MULTIPLICATIVE CHOW-KÜNNETH DECOMPOSITION AND HOMOLOGY SPLITTING OF CONFIGURATION SPACES

DAN PETERSEN AND ORSOLA TOMMASI

Abstract. We construct a splitting of the cohomology of configuration spaces of points on a smooth proper variety with a multiplicative Chow–Künneth decomposition. Applied to hyperelliptic curves, this shows that the hyperelliptic Torelli group acts trivially on the rational cohomology of ordered configuration spaces of points. Moreover, if $H_{g,n}$ denotes the moduli space of n-pointed hyperelliptic curves, the Leray spectral sequence for the forgetful map $H_{g,n} \to H_g$ degenerates immediately, in sharp contrast to the forgetful map from $M_{g,n}$ to M_g. This allows for new detailed calculations of the cohomology of $M_{2,n}$ for $n \leq 5$, and the stable cohomology of $H_{g,n}$ for $n \leq 5$. We also give a detailed study of the cohomology of symplectic local systems on M_2.

1. Introduction

1.1. Let X be a smooth projective complex algebraic variety. The famous theorem of Deligne–Griffiths–Morgan–Sullivan [DGMS75] says that X is *formal* in the sense of rational homotopy theory, meaning that if $C^\bullet(X, \mathbf{Q})$ denotes a cdga model for the algebra of cochains on X, then $C^\bullet(X, \mathbf{Q})$ and $H^\bullet(X, \mathbf{Q})$ are quasi-isomorphic; that is, there is a zig-zag of cdga quasi-isomorphisms connecting $C^\bullet(X, \mathbf{Q})$ and $H^\bullet(X, \mathbf{Q})$.

1.2. The algebra $C^\bullet(X, \mathbf{Q})$ may be equipped with additional structure, in particular it may be promoted to a commutative dg algebra in the category of cohomological mixed Hodge complexes, or ind-mixed Hodge structures, cf. [Mor78, Hai87, NA87, Cir15, Dre15, Tub23]. One may ask whether $C^\bullet(X, \mathbf{Q})$ and $H^\bullet(X, \mathbf{Q})$ are quasi-isomorphic as commutative dg algebras in mixed Hodge complexes. The answer is that this very rarely holds, as follows for example from [CCM81]. Similarly one can ask over more general base fields whether the étale cochains are quasi-isomorphic to the étale cohomology as ℓ-adic Galois representations, and again it is a rare property.

1.3. Yet highly nontrivial examples *do* exist. In all examples known to us, in fact a stronger property is satisfied. Recall that the Künneth type standard conjecture for X asserts the existence of projectors π_i acting on the Chow motive $h(X)$ decomposing the motive into summands $h^i(X)$, $i = 0, \ldots, 2d$, $d = \dim(X)$:

$$h(X) \cong \bigoplus_{i=0}^{2d} h^i(X).$$

The diagonal defines a map $h(X) \otimes h(X) \to h(X)$, the cup-product, and we say that X admits a *multiplicative decomposition* if the Künneth type standard conjecture holds for X and there exists a Künneth decomposition compatible with the cup-product: if we decompose the cup-product into components

$$\bigoplus_{i,j} h^i(X) \otimes h^j(X) \to \bigoplus_k h^k(X)$$

then $h^i(X) \otimes h^j(X) \to h^k(X)$ vanishes unless $i + j = k$. Equivalently, the class of the small diagonal $\Delta_{(3)} \subset X \times X \times X$ satisfies

$$[\Delta_{(3)}] = \sum_{i+j+k=2d} (\pi_i \times \pi_j \times \pi_k)_*[\Delta_{(3)}] \in \mathrm{CH}_d(X^3).$$

The notion of a multiplicative decomposition was isolated as an object of study in work of Voisin [Voi14] and explicitly by Shen–Vial [SV16]. Shen and Vial refer to this as a ***multiplicative Chow–Künneth decomposition***, but we believe that this property is natural enough to have a more concise name.

1.4. Let us mention some known and conjectural examples, beginning with examples predating the actual definition. One can show using the Fourier transform that abelian varieties always admit multiplicative decompositions [DM91, Kün94]. Hyperelliptic curves admit multiplicative decompositions, as we elaborate on shortly, but a very general curve of genus $g \geq 3$ does not, essentially due to [GS95, Cer83]. The results of Beauville–Voisin on the Chow ring of a $K3$ surface [BV04] can be interpreted as saying that $K3$ surfaces admit multiplicative decompositions. It is conjectured more generally that any hyperkähler variety admits a multiplicative decomposition [SV16]. This has been proven for Hilbert schemes of points on $K3$ surfaces [Via17, NOY21] and generalized Kummer varieties [FTV19]. According to a conjecture of Murre [Mur93], the conjectural Bloch–Beilinson filtration on Chow groups $\mathrm{CH}(X)$ is naturally induced by a Künneth decomposition of the Chow motive $h(X)$. Asking for this Künneth decomposition to be multiplicative would then imply a multiplicativity property of the Bloch–Beilinson filtration, so that the conjecture that hyperkähler varieties admit multiplicative decompositions is a a refinement of Beauville's conjecture [Bea07] that the Bloch–Beilinson filtration admits a multiplicative splitting for hyperkähler varieties. For more recent work and many pointers to the literature see [FLV21a, FLV21b].

1.5. If X is a curve, then there is a noncanonical isomorphism

$$h(X) \cong h^0(X) \oplus h^1(X) \oplus h^2(X)$$

depending on the choice of a degree 1 zero-cycle $o_X \in \mathrm{CH}_0(X)$, with corresponding projectors π_0, π_1 and π_2 given by

$$\pi_0 = [X] \times o_X, \qquad \pi_1 = [\Delta_{(2)}] - \pi_0 - \pi_2, \qquad \pi_2 = o_X \times [X].$$

The condition that this decomposition is multiplicative becomes that

(1) $$[\Delta_{(3)}] = \Delta_{(2)} \times o_X + (\text{perm.}) - [X] \times o_X \times o_X - (\text{perm.})$$

since $\Delta_{(2)} \times o_X = \sum_{i+j=2}(\pi_i \times \pi_j \times \pi_2)_*[\Delta_{(3)}]$ and $[X] \times o_X \times o_X = (\pi_0 \times \pi_2 \times \pi_2)_*[\Delta_{(3)}]$. This condition is exactly the vanishing of the Gross–Schoen cycle [GS95] associated to o_X. One can show the Gross–Schoen cycle associated to o_X can only vanish when $o_X = \frac{1}{2g-2}K_X$ [FLV21b, Section 3], and that in this case vanishing of the Gross–Schoen cycle is equivalent to vanishing of the Ceresa cycle [Cer83].

1.6. One can repeat the same analysis for an algebraic surface S with vanishing irregularity, where a degree 1 zero-cycle o_S induces similarly $h(S) \cong h^0(S) \oplus h^2(S) \oplus h^4(S)$. The vanishing of the analogue of the Gross–Schoen cycle is precisely what Beauville–Voisin [BV04, Proposition 2.2] proved in the case that S is a $K3$ surface, and o_S is the class of a point on a rational curve on S.

1.7. In this paper we investigate configuration spaces of points on varieties with a multiplicative decomposition. We define $F(X, n) = \{(x_1, \ldots, x_n) \in X^n : x_i \neq x_j \text{ for } i \neq j\}$.

1.8. Theorem. *Let X be a smooth proper complex algebraic variety admitting a multiplicative decomposition. The mixed Hodge structure on $H^\bullet(F(X, n), \mathbf{Q})$ is a direct sum of pure Hodge structures, for all n.*

1.9. The property that $H^\bullet(F(X,n),\mathbf{Q})$ is a direct sum of pure Hodge structures was recently investigated by Looijenga [Loo23] in the case that X is a curve. Looijenga proved that if X is a general curve of genus ≥ 3, then $H^\bullet(F(X,n),\mathbf{Q})$ does not decompose in this way as a direct sum of pure Hodge structures when $n \geq 3$. This is equivalent to the purely topological assertion that the Torelli group of a topological (closed oriented) surface Σ of genus ≥ 3 acts nontrivially on $H^\bullet(F(\Sigma,n),\mathbf{Q})$ for $n \geq 3$. Our result implies similarly the following purely topological statement:

1.10. **Corollary**. *Let Σ be a closed hyperelliptic surface. The hyperelliptic Torelli group acts trivially on $H^\bullet(F(\Sigma,n),\mathbf{Q})$ for all n.*

1.11. The action of the Torelli group of a surface on the cohomology of configuration spaces of points has received significant recent interest [Mor07, Bia20, BMW22, Sta23, BS22]. In particular, Bianchi and Stavrou [BS22] have conjectured that the genus 2 Torelli group acts trivially on the cohomology of configuration space, and our results confirm this conjecture (at least rationally[1]) since in genus two the hyperelliptic mapping class group coincides with the usual mapping class group.

