From Tripods to Bipods: Reducing the Queue Number of Planar Graphs Costs Just One Leg

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Abstract

As an alternative to previously existing planar graph product structure theorems, we prove that every planar graph G is a subgraph of the strong product of K_2, a path and a planar subgraph of a 4-tree. As an application, we show that the queue number of planar graphs is at most 38 whereas the queue number of planar bipartite graphs is at most 25.

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1 Introduction

Linear layouts have been a popular research topic in topological graph theory since Bernhart and Kainen [8] introduced *book embeddings*, which are also known as *stack layouts*. In a stack layout of a graph, its vertices are placed on a line inducing a linear order $\prec$ of the vertices, whereas edges are partitioned into sets referred to as *pages* or *stack*. As an additional constraint, edges (u, v), (u', v') assigned to the same page are forbidden to *cross*, i.e., we are forbidden to have $u \prec u' \prec v \prec v'$. Stack layouts can be used to model chip design processes [11] and thus it is no surprise, that within the decade following their introduction, a tight upper bound for the number of stacks required for embedding planar graphs was found [32]. It is worth remarking that the corresponding lower bound has been established only a few years ago [6, 33] indicating the difficulty of the problem.

As a dual concept to stack layouts, Heath, Leighton and Rosenberg [21, 22] introduced queue layouts. Here, vertices are again placed on a line inducing a linear order $\prec$, however,

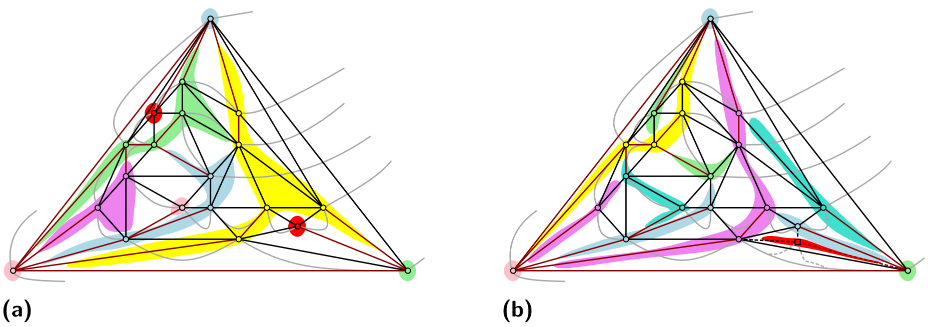


Figure 1 Two layered H-partitions of a graph. (a) Partition into tripods according to [16] where the red tripods are selected by degenerate steps of the algorithm. (b) Partition into bipods according to Theorem 1.2 were the red bipod is a dummy bipod inserted in a degenerate step. In particular note that the yellow bag (first step in the decomposition) is more complex in (a) compared to (b). The gray layers show the BFS-layering.

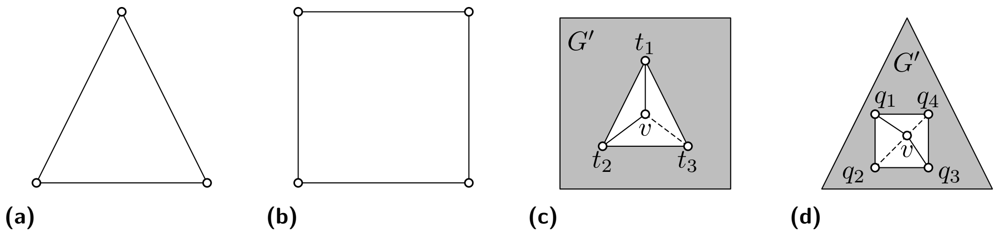


Figure 2 Illustration of (a) **BC1**, (b) **BC2**, (c) **CR1** and (d) **CR2**. The embedding shown corresponds to the canonical embedding of H. Dashed edges may be present or absent.

edges are partitioned into *queues* in which *nesting* edges (u, v), (u', v') are forbidden, i.e., we do not allow $u \prec u' \prec v' \prec v$. For a graph G, the minimum number of queues required by any of its queue layouts is called the *queue number* $\mathrm{qn}(G)$. Despite often being highly non-planar, queue layouts still are useful in practical applications such as VLSI [26] and 3D graph drawing [13, 17]. Already Heath, Leighton and Rosenberg [21] conjectured that every planar graph has constant queue number. Despite serious attempts, this conjecture proved elusive for almost 3 decades during which only improved superconstant upper bounds where found [3, 12, 15]. On the other, constant upper bounds for important subclasses of planar graphs [2, 4] provided reasons to believe that a constant upper bound may be possible.

Finally, in 2019, a breakthrough result by Dujmović, Joret, Micek, Morin, Ueckerdt and Wood [16] proved the correctness of Heath, Leighton and Rosenberg's conjecture. Namely, they proved that the queue number of planar graphs is at most 49. Potentially even more importantly, the result was obtained using a technique that provided deep insights into the structure of planar graphs. More precisely, they discovered that every planar graph can be expressed as a subgraph of the *strong product* of a K_3, a path of arbitrary length and a planar graph H of treewidth at most 3. Intuitively, this result shows, that given a BFS-layering of a planar graph, it is possible to partition vertices into a set of *bags* where **1.** the vertices of each bag consist of a so-called *tripod*, that is, a triangle which is root of three upward paths in the BFS-layering, and **2.** the graph obtained by contracting each bag into a single vertex is a planar graph of treewidth at most 3; refer also to Fig. 1a for an illustration. This *planar graph product structure theorem* also has been successfully applied for coloring applications [7, 18] and already inspired a plethora of follow-up works [7, 10, 14, 18, **?**, 24, 27, 29, 30].

Our Contribution. We provide new insights into the structure of planar graphs. An important subclass of planar graphs in our analysis will be the *planar quasi-4-trees.*

▶ **Definition 1.1.** *A graph G is a* planar quasi-4-tree *if G is one of the two base cases*

BC1 *G is a C_3.*
BC2 *G is a C_4.*

or if G can be constructed from one of the base cases by applying a sequence of the following construction rules:

CR1 *G is obtained from a planar quasi-4-tree $G' = (V', E')$ by inserting a new vertex $v \notin V'$ and edges (v, t_1) and (v, t_2) and possibly (v, t_3) where (t_1, t_2, t_3) is a triangular face of G'.*

CR2 *G is obtained from a planar quasi-4-tree $G' = (V', E')$ by inserting a new vertex $v \notin V'$ and edges (v, q_1) and (v, q_3) and possibly (v, q_2) and/or (v, q_4) where (q_1, q_2, q_3, q_4) is a quadrangular face of G'.*

We remark that it is easy to see from the definition, that planar quasi-4-trees have treewidth at most 4. Moreover, planar 3-trees are a proper subclass of planar quasi-4-trees. Also refer to Figure 2 for an illustration.

As our core result, we prove the following; see Figure 1b:

▶ **Theorem 1.2.** *Let G be a planar graph. Then G admits a H-partition with layered width 2 such that H is a subgraph of a planar quasi-4-tree.*

Note that we will formally define H-partitions in Section 2. Theorem 1.2 immediately implies the following graph product structure theorem:

▶ **Theorem 1.3.** *Let G be a planar graph. Then G is subgraph of $K_2 \boxtimes P \boxtimes H$ where P is a path of arbitrary length and H is a planar quasi-4-tree.*

Intuitively, Theorem 1.3 provides an alternative to the result of Dujmović, Joret, Micek, Morin, Ueckerdt and Wood [16] where each bag is of simpler structure, namely, a *bipod*, consisting of an edge which is root of *only two* upward paths in the BFS-layering. This comes at the expense of increasing the treewidth of graph H from 3 to 4. Thus, it depends on the application whether or not our variant of the planar graph structure product theorem proves more useful.

In particular, we show the usefulness of Theorem 1.3 by investigating the queue number of planar graphs. As a central tool, we prove the following result on the queue number of H, which improves upon a result by Wiechert [31] on the queue number of *not necessarily* planar graphs of treewidth 4 for the class of planar quasi-4-trees:

▶ **Theorem 1.4.** *Let G be a planar quasi-4-tree. Then,* $\mathrm{qn}(G) \leq 6$.

We believe that this result may be of independent interest as possibly *every* planar graph of treewidth at most 4 is a subgraph of a planar quasi-4-tree. Combining Theorems 1.2 and 1.4, we slightly improve the best known upper bound [5] on the queue number of planar graphs.

▶ **Theorem 1.5.** *Let G be a planar graph. Then* $\mathrm{qn}(G) \leq 38$.

Finally, we generalize our results to bipartite planar graphs and obtain the following:

▶ **Theorem 1.6.** *Let G be a planar bipartite graph. Then G admits a H-partition with bichromatic layered width 2 such that H is a subgraph of a planar quasi-4-tree.*

This allows us to also improve the best known upper bound [20] for the queue number of bipartite planar graphs:

▶ **Theorem 1.7.** *Let G be a planar bipartite graph. Then* $\mathrm{qn}(G) \leq 25$.

Our improvements require adjustments to basically all parts of the proof strategy of Dujmović, Joret, Micek, Morin, Ueckerdt and Wood [16] that are on the other hand not too technically difficult, i.e., they “cost just a leg” as the title of our paper suggests. We believe that another nice aspect of our paper - in particular for readers new to the research field - is that consequently it holistically describes the entire proof process whereas [16] was partially using known results [1, 2, 19] as a black box.

Organization of the Paper. First, we review necessary definitions and techniques as well as the state-of-the-art in Section 2. Then, in Section 3, we provide our refined approach for layered H-partitions culminating in Theorems 1.2 and 1.3. In Section 4, we investigate the queue number of planar quasi-4-trees yielding Theorem 1.4 which then allows us to prove Theorem 1.5. Finally, we extend our results to bipartite planar graphs in Section 5 obtaining Theorems 1.6 and 1.7. We conclude the paper with open problems in Section 6.

2 Preliminaries

Planar Graphs and their Drawings. We require some basic understanding of planar graph drawings. A *drawing* of a graph $G = (V, E)$ is a function $\Gamma : G \to \mathbb{R}^2$ that maps each vertex $v \in V$ to a distinct point $\Gamma(v)$ of the Euclidean plane and each edge $(u, v) \in E$ to an open Jordan arc $\Gamma(u, v)$ connecting $\Gamma(u)$ and $\Gamma(v)$. We call a drawing Γ *planar* if and only if for each pair of edges $e_1, e_2 \in E$, $\Gamma(e_1)$ and $\Gamma(e_2)$ are interior-point disjoint. We call a graph *planar* if it admits a planar drawing.

Given a planar drawing, the open Jordan arcs representing E partition the Euclidean plane into a set of regions which we call *faces*. Planar drawings can be partitioned into equivalence classes called *embeddings* such that each drawing of the same embedding induces the same set of faces defined by the counter-clockwise cyclic sequence of edges along its boundary. We say a graph is *plane* if it is associated with a prespecified planar embedding. We also say that we *embed* a graph, if we compute one of its embeddings.

One face of a planar drawing necessarily is unbounded, we call this face the *outer face* and the remaining faces *internal faces*. We call a planar drawing *outerplanar* if all vertices occur on the outer face. Similarly, we call a graph *outerplanar* if it admits an outerplanar drawing and *outerplane* if it is associated with an outerplanar embedding, i.e., a planar embedding where all vertices are incident to the outer face.

Let $G = (V, E)$ be a plane graph. Given a set $S \subset V$, the *subgraph* of $G = (V, E)$ *induced by* S is defined as $G_S = (S, E \cap (S \times S))$. An induced subgraph G_S naturally inherits an *induced embedding* of G by considering only the part of the embedding of G describing G_S.

The *dual graph* $D = (V_D, E_D)$ of graph G contains a vertex for each face in the associated planar embedding of G and an edge $(d_1, d_2) \in E_D$ between two vertices $d_1, d_2 \in V_D$ if the two faces f_1 and f_2 of G corresponding to d_1 and d_2 share an edge on their boundaries. The *weak* dual graph D^* is obtained from D by removing the vertex corresponding to the outer face. In particular, the weak dual graph of an outerplane graph is a tree.

