Efficient Fault-Tolerant Single Qubit Gate Approximation And Universal Quantum Computation Without Using The Solovay-Kitaev Theorem

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Arbitrarily accurate fault-tolerant (FT) universal quantum computation can be carried out using the Clifford gates Z, S, CNOT plus the non-Clifford T gate. Moreover, a recent improvement of the Solovay-Kitaev theorem by Kuperberg implies that to approximate any single-qubit gate to an accuracy of $\epsilon > 0$ requires $O(\log^c[1/\epsilon])$ quantum gates with c > 1.44042. Can one do better? That was the question asked by Nielsen and Chuang in their quantum computation textbook. Specifically, they posted a challenge to efficiently approximate single-qubit gate, fault-tolerantly or otherwise, using $O(\log[1/\epsilon])$ gates chosen from a finite set. Here I give a partial answer to this question by showing that this is possible using $O(\log[1/\epsilon]\log\log[1/\epsilon]\log\log\log[1/\epsilon]\cdots)$ FT gates chosen from a finite set depending on the value of ϵ . The key idea is to construct an approximation of any phase gate in a FT way by recursion to any given accuracy $\epsilon > 0$. This method is straightforward to implement, easy to understand, and interestingly does not involve the Solovay-Kitaev theorem.

Introduction — While Clifford gates alone cannot perform universal quantum computation, adding one more non-Clifford gate, namely, the T gate, to the list can. (Definitions of the gates mentioned here will be given later.) For instance, using single-qubit Z, S and T gates together with two-qubit CNOT gate are sufficient to approximate any unitary operator in U(2) to arbitrary precision as well as performing universal computation [1–4]. As the operator $e^{i\alpha}$ can be executed fault-tolerantly, I will restrict most of the discussions below to SU(2). This approach of approximating any (special) unitary operator to arbitrary precision using a fixed finite set of quantum gates is analogous to hardware gates and machine language software in classical computers — all classical digital computer hardware and programs are composed of a finite set of classical gates and low-level machine codes that is powerful enough to perform universal classical computation.

There is an important difference between the classical and quantum situations. As SU(2) is a continuous group, any unitary operator generated by a finite set of quantum gates is at most dense in rather than equals SU(2). Thanks to the Solovay-Kitaev theorem [1, 2], one may approximate any SU(2) operator to arbitrary precision by composition of a finite set of quantum gates obeying certain technical conditions. Specifically, let \mathfrak{S} be a subset of SU(2) such that the set generated by \mathfrak{S} is dense in SU(2) and that $G \in \mathfrak{S}$ implies $G^{-1} \in \mathfrak{S}$. Then, for any $\epsilon > 0$ and $U \in SU(2)$, there exists a $S \in SU(2)$ composing of a finite product of gates taken from \mathfrak{S} such that the operator norm $||U - S|| < \epsilon$ [1-5]. Since CNOT plus the set of all SU(2) gates can simulate any special unitary gate [6], it means that all special unitary operators can be approximated to arbitrary accuracy. (Throughout this paper, the terms accuracy or precision between two unitary operators U and S always refer to the operator norm ||U - S||. Besides, the symbol ϵ is implicitly assumed to be in the interval (0,1].)

The Solovay-Kitaev theorem further tells us that the approximation can be done very efficiently. The version of Solovay-Kitaev theorem reported in Ref. [2] showed that for any fixed $\epsilon, \delta > 0$, there is an algorithm returning S that makes up of an instruction sequence of length $O(\log^{3+\delta}[1/\epsilon])$ such that $||U-S|| < \epsilon$. The run time of this algorithm is $O(\log^{3+\delta}[1/\epsilon])$ and the approximation quantum circuit does not involve ancilla. Furthermore, if ancillas are allowed, both the instruction sequence length and the run time scale like $O(\log^2[1/\epsilon]\log\log[1/\epsilon])$. Recently, Kuperberg generalized the Solovay-Kitaev theorem in a several directions. In particular, he proved that the instruction sequence length can be reduced to $O(\log^c[1/\epsilon])$ with $c = \log_{(1+\sqrt{5})/2} 2 + \delta \approx 1.44042 + \delta$ without using ancilla [7].