1.12. Our argument for Theorem 1.8 is robust enough to work in families. For a morphism $X \to S$, define the relative configuration space $F(X/S,n)$ to be the complement of the "big diagonal" in the n-fold fibered product of X over S. For example, if $C \to M_g$ is the universal curve over the moduli space of curves, then $F(C/M_g,n) = M_{g,n}$. The argument for Theorem 1.8 shows with only minor modifications the following result.

1.13. **Theorem**. *Let $X \to S$ be a smooth proper morphism of smooth complex algebraic varieties. Let $f^{(n)} : F(X/S,n) \to S$ be the projection to the base from the relative configuration space. Suppose that $X \to S$ admits a fiberwise multiplicative decomposition. Then in Saito's category $\mathcal{D}^b(\mathsf{MHM}(S))$ there is a decomposition of $Rf^{(n)}_*\mathbf{Q}$ into a direct sum of degree-shifted polarizable variations of Hodge structure. In particular, the Leray spectral sequence for $f^{(n)}$ degenerates at E_2.*

1.14. Nontrivial examples of fiberwise multiplicative decompositions are even harder to come by. Abelian schemes admit multiplicative decompositions by the same Fourier transform arguments as in the absolute case. Voisin [Voi14] shows that if $X \to S$ is a family of $K3$ surfaces, then there is a Zariski dense open subset of S over which there exists a fiberwise multiplicative decomposition. Tavakol [Tav18] (see also [PTY21, §5.2.3]) showed that any family of hyperelliptic curves admits a fiberwise multiplicative decomposition. This gives the following corollary.

1.15. **Corollary**. *Let $H_{g,n}$ denote the moduli stack of smooth genus g hyperelliptic curves with n distinct ordered marked points. The Leray–Serre spectral sequence for the forgetful map $H_{g,n} \to H_g$ degenerates rationally.*

1.16. By contrast, the Leray–Serre spectral sequence for $f^{(n)} : M_{g,n} \to M_g$ is highly nondegenerate in general. Indeed, consider a closed oriented surface Σ, and consider $H^\bullet(F(\Sigma,n),\mathbf{Q})$ as a representation of the mapping class group $\mathrm{Mod}(\Sigma)$. One can show using the Totaro spectral sequence [Tot96] that if $g \geq 1$, then as a representation of $\mathrm{Mod}(\Sigma) \times S_3$ one has

$$H^4(F(\Sigma,3),\mathbf{Q}) \cong V_2 \otimes \sigma_{111} \oplus V_1 \otimes \sigma_3$$

where V_λ denotes an irreducible representation of Sp_{2g}, and σ_μ denotes a Specht module (irreducible representation of S_3). If $g \gg 0$, then $H^3(M_g, V_1) \cong \mathbf{Q}$ [Loo96]. Thus, one knows that in the Leray spectral sequence

$$E_2^{p,q} = H^p(M_g, R^q f^{(3)}_* \mathbf{Q}) \implies H^{p+q}(M_{g,3},\mathbf{Q})$$

[1]We do not know whether there is torsion in $H^\bullet(F(\Sigma,n),\mathbf{Z})$ when Σ is a genus two surface.

one has $E_2^{3,4} \cong \mathbf{Q}$ for sufficiently large g. On the other hand the stable cohomology of $M_{g,n}$ is concentrated in even degrees, so for g sufficiently large this class must be killed by a differential.[2]

1.17. Let Σ be a genus g closed surface. The Totaro spectral sequence [Tot96] can be used to calculate $H^\bullet(F(\Sigma,n),\mathbf{Q})$, and the spectral sequence is $\mathrm{Mod}(\Sigma)$-equivariant. The action of $\mathrm{Mod}(\Sigma)$ on the E_2-page factors through the symplectic group $\mathrm{Sp}_{2g}(\mathbf{Z})$, where it is given by an algebraic representation of Sp_{2g}. Consequently, there is an induced filtration on the cohomology of the configuration space, such that $\mathrm{Gr}\, H^\bullet(F(\Sigma,n),\mathbf{Q})$ is naturally an algebraic representation of Sp_{2g}. According to Corollary 1.10, determining $\mathrm{Gr}\, H^\bullet(F(\Sigma,n),\mathbf{Q})$ as an Sp_{2g}-module is equivalent to determining the action of the hyperelliptic mapping class group on $H^\bullet(F(\Sigma,n),\mathbf{Q})$. It thus follows from Corollary 1.15 that in order to calculate the cohomology groups $H^\bullet(H_{g,n},\mathbf{Q})$, it suffices to determine $\mathrm{Gr}\, H^\bullet(F(\Sigma,n),\mathbf{Q})$ as an Sp_{2g}-module, and to calculate $H^\bullet(H_g,V_\lambda)$ for any irreducible representation V_λ of Sp_{2g}.

1.18. There are two cases in which much is known about the cohomology of H_g with symplectic coefficients: namely, when $g=2$, and $g=\infty$.

- All genus two curves are hyperelliptic, and $H_2 = M_2$, which is an open substack of the moduli space A_2 of principally polarized abelian surfaces. All cohomology groups of A_2 with symplectic coefficients are known [Pet15], and the cohomology of $A_2 \setminus M_2$ with symplectic coefficients is likewise known. From this one may derive an almost complete description of the cohomology groups $H^\bullet(M_2,V_\lambda)$; in particular, all cohomology groups can be computed for $|\lambda| \leq 12$, and a conjectural formula exists for all λ. This is explained in Section 3 of the paper.
- As g grows and λ is fixed, the cohomology groups $H^\bullet(H_g,V_\lambda)$ enjoy homological stability. The stable cohomology of H_g with symplectic coefficients was recently calculated for all λ in [BDPW23].

1.19. In spite of the explicitness of the Totaro spectral sequence, determining $\mathrm{Gr}\, H^\bullet(F(\Sigma,n),\mathbf{Q})$ as an Sp_{2g}-module is in general very complicated. These groups were calculated for $g=2$ and $n \leq 5$ by the second author around 2007, see [Tom24] for a much later published version. The idea is that the differentials are highly constrained by the condition that they are equivariant for the natural action of $\mathrm{Sp}_4 \times S_n$ on the E_2-page, and that they are compatible with the various pullback maps $H^\bullet(F(\Sigma,n),\mathbf{Q}) \to H^\bullet(F(\Sigma,n+k),\mathbf{Q})$ induced by forgetting points, and for small n such considerations are enough to determine the differentials completely. These cohomology groups are also the subject of work in progress of Louis Hainaut [Hai24], and we have learned a complete answer for $n \leq 5$ from him for any genus, as well as conjectural answers for larger values of n. Combining the calculations of [Tom24] and Hainaut with the knowledge of cohomology with symplectic coefficients of §1.18, we are able to determine all rational cohomology groups of $M_{2,n}$ and $H_{\infty,n}$, for $n \leq 5$, both as representation of S_n and as a mixed Hodge structure up to semisimplification. (The mixed Hodge structure is mixed Tate.)

1.20. The determination of the cohomology groups of $M_{2,n}$ for $n \leq 5$ is not new — computing these groups was precisely the goal of [Tom24]. The results of [Tom24] were obtained using the Vassiliev–Gorinov method. The method used here seems much more likely to extend to larger values of n. Indeed, for $n \leq 12$ we have complete knowledge of the cohomology of the relevant local systems on M_2, so in this range one could compute the cohomology of $M_{2,n}$ if one knew the rational cohomology of $F(\Sigma,n)$ as a representation of Sp_4. Note that Corollary 1.10 shows that the natural action of $\mathrm{Mod}(\Sigma)$ factors through Sp_4 when $g=2$.

[2] We have omitted Tate twists in the above discussion, since they are not needed. More precisely one has $H^4(F(\Sigma,3),\mathbf{Q}) \cong V_2(-1)\otimes\sigma_{111} \oplus V_1(-2)\otimes\sigma_3$ and $H^3(M_g,V_1) \cong \mathbf{Q}(-2)$.

1.21. It is interesting to compare $H_{\infty,n}$ and $M_{\infty,n}$. The stable cohomology of $M_{\infty,n}$ is algebraic, coincides with the tautological ring, and conjecturally the cycle map from Chow to cohomology is an isomorphism in a stable range (this is however widely open). By contrast, the stable cohomology of $H_{\infty,n}$ is very complicated, not pure, and algebraic classes make up only a miniscule fragment of the cohomology. The Chow groups of $H_{g,n}$ have been completely determined by Canning and Larson in the range $n \leq 2g+6$ [CL22] — in this range they coincide with the tautological rings, calculated by Tavakol [Tav18].

