , and can easily find that $\theta = \beta - \alpha$, which leads to the equality of (15), see Figure 2. In general, $\theta = |\beta - \alpha|$ holds, which does not influence the cosine value. The expression of $A(r)$ is obtained directly. \square

Remark 1. In the proof, we derive (11) by replacing r_{13}^2 with the expression in (12). It is almost the same if we eliminate r_{24}^2 instead, with $A(r)/(2r_{24})$ and $B(r)$ changing correspondingly. Things will be a little different if we do the same thing to r_{14}^2 or r_{23}^2 since one can easily check that the relationship of the angles becomes $\theta + \alpha + \beta = \pi$.

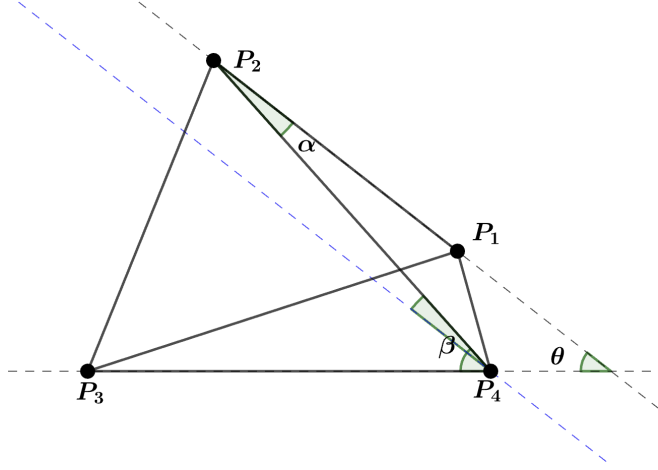
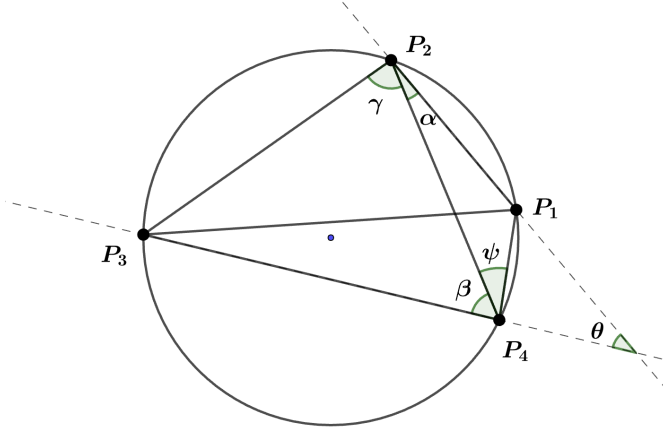
FIGURE 2. $\theta = |\beta - \alpha|$.

FIGURE 3. A co-circular configuration.

Remark 2. Here, we mention some special cases of convex quadrilateral configurations.

1. For a trapezoid configuration, it is easy to see that $\theta = 0 \Leftrightarrow \alpha = \beta$, then (10) becomes the trapezoidal constraint used in [19, 29]. One can refer to [16, 27] for more details of the characteristics of trapezoidal quadrilaterals.
2. Let $P(r) = r_{12}r_{34} + r_{14}r_{23} - r_{13}r_{24}$. For a co-circular configuration $r \in \mathcal{G}$ with the given order in Figure 1 (also see Figure 3), $C(r)=0$ holds since any co-circular quadrilateral r is strictly convex. We further denote by $\psi = \angle P_1P_3P_2 = \angle P_1P_4P_2$, $\gamma = \angle P_3P_2P_4 = \angle P_3P_1P_4$ and d the diameter of the circle. We calculate $C(r) - 2\cos\theta P(r)$ directly with the complementarity of the opposite

angles $(\alpha + \gamma) + (\beta + \psi) = \pi$ and relationships derived from the law of sines in triangles, i.e.,