Let G be a planar quasi-4-tree with a known construction sequence using the base cases and construction rules from Definition 1.1. A *canonical embedding* of G is constructed as follows. According to the base cases of Definition 1.1 there is an initial C_3 or C_4 (Base Cases **BC1** and **BC2** of Definition 1.1) which serves as the outer face of G. In all remaining steps, we introduce a new vertex v and connect it to the vertices of a face f of a graph G' constructed in a previous step (see Construction Rules **CR1** and **CR2** of Definition 1.1). In such steps, we embed v in the interior of f.

Graph Product Structure. The first goal of our paper is to express a given planar graph as a subgraph of the *strong product* of graphs exhibiting an easy-to-describe structure. The *strong product* $G \boxtimes H$ of graphs $G = (V_G, E_G)$ and $H = (V_H, E_H)$ is formally defined as a graph $P = (V_P, E_P)$ on vertex set $V_P = V_G \times V_H$. Its edge set E_P contains an edge between vertices (g_1, h_1) and (g_2, h_2) if and only if **1.** $(g_1, g_2) \in E_G$ and $h_1 = h_2$, or, **2.** $g_1 = g_2$ and $(h_1, h_2) \in E_H$, or, **3.** $(g_1, g_2) \in E_G$ and $(h_1, h_2) \in E_H$. Notably, graphs of rather complex structure may be described as products of graphs of easier structure as seen in the example of Figure 3a. Most notably in this regard are the following two theorems known also as *Planar Graph Product Structure Theorems*:

▶ **Theorem 2.1** (Dujmović, Joret, Micek, Morin, Ueckerdt, Wood [16]). *Let G be a planar graph. Then G is subgraph of $K_3 \boxtimes P \boxtimes H$ where P is a path of arbitrary length and H is a planar graph of tree-width at most* 3.

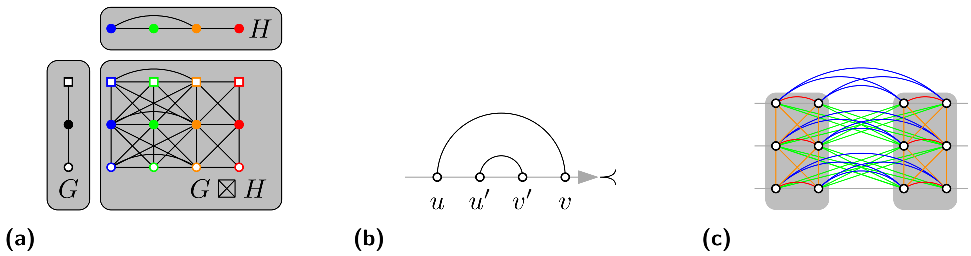


Figure 3 (a) The strong product of two graphs. (b) A pair of nesting edges (u, v) and (u', v') w.r.t. to a linear order $\prec$. (c) The subgraph of a graph G with layered width 2 between two bags and three layers. Red and orange edges are intra-bag edges whereas blue and green edges are inter-bag. Red and blue edges are intra-layer edges whereas orange and green edges are inter-layer.

▶ **Theorem 2.2** (Ueckerdt, Wood, Yi [30]). *Let G be a planar graph. Then G is subgraph of $H \boxtimes P$ where P is a path of arbitrary length and H is a planar graph of simple tree-width at most* 6*.*

As stated above in Theorem 1.3, we provide yet another such theorem. In comparison to Theorem 2.1, one may at first glance wonder if our result is actually an improvement - namely, we decrease the complexity of the first factor graph while increasing the complexity of the third one. We answer this question positively by showing that our variant of the graph product structure theorem provides an improved result regarding queue layouts.

Queue Layouts. Thus, our second goal is to compute *queue layouts* of planar graphs that use few queues. A *k-queue layout* of a graph $G = (V, E)$ is a 2-tuple $(\prec, \mathfrak{Q})$ where $\prec$ is a total order of V and $\mathfrak{Q}$ is a partition of E with $|\mathfrak{Q}| = k$ such that for each edge set $\mathcal{Q} \in \mathfrak{Q}$ for each $(u, v), (u', v') \in \mathcal{Q}$ it holds, that (u, v) and (u', v') do not *nest*, i.e., we do not have $u \prec u' \prec v' \prec v$ (see also Figure 3b). For a graph G, the *queue number $qn(G)$* is the minimum number k such that G has a k-queue layout. A recent break-through result in this field has been obtained by Dujmović, Joret, Micek, Morin, Ueckerdt and Wood [16] and improved as follows:

▶ **Theorem 2.3** (Bekos, Gronemann, Raftopoulou [5]). *Let G be a planar graph. Then* $\mathrm{qn}(G) \leq 42$.

▶ **Theorem 2.4** (Förster, Kaufmann, Merker, Pupyrev, Raftopoulou [20]). *Let G be a bipartite planar graph. Then* $\mathrm{qn}(G) \leq 28$.

As can be seen from the discussion of our results above, we slightly improve upon these results with Theorems 1.5 and 1.7.

Layered H-Partitions. The central technique used in previous studies [16, 5, 20] for proving graph product structure theorems and constant upper bounds on the queue number of planar graphs are *layered H-partitions* of graphs; see also Fig 3c.

Let $G = (V_G, E_G)$ and $H = (V_H, E_H)$. Further let $\mathcal{B} = \{B_x \subset V_G | x \in V_H\}$ be a partition of V_G into sets called *bag*. We say that the 2-tuple $(H, \mathcal{B})$ is a *H-partition* of G if for each edge $(u, v) \in E_G$ one of the following holds: **1.** $u, v \in B_x$ for some $x \in V_H$, or **2.** $u \in B_x$ and $v \in B_y$ for some $x, y \in V_H$ with $x \neq y$ but $(x, y) \in E_H$. Note that in Case 1, we call (u, v) an *intra-bag* edge whereas we call (u, v) an *inter-bag* edge in Case 2. We say that an H-partition has *width w*, if each bag V_x with $x \in V_H$ contains at most w vertices.

It turns out that this basic definition of width is often too restrictive in order to abuse it for algorithmic applications. Thus, we define a relaxation using a *BFS-layering*. A partition $\mathcal{L} = (L_0, L_1, \ldots)$ of V_G is called a *BFS-layering* of G if L_i contains exactly the vertices with graph-theoretic distance i of a specified vertex $r \in V_G$. Observe that $\mathcal{L}$ also partitions E in the sense that each edge $(u, v) \in E$ with $u \in L_i$ and $v \in L_j$ is either an *intra-layer* edge if $i = j$ or an *inter-layer* edge if $|i - j| = 1$. Note that no other edges exist as vertices are layered by graph-theoretic distance. Given, a BFS-layering $\mathcal{L}$, we say that an H-partition $(H, \mathcal{B})$ has *layered-width* ℓ w.r.t. $\mathcal{L}$ if $|B_x \cap L_i| \leq \ell$ for every $x \in V_H$ and $i \geq 0$. If additional there exists no *intra-layer* edges, we say that H has *bichromatic layered width* ℓ w.r.t. $\mathcal{L}$.

Intuitively, in order to compute good graph product structure theorem and queue layouts, the goal is to find *layered* H-partitions that partition G into bags with few vertices on each layer of a BFS-layering such that the bags are connected by a graph H showing nice properties. To this end, all of Theorems 2.1, 2.3, and 2.4 rely on the following result; see also Fig. 1a:

▶ **Theorem 2.5** (Dujmović, Joret, Micek, Morin, Ueckerdt, Wood [16]). *Let G be a planar graph. Then G admits a H-partition with layered width 3 such that H is a planar graph of treewidth at most 3.*

Note that in scope of Theorem 2.5, the content of a bag of the H-partition is known as a *tripod*. Namely, Dujmović, Joret, Micek, Morin, Ueckerdt and Wood [16] construct the bags of an H-partition by finding a triangle t incident to three *vertical paths* of t in the BFS-layering using Sperner's lemma [1]. A *vertical path* P of t here is a path of G containing exactly one vertex $v \in L_i$ of t such that either **1.** for all vertices $w \in P \setminus \{v\}$ with $w \in L_j$, we have $i > j$, or **2.** for all vertices $w \in P \setminus \{v\}$ with $w \in L_j$, we have $i < j$. As the title of our paper suggests, we will instead compute a layered H-partition such that each bag contains a *bipod*, that is, an edge e incident to two vertical paths of e in the BFS-layering. For this approach to work, we have to carefully adapt all steps in the proof of Theorem 2.5 to yield Theorem 1.2.

3 Graph Product Structure of Planar Graphs

3.1 A Generalization of Sperner's Lemma

A crucial ingredient in the proof of the graph product structure theorem by Dujmović, Joret, Micek, Morin, Ueckerdt and Wood [16] is Sperner's lemma about (not necessarily proper) colorings of internally triangulated graphs. For computing our layered H-partition, we will require a similar result, that allows for more than three colors. We find it worth remarking that a similar result for quadrangulations has been presented by Musin [28, Theorem A] which initially inspired this research until we found the following lemma applicable for triangulations.

▶ **Lemma 3.1.** *Let $G = (V, E)$ be an internally triangulated graph and $\chi : V \to \{1, 2, \ldots, k\}$ a k-coloring of the vertices with $k \geq 3$ so that the outer face f_o of G is $(P_1, P_2, \ldots, P_k)$ where P_i is a path whose vertices are colored with color i. Then, G contains a triangle $(t_1, t_2, t_3) \neq f_o$ such that $\chi(t_1) = 1, \chi(t_2) = 2$ and $\chi(t_3) \notin \{1, 2\}$.*

Proof. We closely follow the proof of Sperner's Lemma from The Book [1]. Let $E_{12} \subset E$ denote the set of edges whose endvertices have been colored with colors 1 and 2, i.e., for $(u, v) \in E_{12}$ either $\chi(u) = 1, \chi(v) = 2$ or $\chi(u) = 2, \chi(v) = 1$. Now consider the dual graph

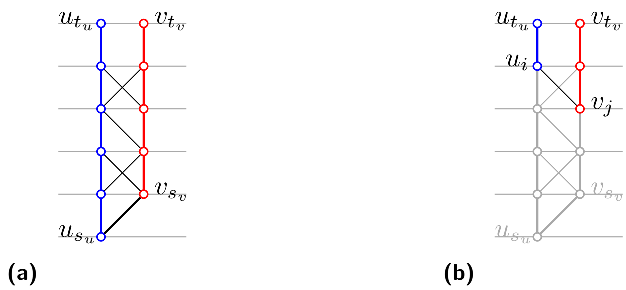


Figure 4 (a) A bipod. (b) The result of Lemma 3.3 applied to (a).

$G^* = (V^*, E^*)$ of G and let $E^*_{12} \subset E^*$ denote the set of edges of the dual graph that are dual to edges in E_{12}. For $v^* \in V^*$ let $\deg_{12}(v^*)$ denote the number of edges of E^*_{12} adjacent to v^*.

Note that for vertex v^*_o dual to the outer face of G we have $\deg_{12}(v^*_o) = 1$ by the coloring of the outer face according to the conditions of the lemma. Since the sum $\sum_{v^* \in V^*} \deg_{12}(v^*)$ must be even, we have that there exists a vertex $v^*_1 \in V^*_1 \setminus \{v^*_o\}$ with $\deg_{12}(v^*_1)$ being odd. More precisely, since each vertex $v^* \in V^* \setminus \{v^*_o\}$ is dual to a triangle t, we have $\deg_{12}(v^*) \leq 3$, i.e., $\deg_{12}(v^*_1) \in \{1, 3\}$.

