ALGEBRAIC DESCRIPTION OF COMPLEX CONJUGATION ON COHOMOLOGY OF A SMOOTH PROJECTIVE HYPERSURFACE

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ABSTRACT. We describe complex conjugation on the primitive middle-dimensional algebraic de Rham cohomology of a smooth projective hypersurface defined over a number field that admits a real embedding. We use Griffiths' description of the cohomology in terms of a Jacobian ring. The resulting description is algebraic up to transcendental factors explicitly given by certain periods.

Contents

1.	Introduction	1
2.	A description of complex conjugation on algebraic de Rham cohomology	2
2.1.	. A review on Griffiths' theory and its refinement	2
2.2.	. The main result	4
2.3.	. A proof of the main result	5
2.4.	. The case of odd n	7
2.5.	. Pure \mathbb{R} -Hodge structure	8
References		8

1. Introduction

Let V be a smooth projective variety over a number field \mathbb{k} . For each embedding $\sigma : \mathbb{k} \hookrightarrow \mathbb{C}$, we consider the associated complex manifold $V_{\sigma} := (V \times_{\mathbb{k}, \sigma} \mathbb{C})(\mathbb{C})$ and its Betti (singular) cohomology

$$H_{\mathrm{B}}^{i}(V_{\sigma},\mathbb{Q}) = H^{i}((V \times_{\mathbb{k},\sigma} \mathbb{C})(\mathbb{C}),\mathbb{Q}).$$

After tensoring with \mathbb{C} , we have the Hodge decomposition

$$\mathcal{P}: H^i_{\mathrm{B}}(V_{\sigma}, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\cong} \bigoplus_{p+q=i} H^{p,q}$$

where $H^{p,q}$ stands for the Dolbeault cohomology group of V_{σ} . Here and elsewhere, we always identify Betti cohomology and analytic de Rham cohomology after tensoring with \mathbb{C} via the de Rham theorem.

Now we note that if σ is a real embedding, then the complex conjugation $c \in \operatorname{Gal}(\mathbb{C}/\mathbb{R})$ acts on V_{σ} which induces an involution ι_{σ} on $H^{i}_{\mathrm{B}}(V_{\sigma},\mathbb{Q})$. On the other hand, clearly $1 \otimes c$ is also an involution on $H^{i}_{\mathrm{B}}(V_{\sigma},\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}$. It is well-known that both involutions $\iota_{\sigma} \otimes 1$ and $1 \otimes c$ switch the Hodge components $H^{p,q}$ and $H^{q,p}$ under \mathfrak{P} . In particular, $1 \otimes c$ corresponds to the complex conjugation \bullet on $\bigoplus_{p+q=i} H^{p,q}$, which makes $(H^{i}_{\mathrm{B}}(V_{\sigma},\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C})^{1 \otimes c} \cong H^{i}_{\mathrm{B}}(V_{\sigma},\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{R} \cong H^{i}_{\mathrm{B}}(V_{\sigma},\mathbb{R})$ into a pure \mathbb{R} -Hodge structure of weight i.

Example 1.1. Let E be an elliptic curve defined over \mathbb{Q} with the canonical embedding $\sigma \colon \mathbb{Q} \hookrightarrow \mathbb{R} \subset \mathbb{C}$. Then one can think of an element of $H^1_{\mathrm{B}}(E_{\sigma},\mathbb{Q}) \cong H^1(\mathbb{C}/\mathbb{Z}^2,\mathbb{Q})$ as of the form $\mathbb{Q}dx + \mathbb{Q}dy$, where z = x + iy is a coordinate for \mathbb{C} . Because σ is a real embedding, the map ι_{σ} satisfies $\iota_{\sigma}(dx) = dx$ and $\iota_{\sigma}(dy) = -dy$. For the Hodge decomposition, we have

$$H^1_{\mathrm{B}}(E_{\sigma},\mathbb{Q})\otimes\mathbb{C}\cong H^{1,0}\oplus H^{0,1}\cong\mathbb{C}dz\oplus\mathbb{C}d\bar{z},$$

where dz = dx + idy and $d\bar{z} = dx - idy$. Then both $\iota_{\sigma} \otimes 1$ and $1 \otimes c$ clearly swap $H^{1,0}$ and $H^{0,1}$, because $(\iota_{\sigma} \otimes 1)(dx \pm idy) = dx \mp idy$ and $(1 \otimes c)(dx \pm idy) = dx \mp idy$.

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On the other hand, we consider the algebraic de Rham cohomology

$$H^i_{\mathrm{dR}}(V) := \mathbb{H}^i(V, \Omega^{\bullet}_{V/\Bbbk})$$

where $\Omega_{V/\Bbbk}^{\bullet}$ stands for the algebraic de Rham complex of V. This is a finite-dimensional \Bbbk -vector space with a decreasing filtration defined by

$$F^k H^i_{\mathrm{dR}}(V) = \mathbb{H}^i(V, \Omega^{\geq k}_{V/\mathbb{k}})$$

Between the two cohomology groups, the algebraic de Rham theorem provides an isomorphism

$$I_{\sigma}: H^{i}_{\mathrm{dR}}(V) \otimes_{\Bbbk, \sigma} \mathbb{C} \xrightarrow{\cong} H^{i}_{\mathrm{B}}(V_{\sigma}, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}.$$

such that $F^kH^i_{\mathrm{dR}}(V)\otimes_{\Bbbk,\sigma}\mathbb{C}$ on the de Rham side corresponds to $\bigoplus_{p\geq k}H^{p,q}$ on the Betti side. If $\sigma\colon \Bbbk\hookrightarrow\mathbb{C}$ is a real embedding, then the conjugation $1\otimes c$ on the de Rham side corresponds to $\iota_\sigma\otimes c$ on the Betti side so that we have an isomorphism $H^i_{\mathrm{dR}}(V)\otimes_{\Bbbk,\sigma}\mathbb{R}\cong (H^i_{\mathrm{B}}(V_\sigma,\mathbb{Q})\otimes_{\mathbb{Q}}\mathbb{C})^{\iota_\sigma\otimes c}$ (see [2, Proposition 1.4]).

