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Abstract

The study of symmetric structures is a new trend in Ramsey theory. Recently in [7], Di Nasso initiated a systematic study of symmetrization of classical Ramsey theoretical results, and proved a symmetric version of several Ramsey theoretic results. In this paper Di Nasso asked if his method could be adapted to find new non-linear Diophantine equations that are partition regular [7, Final remarks (4)]. By analyzing additive, multiplicative, and symmetric large sets, we construct new partition regular equations that give a first affirmative answer to this question. A special case of our result shows that if P is a polynomial with no constant term then the equation $x+P(y-x)=z+w+zw$, where $y\neq x$ is partition regular. Also we prove several new monochromatic patterns involving additive, multiplicative, and symmetric structures. Throughout our work, we use tools from the Algebra of the Stone-Čech Compactifications of discrete semigroups.

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1 Introduction

Throughout our article by $\mathbb{N}$, we denote the set of all positive integers. For any $r\in\mathbb{N}$, a r-coloring of a set X is a partition of X into r disjoint sets. By finite coloring we mean any r-coloring, for any $r\in\mathbb{N}$. Let $(S,\cdot)$ be a semigroup, and $\mathcal{F}$ be a family of subsets of S. We say $\mathcal{F}$ is partition regular (P.R.) if, for any finite coloring of S, there exists a member of $F\in\mathcal{F}$ in the same color, i.e. F is monochromatic. Given a function $F:\mathbb{Z}^d\to\mathbb{Z}$ is called P.R. if for every finite coloring of $\mathbb{Z}$, there exists monochromatic $x_1,x_2,\ldots,x_d$ such that $F(x_1,x_2,\ldots,x_d)=0$. A core problem in arithmetic Ramsey theory is to characterize of the partition regular families and functions.

A cornerstone result in arithmetic Ramsey theory is the van der Waerden's theorem [19], which states that for any $l,r\in\mathbb{N}$, and for any r coloring of $\mathbb{Z}$, there exists a monochromatic arithmetic progression (A.P.) of length l, in other words, the collection of l length A.P.'s is P.R. Another important theorem was due to Schur [18] which states that the collection $\{\{x,y,x+y\}:x,y\in\mathbb{Z}\}$ is P.R. Passing to the map $n\to 2^n$, one can immediately prove the multiplicative version of the van der Waerden's (which says the collection of l length geometric progressions (G.P.) is P.R.), and the Schur theorem (which says $\{\{x,y,x\cdot y\}:x,y\in\mathbb{Z}\}$ is partition regular). However, the combined extension of additive and multiplicative Schur's theorem is one of the hardest and most investigated conjectures in arithmetic Ramsey theory. The following question is a long-standing open problem and motivates us to study combined additive and multiplicative patterns.

Question 1.1. [14, Question 3] *Does the collection* $\{\{x,y,x+y,x\cdot y\}:x,y\in\mathbb{N}\}$ *P.R.?*

In 1979 by N. Hindman [12, Section 4] and R. Graham [9, pages 68-69] proved this pattern is monochromatic for two colorings. In [17], J. Moreira proved the collection $\{x,x+y,xy\}$ is P.R. Later in [1], J.M. Barrett, M. Lupini, and J. Moreira proved the collection $\{x,x+y+xy,xy\}$ is P.R. In [7], M. Di Nasso proved the collection $\{x,y,x+y+xy\}$ is P.R. and then he extended this result in several directions. In this article, he introduces a new operation to study "symmetric structure".

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Definition 1.2 (Symmetric structure)**.** For any $n \in \mathbb{N}$, and any $(x_1, x_2, \ldots, x_n) \in \mathbb{N}^n$ a structure $P(x_1, x_2, \ldots, x_n)$ is called symmetric structure if $P(x_1, x_2, \ldots, x_n) = P(x_{i_1}, x_{i_2}, \ldots, x_{i_n})$ for any $(i_1, i_2, \ldots, i_n) \in S_n$, where S_n is the set of all permutation of $\{1, 2, \ldots, n\}$.

If $P(x_1, x_2, \ldots, x_n)$ is a polynomial, then this structure is called a symmetric polynomial.

For example, $P(x, y) = x + y + xy$ is a symmetric polynomial. In [7], M. Di Nasso found monochromatic patterns arising from symmetric polynomials. Later in [6], A. Chakraborty and the second author found polynomial extensions of several symmetric patterns.

In this article, we analyze the algebraic structure of the new symmetric operation introduced by M. Di Nasso [7]. In section 2, we recall the algebraic structure of the ultrafilters over discrete semigroups, and some basic notions from [15]. Then in section 3, we study the interplay between the additive, multiplicative, and symmetric large set. This is an analogous study in the direction of [3]. Then we deduce several new combinatorial applications of our results. Also in Corollary 4.12, we find new partition regular families which gives an answer to [7, Final remarks (4)].

2 Preliminaries

In this section, we recall the algebraic structure of the Stone-Čech compactification of discrete semigroups that we use in our work. Then we recall the newly introduced symmetric operation by M. Di Nasso in [7], which will help us to study symmetric patterns.

2.1 A brief introduction to the algebra of ultrafilters

In this section we recall some basic prerequisites from the ultrafilter theory. For details we refer the book [15] to the readers.

A filter $\mathcal{F}$ over any nonempty set X is a collection of subsets of X such that

1. $\emptyset \notin \mathcal{F}$, and $X \in \mathcal{F}$,
2. $A \in \mathcal{F}$, and $A \subseteq B$ implies $B \in \mathcal{F}$,
3. $A, B \in \mathcal{F}$ implies $A \cap B \in \mathcal{F}$.

Using Zorn's lemma we can guarantee the existence of maximal filters which are called ultrafilters. Any ultrafilter p has the following property:

- if $X = \cup_{i=1}^{r} A_i$ is any finite partition of X, then there exists $i \in \{1, 2, \ldots, r\}$ such that $A_i \in p$.

