# Shafarevich—Tate groups of holomorphic Lagrangian fibrations II

#### Anna Abasheva

#### Abstract

Let X be a compact hyperkähler manifold with a Lagrangian fibration  $\pi\colon X\to B$ . A Shafarevich—Tate twist of X is a holomorphic symplectic manifold with a Lagrangian fibration  $\pi^\varphi\colon X^\varphi\to B$  which is isomorphic to  $\pi$  locally over the base. In particular,  $\pi^\varphi$  has the same fibers as  $\pi$ . A twist  $X^\varphi$  corresponds to an element  $\varphi$  in the Shafarevich–Tate group III of X. We show that  $X^\varphi$  is Kähler when a multiple of  $\varphi$  lies in the connected component of unity of III and give a necessary condition for  $X^\varphi$  to be bimeromorphic to a Kähler manifold.

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# 1 Introduction

#### 1.1 Definitions

**Definition 1.1.1.** An *irreducible holomorphic symplectic* manifold X is a compact complex simply connected manifold admitting a closed holomorphic symplectic form  $\sigma$  such that  $H^0(\Omega_X^2) = \mathbb{C} \cdot \sigma$ . If X is Kähler, then we will call X irreducible  $hyperk\"{a}hler^1$ .

 $<sup>^{1}</sup>$ Most algebraic geometers use terms holomorphic symplectic manifold and hyperkähler manifolds interchangeably. However, it is important for us to make this distinction because we will encounter non-Kähler holomorphic symplectic manifolds in this paper

**Definition 1.1.2.** A Lagrangian fibration on an irreducible holomorphic symplectic manifold X is a morphism  $\pi \colon X \to B$  with connected fibers to a normal variety B such that the restriction of  $\sigma$  to a smooth fiber is zero.

If X is hyperkähler and the base B is smooth, then B is necessarily isomorphic to  $\mathbb{P}^n$  [Hwa08]. No examples of Lagrangian fibrations on irreducible holomorphic symplectic manifolds over a base other than  $\mathbb{P}^n$  have been discovered and conjecturally the base should always be  $\mathbb{P}^n$ .

**Definition 1.1.3.** Define the sheaf  $T_{X/B}$  of vertical vector fields on X as the kernel of the map  $T_X \to \pi^*T_B/\operatorname{Tors}(\pi^*T_B)$ , where  $T_B := (\Omega_B)^{\vee}$  and  $\operatorname{Tors}(\pi^*T_B)$  is the torsion subsheaf of  $\pi^*T_B$ .

The flow of a vertical vector field v induces a vertical automorphism  $\exp(v)$  of X.

**Definition 1.1.4.** Consider the sheaf  $Aut_{X/B}^0$  on B consisting of all vertical automorphisms that are of the form  $\exp(v)$  for some vertical vector field v locally over B. The Shafarevich-Tate group of the fibration  $\pi \colon X \to B$  is defined to be the group  $\coprod = H^1(B, Aut_{X/B}^0)^3$ .

The group III has a beautiful geometric interpretation. Cover B by open disks so that  $B = \bigcup U_i$ . For each subset I of indices, we denote  $\bigcap_{i \in I} U_i$  by  $U_I$  and  $\pi^{-1}(U_I)$  by  $X_I$ . Every class  $\varphi \in III$  can be represented

by a Čech 1-cocycle with coefficients in  $Aut_{X/B}^0$ . In other words, we have a vertical automorphism  $\varphi_{ij}$  of  $X_{ij}$  for each pair of indices i, j, and

$$\varphi_{ik} = \varphi_{jk} \circ \varphi_{ij}. \tag{1}$$

For each i, j glue  $X_i$  to  $X_j$  by the automorphism  $\varphi_{ij}$  to get a new variety  $X^{\varphi}$ . By the cocycle condition (1) the variety  $X^{\varphi}$  is a smooth Hausdorff complex manifold admitting a fibration

$$\pi^{\varphi} \colon X^{\varphi} \to B.$$

**Definition 1.1.5.** The manifold  $X^{\varphi}$  constructed above is called the *Shafarevich-Tate twist* of X with respect to the class  $\varphi \in III$ .

Note that the sheaves  $Aut^0_{X/B}$  and  $Aut^0_{X^\varphi/B}$  are isomorphic. Hence the Shafarevich–Tate group of  $\pi\colon X\to B$  is the same as the Shafarevich–Tate group of  $\pi^\varphi\colon X^\varphi\to B$ . The Shafarevich–Tate twist of  $X^\varphi$  with respect to  $\psi\in \mathrm{III}$  is isomorphic to  $X^{\varphi+\psi}$ .

The Shafarevich–Tate group  $\coprod = H^1(B, Aut^0_{X/B})$  has a structure of a topological group, possibly non-Hausdorff [AR21, Subsection 3.1]. Denote its connected component of unity by  $\coprod^0$ . By Theorem 2.2.7 the group  $\coprod^0$  is a quotient of  $\mathbb C$  by a finitely generated subgroup. By [AR21, Subsection 6.3] the discrete part  $\coprod/\coprod^0$  of  $\coprod$  satisfies:

$$(\coprod/\coprod^0) \otimes \mathbb{Q} \simeq H^2(R^1\pi_*\mathbb{Q}).$$

For a class  $\varphi \in \coprod$ , we will denote by  $\overline{\varphi}$  its image in  $\coprod / \coprod^0 \otimes \mathbb{Q}$ . We will denote by  $\coprod'$  the set of classes  $\varphi \in \coprod$  such that  $\overline{\varphi} = 0$ .

**Definition 1.1.6.** A Shafarevich-Tate deformation is a Shafarevich-Tate twist  $X^{\varphi}$  of X with respect to an element  $\varphi \in \coprod^0$ .

#### 1.2 Statement of the results

**Theorem A** (3.0.6, Theorem 3.0.7). Let  $\pi: X \to B$  be a Lagrangian fibration on an irreducible hyperkähler manifold X. Pick a class  $\varphi \in \mathrm{III'}$ , i.e., a class  $\varphi$  such that  $r\varphi$  lies in  $\mathrm{III^0}$  for some positive integer r. Then the following holds.

- 1. The twist  $X^{\varphi}$  is Kähler.
- 2. A twist  $X^{\varphi}$  with respect to  $\varphi \in \coprod'$  is projective if and only if  $\varphi$  is torsion.

We proved a version of this theorem in [AR21, Theorem B] for a general hyperkähler manifold, assuming  $\pi$  has no multiple fibers in codimension one [AR21, Definition 2.3]and  $\varphi \in \text{III}^0$ . The new proof does not require any of this assumptions.

**Remark 1.2.1.** A weaker version of Theorem A recently appeared in [SV24]. However, our arguments are different, and we prove a more general statement.

<sup>&</sup>lt;sup>2</sup>When B is smooth, the sheaf  $\pi^*T_B$  is clearly locally free, hence torsion free. We do not know whether  $\pi^*T_B$  is torsion free in general

<sup>&</sup>lt;sup>3</sup>Ш is a letter of the Russian alphabet pronounced as "Sha". It is the first letter in the last name Шафаревич (Shafare-vich).

**1.2.2.** A Shafarevich–Tate twist  $X^{\varphi}$  of a holomorphic symplectic manifold is holomorphic symplectic, and the fibration  $\pi^{\varphi}$  is a Lagrangian fibration [AR21, Corollary 3.7]. We can show more.

**Theorem B** (5.2.9). Let  $\pi: X \to B$  be a Lagrangian fibration on an irreducible hyperkähler manifold. Then for any  $\varphi \in \coprod$  we have  $H^0(X^{\varphi}, \Omega^2_{X^{\varphi}}) = \mathbb{C} \cdot \sigma$ , where  $\sigma$  is a holomorphic symplectic form on  $X^{\varphi}$ . Moreover,  $H^1(X^{\varphi}, \mathbb{Q}) = 0$ .

**1.2.3.** In the next theorem we compute the second Betti number of Shafarevich–Tate twists. Note that the differential  $d_2$  on the second page of the Leray spectral sequence of  $\mathbb{Q}_X$  for the map  $\pi$  maps  $H^0(B, R^2\pi_*\mathbb{Q})$  to  $H^2(B, R^1\pi_*\mathbb{Q}) \simeq (\mathrm{III}/\mathrm{III}^0) \otimes \mathbb{Q}$ .

**Theorem C** (5.3.11). Let  $\pi: X \to B$  be a Lagrangian fibration on an irreducible hyperkähler manifold X and  $\varphi \in \coprod$ . Then exactly one of the following two cases occurs.

- 1. If the image  $\overline{\varphi}$  of  $\varphi$  in  $H^2(R^1\pi_*\mathbb{Q})$  lies in the image of  $d_2$ , then  $b_2(X^{\varphi}) = b_2(X)$ . Moreover, there is a cohomology class  $h \in H^2(X^{\varphi})$  which restricts to an ample class on a smooth fiber.
- 2. If  $\overline{\varphi}$  is not in the image of  $d_2$ , then  $b_2(X^{\varphi}) = b_2(X) 1$ . In this case all cohomology classes  $h \in H^2(X^{\varphi})$  restrict trivially to a smooth fiber.

**Definition 1.2.4.** A complex manifold is said to be of  $Fujiki\ class\ C$  if it is bimeromorphic to a Kähler manifold.

We will derive the following criterion for non-Kählerness of Shafarevich—Tate twists as an easy corollary of Theorem C.

**Theorem D** (5.3.12). Let  $\pi: X \to B$  be a Lagrangian fibration on a hyperkähler manifold X. Pick  $\varphi \in \coprod$  such that  $\overline{\varphi}$  is not in the image of  $d_2$ . Then  $X^{\varphi}$  is not of Fujiki class C, in particular, not Kähler.

- 1.2.5. Outline of the paper. We start by recalling basic facts about Lagrangian fibrations and their Shafarevich–Tate twists in Section 2. Many results in Section 2 were contained our previous work [AR21] but were stated assuming that the base B of a Lagrangian fibration  $\pi$  is smooth and  $\pi$  has no multiple fibers in codimension one. We show that these assumptions are not necessary. In Section 3 we will prove the second part of Theorem A, which is easier than the first part. The first part of Theorem A will be proven in Section 4. In Section 5 we study cohomological properties of Shafarevich–Tate twists. We will see that Shafarevich–Tate twists have trivial first cohomology in Subsection 5.1 and prove that  $H^0(\Omega^2_{X^{\varphi}})$  is one-dimensional in Subsection 5.2. These two statements immediately imply Theorem B. Finally, in Subsection 5.3 we prove Theorem C and then show how to derive Theorem D from Theorem C.
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#### 2 Preliminaries

#### 2.1 Lagrangian fibrations

**2.1.1.** Beauville-Bogomolov-Fujiki form. One of the key cohomological features of hyperkähler manifold is the existence of a quadratic form on their second cohomology called *Beauville-Bogomolov-Fujiki* form (BBF form).

**Theorem 2.1.2** ([GJH03, Part III, Corollary 23.11 & Proposition 23.14]). Let X be an irreducible hyperkähler manifold of dimension 2n. Then there exists an integral symmetric non-degenerate form q on  $H^2(X)$  such that  $\forall \alpha \in H^2(X, \mathbb{Z})$ ,

$$c_X q(\alpha)^n = \int_X \alpha^{2n}.$$

The constant  $c_X$  is positive and depends only on the deformation type of X.

**Remark 2.1.3.** The integral form q from Theorem 2.1.2 is uniquely defined if we require it to be non-divisible.

**Definition 2.1.4.** The form q from Theorem 2.1.2 is called the *Beauville–Bogomolov–Fujiki form* or BBF form.

**2.1.5. Fibers of Lagrangian fibrations are abelian varieties.** Consider a Lagrangian fibration  $\pi \colon X \to B$  (Definition 1.1.2) on an irreducible holomorphic symplectic manifold X. A general fiber of  $\pi$  is a complex torus and even an abelian variety [Cam06]. The projectivity of smooth fibers follows easily from the theorem below.

**Theorem 2.1.6** ([Voi92, Mat99]). Let  $\pi: X \to B$  be a Lagrangian fibration on a hyperkähler manifold and F its smooth fiber. Then the restriction map

$$H^2(X,\mathbb{Q}) \to H^2(F,\mathbb{Q})$$

has rank one.

Thanks to Theorem 2.1.6, for any Kähler class  $h \in H^2(X, \mathbb{R})$  some real multiple  $c \cdot h$  of h restricts to an integral class on F. The class  $c \cdot h|_F$  is Kähler and integral, hence ample. It follows that F is indeed an abelian variety.

- **2.1.7. Discriminant.** The image in B of singular fibers of  $\pi$  is called the *discriminant* of the Lagrangian fibration and will be denoted by  $\Delta$ . It is known to be a divisor [HO09, Proposition 3.1]. We define  $B^{\circ}$  to be the complement of  $\Delta$  and  $X^{\circ} := \pi^{-1}(B^{\circ})$ .
- **2.1.8. Vertical vector fields.** The holomorphic symplectic form  $\sigma$  enables us to construct a lot of vertical vector fields on X. First, it induces an isomorphism  $\Omega_X \xrightarrow{\iota_{\sigma}} T_X$ . Let X' denote the subset  $\pi^{-1}(B^{reg}) \subset X$ . Consider the composition of maps

$$\pi^*\Omega_{B^{reg}} \hookrightarrow \Omega_{X'} \xrightarrow{\iota_{\sigma}} T_{X'} \to \pi^*T_{B^{reg}}.$$

It is easy to see that it vanishes on  $X^{\circ}$ . Indeed, for every form  $\alpha$  on an open subset of the base, the vector field dual to  $\pi^*\alpha$  is tangent to smooth fibers of  $\pi$ . Since  $\pi^*T_{B^{reg}}$  is locally free, the map  $\pi^*\Omega_{B^{reg}} \to \pi^*T_{B^{reg}}$  vanishes on X'. Therefore, the map  $\iota_{\sigma}$  sends  $\pi^*\Omega_{B^{reg}}$  into  $T_{X'/B^{reg}}$  (Definition 1.1.3). By taking pushforwards to  $T_{X'/B^{reg}}$  we obtain a map

$$\pi_*\pi^*\Omega_{B^{reg}} \hookrightarrow \pi_*T_{X'/B^{reg}}.$$

By [Mat99]  $\pi_*\Theta_X \simeq \Theta_B$ . The projection formula implies that  $\pi^*\pi_*\Omega_{B^{reg}} \simeq \Omega_{B^{reg}}$ , and we get a map:

$$\Omega_{B^{reg}} \hookrightarrow \pi_* T_{X'/B^{reg}}.$$
 (2)

The sheaf  $T_{X/B}$  is the kernel of the map  $T_X \to \pi^* T_B / \text{Tors}(\pi^* T_B)$  (Definition 1.1.3). The kernel of a map of a reflexive sheaf to a torsion-free sheaf is a reflexive sheaf, hence  $T_{X/B}$  is reflexive. The pushforward of a reflexive sheaf along an equidimensional morphism is reflexive [Har80, Corollary 1.7], hence  $\pi_* T_{X/B}$  is reflexive as well. Therefore, the map (2) extends to a map

$$\iota_{\sigma} \colon \Omega_B^{[1]} \hookrightarrow \pi_* T_{X/B}.$$
 (3)

Here  $\Omega_B^{[1]}$  denotes the sheaf of reflexive differentials on B, i.e., the double dual of  $\Omega_B$ . Equivalently,  $\Omega_B^{[1]} := j_* \Omega_{B^{reg}}$ , where  $j : B^{reg} \hookrightarrow B$  is the embedding of the smooth locus of B into B. Similarly, we define  $\Omega_B^{[i]}$  as  $j_* \Omega_{B^{reg}}^i$ .