A natural question is if one can describe complex conjugation in terms of algebraic de Rham cohomology. In view of this, let us introduce an involution c_{dR} on $H^i_{\mathrm{dR}}(V) \otimes_{\Bbbk,\sigma} \mathbb{C}$ corresponding to $1 \otimes c$ under I_{σ} (and hence $\overline{\bullet}$ under $\mathcal{P} \circ I_{\sigma}$). Our main question is

Is there a computable description of c_{dR} ?

The goal of this article is to explicitly describe c_{dR} on the primitive part of the de Rham cohomology of V, when V is a smooth projective hypersurface over \mathbb{k} , using the Jacobian description of the primitive de Rham cohomology ([4] and [5]) and the result of Carlson and Griffiths ([1]).

We briefly explain the structure of the paper. We review Griffiths' theory of primitive middle-dimensional cohomology of a smooth projective hypersurface in Subsection 2.1 and state the main result for odd dimensional hypersurfaces in Subsection 2.2. In Subsection 2.3, we provide a proof of Theorem 2.3. We give a statement for even dimensional hypersurfaces and its proof in Subsection 2.4. In Subsection 2.5 we deduce a corollary for an explicit description of real primitive cohomology in terms of the Jacobian ring.

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2. A description of complex conjugation on algebraic de Rham cohomology

2.1. A review on Griffiths' theory and its refinement. Let \mathbbm{k} be a number field and n be a positive integer. Let $\mathbbm{k}[\underline{x}] = \mathbbm{k}[x_0, \cdots, x_n]$ be a polynomial ring. We fix a homogeneous polynomial $G(\underline{x}) \in \mathbbm{k}[\underline{x}]$ of degree $e \geq 1$ so that the corresponding hypersurface $X_G \subset \mathbf{P}^n_{\mathbbm{k}}$ is nonempty and defines an irreducible smooth projective hypersurface of dimension n-1 over \mathbbm{k} .

For each embedding $\sigma: \Bbbk \hookrightarrow \mathbb{C}$, which we assume to factor through \mathbb{R} for the main result of the paper, we consider the complex manifold $X = (X_G \times_{\Bbbk,\sigma} \mathbb{C})(\mathbb{C})$ and its middle-dimensional primitive singular cohomology $H^{n-1}_{\text{prim}}(X,\mathbb{Q})$. Then $H^{n-1}_{\text{prim}}(X,\mathbb{Q})$ is a pure \mathbb{Q} -Hodge structure of weight n-1, as we have the Hodge decomposition

$$(2.1) \mathcal{P}: H^{n-1}_{\text{prim}}(X, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\cong} H_0 := \bigoplus_{p+q=n-1} H_0^{p,q}$$

where $H_0^{p,q}$ is the primitive part of $H^{p,q}$. We also consider the primitive de Rham cohomology $H_{\mathrm{dR,prim}}^{n-1}(X_G)$ of X_G over k. The standard comparison isomorphism

$$(2.2) I_{\sigma}: H^{n-1}_{\mathrm{dR,prim}}(X_G) \otimes_{\mathbb{k},\sigma} \mathbb{C} \xrightarrow{\cong} H^{n-1}_{\mathrm{prim}}(X,\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}$$

exists for primitive parts (in fact, $H_{\text{prim}}^{2m+1} = H^{2m+1}$ for n = 2m + 2 and $H_{\text{prim}}^{2m} \neq H^{2m}$ for n = 2m + 1). Now we give a brief review of Griffiths' theory ([4]) and its refined version over \mathbb{k} ([5]). We introduce a new variable y and consider the polynomial ring A and the polynomial $S(y, x) \in A$ as follows:

$$A := \mathbb{k}[y, \underline{x}], \quad S(y, \underline{x}) := yG(\underline{x}).$$

We also introduce two new gradings, which we call charge ch and weight wt:

$$ch(y) := -e,$$
 $ch(x_i) := \deg(x_i) = 1,$ $wt(y) = 1,$ $wt(x_i) = 0$

for $i = 0, \dots, n$. For each $m \in \mathbb{Z}$, we denote the *ch* m (respectively, wt m) homogeneous component by A_m (respectively, $A_{(m)}$).

For each $i=0,\dots,n$, let us set $G_i(\underline{x}):=\frac{\partial G(\underline{x})}{\partial x_i}$. Note that the smoothness of X_G implies that G_0,\dots,G_n is a regular sequence. Let $\operatorname{Jac}(S)$ be the Jacobian ideal in A, i.e. the homogeneous ideal of A generated by $yG_0=\frac{\partial S}{\partial x_0},\dots,yG_n=\frac{\partial S}{\partial x_n}$, and $G=\frac{\partial S}{\partial y}$. Define the background charge as follows:

$$c_X := e - (n+1).$$

Under these notations, P. Griffiths proved the following theorem.