First, assume that $\deg_{12}(v^*_1) = 3$. Then, all vertices incident to triangle t must be colored with colors 1 and 2. However, then we have necessarily a monocolored edge which contradicts $\deg_{12}(v^*_1) = 3$. Thus, we must have $\deg_{12}(v^*_1) = 1$. Thus, w.l.o.g. $t = (t_1, t_2, t_3)$ is colored with $\chi(t_1) = 1$ and $\chi(t_2) = 2$. Now, if $\chi(t_3) \in \{1, 2\}$, it follows that $\deg_{12}(t^*_1) = 2$; a contradiction. Thus, $t_3 \notin \{1, 2\}$. ◀

3.2 Decomposition of Planar Graphs into Bipods

In this section, we prove a stronger version of Theorem 1.2 providing us with the required tools to prove Theorems 1.3 and 1.5. To this end, we more precisely define the following notion of *bipods*.

▶ **Definition 3.2.** *Let $\mathcal{L} = (L_0, L_1, \ldots)$ be a BFS-layering of a graph $G = (V, E)$. We call G* bipod *if*

P1 *$|L_i| \leq 2$ for $i \geq 0$, and*
P2 *there are two disjoint vertical paths $P_1 = (u_{s_u}, u_{s_u+1}, \ldots, u_{t_u})$ and $P_2 = (v_{s_v}, \ldots, v_{t_v})$ in G such that* **a.** *$u_i, v_i \in L_i$,* **b.** *if both P_1 and P_2 are non-empty $(u_{s_u}, v_{s_v}) \in E$ and* **c.** *for each $v \in V$, v either belongs to P_1 or P_2.*

Intuitively, we arrive at this definition as follows. In our proof of Theorem 1.2, we will identify bipods by finding a suitable edge e incident to two vertical paths of e. Now, by definition, a bipod contains at most two vertices on each BFS layer, i.e., the vertices of two consecutive layers may induce a $K_{2,2}$.

It turns out, that we can assume a simplified structure for our bipods. Namely, we will only consider *acyclic bipods*, that is bipods containing no cycles. The following lemma proves the validity of this assumption:

▶ **Lemma 3.3.** *Let $\mathcal{L} = (L_0, L_1, \ldots)$ be a BFS-layering of a graph $G = (V, E)$ that is a bipod w.r.t. $\mathcal{L}$ with disjoint vertical paths $P_1 = (u_{s_u}, u_{s_u+1}, \ldots, u_{t_u})$ and $P_2 = (v_{s_v}, v_{s_v+1} \ldots, v_{t_v})$ as in Definition 3.2. Then, there exists a non-empty subgraph G' of G induced by paths $P'_1 = (u_i, u_{i+1}, \ldots, u_{t_u})$ and $P'_2 = (v_j, v_{j+1}, \ldots, v_{t_v})$ with $i \geq s_u$ and $j \geq s_v$ that is an acyclic bipod w.r.t. $\mathcal{L}$.*

Proof. By Property **P2** of Definition 3.2, we can partition V into two disjoint paths P_1 and P_2. Assume w.l.o.g. that $P_1 \neq \emptyset$. If $P_2 = \emptyset$, we observe that G is already acyclic as then by Property **P2** of Definition 3.2 we actually have $|L_i| \leq 1$ for $i \geq 0$.

Next, assume that $P_2 \neq \emptyset$. Choose first i and then j maximally, such that $(u_i, v_j) \in E$, i.e., $u_i \in L_i$ and $v_j \in L_j$ and it may be that $u_i = u_{t_u}$ or $v_j = v_{t_v}$. Note that (u_i, v_j) exists by Property **P2** of Definition 3.2. We now define G' as the subgraph induced by paths $P_1' = (u_i, u_{i+1}, \ldots, u_{t_u})$ and $P_2' = (v_j, v_{j+1}, \ldots, v_{t_v})$. As G' is a subgraph of G, it clearly fulfills Property **P1** of Definition 3.2. In addition, since $(u_i, v_j) \in E$ and since P_1' and P_2' are vertical subpaths of P_1 and P_2 respectively, G' also fulfills Property **P2** of Definition 3.2 and thus is a bipod. By the maximal choice of i and j, it follows that (u_i, v_j) is the only edge connecting a vertex $u \in P_1'$ and a vertex in $v \in P_2'$, thus acyclicity is obtained. ◀

We obtain the following stronger version of Theorem 1.2 where we assume that G is already triangulated and embedded. Note that this assumption is not a loss of generality as we can augment our input graph suitably.

▶ **Theorem 3.4.** *Let $G = (V_G, E_G)$ be a triangulated plane graph such that (v_1, v_2, v_3) denotes the outer face. Further, let T be a BFS-tree of G rooted at v_1 inducing a BFS-layering $\mathcal{L} = (L_0, L_1, \ldots)$. Then G admits a H-partition $(H, \mathcal{B})$ with layered width 2 w.r.t. $\mathcal{L}$ such that H is a subgraph of a planar quasi-4-tree and each bag $B \in \mathcal{B}$ induces an acyclic bipod.*

Proof. We follow a similar proof strategy as Dujmović, Joret, Micek, Morin, Ueckerdt and Wood [16]. We describe how to incrementally construct an H-partition $(H, \mathcal{B})$ of G of layered width 2 such that each bag $B \in \mathcal{B}$ is an upward-planar bipod and such that $H = (V_H, E_H)$ is a planar quasi-4-tree. Namely, after the k-th incremental step, we will have found such a decomposition for a subgraph G_k of G induced by the vertices contained in bags of $\mathcal{B}$. We maintain the following invariants throughout our algorithm:

- **I3.4.1** H is a planar quasi-4-tree.
- **I3.4.2** The vertices of each bag B_x with $x \in H$ induce an acyclic bipod.
- **I3.4.3** Let $t = (t_1, t_2, t_3)$ be an internal triangular face in the canonical embedding of H. Then, in G there exists a specified cycle $C_t = (P_1, P_2, P_3)$ such that path P_i contains only vertices of B_{t_i} for $1 \leq i \leq 3$.
- **I3.4.4** Let $q = (q_1, q_2, q_3, q_4)$ be an internal quadrangular face in the canonical embedding of H. Then, in G there exists a specified cycle $C_q = (P_1, P_2, P_3, P_4)$ such that path P_i contains only vertices of B_{q_i} for $1 \leq i \leq 4$.
- **I3.4.5** Let $v \in V_G \setminus (\bigcup_{x \in H} B_x)$. Then in the embedding of G, vertex v is embedded inside cycle C_f with f being a face of the canonical embedding of H. Moreover, v is part of a subtree of T rooted at a vertex of B_x such that x is on the boundary of f.
- **I3.4.6** If $v_x \in B_x$ and $v_y \in B_y$ are adjacent, i.e., $(v_x, v_y) \in E_G$, we also have $(x, y) \in E_H$.

Initialization. We start by computing the H-partition for subgraph G_0 consisting only of vertices v_1, v_2, v_3. We initialize H as a triangle (x_1, x_2, x_3) with bags $B_{x_i} = \{v_i\}$ for $1 \leq i \leq 3$. By Base Case **BC1** of Definition 1.1, H is a planar quasi-4-tree, i.e. Invariant **I3.4.1** holds. Moreover, each bag B_{x_i} for $1 \leq i \leq 3$ consists of exactly one vertex. Thus it contains less than 2 vertices on each BFS-layer L_i and its induced subgraph is acyclic, i.e., B_{x_i} induces an acyclic bipod and Invariant **I3.4.2** is established. In its canonical embedding, H contains exactly one face, namely $f = (x_1, x_2, x_3)$. We specify $C_f = (P_1, P_2, P_3)$ with P_i being the path of length 0 consisting only of vertex v_i for $1 \leq i \leq 3$. As $v_i \in B_{x_i}$, we conclude that

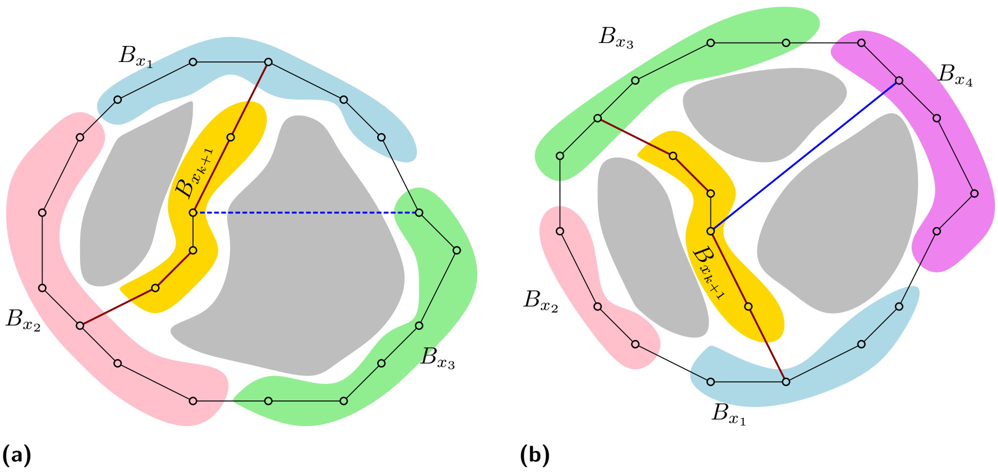


Figure 5 (a) Iterative Step where f is triangular. The dashed blue edge may be present. (b) Iterative Step where f is quadrangular. Here, the right region separated by $B_{x_{k+1}}$ is split into two parts by an edge between $B_{x_{k+1}}$ and B_{x_4}. Dark red edges belong to the BFS tree T.

Invariant **I3.4.3** is established. Further, **I3.4.4** hold trivially as H contains no quadrangular face in its canonical embedding. Next, all vertices $V \setminus \{v_1, v_2, v_3\}$ of G are located inside the triangle $C_f = (v_1, v_2, v_3)$ as it is cycle bounding the outer face. Since the root of T is located on the outer face, we also have that every vertex $v \in V \setminus \{v_1, v_2, v_3\}$ is part of a subtree of T rooted at one of v_1, v_2, v_3. Thus, we also have Invariant **I3.4.5**. Finally, Invariant **I3.4.6** follows as both (v_1, v_2, v_3) and (x_1, x_2, x_3) form a clique.

Iterative Step. Here, our aim is to identify another bipod $B \neq \emptyset$ in G that can be added to our H-partition such that Invariants **I3.4.1**–**I3.4.5** are maintained when G_{k+1} is obtained from G_k by adding the vertices of B and all induced edges. We begin by selecting a face f of the canonical embedding of H such that a vertex of $V \setminus \{\bigcup_{x \in H} B_x\}$ (i.e., a vertex not assigned to any bipod yet) is contained inside C_f. We then identify a bipod B of G whose vertices are located inside C_f such that **1.** if $f = (x_1, x_2, x_3)$ is triangular, there is a vertex b_1 of B incident to a vertex in B_{x_1} and a vertex b_2 of B incident to a vertex in B_{x_2}, **2.** if $f = (x_1, x_2, x_3, x_4)$ is quadrangular, there is a vertex b_1 of B incident to a vertex in B_{x_1} and a vertex b_3 of B incident to a vertex in B_{x_3}. To accomplish that, aside from a special degenerate case, we will make use of our generalization of Sperner's lemma in Lemma 3.1 to identify a triangular face (t_1, t_2, t_3) of G such that t_i is part of a subtree of T rooted at a vertex of B_{x_i} on the boundary of f. In the following, we describe the process in more detail depending on whether f is triangular or quadrangular.

$f = (x_1, x_2, x_3)$ **is triangular.** For an illustration, refer to Figure 5a. We color each vertex inside C_f and the vertices of C_f such that all vertices of B_{x_i} and all vertices inside C_f which are part of a subtree of T rooted at a vertex of B_{x_i} are colored with color i for $1 \leq i \leq 3$. Note that by Invariant **I3.4.5**, this colors all vertices. We prove the following claim:

▷ Claim 3.5. There is an edge $(u, v) \in E$ such that w.l.o.g. u is located inside C_f and u and v have different colors.