**2.1.9.** The map (3) turns out to be an isomorphism. We showed this fact in [AR21, Theorem 2.6] assuming that B is smooth and the fibration  $\pi$  has no multiple fibers in codimension one. These assumptions are not necessary, as we will see very soon. The proof relies on the following elementary lemma.

**Lemma 2.1.10.** Let  $\pi: Y \to S$  be a flat morphism of finite type of possibly non-compact complex manifolds. As before, denote by  $S^{\circ}$  the image of smooth fibers of  $\pi$  and by  $Y^{\circ}$  the preimage of  $S^{\circ}$  in Y. Let  $\Delta := S \setminus S^{\circ}$  be the discriminant locus of  $\pi$ . Suppose that  $\alpha$  is a holomorphic k-form on Y such that the restriction of  $\alpha$  to  $Y^{\circ}$  satisfies

$$\alpha|_{\mathbf{v}^{\circ}} = \pi^* \beta^{\circ}$$

for some holomorphic k-form  $\beta^{\circ}$  on  $S^{\circ}$ . Then the form  $\beta^{\circ}$  extends to a holomorphic k-form  $\beta$  on S and  $\alpha = \pi^*\beta$ .

*Proof.* Suppose that  $\alpha\Big|_{\pi^{-1}(S')} = \pi^*\beta'$  for some form  $\beta'$  on an open subset  $S' \subset S$  with complement of codimension at least two. Then we are done. Indeed, by Hartogs theorem  $\beta'$  extends to a holomorphic form  $\beta$  on S. The forms  $\pi^*\beta$  and  $\alpha$  coincide on an open subset, hence they coincide on Y. Therefore, it is enough to prove the statement for some  $S' \subset S$  as above.

If  $\operatorname{codim} \Delta \geq 2$ , then we are done, so let us assume that  $\operatorname{codim} \Delta = 1$ . Pick a general point  $b \in \Delta$ . Let U be a neighborhood of b. It is enough to show that  $\beta^{\circ}$  extends to a holomorphic form on U. The fibration  $\pi$  might not admit a local section in a neighborhood U of b; yet, for some finite cover  $f: V \to U$  ramified in  $\Delta \cap U$ , the base change morphism  $\pi_V: X_V \to V$  of  $\pi$  to V admits a section. Call this section  $s: V \to X_V$  and denote the map  $X_V \to X$  by F. We obtain the following diagram

$$X_{V} \xrightarrow{F} X_{U} \longleftrightarrow X$$

$$s \downarrow^{\pi_{V}} \qquad \downarrow^{\pi_{U}} \qquad \downarrow^{\pi}$$

$$V \xrightarrow{f} U \longleftrightarrow B$$

The following equality of forms on  $F^{-1}((X_U)^{\circ})$  holds:

$$F^*\alpha\Big|_{F^{-1}((X_U)^\circ)} = \pi_V^* f^*\beta^\circ.$$

It follows that the form  $s^*F^*\alpha$  coincides with  $f^*\beta^\circ$  on  $V^\circ$ . Therefore  $f^*\beta^\circ$  can be extended to a form  $\beta_V := s^*F^*\alpha$  on V. As we will see in a moment, this implies that  $\beta^\circ$  extends to a holomorphic form on U. Indeed, choose coordinates  $(t, z_1, \ldots z_{n-1})$  on U and  $(s, z_1, \ldots z_{n-1})$  on V such that  $\Delta \cap U = \{t = 0\}$  and the map f sends  $(s, z_1, \ldots z_{n-1})$  to  $(s^k, z_1, \ldots z_{n-1})$ . Write

$$\beta^{\circ} = hdt + \sum_{i=1}^{k-1} h_i dz_i$$

for some functions h and  $h_i$  on  $U^{\circ}$ . Then

$$f^*\beta^\circ = kh(s^k, z)s^{n-1}ds + \sum_{i=1}^{n-1} h_i dz_i.$$

The form  $f^*\beta^\circ$  extends to a holomorphic form on V. Hence the functions  $h_i$ 's extend to holomorphic functions on V. They are bounded on V, hence bounded on U. Therefore,  $h_i$ 's extend to holomorphic functions on U. The function  $h(s^k, z)s^{k-1}$  is also bounded, hence so is

$$h(t,z)t = h(s^k, z)s^k.$$

Therefore, h has at worst a simple pole at  $\Delta$ . But the form

$$f^* \frac{dt}{t} = k \frac{ds}{s},$$

is not holomorphic. Hence h is actually holomorphic on U. It follows that  $\beta^{\circ}$  extends to a holomorphic form on U.

**Theorem 2.1.11.** The map  $\iota_{\sigma} \colon \Omega_B^{[1]} \to \pi_* T_{X/B}$  is an isomorphism.

Proof. This map is definitely an isomorphism over  $B^{\circ}$  and is injective (2.1.8). It is enough to show that it is surjective. Let v be a vertical vector field over an open subset  $U \subset B$ . Then the form  $\iota_v \sigma$  equals  $\pi^* \beta^{\circ}$  for some holomorphic 1-form  $\beta^{\circ}$  on  $B^{\circ} \cap U$ . By Lemma 2.1.10, the form  $\beta^{\circ}$  extends to a holomorphic form  $\beta$  on  $U^{reg}$  and  $\iota_v \sigma \Big|_{\pi^{-1}(U^{reg})}$  coincides with  $\pi^* \beta$ . Hence the map  $\iota_\sigma$  sends the form  $\beta$ , considered as a section of  $\Omega_B^{[1]}$  over U, to v.

**2.1.12. Higher pushforwards of**  $\mathcal{O}_X$  When the base B of a Lagrangian fibration is smooth, the higher pushforward sheaves  $R^i\pi_*\mathcal{O}_X$  are locally free [Mat05]. Without the smoothness assumption one can show that the sheaves  $R^i\pi_*\mathcal{O}_X$  are reflexive for all  $i \geq 0$  [Ou19, Proposition 3.6]. Let  $\omega$  a Kähler form on X. Consider the composition of maps

$$\Omega_B^{[1]} \xrightarrow{\iota_\sigma} \pi_* T_{X/B} \xrightarrow{f_\omega} R^1 \pi_* \Theta_X.$$

Here  $f_{\omega}$  sends a vertical vector field v to the class  $[\iota_v \omega_{\bar{\partial}}]$  of the  $\bar{\partial}$ -closed (0,1)-form  $\iota_v \omega$  under the  $\bar{\partial}$ -differential

**Theorem 2.1.13** ([Ou19],[Mat05]). Let  $\pi: X \to B$  be a Lagrangian fibration on a projective manifold. Then the map  $\Omega_B^{[1]} \to R^1\pi_*\mathcal{O}_X$  and the induced maps  $\Omega_B^{[i]} \to R^i\pi_*\mathcal{O}_X$  are isomorphisms.

Corollary 2.1.14. Let  $\pi\colon X\to B$  be a Lagrangian fibration on an irreducible hyperkähler manifold, not necessarily projective. Then Theorem 2.1.13 holds for any Shafarevich-Tate twist  $X^\varphi$  of X, in particular for X itself, i.e.,

$$R^i \pi^{\varphi}_* \mathcal{O}_{X^{\varphi}} \simeq \Omega_B^{[i]}$$

*Proof.* By [Huy99, Theorem 3.5] any non-trivial family of deformations of an irreducible hyperkähler manifolds contains a projective deformation. Therefore there exists a projective Shafarevich–Tate deformation  $\pi^{\psi} \colon X^{\psi} \to B$  of the Lagrangian fibration  $\pi \colon X \to B$ . It follows from Theorem 2.1.13 that

$$R^i \pi^{\psi}_* \mathcal{O}_{X^{\psi}} \simeq \Omega_B^{[i]}.$$

The sheaf of groups  $Aut_{X/B}^0$  acts trivially on  $R^i\pi_*\mathcal{O}_X$ . Indeed, the restriction of  $R^i\pi_*\mathcal{O}_X$  to  $B^\circ$  is a vector bundle with fibers  $H^{0,i}(F)$ . Automorphisms in  $Aut_{X/B}^0$  act trivially on  $H^{0,i}(F)$  for any smooth fiber F. Thus the action of  $Aut_{X/B}^0$  on  $R^i\pi_*\mathcal{O}_X$  is trivial over  $B^\circ$ . The sheaf  $R^i\pi_*\mathcal{O}_X$  is torsion-free, hence the action of  $Aut_{X/B}^0$  is trivial everywhere.

We obtain that for any  $\varphi \in \coprod$ 

$$R^i \pi_*^{\varphi} \mathcal{O}_{X^{\varphi}} \simeq R^i \pi_*^{\psi} \mathcal{O}_{X^{\psi}} \simeq \Omega_B^{[i]}.$$

**Remark 2.1.15.** It follows from Corollary 2.1.14 that the sheaves  $R^i\pi_*\mathcal{O}_X$  are locally free on  $B^{reg}$ . The base change theorem [Mum12, Chapter 5, Corollary 2&3] implies that for all points  $b \in B^{reg}$  the dimension of  $H^i(\mathcal{O}_{\pi^{-1}(b)})$  does not depend on b. In particular,  $h^0(\mathcal{O}_{\pi^{-1}(b)}) = 1$  for every  $b \in B^{reg}$ .

**Theorem 2.1.16.** Let B be the base of a Lagrangian fibration on an irreducible hyperkähler manifold X. Then the cohomology groups  $H^j(B, \Omega_B^{[i]})$  are the same as for  $B = \mathbb{P}^n$ .

*Proof.* Step 1. By Corollary 2.1.14,  $H^j(\Omega_B^{[i]}) \simeq H^j(R^i\pi_*\Theta_X)$ . It follows from a result by Kollár [Kol86, p.172] that

$$R\pi_*\mathcal{O}_X \simeq \bigoplus R^i\pi_*\mathcal{O}_X[-i].$$

Therefore the Leray spectral sequence for  $\mathcal{O}_X$  degenerates on  $E^2$  and

$$h^{0,k}(X) = \sum_{i=0}^{k} h^{k-i}(R^i \pi_* \mathcal{O}_X).$$

When k is odd  $h^{0,k} = 0$  and when k is even  $h^{0,k} = 1$ . We see immediately that  $H^j(R^i\pi_*\mathcal{O}_X) = 0$  when i+j is odd. When k is even, there is exactly one  $i \leq k$  such that  $H^{k-i}(R^i\pi_*\mathcal{O}_X)$  is non-zero.

Step 2. We will show that  $H^i(R^i\pi_*\mathcal{O}_X)$  does not vanish. This will complete the proof. Consider the filtration  $F^iH^{0,k}(X)$  on  $H^{0,k}(X)$  induced by the Leray spectral sequence. First, consider the case k=2. The cohomology group  $H^{0,2}(X)$  is generated by  $\overline{\sigma}$ . The restriction of  $\overline{\sigma}$  to a smooth fiber is zero, hence the image of  $\overline{\sigma}$  in  $H^0(B, R^2\pi_*\mathcal{O}_X)$  vanishes. The form  $\overline{\sigma}$  is non-degenerate, hence not the pullback of a (0,2)-form on the base even locally. Therefore  $F^0H^{0,2}(X)=0$  and  $F^1H^{0,2}(X)=F^2H^{0,2}(X)=H^{0,2}(X)$ .

It follows that  $\overline{\sigma}^i \in F^iH^{0,2i}(X)$  for all i. Suppose that we know that  $\overline{\sigma}^i \notin F^{i-1}H^{0,2i}(X)$ . Then  $H^i(R^i\pi_*\mathcal{O}_X) = F^iH^{0,2i}(X)/F^{i-1}H^{0,2i}(X)$  is non-zero, and we are done. If  $\overline{\sigma}^i$  happens to be contained in  $F^{i-1}H^{0,2i}(X)$ , then  $\overline{\sigma}^n$  is contained in  $F^{n-1}H^{0,2n}(X)$ . However,  $F^{n-1}H^{0,2n}(X)$  vanishes for dimension reasons. Indeed,  $H^{n+k}(R^{n-k}\pi_*\mathcal{O}_X) = 0$  for k > 0. Hence  $\overline{\sigma}^n = 0$ , contradiction.

**Remark 2.1.17.** A base of a Lagrangian fibration behaves like  $\mathbb{P}^n$  from many points of view (conjecturally because it is always  $\mathbb{P}^n$ ). We encourage an interested reader to look into the wonderful survey [HM22] for details.

# 2.2 Shafarevich-Tate group

**2.2.1. Structure of the Shafarevich–Tate group.** Recall that the sheaf of groups  $Aut_{X/B}^0$  is defined as the image of the exponential map  $\pi_*T_{X/B} \to Aut_{X/B}$  (Definition 1.1.4). Define  $\Gamma$  to be the kernel of this map. The short exact sequence

$$0 \to \Gamma \to \pi_* T_{X/B} \to Aut^0_{X/B} \to 0$$

induces the long exact sequence of cohomology groups:

$$H^1(\Gamma) \to H^1(\pi_* T_{X/B}) \to \coprod \to H^2(\Gamma).$$
 (4)

We will call the image of  $H^1(\pi_*T_{X/B})$  in III the connected component of unity of III and will denote it by III<sup>0</sup>. The quotient III/III<sup>0</sup> is the discrete part of III.

The sequence (4) is exact on the right. Indeed, the cohomology group  $H^2(\pi_*T_{X/B})$  is isomorphic to  $H^2(\Omega_B^{[1]})$  by Theorem 2.1.11. By Theorem 2.1.16 this cohomology group vanishes. Similarly, the vector space  $H^1(\pi_*T_{X/B})$  is isomorphic to  $H^1(B,\Omega_B^{[1]})$  and is one-dimensional.

**2.2.2. Degenerate twistor deformations.** There is a useful differential geometric point of view on Shafarevich–Tate deformations [AR21, Subsection 2.3]. Let  $\sigma$  be a holomorphic symplectic form on X and  $\alpha$  be a closed (1,1)-form on B. The form  $\sigma + t\pi^*\alpha$  is obviously not holomorphic, but it turns out that there exists a *different* complex structure  $I_t$  on X making  $\sigma + t\pi^*\alpha$  holomorphic symplectic [SV24, Section 2.2]. Moreover, such a complex structure is unique.

**Definition 2.2.3.** Denote by  $X_t$  the manifold X with the new complex structure  $I_t$ . It is called a degenerate twistor deformation of X.

It is not hard to see that the fibration  $\pi\colon X_t\to B$  is holomorphic and Lagrangian with respect to the new complex structure.

Degenerate twistor deformations form a family

$$\Pi: \mathcal{X} \to \mathbb{A}^1$$
.

and the fiber of  $\Pi$  over  $t \in \mathbb{A}^1$  is isomorphic to the degenerate twistor deformation  $X_t$ .

**Definition 2.2.4** ([AR21, Definition 2.18, Definition 3.4]). The family  $\Pi: \mathcal{X} \to \mathbb{A}^1$  is called the *degenerate twistor family* or the *Shafarevich-Tate family*.

We will see in Theorem 2.2.10 that all degenerate twistor deformations are Shafarevich–Tate deformations (Definition 1.1.6). That justifies the use of the term Shafarevich–Tate family.