Theorem 2.1. [4] There is an isomorphism

$$\phi_{\mathbb{C}}: \frac{A_{c_X} \otimes_{\Bbbk, \sigma} \mathbb{C}}{Jac(S) \cap (A_{c_X} \otimes_{\Bbbk, \sigma} \mathbb{C})} \xrightarrow{\cong} H_0 = \bigoplus_{p+q=n-1} H_0^{p,q}$$

which sends the wt r homogeneous part $A_{c_X,(r)} \otimes_{\mathbb{k},\sigma} \mathbb{C}$ (for $r \leq n-1$)¹ to $H_0^{n-1-r,r}$.

Let us briefly describe $\phi_{\mathbb{C}}$. Consider the relative long exact sequence for the pair $(\mathbf{P}^n, \mathbf{P}^n \setminus X)$

$$\cdots \longrightarrow H_{n-1}(\mathbf{P}^n, \mathbf{P}^n \setminus X) \longrightarrow H_n(\mathbf{P}^n \setminus X) \longrightarrow H_n(\mathbf{P}^n) \longrightarrow H_n(\mathbf{P}^n, \mathbf{P}^n \setminus X) \longrightarrow \cdots$$

As we have an isomorphism $H_{n-1}(\mathbf{P}^n, \mathbf{P}^n \setminus X) \cong H_{n-1}(X)$ using a tubular neighborhood of X in \mathbf{P}^n , it induces a tubular neighborhood map

$$T: H_{n-1}(X,\mathbb{C}) \to H_n(\mathbf{P}^n \setminus X,\mathbb{C})$$

On the other hand, the residue map

res :
$$H^n(\mathbf{P}^n \setminus X, \mathbb{C}) \to H^{n-1}(X, \mathbb{C})$$

is locally defined by sending $\omega = \frac{dG}{G} \wedge \omega_1 + \omega_2$, where ω_1, ω_2 are holomorphic, to ω_1 , so it can be shown to respect the Hodge filtration and induces

res :
$$F^pH^n(\mathbf{P}^n \setminus X, \mathbb{C}) \to F^{p-1}H^{n-1}(X, \mathbb{C})$$
.

These two maps form the following commutative diagram

$$\begin{array}{cccc} H_{n-1}(X,\mathbb{C}) & \otimes & H^{n-1}(X,\mathbb{C}) \\ \downarrow & & & \uparrow^{\mathrm{res}} & & \downarrow^{\mathrm{res}} \\ H_{n}(\mathbf{P}^{n} \setminus X,\mathbb{C}) & \otimes & H^{n}(\mathbf{P}^{n} \setminus X,\mathbb{C}) \end{array}$$

in that

$$\int_{[\gamma]} \operatorname{res}([\omega]) = \frac{1}{2\pi\sqrt{-1}} \int_{T([\gamma])} [\omega]$$

holds for any $[\gamma] \in H_{n-1}(X,\mathbb{C})$ and $[\omega] \in H^n(\mathbf{P}^n \setminus X,\mathbb{C})$.

Let

$$\Omega_{\underline{x}} = \sum_{i=0}^{n} (-1)^{i} x_{i} dx_{0} \wedge \dots \wedge \widehat{dx_{i}} \wedge \dots \wedge dx_{n}$$

Then the Griffiths map $\phi_{\mathbb{C}}$ is given by

(2.4)
$$\phi_{\mathbb{C}}([y^i f(\underline{x})]) = \operatorname{res}\left(\left[\frac{(-1)^i i! f(\underline{x})\Omega_{\underline{x}}}{G(\underline{x})^{i+1}}\right]\right)$$

where $\left[\frac{(-1)^i i! f(\underline{x}) \Omega_x}{G(\underline{x})^{i+1}}\right]$ represents the cohomology class in $H^n(\mathbf{P}^n \setminus X, \mathbb{C})$.

Griffiths' theorem provides us an explicitly computable description of $H^{n-1}_{\text{prim}}(X,\mathbb{C})$ and its Hodge decomposition $\bigoplus_{p+q=n-1} H^{p,q}_0$. In [5], the authors provided a refined version of $\phi_{\mathbb{C}}$, that is, a \mathbb{k} -linear isomorphism

(2.5)
$$\phi: \frac{A_{c_X}}{\operatorname{Jac}(S) \cap A_{c_X}} \xrightarrow{\cong} H^{n-1}_{\operatorname{dR,prim}}(X_G)$$

¹If $r \ge n$, then the wt r homogeneous part belongs to the Jacobian ideal, i.e., $A_{c_X,(r)} \subset \text{Jac}(S) \cap A_{c_X,(r)}$.

which induces the isomorphism $\phi_{\mathbb{C}}$ after tensoring with \mathbb{C} and composing with $\mathcal{P} \circ I_{\sigma}$. Thus our question regarding c_{dR} can be reformulated as follows:

For a given homogeneous polynomial $U(y,\underline{x}) \in A_{c_X}$, can we find a polynomial $V(y,\underline{x}) \in A_{c_X} \otimes_{\Bbbk,\sigma} \mathbb{C}$ such that the cohomology class $(\mathcal{P} \circ I_{\sigma} \circ \phi)([V(y,\underline{x})])$ is same as the complex conjugation $(\mathcal{P} \circ I_{\sigma} \circ \phi)([U(y,\underline{x})])$ in H_0 ?