Let $(S, \cdot)$ be any discrete semigroup. Let βS be the collection of all ultrafilters. For every $A \subseteq S$, define $\overline{A} = \{p \in \beta S : A \in p\}$. Now one can check that the collection $\{\overline{A} : A \subseteq S\}$ forms a basis for a topology. This basis generates a topology over βS. We can extend the operation "$\cdot$" of S over βS as: for any $p, q \in \beta S$, $A \in p \cdot q$ if and only if $\{x : x^{-1}A \in q\} \in p$. With this operation "$\cdot$", $(\beta S, \cdot)$ becomes a compact Hausdorff right topological semigroup. One can show that βS is nothing but the Stone-Čech compactification of S. Hence Ellis's theorem guarantees that there exist idempotents in $(\beta S, \cdot)$.

Before we proceed, let us recall the notions of IP sets. For any set X, let $\mathcal{P}_f(X)$ be the collection of all nonempty finite subsets of X.

Definition 2.1 (IP Sets)**.** Let $(S, +)$ be a commutative semigroup. A set $A \subseteq S$ is said to be an IP set if there exists an injective sequence $\langle x_n \rangle_{n \in \mathbb{N}}$ in S such that

$$A = FS(\langle x_n \rangle_{n \in \mathbb{N}}) = \left\{ \sum_{n \in \alpha} x_n : \alpha \in \mathcal{P}_f(\mathbb{N}) \right\}.$$

Abbreviately for $\alpha \in \mathcal{P}_f(\mathbb{N})$, we write $x_\alpha = \sum_{n \in \alpha} x_n$.

The following theorem is a cornerstone theorem in Ramsey theory, often known as the Hindman theorem [11].

Theorem 2.2 (Hindman Theorem)**.** *If $(S, +)$ is a commutative semigroup, then for every finite coloring of S, there exists a monochromatic IP set.*

It can be shown that every member of the idempotents of $(\beta S, \cdot)$ contains an IP set, which means every idempotent witnesses Hindman's theorem. Using Zorn's lemma one can show that $(\beta S, \cdot)$ contains minimal left ideals (minimal w.r.t. the inclusion). A well-known fact is that the union of such minimal left ideals is a minimal two-sided ideal, denoted by $K(\beta S, \cdot)$. Here we recall a few well-known classes of sets that are relevant to our work.

Definition 2.3. Let $(S, \cdot)$ be a semigroup, let $n \in \mathbb{N}$ and let $A \subseteq S$. We say that

- A is a thick set if for any finite subset $F \subset S$, there exists an element $x \in S$ such that $Fx = \{fx : f \in F\} \subset A$;
- A is a syndetic set if there exists a finite set $F \subset S$ such that $S = \bigcup_{x \in F} x^{-1}A$, where $x^{-1}A = \{y : xy \in A\}$;
- A is piecewise syndetic if there exists a finite set $F \subset S$ such that $\bigcup_{x \in F} x^{-1}A$ is a thick set. It is well known that A is piecewise syndetic if and only if there exists $p \in K(\beta S, \cdot)$ such that $A \in p$.
- A is central if it belongs to a minimal idempotent in βS.

It can be proved that a set A is thick iff there exists a left ideal L such that $L \subseteq \overline{A}$. And a set A is syndetic iff for every left ideal L, $L \cap \overline{A} \neq \emptyset$. Note that if $f : (S, \cdot) \to (T, \cdot)$ be any map between discrete semigroups.Then $\tilde{f} : \beta S \to \beta T$ be the continuous extension of f defined by $\tilde{f}(p) = \tilde{f}(p - \lim x) = p - \lim f(x)$, where $p - \lim f(x) = y$ if and only if for every neighbourhood U of y, $\{x : f(x) \in U\} \in p$. For details about the theory of ultrafilters, we refer to the book [15].

2.2 A brief review over the symmetric operation

After lifting the multiplicative operation over the affine space via an isomorphism, in [7], Di Nasso introduced a new symmetric operation over $\mathbb{Z}$, that we will discuss soon. In this subsection, we describe this operation briefly. First, we need to recall some basic facts from [7].

Definition 2.4 (Elementary symmetric polynomial)**.** For $j = 1, 2, \ldots, n$, the elementary symmetric polynomial in n variables is the polynomial:

$$e_j (X_1, X_2, \ldots, X_n) = \sum_{1 \leq i_1 \leq \ldots \leq i_j \leq n} X_{i_1} X_{i_2} \cdots X_{i_j} = \sum_{\emptyset \neq G \subseteq \{1, \ldots, n\}} \prod_{s \in G} X_s.$$

For all $a_1, \ldots, a_n$, the product $\prod_{j=1}^{n} (a_j + 1) = \sum_{j=1}^{n} e_j (a_1, \ldots, a_n) + 1$, and so

$$c = \sum_{j=1}^{n} e_j (a_1, \ldots, a_n) \Longleftrightarrow \prod_{j=1}^{n} (a_j + 1) = (c + 1).$$

More generally, for $l, k \neq 0$ it can be easily verified that

$$\prod_{j=1}^{n} (la_j + k) = lc + k \Longleftrightarrow c = \sum_{j=1}^{n} l^{j-1} k^{n-j} e_j (a_1, \ldots, a_n) + \frac{k^n - k}{l}.$$

If $k, l \in \mathbb{N}$, then $c \in \mathbb{Z}$ if and only if $l | k (k - 1)$.

The function $\mathfrak{S}_{l,k} (\cdot)$ in the next definition is precisely the same as the value of c in Definition 2.4, which gives a justification for our attention to this function.