Proof. Assume for a contradiction that there was no such edge (u, v). By assumption, C_f contains vertices of $V \setminus \{\bigcup_{x \in H} B_x\}$. Hence, assume w.l.o.g. that C_f contains vertices

$V_1 \subseteq V \setminus \{\bigcup_{x \in H} B_x\}$ with $V_1 \neq \emptyset$ that are colored with color 1. Since there is no edge (u, v) such that u and v have different colors within C_f, we must have that each vertex $u \in V_1$ is only incident to other vertices of V_1 and to vertices of B_{x_1}.

Since the subgraph of G inside cycle C_f is internally triangulated, it follows that there exists a single edge (u', v') separating the vertices of V_1 from all vertices of different colors, i.e., in particular from B_{x_2} and B_{x_3}. This is only possible if both $u', v' \in B_{x_1}$. But then, B_{x_1} is not acyclic, a contradiction to Invariant **I3.4.2**. $\lhd$

We now apply Claim 3.5 to obtain an edge (v_1, v_2) such that w.l.o.g. v_1 and v_2 have colors 1 and 2, respectively. Further, w.l.o.g. $v_1 \notin B_{x_1}$. By Invariant **I3.4.5**, v_1 is part of a subtree of BFS-tree T rooted at a vertex p_1 of B_{x_1}. Thus, there exists a path P_1 of length at least one between v_1 and p_1 in T. Similarly, we have that v_2 is part of a subtree of BFS-tree T rooted at a vertex p_2 of B_{x_2}, i.e., there is a path P_2 of length at least zero between x_2 and p_2 in T. We introduce a new vertex x_{k+1} to H and define $B_{x_{k+1}}$ as the subgraph induced by the vertices of P_1 and P_2 without p_1 and p_2. Since $v_1 \neq p_1$, we have that $|B_{x_{k+1}}| \geq 1$. We first observe that $B_{x_{k+1}}$ is a bipod. To this end, observe that each of the paths P_1 and P_2 contains at most one vertex of BFS-layer L_i, i.e. they are a vertical path and $B_{x_{k+1}}$ is a bipod. If $B_{x_{k+1}}$ is not acyclic, by Lemma 3.3 we can find a subset of $B_{x_{k+1}}$ inducing a acyclic bipod such that p_1 and p_2 are connected via a path through that bipod. Thus, Invariant **I3.4.2** holds. In H, we connect x_{k+1} to x_1 and x_2 and if there is an edge $e_3 = (v_{k+1}, v_3)$, where $v_{k+1} \in B_{x_{k+1}}$ and $v_3 \in B_{x_3}$, also to x_3. Thus, by Construction Rule **CR1** of Definition 1.1, H is still a planar quasi-4-tree and Invariant **I3.4.1** holds. Moreover, we also immediately yield Invariant **I3.4.6**. Finally, consider the new faces in the canonical embedding of H. One of the new faces is triangle $t_{k+1} = (x_1, x_2, x_{k+1})$. Notice that $P_1 \cup P_2$ cuts the region inside C_f into two separate parts. One is bounded by a cycle consisting of a path of vertices inside B_{x_1}, a path of vertices inside B_{x_2} and B_{k+1}. We define this cycle to be $C_{t_{k+1}} = (x_1, x_2, x_{k+1})$. For the other part, we consider two cases. If e_3 does not exist, it is bounded by a cycle C_4 consisting of a path of vertices inside B_{x_1}, a path of vertices inside B_{x_2}, a path of vertices inside B_{x_3} and B_{k+1}. In addition, we have that the canonical embedding of H contains just one additional new face $q_{k+1} = (x_1, x_{k+1}, x_2, x_3)$. Here, we define $C_4 = C_{q_{k+1}}$. If e_3 does exist, it cuts the cycle C_4 into two subcycles. The first such cycle C_1 consists of a path of vertices inside B_{x_1}, a path of vertices inside $B_{x_{k+1}}$ and a path of vertices inside B_{x_3}. The second such cycle C_2 consists of a path of vertices inside B_{x_2}, a path of vertices inside $B_{x_{k+1}}$ and a path of vertices inside B_{x_3}. Moreover, in this scenario H contains two additional new faces, namely $f_1 = (x_1, x_{k+1}, x_3)$ and $f_2 = (x_2, x_{k+1}, x_3)$. Here, we set $C_{f_i} = C_i$. In both cases, we yield Invariants **I3.4.3** and **I3.4.4**.

Finally, consider any vertex $v \in V \setminus \{\bigcup_{x \in H} B_x\}$ contained previously in C_f. It is now contained in a new face f' of our updated graph H. Moreover, by Invariant **I3.4.5**, it is part of a subtree of T rooted at a vertex p of B_{x_1}, B_{x_2} or B_{x_3}. Consider the path P_v between v and P_v. If P_v does not contain vertices of $B_{x_{k+1}}$, we have that p is contained in a bipod B_x such that x is on the boundary of $C_{f'}$. Otherwise, v now is part of a subtree rooted at a vertex of $B_{x_{k+1}}$. Thus, Invariant **I3.4.5** is maintained.

$f = (x_1, x_2, x_3, x_4)$ **is quadrangular.** For an illustration, refer to Figure 5b. We color each vertex inside C_f and the vertices of C_f such that all vertices of B_{x_i} and all vertices inside C_f which are part of a subtree of T rooted at a vertex of B_{x_i} are colored with color i for $1 \leq i \leq 4$. Note that by Invariant **I3.4.5**, this colors all vertices. Moreover, by Invariant **I3.4.4**, this coloring fulfills the preconditions of Lemma 3.1. Thus, w.l.o.g. there exists a triangle

$t = (t_1, t_2, t_3)$ of G inside C_f with t_i being colored i for $1 \le i \le 3$. Momentarily assume the following:

▶ **Assumption 3.6.** *There is no pair of vertices $v_1 \in B_{x_1}$ and $v_3 \in B_{x_3}$ such that $(v_1, v_3) \in E$ and (v_1, v_3) is drawn inside C_f.*

Assuming Assumption 3.6, we have $t_1 \notin B_{x_1}$ or $t_3 \notin B_{x_3}$.

Then, assume w.l.o.g. that $t_1 \notin B_{x_1}$. By Invariant **I3.4.5**, t_1 is part of a subtree of BFS-tree T rooted at a vertex p_1 of x_1. Thus, there exists a vertical path P_1 of length at least one between t_1 and p_1 in T. Similarly, we have that t_3 is part of a subtree of BFS-tree T rooted at a vertex p_3 of x_3, i.e., there is a vertical path P_3 of length at least zero between t_3 and p_3 in T. We introduce a new vertex x_{k+1} to H and define $B_{x_{k+1}}$ as the subgraph induced by the vertices of P_1 and P_3 without p_1 and p_3. Since $t_1 \neq p_1$, we have that $|B_{x_{k+1}}| \geq 1$. We first observe that $B_{x_{k+1}}$ is a bipod. To this end, observe that each of paths P_1 and P_3 contains at most one vertex of BFS-layer L_i, i.e., $B_{x_{k+1}}$ induces a bipod. By Lemma 3.3, we can further assume $B_{x_{k+1}}$ to induce an acyclic bipod as otherwise we can select a subset of $B_{x_{k+1}}$ that fulfills this property. In particular, such an acyclic bipod still induces a path that can be extended by p_1 and p_3 to obtain a path connecting p_1 and p_3. We conclude that Invariant **I3.4.2** holds. In H, we connect x_{k+1} to x_1 and x_3. Moreover, if there is an edge $e_2 = (v_{k+1}, v_2)$, where $v_{k+1} \in B_{x_{k+1}}$ and $v_2 \in B_{x_3}$, we also connect x_{k+1} to x_2. Similarly, if there is an edge $e_4 = (v_{k+1}, v_2)$, where $v_{k+1} \in B_{x_{k+1}}$ and $v_2 \in B_{x_3}$, we also connect x_{k+1} to x_2. Thus, by Construction Rule **CR2** of Definition 1.1, H is still a planar quasi-4-tree and Invariant **I3.4.1** holds. Moreover, we also immediately yield Invariant **I3.4.6**. Finally, consider the new faces in the canonical embedding of H. Notice that $P_1 \cup P_3$ cuts the region inside C_f into two separate parts.

One is bounded by a cycle C_{123} consisting of a path of vertices inside B_{x_1}, a path of vertices inside B_{x_2}, a path of vertices inside B_{x_3} and B_{k+1}. If e_2 does not exist, we have that the canonical embedding of H contains a new face $q_{k+1} = (x_1, x_2, x_3, x_{k+1})$ and we define $C_{q_{k+1}} = C_{123}$. If e_2 does exist, it cuts the cycle C_{123} into two subcycles. The first such cycle C_{12} consists of a path of vertices inside B_{x_1}, a path of vertices inside B_{x_2} and a path of vertices inside $B_{x_{k+1}}$. The second such cycle C_{23} consists of a path of vertices inside B_{x_2}, a path of vertices inside B_{x_3} and a path of vertices inside $B_{x_{k+1}}$. Moreover, in this scenario H contains two new faces, namely $f_{12} = (x_1, x_2, x_{k+1})$ and $f_{23} = (x_2, x_3, x_{k+1})$. Here, we set $C_{f_{12}} = C_{12}$ and $C_{f_{23}} = C_{23}$.

The second part is bounded by a cycle C_{143} consisting of a path of vertices inside B_{x_1}, a path of vertices inside B_{x_4}, a path of vertices inside B_{x_3} and B_{k+1}. If e_4 does not exist, we have that the canonical embedding of H contains a new face $q_{k+1} = (x_1, x_4, x_3, x_{k+1})$ and we define $C_{q_{k+1}} = C_{143}$. If e_4 does exist, it cuts the cycle C_{143} into two subcycles. The first such cycle C_{14} consists of a path of vertices inside B_{x_1}, a path of vertices inside B_{x_4} and a path of vertices inside $B_{x_{k+1}}$. The second such cycle C_{43} consists of a path of vertices inside B_{x_4}, a path of vertices inside B_{x_3} and a path of vertices inside $B_{x_{k+1}}$. Moreover, in this scenario H contains two new faces, namely $f_{14} = (x_1, x_4, x_{k+1})$ and $f_{43} = (x_4, x_3, x_{k+1})$. Here, we set $C_{f_{14}} = C_1$ and $C_{f_{43}} = C_{43}$.

In all cases, we yield Invariants **I3.4.3** and **I3.4.4**. Finally, consider any vertex $v \in V \setminus \{\bigcup_{x \in H} B_x\}$ contained previously in C_f. It is now contained in a new face f' of our updated graph H. Moreover, by Invariant **I3.4.5**, it is part of a subtree of T rooted at a vertex p of B_{x_1}, B_{x_2}, B_{x_3} or B_{x_4}. Consider the path P_v between v and P_v. If P_v does not contain vertices of $B_{x_{k+1}}$, we have that p is contained in a bipod B_x such that x is on the boundary of $C_{f'}$. Otherwise, v now is part of a subtree rooted at a vertex of $B_{x_{k+1}}$. Thus,

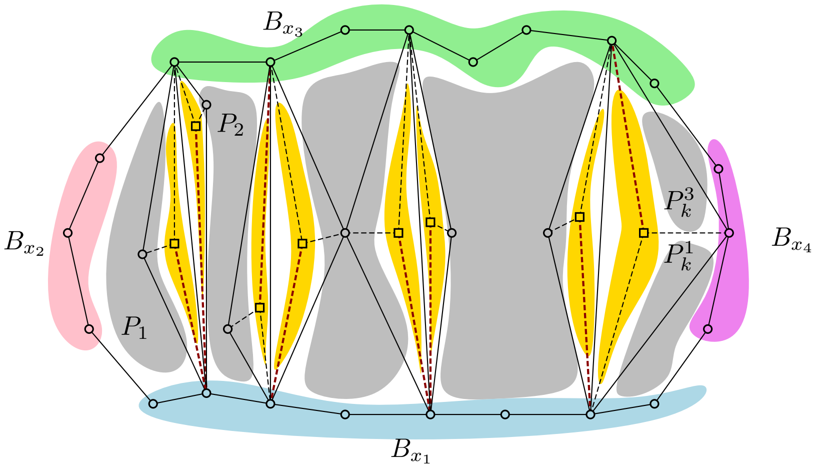


Figure 6 A degenerate step of our algorithm. Dashed edges and square-shaped vertices are inserted to augment G in this step, dark red edges belong to the augmented BFS tree.