2.2. The main result. Let μ be the \mathbb{C} -dimension of $H^{n-1}_{\text{prim}}(X,\mathbb{C})$. We fix an index set I of cardinality μ and decompose

$$I = I_0 \sqcup \cdots \sqcup I_{n-1}$$

such that the cardinality $|I_q|$ of I_q is the \mathbb{C} -dimension of $H_0^{n-q-1,q}$. It is well-known (see, for example, [3, 5.3]) that the diffeomorphism type of such a hypersurface X is determined by the degree e and the dimension n-1, and indeed that

$$\mu = \frac{e-1}{e} \left((e-1)^n + (-1)^{n+1} \right)$$

Note that if e = 1, then indeed $H^{n-1}_{\text{prim}}(X, \mathbb{C}) = 0$ holds and everything vacuously works out. Henceforth, we assume e > 1 and accordingly $\mu > 0$.

For simplicity of presentation, we first assume n=2m+2 $(m\geq 0)$ is even. In this case, the *intersection pairing* $H_{2m+1}(X,\mathbb{Z})\otimes H_{2m+1}(X,\mathbb{Z})\to \mathbb{Z}$ is skew-symmetric and non-degenerate. Thus μ is an even number 2g for some positive integer g. We choose a *canonical basis* $\{\gamma_1,\cdots,\gamma_{2g}\}$ of $H_{2m+1}(X,\mathbb{Z})$ (in the sense of [4, page 478]), that is, a \mathbb{Z} -basis of $H_{2m+1}(X,\mathbb{Z})$ such that the intersection numbers $\gamma_i \bullet \gamma_j$ give an integral skew-symmetric $2g \times 2g$ matrix

$$Q := \begin{pmatrix} 0 & \mathrm{id}_g \\ -\mathrm{id}_g & 0 \end{pmatrix}, \quad \mathrm{id}_g = \mathrm{the} \ g \times g \ \mathrm{identity} \ \mathrm{matrix}$$

On the other hand, given $I = I_0 \sqcup \cdots \sqcup I_{2m+1}$, Hodge symmetry implies that the cardinality of $I_0 \sqcup \cdots \sqcup I_m$ is g. Then by (2.5) we can choose g elements of the form $y^i f_{j_i}(\underline{x}) \in A_{c_X}$, where $i = 0, \cdots, m$ and $j_i = 1, \cdots, |I_i|$, such that they are k-linearly independent modulo $\operatorname{Jac}(S) \cap A_{c_X}$. Note that one has to have

$$\deg f_{j_i} = c_X + ie = e - (n+1) + ie = e(i+1) - (n+1)$$

We write these basis elements as

$$\{U_1, U_2, \cdots, U_g\} := \{y^i f_{j_i} \mid i = 0, \cdots, m, \ j_i = 1, \cdots, |I_i|\}$$

Having so chosen bases for

$$H_{2m+1}(X,\mathbb{Z})$$
 and $\bigoplus_{i=0}^{m} \frac{A_{c_X,(i)}}{\operatorname{Jac}(S) \cap A_{c_X,(i)}}$

we form the $g \times 2g$ period matrix

(2.7)
$$P_X = \begin{pmatrix} \int_{\gamma_1} \phi_{\mathbb{C}}(U_1) & \cdots & \int_{\gamma_{2g}} \phi_{\mathbb{C}}(U_1) \\ \vdots & \ddots & \vdots \\ \int_{\gamma_1} \phi_{\mathbb{C}}(U_g) & \cdots & \int_{\gamma_{2g}} \phi_{\mathbb{C}}(U_g) \end{pmatrix}$$

Definition 2.2. For n=2m+2 and each $k=1,2,\cdots,g=\frac{\mu}{2}$, let

$$U_{k,\underline{\gamma}} = \sum_{i,j=1,\dots,2g} \int_{\gamma_i} \phi_{\mathbb{C}}(U_k) \cdot (\gamma_i \bullet \gamma_j) \cdot \overline{\int_{\gamma_j} \phi_{\mathbb{C}}(U_k)} \in \mathbb{C}$$

be the k-th diagonal entry of the following $g \times g$ matrix

$$P_X \cdot Q \cdot \overline{P_X}^T$$

where $\overline{P_X}^T$ is the transpose of the complex conjugation of the matrix P_X .

To state the main result, we introduce

(2.8)
$$\mathbb{D} = \mathbb{D}(\underline{x}) = \det\left(\frac{\partial^2 G(\underline{x})}{\partial x_i \partial x_j}\right), \quad i, j = 0, \dots, n$$

Then $\mathbb{D} \in \mathbb{k}[\underline{x}]$ is a homogeneous polynomial of degree (n+1)(e-2), because X_G is assumed to be smooth. For a homogeneous polynomial $h(\underline{x})$ of degree (n+1)(e-2), we define r_h to be the unique number in \mathbb{k} (which exists by [6, (12.6) (ii)]) such that

(2.9)
$$h(\underline{x}) \equiv r_h \cdot \mathbb{D}(\underline{x}) \mod \operatorname{Jac}(G) = \langle G_0, \cdots, G_n \rangle$$

in $\mathbb{k}[\underline{x}]$. Note that $h(\underline{x})$ belongs to the Jacobian ideal Jac(G) if and only if $r_h = 0$.