**2.2.5.** The connected component of unity of III. The isomorphism  $f_{\omega} \colon \pi_* T_{X/B} \to R^1 \pi_* \mathcal{O}_X$  from 2.1.12 sends the subsheaf  $\Gamma \subset \pi_* T_{X/B}$  into  $R^1 \pi_* \mathbb{Q}$  [AR21, Proposition 4.4]. In the same paper we showed that the sheaf  $\Gamma_{\mathbb{Q}} := \Gamma \otimes \mathbb{Q}$  is isomorphic to  $R^1 \pi_* \mathbb{Q}$ . The exact sequence (4) implies that

$$\coprod^{0} = H^{1}(B, \pi_{*}T_{X/B}) / \operatorname{im} H^{1}(B, \Gamma).$$

The isomorphism  $f_{\omega} \colon \pi_* T_{X/B} \to R^1 \pi_* \mathcal{O}_X$  identifies  $\coprod^0$  with a quotient of

$$H^{1}(B, R^{1}\pi_{*}\mathcal{O}_{X})/\operatorname{im} H^{1}(R^{1}\pi_{*}\mathbb{Z})$$
 (5)

by a finite subgroup. In Theorem 2.2.7 we will describe  $\mathrm{III}^0$  in terms of cohomology of X. First, let us introduce some notation. Let  $W_{\mathbb{Z}} \subset H^2(X,\mathbb{Z})$  be the subgroup of cohomology classes on X that restrict trivially to all fibers. By [Mat99]  $\mathrm{Pic}(B)$  has rank one. Denote by  $\eta$  the class of the pullback of the ample generator of  $\mathrm{Pic}(B)/\mathrm{Tors}(\mathrm{Pic}(B))$  to X.

**Definition 2.2.6.** Let  $G_i$ , i=1,2 be two abelian groups of the form  $G_i=\mathbb{C}^k/\Lambda_i$ , where  $\Lambda_i$  is a finitely generated subgroups of  $\mathbb{C}^k$ . We will call  $G_1$  and  $G_2$  isogenous if the subgroup  $\Lambda_1 \cap \Lambda_2$  is of finite index in both  $\Lambda_1$  and  $\Lambda_2$ . Equivalently, the subspace  $\Lambda_1 \otimes \mathbb{Q} \subset \mathbb{C}^k$  coincides with  $\Lambda_2 \otimes \mathbb{Q}$ 

**Theorem 2.2.7.** Let  $\pi: X \to B$  be a Lagrangian fibration on an irreducible hyperkähler manifold X. Then the group  $\mathrm{III}^0$  is isogenous to

$$H^{0,2}(X)/p(H^2(X,\mathbb{Z})),$$

where  $p: H^2(X, \mathbb{Z}) \to H^{0,2}(X)$  is the Hodge projection.

*Proof.* By [AR21, Proposition 4.7], the Leray spectral sequence induces the following isomorphisms:

$$H^{1}(B, R^{1}\pi_{*}\mathcal{O}_{X}) \simeq H^{0,2}(X), \text{ and } H^{1}(B, R^{1}\pi_{*}\mathbb{Z}) = W_{\mathbb{Z}}/\eta.$$

It follows from (5) that  $\coprod^0$  is isogenous to

$$H^{0,2}(X)/p(W_{\mathbb{Z}}).$$

For every ring  $\mathcal{R}$  define  $W_{\mathcal{R}} := W_{\mathbb{Z}} \otimes \mathcal{R}$ . It is enough to show that  $p(W_{\mathbb{Q}}) = p(H^2(X, \mathbb{Q}))$ . The inclusion  $p(W_{\mathbb{Q}}) \subset p(H^2(X, \mathbb{Q}))$  is clear. For the opposite inclusion, note that  $W_{\mathbb{C}}$  contains  $\sigma$  and  $\overline{\sigma}$  [AR21, Lemma 3.5]. Therefore  $(W_{\mathbb{Q}})^{\perp}$  is contained in  $H^{1,1}(X)$ . It is a rational subspace, hence  $(W_{\mathbb{Q}})^{\perp} \subset NS_{\mathbb{Q}}(X)$ . It follows that

$$T_{\mathbb{Q}}(X) := NS_{\mathbb{Q}}(X)^{\perp} \subset W_{\mathbb{Q}}.$$

The image of  $T_{\mathbb{Q}}(X)$  under the Hodge projection coincides with the image of  $H^2(X,\mathbb{Q})$ . Indeed, the kernel of  $p \colon H^2(X,\mathbb{Q}) \to H^{0,2}(X)$  is  $NS_{\mathbb{Q}}(X)$ . Therefore,

$$p(H^2(X,\mathbb{Q})) = p(T_{\mathbb{Q}}(X)) \subset p(W_{\mathbb{Q}}),$$

and we are done.  $\Box$ 

As an immediate corollary we obtain:

Corollary 2.2.8. The set of torsion elements of  $\coprod^0$  is dense in  $\coprod^0$ .

*Proof.* By Theorem 2.2.7 it is enough to prove the same statement for the group  $H^{0,2}(X)/p(H^2(X,\mathbb{Z}))$ . The subgroup of torsion elements of this group is  $p(H^2(X,\mathbb{Q}))/p(H^2(X,\mathbb{Z}))$ . The projection  $H^2(X,\mathbb{R}) \to H^{0,2}(X)$  is surjective and  $H^2(X,\mathbb{Q})$  is dense in  $H^2(X,\mathbb{R})$ , hence the claim.

2.2.9. Degenerate twistor deformations are Shafarevich—Tate twists. By Theorem 2.1.11 and Corollary 2.1.14 the following one-dimensional vector spaces are isomorphic

$$H^1(\pi_* T_{X/B}) \simeq H^1(\Omega_B^{[1]}) \simeq H^{1,1}(R^1 \pi_* \mathcal{O}_X) \simeq H^{0,2}(X) \simeq \mathbb{C}.$$
 (6)

Let  $\sigma$  be a holomorphic symplectic form on X. Pick a d-closed (1,1)-form  $\alpha$  on B, whose class in  $H^1(\Omega_B^{[1]})$  is non-trivial. We may and will choose the isomorphisms (6) in such a way that  $[\alpha] \in H^1(\Omega_B^{[1]})$  is identified with  $\overline{\sigma} \in H^{0,2}(X)$ , which is identified with  $1 \in \mathbb{C}$ .

**Theorem 2.2.10.** Let  $\pi: X \to B$  be a Lagrangian fibration on a hyperkähler manifold. For every  $t \in H^1(\pi_*T_{X/B}) \simeq \mathbb{C}$  consider its image  $\varphi_t \in \coprod$  by the map (4). Then the degenerate twistor deformation  $X_t$  is isomorphic to the Shafarevich-Tate twist  $X^{\varphi_t}$  of X by  $\varphi_t$ . This isomorphism preserves the Lagrangian fibrations.

*Proof.* In [AR21, Theorem 3.9 = Theorem A] this result was proven under the additional assumptions that B is smooth and  $\pi$  has no multiple fibers in codimension one. The proof actually does not use these assumptions. The reader can mentally replace  $\Omega_B^1$  in the proof of [AR21, Theorem 3.9] with  $\Omega_B^{[1]}$  and keep in mind that thanks to Theorem 2.1.16

$$H^1(\Omega_B^{[1]}) \simeq H^1(\pi_* T_{X/B}) \simeq \mathbb{C}$$

regardless of whether the base is smooth or fibers in codimension one are non-multiple.  $\Box$ 

**2.2.11. The discrete part of** III. The isomorphism  $\Gamma_{\mathbb{Q}} \simeq R^1 \pi_* \mathbb{Q}$  gives an easy description of the discrete part of III. By the exact sequence (4), the discrete part III/III<sup>0</sup> satisfies

$$(\mathrm{III}/\mathrm{III}^0)\otimes \mathbb{Q}\simeq H^2(\Gamma_{\mathbb{Q}})\simeq H^2(R^1\pi_*\mathbb{Q}).$$

# 3 Projective twists

The goal of this section is to prove the second part of Theorem A. It will follow from the statement below:

**Theorem 3.0.1.** Let  $\pi: X \to B$  be a Lagrangian fibration on a holomorphic symplectic manifold, and  $\varphi \in \coprod$  a torsion element. Then there is a natural isomorphism

$$NS_{\mathbb{O}}(X)/\eta \to NS_{\mathbb{O}}(X^{\varphi})/\eta,$$
 (7)

where  $\eta$  is the pullback of an ample class on B. Moreover, the isomorphism (7) sends

- classes on X with cohomologically trivial restriction to smooth fibers to classes with cohomologically trivial restriction to smooth fibers;
- relatively ample classes to relatively ample classes.

**Lemma 3.0.2.** Let  $\varphi$  be an r-torsion element in III. Cover B by small open subsets  $U_i$  and represent  $\varphi$  by a Čech cocycle  $(\varphi_{ij})$ ,  $\varphi_{ij} \in Aut^0_{X/B}(U_{ij})$ . Then we can choose  $\varphi_{ij}$  in such a way that  $r\varphi_{ij}$  is the identity automorphism of  $X_{ij}$  for each i, j.

*Proof.* Since the class of  $r\varphi$  is trivial in III, we can find automorphisms  $\beta_i \in Aut^0_{X/B}(U_i)$  such that

$$r\varphi_{ij} = \beta_i - \beta_i$$
.

There exist automorphisms  $\gamma_i$  such that  $r\gamma_i = \beta_i$ . Indeed, we can write  $\beta_i = \exp(v_i)$  for some vertical vector field  $v_i$ . The automorphism  $\gamma_i := \exp(v_i/r)$  will do the job. Replace  $\varphi_{ij}$  with  $\varphi_{ij} + \gamma_i - \gamma_j$ . The new set of automorphisms satisfies the condition of the lemma.

**3.0.3. Gluing a line bundle.** The proof of Theorem 3.0.1 relies on the following idea. Pick a line bundle L on X and cover B by open disks  $U_i$ . Let  $L_i$  denote the restriction of  $L_i$  to  $X_i$ . We will see that for some  $s \in \mathbb{Z}_{>0}$ , the line bundles  $L_i^s$  can be glued into a line bundle on  $X^{\varphi}$ . This result will eventually follow from the lemma below.

**Lemma 3.0.4.** Let L be a line bundle on an abelian variety A and t an r-torsion element of A. Then

$$t^*L^r \simeq L^r$$
.

*Proof.* Consider the morphism  $\varphi_L \colon A \to A^{\vee}$  sending x to  $x^*L \otimes L^{-1}$ . The map  $\varphi_L$  is a homomorphism because any morphism of abelian varieties sending zero to zero is a homomorphism [Mum12, Section 4, Corollary 1]. Therefore  $\varphi_L(t)$  is an r-torsion line bundle, i.e.,

$$(\varphi_L(t))^r = t^*L^r \otimes L^{-r} \simeq \mathcal{O}_A.$$

**Lemma 3.0.5.** Let  $\pi: Y \to S$  be a proper flat morphism between varieties such that  $h^0(\Theta_{Y_b}) = 1$  for all  $b \in S$  outside a codimension at least two subset of S. Consider a line bundle M on Y with the following properties:

- 1. the restriction of M to any smooth fiber is trivial;
- 2. the restriction of M to any fiber  $Y_b$  lies in  $Pic^0(Y_b)$ . Here  $Pic^0(Y_b)$  is the connected component of unity of  $Pic(Y_b)$ .

Then some positive multiple  $M^s$  of M for  $s \in \mathbb{Z}_{>0}$  is isomorphic to the pullback of a line bundle from S.

*Proof.* Step 1. It is enough to show this statement for some  $S' \subset S$  with complement of codimension at least two. Indeed, suppose that  $M^s|_{\pi^{-1}(S')}$  is isomorphic to  $\pi^*K'$  for a line bundle K' on S'. We can extend K' to a line bundle K on S. The line bundles  $\pi^*K$  and  $M^s$  are isomorphic outside a codimension at least two subset of Y, hence they are isomorphic.

Step 2. Denote by  $\Delta$  the discriminant locus of  $\pi$ . If  $\operatorname{codim} \Delta \geq 2$ , then we are done thanks to Step 1. So we may assume  $\operatorname{codim} \Delta = 1$ . Consider the group  $E_b \subset \operatorname{Pic}(Y_b)$  of line bundles L on  $Y_b$  with the following property: there exists a line bundle  $\tilde{L}$  on Y which is trivial on smooth fibers and restricts

to L on  $Y_b$ . By Raynaud's theorem [Ray70, Introduction],  $E_b$  has dimension  $h^0(\mathcal{O}_{Y_b}) - 1$  for a general point  $b \in \Delta$ . The assumption that  $h^0(\mathcal{O}_{Y_b}) = 1$  for a general point  $b \in \Delta$  implies that  $E_b$  is discrete for any fiber of  $\pi$  over a general point  $b \in \Delta$ . The line bundle  $M_b := M\Big|_{Y_b}$  is in  $E_b$  by the first property. By the second assumption,  $M_b \in Pic^0(Y_b)$ . Consider the group  $\langle M_b \rangle$  generated by  $M_b$  inside  $Pic^0(Y_b)$ . It is contained inside  $E_b$ , hence is discrete. Since the group space  $Pic^0(Y_b)$  is of finite type [FGI+05, Proposition 9.5.3], the group  $\langle M_b \rangle$  is of finite type as well. Hence  $\langle M_b \rangle$  is finite, in other words,  $M_b$  is torsion. Therefore, some power  $M^s$  of M restricts trivially to all fibers over  $S' \subset S$  with complement of codimension at least two. Define a line bundle  $K' := \pi_* M^s \Big|_{\pi^{-1}(S')}$ . The natural map  $\pi^* K' \to M^s \Big|_{\pi^{-1}(S')}$  is an isomorphism.

**3.0.6.** We are now ready to prove of Theorem 3.0.1.

Proof of Theorem A (2). Step 1. Pick a line bundle L on X. As before, choose a Čech cocycle  $(\varphi_{ij})$  with  $r\varphi_{ij} = 0$  representing an r-torsion class  $\varphi \in \coprod$ . We will construct an isomorphism:

$$f_{ij} \colon \varphi_{ij}^* L^s \Big|_{\pi^{-1}(U_{ij})} \to L^s \Big|_{\pi^{-1}(U_{ij})}.$$

for some  $s \in \mathbb{Z}_{>0}$ . The line bundle  $\varphi_{ij}^*L_j^r \otimes L_i^{-r}$  on  $X_{ij}$  restricts trivially to smooth fibers by Lemma 3.0.4. Moreover, it satisfies the second condition of Lemma 3.0.5 because  $\varphi_{ij} \in Aut_{X/B}^0$ . For every  $b \in B^{reg}$  the fibers  $\pi^{-1}(b)$  satisfy  $h^0(\mathcal{O}_{\pi^{-1}(b)}) = 1$ , see 2.1.12. Hence the restriction of some multiple of  $\varphi_{ij}^*L^r \otimes L^{-r}$  to  $\pi^{-1}(B^{reg} \cap U_{ij})$  is the pullback of a line bundle on  $B^{ref} \cap U_{ij}$  (Lemma 3.0.5). Since the codimension of the complement of  $B^{reg}$  in B is at least two, some multiple of  $\varphi_{ij}^*L^r \otimes L^{-r}$  is actually the pullback of a line bundle on  $U_{ij}$ . When the subsets  $U_i$ 's are sufficiently small, all line bundles on  $U_{ij}$ 's are trivial. Therefore the sheaves  $\varphi_{ij}^*L^s\Big|_{p_i=1(U_{ij})}$  and  $L^s\Big|_{\pi^{-1}(U_{ij})}$  are isomorphic.

**Step 2.** The isomorphisms  $f_{ij}$  might not a priori satisfy the cocycle condition. In other words, the following map

$$f_{ij}^{-1} \circ \varphi_{ij}^* f_{jk}^{-1} \circ f_{ik}$$

is some automorphism of  $L\Big|_{U_{ijk}}$ , which might not be trivial. Denote it by  $\lambda_{ijk}$ . The automorphism  $\lambda_{ijk}$  is a multiplication by a non-zero holomorphic function on  $X_{ijk}$ , which must be the pullback of a function on the base. Therefore the automorphisms  $\lambda_{ijk}$  define a Čech 2-cocycle on B with coefficients in  $\mathcal{O}_B^{\times}$ .