Invariant I3.4.5 is maintained.

Degenerate Step: Assumption 3.6 does not hold. It remains only to consider the scenario where $f = (x_1, x_2, x_3, x_4)$ such that Assumption 3.6 does not hold true. In this case, let E_{13} be the set of edges (v_1, v_3) drawn inside f such that $v_1 \in B_{x_1}$ and $v_3 \in B_{x_3}$. For an illustration, refer to Fig. 6

Observe that E_{13}, which is a crossing-free set of edges, partitions the part of the Euclidean plane inside C_f into disjoint parts $P_1, P_2, \ldots, P_p$ with $|E_{13}| = p - 1$ where **1.** part P_1 is bounded by a subpath of B_{x_1}, a subpath of B_{x_3} and a subpath of B_{x_2} and one edge of E_{13}, **2.** part P_p is bounded by a subpath of B_{x_1}, a subpath of B_{x_3} and a a subpath ofB_{x_4} and one edge of E_{13}, and **3.** part P_i with $2 \le i \le p-1$ is bounded by a subpath of B_{x_1}, a subpath of B_{x_3}, an edge of E_{13} shared with the boundary of P_{i-1} and an edge of E_{13} shared with the boundary of P_{i+1}.

Let P_i be a *non-empty* part, i.e., there exists at least one vertex of $V \setminus \{\bigcup_{x \in H} B_x\}$ inside P_i. Let $e_{13} \in E_{13}$ be an edge on the boundary of P_i. Since G is triangulated, there exists a triangle $t = (t_1, t_2, t_3)$ inside P_i such that $(t_1, t_3) = e_{13}$. Because P_i was chosen to not contain any edge of E_{13} in its interior and because P_i is not empty, we conclude that $t_2 \in V \setminus \{\bigcup_{x \in H} B_x\}$ or if $i = 1$ or $i = p$, we may have that $t_2 \in B_{x_2}$ or $t_2 \in B_{x_4}$, respectively. Note that if $t_2 \in B_{x_2}$ or $t_2 \in B_{x_4}$, edge (t_2, v_y) partitions the region P_i into a region P_i^1 and a region P_i^3 such that vertices in P_i^1 can be incident to vertices from B_{x_1} but not to vertices from B_{x_3} whereas vertices in P_i^3 can be incident to vertices from B_{x_3} but not to vertices from B_{x_1}. We now augment graph G by adding a vertex v_y inside t and connect y to all of t_1, t_2 and t_3. Clearly, the resulting graph remains a triangulation. By Invariant I3.4.5, we have $t_1 \in L_{i_1}$, $t_2 \in L_{i_2}$ and $t_3 \in L_{i_3}$ such that $i_2 \ge i_3$ and $i_2 \ge i_1$. Assume w.l.o.g. that $i_1 \le i_3$. Then, we augment T by edge (t_1, v_y). It is easy to see that T remains a BFS-tree. Moreover, since t is a triangle and t_1 the vertex of t on the lowest layer, we have $i_2 - 1, i_3 - 1 \le i_1 \le i_2, i_3$. As v_y is now added to layer L_{i_1+1}, we observe that edges (t_2, v_y) and (t_3, v_y) respect the BFS-layering.

We now create a new bag $B_y = \{v_y\}$. As it contains only one vertex, we trivially fulfill Invariant I3.4.2. Further, in H we connect y with x_1 and x_3 and if $t_2 \in B_{x_j}$ also with x_i for $j \in \{2, 4\}$. This modification ensures Invariant I3.4.1 as it corresponds to Construction

Rule **CR2** of Definition 1.1. Moreover, we have also Invariant **I3.4.6** as $(t_1, v_y) \in E_G$ and $(t_3, v_y) \in E_G$ but also $(x_1, y) \in E_H$ as well as $(x_3, y) \in E_H$ while in case that $t_2 \in B_{x_j}$ for $j \in \{2, 4\}$, we also have $(x_j, y) \in E_H$.

After performing our augmentation for all triangles at the boundary non-empty regions starting with region P_1, followed by regions P_i in increasing order of i. Moreover, if $2 \leq i \leq p-1$, we first process the edge of E_{13} shared with P_{i-1}. In the canonical embedding of H, this partitions face f such that there exists a face f_i containing the bounding bipods of P_i. In particular, we yield Invariant **I3.4.4** as the vertices of non-empty part P_i are now located in a cycle C_{f_i} **1.** bounded by a subpath of B_{x_1}, a subpath of B_{x_3} and a subpath of B_{x_2} and a newly inserted bipod B_y if $i = 1$, **2.** bounded by a subpath of B_{x_1}, a subpath of B_{x_3} and a subpath of B_{x_4} and a newly inserted bipod B_y if $i = p$, and **3.** bounded by a subpath of B_{x_1}, a subpath of B_{x_3} and two newly inserted bipod $B_y, B_{y'}$ if $2 \leq i \leq p-1$.

Note that if $i = 1$ or $i = p$ and the new bipod B_y in P_i shares an edge with B_{x_j} where $j = 2$ or $j = 4$, respectively, instead of f_i we yield two triangular faces f_i^1 and f_i^3 such that f_i^1 is bounded by a subpath of B_{x_1}, a subpath of B_{x_j} and the newly inserted bipod B_y whereas f_i^3 is bounded by a subpath of B_{x_3}, a subpath of B_{x_j} and the newly inserted bipod B_y. Since we create no other triangular faces, Invariant **I3.4.3** follows. Moreover, Invariant **I3.4.5** holds as we do not change how vertices of $V_G \setminus (\bigcup_{x \in H} B_x)$ are connected to T. Thus, after the degenerate step all invariants are again fulfilled.

Finally, we argue that all of the faces f_i for a non-empty part P_i fulfill Assumption 3.6. To this end, first observe that there is no edge of E_{13} inside f_i as otherwise we would have not defined P_i the way that we did. Thus, it remains to show that the other two bipods bounding C_f are not connected. If $i = 1$ or $i = k$, it is trivial to see that the newly inserted bipod B_y is not incident to a vertex from B_{x_2} or B_{x_4}, respectively, as the triangle into which v_y was inserted contains no vertex of B_{x_2} or B_{x_4}. If $2 \leq i \leq k-1$, we observe that the two triangles into which $v_y \in B_y$ and $v_{y'} \in B_{y'}$ are inserted must be disjoint as each such triangle contains exactly one edge of E_{13}. Thus, we also have in this scenario that $(v_y, v_{y'}) \notin E$.

Conclusion. In the paragraphs above, we have described how to perform iterative steps. In particular, we concluded that we can always perform a valid iterative step that satisfies all of our Invariants **I3.4.1**–**I3.4.6**. If a degenerate step was performed, we concluded that the number of faces f of H that may cause a degenerate step to happen, decreases. On the other hand, in non-degenerate steps we create at most 2 new quadrangular faces in the canonical embedding of H that may cause a degenerate step. Since in the k-th non-degenerate step, at least one vertex of $v \in V \setminus \{\bigcup_{x \in H} B_x\}$ is assigned to a new bipod B_k, we conclude that our algorithm assigns all vertices in a linear number of iterations and the theorem follows. ◀

3.3 An Alternative Planar Graph Product Structure Theorem

Having established Theorem 1.2, it remains to apply the following lemma, stated as an observation by Dujmović, Joret, Micek, Morin, Ueckerdt and Wood [16].

▶ **Lemma 3.7** (Dujmović, Joret, Micek, Morin, Ueckerdt and Wood [16]). *Let $G = (V_G, E_G)$ be a graph such that there exists an H-partition $(H, \mathcal{B})$ of G with layered width ℓ with respect to a BFS-layering $\mathcal{L} = (L_0, L_1, \ldots)$ and $H = (V_H, E_H)$. Then, G is a subgraph of $K_\ell \boxtimes P \boxtimes H$ for a path P.*

For the convenience of the reader we provide a short proof.

Proof of Lemma 3.7. Let d be chosen maximally so that $L_d \neq \emptyset$ and let $P = (p_1, p_2, \ldots, p_d)$ and $K_\ell = (V_\ell, V_\ell \times V_\ell)$ with $V_\ell = \{v_1, \ldots, v_\ell\}$. We first show that the graph induced by each bag is a subgraph of $(K_\ell \boxtimes P) \times (\{x\}, \emptyset)$ for $x \in V_H$. For a fixed layer $i \in \{0, \dots, d\}$, $K_\ell \boxtimes P$ contains an ℓ-clique on $((v_1, p_i), x), ((v_2, p_i), x), \dots, ((v_\ell, p_i), x)$. Moreover, between layers $i \in \{0, \dots, d-1\}$ and $i+1$, there is a complete bipartite graph between parts $\{((v_1, p_i), x), ((v_2, p_i), x), \dots, ((v_\ell, p_i), x)\}$ and $\{((v_1, p_{i+1}), x), \dots, ((v_\ell, p_{i+1}), x)\}$. Therefore $(K_\ell \boxtimes P) \times (\{x\}, \emptyset)$ includes all possible connections in the bag x.

Next, consider inter-bag edges. According to the definition of H-partitions, there are only connections between two bags B_x and B_y, if $(x, y) \in E_H$. On the other hand, for $(x, y) \in E_H$, we get from the definition of the strong product $(K_\ell \boxtimes P) \boxtimes H$, that it contains all edges between $((v_i, p_j), x)$ and $((v_{i'}, p_{j'}), y)$ for $1 \le i, i' \le \ell$ and $|j - j'| \le 1$. Thus, all inter-bag edges are included in the product. ◀

Theorem 1.2 and Lemma 3.7 now immediately prove Theorem 1.3.

4 Queue Number of Planar Graphs

4.1 Planar Quasi-4-Trees

This section is devoted to proving Theorem 1.4. We remark that Wiechert [31] proved that not necessarily planar graphs of treewidth 4 have queue number at most 15. In order to do improve this result, we adapt the *peeling-into-levels* approach used by Alam et al. [2] for computing 5-queue layouts for planar 3-trees to work for planar quasi-4-trees. As a central ingredient, we have to refine a lemma by Dujmović et al. [19, Lemma 22] that discusses layered drawings of triangulated outerplanar graphs for our purposes. To this end, recall that in a layered drawing, each vertex v is placed on a layer $\ell(v) \in \mathbb{N}$ such that vertices on the same layer occur on a horizontal line. In a layered drawing, we say that an edge (u, v) has span k if $|\ell(u) - \ell(v)| = k$. Note that while we assume integer y-coordinates, x-coordinates can be real.

▶ **Lemma 4.1.** *Let G be an outerplane graph such that every internal face is either a triangle or a quadrangle. Then, G admits a straight-line layered drawing where*

P1 *each edge (u, v) of G has span 1 or 2,*
P2 *the three vertices of a triangle (t_1, t_2, t_3) are placed on three consecutive layers, i.e., $\ell(t_1) = \ell(t_2) - 1 = \ell(t_3) - 2$, and*
P3 *the four vertices of a triangle (q_1, q_2, q_4, q_3) are placed on four consecutive layers, more precisely, $\ell(q_1) = \ell(q_2) - 1 = \ell(q_3) - 2 = \ell(q_4) - 3$.*

Proof. We prove the statement constructively by iteratively drawing a subgraph that contains n_f internal faces of G. We assume w.l.o.g. that G is biconnected as otherwise we can augment G to fulfill our assumption. In addition to the requirements stated in the lemma, we additionally maintain the following invariants:

I4.1.1 Each vertical line intersecting the boundary of the drawing intersects it exactly twice, i.e., it is x-monotone.
I4.1.2 The *lower envelope* containing the edges visible from $(0, -\infty)$ consists of a single edge that belongs to the outer face of G.