Theorem 2.3. Let n=2m+2 and let $U_k=y^if_{j_i}(\underline{x})\in\{U_1,\cdots,U_g\}$ in (2.6). Let us choose any polynomial $y^{n-1-i}g_{j_i}(\underline{x})\in A_{c_X}$ such that $f_{j_i}(\underline{x})\cdot g_{j_i}(\underline{x})$ does not belong to the Jacobian ideal Jac(G). If we define a homogeneous polynomial $\tilde{f}_{j_i}(\underline{x})\in(\mathbb{k}\otimes_{\mathbb{k},\sigma}\mathbb{C})[\underline{x}]$ of degree e(n-i)-(n+1) by

(2.10)
$$\tilde{f}_{j_i}(\underline{x}) := b_{i,n-1-i} \frac{U_{k,\underline{\gamma}}}{r_{f_{j_i}g_{j_i}}(2\pi\sqrt{-1})^{n-1}e(e-1)^n} \cdot g_{j_i}(\underline{x})$$

where $U_{k,\underline{\gamma}}$ is given in Definition 2.2, $r_{f_{j_i}g_{j_i}}$ is a non-zero number given in (2.9) by the assumption, and

$$(2.11) b_{i,j} = (-1)^{\frac{i(i+1)}{2} + \frac{j(j+1)}{2} + j^2},$$

then we have

$$\overline{\phi_{\mathbb{C}}([y^{i}f_{j_{i}}(\underline{x})])} = \phi_{\mathbb{C}}([y^{n-1-i}\tilde{f}_{j_{i}}(\underline{x})]) \quad \text{in } H_{0} = \bigoplus_{p+q=n-1} H_{0}^{p,q}$$

where $[\bullet]$ is the equivalence class of \bullet modulo $Jac(S) \cap (A_{c_X} \otimes_{\Bbbk,\sigma} \mathbb{C})$ and $\overline{\bullet}$ is the complex conjugation $1 \otimes c$ on $\bigoplus_{p+q=n-1} H_0^{p,q}$ satisfying $\overline{H_0^{p,q}} = H_0^{q,p}$.

Note that the same formula would realize the "complex conjugation" of $y^{n-1-i}\tilde{f}_{j_i}$ to be $y^if_{j_i}$ as it should be; the asymmetry in $b_{i,j}$ cancels out the skew-symmetricity of Q.

Example 2.4 (elliptic curves). Let $G(\underline{x}) = G(x_0, x_1, x_2) \in \mathbb{Q}[x_0, x_1, x_2]$ be a homogeneous polynomial of degree 3 which defines a elliptic curve in $\mathbf{P}^2_{\mathbb{Q}}$, that is, n=2 and e=3. We fix a canonical \mathbb{Z} -basis $\{\gamma_1, \gamma_2\}$ of the homology group $H_1(X, \mathbb{Z})$. On the other hand, we have $I = I_0 \sqcup I_1$ with $|I_0| = |I_1| = 1$. Then for any homogeneous polynomial $u(x_0, x_1, x_2)$ of degree 3 which does not belong to $\mathrm{Jac}(S)$, the set $\{[1], [yu(x_0, x_1, x_2)]\}$ is a \mathbb{Q} -basis of $\mathbb{Q}[y, x_0, x_1, x_2]_0/\mathrm{Jac}(S)$. Note that $\mathbb{D}(\underline{x}) = \mathbb{D}(x_0, x_1, x_2) = \det\left(\frac{\partial^2 G(\underline{x})}{\partial x_i \partial x_j}\right)$ is homogeneous of degree 3, which by definition yields $r_{\mathbb{D}(\underline{x})} = 1$ and hence does not belong to $\mathrm{Jac}(S)$. Therefore Theorem 2.3 says that

$$\overline{\phi_{\mathbb{C}}([1])} = \phi_{\mathbb{C}}([\mu \cdot y \mathbb{D}(\underline{x}))]$$

where

$$\mu := \frac{\int_{\gamma_1} \omega \cdot \overline{\int_{\gamma_2} \omega} - \int_{\gamma_2} \omega \cdot \overline{\int_{\gamma_1} \omega}}{24\pi\sqrt{-1}} \in \mathbb{C}, \quad \omega = \phi_{\mathbb{C}}([1]).$$

2.3. A proof of the main result. We define a bilinear pairing

$$\mathcal{C}:\bigoplus_{p+q=n-1}H_0^{p,q}\times\bigoplus_{p+q=n-1}H_0^{p,q}\to\mathbb{C}$$

by

$$\mathfrak{C}(w,w') := \int_X w \wedge w', \quad w,w' \in \bigoplus_{p+q=n-1} H_0^{p,q}$$

Note that unless $w \in H_0^{p,q}$ and $w' \in H_0^{p',q'}$ satisfy both p+p'=n-1 and q+q'=n-1, the value $\mathcal{C}(w,w')$ is zero. We also define a bilinear pairing

$$\mathcal{R}: \frac{A_{c_X} \otimes_{\Bbbk,\sigma} \mathbb{C}}{\operatorname{Jac}(S) \cap (A_{c_X} \otimes_{\Bbbk,\sigma} \mathbb{C})} \times \frac{A_{c_X} \otimes_{\Bbbk,\sigma} \mathbb{C}}{\operatorname{Jac}(S) \cap (A_{c_X} \otimes_{\Bbbk,\sigma} \mathbb{C})} \to \mathbb{C}$$

by

$$\mathcal{R}([y^i f(\underline{x})], [y^j g(\underline{x})]) = \begin{cases} b_{i,j} (2\pi\sqrt{-1})^{n-1} e(e-1)^n r_{f(\underline{x})g(\underline{x})} & \text{if } i+j=n-1\\ 0 & \text{if } i+j\neq n-1 \end{cases}$$

where $b_{i,j}$ is given in (2.11) and r_h is defined in (2.9).

Lemma 2.5. The following diagram commutes

Proof. This essentially follows from [1, Theorem 3] and results of [6].