How to execute gates in the set \mathfrak{S} in a fault-tolerant (FT) manner? Although this is an open question for a general set \mathfrak{S} , FT implementations of Z, H, X, S, T and CNOT are known using the so-called Steane-like code [3, 4]. Consider a classical linear self-orthogonal binary code \mathcal{C} , namely, the one whose codeword space is a subset of its dual. Then, the CSS code constructed from \mathcal{C} , namely,

$$|\ell\rangle_{\rm L} = \frac{1}{|\mathcal{C}|^{1/2}} \sum_{a \in \mathcal{C}} |\ell + a\rangle$$
 (1)

for all ℓ in the coset $\mathcal{C}^{\perp}/\mathcal{C}$ [3, 4, 8–10] is called Steane-like because the Steane's seven qubit code [10] is perhaps the most famous example of this type of code. (In what follows, a state ket with the subscript L denotes a logical state of a quantum code.) Using Steane-like code, the logical Z, H, X, S and CNOT are automatically FT. Besides, T gate has a FT implementation using gate teleportation [3, 4]. Here I remark that Harrow *et al.* gave a non-constructive proof showing that the instruction sequence length can be lowered to $O(\log[1/\epsilon])$ using a certain instruction set making up of three elements [11]. Nonetheless, no FT execution of the gates used in their construction is known to date.

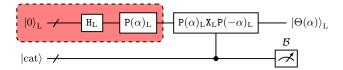


FIG. 1. FT preparation of the state $|\Theta(\alpha)\rangle_{L}$ defined in the text. Here $|\text{cat}\rangle$ is the cat state $(|0\rangle^{\otimes n} + |1\rangle^{\otimes n})/\sqrt{2}$, where n is the codeword length. The circuit in the red zone creates $|\Theta(\alpha)\rangle_{L}$ and the circuit outside the red zone performs the stabilizer measurement $P(\alpha)_{L}X_{L}P(-\alpha)_{L}$. The state is kept only if the measurement result corresponds to the +1 eigenvalue of the stabilizer. Here the measurement basis $\mathcal{B} = \left\{\sum_{\ell=0}^{1} (-1)^{a\ell} |\ell,\ell+b_1,\cdots,\ell+b_n\rangle/\sqrt{2}\right\}$ where $a,b_1,\cdots,b_n\in\{0,1\}$.

The length of instruction sequence required to approximate a general SU(2) operator to a precision of ϵ using a fixed finite set of generators is at least $\Omega(\log[1/\epsilon])$ [3]. Can this lower bound be attained using a computationally efficient procedure? This was the challenge posted by Nielsen and Chuang in their book [3]. Here I give a partial answer to this question by reporting an efficient way to approximate any SU(2) operator to an accuracy of ϵ using $O(\log[1/\epsilon]\log\log[1/\epsilon]\log\log[1/\epsilon]\log\log[1/\epsilon]\cdots)$ gates with the product terminated when $\log\log\cdots\log(1/\epsilon)$ becomes O(1). (In what follows, all products in the form $\log(1/\epsilon)\log\log(1/\epsilon)\cdots$ are implicitly assumed to be terminated when $\log\log\cdots\log(1/\epsilon)$ becomes O(1).) More importantly, this approximation can be performed in a FT manner.

Method — I begin by denoting the computational basis of a two-dimensional Hilbert space by $\{|0\rangle\,,|1\rangle\}.$ Recall that the set of all single qubit gates plus CNOT gate is universal, where CNOT $|a,b\rangle=|a,a+b\rangle$ for a,b=0,1. Besides, up to a global phase, any single qubit gate can be decomposed as two rotations along z-axis and one rotation along x-axis, or equivalently generated using two Hadamard gates H and three phase gates $P(\alpha)$ where $H\,|a\rangle=\sum_{b=0}^1(-1)^{ab}\,|b\rangle\,/\sqrt{2}$ for a=0,1 and [6,12]