2. Splitting the cohomology of configuration spaces

2.1. Let us first justify the claim made in the introduction, that if a variety X admits a multiplicative decomposition then $C^\bullet(X,\mathbf{Q}) \simeq H^\bullet(X,\mathbf{Q})$ compatibly with mixed Hodge structures. The key point is the following commutative diagram:

$$\begin{tikzcd}
\mathsf{SmComp}_{\mathbf{C}}^{\mathrm{op}} \arrow[rr, "h"] \arrow[dr, "C^\bullet"'] & & \mathsf{ChowMotives} \arrow[dl] \\
& \mathcal{D}^b(\mathsf{MHS}_{\mathbf{Q}}) &
\end{tikzcd}$$

Here $\mathsf{SmComp}_{\mathbf{C}}$ and $\mathsf{ChowMotives}$ are the 1-categories of smooth proper varieties and Chow motives over $\mathbf{C}$, and $\mathcal{D}^b(\mathsf{MHS}_{\mathbf{Q}})$ is the derived ∞-category of mixed Hodge structures. Importantly, all functors in the diagram are symmetric monoidal. For the two vertical arrows monoidality follows from recent work of Tubach [Tub23]. The functor $\mathsf{SmComp}_{\mathbf{C}}^{\mathrm{op}} \to \mathcal{D}^b(\mathsf{MHS}_{\mathbf{Q}})$ sends X to a Hodge-theoretically enhanced version of $C^\bullet(X,\mathbf{Q})$; in terms of the constructions of Tubach's paper, we are rather considering the object $(a_X)_*(a_X)^*\mathbf{Q}$ in $\mathcal{D}^b(\mathsf{MHM}(\mathrm{Spec}(\mathbf{C}))) \simeq \mathcal{D}^b(\mathsf{MHS}_{\mathbf{Q}})$, where a_X is the structural morphism of X. Now any X in $\mathsf{SmComp}_{\mathbf{C}}$ is a cocommutative comonoid object via the diagonal $X \to X \times X$. This fact, and monoidality of the functors in the diagram, makes $C^\bullet(X,\mathbf{Q})$ into a commutative monoid object, i.e. an $\mathbb{E}_\infty$-algebra, in the ∞-category $\mathcal{D}^b(\mathsf{MHS}_{\mathbf{Q}})$. Since the category is $\mathbf{Q}$-linear, such an $\mathbb{E}_\infty$-algebra can be rectified to an actual commutative dg algebra in mixed Hodge structures, unique up to contractible choice. But now the assumption that X admits a multiplicative decomposition implies that $h(X)$ is isomorphic as a commutative monoid to the direct sum $\bigoplus_i h^i(X)$, and commutativity of the diagram then implies the same for $C^\bullet(X,\mathbf{Q})$.

2.2. We can now prove Theorem 1.8.

Proof of Theorem 1.8. The first author has introduced an explicit functor $A \mapsto \mathcal{CF}(A,n)$ from non-unital commutative dg algebras to S_n-equivariant cochain complexes [Pet20]. It has the following properties:

(i) The functor $\mathcal{CF}(-,n)$ takes quasi-isomorphisms to quasi-isomorphisms.
(ii) If A is a cdga model for $C_c^\bullet(X,\mathbf{Q})$, where X is a locally compact Hausdorff topological space, then $H^\bullet(\mathcal{CF}(A,n)) \cong H_c^\bullet(F(X,n),\mathbf{Q})$.

The verification of these properties is sheaf-theoretic and uses only the six functors formalism, which implies that it makes sense purely algebraically and is compatible with all sorts of "enhanced" structure.

Now take X to be a smooth proper complex variety. We promote (a cdga model of) $C^\bullet(X,\mathbf{Q})$ to a cdga in mixed Hodge structures and plug it into the functor $\mathcal{CF}(-,n)$. Then the result computes $H_c^\bullet(F(X,n),\mathbf{Q})$ with its mixed Hodge structure. Now if X admits a multiplicative decomposition then $C^\bullet(X,\mathbf{Q}) \simeq H^\bullet(X,\mathbf{Q})$, and we may instead plug $H^\bullet(X,\mathbf{Q})$ into the functor, with its usual pure Hodge structure. But then the output is simply a cochain complex of pure polarizable Hodge structures, with differentials preserving the Hodge structure. Since this category is semisimple, the resulting cochain complex is quasi-isomorphic to its cohomology, and in particular $H_c^\bullet(F(X,n),\mathbf{Q})$ is a direct sum of pure Hodge structures. Then the same is true for $H^\bullet(F(X,n),\mathbf{Q})$ by Poincaré duality. □

2.3. A reader may be hesitant about the assertion that since the construction of $\mathcal{CF}$ only uses the six functors, it is compatible with mixed Hodge structures. One way to think about the functor $\mathcal{CF}$ is as follows. Fix a space X and a coefficient ring R. Let Π_n denote the poset of partitions of the set $\{1,\ldots,n\}$, ordered by refinement. There is a functor from Π_n to cochain complexes, assigning to a partition with k blocks the complex $C_c^\bullet(X^k,R)$. A morphism in Π_n corresponding to blocks being merged is mapped to a diagonal inclusion between cartesian powers of X. The total homotopy fiber[3] of this diagram is quasi-isomorphic to $C_c^\bullet(F(X,n),R)$. The point is that when $R=\mathbf{Q}$, the whole diagram $\Pi_n \to \mathrm{Ch}_{\mathbf{Q}}$ can be reconstructed up to quasi-isomorphism from a cdga model[4] for X and the Künneth formula, and the functor $\mathcal{CF}(-)$ is a Bousfield-Kan type resolution computing this total homotopy fiber. This is really a special case of a construction introduced in [Pet17] for a general stratified space. Now the main construction of [Pet17] has been generalized to the setting of étale motivic sheaves by Dupont and Juteau [DJ20], meaning that we obtain a model of $\mathcal{CF}$ in étale motives. With such a model in place one may then apply the realization functor to the derived category of mixed Hodge modules [Ivo16, Tub23]. This can also be carried out in families:

Proof of Theorem 1.13. We suppose as in the theorem that we are given a smooth proper family $X \to S$ admitting a fiberwise multiplicative decomposition.

Denote the n-fold fibered power of X over S by X^n. The space X^n is stratified with strata indexed by the partition lattice Π_n. By assigning to every element of Π_n the closure of the corresponding stratum we get a contravariant functor from Π_n to smooth proper varieties over S, taking a partition with k blocks to X^k. We may think of this as a diagram in Chow motives over S. By our assumption that there exists a fiberwise multiplicative decomposition, this diagram is isomorphic to a diagram whose values are pointwise given by direct sums of pure motives. Morphisms are morphisms of Chow motives and in particular preserving weights.

Now apply the realization to get a Π_n-shaped homotopy coherent diagram in the category $\mathrm{DA}_c(S)$ of étale motives over S of geometric origin [Ayo07a, Ayo07b, Ayo14, CD16]. By [DJ20] we can form the total homotopy fiber of this diagram by an explicit procedure which outputs a Postnikov system [GM03, Chapter 4] in $\mathrm{DA}_c(S)$, such that the output is weakly equivalent to the compactly supported motive of $F(X/S,n)$. From the explicit construction we see that all terms in the Postnikov system are still sums of pure motives, and all morphisms preserve weights.

Now apply the realization to mixed Hodge modules [Ivo16, Tub23]. In the resulting Postnikov system in $\mathcal{D}^b(\mathsf{MHM}(S))$, all terms are now sums of shifted polarizable variations of Hodge structures, with morphisms preserving the weights. By semisimplicity of the category of pure mixed Hodge modules of fixed weights, this Postnikov system must be isomorphic to one in which all objects are equal to their homology and all morphisms are zero. In particular the object of the Postnikov

[3]Let D be a category with an initial object $*$. The total homotopy fiber of a functor $F : D \to M$ taking values e.g. in a model category is the homotopy fiber of the map $F(*) \to \operatorname{holim}_{x\in D,\, x\not\simeq *} F(x)$.

[4]Over a more general ground ring one would need to know $C_c^\bullet(X,R)$ as an $\mathbb{E}_\infty$-algebra.

system which represents the total homotopy fiber has this property, which means that the object $Rf_!^{(n)}\mathbf{Q}$ is a direct sum of degree-shifted polarizable variations of Hodge structure. Then the same is true for $Rf_*^{(n)}$ by duality. $\square$

2.4. Remark. In our arguments it was critical that the six operations on mixed Hodge modules constructed by Saito [Sai90] can be lifted to the level of ∞-categories, as was recently proven by Tubach [Tub23]. Saito only constructed these operations as functors between triangulated categories, meaning in particular that commutative diagrams involving the six operations in Saito's formalism only commute up to non-specified, not necessarily coherent, homotopy. Such a diagram does not contain enough information to form e.g. a homotopy limit. The proof of Theorem 1.13 might not at first seem to involve any higher category theory — however, the construction of [DJ20] uses that $\mathrm{DA}(-)$ admits more structure than merely a system of triangulated categories; it is a stable derivator. Moreover, we need to know that the realization functor $\mathrm{DA}_c(S) \to \mathcal{D}^b(\mathsf{MHM}(S))$ commutes with the six operations, which was not known until [Tub23].