Consider the following chunk of the long exact sequence of cohomology of the exponential exact sequence on B:

$$H^2(B, \mathcal{O}_B) \to H^2(B, \mathcal{O}_B^{\times}) \to H^3(B, \mathbb{Z}) \to H^3(B, \mathcal{O}_B).$$

By Theorem 2.1.16, the cohomology groups  $H^2(B, \mathcal{O}_B)$  and  $H^3(B, \mathcal{O}_B)$  vanish. Hence  $H^2(B, \mathcal{O}_B^{\times}) \simeq H^3(B, \mathbb{Z})$ . The cohomology groups  $H^i(B, \mathbb{Q})$  are the same as for  $\mathbb{P}^n$  [SY22, Theorem 0.2], in particular  $H^3(B, \mathbb{Z})$  is torsion. Hence some power, say s', of the cocycle  $(\lambda_{ijk})$  vanishes. Replace the line bundle  $L^s$  with  $L^{ss'}$  and the isomorphisms  $f_{ij}$  with  $f_{ij}^{\otimes s'}$ . Then  $\lambda_{ijk}$  gets replaced with  $\lambda_{ijk}^{s'}$ , which is a coboundary. Write  $\lambda_{ijk}^{s'} = \mu_{ij}\mu_{jk}\mu_{ki}$  for some  $\mu_{ij} \in \mathcal{O}_B^{\times}(U_{ij})$ . Then the isomorphisms  $(\mu_{ij}^{-1} \cdot f_{ij})$  satisfy the cocycle condition. It follows that we can glue the line bundles  $L^s|_{\pi^{-1}(U_{ij})}$  into a global line bundle  $L^{\varphi}$  on  $X^{\varphi}$ .

The line bundle  $L^{\varphi}$  depends only on the choice of  $\mu_{ij} \in \mathcal{O}_B^{\times}(U_{ij})$ . Different choices of  $\mu_{ij}$  differ by a 1-cocycle with coefficients in  $\mathcal{O}_B^{\times}(U_{ij})$ . Therefore,  $L^{\varphi}$  is well-defined up to the pullback of a line bundle on B. We construct a map

$$NS_{\mathbb{O}}(X)/\eta \to NS_{\mathbb{O}}(X^{\varphi})/\eta$$

by sending the class of L in  $NS_{\mathbb{Q}}(X)/\eta$  to the class  $[L^{\varphi}]/(ss') \in NS_{\mathbb{Q}}(X^{\varphi})/\eta$ .

**Step 3.** The restriction of  $L^{\varphi}$  to  $X_i^{\varphi}$  coincides with a power of  $L_i$ . Therefore the class of  $L^{\varphi}$  in  $NS_{\mathbb{Q}}(X)$  has trivial restriction to smooth fibers if and only if this is true for L, and  $L^{\varphi}$  is relatively ample if and only if so is L.

Instead of proving Theorem A directly, we will show a more general statement.

**Theorem 3.0.7.** Let  $\pi: X \to B$  be a Lagrangian fibration on a projective hyperkähler manifold, and  $\varphi \in \text{III'}$ . Then the following are equivalent:

- 1.  $\varphi$  is torsion;
- 2.  $X^{\varphi}$  is projective;
- 3. there is a class  $\alpha \in H^{1,1}(X^{\varphi}, \mathbb{Q})$  such that  $q(\alpha, \eta) \neq 0$ .

*Proof.* (1)  $\Rightarrow$  (2). By Theorem 3.0.1 there is a relatively ample class on  $X^{\varphi}$ . Hence  $X^{\varphi}$  is projective.

- (2)  $\Rightarrow$  (3). An ample class  $\alpha$  on  $X^{\varphi}$  will do the job.
- (3)  $\Rightarrow$  (1). We can find a torsion element  $\psi \in \text{III}$  such that  $\varphi \psi$  is arbitrarily close to 0. In particular, we may assume that  $X^{\varphi-\psi}$  is Kähler. A cohomology class  $\alpha$  has non-zero intersection with  $\eta$  if and only if the restriction of  $\alpha$  to a smooth fiber is non-trivial (Theorem 2.1.6). By Theorem 3.0.1 the manifold  $X^{\varphi-\psi}$  carries a rational (1,1)-class  $\alpha'$  such that  $q(\alpha',\eta) \neq 0$  as well. By [AR21, Lemma 5.15], a hyperkähler manifold with a Lagrangian fibration admitting a (1,1)-class not orthogonal to  $\eta$  is projective. Hence  $X^{\varphi-\psi}$  is projective. By the implication (1)  $\Rightarrow$  (2), the manifold  $X^{\varphi}$  is projective as well.

Corollary 3.0.8. Let  $\pi: X \to B$  be a Lagrangian fibration on a hyperkähler manifold. As before, denote by  $\coprod'$  the subset of  $\varphi \in \coprod'$ , s.t.,  $N\varphi \in \coprod^0$  for some  $N \in \mathbb{Z}_{>0}$ . Then the set of  $\varphi \in \coprod'$  such that  $X^{\varphi}$  is Kähler is open and dense in  $\coprod'$ .

*Proof.* First, it is enough to prove this corollary for a projective X. Indeed,  $X^{\psi}$  is projective for some  $\psi \in \mathrm{III^0}$  by the same argument as the one used in the proof of Corollary 2.1.14. If we manage to prove Corollary 3.0.8 for  $X^{\psi}$ , then the same result for X will follow because every Shafarevich-Tate twist of  $X^{\psi}$  is a Shafarevich-Tate twist of X.

Let us assume that X is projective. Twists  $X^{\varphi}$  of X with respect to torsion elements  $\varphi \in \coprod'$  are projective (Theorem 3.0.7). Moreover, the set of torsion elements is dense in  $\coprod'$  (Corollary 2.2.8). Hence the set of Kähler twists with respect to  $\varphi \in \coprod'$  is dense in  $\coprod'$ . Kählerness is open in a space of deformations, therefore, this set is also open.

#### 4 Kähler twists

As we showed in Corollary 3.0.8, all twist  $X^{\varphi}$  with respect to  $\varphi \in \operatorname{III'}$  are Kähler except maybe for a nowhere dense subset of  $\operatorname{III'}$ . In this section we will show that  $X^{\varphi}$  is actually Kähler for all  $\varphi \in \operatorname{III'}$ , and thus we prove Theorem A(1). Note that Theorem A(1) will immediately follow from the statement below by applying it to  $X^{\varphi}$  for some  $\varphi \in \operatorname{III'}$ .

**Proposition 4.0.1.** Let  $\pi\colon X\to B$  be a Lagrangian fibration on an irreducible holomorphic symplectic manifold. Consider the restriction  $\mathcal{X}\to\mathbb{D}$  of its Shafarevich-Tate family to a disk  $\mathbb{D}\subset\mathbb{A}^1$ . Suppose that the set  $U\subset\mathbb{D}$  parametrizing Kähler Shafarevich-Tate deformations of X is non-empty and  $0\in\overline{U}$ . Then X is hyperkähler.

#### 4.1 Limits of hyperkähler manifolds

It follows from Corollary 3.0.8 that every Shafarevich-Tate twist  $X^{\varphi}$  with respect to  $\varphi \in \text{III}'$  is a *limit* of hyperkähler manifolds in the sense of the following definition.

**Definition 4.1.1.** Let X be a compact complex manifold. Consider a family of deformations  $\mathcal{X} \to T$  of X, and let  $0 \in T$  be the point corresponding to X. The manifold X is said to be a *limit of Kähler manifolds* if for some family of deformations  $\mathcal{X} \to T$  there is a sequence of points  $t_n \in T$  converging to 0 such that the deformation  $X_{t_n}$  is a Kähler manifold.

A limit of Kähler manifolds does not have to be Kähler, however the following is expected to be true.

Conjecture 1. [Pop11] A limit of Kähler manifolds is of Fujiki class C, i.e., is bimeromorphic to a Kähler manifold.

Arvid Perego in [Per17] showed that this conjecture holds for holomorphic symplectic manifolds with some additional assumptions.

**Theorem 4.1.2** ([Per17, Theorem 1.14]). Let  $(X, \sigma)$  be a compact holomorphic symplectic  $b_2$ -manifold, i.e.,

$$h^{2,0}(X) + h^{1,1}(X) + h^{0,2}(X) = b_2(X).$$

with a holomorphic symplectic form  $\sigma$ , which is a limit of irreducible hyperkähler manifolds. Then X is bimeromorphic to an irreducible hyperkähler manifold, in particular, it is of Fujiki class C.

We will use some of Perego's ideas in the proof of Theorem A(1).

#### 4.2 Idea of the proof

Before we get started with the proof of Proposition 4.0.1, we will sketch its main steps below.

Step 1. Period map and Torelli theorems. (Subsection 4.3). Using Local and Global Torelli Theorems (Theorem 4.3.4), we construct a family

$$\mathcal{U} \to \mathbb{D}$$

such that  $Y_t$  is hyperkähler for all  $t \in \mathbb{D}$  and  $X_t$  is bimeromorphic to  $Y_t$  for all  $t \in U \subset \mathbb{D}$  (Lemma 4.3.6).

Step 2. Lagrangian fibration on  $Y_t$ . (Subsection 4.4). Let t be a very general point in U. We will show in Corollary 4.4.3 that  $Y_t$  admits a Lagrangian fibration  $p_t \colon Y_t \to B'$ , and every bimeromorphism  $f_t \colon X_t \dashrightarrow Y_t$  commutes with the Lagrangian fibrations on  $X_t$  and  $Y_t$ . This step relies on a result by Greb-Lehn-Rollenske [GLR13]. Namely, they proved that a non-projective hyperkähler manifold containing a Lagrangian torus admits a Lagrangian fibration.

Step 3.  $\mathcal{Y} \to \mathbb{D}$  is almost a Shafarevich-Tate family. (Subsection 4.5). We will see in Proposition 4.5.1 that the family  $\mathcal{Y} \to \mathbb{D}$  is a Shafarevich-Tate family after restriction to some open dense subset  $V \subset U$ . Moreover, it will turn out that the base B' of the Lagrangian fibration  $p_t \colon Y_t \to B'$  for  $t \in V$  is isomorphic to B (Proposition 4.5.2).

Step 4.  $Y_0$  is bimeromorphic to a degenerate twistor deformation of  $Y_t$ . (Subsection 4.6). Let  $\mathcal{Y}' \to \mathbb{D}$  be the Shafarevich-Tate family of a Lagrangian fibration  $p_\tau \colon Y_\tau \to B$  for some  $\tau \in V$ . By the previous step,  $Y_t \simeq Y_t'$  for all  $t \in V$ . Essentially the same argument as the one used by Perego in his proof of [Per17, Lemma 2.6] will show that  $Y := Y_0$  is bimeromorphic to  $Y' := Y_0'$  (Lemma 4.6.1). Therefore, Y' is of Fujiki class  $\mathcal{C}$ .

Step 5. Shafarevich—Tate deformations of bimeromorphic Lagrangian fibrations are bimeromorphic. (Subsection 4.7). We saw in Step 2 that the Lagrangian fibrations  $X_t$  and  $Y_t = Y_t'$  are bimeromorphic for some  $t \in V$ . We will see in 4.7.1 that all Shafarevich—Tate deformations of  $X_t$  and  $Y_t'$  are bimeromorphic. Therefore, X is bimeromorphic to Y', which is in its turn bimeromorphic to a hyperkähler manifold Y (Corollary 4.7.3). Hence X is of Fujiki class C.

Step 6. Criterion for Kählerness. (Subsection 4.8). Perego discovered in [Per17, Theorem 1.15] a cohomological criterion for Kählerness of limits of hyperkähler manifolds which are of Fujiki class  $\mathcal{C}$ . We will check that the assumptions of Perego's criterion are satisfied for Shafarevich-Tate twists and will conclude that X is hyperkähler (Proposition 4.8.3).

#### 4.3 Period map and Torelli theorems

**4.3.1. Period map for hyperkähler manifolds.** Let X be a hyperkähler manifold and  $\Lambda$  be a lattice isomorphic to the lattice  $(H^2(X,\mathbb{Z}),q_X)$ , where  $q_X$  is the BBF form (Definition 2.1.4). Denote  $\Lambda_{\mathbb{C}} := \Lambda \otimes \mathbb{C}$ .

**Definition 4.3.2.** The moduli space  $\mathcal{M}_{\Lambda}$  of  $\Lambda$ -marked hyperkähler manifolds is the moduli space of pairs (Y,g) where Y is a hyperkähler manifold and  $g: H^2(Y,\mathbb{Z}) \to \Lambda$  is an isomorphism of lattices.

**Definition 4.3.3.** The period map

Per: 
$$M_{\Lambda} \to \mathbb{P}(\Lambda_{\mathbb{C}})$$

sends the point of  $\mathcal{M}_{\Lambda}$  corresponding to a pair (Y,g) to the class of the line  $g(H^{2,0}(Y)) \subset \Lambda_{\mathbb{C}}$ . The image of a pair (Y,g) under the period map is called its *period*.

**Theorem 4.3.4.** 1. The image of the period map is contained in the subset  $\Omega_{\Lambda}$  consisting of  $[\sigma] \in \mathbb{P}(\Lambda_{\mathbb{C}})$  such that

$$q(\sigma) = 0$$
 and  $q(\sigma, \overline{\sigma}) > 0$ .

- 2. (Local Torelli Theorem [Bea83]) The period map is a local biholomorphism onto  $\Omega_{\mathbb{C}}$ .
- 3. (Global Torelli Theorem [Huy99, Theorem 8.1], [Huy11, Corollary 6.1]). Let  $\mathcal{M}_{\Lambda}^0$  be a connected component of  $\mathcal{M}_{\Lambda}$ . Then the period map

Per: 
$$\mathcal{M}^0_{\Lambda} \to \Omega_{\Lambda}$$

is surjective. Moreover, two points (X,g) and (X',g') of  $\mathcal{M}^0_{\Lambda}$  have the same periods if and only if there exists a bimeromorphism  $f\colon X \dashrightarrow X'$  such that the pullback map  $f^*\colon H^2(X') \to H^2(X)$  coincides with  $g^{-1}\circ g$ .

**4.3.5. Period map for Shafarevich–Tate deformations.** Assume that X admits a Lagrangian fibration  $\pi: X \to B$ . Consider its Shafarevich–Tate family (Definition 2.2.4)

$$\Pi: \mathcal{X} \to \mathbb{A}^1$$
.

We can construct a period map

$$\operatorname{Per}_{\operatorname{III} T} \colon \mathbb{A}^1 \to \mathbb{P}(H^2(X,\mathbb{C})).$$

exactly as in Definition 4.3.3 by sending the class of  $t \in \mathbb{A}^1$  to the class of the holomorphic symplectic form  $\sigma_t$  on  $X_t$ . Denote by  $\eta$  the class of the pullback of an ample class on B to X. It is easy to see [AR21, Proposition 3.10] that the map  $Per_{IIIT}$  is an isomorphism onto the affine line

$$\{[\sigma+t\eta]\mid t\in\mathbb{C}\}\subset\mathbb{P}(H^2(X,\mathbb{C})).$$

In particular, the image of  $Per_{IIIT}$  lies in  $\Omega_{\Lambda}$ .