Definition 2.5 ((l, k)-symmetric polynomial)**.** For $l, k \in \mathbb{Z}$ with $l, k \neq 0$ the (l, k)-symmetric polynomial in n variables is

$$\begin{aligned} \mathfrak{S}_{l,k} (X_1, X_2, \ldots, X_n) &= \sum_{j=1}^{n} l^{j-1} k^{n-j} e_j (X_1, X_2, \ldots, X_n) + \frac{k^n - k}{l} \\ &= \sum_{\emptyset \neq G \subseteq \{1, \ldots, n\}} \left(l^{|G|-1} k^{n-|G|} \cdot \prod_{s \in G} X_s \right) + \frac{k^n - k}{l}. \end{aligned}$$

Definition 2.6. For $l, k \in \mathbb{Z}$ where $l \neq 0$ divides $k\,(k-1)$, define

$$a \circledast_{l,k} b = c \Longleftrightarrow (la+k)\,(lb+k) = (lc+k)\,.$$

So,

$$c = a \circledast_{l,k} b = \frac{1}{l}\left[(la+k)\,(lb+k) - k\right] = lab + k\,(a+b) + \frac{k^2-k}{l}.$$

Clearly, $c \in \mathbb{Z}$ if and only if l divides $k^2 - k = k\,(k-1)$.

It is easy to verify that $(\mathbb{Z}, \circledast_{l,k})$ is an abelian group when $l|k(k-1)$. In [7], Di Nasso analyzed the structure of this group to deduce new Ramsey theoretic patterns. In our work we analyze the algebraic structure of this group to deduce new patterns and new partition regular equations. Throughout our work, we will be concerned about the group $(\mathbb{Z}, \circledast_{1,t})$, where $t \in \mathbb{Z}$.

3 Interplay between additive, multiplicative, and symmetric large sets

Let $t \in \mathbb{Z}$, $i : \mathbb{Z} \to \mathbb{Z}$ be the identity map defined by $i(x) = x$ for all $x \in \mathbb{Z}$. Let $h_t : \mathbb{Z} \to \mathbb{Z}$ be the map defined by $h_t(x) = x + t$ for all $x \in \mathbb{Z}$. When t is fixed, we use h to denote h_t. Let $\tilde{i}$, and $\tilde{h}$ are the map defined from $\beta\mathbb{Z}$ to $\beta\mathbb{Z}$, are the continuous extensions of i, and h respectively. In this section, we study how much additive and multiplicative large sets behave like a symmetric large set. Basically we test the compatibility of the following diagram.

$$\begin{tikzcd}
(\beta\mathbb{Z}, \cdot) \arrow[rr, "\widetilde{h}"] \arrow[d, leftrightarrow, "\widetilde{i}"'] & & (\beta\mathbb{Z}, \odot_t) \arrow[d, leftrightarrow, "\widetilde{i}"] \\
(\beta\mathbb{Z}, +) \arrow[rr, "\widetilde{h}"'] & & (\beta\mathbb{Z}, \oplus_t)
\end{tikzcd}$$

For every $t \in \mathbb{Z}$, and $a, b \in \mathbb{Z}$, define $a \odot_t b = a \circledast_{1,-t} b$. Define $a \oplus_t b = a + b - t$. One can check that $(\mathbb{Z}, \oplus_t, \odot_t)$ forms a ring, and the map $h : (\mathbb{Z}, +, \cdot) \to (\mathbb{Z}, \oplus_t, \odot_t)$ defined by $h(x) = x + t$ for all $x \in \mathbb{Z}$ is a ring isomorphism. Surprisingly it shows that a translation of additive and multiplicative large sets are "large" (in some sense) in $(\mathbb{Z}, \oplus_t, \odot_t)$. As for $t = 0$, $x \odot_0 y = x \cdot y$, we can stress that: *the operation $\odot_t$ clearly generalizes the multiplicative operation.* As $\odot_t$ is also a symmetric operation, this leads to a natural question how many additive large sets are symmetrically large, and how many symmetrically large sets are additive large? This immediately generalizes a large studied theory on the interplay between "additive and multiplicative large sets"[1]. For each $t \in \mathbb{Z}$, the following theorem shows the interplay between the multiplicative and $\odot_t$ syndetic, thick, and piecewise syndetic sets.

Theorem 3.1. *If $t \in \mathbb{Z}$, then*

1. *$A \subseteq \mathbb{Z}$ multiplicative thick implies $A + t$ is thick in $(\mathbb{Z}, \odot_t)$, and*

2. *$A \subseteq \mathbb{Z}$ multiplicative syndetic implies $A + t$ is syndetic in $(\mathbb{Z}, \odot_t)$.*

3. *$A \subseteq \mathbb{Z}$ multiplicative piecewise syndetic/ Central implies $A + t$ is piecewise syndetic/ Central in $(\mathbb{Z}, \odot_t)$.*

Proof. Let $h : \mathbb{Z} \to \mathbb{Z}$ be defined by $h(x) = x + t$. Then h is an isomorphism between $(\mathbb{Z}, \cdot)$ and $(\mathbb{Z}, \odot_t)$. Now the extension of h over $\beta\mathbb{Z}$ is defined by $\tilde{h} : \beta\mathbb{Z} \to \beta\mathbb{Z}$ be defined by $\tilde{h}(p) = \tilde{h}(p - \lim x) = p - \lim h(x) = p - \lim x + t = p + t$. As h is a homomorphism, $\tilde{h}$ is also a homomorphism from $(\beta\mathbb{Z}, \cdot)$ and $(\beta\mathbb{Z}, \odot_t)$. Now for every $p \in \beta\mathbb{Z}$, $\tilde{h}(p - t) = p$, and hence $\tilde{h}$ is surjective. Again it is easy to verify that $\tilde{h}(p) = \tilde{h}(q)$ implies $p = q$, proving $\tilde{h}$ is injective. Hence $\tilde{h}$ is an isomorphism, and we have for every left ideal L in $(\beta\mathbb{Z}, \cdot)$, $\tilde{h}(L) = L + t$ is a left ideal of $(\beta\mathbb{Z}, \odot_t)$.