$$\frac{r_{12}}{\sin \psi} = \frac{r_{13}}{\sin(\gamma + \alpha)} = \frac{r_{14}}{\sin \alpha} = \frac{r_{23}}{\sin \beta} = \frac{r_{24}}{\sin(\psi + \alpha)} = \frac{r_{34}}{\sin \gamma} = d$$

to replace all the mutual distances with d and sines, then we get $C(r) - 2 \cos \theta P(r) = 0$. Since $\cos \theta \neq 0$, this implies that $P(r) = 0$, i.e., the well-known Ptolemy's theorem for this co-circular quadrilateral. In addition, according to Theorem 6 in [30], Ptolemy's theorem also holds for the 4-body collinear case. For an arbitrary quadrilateral, the inequality $P \geq 0$ holds; see [25].

3. A classical kite shape (concave or convex) is a quadrilateral with two couple of equal-length adjacent sides. A generalized version has at least one pair of equal opposite angles. One can refer to the characteristics given by Santoprete [30] and Josefsson [17] for more details.
4. If P_1, P_2 and P_3 are collinear, one can check that $A(r) \neq 0$ still holds, while it vanishes for the remaining three 3-body collinear cases. If we use $C(r) = 0$ as constraint instead of $F_4(r) = 0$, the case that P_1, P_2 and P_3 are collinear cannot be excluded.

Let $\mathcal{M} = \{r \in \mathbb{R}^6 | I(r) - I_0 = 0, C(r) = 0\}$, and denote by

$$\mathcal{M}^+ = \{r \in (\mathbb{R}^+)^6 | I(r) - I_0 = 0, C(r) = 0\}$$

the subset of \mathcal{M} with respect to positive r_{ij} s. Let

$$\mathcal{C} = \{r \in \mathcal{G} \cap \mathcal{M}^+ | F_4(r) = 0\},$$

which is the collection of configurations we care about. The relationship of these sets is shown in Figure 4. Since the topology of \mathcal{C} is complicated, we turn to study a larger region \mathcal{M}^+ . In what follows, we will see that the Eulerian characteristic of \mathcal{M}^+ is 1.

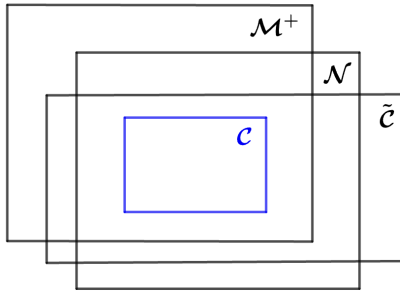


FIGURE 4. $\mathcal{C} \subset (\mathcal{M}^+ \cap \mathcal{N} \cap \tilde{\mathcal{C}})$.

With the decomposition of $F_4(r)$ in (11), we have

Lemma 1. *For any $r \in \tilde{\mathcal{C}}$, we have $\nabla_r F_4(r) = \nabla_r C(r) \cdot A(r)$.*

Proof. From (11) we have

$$\nabla_r F_4(r) = \nabla_r C(r) \cdot A(r) + C(r) \cdot \nabla_r A(r).$$

For any $r \in \tilde{\mathcal{C}}$, we have $C(r) = 0$, and the equality in the lemma holds obviously. \square

Then we conclude that

Proposition 3. *Let $r \in \mathcal{C}$. Then r is the critical point of $\tilde{W}(r)$ if and only if it is the critical point of*

$$W(r) = U(r) + \lambda(I(r) - I_0) + \eta C(r),$$

where λ and $\eta = \sigma \cdot A(r)$ are multipliers.

Proof. We can derive this result by calculating the critical point equations of $\tilde{W}(r)$ and $W(r)$. \square

Here, we need to point out that $\eta \neq 0$ if and only if $r \in \mathcal{G}$ is strictly convex or only P_1, P_2 and P_3 are collinear.