**Lemma 4.3.6.** As in Proposition 4.0.1, let  $\mathcal{X} \to \mathbb{D}$  be a Shafarevich-Tate deformation over a disk  $\mathbb{D} \subset \mathbb{A}^1$ . Assume that  $0 \in \overline{U}$ , where  $U \subset \mathbb{D}$  is the set of Kähler Shafarevich-Tate twists. Then there exists a family  $\mathcal{Y} \to \mathbb{D}$  such that

- $\forall t \in \mathbb{D}, Y_t \text{ is hyperk\"ahler};$
- $\forall t \in U$ , the manifolds  $X_t$  and  $Y_t$  are bimeromorphic

*Proof.* Let us apply the Global Torelli theorem (Theorem 4.3.4(3)) to some hyperkähler Shafarevich–Tate deformation of X. We obtain that there exists a hyperkähler manifold  $Y_0$  deformation equivalent to  $X_0$  whose period coincides with the period of  $X_0$ . The period map is a biholomorphism in a neighborhood of  $Y_0$  in  $\mathcal{M}_{\Lambda}$  (Theorem B(2)). Hence we can find a family

$$\mathcal{Y} \to \mathbb{D}$$

of hyperkähler manifolds such that its image under the period map coincides with the image of  $\mathcal{X} \to \mathbb{D}$ . For every  $t \in U \subset \mathbb{D}$ , the manifolds  $X_t$  and  $Y_t$  are deformation equivalent hyperkähler manifolds whose periods coincide. Hence they are bimeromorphic (Theorem 4.3.4(3)).

We are done with Step 1 (4.2) of the proof of Proposition 4.0.1.

#### 4.4 Lagrangian fibrations on non-projective hyperkähler manifolds

**Lemma 4.4.1.** Let  $\pi: X \to B$  be a Lagrangian fibration on a hyperkähler manifold. Assume that  $NS(X) \subset \eta^{\perp}$ , where  $\eta = \pi^* h$  is the pullback of an ample class h of B. Then all curves on X lie in fibers of  $\pi$ .

*Proof.* Let  $C \subset X$  be a curve. Denote by  $c \in H^2(X,\mathbb{Q})$  the class BBF dual to C. Then

$$\eta \cdot C = q(\eta, c) = 0.$$

Therefore,

$$h \cdot \pi_* C = 0.$$

The class h is ample, hence  $\pi_*C$  is a trivial cycle. Therefore, C is contained in a fiber of  $\pi$ .

**Proposition 4.4.2.** Let  $f: X \dashrightarrow Y$  be a bimeromophism of hyperkähler manifolds. Suppose that X admits a Lagrangian fibration  $\pi: X \to B$  and  $NS(X) \subset \eta^{\perp}$ . Then the following holds.

- 1. The hyperkähler manifold Y admits a Lagrangian fibration  $p: Y \to B'$ .
- 2. There exists a birational map  $g: B \longrightarrow B'$  making the diagram

$$X \xrightarrow{f} Y$$

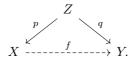
$$\downarrow^{\pi} \qquad \downarrow^{p}$$

$$B \xrightarrow{g} B'$$

commutative.

3. The meromorphic map f is holomorphic on  $X^{\circ}$  and induces an isomorphism  $X^{\circ} \to Y^{\circ}$ . As before,  $X^{\circ}$  (resp.  $Y^{\circ}$ ) denotes the union of smooth fibers of  $\pi$  (resp. p).

*Proof.* **Step 1.** First, we will show that f is defined on  $X^{\circ}$  and sends it isomorphically onto its image. Since X and Y are bimeromorphic, we can find a complex manifold Z together with bimeromorphic maps  $p \colon Z \to X$  and  $q \colon Z \to Y$  making the following diagram commutative



For every  $y \in Y$ , the preimage  $q^{-1}(y) \subset Y$  is rationally chain connected ([HM07]). Let  $C \subset q^{-1}(y)$  be a rational curve. Then either C is contracted by p or p(C) is contained in  $\pi^{-1}(\Delta)$ . Indeed, by Lemma 4.4.1 there are no rational curves in X passing through a point in  $X^{\circ} = X \setminus \pi^{-1}(\Delta)$ . If for some rational curve  $C \subset q^{-1}(y)$ , the image of C in X lies in  $\pi^{-1}(\Delta)$ , then the image of  $q^{-1}(y)$  under p lies in  $q^{-1}(\Delta)$  because  $q^{-1}(y)$  is rationally chain connected. Similarly, if some rational curve  $Q \subset q^{-1}(y)$  is contracted by  $Q \subset q^{-1}(y)$  is contracted to this point.

Denote by  $N \subset Y$  the image of  $p^{-1}(\pi^{-1}(\Delta))$  in Y. We have just shown that  $p(q^{-1}(N)) = \pi^{-1}(\Delta)$  and  $p(q^{-1}(Y \setminus N)) = X^{\circ}$ . Moreover, all fibers of  $q\Big|_{q^{-1}(Y \setminus N)}$  are contracted by p. Therefore, the inverse rational map  $f^{-1}: Y \dashrightarrow X$  is defined on  $Y \setminus N$  and maps it to  $X^{\circ}$ .

We can choose holomorphic symplectic forms  $\sigma_X$  and  $\sigma_Y$  on X and Y respectively in such a way that  $(f^{-1})^*\sigma_X = \sigma_Y$ . Since both forms  $\sigma_X$  and  $\sigma_Y$  are non-degenerate, the morphism  $f^{-1}\Big|_{Y\setminus N}: Y\setminus N\to X^\circ$  has 0-dimensional fibers, hence is an isomorphism. That clearly implies that the map  $f\Big|_{X^\circ}$  is an isomorphism from  $X^\circ$  onto  $Y\setminus N$ .

**Step 2.** The manifold Y contains an open subset isomorphic to  $X^{\circ}$ , in particular, it contains a Lagrangian torus. Moreover, Y is non-projective because it is bimeromorphic to a non-projective Kähler manifold. By Greb-Lehn-Rollenske theorem [GLR13], the Lagrangian fibration  $Y \setminus N \simeq X^{\circ} \to B^{\circ}$  extends to a Lagrangian fibration

$$p: Y \to B'$$
.

Moreover, the base B' is birational to B. The statement is proven.

**Corollary 4.4.3.** In the notation of Lemma 4.3.6, let  $Y_t$  be the fiber of  $\mathcal{Y} \to \mathbb{D}$  over a very general  $t \in U$ . Then  $Y_t$  admits a Lagrangian fibration  $p_t \colon Y_t \to B'$  bimeromorphic to the Lagrangian fibration  $\pi_t \colon X_t \to B$ .

*Proof.* By Theorem 3.0.7 for a very general  $t \in U$ ,  $NS(X_t) \subset \eta^{\perp}$ . By construction of  $\mathcal{Y}$  (Lemma 4.3.6), the manifolds  $X_t$  and  $Y_t$  are bimeromorphic for every  $t \in U$ . The statement of the corollary follows by applying Proposition 4.4.2 to  $X_t$  and  $Y_t$ .

We are done with Step 2 (4.2).

#### $\mathcal{U} \to \mathbb{D}$ is almost a Shafarevich-Tate family 4.5

**Proposition 4.5.1.** In the notation of Lemma 4.3.6 there exists an open dense subset  $V \subset U$  such that the restriction of V to V is a Shafarevich-Tate family.

*Proof.* For a very general  $t \in U$ , the manifold  $Y_t$  admits a Lagrangian fibration (Corollary 4.4.3). We claim that the family  $\mathcal{U}$  coincides with the Shafarevich-Tate family of  $Y_t$  in a neighborhood of t. Indeed, the images under the period map of the Shafarevich-Tate family and of  $\mathcal{U} \to \mathbb{D}$  coincide. By the Local Torelli theorem (Theorem 4.3.4(2)), these families must coincide in a neighborhood of  $t \in \mathbb{D}$ . Denote by  $V \subset U$  the set of  $t \in U$  such that  $Y_t$  admits a Lagrangian fibration  $p_t \colon Y_t \to B'$ . We have just shown that V is open and dense in U. Moreover, the restriction of  $\mathcal{Y} \to \mathbb{D}$  to V is a Shafarevich-Tate family in a neighborhood of a very general point of V.

**Proposition 4.5.2.** The bases B and B' of Lagrangian fibrations  $\pi_t \colon X_t \to B$  and  $p_t \colon Y_t \to B'$  are isomorphic for all  $t \in V$ .

*Proof.* Let  $t \in V \subset \mathbb{D}$  be such that  $NS(X_t) \not\subset \eta^{\perp}$ . By Theorem 3.0.7  $X_t$ , and hence also  $Y_t$ , is projective. The manifolds  $X_t$  and  $Y_t$  are birational by Global Torelli Theorem (Theorem 4.3.4(3)). This birational isomorphism preserves the class  $\eta$ , hence commutes with Lagrangian fibrations. By [Mat14, Corollary 2], the bases of birational Lagrangian fibrations on projective irreducible holomorphic symplectic manifolds are isomorphic. Hence  $B \simeq B'$ . 

The two propositions of this subsection complete the proof of Step 3 (4.2) of Proposition 4.0.1.

#### Limits of isomorphisms 4.6

Consider the following three families of irreducible holomorphic symplectic manifolds:

- 1.  $\mathcal{X} \to \mathbb{D}$ . A Shafarevich-Tate family over a disk  $\mathbb{D} \subset \mathbb{A}^1$ . We assume that there exists an open subset  $U \subset \mathbb{D}$  such that  $\forall t \in U, X_t$  is hyperkähler and  $0 \in \overline{U}$ .
- 2.  $\mathcal{Y} \to \mathbb{D}$ . A family of hyperkähler manifolds such that for all  $t \in U$ ,  $Y_t$  is bimeromorphic to  $X_t$ constructed in Lemma 4.3.6.
- 3.  $\mathcal{Y}' \to \mathbb{D}$ . The family of Shafarevich-Tate deformations of a Lagrangian fibration  $p_t \colon Y_t \to B$  for some  $t \in V$ . Its restriction to V coincides with the restriction of Y to V (Proposition 4.5.1).

The images under the period map (Definition 4.3.3) of all three families coincide.

**Lemma 4.6.1.** The holomorphic symplectic manifolds  $Y := Y_0$  and  $Y' := Y'_0$  are bimeromorphic.

*Proof.* The proof of this lemma follows closely the second part of the proof of [Per17, Lemma 2.6]. First, since  $Y_t$  is Kähler for all  $t \in \mathbb{D}$ , we can find a family  $\{\beta_t\}_{t \in \mathbb{D}}$  of Kähler forms on  $Y_t$ . Second, we can find a family of d-closed (1,1)-forms  $\{\alpha_t\}_{t\in\mathbb{D}}$  such that  $[\alpha_t]$  interesects positively all rational curves in fibers of the Lagrangian fibration  $p_t \colon Y'_t \to B'$ . It is possible to find such  $\{\alpha_t\}_{t \in \mathbb{D}}$  because  $Y'_t$  is a Shafarevich–Tate deformation and fibers of  $p_t$  are the same for all t. Moreover, we can suppose that  $q([\alpha_t]) > 0 \ \forall t$  up to possibly shrinking  $\mathbb{D}$ .

By Lemma 4.4.1, for a very general  $t \in V$ , all rational curves on  $Y'_t$  are contained in fibers of  $p_t$ . Hence the class  $[\alpha_t]$  intersects all rational curves on  $Y'_t$  positively and has positive square with respect to the BBF form. By [Bou01],  $[\alpha_t]$  or  $-[\alpha_t]$  is a Kähler class.

Up to changing the sign of  $\alpha_t$  we can assume that  $[\alpha_t]$  is a Kähler class on  $Y'_t$  for a very general  $t \in V$ . Since Kähleness is an open property, we conclude that  $[\alpha_t]$  is a Kähler class for all  $t \in V'$ , where  $V' \subset V$ is a dense open subset of V. Therefore, there exists a family of forms  $\{\alpha_t\}$  on  $Y'_t$  such that the form  $\alpha_t$ is Kähler for all  $t \in V'$ .

We conclude that there exists a sequence  $\{t_m\}_{m\in\mathbb{N}}$  of points in V which converges to 0, and such that for every  $m \in \mathbb{N}$ , we have a Kähler form  $\alpha_m := \alpha_{t_m}$  on  $Y'_m := Y'_{t_m}$  and a Kähler form  $\beta_m := \beta_{t_m}$  on  $Y_m := Y_{t_m}$ , such that the sequence  $\{\alpha_m\}$  converges to  $\alpha_0$  and  $\{\beta_m\}$  converges to  $\beta_0$ .

Introduce  $\Lambda$ -markings  $g_t : H^2(Y_t) \to \Lambda$  and  $g'_t : H^2(Y'_t) \to \Lambda$  on  $Y_t$  and  $Y'_t$  respectively. We can assume

that  $\forall t \in V$ , the isomorphism  $f_t \colon Y'_t \to Y_t$  satisfies:

$$f_t^* = (g_t')^{-1} \circ g_t.$$

Let  $\Gamma_m \subset Y'_m \times Y_m$  be the graph of the isomorphism  $f_m \colon Y'_m \to Y_m$ . Let us compute its volume with respect to the Kähler form  $P_1^*\alpha_m + P_2^*\beta_m$ , where  $P_1$  and  $P_2$  are the projections of  $Y'_m \times Y_m$  to  $Y'_m$  and  $Y_m$  respectively. We have

$$vol(\Gamma_m) = \int_{Y_m} (\beta_m + f_m^* \alpha_m)^{2n} = \int_{Y_m} ([\beta_m] + f_m^* [\alpha_m])^{2n}$$

Taking the limit as m goes to infinity, we get

$$\lim_{m \to \infty} \text{vol}(\Gamma_m) = \int_{Y_0} ([\beta_0] + (g_0')^{-1} \circ g_0([\alpha_0])^{2n} < \infty.$$

Hence, the volumes of the graphs  $\Gamma_m$  are bounded. By Bishop's Theorem [Bis64] (see also [BR75, Lemma 5.1]), the cycles  $\Gamma_m$  converge to a cycle  $\Gamma \subset Y_0' \times Y_0$ . Next, we need to show that  $\Gamma$  contains an irreducible component of a graph of a bimeromorphism. The proof follows word by word the argument in [Per17, Lemma 2.6] (see also the proof of [Huy99, Theorem 4.3])

Lemma 4.6.1 concludes Step 4 (4.2) of the proof of Proposition 4.0.1.

### 4.7 Shafarevich–Tate deformations of bimeromorphic Lagrangian fibrations

**Proposition 4.7.1.** Let  $\pi\colon X\to B$  and  $p\colon Y\to B$  be two Lagrangian fibrations on irreducible holomorphic symplectic manifolds X and Y. Suppose that there is a bimeromorphic map  $f\colon X\dashrightarrow Y$  which commutes with the Lagrangian fibrations. Fix a Kähler form  $\alpha$  on B and consider the degenerate twistor deformations  $X_t$  and  $Y_t$  corresponding to  $\alpha$  (Definition 2.2.3). Then there exists a bimeromorphism  $f_t\colon X_t\dashrightarrow Y_t$  which commutes with the Lagrangian fibrations on  $X_t$  and  $Y_t$ .