[1] for history on this topic authors are referred to the article [3]

1. Now if A is multiplicatively thick, then there exist a left ideal L such that $L \subset \overline{A}$. But then, $\tilde{h}(L) = L + t \subseteq \overline{A+t}$, which implies $A + t$ is thick in $(\mathbb{Z}, \odot_t)$.

2. Again if A is multiplicative syndetic, then for every left ideal L, we have $L \cap \overline{A} \neq \emptyset$. But $\tilde{h}$ is an isomorphism. Hence for every left ideal M for $(\beta\mathbb{Z}, \odot_t)$, $\tilde{h}^{-1}(M)$ is a left ideal of $(\beta\mathbb{Z}, \cdot)$. Hence, $\tilde{h}^{-1}(M) \cap \overline{A} \neq \emptyset$, which implies $M \cap (A + t) \neq \emptyset$. This shows that $A + t$ is syndetic in $(\mathbb{Z}, \odot_t)$.

The proof of 3 is similar, so we leave it to the reader. □

The following theorem says how additive large sets are preserved in the group $(\mathbb{Z}, \oplus_t)$.

Theorem 3.2. *For every $t \in \mathbb{Z}$, each piecewise syndetic set in $(\mathbb{Z}, \oplus_t)$ is piecewise syndetic in $(\mathbb{Z}, +)$.*

Proof. It will be sufficient to prove that $K(\beta\mathbb{Z}, \oplus_t) \subseteq K(\beta\mathbb{Z}, +)$. Let $h : \mathbb{Z} \to \mathbb{Z}$ defined by $h(x) = x + t$ for all $x \in \mathbb{Z}$. Then h is ring isomorphism between $(\mathbb{Z}, +, \cdot)$ into $(\mathbb{Z}, \oplus_t, \odot_t)$. Now

$$K(\beta\mathbb{Z}, \oplus_t) = \tilde{h}[K(\beta\mathbb{Z}, +)] = K(\beta\mathbb{Z}, +) + t \subseteq K(\beta\mathbb{Z}, +).$$

This completes the proof. □

In [15], authors proved that $K(\beta\mathbb{Z}, +) \cap K(\beta\mathbb{Z}, \cdot) = \emptyset$. The above Proposition immediately implies that for every $t \in \mathbb{Z}$, $K(\beta\mathbb{Z}, \oplus_t) \cap K(\beta\mathbb{Z}, \cdot) = \emptyset$. It was proved in [4], that $\mathrm{cl}((K(\beta\mathbb{Z}, +))) \cap K(\beta\mathbb{Z}, \cdot) \neq \emptyset$. The following theorem is the symmetric version of this result. In the following theorem $\mathbb{Z}^\star = \mathbb{Z} \setminus \{0\}$.

Theorem 3.3. *For every $t \in \mathbb{Z}$, $E_t = cl(K(\beta\mathbb{Z}, +)) \cap K((\beta\mathbb{Z}, \odot_t) \neq \emptyset$. In fact, E_t is a left ideal of $K(\beta\mathbb{Z}^\star, \odot_t)$.*

Proof. For $t = 0$, the result is immediate from [4]. So we may assume $t \neq 0$. We claim that $\mathrm{cl}(K(\beta\mathbb{Z}, +))$ is a left ideal of $(\beta\mathbb{Z}^\star, \odot_t)$. Then our proposition directly follows from our claim.

To prove our claim, let $p \in \beta\mathbb{Z}^\star$, and $q \in \mathrm{cl}(K(\beta\mathbb{Z}, +))$. Let $A \in p \odot_t q$. Hence $\{n \in \mathbb{Z} : n^{-1}A \in q\} \in p$. Choose $n \in \mathbb{Z}^\star$ such that $n^{-1}A \in q$, hence $B = n^{-1}A$ is an additive piecewise syndetic set. But then $A = n \odot_t B = \{t(t+1) - tn + nx - tx : x \in B\} = \{t(t+1) - tn + (n-t)x : x \in B\}$ is again an additive piecewise syndetic set. Hence $p \odot_t q \in \mathrm{cl}(K(\beta\mathbb{Z}, +))$.

□

In the next section, we use the following theorem to find a new class of equations that are partition regular.

Theorem 3.4. *For every $t \in \mathbb{Z}$,*

$$\tilde{h}[(cl(E(K(\beta\mathbb{Z}, +))) \cap E(K(\beta\mathbb{Z}, \cdot))] = cl(E(K(\beta\mathbb{Z}, \oplus_t))) \cap E(K(\beta\mathbb{Z}, \odot_t)).$$

Proof. Let $h : (\mathbb{Z}, +, \cdot) \to (\mathbb{Z}, \oplus_t, \odot_t)$ be the ring isomorphism defined by $h(x) = x + t$ for all $x \in \mathbb{Z}$. And let $\tilde{h} : \beta\mathbb{Z} \to \beta\mathbb{Z}$ be the continuous extension of h. As h is a ring homomorphism, $\tilde{h} : (\beta\mathbb{Z}, +) \to (\beta\mathbb{Z}, \oplus_t)$ and $\tilde{h} : (\beta\mathbb{Z}, \cdot) \to (\beta\mathbb{Z}, \odot_t)$ are semi-group homomophisms. In fact, $\tilde{h}(p) = p + t$ for all $p \in \beta\mathbb{Z}$ shows that $\tilde{h}$ is also bijective. Hence, $\tilde{h}$ is a semi-group isomorphism from $(\beta\mathbb{Z}, +)$ to $(\beta\mathbb{Z}, \oplus_t)$, and $(\beta\mathbb{Z}, \cdot)$ to $(\beta\mathbb{Z}, \odot_t)$. Now we claim that $\tilde{h}(cl((E(K(\beta\mathbb{Z}, +))) = cl((E(K(\beta\mathbb{Z}, \oplus_t)))$.