2.3. The type of the critical point of $W(r)$. The critical point equations of $W(r)$ are

$$(18) \quad \begin{cases} S_{12} + 2\eta \cdot r_{34} \cos \theta = 0, \\ S_{13} - 2\eta \cdot r_{13} = 0, \\ S_{14} + 2\eta \cdot r_{14} = 0, \\ S_{23} + 2\eta \cdot r_{23} = 0, \\ S_{24} - 2\eta \cdot r_{24} = 0, \\ S_{34} + 2\eta \cdot r_{12} \cos \theta = 0, \end{cases}$$

where

$$\begin{cases} S_{ij} = m_i m_j r_{ij} (\delta - r_{ij}^{-3}), & 1 \leq i < j \leq 4, \\ \delta = \frac{\lambda}{m} > 0. \end{cases}$$

We continue to compute the Hessian of $W(r)$ and obtain

$$D^2 W(r) = \begin{bmatrix} R_{12} & 0 & 0 & 0 & 0 & 2\eta \cos \theta \\ 0 & R_{13} - 2\eta & 0 & 0 & 0 & 0 \\ 0 & 0 & R_{14} + 2\eta & 0 & 0 & 0 \\ 0 & 0 & 0 & R_{23} + 2\eta & 0 & 0 \\ 0 & 0 & 0 & 0 & R_{24} - 2\eta & 0 \\ 2\eta \cos \theta & 0 & 0 & 0 & 0 & R_{34} \end{bmatrix},$$

where

$$R_{ij} = m_i m_j \left(\delta + \frac{2}{r_{ij}^3} \right) > 0, \quad 1 \leq i < j \leq 4.$$

Lemma 2. *Suppose that $\tilde{r} \in \mathcal{M}^+$ is a critical point of $W(r)$. Then \tilde{r} is a non-degenerate local minimum of $W(r)$.*

Proof. We can easily get four positive eigenvalues of the six of $D^2W(r)$ using the corresponding expressions of 2η in (18)

$$\begin{cases} \zeta_1 = R_{13} - 2\eta = \frac{m_1 m_3}{r_{13}^3} > 0, \\ \zeta_2 = R_{14} + 2\eta = \frac{m_1 m_4}{r_{14}^3} > 0, \\ \zeta_3 = R_{23} + 2\eta = \frac{m_2 m_3}{r_{23}^3} > 0, \\ \zeta_4 = R_{24} - 2\eta = \frac{m_2 m_4}{r_{24}^3} > 0. \end{cases}$$

The left two ζ_5 and ζ_6 are the roots of the following quadratic equation of ζ

$$\zeta^2 + a_1 \zeta + a_0 = 0,$$

where

$$\begin{cases} a_1 = -(\zeta_5 + \zeta_6) = -(R_{12} + R_{34}) < 0, \\ a_0 = \zeta_5 \cdot \zeta_6 = 4r_{12}^3 r_{13}^6 r_{14}^6 r_{23}^6 r_{24}^3 r_{34}^3 (R_{12} R_{34} - 4\eta^2 r_{12}^3 r_{34}^3 \cos^2 \theta). \end{cases}$$

To show ζ_5 and ζ_6 are both positive, it suffices to show $a_0 > 0$. In fact, from the first and the last equations in (18) and the relationship of S_{ij} and R_{ij} , we have

$$\begin{aligned} \cos \theta &= -\frac{S_{12}}{2\eta r_{34}} = \frac{r_{12}(R_{12} - 3/r_{12}^3)}{2\eta r_{34}}, \\ &= -\frac{S_{34}}{2\eta r_{12}} = \frac{r_{34}(R_{34} - 3/r_{34}^3)}{2\eta r_{12}}. \end{aligned}$$

Substituting the above two equalities at the same time into the expression of a_0 to replace each $\cos \theta$, we have

$$a_0 = 3m_1 m_2 m_3 m_4 (1 + (r_{12}^3 + r_{34}^3)\delta) > 0,$$

which implies that ζ_5, ζ_6 are both positive. From the above analysis, we derive that the critical point of $W(r)$, if it exists, is a local minimum with Morse index 0. \square

2.4. The topology of \mathcal{M}^+ .

Lemma 3. $\chi(\mathcal{M}^+) = 1$.