3. Local systems on M_2

3.1. In this section we shall explain what is known about the cohomology of the local systems V_λ on M_2, summarizing and slightly improving on results of [Pet15, Pet16, Wat18]. Our approach is to use the Torelli map, which in genus 2 is an open embedding

$$M_2 \hookrightarrow A_2$$

of the moduli space of genus two curves into the moduli space of principally polarized abelian surfaces. The complement $A_2 \setminus M_2$ is the locus $A_{1,1}$ of products of elliptic curves, and $A_{1,1} \cong \mathrm{Sym}^2 M_{1,1}$. We thus get a Gysin long exact sequence:

$$\ldots \to H^k(A_{1,1}, V_\lambda)(-1) \to H^{k+2}(A_2, V_\lambda) \to H^{k+2}(M_2, V_\lambda) \to H^{k+1}(A_{1,1}, V_\lambda)(-1) \to \ldots$$

As we explain shortly, all cohomology groups $H^\bullet(A_2, V_\lambda)$ and $H^\bullet(A_{1,1}, V_\lambda)$ are understood, and hence if we can determine all Gysin homomorphisms

$$H^k(A_{1,1}, V_\lambda)(-1) \to H^{k+2}(A_2, V_\lambda)$$

then we will have understood the cohomology of M_2 with symplectic coefficients (at least up to semisimplification).

3.2. A first observation is that (for $\lambda \neq 0$), $H^k(M_2, V_\lambda)$ is nontrivial only for $k \in \{1, 2, 3\}$. This is because the virtual cohomological dimension of the genus 2 mapping class group is 3, and H^0 vanishes since the genus two mapping class group surjects onto $\mathrm{Sp}_4(\mathbf{Z})$. We therefore only need to consider two Gysin homomorphisms:

$$H^0(A_{1,1}, V_\lambda)(-1) \to H^2(A_2, V_\lambda)$$

and

$$H^1(A_{1,1}, V_\lambda)(-1) \to H^3(A_2, V_\lambda).$$

3.3. Let us first discuss the cohomology of $A_{1,1}$ with symplectic coefficients. The computation of $H^\bullet(A_{1,1}, V_\lambda)$ reduces to computing the cohomology of $M_{1,1}$ with coefficient in an irreducible representation of SL_2, once one knows a branching formula for the restriction from Sp_4 to $\mathrm{SL}_2^2 \rtimes \Sigma_2$, which is determined in [Pet13, Proposition 3.4]. The restriction of the local systems V_λ to $A_{1,1}$

decomposes as a direct sum of local systems denoted $U_{a,b}$, U_a^+ and U_a^-. The cohomologies of these local systems are given by

$$\begin{aligned} H^\bullet(A_{1,1}, U_{a,b}) &\cong H^\bullet(M_{1,1}, V_a) \otimes H^\bullet(M_{1,1}, V_b) \\ H^\bullet(A_{1,1}, U_a^+) &\cong \operatorname{Sym}^2 H^\bullet(M_{1,1}, V_a) \\ H^\bullet(A_{1,1}, U_a^-) &\cong \wedge^2 H^\bullet(M_{1,1}, V_a) \end{aligned}$$

where V_ℓ denotes the local system on $M_{1,1}$ associated to the ℓth symmetric power representation of SL_2. The cohomology $H^\bullet(M_{1,1}, V_\ell)$ is determined by the Eichler–Shimura isomorphism:

$$H^k(M_{1,1}, V_\ell) \cong \begin{cases} \mathbf{Q} & k = \ell = 0 \\ \mathsf{S}_{\ell+2} \oplus \mathbf{Q}(-1-\ell) & k = 1, \ell > 0 \text{ even} \\ 0 & \text{otherwise.} \end{cases}$$

See [Ver61, Del69, Zuc79]. Here $\mathsf{S}_{\ell+2}$ denotes the "motive" (or Galois representation, or Hodge structure) attached to cusp forms for $\mathrm{SL}(2, \mathbf{Z})$ of weight $\ell + 2$. Hence $H^\bullet(A_{1,1}, V_\lambda)$ is completely known in all degrees for all λ.

3.4. From the preceding paragraph we see in particular that the only local systems on $A_{1,1}$ with nontrivial H^1 are the ones of the form $U_{a,0}$ with $a > 0$ even. From the branching formula [Pet13, Proposition 3.4], each V_λ (where $\lambda = (a, b)$) contains exactly one summand of this form when restricted to $A_{1,1}$, namely the local system $U_{a-b,0}(-b)$, except if $a = b$, in which case there is no such summand. Similarly $H^0(A_{1,1}, V_\lambda)$ is nontrivial only when the restriction of V_λ contains a copy of the trivial representation U_0^+ as a summand, and from the branching formula this happens if and only only if $\lambda = (a, b)$ satisfies $a = b \equiv 0 \pmod 2$. We obtain:

3.5. Proposition. *Let $\lambda = (a, b)$ be even. The cohomology $H^k(A_{1,1}, V_\lambda)$ in degrees $k = 0$ and $k = 1$ is given by*

$$H^0(A_{1,1}, V_\lambda) = \begin{cases} \mathbf{Q}(-b) & a = b \equiv 0 \pmod 2 \\ 0 & \textit{otherwise,} \end{cases}$$

and

$$H^1(A_{1,1}, V_\lambda) = \begin{cases} \mathsf{S}_{a-b+2}(-b) \oplus \mathbf{Q}(-1-a) & a > b \\ 0 & a = b, \end{cases}$$

respectively.

3.6. We now discuss the cohomology groups $H^\bullet(A_2, V_\lambda)$. All these cohomology groups were completely determined in [Pet15, Theorem 2.1], building on earlier work of several people e.g. [FvdG04, Har12, Wei09, Fli05, Art04]. We do not give the explicit formula, which is rather complicated; we will be content to describe the only as much of the cohomology as is needed for the computation of the relevant Gysin homomorphisms.

3.7. Starting with the Gysin map $H^0(A_{1,1}, V_\lambda)(-1) \to H^2(A_2, V_\lambda)$, we read from [Pet15] that if $\lambda = (a, b)$ is even then

$$H^2(A_2, V_\lambda) = \begin{cases} \mathbf{Q}(-1-b)^{\oplus s_{a+b+4}} & a = b \equiv 0 \pmod 2 \\ 0 & \text{otherwise.} \end{cases}$$

Here s_k denotes the dimension of the space of cusp forms for $\mathrm{SL}(2, \mathbf{Z})$ of weight k. Comparing with Proposition 3.5 we see that there is a possibility of a nonzero Gysin map precisely when $a = b \equiv 0 \pmod 2$ and $s_{a+b+4} > 0$. Since the domain of this Gysin map is one-dimensional we only need to determine if the homomorphism is zero or not. The Gysin map was shown to always be nonzero when $a = b \equiv 0 \pmod 2$ and $s_{a+b+4} > 0$ in [Pet16, Section 5]. Let us outline the structure of the argument. We omit Tate twists in what follows, since they will play no role. We consider the map $f : A_1 \times A_1 \to A_2$, and fix an inclusion of the trivial local system $\mathbf{Q}$ into $f^*V_{b,b}$. It suffices to show $H^0(A_1 \times A_1, \mathbf{Q}) \to H^2(A_2, V_{b,b})$ is nonzero. We consider the Satake compactifications $\overline{A}_1$ and $\overline{A}_2$ and the embeddings $j' : A_1 \times A_1 \to \overline{A}_1 \times \overline{A}_1$, $j : A_2 \to \overline{A}_2$. There is a commutative diagram

$$\begin{tikzcd} H^0(A_1 \times A_1, \mathbf{Q}) \arrow[r] \arrow[d] & H^2(A_2, V_{b,b}) \arrow[d] \\ (R^0 j'_* \mathbf{Q})_\xi \arrow[r] & (R^2 j_* V_{b,b})_{f(\xi)} \end{tikzcd}$$

where ξ denotes the maximally degenerate point of $\overline{A}_1 \times \overline{A}_1$, $f(\xi)$ is its image in $\overline{A}_2$ (which again is the unique maximally degenerate point), and the subscripts denote talking stalks of the respective sheaves. Since $R^0 j'_* \mathbf{Q} \cong \mathbf{Q}$ the left vertical arrow is an isomorphism and it suffices to show that the Gysin map between stalks is nonzero. A formula due to Harder gives an expression for these stalks that can eventually be used to prove the result. The outcome is the following theorem.