Proof of the Claim: Let $q \in \tilde{h}(cl((E(K(\beta\mathbb{Z}, +)))$. Hence $q = \tilde{h}(p)$ where $p \in cl((E(K(\beta\mathbb{Z}, +))$. Observe that $h(\mathbb{Z}) = \mathbb{Z}$ is central in $(\mathbb{Z}, \oplus_t)$. Hence from [3, Lemma 4.6.], [10, Theorem 2.6.], if $B \in \mathbb{Z}$ is central in $(\mathbb{Z}, +)$, then $h(B)$ is central in $(\mathbb{Z}, \oplus_t)$. Let us assume $A \in q$. It implies that there exists $B \in p$ such that $h(B) \subseteq A$. But $B \in p$ implies that B is central in $(\mathbb{Z}, +)$. Hence, A is central in $(\mathbb{Z}, \oplus_t)$. So $q \in cl(E(K(\beta\mathbb{Z}, \oplus_t)))$. To prove the converse, we assume $q \in cl(E(K(\beta\mathbb{Z}, \oplus_t)))$. We define $h_1 : (\mathbb{Z}, \oplus_t) \to (\mathbb{Z}, +)$ by $h_1(x) = x - t$ for all $x \in \mathbb{Z}$. Here h_1 is a semigroup isomorphism and its' extension $\tilde{h_1} : (\beta\mathbb{Z}, \oplus_t) \to (\beta\mathbb{Z}, +)$ defined by $\tilde{h_1}(p) = p - t$ is an isomorphism too. Clearly for every $p \in \beta\mathbb{Z}$, $\tilde{h_1}[\tilde{h}(p)] = \tilde{h}[\tilde{h_1}(p)] = p$. Again by similar reasoning, we have $q \in \tilde{h}(cl((E(K(\beta\mathbb{Z}, +)))$. This gives the reverse inclusion and proves our claim.

Now again $\tilde{h} : (\beta\mathbb{Z}, \cdot) \to (\beta\mathbb{Z}, \odot_t)$ is an isomorphism. Hence, we have $\tilde{h}[(K(\beta\mathbb{Z}, \cdot)] = K(\beta\mathbb{Z}, \odot_t)$, which implies $\tilde{h}[E((K(\beta\mathbb{Z}, \cdot))] = E(K(\beta\mathbb{Z}, \odot_t))$, as idempotents go to idempotents. Hence we have $\tilde{h}[(\overline{(E(K(\beta\mathbb{Z}, +))} \cap E(K(\beta\mathbb{Z}, \cdot))] = \overline{(E(K(\beta\mathbb{Z}, \oplus_t))} \cap E(K(\beta\mathbb{Z}, \odot_t))$.

□

From [3, Theorem 6.1], we know that every multiplicative syndetic set is additively central. However for general $(\mathbb{Z}, \odot_t)$, we have the following weaker result.

Theorem 3.5. *Let $t \in \mathbb{Z}$, and A be a syndetic set in $(\mathbb{Z}, \odot_t)$. Then A is additively piecewise syndetic.*

Proof. Let $\{t_1, t_2, \ldots, t_n\}$ be a finite set such that $\mathbb{Z} = \cup_{i=1}^{n} t_i^{-1} A$. Then at least one of them is additively central. For some $s \in \{t_1, t_2, \ldots, t_n\}$, let $C = s^{-1}A$ is additively central. Then

$$A = s \odot_t C = t(t+1) - ts + \{(s-t) \cdot y : y \in C\} = t(t+1) - ts + (s-t) \cdot C.$$

As C is central, $(s-t) \cdot C$ is central, and so A being a translation of a central set, is a piecewise syndetic set. $\square$

4 Combinatorial applications

In this section, we find several new monochromatic patterns. We prove for every $t \in \mathbb{Z}$, certain $\odot_t$ large sets contain arithmetic progressions of arbitrary length, and its polynomial extensions which were known for $t = 0$. Then we find new equations which are partition regular.

4.1 New monochromatic patterns

In this section, we study how much symmetric structures are preserved in additive and multiplicative large sets. We show in most of the cases they contain symmetric patterns up to an additive translation.

Theorem 4.1. *Let $t \in \mathbb{Z}$, and $A \in p$ for some $p \in E(\beta\mathbb{Z}, \cdot)$. Then there exists a sequence $\langle x_n \rangle_n$ such that $\sigma_t(\langle x_n \rangle_n) - t \subseteq A$.*

Proof. From the previous discussion, it is clear that $\tilde{h}(p) \in E(\beta\mathbb{Z}, \odot_t)$. Then if $A \in p$, $h(A) \in \tilde{h}(p)$, as h is an isomorphism. Hence there exists a sequence $\langle x_n \rangle_n$ such that $\sigma_t(\langle x_n \rangle_n) \subseteq A + t$. This completes the proof. $\square$

As arithmetic progressions are invariant under multiplication, using van der Waerden's theorem, one can easily deduce that every multiplicatively piecewise syndetic set contains A.P of arbitrary length. The following theorem says that symmetric large sets also contain A.P of arbitrary length.

Theorem 4.2. *For every $t \in \mathbb{Z}$, $\odot_t$-piecewise syndetic sets contain A.P of arbitrary length.*

Proof. Let A be a $\odot_t$ piecewise syndetic set. Hence there exists $p \in K(\beta\mathbb{Z}, \odot_t)$ such that $A \in p$. But then $A - t$ is a multiplicative piecewise syndetic set. Hence for every $l \in \mathbb{Z}$, there exists $a, d \in \mathbb{Z}$ such that $\{a, a+d, \ldots, a+ld\} \subset A - t$. Hence A contains AP of arbitrary length. This completes the proof. $\square$

Let $\mathcal{AP} = \{p \in \beta\mathbb{Z} : (\forall A \in p) A$ contains A.P. of arbitrary length$\}$. Clearly the above theorem says that $\cup_{t \in \mathbb{Z}} cl\left(K(\beta\mathbb{Z}, \odot_t)\right) \subseteq \mathcal{AP}$. But for the ultrafilters witnessing polynomial extensions of van der Waerden's theorem, things are different. Let $\mathbb{P}$ be the set of all polynomials with rational coefficients, and with zero constant term. Let $\mathcal{L} = \{p \in \beta\mathbb{Z} : (\forall A \in p)(\forall F \in \mathcal{P}_f(\mathbb{P})(\exists\, a, d \in \mathbb{Z})\ (\forall\, p \in F)\ a + p(d) \in A\}$. From [5], we know that the following set is non-empty.