Proof. Consider the graph  $\Gamma \subset X \times Y$  of the bimeromorphism f. It is a Lagrangian subvariety of  $X \times Y$  with respect to the holomorphic symplectic form  $P_X^*\sigma_X - P_Y^*\sigma_Y$ , where  $P_X$  and  $P_Y$  are projection of  $X \times Y$  on X and Y respectively and  $\sigma_X$ ,  $\sigma_Y$  are holomorphic symplectic forms on X and Y respectively. It is assumed that  $f^*\sigma_Y = \sigma_X$ . The form  $P_X^*\pi_X^*\alpha - P_Y^*\pi_Y^*\alpha$  vanishes on  $\Gamma$ . Therefore  $\Gamma$  is Lagrangian with respect to a form

$$P_X^*(\sigma_X + t\pi_X^*\alpha) - P_Y^*(\sigma_Y + t\pi_Y^*\alpha) \tag{8}$$

for any  $t \in \mathbb{C}$ . Consider the complex structures  $I_t$  on  $X_t$  and  $J_t$  on  $Y_t$  induced by holomorphic symplectic forms  $\sigma_X + t\pi_X^*\alpha$  and  $\sigma_Y + t\pi_Y^*\alpha$  (see 2.2.2). The form (8) is holomorphic symplectic with respect to the complex structure  $(I_t, J_t)$  on  $X \times Y$ . In other words, the form (8) is holomorphic symplectic on  $X_t \times Y_t$ . A Lagrangian submanifold of a holomorphic symplectic manifold is necessarily complex. This is an immediate consequence of the following linear algebraic fact: a real subspace of a complex vector space which is Lagrangian with respect to a holomorphic symplectic form is complex. A priori  $\Gamma$  is only a real analytic subvariety of  $X_t \times Y_t$ , but it must be complex analytic in its smooth points because it is Lagrangian. By [Rei70] (see also [Kur])  $\Gamma_t$  is a complex analytic subvariety of  $X_t \times Y_t$ . It induces a desired bimeromorphism  $f_t \colon X_t \dashrightarrow Y_t$ .

**Remark 4.7.2.** The proof of Proposition 4.7.1 shows that the bimeromorphism  $f_t: X_t \dashrightarrow Y_t$  is the same as  $f: X \dashrightarrow Y$  real analytically.

Corollary 4.7.3. Let  $\mathcal{X} \to \mathbb{D}$  be a Shafarevich-Tate family as in Proposition 4.0.1. Then  $X = X_0$  is bimeromorphic to a hyperkähler manifold, in particular, is of Fujiki class  $\mathcal{C}$ .

Proof. Consider the Shafarevich-Tate family  $Y' \to \mathbb{D}$  introduced in Subsection 4.6. For every  $t \in V \subset \mathbb{D}$  the manifold  $Y'_t$  admits a Lagrangian fibration  $p'_t \colon Y'_t \to B$  over the same base as  $X_t$  (Proposition 4.5.2) and the manifolds  $Y'_t$  and  $X_t$  are bimeromorphic as Lagrangian fibrations. Proposition 4.7.1 implies that all degenerate twistor deformations of  $X_t$  and  $Y'_t$  are bimeromorphic. In particular,  $X = X_0$  is bimeromorphic to  $Y' := Y'_0$ . By Lemma 4.6.1 the manifold Y' is bimeromorphic to Y. Hence X is bimeromorphic to the hyperkähler manifold Y.

We completed the proof of Step 5 (4.2).

#### 4.8 Criterion for Kählerness

The last step of the proof of Proposition 4.0.1 will rely on the following theorem by Perego.

**Theorem 4.8.1** ([Per17, Theorem 1.15]). Let X be a compact holomorphic symplectic manifold of Fujiki class C which is a limit of hyperkähler manifolds. Assume that there is a class  $\beta \in H^{1,1}(X)$  satisfying the following properties:

- 1.  $q(\beta) > 0$ ;
- 2.  $\beta \cdot C > 0$  for any rational curve  $C \subset X$ ;
- 3.  $q(\beta, \xi) \neq 0$  for any non-zero  $\xi \in NS(X)$ .

Then X is hyperkähler and  $\beta$  is a Kähler class on  $X^4$ .

Perego's result easily implies the following criterion for Kählerness. Before stating it, let us recall that the *Mori cone* of a compact complex manifold X is the cone  $NE(X) \subset H_2(X,\mathbb{R})$  generated by classes of curves on X. For any morphism  $X \to Y$  we define the *relative Mori cone*  $NE(X/Y) \subset H_2(X,\mathbb{R})$  as the cone generated by classes of curves contained in fibers of  $X \to Y$ .

Corollary 4.8.2. Let X be a compact holomorphic symplectic manifold of Fujiki class C which is a limit of hyperkähler manifolds. Assume that there is a class  $\beta \in H^{1,1}(X)$  satisfying the following two properties:

- 1.  $q(\beta) > 0$ ;
- 2.  $\beta \cdot c > 0$  for any class  $c \in \overline{NE(X)}$ .

Then X is hyperkähler, and  $\beta$  is a Kähler class on X.

*Proof.* The class  $\beta$  obviously satisfies the first two assumptions of Theorem 4.8.1. Consider the set  $\mathcal{W} \subset H^{1,1}(X)$  defined as

$$\mathcal{W} = \bigcup_{\xi \in NS(X) \setminus \{0\}} \left( \xi^{\perp} \cap H^{1,1}(X) \right).$$

The set  $\mathcal{W}$  is a union of a countable number of hyperplanes. If  $\beta \notin \mathcal{W}$ , then we are done. Assume that  $\beta \in \mathcal{W}$ . There is a neighborhood U of  $\beta$  inside  $H^{1,1}(X)$  such that every  $\beta' \in U$  satisfies the assumptions of the corollary. A very general  $\beta' \in U$  does not lie in  $\mathcal{W}$ . Theorem 4.8.1 implies that X is hyperkähler. A class  $\beta \in H^{1,1}(X)$  on a hyperkähler manifold X is Kähler if and only if it satisfies the two assumptions of the corollary [Bou01, Théorème 1.2] (see also [Huy03, Proposition 3.2]), hence  $\beta$  is a Kähler class.  $\square$ 

**Proposition 4.8.3.** As in Proposition 4.0.1 let  $\mathcal{X} \to \mathbb{D}$  be a Shafarevich-Tate family over a disk such that  $X_t$  is Kähler for all  $t \in U$  and  $0 \in \overline{U}$ . Assume that  $X = X_0$  is of Fujiki class C. Then X is Kähler.

*Proof.* As before, we denote by  $\eta$  the pullback of an ample class on B to X. If  $NS(X) \not\subset \eta^{\perp}$ , then X is projective (Theorem 3.0.7), and we are done. Hence we may assume that  $NS(X) \subset \eta^{\perp}$ . In this case NE(X/Y) = NE(X) by Lemma 4.4.1. By Corollary 4.8.2 it is enough to construct a class  $\beta \in H^{1,1}(X)$  such that  $q(\beta) > 0$  and  $\beta \cdot c > 0$  for any class  $c \in NE(X/Y)$ .

Pick a Kähler class  $\beta''$  on  $H^2(X_t)$  for some  $t \in U$ . Then  $\beta'' \cdot c > 0$  for any class  $c \in \overline{NE(X/Y)}$ . Identify  $H^2(X_t)$  and  $H^2(X)$  using the Gauss-Manin connection. The classes of horizontal curves on  $X_t$  will get identified with classes of horizontal curves on X. Let  $\beta'$  be the (1,1)-part of  $\beta''$  considered as a class in  $H^2(X)$ . Since every class in  $H^{2,0}(X)$  and  $H^{0,2}(X)$  restricts trivially to any curve on X, the class  $\beta'$  satisfies the second condition of Corollary 4.8.2. Define  $\beta := \beta' + k\eta$  for  $k \gg 0$ . Then

$$q(\beta) = q(\beta') + 2kq(\beta', \eta),$$

which is positive for sufficiently big k. The class  $\beta \in H^2(X)$  satisfies both condition of Corollary 4.8.2, hence X is hyperkähler.

<sup>&</sup>lt;sup>4</sup>Perego states this result in a greater generality assuming only that X is a  $b_2$ -manifold, i.e.,  $h^{2,0}(X) + h^{1,1}(X) + h^{0,2}(X) = b_2(X)$ . Manifolds of Fujiki class C admit Hodge decomposition, hence are  $b_2$ -manifolds.

#### **4.8.4.** We are ready to prove Proposition 4.0.1 and Theorem A.

Proof of Theorem A and Proposition 4.0.1. As explained in the beginning of Section 4, Theorem A follows easily from Proposition 4.0.1. The proof of Proposition 4.0.1 follows the steps outlined in Subsection 4.2. We are done with all of them by now. We started with a Shafarevich-Tate family  $\mathcal{X} \to \mathbb{D}$  satisfying the conditions of Proposition 4.0.1. Then in Lemma 4.3.6 we constructed a family of hyperkähler manifolds  $Y \to \mathbb{D}$  with the same period as  $X \to \mathbb{D}$  such that  $X_t$  is bimeromorphic to  $Y_t$  for  $t \in U$ . Next, we proved that for a very general  $t \in U$ , the manifold  $Y_t$  admits a Lagrangian fibration  $p_t \colon Y_t \to B'$  (Proposition 4.4.2). After that we showed that actually  $Y_t$  admits a Lagrangian fibration for any  $t \in V$  for some open dense  $V \subset U$  and the restriction of  $Y_t$  to  $Y_t$  is a Shafarevich-Tate family (Proposition 4.5.1). Moreover the base of the Lagrangian fibration on  $Y_t$  is actually isomorphic to  $Y_t$  (Proposition 4.5.2). In the next step, we showed that  $Y_t := Y_0$  is bimeromorphic to  $Y_t := Y_0$ , which is a Shafarevich-Tate deformation of  $Y_t$  for  $t \in V$  (Lemma 4.6.1). Corollary 4.7.3 implies that  $X_t$  is bimeromorphic to  $Y_t$  and hence to  $Y_t$ . Finally, we use a version of [Per17, Theorem 1.15] in Lemma 4.6.1 to conclude that a Shafarevich-Tate deformation of Fujiki class  $C_t$  must be hyperkähler. That finishes the proof.

# 5 Topology of Shafarevich–Tate twists

In this Section we will prove Theorems B, C and D.

**5.0.1. Higher pushforwards of**  $\mathbb{Q}_X$  **do not depend on a twist.** Let  $\pi: X \to B$  be a Lagrangian fibration and  $\pi^{\varphi}: X^{\varphi} \to B$  its Shafarevich–Tate twist. Then the sheaves  $R^k \pi_* \mathbb{Z}$  and  $R^k \pi_*^{\varphi} \mathbb{Z}$  are canonically identified. Indeed, represent  $\varphi$  as a Čech cocycle  $(\varphi_{ij})$ , where  $\varphi_{ij} \in Aut_{X/B}^0(U_{ij})$ . The automorphisms  $\varphi_{ij}$  are flows of vector fields, hence they act trivially on  $H^k(X_{ij})$ .

In particular the vector spaces  $H^0(R^2\pi_*\mathbb{Q})$  and  $H^0(R^2\pi_*^{\varphi}\mathbb{Q})$  are canonically identified. However, the differentials

$$d_2: H^0(R^2\pi_*\mathbb{Q}) \to H^2(R^1\pi_*\mathbb{Q})$$
 and  $d_2^{\varphi}: H^0(R^2\pi_*\mathbb{Q}) \to H^2(R^1\pi_*\mathbb{Q})$ 

from the Leray spectral sequence of X and  $X^{\varphi}$  respectively may be different.

**5.0.2.** The restriction map  $H^2(X) \to H^2(F)$  has rank at most one. Suppose that X is hyperkähler, and let F be a smooth fiber of  $\pi \colon X \to B$ . By Theorem 2.1.6, the restriction map  $H^2(X) \to H^2(F)$  has a one-dimensional image generated by an ample class. Global invariant cycle theorem implies that

$$\operatorname{im}\left(H^2(X,\mathbb{Q})\to H^2(F,\mathbb{Q})\right)=H^2(F,\mathbb{Q})^{\pi_1(B^\circ)}=H^0(B^\circ,R^2\pi_*\mathbb{Q}\big|_{_{B^\circ}}).$$

Here  $H^2(F,\mathbb{Q})^{\pi_1(B^\circ)}$  denotes the subspace of  $H^2(F)$  invariant under the monodromy action of  $\pi_1(B^\circ)$ . It follows that  $H^2(F,\mathbb{Q})^{\pi_1(B^\circ)}$  is one-dimensional and generated by an ample class.

Let  $X^{\varphi}$  be a Shafarevich–Tate twist of X, not necessarily Kähler. Then the image of the map  $H^2(X^{\varphi},\mathbb{Q}) \to H^2(F,\mathbb{Q})$  still lies in  $H^2(F,\mathbb{Q})^{\pi_1(B^{\circ})}$ . The latter space is isomorphic to  $H^0(B^{\circ},R^2\pi_*\mathbb{Q}\big|_{B^{\circ}})$ , hence does not depend on a twist. We obtain the following statement.

**Proposition 5.0.3.** Let  $\pi: X \to B$  be a Lagrangian fibration on an irreducible hyperkähler manifold X and  $X^{\varphi}$  be its Shafarevich-Tate twist. Then the restriction map

$$H^2(X^{\varphi}) \to H^2(F)$$

is either trivial or has a one-dimensional image generated by an ample class of F.

#### 5.1 First cohomology of twists

**Lemma 5.1.1.** Let  $\pi: X \to B$  be a Lagrangian fibration on an irreducible hyperkähler manifold. Then B is simply connected.

*Proof.* For some  $\psi \in \text{III}^0$ , the twist  $X^{\psi}$  is projective. Hence we may and will assume that X is projective. If  $f: M \to N$  is a dominant map of normal algebraic varieties such that the general fiber of f is irreducible, then  $f(\pi_1(M)) = \pi_1(N)$  [Kol95, Proposition 2.10.2]. Therefore,  $\pi_1(B) = \pi_1(X) = 0$ .

**Proposition 5.1.2.** Let  $\pi: X \to B$  be a Lagrangian fibration on an irreducible hyperkähler manifold X and  $X^{\varphi}$  its Shafarevich-Tate twist. Then  $H^1(X^{\varphi}, \mathbb{Q}) = 0$ .

*Proof.* For any Lagrangian fibration  $\pi: X \to B$  on a hyperkähler manifold, the pullback map  $H^2(B, \mathbb{Q}) \to H^2(X, \mathbb{Q})$  is injective [HM22, Corollary 1.13]. It follows from Leray spectral sequence that the sequence

$$0 \to H^1(B, \mathbb{Q}) \to H^1(X, \mathbb{Q}) \to H^0(B, R^1 \pi_* \mathbb{Q}) \to 0$$

$$\tag{9}$$

is exact. Since B and X are simply connected (Lemma 5.1.1), the group  $H^0(B, R^1\pi_*\mathbb{Q})$  vanishes. The exact sequence (9) for  $X^{\varphi}$  implies that for any Shafarevich–Tate twist  $X^{\varphi}$ ,

$$H^1(X^{\varphi}, \mathbb{Q}) \simeq H^1(B, \mathbb{Q}) = 0.$$

# 5.2 Hodge numbers of twists

Recall that by [AR21, Corollary 3.7] a Shafarevich–Tate twist  $X^{\varphi}$  inherits a holomorphic symplectic form  $\sigma$  from X.

**Proposition 5.2.1.** Let  $\pi: X \to B$  be a Lagrangian fibration on an irreducible hyperkähler manifold X. Then  $\forall \varphi \in \coprod$ :

$$H^{0,k}(X^{\varphi}) := H^k(X^{\varphi}, \mathcal{O}_{X^{\varphi}}) = \begin{cases} 0, & \text{if } k \text{ is odd;} \\ \mathbb{C} \cdot \overline{\sigma}^{k/2}, & \text{if } k \text{ is even.} \end{cases}$$

*Proof.* For any  $\varphi \in III$ , the sheaf  $R^i \pi_*^{\varphi} \mathcal{O}_{X^{\varphi}}$  is isomorphic to  $\Omega_B^{[i]}$  by Corollary 2.1.14. The Leray spectral sequence for  $\mathcal{O}_{X^{\varphi}}$  has the form

$$E_2^{p,q} = H^q(R^p \pi_* \mathcal{O}_{X^{\varphi}}) \simeq H^q(\Omega_B^{[p]}) = \begin{cases} 0, & \text{if } p \neq q; \\ \mathbb{C}, & \text{otherwise.} \end{cases}$$
 (10)

This computation follows from Theorem 2.1.16. The spectral sequence (10) degenerates at  $E_2$ , hence  $H^{0,k}(X^{\varphi}) = 0$  for k odd and  $H^{0,k}(X^{\varphi}) = H^{k/2,k/2}(B) = \mathbb{C}$  when k is even.