Proof. We can only consider the equal mass case since the Eulerian characteristic number is invariant under continuous deformations. Suppose that $I_0 = \frac{1}{2m}$ and $m_1 = \dots = m_4 = 1$. Let \mathcal{M}_0 and \mathcal{M}_0^+ be the sets corresponding to the equal mass cases of \mathcal{M} and \mathcal{M}^+ , respectively. We denote by $\mathcal{S}^n(\rho)$ the n -dimension sphere with radius ρ and

$$\mathcal{S}^5(1) = \{r \in \mathbb{R}^6 \mid r_{12}^2 + r_{13}^2 + r_{14}^2 + r_{23}^2 + r_{24}^2 + r_{34}^2 = 1\}.$$

Then, with $r_{ij} \geq 0$, the constraints in \mathcal{M}_0^+ are

$$(19) \quad \begin{cases} r_{12}^2 + r_{13}^2 + r_{14}^2 + r_{23}^2 + r_{24}^2 + r_{34}^2 = 1, \\ 2r_{12}r_{34} \cos \theta + r_{23}^2 + r_{14}^2 - r_{24}^2 - r_{13}^2 = 0. \end{cases}$$

We first assume that $\cos \theta \neq 0$. Divide both sides of the second equation by $\cos \theta$ and then use the first equality to add and subtract it separately. We have

$$(20) \quad \begin{cases} (r_{12} + r_{34})^2 + \left(\frac{1}{\cos \theta} + 1\right)(r_{14}^2 + r_{23}^2) = 1 + \left(\frac{1}{\cos \theta} - 1\right)(r_{24}^2 + r_{13}^2), \\ (r_{12} - r_{34})^2 + \left(\frac{1}{\cos \theta} + 1\right)(r_{24}^2 + r_{13}^2) = 1 + \left(\frac{1}{\cos \theta} - 1\right)(r_{14}^2 + r_{23}^2). \end{cases}$$

If we denote by

$$(21) \quad \begin{cases} v_1 = r_{12} + r_{34} \geq 0, \\ w_1 = r_{12} - r_{34} \leq v_1, \\ \Theta(\theta) = \frac{1}{\cos \theta} + 1 \geq 2, \\ G_1 = r_{14}^2 + r_{23}^2 \geq 0, \\ G_2 = r_{24}^2 + r_{13}^2 \geq G_1, \end{cases}$$

then (20) becomes

$$(22) \quad \begin{cases} v_1^2 + \Theta G_1 = 1 + (\Theta - 2) G_2, \\ w_1^2 + \Theta G_2 = 1 + (\Theta - 2) G_1. \end{cases}$$

Furthermore $0 \leq G_1 \leq \frac{1}{2}$ holds since from the first equation in (19) we have $G_1 + G_2 \leq 1$. Noticing that $v_1^2 - w_1^2 = 2(\Theta - 1)(G_2 - G_1) \geq 0$ implies $|w_1| \leq |v_1|$. Then $w_1^2 = 1 + (\Theta - 2) G_1 - \Theta G_2 \geq 0$ must hold to make sense. This leads to the following inequalities