Let $\eta: X \to \mathbf{P}^n$ be a given closed embedding. Let us consider the following exact sequence of complexes of sheaves on \mathbf{P}^n :

$$(2.12) 0 \to \Omega_{\mathbf{P}^n}^{\bullet} \to \Omega_{\mathbf{P}^n}^{\bullet}(\log X) \stackrel{\mathrm{res}}{\to} \eta_* \Omega_X^{\bullet - 1} \to 0,$$

where Ω_Y^p is a sheaf of holomorphic *p*-forms on Y and $\Omega_{\mathbf{p}_n}^p(\log X)$ is a sheaf of meromorphic *p*-forms ω on \mathbf{P}^n such that ω and $d\omega$ are regular on $\mathbf{P}^n \setminus X$ and have at most a pole of order one along X. See [6, page 444] for more details. Let δ be the coboundary map in the Poincaré residue sequence induced from (2.12):

(2.13)
$$\delta: H^{n-1}(X, \Omega_X^{n-1}) \to H^n(\mathbf{P}^n, \Omega_{\mathbf{P}^n}^n).$$

A main result of [1] gives an explicit Cech-type formula for the coboundary map δ of the cup product of two cohomology classes: let i and j be non-negative integers such that i + j = n - 1. For homogeneous polynomials $A(\underline{x}), B(\underline{x}) \in \mathbb{C}[\underline{x}]$ such that $\deg(A) = e(i+1) - (n+1)$ and $\deg(B) = e(j+1) - (n+1)$,

(2.14)
$$\delta\left(\operatorname{res}\left(\frac{A(\underline{x})\Omega_{\underline{x}}}{G(\underline{x})^{i+1}}\right)\cdot\operatorname{res}\left(\frac{B(\underline{x})\Omega_{\underline{x}}}{G(\underline{x})^{j+1}}\right)\right) = c_{i,j}\frac{A(\underline{x})B(\underline{x})\Omega_{\underline{x}}}{\frac{\partial G}{\partial x_0}\frac{\partial G}{\partial x_1}\cdots\frac{\partial G}{\partial x_n}},$$

where \cdot is the cup product of the singular cocycles and

(2.15)
$$c_{i,j} = \frac{(-1)^{\frac{i(i+1)}{2} + \frac{j(j+1)}{2} + n - 1 + j^2}}{i!j!} e.$$

See [1, Theorem 3] for details.

For any $h(\underline{x}) \in \mathbb{C}[\underline{x}]$, we have the following formula:

$$\int_{\mathbf{P}^n} \left[\frac{h(\underline{x})\Omega_{\underline{x}}}{\frac{\partial G}{\partial x_0} \frac{\partial G}{\partial x_1} \cdots \frac{\partial G}{\partial x_n}} \right] = \frac{(2\pi\sqrt{-1})^n}{e-1} \operatorname{Res}_0 \left\{ \frac{h(\underline{x})}{\frac{\partial G}{\partial x_0} \frac{\partial G}{\partial x_1} \cdots \frac{\partial G}{\partial x_n}} \right\}.$$

where Res₀ is the Grothendieck residue map (see [6, (12.3)] for its definition and well-definedness). This follows from [6, Remark (12.10)], where we set $d_0 = e - 1$, m = n, and $F_i = \frac{\partial G}{\partial x_i}$ in their notations. Moreover, we use the following result [6, (12.5)] on the computation of the Grothendieck residue:

$$\operatorname{Res}_{0} \left\{ \frac{h(\underline{x})}{\frac{\partial G}{\partial x_{0}} \cdots \frac{\partial G}{\partial x_{n}}} \right\} = (e-1)^{n+1} r_{h}$$

where $r_h \in \mathbb{C}$ is defined in (2.9) for a homogeneous polynomial h of degree (n+1)(e-2) and otherwise set to be zero. Thus we have

(2.16)
$$\int_{\mathbf{P}^n} \left[\frac{h(\underline{x})\Omega_{\underline{x}}}{\frac{\partial G}{\partial x_0} \frac{\partial G}{\partial x_1} \cdots \frac{\partial G}{\partial x}} \right] = (2\pi\sqrt{-1})^n (e-1)^n r_{h(\underline{x})}$$

The relationship between the long exact sequence of hypercohomology of (2.12) and the Gysin sequence is given by the following commutative diagram (see, for instance, [6, Proposition (11.4)]):

$$\cdots \longrightarrow \mathbb{H}^{k-2}(\Omega_X^{\bullet}) \xrightarrow{\widetilde{\delta}} \mathbb{H}^k(\Omega_{\mathbf{P}^n}^{\bullet}) \longrightarrow \mathbb{H}^k(\Omega_{\mathbf{P}^n}^{\bullet}(\log X)) \xrightarrow{\mathrm{res}} \mathbb{H}^{k-1}(\Omega_X^{\bullet}) \longrightarrow \cdots$$

$$\cong \uparrow \qquad \qquad \qquad \uparrow \cong \qquad \qquad \uparrow \cong \qquad \qquad \uparrow \cong$$

$$\cdots \longrightarrow H^{k-2}(X,\mathbb{C}) \xrightarrow{2\pi\sqrt{-1}\cdot \eta_!} H^k(\mathbf{P}^n,\mathbb{C}) \longrightarrow H^k(\mathbf{P}^n - X,\mathbb{C}) \longrightarrow H^{k-1}(X,\mathbb{C}) \longrightarrow \cdots$$