The form  $\overline{\sigma}^r$  is d-closed and not  $\bar{\partial}$ -exact. Indeed, if  $\overline{\sigma}^r = \bar{\partial}\alpha$ , then

$$0 = \int_{V} \overline{\partial}(\alpha \overline{\sigma}^{(n-r)} \sigma^{n}) = \int_{V} d(\alpha \overline{\sigma}^{(n-r)} \sigma^{n}) = \int_{V} \overline{\sigma}^{n} \sigma^{n} \neq 0,$$

contradiction. Hence the class of  $\overline{\sigma}^r$  is non-trivial in  $H^{0,2r}(X^{\varphi})$  and thus generates  $H^{0,2r}(X^{\varphi})$ .

Next we will compute  $H^0(\Omega^2_{X^{\varphi}})$  for a Shafarevich-Tate twist  $X^{\varphi}$ . We will start with a few preliminary lemmas.

**Lemma 5.2.2.** Let  $\xi$  be a holomorphic 2-form on  $X^{\varphi}$ . Then  $\xi$  restricts trivially to all smooth fibers.

*Proof.* The restriction of  $\xi$  to every smooth fiber is d-closed because all holomorphic forms on Kähler manifolds are closed. Therefore,  $\xi$  defines a section of the local system  $R^2\pi_*\mathbb{C}\big|_{B^\circ}$ . By 5.0.2 this local system has just one non-trivial section, which is the class of a form of type (1,1). The class  $[\xi]_F$  is of type (2,0), hence it must be trivial. There are no non-trivial exact holomorphic forms on F, hence  $\xi|_F = 0$  for every smooth fiber.

**Lemma 5.2.3.** Let  $\pi: Y \to S$  be a proper Lagrangian fibration over a not necessarily compact base. Consider a holomorphic 2-form  $\xi$  on Y with trivial restriction to every smooth fiber. Then  $\xi$  induces a map

$$\iota_{\xi} \colon \pi_* T_{Y/S} \to \Omega_S^{[1]}.$$

Proof. Consider the map

$$\iota_{\xi} \colon T_{Y/S} \to \Omega_Y$$

sending a vector field v to  $\iota_v \xi$ . As before, denote by  $Y^\circ$  the union of smooth fibers of  $\pi$  and  $S^\circ := \pi(Y^\circ)$ . For every vertical vector field v, the restriction of  $\iota_v \xi$  to  $Y^\circ$  lies in  $\pi^* \Omega_{S^\circ}$  because  $\xi\big|_F = 0$  for every smooth fiber F. It follows that the image of  $\iota_\xi$  lies in the sheaf  $(\pi^* \Omega_S)^{sat}$  consisting of 1-forms  $\alpha$  such that  $\alpha\big|_{Y^\circ} \in \pi^* \Omega_S^\circ$ . By taking pushforwards, we obtain a map

$$\iota_{\xi} \colon \pi_* T_{Y/S} \to \pi_* (\pi^* \Omega^1_S)^{sat}.$$

We will show that  $(\pi^*\Omega_S)^{sat} \simeq \Omega_S^{[1]}$ . Indeed, this is definitely true over  $S^{\circ}$ . Let  $\alpha$  be a local section of  $(\pi^*\Omega_S)^{sat}$ . Then the restriction of  $\alpha$  to  $Y^{\circ}$  is the pullback of a form from  $S^{\circ}$ . By Lemma 2.1.10 the form  $\alpha$  must be the pullback of a reflexive form from  $S^{\circ}$ .

**5.2.4.** Let  $\pi: X \to B$  be a Lagrangian fibration. Consider the subsheaf  $(\pi_*\Omega_X^2)'$  of  $\pi_*\Omega_X^2$  consisting of holomorphic 2-forms  $\xi$  with trivial restriction to all smooth fibers. Thanks to Lemma 5.2.3, there is a natural map

$$(\pi_*\Omega_X^2)' \to \mathcal{H}om(\pi_*T_{X/B}, \Omega_B^{[1]}). \tag{11}$$

The holomorphic symplectic form  $\sigma$  on X induces an isomorphism  $\pi_* T_{X/B} \simeq \Omega_B^{[1]}$  (Theorem 2.1.11). Composing the map (11) with this isomorphism, we obtain a map of sheaves

$$\rho \colon (\pi_*\Omega_X^2)' \to \mathcal{E}nd(\Omega_B^{[1]}).$$

**Lemma 5.2.5.** Define the sheaf  $\operatorname{End}'_X(\Omega_B^{[1]})$  as the image of  $\rho$ . Then for any Shafarevich-Tate twist  $X^{\varphi}$  the sheaf  $\operatorname{End}'_{X_{\varphi}}(\Omega_B^{[1]})$  coincides with  $\operatorname{End}'_X(\Omega_B^{[1]})$ .

*Proof.* The statement is local on B. For every sufficiently small open disk  $U \subset B$ , the manifolds  $\pi^{-1}(U)$  and  $(\pi^{\varphi})^{-1}(U)$  are isomorphic as Lagrangian fibrations, hence the claim.

Lemma 5.2.6. The sequence of sheaves on B

$$0 \to \Omega_B^{[2]} \to (\pi_* \Omega_X^2)' \to \mathcal{E} nd_X'(\Omega_B^{[1]}) \to 0$$

is exact.

*Proof.* Note that the map  $(\pi_*\Omega_X^2)' \to \mathcal{E}nd_X'(\Omega_B^{[1]})$  is surjective by the definition of  $\mathcal{E}nd_X'(\Omega_B^{[1]})$ . The first map  $\Omega_B^{[2]} \to (\pi_*\Omega_X^2)'$  is clearly injective as well.

The composite map  $\Omega_B^{[2]} \to \mathcal{E}nd'_X(\Omega_B^{[1]})$  vanishes. Indeed, let  $\alpha$  be a local section of  $\Omega_B^{[2]}$ . Then for any vertical vector field v, the form  $\iota_v \pi^* \alpha$  vanishes on  $X^{\circ}$ , hence vanishes everywhere. Therefore,  $\rho(\alpha) = 0$ .

It remains to prove exactness in the middle term. Let  $U \subset B$  be an open subset and  $\xi$  a holomorphic 2-form on  $\pi^{-1}(U)$  such that  $\rho(\xi) = 0$ . Consider the restriction of  $\xi$  to  $X^{\circ}$ . Since  $\iota_v \xi = 0$  for every vertical vector field v, the form  $\xi$  is contained in  $\pi^*\Omega^2_B(\pi^{-1}(U \cap B^{\circ}))$ . The projection formula together with the fact that  $\pi_*\mathcal{O}_{X^{\circ}} \simeq \mathcal{O}_{B^{\circ}}$  implies that

$$\pi^*\Omega^2_B(\pi^{-1}(U\cap B^\circ))=\pi_*\pi^*\Omega^2_B(U\cap B^\circ)=\Omega^2_B(U\cap B^\circ).$$

Hence there exists a holomorphic 2-form  $\alpha^{\circ}$  on  $U \cap B^{\circ}$  such that  $\xi \Big|_{\pi^{-1}(U \cap B^{\circ})} = \pi^* \alpha^{\circ}$ . By Lemma 2.1.10,  $\xi = \pi^* \alpha$  for some reflexive holomorphic 2-form  $\alpha$  on U.

We are finally ready to show that all holomorphic 2-forms on  $X^{\varphi}$  are multiples of  $\sigma$ .

**Theorem 5.2.7.** Let  $\pi\colon X\to B$  be a Lagrangian fibration on an irreducible hyperkähler manifold X. Then  $H^0(\Omega^2_{X\varphi})$  is generated by the holomorphic symplectic form  $\sigma$  for all  $\varphi\in \coprod$ .

*Proof.* By Lemma 5.2.2 every holomorphic 2-form  $\xi$  on  $X^{\varphi}$  restricts trivially to every smooth fiber. Therefore,

$$H^0(B,(\pi_*\Omega^2_{X^\varphi})')=H^0(X^\varphi,\Omega^2_{X^\varphi}).$$

Lemma 5.2.6 shows that the sequence

$$0 \to \Omega_B^{[2]} \to (\pi_*\Omega_{X^\varphi}^2)' \to \operatorname{End}_{X^\varphi}'(\Omega_B^{[1]}) \to 0$$

is exact. Consider its long exact sequence of cohomology

$$0 \to H^0(\Omega_B^{[2]}) \to H^{2,0}(X^\varphi) \to H^0(\operatorname{End}_{X^\varphi}'(\Omega_B^{[1]})) \to H^1(\Omega_B^{[2]})$$

The cohomology groups  $H^i(\Omega_B^{[2]})$  vanish for i=0,1 (Theorem 2.1.16). Therefore,

$$H^{2,0}(X^\varphi) \simeq H^0(\operatorname{End}_{X^\varphi}'(\Omega_B^{[1]})).$$

The sheaf  $\operatorname{\mathcal{E}\!\mathit{nd}}_{X^{\varphi}}'(\Omega_B^{[1]})$  does not depend on a twist by Lemma 5.2.5, therefore  $H^{2,0}(X^{\varphi})$  does not depend on a twist.

Remark 5.2.8. When  $B = \mathbb{P}^n$  the proof of Theorem 5.2.7 can be simplified because  $\operatorname{End}(\Omega^1_{\mathbb{P}^n}) \simeq \mathbb{C}$ . By Lemma 5.2.2, every holomorphic 2-form  $\xi$  on  $X^{\varphi}$  restricts trivially to smooth fibers. Hence  $\xi$  induces an endomorphism  $\rho(\xi)$  of  $\Omega^1_{\mathbb{P}^n}$  (Lemma 5.2.3). Since  $\operatorname{End}(\Omega^1_{\mathbb{P}^n}) = \mathbb{C}$ , there exists a number  $\lambda \in \mathbb{C}$  such that  $\rho(\xi - \lambda \sigma) = 0$ . The contraction of every vertical vector field on X with  $\xi - \lambda \sigma$  vanishes, hence

$$(\xi - \lambda \sigma)|_{\mathbf{x}^{\circ}} \in \pi^* \Omega^2_{(\mathbb{P}^n)^{\circ}}.$$

Since  $\pi_*\pi^*\Omega^2_{\mathbb{P}^n}=\Omega^2_{\mathbb{P}^n}$ , we have  $(\xi-\lambda\sigma)\big|_{X^\circ}=\pi^*\alpha^\circ$  for some holomorphic 2-form  $\alpha^\circ$  on  $(\mathbb{P}^n)^\circ$ . By Lemma 2.1.10,  $\alpha^\circ$  extends to a holomorphic form  $\alpha$  on  $\mathbb{P}^n$  and  $\xi-\lambda\sigma=\pi^*\alpha$ . There are no holomorphic forms on  $\mathbb{P}^n$ , hence  $\xi=\lambda\sigma$ .

We were unable to show that  $\operatorname{End}(\Omega_B^{[1]}) \simeq \mathbb{C}$  for any base of a Lagrangian fibration, although we expect it to be true.

**5.2.9.** Proof of Theorem B. The statement immediately follows from Proposition 5.1.2 and Theorem 5.2.7.

#### 5.3 Second cohomology of a twist

Our goal now is to prove Theorems C and Theorem D.

**Lemma 5.3.1.** Let  $\pi\colon X\to B$  be a Lagrangian fibration on an irreducible hyperkähler manifold X. Define the sheaf  $\mathcal{N}S$  on B as the image of the Chern class map  $R^1\pi_*\mathbb{O}_X^\times\to R^2\pi_*\mathbb{Z}$ . Then

$$H^0(B, \mathcal{NS}) = H^0(R^2 \pi_* \mathbb{Z}),$$

In other words, for every section  $\xi$  of  $R^2\pi_*\mathbb{Z}$  and a sufficiently fine open cover  $B = \bigcup U_i$ , there are line bundles  $L_i$  on  $X_i$  such that  $\xi\Big|_{U_i} = c_1(L_i)$ . In particular, every section  $\xi$  of  $R^1\pi_*\mathbb{Z}$  is locally the class of a closed (1,1)-form.

*Proof.* Consider the exponential exact sequence

$$0 \to \mathbb{Z}_X \to \mathcal{O}_X \to \mathcal{O}_X^{\times} \to 0.$$

It induces a long exact sequence of pushforward sheaves:

$$R^1\pi_*\mathcal{O}_X^{\times} \to R^2\pi_*\mathbb{Z} \to R^2\pi_*\mathcal{O}_X.$$

The sheaf  $R^2\pi_*\mathbb{Z}/\operatorname{im}(R^1\pi_*\mathcal{O}_X^{\times})=R^2\pi_*\mathbb{Z}/\mathcal{N}\mathcal{S}$  is a subsheaf of  $R^2\pi_*\mathcal{O}_X\simeq\Omega_B^{[2]}$ . Since  $H^0(\Omega_B^2)=0$  (Theorem 2.1.16), the sheaf  $R^2\pi_*\mathbb{Z}/\mathcal{N}\mathcal{S}$  has no global sections. Hence the natural inclusion  $H^0(B,\mathcal{N}\mathcal{S})\to H^0(R^2\pi_*\mathbb{Z})$  is an isomorphism.

**5.3.2.** Isomorphisms between  $T_{X/B}$  and  $R^1\pi_*\mathcal{O}_X$ . Let  $\xi$  be a global section of  $H^0(R^2\pi_*\mathbb{Q})$ . It defines a map  $f_{\xi} \colon \pi_*T_{X/B} \to R^1\pi_*\mathcal{O}_X$  in a similar way that a class  $\omega \in H^2(X,\mathbb{Q})$  defines a map  $f_{\omega}$  in 2.1.12. Namely, by Lemma 5.3.1 we can represent  $\xi\Big|_{U_i}$  by a closed (1,1)-form  $\xi_i$  on  $X_i$ . Consider the map

$$f_{\xi_i} \colon \pi_* T_{X_i/U_i} \to R^1 \pi_* \mathcal{O}_X \Big|_{U_i}$$

sending v to the class of  $[\iota_v \xi_i]$  under the  $\bar{\partial}$ -differential. Since the sheaves  $\pi_* T_{X/B}$  and  $R^1 \pi_* \mathcal{O}_X$  are torsion-free (Theorem 2.1.13), the map  $f_{\xi_i}$  is determined completely by its restriction to  $B^{\circ} \cap U_i$ . Therefore, the map  $f_{\xi_i}$  depends only on the class  $[\xi|_F]$  of the restriction of  $\xi$  to smooth fibers. In particular, maps  $f_{\xi_i}$  do not depend on the choice of  $\xi_i$  representing  $\xi \in H^0(R^2 \pi_* \mathbb{Q})$  and glue into a map

$$f_{\xi} \colon \pi_* T_{X/B} \to R^1 \pi_* \mathcal{O}_X.$$

**5.3.3.** The map  $H^0(B, R^2\pi_*\mathbb{C}) \to \operatorname{Hom}(\pi_*T_{X/B}, R^1\pi_*\mathcal{O}_X)$  sending  $\xi$  to  $f_{\xi}$  factors through the restriction to a smooth fiber F:

$$H^0(B, R^2\pi_*\mathbb{C}) \to H^0(B^\circ, R^2\pi_*\mathbb{C}) = H^2(F)^{\pi_1(B^\circ)} \to \mathrm{Hom}(\pi_*T_{X/B}, R^1\pi_*\mathbb{O}_X).$$

The vector space  $H^2(F)^{\pi_1(B^\circ)}$  is one-dimensional by 5.0.2. Fix an element  $\xi_0 \in H^0(B, R^2\pi_*\mathbb{Z})$  which restricts non-trivially to F and let  $f_0 := f_{\xi_0}$  be the induced isomorphism  $\pi_*T_{X/B} \to R^1\pi_*\mathcal{O}_X$ . For every  $\xi \in H^0(B, R^2\pi_*\mathbb{C})$  we define a number  $\lambda_{\xi}$  by the identity

$$f_{\xi} = \lambda_{\xi} f_0.$$

Recall that the isomorphism  $f_0: \pi_*T_{X/B} \to R^1\pi_*\mathcal{O}_X$  sends  $\Gamma_{\mathbb{Q}} = \Gamma \otimes \mathbb{Q} \subset \pi_*T_{X/B}$  isomorphically onto  $R^1\pi_*\mathbb{Q} \subset R^1\pi_*\mathcal{O}_X$  (2.2.5). We identify the group  $H^2(\Gamma_{\mathbb{Q}}) = (\mathbb{III}/\mathbb{III}^0) \otimes \mathbb{Q}$  with  $H^2(R^1\pi_*\mathbb{Q})$  using the isomorphism  $f_0$ .