One can show that $\mathcal{L}$ is a left ideal of $(\beta\mathbb{Z}^*, \cdot)$. The following Theorem shows that for every $t \in \mathbb{Z}$, $\mathcal{L}$ is a left-ideal in $(\beta\mathbb{Z} \setminus \mathbb{Z}, \odot_t)$.

Theorem 4.3. *For every $t \in \mathbb{Z}$, $\mathcal{L}$ is a left ideal of $(\beta\mathbb{Z} \setminus \mathbb{Z}, \odot_t)$.*

Proof. Let $p \in \beta\mathbb{Z} \setminus \mathbb{Z}$, $q \in \mathcal{L}$, and $A \in p \odot_t q$. Then $\{n : n^{-1}A \in q\} \subset p$. Now we choose any $n \in \mathbb{Z} \setminus \{t\}$ and let $F \in \mathcal{P}_f(\mathbb{P})$. Let $F' = \{\frac{1}{n-t} f : f \in F\}$. Then there exists $a, d \in \mathbb{Z}$ such that $\forall P \in F'$,

$$a + \frac{1}{n} P(d) \in n^{-1} A.$$

i.e., $n \odot_t (a + P(d)) = c + P(d) \in A$ for some $c \in \mathbb{Z}$. As $F \in \mathcal{P}_f(\mathbb{P})$ is arbitrary, so we have $p \odot_t q \in \mathcal{L}$. Hence, $\mathcal{L}$ is a left ideal. $\square$

Before we proceed to our next result, we need the following technical definition.

Definition 4.4. For every $t \in \mathbb{Z}$, define the followings:

1. For each $A_1, A_2, \dots, A_n \subset \mathbb{Z}$, define

$$\sigma_t(A_1, A_2, \dots, A_n) = \{a_1 \odot_t a_2 \odot_t \dots \odot_t a_n : a_i \in A_i \text{ for each } 1 \leq i \leq n\}.$$

2. For any sequence $\langle A_n \rangle_n$ of subsets of $\mathbb{Z}$, define

$$\sigma_t(\langle A_n \rangle_n) = \left\{ \sigma_t(A_{i_1}, A_{i_2}, \dots, A_{i_n}) : n \in \mathbb{N}, A_{i_j} \in \langle A_n \rangle_n \text{ for all } n \right\}.$$

Let $\mathcal{F}$ be a family of finite subsets of $\mathbb{Z}$. We call such a family, a Rado family if every additive or multiplicative piecewise syndetic subset of $\mathbb{Z}$ contains a member of $\mathbb{Z}$. For example, a collection of finite-length arithmetic progressions are Rado family. The following theorem mixes symmetric patterns with other monochromatic patterns.

Theorem 4.5. *Let $t \in \mathbb{Z}$ and $p \in cl(E(K(\beta\mathbb{Z}, \oplus_t))) \cap E(K(\beta\mathbb{Z}, \odot_t))$. Then for every $A \in p$, there exist a sequence $\langle F_n \rangle_n \subset \mathcal{F}$, and a sequence $\langle x_n \rangle_n$ such that*

1. $\langle F_n \rangle_n \subset A$,

2. $\sigma_t(\langle x_n \rangle_n) \cup \sigma_t(\langle F_n \cdot x_n \rangle_n) \cup \sigma_t(\langle F_n \rangle_n) \subset A$.

Proof. As $A \in p$ and $p \odot_t p = p$, we have $A^\star = \{x \in A : x^{-1}A \in p\} \in p$. From [15, Lemma 4.14], we know if $x \in A^\star$, then $x^{-1}A^\star \in p$. Let $F_1 \in \mathcal{F}$ such that $F_1 \subset A^\star$. Then $B = A^\star \bigcap_{f \in F_1} f^{-1}A^\star \in p$. Now choose $x_1 \in B$. Choose $F_1 \cdot x_1 \subset A^\star$. Suppose for some $N \in \mathbb{N}$, we have a sequence $\langle F_n \rangle_{n=1}^N \subset \mathcal{F}$, and $\langle x_n \rangle_{n=1}^N \subset A^\star$ such that $C = \sigma_t(\langle x_n \rangle_n) \cup \sigma_t(\langle F_n \cdot x_n \rangle_n) \cup \sigma_t(\langle F_n \rangle_n) \subset A^\star$. Let $D = A^\star \cap_{y \in C} y^{-1}A^\star \in p$. Then choose $F_{N+1} \in D^\star$, and then choose $x_{N+1} \in D^\star \cap F_{N+1}^{-1}D^\star$. This completes the induction. a sequence such that $\sigma_t(\langle x_n \rangle_n) \subset B$. This gives us the desired result. $\square$

The following corollary is an immediate application of the above theorem.

Corollary 4.6. *Let $p \in (cl(E(K(\beta\mathbb{Z}, +))) \cap E(K(\beta\mathbb{Z}, \cdot))$, and $A \in p$. Then there exist $F \in \mathcal{F}$, and a sequence $\langle x_n \rangle_n$ such that $F \cup \sigma_t(\langle x_n \rangle_n) \cup \{f \odot_t y; f \in F \text{ and } y \in \sigma_t(\langle x_n \rangle_n)\} \subset A + t$*

Proof. Let $A \in p$, and so $h(A) \in \tilde{h}(p)$ where $\tilde{h}(p) \in cl(E(K(\beta\mathbb{Z}, \oplus_t))) \cap E(K(\beta\mathbb{Z}, \odot_t))$. Now our corollary follows from Theorem 4.5. $\square$

In [4], authors proved for every finite partition of $\mathbb{Z}$, there exists two sequences $\langle x_n \rangle_n$, $\langle y_n \rangle_n$ such that $FS(\langle x_n \rangle_n) \cup FP(\langle x_n \rangle_n)$ is monochromatic. The following theorem is a symmetric variant, as well as an extension of this result.