3.8. Theorem.

$$H^1(M_2, V_\lambda) \cong \begin{cases} \mathbf{Q}(-3) & \lambda = (2,2) \\ 0 & \textit{otherwise.} \end{cases}$$

3.9. We may now similarly consider the Gysin map $H^1(A_{1,1}, V_\lambda)(-1) \to H^3(A_2, V_\lambda)$. We will not state the full expression for $H^3(A_2, V_\lambda)$ from [Pet15] as it is a bit cumbersome. What will be relevant is that (up to semisimplification) $H^3(A_2, V_{a,b})$ contains as a direct summand the terms

$$\mathsf{S}_{a-b+2}(-1-b)^{\oplus s_{a+b+4}} + \mathbf{Q}(-2-a)^{\oplus s_{a+b+4}};$$

moreover, if $b = 0$ it contains an additional term $\mathbf{Q}(-2-a)$. The remaining terms occuring in $H^3(A_2, V_\lambda)$ do not occur in $H^1(A_{1,1}, V_\lambda)(-1)$ and are therefore irrelevant for the question of determining the Gysin map. The natural expectation is that if $a > b$ and $s_{a+b+4} > 0$ then $H^1(A_{1,1}, V_{a,b})(-1) \cong \mathsf{S}_{a-b+2}(-b) \oplus \mathbf{Q}(-1-a)$ injects into $H^3(A_2, V_{a,b})$, and if $b = 0$ then at least the summand $\mathbf{Q}(-1-a)$ injects. Part of this was proven by Watanabe [Wat18], who showed that if $a > b$ and $s_{a+b+4} > 0$ then the Gysin map is injective on the summand $\mathbf{Q}(-2-a)$ of $H^1(A_{1,1}, V_{a,b})$. His argument was an adaptation of the one used in [Pet16], considering instead the commutative diagram

$$\begin{tikzcd} H^1(A_1 \times A_1, V_{a-b} \boxtimes \mathbf{Q}) \arrow[r] \arrow[d] & H^3(A_2, V_{a,b}) \arrow[d] \\ (R^1 j'_*(V_{a-b} \boxtimes \mathbf{Q}))_\xi \arrow[r] & (R^3 j_* V_{a,b})_{f(\xi)} \end{tikzcd}$$

where we use the same notation as before. For the embedding $i : A_1 \to \overline{A}_1$ the stalk of $R^1 i_* V_a$ at infinity is one-dimensional for all even $a \geq 0$, so also $R^1 j'_*(V_{a-b} \boxtimes \mathbf{Q}))_\xi$ is one-dimensional.

3.10. We will now improve on this by showing moreover that if $b = 0$ then the summand $\mathbf{Q}(-2-a)$ of $H^1(A_{1,1}, V_{a,0})(-1)$ injects into $H^3(A_2, V_{a,0})$ under the Gysin map. (This will in fact be simpler than the arguments of [Pet16] and [Wat18].) We fix an embedding of $V_a \boxtimes \mathbf{Q}$ into $f^* V_{a,0}$ and consider instead the commutative diagram

$$\begin{tikzcd} H^1(A_1 \times A_1, V_a \boxtimes \mathbf{Q}) \arrow[r] \arrow[d] & H^3(A_2, V_{a,0}) \arrow[d] \\ (R^1 j'_*(V_a \boxtimes \mathbf{Q}))_\eta \arrow[r] & (R^3 j_* V_{a,0})_{f(\eta)} \end{tikzcd}$$

where we now let $\eta \in \overline{A}_1 \times \overline{A}_1$ be the point at infinity of the first factor, but an arbitrary point in the interior of the second factor. Then again the stalk $(R^1 j'_*(V_a \boxtimes \mathbf{Q}))_\eta$ is one-dimensional and the left vertical map is an isomorphism. The stalks occuring here are determined by Harder's formula as discussed in [Pet16, Section 5]. (From now on we freely suppose that the reader has read [Pet16, Section 5]). Let us begin by considering the stalk at $f(\eta)$. Since $f(\eta)$ lies on the one-dimensional boundary stratum of $\overline{A}_2$ (as opposed to the zero-dimensional stratum) we need to replace the Siegel parabolic in Harder's formula with the Klingen parabolic, which we take to consist of matrices of the form

$$\begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & 0 \\ 0 & * & * & * \end{pmatrix}.$$

We write MU for the Levi decomposition of the Klingen parabolic, where $M \cong \mathrm{SL}_2 \times \mathbb{G}_m$ is given by matrices of the form

$$\begin{pmatrix} * & 0 & 0 & 0 \\ 0 & * & 0 & * \\ 0 & 0 & * & 0 \\ 0 & * & 0 & * \end{pmatrix},$$

and denote by $\mathfrak{u}$ the Lie algebra of U. Harder's formula gives

$$(R^3 j_* V_{a,0})_{f(\eta)} \cong \bigoplus_{p+q=3} H^p(M(\mathbf{Z}), H^q(\mathfrak{u}, V_{a,0})).$$

We claim that this expression equals $H^3(\mathfrak{u}, V_{a,0})$. Indeed as discussed in [Pet16, Section 5] a theorem of Kostant calculates $H^q(\mathfrak{u}, V_{a,b})$ as an M-module, and Kostant's formula shows that as an SL_2-module it is irreducible, given by the bth symmetric power representation when $q = 0, 3$ and the $(a+1)$st symmetric power when $q = 1, 2$ [Sch95, Table 2.3.4]. The result follows. Now we consider the intersection of the Klingen parabolic with the standard embedded copy of $H = \mathrm{SL}_2 \times \mathrm{SL}_2$ in Sp_4, i.e.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \times \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \mapsto \begin{pmatrix} a & 0 & b & 0 \\ 0 & a' & 0 & b' \\ c & 0 & d & 0 \\ 0 & c' & 0 & d' \end{pmatrix}.$$

This intersection gives the standard Borel parabolic in the first factor and the whole group in the second factor, which means that we chose our Klingen parabolic wisely as this corresponds precisely to the stratum of the point η. We may then repeat the calculation: we get the same Levi but a smaller unipotent radical $\mathfrak{u}_H$ which is now one-dimensional, and we find similarly $(R^1 j'_*(V_a \boxtimes \mathbf{Q}))_\eta \cong H^1(\mathfrak{u}_H, V_a \boxtimes \mathbf{Q})$. Now we should determine the nonvanishing of the Gysin map $H^1(\mathfrak{u}_H, V_a \boxtimes \mathbf{Q}) \to H^3(\mathfrak{u}, V_{a,0})$. But this map is nothing by the Poincaré dual of the restriction

$$\mathbf{Q} \cong H^0(\mathfrak{u}, V_{a,0}) \to H^0(\mathfrak{u}_H, V_a \boxtimes \mathbf{Q}) \cong \mathbf{Q}$$

which is evidently an isomorphism.

3.11. To recapitulate, we know that $H^1(A_{1,1}, V_{a,b})(-1) \cong \mathsf{S}_{a-b+2}(-b) \oplus \mathbf{Q}(-1-a)$ if $a > b$, and the group vanishes for $a = b$. We know that the Gysin map is always nonzero on the second summand $\mathbf{Q}(-1-a)$. Our expectation is that the Gysin map restricted to the first summand $\mathsf{S}_{a-b+2}(-b)$ is nonzero whenever it can be, i.e. whenever $s_{a-b+2} > 0$ and $s_{a+b+4} > 0$. The smallest (a, b) satisfying both these conditions is $(a, b) = (11, 1)$, which makes the local system $V_{11,1}$ the unique local system of weight ≤ 12 for which there is an ambiguity.

λ	$H^1(M_2, V_\lambda)$	$H^2(M_2, V_\lambda)$	$H^3(M_2, V_\lambda)$
$(1,1)$	0	0	L^5
$(2,0)$	0	0	0
$(2,2)$	L^3	0	0
$(3,1)$	0	L^5	L^7
$(4,0)$	0	0	0
$(3,3)$	0	0	$L^9 + L^8$
$(4,2)$	0	L^6	0
$(5,1)$	0	L^7	$L^9 + L^8$
$(6,0)$	0	0	L^9
$(4,4)$	0	0	0
$(5,3)$	0	0	$L^{11} + L^{10} + L^9$
$(6,2)$	0	0	L^{10}
$(7,1)$	0	0	$2L^{11} + L^{10}$
$(8,0)$	0	0	$L^{11} + L^{10}$
$(5,5)$	0	0	$L^{13} + L^{12} + L^{10}$
$(6,4)$	0	L^8	L^{11}
$(7,3)$	0	L^9	$L^{13} + 2L^{12} + L^{11} + L^{10}$
$(8,2)$	0	L^{10}	$L^{13} + L^{12} + L^{11}$
$(9,1)$	0	L^{11}	$2L^{13} + 2L^{12} + L^2\mathsf{S}_{12}$
$(10,0)$	0	$L\mathsf{S}_{12}$	L^{13}
$(6,6)$	0	0	0
$(7,5)$	0	0	$L^{15} + L^{14} + L^{13} + L^{12} + L^{11}$
$(8,4)$	0	0	$L^{14} + L^{13} + L^{12}$
$(9,3)$	0	0	$2L^{15} + 2L^{14} + 2L^{13} + L^{12} + L^4\mathsf{S}_{12}$
$(10,2)$	0	0	$L^{15} + 2L^{14} + L^{13} + L^4\mathsf{S}_{12}$
$(11,1)$	0	0	$2L^{15} + 2L^{14} + L^4\mathsf{S}_{12}$
$(12,0)$	0	0	$2L^{15} + L^{14} + L^4\mathsf{S}_{12}$

Table 1. Cohomology of symplectic local systems on M_2 of weight at most 12.