Here all the vertical maps are isomorphisms, the map δ in (2.13) can be regarded as a part of $\widetilde{\delta}$, and the lower shriek $\eta_!$ of η can be constructed as the composition of three maps:

$$H^{k-2}(X,\mathbb{C}) \stackrel{\cong}{\to} H_{2n-k}(X,\mathbb{C}) \stackrel{\eta_*}{\to} H_{2n-k}(\mathbf{P}^n,\mathbb{C}) \stackrel{\cong}{\to} H^k(\mathbf{P}^n,\mathbb{C}),$$

where the first and third maps are the Poincaré duality isomorphisms. Thus when k=2n, we have $\int_X \mu = \int_{\mathbf{P}^n} \eta_!(\mu)$ for $\mu \in H^{2n-2}(X,\mathbb{C}) \cong H^{n-1}(X,\Omega_X^{n-1})$. Moreover, because $\delta = 2\pi \sqrt{-1} \cdot \eta_!$, we have

(2.17)
$$\int_{X} \mu = \frac{1}{2\pi\sqrt{-1}} \int_{\mathbf{P}^{n}} \delta(\mu).$$

Therefore we have

$$\begin{split} \int_{X} \operatorname{res} \left(\frac{(-1)^{i} i! f(\underline{x}) \Omega_{\underline{x}}}{G(\underline{x})^{i+1}} \right) \cdot \operatorname{res} \left(\frac{(-1)^{j} j! g(\underline{x}) \Omega_{\underline{x}}}{G(\underline{x})^{j+1}} \right) \\ \stackrel{\text{(2.17)}}{=} & \frac{1}{2\pi \sqrt{-1}} \int_{\mathbf{P}^{n}} \delta \left(\operatorname{res} \left(\frac{(-1)^{i} i! f(\underline{x}) \Omega_{\underline{x}}}{G(\underline{x})^{i+1}} \right) \cdot \operatorname{res} \left(\frac{(-1)^{j} j! g(\underline{x}) \Omega_{\underline{x}}}{G(\underline{x})^{j+1}} \right) \right) \\ \stackrel{\text{(2.14)}}{=} & \frac{1}{2\pi \sqrt{-1}} \int_{\mathbf{P}^{n}} c_{i,j} \left[\frac{(-1)^{i+j} i! j! f(\underline{x}) g(\underline{x}) \Omega_{\underline{x}}}{\frac{\partial G}{\partial x_{0}} \frac{\partial G}{\partial x_{1}} \cdots \frac{\partial G}{\partial x_{n}}} \right] \\ \stackrel{\text{(2.15)}}{=} & \frac{b_{i,j} \cdot e}{2\pi \sqrt{-1}} \int_{\mathbf{P}^{n}} \left[\frac{f(\underline{x}) g(\underline{x}) \Omega_{\underline{x}}}{\frac{\partial G}{\partial x_{0}} \frac{\partial G}{\partial x_{1}} \cdots \frac{\partial G}{\partial x_{n}}} \right] \\ \stackrel{\text{(2.16)}}{=} & b_{i,j} (2\pi \sqrt{-1})^{n-1} e(e-1)^{n} r_{f(\underline{x}) g(\underline{x})}. \end{split}$$

If we use (2.4), then the lemma follows from the above computation.

Proof of Theorem 2.3. The following diagram commutes

$$\bigoplus_{p+q=2m+1} H_0^{p,q} \times \bigoplus_{p+q=2m+1} H_0^{p,q}$$

$$\cong \downarrow \qquad \qquad \cong \downarrow$$

$$H_{2m+1}(X,\mathbb{C}) \times H_{2m+1}(X,\mathbb{C}) \xrightarrow{\mathcal{B}} \mathbb{C}$$

where the bottom pairing \mathcal{B} is the intersection pairing, and the isomorphism $H_0 \xrightarrow{\cong} H_{2m+1}(X,\mathbb{C})$ is given by

$$\omega \mapsto \left(\int_{\gamma_1} \omega\right) \cdot \gamma_1 + \dots + \left(\int_{\gamma_{2g}} \omega\right) \cdot \gamma_{2g}.$$

Therefore with respect to the complex conjugation $1 \otimes c$, the value $\mathcal{C}(\phi_{\mathbb{C}}(U_k), \overline{\phi_{\mathbb{C}}(U_k)}) = \int_X \phi_{\mathbb{C}}(U_k) \wedge \overline{\phi_{\mathbb{C}}(U_k)}$ is the same as

$$\mathcal{B}\left(\sum_{i=1}^{2g} \left(\int_{\gamma_i} \phi_{\mathbb{C}}(U_k)\right) \cdot \gamma_i, \sum_{j=1}^{2g} \left(\overline{\int_{\gamma_j} \phi_{\mathbb{C}}(U_k)}\right) \cdot \gamma_j\right)$$

which is $U_{k,\underline{\gamma}}$ in Definition 2.2. Then in view of Lemma 2.5, finding $\phi_{\mathbb{C}}^{-1}(\overline{\phi_{\mathbb{C}}(U_k)})$ for $U_k = y^i f_{j_i}(\underline{x})$ amounts to finding $h_{j_i}(\underline{x})$ such that $\Re(y^i f_{j_i}, y^{n-1-i} h_{j_i}) = U_{k,\underline{\gamma}}$. Now it is clear that $h_{j_i} = \tilde{f}_{j_i}$ as defined in (2.10) satisfies the desired condition.