$$P(\alpha) = |0\rangle\langle 0| + \exp(i\alpha) |1\rangle\langle 1|. \tag{2}$$

So it remains to show how to approximate any $P(\alpha)$ in a FT way through a computationally efficient algorithm. Observe that $\|P(\alpha) - P(\alpha + \Delta \alpha)\| = [(1 - \cos \Delta \alpha)^2 + \sin^2 \Delta \alpha]^{1/2} \le |\Delta \alpha|$. Thus, to approximate the $P(\alpha)$ gate in a FT way to an accuracy of ϵ , it suffices to find a FT implementation of the gate $P(\alpha')$ with $|\alpha - \alpha'| < \epsilon$. (Here I mention on passing that the Z, S and T gates are nothing but the special cases of the phase gate with $\alpha = \pi$, $\pi/2$ and $\pi/4$, respectively.)

Pick a Steane-like code and by circuit tracing, it is clear that Fig. 1 prepares the logical state

$$\left|\Theta(\alpha)\right\rangle_{\rm L} \equiv (\left|0\right\rangle_{\rm L} + e^{i\alpha} \left|1\right\rangle_{\rm L})/\sqrt{2} \tag{3}$$

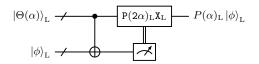


FIG. 2. FT execution of the logical $P(\alpha)$ gate through gate teleportation provided that there is a FT implementation of $P(2\alpha)$. Here the measurement is along the logical computational basis.

in a FT way using a procedure introduced in Ref. [3]. (See also Ref. [13] for a more effective alternative procedure discovered recently.) Specifically, one prepares copies of high fidelity $|\text{cat}\rangle = (|0\rangle^{\otimes n} + |1\rangle^{\otimes n})/\sqrt{2}$ with n being the length of the Steane-like code using noisy state preparation and stabilizer measurement apparatus. Then these high fidelity |cat| states are used to check if the states prepared by the circuit in the red region of Fig. 1 are error-free by measuring their stabilizer $P(\alpha)_L X_L P(-\alpha)_L$ using noisy apparatus. This checking is carried out a few times and only those states that pass the test more often than fail the test are kept [3]. By using these high fidelity $|\Theta(\alpha)\rangle_{L}$ states as inputs, it is straightforward to verify the validity of the gate teleportation circuit in Fig. 2. More importantly, this gate teleportation circuit is FT if $P(2\alpha)_T$ is FT as well. (Actually, for $\alpha = \pi/4$, this circuit is the one used to execute the T gate in a FT manner as there is a FT implementation of $S_L = P(\pi/2)_L [3, 4]$.)

Now I report how to execute $P(2\alpha)_L$. Let G and H be the generator and parity check matrices a classical self-orthogonal [n,k,d] code \mathcal{C} , respectively. Without lost of generality, I write H as $\begin{bmatrix} L \\ G \end{bmatrix}$ where L corresponds to the logical operation of the resultant Steane-like quantum code. Surely, H is of full row rank over \mathbb{F}_2 , the finite field of two elements; and hence it must also be of full row rank over \mathbb{R} . Therefore, for any given $w \in \mathbb{R}^{n-k}$, the equation Hv = w has solution in \mathbb{R}^n though it is not unique for k > 0. In particular, by setting the jth component of w, namely, w_j to $\delta_{j\ell}$ for a fixed $\ell = 1, 2, \cdots, n-2k$, then any one of the solutions v corresponds to the logical operation $P(\alpha)_L$ acting on the ℓ th logical qubit via

$$P(\alpha)_{L} = \bigotimes_{j=1}^{n} P(\alpha v_{j}). \tag{4}$$

Clearly, $P(\alpha)_L$ can be computed in $O(n^3)$ time by Gaussian elimination. To simplify matter, one could amend k more rows to H that are orthogonal to the ℓ th row in L to make the resultant $n \times n$ matrix invertible (both in \mathbb{F}_2 and \mathbb{R}). Then the solution v is unique. Applying this method to the Steane's seven qubit code [10]