3.12. However, this ambiguity in the cohomology of the local system $V_{11,1}$, using recent calculations of Canning–Larson–Payne [CLP23]. Suppose by way of contradiction that the Gysin map were zero. We would obtain a nonzero summand[5] $\mathsf{S}_{12}(-2)$ inside $H^3(M_2, V_{11,1})$, which under the Leray–Serre spectral sequence for the forgetful map $M_{2,12} \to M_2$ would give rise to a nonzero summand $\mathsf{S}_{12}(-2)$ inside $H^{15}(M_{2,12}, \mathbf{Q})$. This summand is of weight 15, i.e. of *pure* weight. This means [Del71, Corollaire 3.2.17] that it is in the image of the restriction map from $H^{15}(\overline{M}_{2,12}, \mathbf{Q})$, implying the existence of a class in $H^{15}(\overline{M}_{2,12}, \mathbf{Q})$ not pushed forward from the Deligne–Mumford boundary. But [CLP23, Lemma 7.3] says precisely that the group $H^{15}(\overline{M}_{2,12}, \mathbf{Q})$ is generated by cycles pushed forward from the boundary! As a consequence we obtain the following theorem:

3.13. Theorem. *Let $\lambda = (a,b)$ be even, and suppose that $a+b \leq 12$. Then*

$$H^2(M_2, V_\lambda) \cong \begin{cases} \mathbf{Q}(-a-2) & a > b > 0 \text{ and } a+b \notin \{8,12\} \\ \mathsf{S}_{12}(-1) & (a,b) = (10,0) \\ 0 & \text{otherwise.} \end{cases}$$

[5] Or rather, a summand up to semisimplification.

3.14. Once $H^1(M_2, V_\lambda)$ and $H^2(M_2, V_\lambda)$ are known one can of course leverage the knowledge of $H^\bullet(A_{1,1}, V_\lambda)$ and $H^\bullet(A_2, V_\lambda)$ to determine also $H^3(M_2, V_\lambda)$. In carrying this out in practice it is easier to work with the Euler characteristics of the spaces involved (taken in a suitable Grothendieck group of Galois representations/Hodge structures) than to work with the main theorem of [Pet15], as the Euler characteristics involved admit much simpler formulae (cf. [FvdG04]). For the reader's edification we compile in Table 1 all cohomology groups $H^\bullet(M_2, V_\lambda)$ for $|\lambda| \le 12$.

3.15. Remark. As mentioned previously, the cohomology $H^\bullet(M_2, V_\lambda)$ was previously determined by the second named author when $|\lambda| \le 4$ [Tom24], by using the Vassiliev–Gorinov method to compute the cohomology of $M_{2,n}$ for small n, and tracing the results backwards through the Leray–Serre spectral sequence.

3.16. Conjecture. *Let $\lambda = (a, b)$ be even, and suppose that $a + b \ge 14$. Then*

$$H^2(M_2, V_\lambda) \cong \begin{cases} \mathbf{Q}(-b-1)^{\oplus(s_{a+b+4}-1)} & a = b \equiv 0 \pmod 2 \\ 0 & \textit{otherwise.} \end{cases}$$

3.17. The precise status of this conjecture, after our discussion above, is the following: $H^2(M_2, V_\lambda)$ is either given by the expression in the conjecture, or it contains an additional summand, which is itself a summand of $\mathsf{S}_{a-b+2}(-b-1)$.

3.18. Remark. The general strategy used in [Pet16, Wat18] and in §3.10 of reducing to a calculation on a deleted neighborhood of a boundary stratum has no chance of working when trying to prove nonvanishing of the Gysin homomorphism on the summand $\mathsf{S}_{a-b+2}(-1-b)$. The summand $\mathsf{S}_{a-b+2}(-1-b)^{\oplus s_{a+b+4}}$ of $H^3(A_2, V_{a,b})$ is precisely the part of cohomology coming from "endoscopy". Unlike the other classes we have considered (which are part of the "Eisenstein cohomology") it is genuinely supported on the interior of the Siegel variety.

3.19. Remark. We see that classes in $H^2(M_2, V_{a,b})$ of pure weight (i.e. weight $a+b+2$) occur only for $a = b \equiv 0 \pmod 2$, and that the first case where such a class occurs is when $\lambda = (10, 10)$, since 24 is the smallest weight for which the space of cusp forms of $\mathrm{SL}(2, \mathbf{Z})$ is of dimension > 1. The fact that $H^2(M_2, V_{10,10}) \cong \mathbf{Q}(-11)$ is what gives rise to the non-tautological class on $\overline{M}_{2,20}$ identified by Graber–Pandharipande [GP03] and the failure of the tautological ring of $\overline{M}_{2,20}$ to have Poincaré duality [PT14, Pet16].

4. Tables of computations

4.1. As a consequence of what we have done in this paper, we are able to compute the cohomology of $M_{2,n}$ and $H_{\infty,n}$ for $n \le 5$. More precisely, we determine the rational cohomology groups, considered both as mixed Hodge structures and as representations of the symmetric groups S_n. The cohomology of $H_{\infty,n}$ is mixed Tate for all n. It is not in general pure, but genuinely mixed; however, the mixed Hodge structure is a direct sum of pure Hodge structures (i.e. there are no nontrivial extensions). The cohomology of $M_{2,n}$ is mixed Tate for $n < 10$, but not for $n \ge 10$, as can be seen from Table 1. When $n < 6$ it is even a direct sum of pure Hodge structures, as there is no room for any nontrivial extensions in the corresponding cohomology groups appearing in Table 1.

4.2. In tabulating these cohomology groups, we will find it convenient to describe them in terms of certain expressions f_λ and h_λ, which we define now. Recall that if λ is a partition of n, then σ_λ denotes the representation of S_n associated to λ. We define $f_\lambda = \sum_i [H^i(M_{2,n}, \mathbf{Q}) \otimes_{S_n} \sigma_\lambda] \cdot t^i$ in $K_0(\mathsf{MHS})[t]$. We write L for the Lefschetz motive, the class of $\mathbf{Q}(-1)$ in $K_0(\mathsf{MHS})$. By what we said in the preceding paragraph, f_λ is a polynomial in t and L when $|\lambda| < 10$. In Table 2 we give the value of f_λ for $|\lambda| \leq 5$.

4.3. We define similarly $h_\lambda = \sum_i [H^i(H_{\infty,n}, \mathbf{Q}) \otimes_{S_n} \sigma_\lambda] \cdot t^i$. For a general λ, the cohomology $H^i(H_\infty, V_\lambda)$ is nonzero for infinitely many i, and in particular h_λ is generally not a polynomial. However, h_λ is always a rational function. Indeed, it turns out that for any λ, the expression

$$\sum_i [H^i(H_\infty, V_\lambda)] \cdot t^i$$

is a rational function of Lt, whose denominator is a product of factors $(1 + L^k t^k)$ $(k \geq 1)$ and $(1 - Lt)$, and the denominator is of degree at most $|\lambda|$ [BDPW23, Remark 10.0.12]. When $|\lambda| < 4$ the cohomology is in fact finite-dimensional, which means that h_λ is a polynomial when $|\lambda| < 4$, tabulated in Table 3. However, h_λ is genuinely a rational function for $|\lambda| \geq 4$. These rational functions for $n = 4, 5$ are tabulated in Table 4, placed on a common denominator.

4.4. Remark. The phenomenon of *representation stability* is visible in these tables, as n grows. That is, the polynomials f_λ associated with the partitions $\lambda = (\lambda_1, \lambda_2, \ldots)$ and $\lambda = (\lambda_1 + 1, \lambda_2, \ldots)$ agree in a growing range of degrees. Similarly for h_λ. Representation stability of the sequence of S_n-representations given by the cohomology of $M_{g,n}$ is due to Jimenez Rolland [JR11]. In our case, we have expressed the cohomology of $M_{2,n}$ resp. $H_{\infty,n}$ directly in terms of the cohomology of the ordered configuration spaces of points on a surface, so one could also see representation stability directly from the results of Church [Chu12].