2.4. The case of odd n. For odd n = 2m + 1 $(m \ge 0)$, according to [4, Proposition 7.1], for

$$H_{2m}(X,\mathbb{Z})_0 = \ker \left(H_{2m}(X,\mathbb{Z}) \to H_{2m}(\mathbf{P}^n,\mathbb{Z}) \right),$$

the intersection pairing

$$H_{2m}(X,\mathbb{Z})_0 \otimes H_{2m}(X,\mathbb{Z})_0 \to \mathbb{Z}$$

is symmetric and non-degenerate, and $\det(\gamma_{\alpha} \bullet \gamma_{\beta}) = \pm e$. We choose a *standard basis* (in the sense of [4, page 485]) $\gamma_1, \dots, \gamma_{\mu}$ for $H_{2m}(X, \mathbb{Z})_0$, that is, a \mathbb{Z} -basis of $H_{2m}(X, \mathbb{Z})_0$ for which the intersection pairing matrix is of the form

$$P = \begin{pmatrix} P_0 & & & 0 \\ & P_1 & & \\ & & \ddots & \\ 0 & & & P_r \end{pmatrix}, \qquad P_1 = \begin{pmatrix} 0 & d_1 \\ d_1 & 0 \end{pmatrix}, \dots, P_r = \begin{pmatrix} 0 & d_r \\ d_r & 0 \end{pmatrix}$$

where P_0 is an anisotropic symmetric integral matrix, that is, $v^T P_0 v \neq 0$ for any integral vector v.

Given $I = I_0 \sqcup \cdots \sqcup I_{2m}$, we choose a set (2.18)

$$\{U_1, U_2, \cdots, U_a\} := \left\{ y^i f_{j_i} \mid i = 0, 1, \cdots, m - 1, \ j_i = 1, \cdots, |I_i| \right\} \sqcup \left\{ y^m f_{j_m} \mid j_m = 1, \cdots, \left| \frac{|I_m| + 1}{2} \right| \right\}$$

which is k-linearly independent modulo $\operatorname{Jac}(S) \cap A_{c_X}$. We also form the $a \times \mu$ period matrix P_X as in (2.7).

Definition 2.6. For n=2m+1 and each $k=1,\cdots,a$, let

$$U_{k,\underline{\gamma}} = \sum_{i,j=1,\cdots,\mu} \int_{\gamma_i} \phi_{\mathbb{C}}(U_k) \cdot (\gamma_i \bullet \gamma_j) \cdot \overline{\int_{\gamma_j} \phi_{\mathbb{C}}(U_k)} \in \mathbb{C}$$

be the k-th diagonal entry of the following $a \times a$ matrix

$$P_X \cdot P \cdot \overline{P_X}^T$$

where $\overline{P_X}^T$ is the transpose of the complex conjugation of the matrix P_X .

Let $U_k = y^i f_{j_i}(\underline{x}) \in \{U_1, U_2, \cdots, U_a\}$ in (2.18). Then the same statement in Theorem 2.3 holds for n = 2m + 1 case by the same argument.

2.5. Pure \mathbb{R} -Hodge structure. For given n, we consider elements of the form $y^i f_{j_i} \in A_{c_X}$ that are \mathbb{k} -linearly independent modulo $\operatorname{Jac}(S) \cap A_{c_X}$ as considered in (2.6) and (2.18). For a notational convenience, we regard j_i as an element of I_i . Then the following corollary is immediate from Theorem 2.3.

Corollary 2.7. For n even, let

(2.19)
$$w_{j_i} = \frac{y^i f_{j_i} + y^{n-1-i} \tilde{f}_{j_i}}{2}$$

for $i = 0, \dots, \frac{n-2}{2}$ and $j_i = 1, \dots, |I_i|$. Then we have

$$(\phi_{\mathbb{C}}^{-1} \circ \mathcal{P})(H^{n-1}_{\text{prim}}(X, \mathbb{R})) = \sum_{j \in I_0 \sqcup \cdots \sqcup I_{\frac{n-2}{2}}} \left(\mathbb{R} \cdot [w_j] + \mathbb{R} \cdot [\sqrt{-1}w_j] \right)$$

For n = 2m + 1 odd, under the same notation (2.19) and

$$w_{j_m} = \frac{y^m f_{j_m} + y^m \tilde{f}_{j_m}}{2}$$

for $j_m = 1, \dots, \lfloor \frac{|I_m|+1}{2} \rfloor$, we have

$$(\phi_{\mathbb{C}}^{-1} \circ \mathcal{P})(H_{\mathrm{prim}}^{n-1}(X, \mathbb{R})) = \begin{cases} \sum_{j \in I_0 \sqcup \cdots \sqcup I_m} \left(\mathbb{R} \cdot [w_j] + \mathbb{R} \cdot [\sqrt{-1}w_j] \right) & \text{if } |I_m| \text{ is even} \\ \sum_{j \in I_0 \sqcup \cdots \sqcup I_m \setminus \{\alpha\}} \left(\mathbb{R} \cdot [w_j] + \mathbb{R} \cdot [\sqrt{-1}w_j] \right) + \mathbb{R} \cdot [w_\alpha] & \text{if } |I_m| \text{ is odd} \end{cases}$$

where we write $\alpha = \lfloor \frac{|I_m|+1}{2} \rfloor \in I_m$.

Example 2.8 (elliptic curves). With the notations of Example 2.4, we have

$$(\phi_{\mathbb{C}}^{-1} \circ \mathcal{P})(H_{\mathrm{prim}}^{n-1}(X, \mathbb{R})) = \mathbb{R}\left[\frac{1 + \mu \cdot y\mathbb{D}(\underline{x})}{2}\right] + \sqrt{-1}\mathbb{R}\left[\frac{1 + \mu \cdot y\mathbb{D}(\underline{x})}{2}\right].$$

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