$$|\ell\rangle_{\mathcal{L}} = \frac{1}{2\sqrt{2}} \sum_{a,b,c=0}^{1} |\ell + a + b + c, \ell + a + b, \ell + a + c\rangle$$

$$\otimes |\ell + a, \ell + b + c, \ell + b, \ell + c\rangle \tag{5}$$

for $\ell = 0, 1$ as an example, I obtain the following logical implementation of the $P(\alpha)$ gate

$$P(\alpha)_{L} = P(-\alpha) \otimes P(\alpha) \otimes \mathbb{1}^{\otimes 4} \otimes P(\alpha). \tag{6}$$

As the $S = P(\pi/2)$ has a FT implementation [3, 4, 13] and the $P(\alpha)_L$ gate execution in Eq. (4) is transversal, by recursively applying the quantum circuits in Figs. 1 and 2, I can implement any $P(\pi\ell/2^m)$ gate in a FT manner with $m, \ell \in \mathcal{Z}^+$ and $\ell < 2^m$ in O(n) times through the recursion $P(\pi\ell/2^m) \Leftarrow P(\pi\ell/2^{m-1}) \Leftarrow P(\pi\ell/2^{m-2}) \Leftarrow \cdots \Leftarrow P(\pi\ell/4)$ that doubles the rotation angle modulo 2π in each iteration. Actually, the idea of recursively generating gates starting from the Clifford gates had been pointed out by Gottesman in Ref. [4]. Nonetheless, he did not develop this idea further.

Conclusions — In summary, to approximate a phase gate $P(\alpha)$ (with $0 < \alpha < 2\pi$) to a precision of at least ϵ , one simply finds non-negative integers m, ℓ such that $|\alpha - \pi \ell/2^m| < \delta$ with $\delta \le \epsilon/\lceil \log_2(1/\epsilon) \rceil$ and apply the above procedure to execute $P(\pi \ell/2^{m})$ in a FT way. In other words, to approximate any SU(2) operator to $\epsilon > 0$ in operator norm, one decomposes the operator as three phase gates plus two H gates and apply FT implementation to each phase gate with angle of rotation accurate to $\epsilon \log_2(1/\epsilon) \log_2 \log_2(1/\epsilon) \cdots$ until $\log_2 \log_2 \cdots \log_2(1/\epsilon)$ becomes O(1). In this way, one obtains a computational efficient method to approximate any single qubit gate to an accuracy of ϵ using $O(\epsilon \log[1/\epsilon] \log \log[1/\epsilon] \cdots)$ gates. In essence, this method is built upon approximating an arbitrary phase gate (or alternatively arbitrary rotation angle) to any given precision in a FT manner rather than the Solovay-Kitaev theorem for specific kind of set of operators in SU(2). For any given precision ϵ , the gates needed are drawn from the set $\{H, P(\pi \ell/2^{m}): m, \ell \in \}$ $\mathbb{Z}^+, \ell < 2^m$ where $m = \lceil \epsilon / \lceil \log_2(1/\epsilon) \log_2 \log_2(1/\epsilon) \cdots \rceil \rceil$. Although it appears that a huge gate set, whose size increases as ϵ decreases, is needed, this collection is very natural in all major noisy intermediate-scale quantum computers to date. Note that superconducting transmon qubits [14], trapped ions [15] and neutral atoms [16] implementation of quantum computers to date, single qubit gates are executed by control pulses in which the angle of rotation on the Bloch sphere is proportional to the integral of effective Hamiltonian over time. For these quantum computers, one can apply $P(\alpha)$ on each qubit and hence $P(\alpha)_{L}$ on the logical qubit relatively easily. Whereas for photonic implementation [3, 17–19], $P(\alpha)$ and hence $P(\alpha)_L$ gates can be easily implemented via phase, polarization or amplitude modulators depending on the qubit representation. Finally, I remark that the construction reported here can be extended to FT implementation of qudits although its effectiveness decreases exponentially with the Hilbert space dimension of the qudit.

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