4.5. Remark. The limiting factor that prevents us from making computations past $n = 5$ is that computing the cohomology of configuration spaces of points is a formidable task. From Totaro [Tot96] one gets an explicit chain complex calculating the cohomology of configuration spaces of points on a surface, but implementing this on a computer quickly becomes infeasible. From Hainaut [Hai24] we have obtained complete answers for $n \leq 5$ and conjectural answers for $n \leq 7$. The second named author has also carried out similar computer-assisted computations in the case $g = 2$, also for $n \leq 5$ [Tom24]. For unordered configuration a complete calculation of the cohomology is known due to Pagaria [Pag23], taking into account both the weight filtration and the Sp_{2g}-action. (This is an "enhanced" version of [DCK17, Knu17, FT05].) His results and our methods allow for the calculation of the cohomology of $M_{2,n}/S_n$ and $H_{\infty,n}/S_n$ for any n, including the mixed Hodge structure.

References

[Art04] James Arthur. Automorphic representations of $\mathrm{GSp}(4)$. In *Contributions to automorphic forms, geometry, and number theory*, pages 65–81. Johns Hopkins Univ. Press, Baltimore, MD, 2004.

[Ayo07a] Joseph Ayoub. Les six opérations de Grothendieck et le formalisme des cycles évanescents dans le monde motivique. I. *Astérisque*, (314):x+466 pp. (2008), 2007.

[Ayo07b] Joseph Ayoub. Les six opérations de Grothendieck et le formalisme des cycles évanescents dans le monde motivique. II. *Astérisque*, (315):vi+364 pp. (2008), 2007.

[Ayo14] Joseph Ayoub. La réalisation étale et les opérations de Grothendieck. *Ann. Sci. Éc. Norm. Supér. (4)*, 47(1):1–145, 2014.

[BDPW23] Jonas Bergström, Adrian Diaconu, Dan Petersen, and Craig Westerland. Hyperelliptic curves, the scanning map, and moments of families of quadratic L-functions. arXiv:2302.07664, 2023.

[Bea07] Arnaud Beauville. On the splitting of the Bloch-Beilinson filtration. In *Algebraic cycles and motives. Vol. 2*, volume 344 of *London Math. Soc. Lecture Note Ser.*, pages 38–53. Cambridge Univ. Press, Cambridge, 2007.

[Bia20] Andrea Bianchi. Splitting of the homology of the punctured mapping class group. *J. Topol.*, 13(3):1230–1260, 2020.

[BMW22] Andrea Bianchi, Jeremy Miller, and Jennifer C. H. Wilson. Mapping class group actions on configuration spaces and the Johnson filtration. *Trans. Amer. Math. Soc.*, 375(8):5461–5489, 2022.

λ	f_λ
$\varnothing$	1
(1)	$1+Lt^2$
(2)	$1+Lt^2+L^5t^5$
(1^2)	Lt^2
(3)	$1+Lt^2+L^2t^3+L^5t^5$
$(2,1)$	$Lt^2+L^5t^5$
(1^3)	0
(4)	$1+Lt^2+L^2t^3+\big(L^5+L^3\big)t^5+L^6t^6+L^7t^8$
$(3,1)$	$Lt^2+L^2t^3+L^5t^5$
(2^2)	$L^3t^4+\big(L^5+L^3\big)t^5$
$(2,1^2)$	$L^5t^6+L^7t^7$
(1^4)	0
(5)	$1+Lt^2+L^2t^3+\big(L^5+L^3\big)t^5+L^6t^6+2L^7t^8$
$(4,1)$	$Lt^2+L^2t^3+\big(L^5+L^3\big)t^5+\big(L^6+L^4\big)t^6+L^6t^7+\big(L^8+2L^7\big)t^8$
$(3,2)$	$L^2t^3+L^3t^4+\big(L^5+L^3\big)t^5+L^4t^6+L^6t^7+L^8t^8+L^8t^9$
$(3,1^2)$	$L^4t^5+\big(L^5+L^4\big)t^6+\big(L^7+L^6\big)t^7+L^8t^8$
$(2^2,1)$	$L^3t^4+L^3t^5+L^5t^6+L^7t^7$
$(2,1^3)$	$L^5t^6+L^7t^7$
(1^5)	0

Table 2. Cohomology of $M_{2,n}$ for $n \leq 5$.

λ	h_λ
$\varnothing$	1
(1)	$1+Lt^2$
(2)	$1-Lt+\big(L^2+L\big)t^2-\big(L^3+L^2\big)t^3+L^3t^4-L^4t^5$
(1^2)	$Lt^2-L^2t^3+L^3t^4-L^4t^5$
(3)	$1-Lt+\big(L^2+L\big)t^2-L^3t^3-L^5t^6$
$(2,1)$	$Lt^2-L^2t^3+L^3t^4-L^4t^5$
(1^3)	0

Table 3. Cohomology of $H_{\infty,n}$ for $n<4$.

λ	$h_\lambda\cdot(1-Lt)(1+L^2t^2)$
(4)	$1-Lt+\big(L^2+L\big)t^2-L^3t^3+L^3t^5-\big(L^5+L^4\big)t^6+L^5t^7$
$(3,1)$	$Lt^2-\big(L^5-L^4\big)t^6$
(2^2)	$L^3t^4-\big(L^4-L^3\big)t^5+\big(L^5-L^4\big)t^6-\big(L^6-L^5\big)t^7$
$(2,1^2)$	0
(1^4)	0
(5)	$1-Lt+\big(L^2+L\big)t^2-L^3t^3+L^3t^5-\big(L^5+L^4\big)t^6+2L^5t^7$
$(4,1)$	$Lt^2+L^3t^5-\big(L^5-L^4\big)t^6+L^5t^7+L^6t^8$
$(3,2)$	$L^2t^3+L^3t^5+L^4t^6-\big(L^6-L^5\big)t^7+L^6t^8$
$(3,1^2)$	$L^4t^5-\big(L^5-2L^4\big)t^6+\big(L^6-L^5\big)t^7-\big(L^7-L^6\big)t^8$
$(2^2,1)$	$L^3t^4-\big(L^4-L^3\big)t^5+\big(L^5-L^4\big)t^6-\big(L^6-2L^5\big)t^7$
$(2,1^3)$	0
(1^5)	0

Table 4. Cohomology of $H_{\infty,4}$ and $H_{\infty,5}$.

[BS22] Andrea Bianchi and Andreas Stavrou. Non-trivial action of the Johnson filtration on the homology of configuration spaces. Preprint available at arXiv:2208.01608, 2022.

[BV04] Arnaud Beauville and Claire Voisin. On the Chow ring of a $K3$ surface. *J. Algebraic Geom.*, 13(3):417–426, 2004.

[CCM81] James Carlson, Herbert Clemens, and John Morgan. On the mixed Hodge structure associated to π_3 of a simply connected complex projective manifold. *Ann. Sci. École Norm. Sup. (4)*, 14(3):323–338, 1981.

[CD16] Denis-Charles Cisinski and Frédéric Déglise. Étale motives. *Compos. Math.*, 152(3):556–666, 2016.

[Cer83] Giuseppe Ceresa. C is not algebraically equivalent to C^- in its Jacobian. *Ann. of Math. (2)*, 117(2):285–291, 1983.

[Chu12] Thomas Church. Homological stability for configuration spaces of manifolds. *Invent. Math.*, 188(2):465–504, 2012.

[Cir15] Joana Cirici. Cofibrant models of diagrams: mixed Hodge structures in rational homotopy. *Trans. Amer. Math. Soc.*, 367(8):5935–5970, 2015.

[CL22] Samir Canning and Hannah Larson. The rational Chow rings of moduli spaces of hyperelliptic curves with marked points. arXiv:2207.10873, 2022.

[CLP23] Samir Canning, Hannah Larson, and Sam Payne. Extensions of tautological rings and motivic structures in the cohomology of $\overline{\mathcal{M}}_{g,n}$. arXiv:2307.08830, 2023.

[DCK17] Gabriel C. Drummond-Cole and Ben Knudsen. Betti numbers of configuration spaces of surfaces. *J. Lond. Math. Soc. (2)*, 96(2):367–393, 2017.

[Del69] Pierre Deligne. Formes modulaires et représentations ℓ-adiques. *Sém. Bourbaki*, 355:139–172, 1969.

[Del71] Pierre Deligne. Théorie de Hodge. II. *Inst. Hautes Études Sci. Publ. Math.*, (40):5–57, 1971.

[DGMS75] Pierre Deligne, Phillip Griffiths, John Morgan, and Dennis Sullivan. Real homotopy theory of Kähler manifolds. *Invent. Math.*, 29(3):245–274, 1975.

[DJ20] Clément Dupont and Daniel Juteau. The localization spectral sequence in the motivic setting. Available at arXiv:2003.04217, 2020.