$$G_1 \leq G_2 \leq \frac{1 + (\Theta - 2) G_1}{\Theta},$$

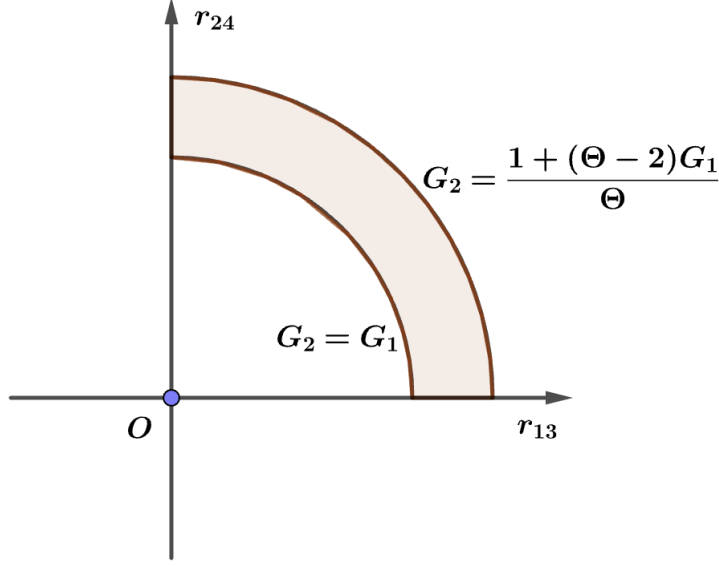
Since $G_1 \leq \frac{1 + (\Theta - 2) G_1}{\Theta} = G_1 + \frac{1 - 2G_1}{\Theta}$ holds for any $0 \leq G_1 \leq \frac{1}{2}$. Denote by

$$\mathcal{D}_1 = \left\{ (r_{14}, r_{23}) \in (\mathbb{R}^+)^2 \mid 0 \leq r_{14}^2 + r_{23}^2 = G_1 \leq \frac{1}{2} \right\},$$

then \mathcal{D}_1 is contractible in the 2-dimension (r_{14}, r_{23}) -plane, i.e., it forms a quarter of a disk with radius $1/2$. Now we fix a point $(r_{14}, r_{23}) \in \mathcal{D}_1$. Then the point (r_{13}, r_{24}) satisfying (2.4) forms a contractible region

$$\mathcal{D}_2 = \left\{ (r_{13}, r_{24}) \in (\mathbb{R}^+)^2 \mid G_1 \leq r_{13}^2 + r_{24}^2 \leq \frac{1 + (\Theta - 2) G_1}{\Theta} \right\},$$

i.e., a quarter of a ring belt in the 2-dimension (r_{13}, r_{24}) -plane, see Figure 5.


 FIGURE 5. The region \mathcal{D}_2 .

To summarize the above, we consider a projection

$$\tilde{p}_1 : (r_{13}, r_{24}, r_{14}, r_{23}) \rightarrow (r_{14}, r_{23}),$$

which induces a fibration $p_1 : \mathcal{D} \rightarrow \mathcal{D}_1$ with \mathcal{D}_2 its fiber and \mathcal{D}_1 its base. Since both \mathcal{D}_1 and \mathcal{D}_2 are contractible, \mathcal{D} is contractible.

For a fixed point $(r_{13}, r_{24}, r_{14}, r_{23}) \in \mathcal{D}$, we find that v_1 and w_1 are determined uniquely by (22). Without loss of generality, we assume that $w_1 = r_{12} - r_{34} \leq 0$ holds. Hence the projection

$$\tilde{p} : (r_{13}, r_{24}, r_{14}, r_{23}, v_1, w_1) \rightarrow (r_{13}, r_{24}, r_{14}, r_{23})$$

induces a fibration $p : \mathcal{M}_0^+ \rightarrow \mathcal{D}$ with \mathcal{D} its base and $\{(v_1, w_1)\}$ its fiber, i.e., a point, and it is contractible. Hence, \mathcal{M}_0^+ is contractible for $\cos \theta \in (0, 1]$.