- **5.3.4. Boundary map** III  $\to H^2(\Gamma)$ . The boundary map III  $\to H^2(\Gamma)$  can be described in terms of Čech cocycles as follows. Pick  $\varphi \in \text{III}$  and represent it by a 1-cocycle  $\varphi_{ij} \in Aut^0_{X/B}(U_{ij})$ . We can find a vertical vector field  $v_{ij}$  on  $X_{ij}$  such that  $\exp(v_{ij}) = \varphi_{ij}$ . The vector field  $v_{ij} + v_{jk} + v_{ki}$  on  $X_{ijk}$  lies in  $\Gamma$  and represents the class  $\overline{\varphi} \in H^2(\Gamma)$ .
- **5.3.5. Boundary map**  $H^0(R^2\pi_*\mathbb{Q}) \to H^2(R^1\pi_*\mathbb{Q})$ . We will describe the boundary map

$$d_2 \colon H^0(R^2\pi_*\mathbb{Q}) \to H^2(R^1\pi_*\mathbb{Q}) \simeq H^2(\Gamma_{\mathbb{Q}})$$

from the Leray spectral sequence of  $\pi$  in terms of Čech cocycles. Let  $\xi$  be a section of  $H^0(R^2\pi_*\mathbb{Q})$ . Represent it locally by (1,1)-forms  $\xi_i$  on  $X_i$ . The difference  $\xi_j - \xi_i$  is an exact form, hence

$$\xi_j - \xi_i = d\rho_{ij}$$

for some 1-form  $\rho_{ij}$  on  $X_{ij}$ . The form  $\rho_{ij}+\rho_{jk}+\rho_{ki}$  is closed on  $X_{ijk}$ , hence defines a cocycle with coefficients in  $R^1\pi_*\mathbb{Q}$ . By 2.2.5 there exists a unique vertical vector field  $w_{ijk}\in\Gamma_{\mathbb{Q}}(U_{ijk})$  on  $X_{ijk}$  such that  $f_0(w_{ijk})$  is equal to the class of the (0,1)-form  $\rho_{ij}^{0,1}+\rho_{jk}^{0,1}+\rho_{ki}^{0,1}$  under the  $\bar{\partial}$ -differential.

**Proposition 5.3.6.** Let  $\pi: X \to B$  be a Lagrangian fibration. Pick a class  $\varphi \in \coprod$ . Let

$$d_2^\varphi \colon H^0(R^2\pi_*\mathbb{Q}) \to H^2(R^1\pi_*\mathbb{Q}) \simeq H^2(\Gamma_\mathbb{Q})$$

be the differential in the Leray spectral sequence for  $\pi^{\varphi}: X^{\varphi} \to B$ . Then for any  $\xi \in H^0(R^2\pi_*\mathbb{Q})$ 

$$d_2^{\varphi}(\xi) = d_2(\xi) + \lambda_{\xi} \overline{\varphi}.$$

*Proof.* Represent  $\xi \in H^0(R^2\pi_*\mathbb{Q})$  by a collection of closed (1,1)-forms  $\xi_i$  on  $X_i$ . When we view  $X_i$  as an open subset of  $X^{\varphi}$ , we will denote the same forms by  $\xi_i^{\varphi}$ . The difference  $\xi_j^{\varphi} - \xi_i^{\varphi}$  is not the same as  $\xi_j - \xi_i$  because there is a twist by  $\varphi_{ij} \in Aut^0_{X/B}(U_{ij})$  involved. Namely

$$\xi_i^{\varphi} - \xi_i^{\varphi} = \varphi_{ij}^* \xi_j - \xi_i = (\varphi_{ij}^* \xi_j - \xi_j) + (\xi_j - \xi_i).$$

As in 5.3.5, write  $\xi_i - \xi_i = d\rho_{ij}$ . Find a vector field  $v_{ij}$  such that  $\varphi_{ij} = \exp(v_{ij})$ . Then

$$\varphi_{ij}^* \xi_j - \xi_j = \int_0^1 \frac{d}{dt} \exp(tv_{ij})^* \xi_j dt = \int_0^1 \exp(tv_{ij})^* \mathcal{L}_{v_{ij}} \xi_j dt =$$

$$= \int_0^1 \exp(tv_{ij})^* (d\iota_{v_{ij}} + \iota_{v_{ij}} d) \xi_j dt = d \int_0^1 \exp(tv_{ij})^* \iota_{v_{ij}} \xi_j dt.$$

Here L denotes the Lie derivative. Set  $\gamma_{ij} = \int_{0}^{1} \exp(tv_{ij})^* \iota_{v_{ij}} \xi_j dt$ . Then

$$\xi_i^{\varphi} - \xi_i^{\varphi} = d(\rho_{ij} + \gamma_{ij}).$$

The form  $\gamma_{ij}$  is of type (0,1) and  $\partial$ -closed. Its class under the  $\partial$ -differential is

$$\int_{0}^{1} [\exp(tv_{ij})^* \iota_{v_{ij}} \xi_j] dt = \int_{0}^{1} [\iota_{v_{ij}} \xi_j] dt = [\iota_{v_{ij}} \xi_j] = f_{\xi}(v_{ij}) = \lambda_{\xi} f_0(v_{ij}).$$

As in 5.3.5, let  $w_{ijk}$  be the element in  $\Gamma_{\mathbb{Q}}(U_{ijk})$  such that  $f_0(w_{ijk}) = [\rho_{ij}^{0,1} + \rho_{jk}^{0,1} + \rho_{ki}^{0,1}]$ . It represents  $d_2(\xi)$ . Then  $d_2^{\varphi}(\xi) - d^2(\xi)$  can be represented by the cocycle

$$\lambda_{\xi}(v_{ij} + v_{jk} + v_{ki}).$$

By 5.3.4 the class of the cocycle  $v_{ij} + v_{jk} + v_{ki}$  in Čech cohomology is  $\overline{\varphi}$ , hence the claim.

Corollary 5.3.7. Let  $\pi: X \to B$  be a Lagrangian fibration. Then for any  $\varphi \in \coprod'$  we have  $d_2^{\varphi} = d_2$ .

*Proof.* For any  $\varphi \in \text{III}'$ , we have  $\overline{\varphi} = 0$ . Proposition 5.3.6 implies that  $d_2^{\varphi} = d_2$ .

Corollary 5.3.8. Let  $\pi: X \to B$  be a Lagrangian fibration. Pick a class  $\varphi \in \coprod$  such that  $\overline{\varphi} \neq 0$ . Consider the restriction maps

$$r: H^2(X) \to H^0(R^2\pi_*\mathbb{Q}), \quad r^{\varphi}: H^2(X^{\varphi}) \to H^0(R^2\pi_*\mathbb{Q}).$$

Let  $H^2(X)^0$  (resp.  $H^2(X^{\varphi})^0$ ) denote the subspace of classes in  $H^2(X)$  (resp.  $H^2(X^{\varphi})$ ) that restrict trivially to a smooth fiber. Then

$$\operatorname{im} r \cap \operatorname{im} r^{\varphi} = r(H^{2}(X)^{0}) = r^{\varphi}(H^{2}(X^{\varphi})^{0}).$$

*Proof.* The image of the restriction map r (resp.  $r^{\varphi}$ ) coincides with the kernel of  $d_2$  (resp.  $d_2^{\varphi}$ ). By Proposition 5.3.6 a class  $\xi$  lies in the kernel of both  $d_2$  and  $d_2^{\varphi}$  if and only if  $\lambda_{\xi} = 0$ , i.e., the restriction of  $\xi$  to a smooth fiber is trivial. The claim follows.

**Proposition 5.3.9.** Let  $\pi: X \to B$  be a Lagrangian fibration on an irreducible hyperkähler manifold and  $X^{\varphi}$  its Shafarevich-Tate twist. Then either  $b_2(X^{\varphi}) = b_2(X)$  or  $b_2(X^{\varphi}) = b_2(X) - 1$ . The first case occurs if and only if there is a class  $h \in H^2(X)$  whose restriction to a smooth fiber is non-trivial.

*Proof.* It follows from the Leray spectral sequence for  $\mathbb{Q}_{X^{\varphi}}$  that

$$b_2(X^{\varphi}) = b_2(B) + \dim H^1(R^1 \pi_* \mathbb{Q}) + \operatorname{rk} r^{\varphi}. \tag{12}$$

Indeed.

$$h^2(X^{\varphi}) = E_{\infty}^{2,0} + E_{\infty}^{1,1} + E_{\infty}^{0,2}.$$

The vector space  $H^0(R^1\pi_*\mathbb{Q})$  vanishes by Proposition 5.1.2, hence the map  $H^2(B) \to H^2(X^{\varphi})$  is injective and  $E^{2,0}_{\infty} = E^{2,0}_2$ . Moreover, the second differential  $d_2 \colon H^1(R^1\pi_*\mathbb{Q}) \to H^3(B,\mathbb{Q})$  is zero because  $H^3(B,\mathbb{Q}) = 0$  [SY22]. Thus  $E^{1,1}_{\infty} = E^{1,1}_2$  The formula (12) implies that

$$b_2(X^{\varphi}) - b_2(X) = \operatorname{rk} r^{\varphi} - \operatorname{rk} r.$$

Since X is hyperkähler, the subspace  $H^2(X)^0$  has codimension 1 in  $H^2(X)$ , hence

$$\operatorname{rk} r = \dim r(H^{2}(X)^{0}) + 1 = \dim r^{\varphi}(H^{2}(X^{\varphi})^{0}) + 1,$$

where the last equality holds by Corollary 5.3.7 if  $\overline{\varphi} = 0$  and by Corollary 5.3.8 if  $\overline{\varphi} \neq 0$ . If there is a class in  $H^2(X^{\varphi})$  restricting non-trivially to a smooth fiber, then  $H^2(X^{\varphi})^0$  is a hyperplane in  $H^2(X^{\varphi})$  (Proposition 5.0.3) and

$$\operatorname{rk} r^{\varphi} = \dim r^{\varphi} (H^{2}(X^{\varphi})^{0}) + 1 = \operatorname{rk} r.$$

In this case,  $b_2(X) = b_2(X^{\varphi})$ . Otherwise,  $H^2(X^{\varphi})^0 = H^2(X^{\varphi})$ , hence

$$\operatorname{rk} r^{\varphi} = \dim r^{\varphi} (H^{2}(X^{\varphi})^{0}) = \operatorname{rk} r - 1,$$

and

$$b_2(X) = b_2(X^{\varphi}) + 1.$$

**Proposition 5.3.10.** Let  $\pi: X \to B$  be a Lagrangian fibration on an irreducible hyperkähler manifold and  $X^{\varphi}$  its Shafarevich-Tate twist. Then there is a class  $h \in H^2(X^{\varphi})$  that restricts non-trivially to a smooth fiber if and only if  $\overline{\varphi} \in H^2(\Gamma_{\mathbb{Q}})$  is in the image of the boundary map  $d_2: H^0(R^2\pi_*\mathbb{Q}) \to H^2(R^1\pi_*\mathbb{Q}) \simeq H^2(\Gamma_{\mathbb{Q}})$ .

*Proof.* Suppose that there is a class  $h \in H^2(X^{\varphi})$  that restricts non-trivially to a smooth fiber. Let  $\overline{h}$  be its image in  $H^0(B, R^2\pi_*\mathbb{Q})$ . By Proposition 5.3.6

$$0 = d_2^{\varphi}(\overline{h}) = d_2(\overline{h}) + \lambda_h \overline{\varphi}.$$

Therefore,  $\overline{\varphi} = -d_2(\overline{h})/\lambda_h$  is in the image of  $d_2$ .

Conversely, suppose that  $\overline{\varphi}$  is in the image of  $\underline{d}_2$ , i.e., there is a class  $\xi \in H^0(R^2\pi_*\mathbb{Q})$  such that  $d_2\xi = \overline{\varphi}$ . Let  $h_0$  be a class on X with  $\lambda_{h_0} = 1$  and  $\overline{h_0}$  its image in  $H^0(R^2\pi_*\mathbb{Q})$ . Consider the class

$$\xi' := (1 + \lambda_{\xi})\overline{h_0} - \xi.$$

Then  $\lambda_{\xi'} = 1$  and

$$d_2^{\varphi}(\xi') = d_2(\xi') + \overline{\varphi} = (1 + \lambda_{\xi})d_2(\overline{h_0}) - d_2(\xi) + \overline{\varphi} = -d_2(\xi) + \overline{\varphi} = 0.$$

Here the first equality holds by Proposition 5.3.6. Therefore,  $\xi'$  lifts to a class in  $H^2(X^{\varphi})$ , which restricts non-trivially to smooth fibers.

**5.3.11.** Proof of Theorem C. Immediately follows from Propositions 5.3.9 and 5.3.10.

**5.3.12.** Proof of Theorem D. Suppose  $X^{\varphi}$  is of Fujiki class C, i.e., there is a rational map  $f: X^{\varphi} \dashrightarrow Y$  to a Kähler manifold Y. Let  $h \in H^2(X,\mathbb{R})$  be the pullback of a Kähler form on Y. The restriction of f to a general fiber F of  $\pi^{\varphi}$  is birational onto its image, hence  $h|_F$  is non-trivial. By Theorem  $\mathbb{C}$ , the twist satisfies  $\overline{\varphi} \in \operatorname{im} d_2$ .

Remark 5.3.13. Consider an abelian surface A which is a product of elliptic curves  $A = E \times F$ . Let  $K^n(A)$  be the generalized Kummer variety of A. It admits a Lagrangian fibration  $\pi \colon K^n(A) \to \mathbb{P}^n$  whose general fiber is isomorphic to  $F^n$ . Let  $p \colon S \to E$  be a primary Kodaira surface which is a principal torsor over F. This is a non-Kähler holomorphic symplectic surface. Consider its associated Bogomolov–Guan manifold  $BG^n(S)$  [Gua95, Bog96]. It admits a Lagrangian fibration  $\pi' \colon BG^n(S) \to \mathbb{P}^n$  whose general fiber is also isomorphic to  $F^n$ . Actually the non-Kähler holomorphic symplectic manifold  $BG^n(S)$  is a Shafarevich–Tate twist of  $K^n(A)$ . As computed in [Gua95, Theorem 2],  $b_2(BG^n(S)) = 6$ , which is exactly  $b_2(K^n(A)) - 1$  in accordance with Theorem C. Theorem C also shows that the rank of the restriction map  $H^2(BG^n(A)) \to H^2(F^n)$  is zero.

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Anna Abasheva Columbia University, Department of Mathematics, 2990 Broadway, New York, NY, USA aa4643(at)columbia(dot)edu