For $n_f = 1$, we select to draw a face of G that corresponds to a leaf in the weak dual of G. Thus, it is either a C_3 or a C_4 and contains at least one edge e_o of the outer face of G. A corresponding drawing where e_o forms the lower envelope is easy to construct; see Figures 7a and 7d, respectively. Hence, assume that $n_f \ge 2$. We select a face f of G sharing an edge e_f

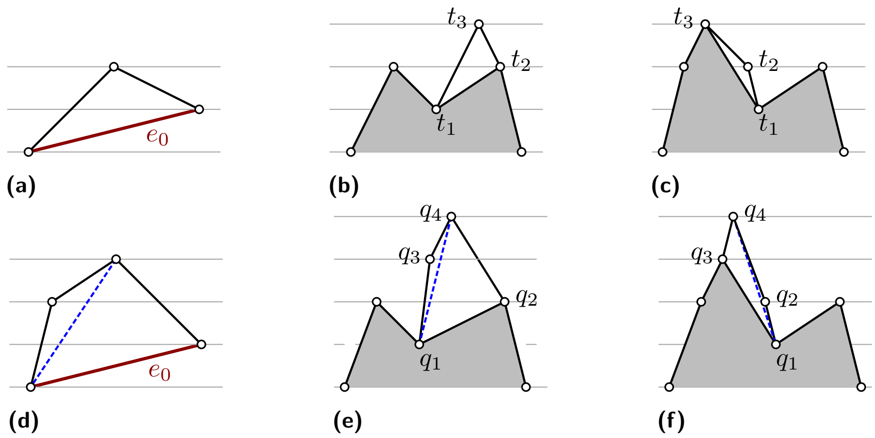


Figure 7 Illustrations for the proof of Lemma 4.1. Observe that dashed blue edges may be added while maintaining planarity.

with the already drawn subgraph G_f of G to be drawn next. Note that f will correspond to a leaf in the weak dual of the subgraph of G induced by the first n_f faces, i.e., it shares exactly the single edge e_f with G_f. Note that we already have obtained a drawing for G_f meeting the requirements of the lemma and Invariants I4.1.1 and I4.1.2.

Observe that Invariant I4.1.1 allows us to partition the edges on the outer cycle of G_f into an upper and lower envelope containing the edges visible from $(0, +\infty)$ and $(0, -\infty)$, respectively. By Invariant I4.1.2, the lower envelope consists of a single edge that belongs to the outer face, hence $e_f = (u, v)$ is on the upper envelope. Four cases may arise.

First, if $\ell(v) - \ell(u) = 1$ and f is triangular, define $t_2 := v$, $t_1 := u$ and the remaining vertex of f not belonging to G_f as t_3. Then, t_3 is placed such that $\ell(t_3) = \ell(t_2) + 1$ and $x(t_3)$ is between $x(t_1)$ and $x(t_2)$; see Figure 7b.

Second, if $\ell(v) - \ell(u) = 2$ and f is triangular, define $t_3 := v$, $t_1 := u$ and the remaining vertex of f not belonging to G_f as t_2. Then, t_2 is placed such that $\ell(t_2) = \ell(t_1) + 1$ and $x(t_2)$ is between $x(t_1)$ and $x(t_3)$; see Figure 7c.

Third, if $\ell(v) - \ell(u) = 1$ and f is quadrangular, define $q_2 := v$, $q_1 := u$ and the remaining vertices of f not belonging to G_f as q_3 and q_4, where q_3 is neighbor of q_1 and q_4 is neighbor of q_2. Then, q_3 is placed such that $\ell(q_3) = \ell(q_2) + 1$ and $x(q_3)$ is between $x(q_1)$ and $x(q_2)$. Moreover, q_4 is placed such that $\ell(q_4) = \ell(q_3) + 1$ and $x(q_4)$ is between $x(q_2)$ and $x(q_3)$; see Figure 7e.

Finally, if $\ell(v) - \ell(u) = 2$ and f is quadrangular, define $q_3 := v$, $q_1 := u$ and the remaining vertices of f not belonging to G_f as q_2 and q_4, where q_2 is neighbor of q_1 and q_4 is neighbor of q_3. Then, q_2 is placed such that $\ell(q_2) = \ell(q_1) + 1$ and $x(q_2)$ is between $x(q_1)$ and $x(q_3)$. Moreover, q_4 is placed such that $\ell(q_4) = \ell(q_3) + 1$ and $x(q_4)$ is between $x(q_2)$ and $x(q_3)$; see Figure 7f.

Observe that in all four cases, the new upper envelope is x-monotone, thus Invariant I4.1.1 holds again. Moreover, the lower envelope remains unchanged, thus Invariant I4.1.2 holds once more. The proof follows. ◀

With Lemma 4.1 established, we describe the peeling-into-levels approach more carefully. Let G be a planar quasi-4-tree embedded with a canonical embedding. Based on our embedding, we partition G as follows:

1. Vertices incident to the outer face belong to *level set* L_0, also called *level zero*.

2. Vertices incident to the outer face of the graph induced by $V \setminus (\bigcup_{k=0}^{i-1} L_0)$ (that is, after deleting level sets L_0 to L_{i-1}) belong to level set L_i, also called *level* i.
3. Edges between two vertices of the same level set are called *level edges*.
4. Edges between two vertices of different levels are called *binding edges*. Note that the levels of endpoints of such an edge differ by 1.

We will denote by G_i the subgraph of G induced by L_i. We now prove that all G_i have a very simple structure:

▶ **Lemma 4.2.** *Let G be a canonically embedded planar quasi-4-tree. Then, each connected component of the graph G_i induced by L_i is outerplane and each of its internal faces is a triangle or a quadrangle. Further, if $i > 0$, it is embedded inside a face of G_{i-1}.*

Proof. We prove the statement inductively by additionally maintaining the following invariant:

I4.2.1 Each interior face of G_i is created by one of the construction rules of Definition 1.1 or part of a base case of Definition 1.1.

For graph G_0 the statement of the lemma clearly holds as G_0 is either a C_3 or a C_4. Moreover, also Invariant **I4.2.1** is fulfilled as the outer face is defined by Base Case **BC1** or **BC2** of Definition 1.1.

Thus, assume now that G_{i-1} has the desired properties and consider a connected component C of graph G_i. Clearly, C must be located inside a face f of G_{i-1} as the outer face of G_{i-1} is connected to G_{i-2} and the embedding of G is planar. Now, by Invariant **I4.2.1**, f is a face constructed by a construction rule of Definition 1.1 or part of a base case of Definition 1.1. Thus, in the fixed construction of G, we insert vertices belonging to L_i inside of f by Construction Rules **CR1** and **CR2** of Definition 1.1. During these operations, we only obtain new faces of G that are either triangular or quadrangular. We now prove the following fact inductively:

▷ Claim 4.3. Each face f' created by inserting a vertex of L_i into f contains only vertices of $L_i \cup L_{i-1}$.

Proof. We induce over the number n_i of vertices of L_i inserted into f. If $n_i = 1$, f is split into two to four faces, each of which containing the one new vertex of L_i and at least one vertex of L_{i-1}.

Now, assume $n_i > 1$ and let v be the n_i-th vertex of L_i inserted into f. By induction hypothesis, after inserting the first $n_i - 1$ vertices of L_i into f, we only created faces F' containing only vertices of L_i and L_{i-1}. As v belongs to L_i and thus must be visible from a vertex of L_{i-1}, it must be inserted into a face $f' \in F'$ that contains a vertex of L_{i-1}. As v is inserted according to Construction Rules **CR1** and **CR2** of Definition 1.1, this procedure only creates faces that only contain vertices of $L_i \cup L_{i-1}$. ◁

The proof follows now immediately from Claim 4.3 as each face created by a Construction Rule of Definition 1.1 is a triangle or quadrangle. ◀

We now have the prerequisite tools for establishing Theorem 1.4.

Proof of Theorem 1.4. We refine the proof of Alam et al. [2] for computing 5-queue layouts for planar 3-trees. A schematic illustration of our approach can be found in Figure 8. Let $L_0, \ldots, L_\lambda$ denote the levels of graph G. We incrementally compute a queue layout $(\prec, \mathfrak{Q})$ of the graph induced by $L_0 \cup \ldots \cup L_i$ with $i \in \{0, \ldots, \lambda\}$. During this process, we maintain the following invariants:

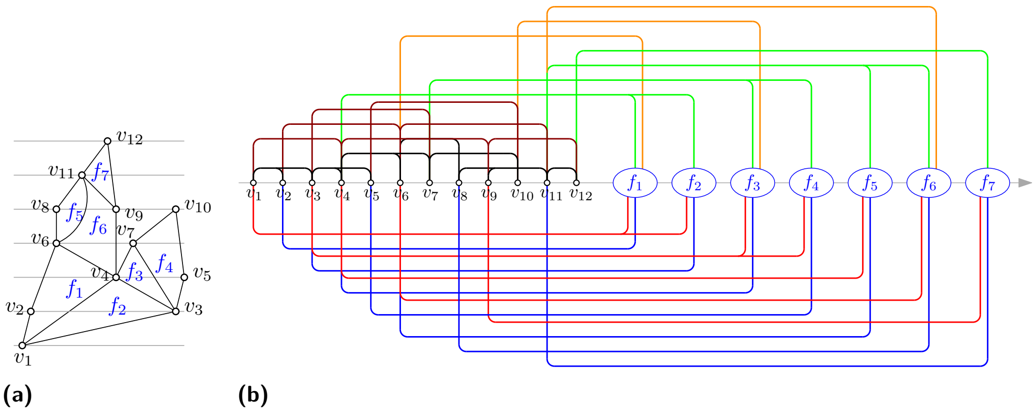


Figure 8 (a) The graph induced by layer L_i with faces f that may contain additional components. (b) Constructed 5-queue layout.

I1.4.1 For $v_j \in L_j$ and $v_{j+1} \in L_{j+1}$ with $j \in \{0, \ldots, i-1\}$ we have $v_j \prec v_{j+1}$.

I1.4.2 There is a linear order $\prec_{c(G_i)}$ of the connected components of G_i such that for each pair C_1 and C_2 of such components with $C_1 \prec_{c(G_i)} C_2$ we have for each pair of vertices $c_1 \in C_1$ and $c_2 \in C_2$ that $c_1 \prec c_2$.

I1.4.3 The level edges between vertices of L_j with $j \in \{0, \ldots, i\}$ are assigned to two queues $\mathcal{L}_1, \mathcal{L}_2 \in \mathfrak{Q}$.

I1.4.4 The binding edges between vertices of L_j and L_{j+1} with $j \in \{0, \ldots, i-1\}$ are assigned to four queues $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \mathcal{B}_4 \in \mathfrak{Q}$.

We begin by showing how to construct a 2-queue layout $(\prec_C, \{\mathcal{L}_1, \mathcal{L}_2\})$ of each connected component C of G_i; see dark red and black edges in Fig. 8b. By Lemma 4.2, we know that C is an outerplanar graph with each internal face being a triangle or a quadrangle. Thus, by Lemma 4.1, we find a layered drawing Γ_C of C. Let $\prec_i$ denote the horizontal left-to-right order of vertices on layer i in Γ_C. We obtain $\prec_C$ by concatenating $\prec_1, \prec_2, \ldots, \prec_h$ where h denotes the height of Γ_C. Further, we assign each edge of C with span 1 to $\mathcal{L}_1$ and each edge of C with span 2 to $\mathcal{L}_2$. By Property **P1** of Γ_C according to Lemma 4.1, we have assigned all edges of C.

▷ Claim 4.4. $(\prec_C, \{\mathcal{L}_1, \mathcal{L}_2\})$ is a valid 2-queue layout.