Theorem 4.7. *For every finite partition of $\mathbb{Z}$, there exist $t \in \mathbb{Z}$, and three sequences $\langle x_n \rangle_n$, $\langle y_n \rangle_n$, $\langle z_n \rangle_n$, and $\langle w_n \rangle_n$, such that $FS(\langle x_n \rangle_n) \cup FP(\langle w_n \rangle_n) \cup (t + FS_{-t}(\langle y_n \rangle_n)) \cup (t + \sigma_{-t}(\langle z_n \rangle_n))$ is monochromatic, where $FS_{-t}(\langle y_n \rangle_n)$ denotes the finite sum over the semigroup $(\mathbb{Z}, \oplus_{-t})$. In fact such a collection of t is infinite.*

Proof. Let $p \in cl(E(K(\beta\mathbb{Z}, +))) \cap E(K(\beta\mathbb{Z}, \cdot))$, and $A \in p$. Choose $q \in E(K(\beta\mathbb{Z}, +))$ be such that $A \in q$. Then $A^\star = \{t \in A : A - t \in q\} \in q$. Let $t \in A^\star$. Now $(\mathbb{Z}, \oplus_{-t}, \odot_{-t})$ is a commutative ring and $h(x) = x - t$ is a ring isomorphism between $(\mathbb{Z}, +, \cdot)$ and $(\mathbb{Z}, \oplus_{-t}, \odot_{-t})$. Now from Theorem 3.4, $\tilde{h}(p) \in cl(E(K(\beta\mathbb{Z}, \oplus_t))) \cap E(K(\beta\mathbb{Z}, \odot_t))$, and $A - t \in \tilde{h}(p)$.

Now $A - t \in q$ implies there exists a sequence $\langle x_n \rangle_n$ such that $FS(\langle x_n \rangle_n) \subseteq A \cap A - t$. And $A - t \in \tilde{h}(p)$ implies there exists two sequences $\langle y_n \rangle_n$ and $\langle z_n \rangle_n$ such that $(t + FS_{-t}(\langle y_n \rangle_n)) \cup (t + \sigma_{-t}(\langle z_n \rangle_n)) \subseteq A - t$. Again from the choice of p, there exists a sequence $\langle w_n \rangle_n$ such that $FP(\langle w_n \rangle_n) \subseteq A$. $\square$

4.2 Generalization to the group $(\mathbb{Z}, \circledast_{l,k})$ for some $l, k \in \mathbb{Z}$

Till now we have discussed how additively large sets are preserved in $(\mathbb{Z}, \odot)$ for every $t \in \mathbb{Z}$. But we know if $l | k \cdot (k-1)$, then $(\mathbb{Z}, \circledast_{l,k})$ forms a group. As in the preceeding sections we were working with ring homomorphisms we restricted ourselves over $(\mathbb{Z}, \odot_t)$ only. In this section, we show that we can generalize some of the results discussed till now over the group $(\mathbb{Z}, \circledast_{l,k})$ for some l, k, satisfying $l | (k-1)$. That means, we can show that some collections of ultrafilters "reach in an additive sense" are large in the group $(\mathbb{Z}, \circledast_{l,k})$, whenever $l | (k-1)$.

Note that if A is syndetic/ piecewise syndetic in $(\mathbb{Z}, +)$, then $n \cdot A$ is also syndetic/ piecewise syndetic for all $n \in \mathbb{Z}^\star$. Again these largeness are preserved under additive translations. Hence for any $n \in \mathbb{Z}$, and if $l | (k-1)$, then $l \nmid k$ (where $l \neq 1$) we have $n \circledast_{l,k} A = (kn + \frac{k^2 - k}{l}) + (ln + k) \cdot A$ is a syndetic/ piecewise syndetic set if A is resp. Hence we have the following theorem.

Theorem 4.8. *If $l(\neq 1), k \in \mathbb{Z}$ such that $l \mid (k-1)$, then for every $n \in \mathbb{Z}$,*

1. *If A is piecewise syndetic/ syndetic in $(\mathbb{Z}, +)$, then $n \circledast_{l,k} A$ is piecewise syndetic/ syndetic in $(\mathbb{Z}, +)$. Hence, we have the following*
   (a) *$cl(K(\beta\mathbb{Z}, +))$ is a left ideal in $(\beta\mathbb{Z}, \circledast_{l,k})$, and*
   (b) *if A is an syndetic set in $(\mathbb{Z}, \circledast_{l,k})$, then A is syndetic in $(\mathbb{Z}, +)$.*
2. *Every $\circledast_{l,k}$-piecewise syndetic sets contain A.P of arbitrary length.*
3. *$\mathcal{L}$ is a left ideals in $(\beta\mathbb{Z}, \circledast_{l,k})$.*

Proof. 1. Proof is similar to the proof of Theorems 3.3, 3.5.

2. Let $p \in \mathcal{AP}$. Then it is easy to verify that for each $n \in \mathbb{Z}$, $n \circledast_{l,k} p \in \mathcal{AP}$. Hence taking closure we have $\beta\mathbb{Z} \circledast_{l,k} \mathcal{AP} \subseteq \mathcal{AP}$. Now choose $p \in \mathcal{AP}$, $q \in \beta\mathbb{Z}$, and $A \in p \circledast q$. Then again it is easy to verify that A contains $A.P.$ of arbitrary length. Hence $\mathcal{AP} \circledast_{l,k} \beta\mathbb{Z} \subseteq \mathcal{AP}$. Hence $\mathcal{AP}$ is a two sided ideal of $(\beta\mathbb{Z}, \circledast_{l,k})$. Hence $K(\beta\mathbb{Z}, \circledast_{l,k}) \subseteq \mathcal{AP}$. This completes the proof.