If $\cos \theta = 0$, we can reduce (19) to

$$\begin{cases} r_{14}^2 + r_{23}^2 = \frac{1}{2}(1 - r_{12}^2 - r_{34}^2), \\ r_{13}^2 + r_{24}^2 = \frac{1}{2}(1 - r_{12}^2 - r_{34}^2). \end{cases}$$

$\mathcal{H} = \{(r_{12}, r_{34}) \in (\mathbb{R}^+)^2 \mid r_{12}^2 + r_{34}^2 \leq 1\}$ is contractible in the 2-dimension (r_{12}, r_{34}) -plane. Now fix a point $(r_{14}, r_{23}) \in \mathcal{H}$. Then

$$\mathcal{H}_1 = \left\{ (r_{14}, r_{23}) \in (\mathbb{R}^+)^2 \mid r_{14}^2 + r_{23}^2 = \frac{1}{2}(1 - r_{12}^2 - r_{34}^2) \right\}$$

forms a contractible quarter of a circle with radius $\sqrt{\frac{1}{2}(1 - r_{12}^2 - r_{34}^2)}$ in the 2-dimension (r_{12}, r_{34}) -plane, and similarly, we have

$$\mathcal{H}_2 = \left\{ (r_{13}, r_{24}) \in (\mathbb{R}^+)^2 \mid r_{13}^2 + r_{24}^2 = \frac{1}{2}(1 - r_{12}^2 - r_{34}^2) \right\}.$$

To sum up, the projection $\tilde{s}_1 : (r_{14}, r_{23}, r_{12}, r_{34}) \rightarrow (r_{12}, r_{34})$ induces a fibration $s_1 : \tilde{\mathcal{H}} \rightarrow \mathcal{H}$, in which both the base \mathcal{H} and the fiber \mathcal{H}_1 are contractible, so $\tilde{\mathcal{H}}$ is contractible. The projection $\tilde{s} : (r_{13}, r_{24}, r_{14}, r_{23}, r_{12}, r_{34}) \rightarrow (r_{14}, r_{23}, r_{12}, r_{34})$ induces a fibration $s : \mathcal{M}_0^+ \rightarrow \tilde{\mathcal{H}}$, in which both the base $\tilde{\mathcal{H}}$ and the fiber \mathcal{H}_2 are contractible, so \mathcal{M}_0^+ is contractible for $\cos \theta = 0$. This lead to that $\chi(\mathcal{M}_0^+) = 1$.

The relationship between \mathcal{M}_0^+ and \mathcal{M}^+ is easy to understand. In fact, \mathcal{M}_0^+ is a contractible subset of $\mathcal{S}^5(1)$, which is homeomorphic to the corresponding

5-dimension ellipsoid $\mathcal{E}^5(1) = \left\{ r \in \mathbb{R}^5 \mid \sum_{i < j}^4 m_i m_j r_{ij}^2 = 1 \right\}$. The cone $C(r) = 0$

passing through the origin intersects $\mathcal{E}^5(1)$ in $\mathcal{M}^+ \subset \mathcal{E}^5(1)$ via the homeomorphic map. Hence we have $\chi(\mathcal{M}^+) = \chi(\mathcal{M}_0^+) = 1$. \square

2.5. The uniqueness.

Lemma 4. *$U(r)$ has a unique critical point on \mathcal{M}^+ .*

Proof. From Lemma 2, the critical points of $U(r)$ restricting on \mathcal{M}^+ are non-degenerate local minimum with Morse index 0, which implies that U is a Morse function. Noticing that $r \rightarrow \partial\mathcal{M}^+$ leads to $U \rightarrow +\infty$, the minimum point exists. Then, from the Morse equation below where α_q denotes the number of the non-degenerate critical points with Morse index q

$$\sum (-1)^q \alpha_q = \chi(\mathcal{M}^+) = 1$$

we have $\alpha_0 = 1$ provided $q = 0$. \square

Finally, we have

The proof of Theorem 1. From Proposition 3, if r is a convex quadrilateral central configuration, it is a critical point of the Lagrangian function $W(r)$, i.e., the potential function U restricted on \mathcal{M}^+ . From Lemma 2, this critical point must be a non-degenerate local minimum. Since $\mathcal{C} \subset \mathcal{M}^+$, we conclude that U has at most one critical point restricted on \mathcal{C} . \square

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DATA AVAILABILITY

All data generated or analyzed during this study are included in this published article.

CONFLICT OF INTEREST STATEMENT

The authors declared that they have no conflicts of interest in this work.

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