Proof. Assume for a contradiction that two edges (u, v), (u', v') assigned to $\mathcal{L}_i$ nest. Assume w.l.o.g. that $\ell(u) = \ell(v) - i$ and $\ell(u') = \ell(v') - i$. First, assume that $\ell(u) \neq \ell(u')$, say w.l.o.g. that $\ell(u) < \ell(u')$. Then we have by construction that $u \prec_C u'$ and since the span of (u, v) and (u', v') are both i also $\ell(v) < \ell(v')$. It follows by construction that $v \prec_C v'$ and (u, v) and (u', v') cannot nest.

Thus, it must be that $\ell(u) = \ell(u')$. Assume w.l.o.g. that $u \prec_C u'$, in particular, we even have that $u \prec_{\ell(u)} u'$. In order for (u, v) and (u', v') to nest, we have $v' \prec_C v$. Since $\ell(v) = \ell(v')$, we also have $v' \prec_{\ell(v)} v$. But this contradicts the planarity of the layered drawing Γ_C as (u, v) and (u', v') are straight-line, start and end at the same layers, respectively, but have to swap horizontal order in-between these layers. ◁

For L_0, we note that there is only a single connected component C and we use the obtained 2-queue layout. Thus, we establish Invariant **I1.4.3** whereas Invariants **I1.4.1**, **I1.4.2** and **I1.4.4** are trivially fulfilled. Moreover, the theorem holds for $\lambda = 0$.

For the inductive step, we first have to reconsider Properties **P2** and **P3** of Γ_C according to Lemma 4.1. Namely, consider an internal face f of Γ_C. All of its incident vertices are located on different layers according to Lemma 4.1. We define the *layer-i-vertex* $v_i(f)$ with $i \in \{1, \dots, 4\}$ as follows.

$$v_i(f) = \begin{cases} t_i & \text{if } f \text{ is a triangle and } i \leq 3 \\ \emptyset & \text{if } f \text{ is a triangle and } i = 4 \\ q_i & \text{if } f \text{ is a quadrangle} \end{cases}$$

The following claim significantly simplifies the following discussion.

▷ Claim 4.5. There exists a total order $\prec_{f(C)}$ of the faces of C such that if $f \prec_{f(C)} f'$ for two faces f and f' of C, we have $v_i(f) \prec_C v_i(f')$ for $i \in \{1, \dots, 4\}$ and $v_i(f), v_i(f') \neq \emptyset$.

Proof. We construct $\prec_{f(C)}$ as follows. We set $f \prec_{f(C)} f'$ if $v_1(f) \prec_C v_1(f')$ or if $v_1(f) = v_1(f')$ and f occurs to the left of f' in the left-to-right order of faces incident to $v_1(f)$ in Γ_C. It remains to discuss that also $v_i(f) \prec_C v_i(f')$ if $f \prec_{f(C)} f'$ for $i \in \{2, \dots, 4\}$.

Note that by definition, we have $\ell(v_1(f)) \leq \ell(v_1(f'))$. If $\ell(v_1(f)) < \ell(v_1(f'))$, we obtain also $\ell(v_i(f)) < \ell(v_i(f'))$ and thus $v_i(f) \prec_C v_i(f')$. Thus, it remains to consider the case where $\ell(v_1(f)) = \ell(v_1(f'))$. In this scenario consider edges $(v_1(f), v_i(f))$ and $(v_1(f'), v_i(f'))$. Note that if $i = 4$ and f and f' are quadrangular, these edges can be inserted as y-monotone curves into face f and f', respectively, and do not occur outside of these faces as C is outerplanar (refer also to the blue dashed edges in Fig. 7). Edges $(v_1(f), v_i(f))$ and $(v_1(f'), v_i(f'))$ occur in between layers $\ell(v_1(f))$ and $\ell(v_i(f))$ as crossing-free y-monotone curves in Γ_C. Thus, if $v_1(f) \neq v_1(f')$, it follows from $v_1(f) \prec_{\ell(v_1(f)} v_1(f')$ that also $v_i(f) \prec_{\ell(v_1(f)+(i-1)} v_i(f')$. Otherwise, $v_1(f) = v_1(f')$ and $(v_1(f), v_i(f))$ and $(v_1(f'), v_i(f'))$ occur in the same left-to-right order around $v_1(f)$ as f and f'. Thus, the claim follows. ◁

We are now ready to describe the incremental step of our algorithm. We process the connected components of graph G_i in the left-to-right order $\prec_C (G_i)$. For the connected component C, we process its faces in the left-to-right order $\prec (f(C))$. For face f of C, we check if it contains connected components of G_{i+1}. If so, we compute a 2-queue layout of each of these components as described above and concatenate its linear order to $\prec$. Then, we assign all binding edges incident to $v_i(f)$ to queue $\mathcal{B}_k$ for $k \in \{1, \dots, 4\}$; see red, blue, green and and orange edges in Fig. 8b, respectively. We then proceed with the next face or component in $\prec (f(C))$ or $\prec_C (G_i)$, respectively. Clearly, this incremental step maintains Invariants **I1.4.1**, **I1.4.2** and **I1.4.3** for each new component inserted. Thus, it remains to discuss that the binding edges between L_i and L_{i+1} maintain Invariant **I1.4.3**. We first consider the case, where different components of G_i are involved.

▷ Claim 4.6. Let (u, v) and (u', v') be two binding edges between L_i and L_{i+1} such that u belongs to a component C of G_i and u' belongs to a component C' of G_i. Then, (u, v) and (u', v') do not nest.

Proof. Assume w.l.o.g. that $C \prec_{C(G_i)} C'$. By Invariant **I1.4.3**, we also have $u \prec u'$. By Lemma 4.2 each connected component C' of G_{i+1} is connected via binding edges to vertices of exactly one component C of G_i. As the components of G_{i+1} are inserted according to the left-to-right order $\prec_C (G_i)$ it also follows that $v \prec v'$ and the statement follows. ◁

Now consider the case where both edges are part of the same connected component.

▷ Claim 4.7. Let (u, v) and (u', v') be two binding edges between L_i and L_{i+1} such that u and u' belong to a component C of G_i. If $(u, v), (u', v') \in \mathcal{B}_k$ for $k \in \{1, \ldots, 4\}$, (u, v) and (u', v') do not nest.

Proof. In this scenario, by Lemma 4.2 we know that each component C' of G_{i+1} is inserted into a face f of G_i. Since $(u, v), (u', v') \in \mathcal{B}_k$, we have $u = v_k(f)$ and $u' = v_k(f')$ for faces f and f' of G_i. Moreover, v is part of a component inserted into f whereas v' is part of a component inserted into f'. If $f = f'$, it follows that $u = u'$ and no nesting occurs. Otherwise, assume w.l.o.g. that $u \prec u'$. Then, also $f \prec_{f(C)} f'$ by Claim 4.5 and therefore $v \prec v'$ since we inserted the components of G_{i+1} that have binding edges incident to vertices of C according to order $\prec (f(C))$. ◁

Claims 4.6 and 4.7 imply Invariant **I1.4.4** and the theorem follows as $\mathfrak{Q} = \{\mathcal{L}_1, \mathcal{L}_2, \mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \mathcal{B}_4\}$ has cardinality six. ◀

4.2 General Planar Graphs

Using Theorem 1.2 and Theorem 1.4, we could immediately apply the following lemma by Dujmović, Joret, Micek, Morin, Ueckerdt and Wood [16] to obtain an upper bound of 39:

▶ **Lemma 4.8** (Dujmović, Joret, Micek, Morin, Ueckerdt and Wood [16]). *Let $G = (V_G, E_G)$ be a graph such that there exists an H-partition $(H, \mathcal{B})$ of G with layered width ℓ with respect to a BFS-layering $\mathcal{L} = (L_0, L_1, \ldots)$ and $H = (V_H, E_H)$. Then,*

$$\mathrm{qn}(G) \leq 3\ell \cdot \mathrm{qn}(H) + \left\lfloor \frac{3}{2}\ell \right\rfloor .$$

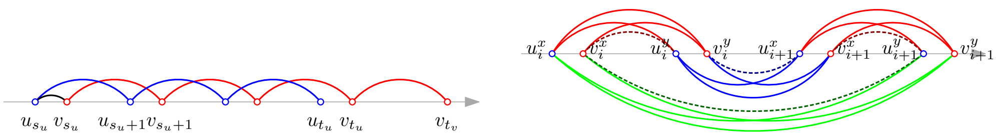
However, we can slightly improve the argumentation used to proof Lemma 4.8 to show Theorem 1.5. In anticipation of Theorem 1.7, we prove the following slightly stronger variant of Theorem 1.5.

▶ **Theorem 4.9.** *Let G be a planar graph. Then $\mathrm{qn}(G) \leq 38$. In particular, if G is a triangulated embedded plane graph with a predetermined BFS-layering $\mathcal{L}$ rooted at a vertex of the outer face, it admits a 38-queue layout $(\prec, \mathfrak{Q})$ such that there is a subset $\mathfrak{L}^* \subset \mathfrak{Q}$ of cardinality $|\mathfrak{L}^* = 13|$ containing only intra-layer edges w.r.t. $\mathcal{L}$.*

Proof. We closely follow the proof of Lemma 4.8 by Dujmović, Joret, Micek, Morin, Ueckerdt and Wood [16]. First, we apply Theorem 3.4 to identify a H-partition $(H, \mathcal{B})$ of layered width 2 where each $B \in \mathcal{B}$ induces an acyclic bipod. We first discuss how to compute a 1-queue layout $(\prec_x, \{\mathcal{Q}_b\})$ of the induced subgraph of each $B_x \in \mathcal{B}$ with $x \in H$ as follows. By Definition 3.2, we can partition the vertices of B_x into two disjoint vertical paths $P_1^x = (u_{s_x}^x, u_{s_x+1}^x, \ldots, u_{t_x}^x)$ and $P_2^x = (v_{\sigma_x}^x, v_{\sigma_x+1} \ldots, v_{\tau_x})$ such that $s_x \leq \sigma_x$. We now define $\prec_x$ such that $u_i^x \prec_x v_i^x$ for $s_x \leq i \leq \max\{t_x, \tau_x\}$ and $\alpha_i \prec_x \beta_j$ with $s_x \leq i \leq j \leq \max\{t_x, \tau_x\}$ and $\alpha, \beta \in \{u^x, v^x\}$. We assign all edges induced by B_x to queue $\mathcal{Q}_b$. See Figure 9a for an illustration.

▷ Claim 4.10. $(\prec_x, \mathcal{Q}_b)$ is a 1-queue layout of the subgraph of G induced by B_x.

Proof. Assume for a contradiction that two edges of (u, v) and (u', v') induced by B_x nest. Assume w.l.o.g. that $u \prec_x u'$, $u \prec_x v$ and $u' \prec_x v'$. We first observe that (u, v) and (u', v') cannot belong to the same vertical path P_i^x with $i \in \{1, 2\}$ as the vertices of P_i^x occur in the order induced by P_i^x in $\prec_x$. Thus, we necessarily obtain $u \prec_x v \prec_x u' \prec_x v'$, i.e. (u, v) and (u', v') do not nest.



(a) **(b)**

Figure 9 Illustrations for the proof of Theorem 4.9.

Second, assume that $(u, v) = (u^x_{s_x}, v^x_{\sigma_x})$. Then, we always have $u \prec_x v \prec_x u' \prec_x v'$ unless $s_x = \sigma_x - 1$ and $(u', v') = (u^x_{s_x}, u^x_{\sigma_x+1})$. In this scenario we obtain $u \prec_x u'v \prec_x \prec_x v'$. Thus, in both cases, (u, v) and (u', v') do not nest.