3. Proof is similar to the proof of Theorem 4.3.

□

4.3 New partition regular equations

In this section we give an affirmative answer to [11, Final remarks (4)]. To do so, we need the following theorem, which is an essential tool to formulate new partition regular equations from old equations.

Theorem 4.9. *[8, Lemma 2.1] Let $\mathcal{U} \in \beta\mathbb{N}$, and let $f(x_1, \ldots, x_n), g(y_1, \ldots, y_m)$ be two functions such that $\mathcal{U} \models (f_1(x_1, \ldots, x_n) = 0) \wedge (g(y_1, \ldots, y_m) = 0)$. Then $\mathcal{U} \models (f_1(x_1, \ldots, x_n) = 0) \wedge (g(y_1, \ldots, y_m) = 0) \wedge (x_1 = y_1)$.*

From [2, 13], we know that every member of $\text{cl}(E(K(\beta\mathbb{Z}, +))) \cap E(K(\beta\mathbb{Z}, \cdot))$ witness the solution of the equation $a + b = c \cdot d$. Later in [8], and [1], the authors extended this result, and proved partition regularity of several new classes of equations. In some upcoming theorems, we produce more new examples of partition regular equations. The following theorem is the first one.

Theorem 4.10. *For every $l, k \in \mathbb{Z}$ with $l|(k-1)$, the equation $\frac{a+b}{2} = c \circledast_{l,k} d$ is P.R. In particular, for $l = 1; k = -1$, $a + b = 2(cd + c + d)$ is P.R.*

Proof. From Theorem 4.8, let $p \in cl(K(\beta\mathbb{Z}, +)) \cap E(K(\beta\mathbb{Z}, \odot_t))$, and $A \in p$. Hence A is central in $(\mathbb{Z}, \circledast_{l,k})$, and it is piecewise syndetic in $(\mathbb{Z}, +)$. As every piecewise syndetic set contains arithmetic progressions of arbitrary length, A contains the solution of the equation $z = \frac{a+b}{2}$. Again A is central in $(\mathbb{Z}, \circledast_{l,k})$. Hence A contains solution of the equation $z = c \circledast_{l,k} d$. Now from the Theorem 4.9, A contains solution of the equation $\frac{a+b}{2} = c \circledast_{l,k} d$. Now expanding this equation we have the desired solution. □

The following theorem is a general version of the above theorem.

Theorem 4.11. *Let $l, k \in \mathbb{Z}$ with $l|(k-1)$, and $G(z_1, x_1, x_2, \ldots, x_n) = 0$, and let $H(z_2, y_1, y_2, \ldots, y_m) = 0$ be two equations such that every additively piecewise syndicate set contains a solution of G, and every $\circledast_{l,k}$ central set contains solution of H. Then $G(z, x_1, x_2, \ldots, x_n) = H(z, y_1, y_2, \ldots, y_m)$ is P.R.*

Proof. From Theorem 3.3, pick $p \in cl(K(\beta\mathbb{Z}, +)) \cap E(K(\beta\mathbb{Z}, \circledast_{l,k}))$. Then each member of p is additively piecewise syndetic, and so p witness the P.R. of $G(z_1, x_1, x_2, \ldots, x_n) = 0$. Again $p \in E(K(\beta\mathbb{Z}, \circledast_{l,k}))$, hence p witness the P.R of $H(z_2, y_1, y_2, \ldots, y_m) = 0$. Hence from the Theorem 4.9, we have $G(z, x_1, x_2, \ldots, x_n) = H(z, y_1, y_2, \ldots, y_m)$ is P.R. □

The following corollary is the most general and covers many new classes of partition regular equations.

Corollary 4.12. *Let $l, k \in \mathbb{Z}$ with $l|(k-1)$, and P be a polynomial with no constant term. Then the equation $x + P(y - x) = z \circledast_{l,k} w$, where $x \neq y$ is P.R.*

In particular for $l = 1; k = -1$, $x + P(y - x) = z + w + zw\,(x \neq y)$ is P.R.

Proof. Every additively piecewise syndetic set contains patterns of the form $\{a, a+d, a+P(d)\}$. Then it witnesses the solution of the equation $z = x + P(y-x)$, where $(x \neq y)$. Hence from Theorem 4.11 we have $x + P(y-x) = z \circledast_{l,k} w$ is P.R. □

Example 4.13. Letting $n \in \mathbb{N}$, and $P(z) = z^n$. Then $x + (y-x)^n = z + w + zw$ is P.R.

Corollary 4.14. *Let $m, n \in \mathbb{N}$. Let $P_1, P_2, \ldots, P_m$ be polynomials such that for every $i \in \{1, 2, \ldots, m\}$ $P_i(0) = 0$. Then the following system of equation is P.R.*

$$\begin{aligned} x_1 - P_1(y-x) &= y_1 \circledast_{l,k} y_2 \circledast_{l,k} \cdots \circledast_{l,k} y_n \\ x_2 - P_2(y-x) &= y_1 \circledast_{l,k} y_2 \circledast_{l,k} \cdots \circledast_{l,k} y_n \\ &\vdots \\ x_m - P_m(y-x) &= y_1 \circledast_{l,k} y_2 \circledast_{l,k} \cdots \circledast_{l,k} y_n \end{aligned}$$

Proof. Every additively piecewise syndetic set contains witnesses the solution of the following system of equations.

$$\begin{aligned} x_1 - P_1(y-x) &= z_1 \\ x_2 - P_2(y-x) &= z_1 \\ &\vdots \\ x_m - P_m(y-x) &= z_1 \end{aligned}$$

Again every central set contains the solutions of the equation $z_2 = y_1 \circledast_{l,k} y_2 \circledast_{l,k} \cdots \circledast_{l,k} y_n$. Hence from Theorem 4.11, our desired conclusion follows. □

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