[DM91] Christopher Deninger and Jacob Murre. Motivic decomposition of abelian schemes and the Fourier transform. *J. Reine Angew. Math.*, 422:201–219, 1991.

[Dre15] Brad Drew. Rectification of Deligne's mixed Hodge structures. Preprint available at arXiv:1511.08288, 2015.

[Fli05] Yuval Z. Flicker. *Automorphic forms and Shimura varieties of* $\mathrm{PGSp}(2)$. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2005.

[FLV21a] Lie Fu, Robert Laterveer, and Charles Vial. The generalized Franchetta conjecture for some hyper-Kähler varieties, II. *J. Éc. polytech. Math.*, 8:1065–1097, 2021.

[FLV21b] Lie Fu, Robert Laterveer, and Charles Vial. Multiplicative Chow-Künneth decompositions and varieties of cohomological K3 type. *Ann. Mat. Pura Appl. (4)*, 200(5):2085–2126, 2021.

[FT05] Yves Félix and Daniel Tanré. The cohomology algebra of unordered configuration spaces. *J. London Math. Soc. (2)*, 72(2):525–544, 2005.

[FTV19] Lie Fu, Zhiyu Tian, and Charles Vial. Motivic hyper-Kähler resolution conjecture, I: generalized Kummer varieties. *Geom. Topol.*, 23(1):427–492, 2019.

[FvdG04] Carel Faber and Gerard van der Geer. Sur la cohomologie des systèmes locaux sur les espaces de modules des courbes de genre 2 et des surfaces abéliennes. I. *C. R. Math. Acad. Sci. Paris*, 338(5):381–384, 2004.

[GM03] Sergei I. Gelfand and Yuri I. Manin. *Methods of homological algebra*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, second edition, 2003.

[GP03] Tom Graber and Rahul Pandharipande. Constructions of nontautological classes on moduli spaces of curves. *Michigan Math. J.*, 51(1):93–109, 2003.

[GS95] Benedict H. Gross and Chad Schoen. The modified diagonal cycle on the triple product of a pointed curve. *Ann. Inst. Fourier (Grenoble)*, 45(3):649–679, 1995.

[Hai87] Richard M. Hain. The de Rham homotopy theory of complex algebraic varieties. I. *K-Theory*, 1(3):271–324, 1987.

[Hai24] Louis Hainaut. Personal communication. 2024.

[Har12] Günter Harder. The Eisenstein motive for the cohomology of $\mathrm{GSp}_2(\mathbb{Z})$. In Carel Faber, Gavril Farkas, and Robin de Jong, editors, *Geometry and arithmetic*, EMS Ser. Congr. Rep., pages 143–164. Eur. Math. Soc., Zürich, 2012.

[Ivo16] Florian Ivorra. Perverse, Hodge and motivic realizations of étale motives. *Compos. Math.*, 152(6):1237–1285, 2016.

[JR11] Rita Jimenez Rolland. Representation stability for the cohomology of the moduli space $\mathscr{M}_g^n$. *Algebr. Geom. Topol.*, 11(5):3011–3041, 2011.

[Knu17] Ben Knudsen. Betti numbers and stability for configuration spaces via factorization homology. *Algebr. Geom. Topol.*, 17(5):3137–3187, 2017.

[Kün94] Klaus Künnemann. On the Chow motive of an abelian scheme. In *Motives (Seattle, WA, 1991)*, volume 55 of *Proc. Sympos. Pure Math.*, pages 189–205. Amer. Math. Soc., Providence, RI, 1994.

[Loo96] Eduard Looijenga. Stable cohomology of the mapping class group with symplectic coefficients and of the universal Abel-Jacobi map. *J. Algebraic Geom.*, 5(1):135–150, 1996.

[Loo23] Eduard Looijenga. Torelli group action on the configuration space of a surface. *J. Topol. Anal.*, 15(1):215–222, 2023.

[Mor78] John W. Morgan. The algebraic topology of smooth algebraic varieties. *Inst. Hautes Études Sci. Publ. Math.*, (48):137–204, 1978.

[Mor07] Tetsuhiro Moriyama. The mapping class group action on the homology of the configuration spaces of surfaces. *J. Lond. Math. Soc. (2)*, 76(2):451–466, 2007.

[Mur93] Jacob P. Murre. On a conjectural filtration on the Chow groups of an algebraic variety. I. The general conjectures and some examples. *Indag. Math. (N.S.)*, 4(2):177–188, 1993.

[NA87] Vicente Navarro Aznar. Sur la théorie de Hodge-Deligne. *Invent. Math.*, 90(1):11–76, 1987.

[NOY21] Andrei Neguţ, Georg Oberdieck, and Qizheng Yin. Motivic decompositions for the Hilbert scheme of points of a K3 surface. *J. Reine Angew. Math.*, 778:65–95, 2021.

[Pag23] Roberto Pagaria. Extra structure on the cohomology of configuration spaces of closed orientable surfaces. *Homology Homotopy Appl.*, 25(1):127–146, 2023.

[Pet13] Dan Petersen. Cohomology of local systems on loci of d-elliptic abelian surfaces. *Michigan Math. J.*, 62(4):705–720, 2013.

[Pet15] Dan Petersen. Cohomology of local systems on the moduli of principally polarized abelian surfaces. *Pacific J. Math.*, 275(1):39–61, 2015.

[Pet16] Dan Petersen. Tautological rings of spaces of pointed genus two curves of compact type. *Compos. Math.*, 152(7):1398–1420, 2016.

[Pet17] Dan Petersen. A spectral sequence for stratified spaces and configuration spaces of points. *Geom. Topol.*, 21(4):2527–2555, 2017.

[Pet20] Dan Petersen. Cohomology of generalized configuration spaces. *Compos. Math.*, 156(2):251–298, 2020.

[PT14] Dan Petersen and Orsola Tommasi. The Gorenstein conjecture fails for the tautological ring of $\overline{\mathcal{M}}_{2,n}$. *Invent. Math.*, 196(1):139–161, 2014.

[PTY21] Dan Petersen, Mehdi Tavakol, and Qizheng Yin. Tautological classes with twisted coefficients. *Ann. Sci. Éc. Norm. Supér. (4)*, 54(5):1179–1236, 2021.

[Sai90] Morihiko Saito. Mixed Hodge modules. *Publ. Res. Inst. Math. Sci.*, 26(2):221–333, 1990.

[Sch95] Joachim Schwermer. On Euler products and residual Eisenstein cohomology classes for Siegel modular varieties. *Forum Math.*, 7(1):1–28, 1995.

[Sta23] Andreas Stavrou. Cohomology of configuration spaces of surfaces as mapping class group representations. *Trans. Amer. Math. Soc.*, 376(4):2821—-2852, 2023.

[SV16] Mingmin Shen and Charles Vial. The Fourier transform for certain hyperkähler fourfolds. *Mem. Amer. Math. Soc.*, 240(1139):vii+163, 2016.

[Tav18] Mehdi Tavakol. Tautological classes on the moduli space of hyperelliptic curves with rational tails. *J. Pure Appl. Algebra*, 222(8):2040–2062, 2018.

[Tom24] Orsola Tommasi. Notes on the cohomology of local systems of small weight on $\mathcal{M}_2$. To appear in the Collino memorial volume *Perspectives on four decades: Algebraic Geometry 1980–2020.* Trends in Mathematics, Birkhäuser, 2024.

[Tot96] Burt Totaro. Configuration spaces of algebraic varieties. *Topology*, 35(4):1057–1067, 1996.

[Tub23] Swann Tubach. On the Nori and Hodge realisations of Voevodsky étale motives. arXiv:2309.11999, 2023.

[Ver61] Jean-Louis Verdier. Sur les intégrales attachées aux formes automorphes (d'après Goro Shimura). In *Séminaire Bourbaki, 13ième année: 1960/61. Textes des conférences. Exposés 205 à 222.* Secrétariat mathématique, Paris, 1961. Fasc. 2, Exp. No. 216.

[Via17] Charles Vial. On the motive of some hyperKähler varieties. *J. Reine Angew. Math.*, 725:235–247, 2017.

[Voi14] Claire Voisin. *Chow rings, decomposition of the diagonal, and the topology of families*, volume 187 of *Annals of Mathematics Studies.* Princeton University Press, Princeton, NJ, 2014.

[Wat18] Tatsunari Watanabe. On the completion of the mapping class group of genus two. *J. Algebra*, 501:303–327, 2018.

[Wei09] Rainer Weissauer. *Endoscopy for* $\mathrm{GSp}(4)$ *and the cohomology of Siegel modular threefolds*, volume 1968 of *Lecture Notes in Mathematics.* Springer-Verlag, Berlin, 2009.

[Zuc79] Steven Zucker. Hodge theory with degenerating coefficients. L_2 cohomology in the Poincaré metric. *Ann. of Math. (2)*, 109(3):415–476, 1979.