Thus, we necessarily have $(u, v) \in P^x_i$ and $(u', v') \in P^x_j$ with $i \neq j$ and $i, j \in \{1, 2\}$. If $i = 2$, $u \prec u'$ implies that the edges are $(u, v) = (v^x_a, v^x_{a+1})$ and $(u', v') = (u^x_b, u^x_{b+1})$ with $s_{u^x} \leq a < b \leq \max\{t_{u^x}, t_{v^x}\}$. Thus, we have either $u \prec_x v \prec_x u' \prec_x v'$ if $a < b - 1$ or $u \prec_x u' \prec_x v \prec_x v'$, i.e. no nesting. Otherwise, we $i = 1$ and $u \prec u'$ implies that the edges are $(u, v) = (u^x_a, u^x_{a+1})$ and $(u', v') = (v^x_b, v^x_{b+1})$ with $s_{u^x} \leq a \leq b \leq \max\{t_{u^x}, t_{v^x}\}$. Thus, we have either $u \prec_x v \prec_x u' \prec_x v'$ if $a < b$ or $u \prec_x u' \prec_x v \prec_x v'$, i.e. no nesting. This contradicts our assumption that (u, v) and (u', v') nest. ◁

We now construct the queue layout $(\prec, \mathfrak{Q})$ of H. To this end, we first compute a 6-queue layout $(\prec_H, \mathfrak{H})$ of H with $\mathfrak{H} = \{\mathcal{H}_1, \mathcal{H}_2, \dots, \mathcal{H}_6\}$. We now define $\prec$ as follows. If $u \in L_i$ and $v \in L_j$ with $j \geq i$, then $u \prec v$. Otherwise, $u, v \in L_i$. Then, if $u \in B_x$ and $v \in B_y$ with $x \neq y$, we set $u \prec v$ if and only if $x \prec_H y$. Otherwise, we also have $u, v \in B_x$ and set $u \prec v$ if and only if $u \prec_x v$. We assign the edges of G to queues of $\mathfrak{Q}$ as follows:

1. Intra-bag intra-layer edges are assigned to queue $\mathcal{Q}^\ell_b \in \mathfrak{Q}$. Claim 4.10 asserts us that there are no nestings between edges in $\mathcal{Q}^\ell_b$.
2. Intra-bag inter-layer edges are assigned to queue $\mathcal{Q}^i_b \in \mathfrak{Q}$.
3. Inter-bag intra-layer edges (v^x_i, u^y_i) with $(x, y) \in \mathcal{H}_h$, $x \prec_H y$ and $i \geq 0$ are assigned to queues $\mathcal{L}^s_h \in \mathfrak{Q}$ and inter-bag intra-layer edges (u^x_i, u^y_i), (u^x_i, v^y_i) and (v^x_i, v^y_i) with $(x, y) \in \mathcal{H}_h$, $x \prec_H y$ and $i \geq 0$ are assigned to queues $\mathcal{L}^\ell_h \in \mathfrak{Q}$, respectively. Observe, that this induces a set of 12 queues $\mathfrak{L} \subset \mathfrak{Q}$ as $1 \leq h \leq 6$. Note that $\mathfrak{L} \cup \{\mathcal{Q}^\ell_b\}$ contains only intra-layer edges. See the red edges in Figure 9b for an illustration.
4. Inter-bag inter-layer edges (v^x_i, u^y_{i+1}) with $(x, y) \in \mathcal{H}_h$, $x \prec_H y$ and $i \geq 0$ are assigned to queues $\mathcal{F}^s_h \in \mathfrak{Q}$ and inter-bag intra-layer edges (u^x_i, u^y_{i+1}), (u^x_i, v^y_{i+1}) and (v^x_i, v^y_{i+1}) with $(x, y) \in \mathcal{H}_h$, $x \prec_H y$ and $i \geq 0$ are assigned to queues $\mathcal{F}^\ell_h \in \mathfrak{Q}$, respectively. Observe, that this induces a set of 12 queues $\mathfrak{F} \subset \mathfrak{Q}$ as $1 \leq h \leq 6$. See the green edges in Figure 9b for an illustration.
5. Inter-bag inter-layer edges (u^x_i, v^y_{i-1}) with $(x, y) \in \mathcal{H}_h$, $x \prec_H y$ and $i \geq 0$ are assigned to queues $\mathcal{R}^s_h \in \mathfrak{Q}$ and inter-bag intra-layer edges (u^x_i, u^y_{i-1}), (v^x_i, u^y_{i-1}) and (v^x_i, v^y_{i-1}) with $(x, y) \in \mathcal{H}_h$, $x \prec_H y$ and $i \geq 0$ are assigned to queues $\mathcal{R}^\ell_h \in \mathfrak{Q}$, respectively. Observe, that this induces a set of 12 queues $\mathfrak{R} \subset \mathfrak{Q}$ as $1 \leq h \leq 6$. See the blue edges in Figure 9b for an illustration.

We claim that each queue contains no nestings:

▷ Claim 4.11. The edges of $\mathcal{Q}^i_b$ do not nest.

Proof. Assume for a contradiction that there are two edges $(u, v), (u', v') \in \mathcal{Q}^i_b$ that nest. By Claim 4.10, we know that two (u, v) and (u', v') belong to different bags, i.e., $u, v \in B_x$ and $u', v' \in B_y$. Assume w.l.o.g. that $u \prec u'$, $u \prec v$ and $u' \prec v'$. If $u, u' \in L_i$, we obtain from

$\prec_H$ that also $v \prec v'$ and (u, v) and (u', v') do not nest. Hence, we have $u \in L_i$, $u' \in L_j$ and $i < j$. But then, we also have that $v \in L_{i'}$ and $v' \in L_{j'}$ with $i' < j'$ and hence $v \prec v'$. Thus (u, v) and (u', v') do not nest, a contradiction. $\lhd$

$\rhd$ Claim 4.12. The edges of $\mathcal{X}_h^\alpha$ do not nest for $1 \le h \le 6$, $\mathcal{X} \in \{\mathcal{L}, \mathcal{F}, \mathcal{B}\}$ and $\alpha \in \{s, \ell\}$.

Proof. Assume for a contradiction that there are edges $(u, v), (u', v') \in \mathcal{X}_h^w$ that nest. W.l.o.g. assume that $u \prec u'$, $u \prec v$ and $u' \prec v'$. First, observe that if $u \in L_i$ and $u' \in L_j$ with $i < j$, we have that $v \in L_i$ and $v' \in L_j$, i.e., $u \prec v \prec u' \prec v'$, i.e. (u, v) and (u', v') do not nest.

Thus, we have $u, u', v, v' \in L_i$. Let $u \in B_x$, $u' \in B_{x'}$, $v \in B_y$ and $v' \in B_{y'}$. Clearly, we have $x \prec_H y$ and $x' \prec_H y'$. If $y = x'$, we have that $v \prec v'$ and no nesting occurs. Thus, $y \neq x'$. Moreover, by $u \prec u'$, we also cannot have $y' = x$.

If $x \neq x'$ and $y \neq y'$, we obtain that the order of u, u', v, v' in $\prec$ corresponds to the order of x, x', y, y' in $\prec_H$. As edges (x, y), (y, y') do not nest in $\mathcal{H}_h$, it follows that neither do (u, v) and (u', v').

If $x = x'$ and $y = y'$, we observe that only edges (u_i^x, v_{i+k}^y) and (v_i^x, u_{i+k}^y) nest where $k = 0$ if $\mathcal{X} = \mathcal{L}$, $k = 1$ if $\mathcal{X} = \mathcal{F}$ and $k = -1$ if $\mathcal{B} = \mathcal{L}$. But one of these edges belongs to $\mathcal{X}_h^\ell$ while the other one belongs to $\mathcal{X}_h^s$.

If $x = x'$ but $y \neq y'$, we necessarily have $u = u' = u_{i+k}^x$ where $k = 0$ if $\mathcal{X} = \mathcal{L}$, $k = 1$ if $\mathcal{X} = \mathcal{F}$ and $k = -1$ if $\mathcal{B} = \mathcal{L}$. Thus we must have $x \neq x'$ but $y = y'$. However, then $v = v' = v_{i+k}^y$; a contradiction to the fact that (u, v) and (u', v') must be non-adjacent. $\lhd$

Since we have shown that no queue contains a nesting, we realize that the obtained linear layout is a 38-queue layout with queues $\mathcal{Q}_b^\ell$, $\mathcal{Q}_b^i$ and for $1 \le h \le 6$ $\mathcal{L}_h^s$, $\mathcal{L}_h^\ell$, $\mathcal{F}_h^s$, $\mathcal{F}_h^\ell$, $\mathcal{R}_h^s$ and $\mathcal{R}_h^\ell$. The proof follows by observing that $\mathfrak{L} \cup \mathcal{Q}_b^\ell$ contain only intra-layer edges where $|\mathfrak{L} \cup \mathcal{Q}_b^\ell| = 13$. ◀

5 Results for Bipartite Planar Graphs

Förster, Kaufmann, Merker, Pupyrev and Raftopoulou [20] made some observations that improved the results of Dujmović, Joret, Micek, Morin, Ueckerdt and Wood [16] for the case where the planar graph is also bipartite. To this end, given a quadrangulation G, they showed how to simultaneously construct a BFS-tree rooted at a vertex r of G and triangulate G such that all edges introduced in the triangulation are intra-layer edges. The following lemma is implicitly proven in [20].

▶ **Lemma 5.1** (Förster, Kaufmann, Merker, Pupyrev and Raftopoulou [20]). *Let $G = (V, E)$ be a quadrangulated plane graph and r a vertex on the outer face of G. Then, there exists a triangulated supergraph $G^* = (V, E \cup E_T)$ of G and a BFS-tree T rooted at r inducing a BFS-layering $\mathcal{L} = (L_0, L_1, \ldots)$ of G^* such that each edge of E_T is a intra-layer edge w.r.t. $\mathcal{L}$.*

For the sake of self-containment of this paper, we explicitly state here the following proof provided in [20]:

Proof of Lemma 5.1 [20]. To compute T we perform a BFS-traversal of G starting from vertex r. Observe that in the BFS-traversal, for each quadrangle $q = (q_1, q_2, q_3, q_4)$ we have that w.l.o.g. $q_1 \in L_i$, $q_2, q_4 \in L_{i+1}$. We add edge (q_2, q_4) to triangulate q, i.e. $q_2, q_4 \in E_T$. Since (q_2, q_4) is an intra-layer edge, the lemma follows. ◀

Finally, we are ready to prove Theorem 1.6 and Theorem 1.7.

Proof of Theorems 1.6 and 1.7. We start by computing a quadrangulated supergraph of G. Then, we apply Lemma 5.1 and obtain a triangulated supergraph G^* of G as well as a BFS-tree T rooted at the outer face. We can then apply Theorems 3.4 and 4.9 and obtain our results since by Lemma 5.1 all intra-layer edges are only part of G^* but not of G. ◀

6 Open Problems

We conclude the paper by stating intriguing open problems motivated by our results.

OP1 Graph product structure theorems have been successfully applied for other applications such as non-repetitive and p-centric colorings [7, 18] and low-treewidth colorings [16]. It remains open to check whether Theorem 1.2 can be a useful tool to improve upon such results in the literature. We remark that no immediate improvement appears to be possible as the treewidth of H increases compared to the result of Dujmović, Joret, Micek, Morin, Ueckerdt and Wood [16], i.e., planarity should be exploited.

OP2 In conjunction with Open Problem **OP1**, we also remark that planar graphs of treewidth 4 received little attention in the literature so far in general. On the other hand, there exist several results on planar 3-trees in the literature; see e.g. [2, 9, 23, 25]. Theorem 1.3 also motivates to study planar graphs of treewidth 4 more closely. A nice concrete problem in this direction would be to settle the following conjecture:

▶ **Conjecture 6.1.** *Let G be a planar graph of treewidth 4. Then, G is subgraph of a planar quasi-4-tree.*

We remark that the $(4 \times k)$-grid graph is not a counterexample for Conjecture 6.1.

OP3 Finally, the upper and lower bounds for queue numbers of graph classes such as planar graphs and k-planar graphs are still far from tight. For instance, for planar graphs the best known lower bound is 4 [2]. Closing these gaps is an important open problem in the field. We believe that proving and generalizing Conjecture 6.1 for graphs of larger treewidth may be an important step in this direction.

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