

# CHARACTERISTIC IDEAL OF THE FINE SELMER GROUP AND RESULTS ON $\mu$ -INVARIANCE UNDER ISOGENY IN THE FUNCTION FIELD CASE.

SOHAN GHOSH AND JISHNU RAY

**ABSTRACT.** Consider a function field  $K$  with characteristic  $p > 0$ . We investigate the  $\Lambda$ -module structure of the Mordell-Weil group of an abelian variety over  $\mathbb{Z}_p$ -extensions of  $K$ , generalizing results due to Lee. Next, we study the algebraic structure and prove a control theorem for the  $S$ -fine Mordell-Weil groups, the function field analogue for Wuthrich's fine Mordell-Weil groups, over a  $\mathbb{Z}_p$ -extension of  $K$ . In case of the unramified  $\mathbb{Z}_p$ -extension,  $K_\infty$ , we compute the characteristic ideal of the Pontryagin dual of the  $S$ -fine Mordell-Weil group. This provides an answer to an analogue of Greenberg's question for the characteristic ideal of the dual fine Selmer group in the function field setup. In the  $\ell \neq p$  case, we prove the triviality of the  $\mu$ -invariant for the Selmer group (same as the fine Selmer group in this case) of an elliptic curve over a non-commutative  $GL_2(\mathbb{Z}_\ell)$ -extension of  $K$  and thus extending Conjecture A. In the  $\ell = p$  case, we compute the change of  $\mu$ -invariants of the dual Selmer groups of elliptic curves under isogeny, giving a lower bound for the  $\mu$ -invariant.

## 1. INTRODUCTION

This article is divided into two parts, namely Part I and Part II dealing with different, yet connected, topics.

**1.1. Part I: Function Field ( $\ell = p$  case).** Let  $K = \mathbb{F}(t)$  be the function field over the finite field  $\mathbb{F}$  of characteristic  $p$ , of cardinality  $p^r$  for some  $r > 0$  and let  $K_\infty$  be the arithmetic  $\mathbb{Z}_p$ -extension of  $K$  (defined in §2.2). Let  $\Lambda$  be the Iwasawa algebra of  $K_\infty$  over  $K$ . In this case, the Iwasawa main conjecture over  $K_\infty$  for semistable abelian varieties  $A$  defined over  $K$ , have been settled by works of [LTT16].

Let  $S$  be a finite set of primes of  $K$  that contains the set of all primes of bad reduction of  $A/K$ . The  $S$ -fine Selmer group  $R^S(E/K_\infty)$  (for definition

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see §2.1) is a subgroup of the classical Selmer group, which is always co-torsion as a  $\Lambda$ -module (cf. [OT09, Theorem 1.7]).

Therefore, a natural question is to find the characteristic ideal of  $R^S(E/K_\infty)$ . The first part of this article addresses this question.

The strategy of the proof is to define the  $S$ -fine Mordell-Weil group,  $\mathcal{M}^S(E/L)$ , for an algebraic extension  $L$  of  $K$ , following [Wut07] (cf. §2.3). The fine Selmer groups fits into the following short exact sequence:

$$0 \longrightarrow \mathcal{M}^S(A/L) \longrightarrow R^S(A/L) \longrightarrow \mathfrak{H}^S(A/L) \longrightarrow 0,$$

where  $\mathfrak{H}^S(A/L)$  denotes the  $S$ -fine Tate-Shafarevich group (defined in §2.3). Note that the  $\mathfrak{H}^S(A/L)$  is the subgroup of the  $\text{III}(A/L)[p^\infty]$ , the  $p$ -primary part of the Tate-Shafarevich group (see Remark 2.7).

For each  $n \geq 1$ , let  $\Phi_n = \frac{(1+T)^{p^n} - 1}{(1+T)^{p^{n-1}} - 1} \in \Lambda$  be the  $p^n$ -th cyclotomic polynomial in  $1+T$ . Let  $K_n$  be the subextension of  $K_\infty$  such that  $[K_n : K] = p^n$ . One of the main result in the first part of our article is the following.

**Theorem 1.1.** *(Theorem 4.9) Let  $L$  be a finite extension of  $K$ . Let  $L_\infty/L$  be the arithmetic  $\mathbb{Z}_p$ -extension. Assume that  $E/L$  has split multiplicative reduction or good reduction at all primes of  $L$  and  $\text{III}(E/L_n)[p^\infty]$  is finite for all  $n \geq 0$ .*

*Also, suppose that  $E$  has split multiplicative reduction at all primes of  $S$  and that  $\mathfrak{H}^S(E/L_\infty)$  is finite. Then,*

$$\text{Char}_\Lambda(R^S(E/L_\infty)^\vee) = \left( \prod_{e_n \geq 1, n \geq 0} \Phi_n^{e_n} \right)$$

$$\text{where } e_n = \frac{\text{rank } E(L_n) - \text{rank } E(L_{n-1})}{\varphi(p^n)}.$$

This theorem has the following applications.

First, under the hypothesis that  $\mathfrak{H}^S(E/K_\infty)$  is finite one can see easily that the  $\mu$ -invariant of the  $S$ -fine Selmer group is trivial confirming the validity of Conjecture A in the function field case (cf. [GJS22, Theorem 3.7]).

Secondly, one can deduce an algebraic functional equation of  $R^S(E/K_\infty)$ . More precisely, one obtains the following result.

**Corollary 1.2.** *Assume that the hypotheses in Theorem 1.1 hold. Then the characteristic ideals of  $R^S(E/K_\infty)^\vee$  and  $R^S(E/K_\infty)^{\vee, \iota}$  as  $\Lambda$ -modules are equal, i.e. there is a pseudo-isomorphism  $R^S(E/K_\infty)^\vee \sim R^S(E/K_\infty)^{\vee, \iota}$ .*

Here  $\iota$  is the involution on  $\Lambda$  sending a group-like element of  $\Gamma = \text{Gal}(K_\infty/K)$  to its inverse. For any  $\Lambda$ -module  $M$ , we write  $M^\iota$  for the  $\Lambda$ -module which coincides with  $M$  as a  $\mathbb{Z}_p$ -module, with the action of  $\Gamma$  given by

$$\gamma \cdot_\iota x = \gamma^{-1}x \text{ for } \gamma \in \Gamma \text{ and } x \in M.$$

**Remark 1.3.** *We don't know how to prove the result in Corollary 1.2 without the assumption that  $\mathfrak{H}^S(E/K_\infty)$  is finite. Even in the number field case over the cyclotomic  $\mathbb{Z}_p$ -extension of  $\mathbb{Q}$ , such a result is not known without this additional hypothesis that  $\mathfrak{H}(E/\mathbb{Q}_{\text{cyc}})$  is finite (cf. [Lei23, discussion after Theorem C]). Infact, it is conjectured by Wuthrich that  $\mathfrak{H}(E/\mathbb{Q}_{\text{cyc}})$  should be finite [Wut07, Question 8.3 and Conjecture 8.4]. For classical  $p^\infty$ -Selmer groups over  $\mathbb{Q}_\infty$ , algebraic functional equations are known but the case of fine Selmer group is considered to be much more difficult and open as of now; partial results are proven in [HKL23, Theorem C].*

**Remark 1.4.** *Let  $K_n$  be the finite subextension of  $K_\infty$  such that  $[K_n : K] = p^n$ . Then we know that  $e_n = \mu \cdot p^n + \lambda \cdot n + O(1)$  where  $|\mathfrak{H}^S(A/K_n)| = p^{e_n}$ . Because of Conjecture A (which is true without the finiteness assumption of  $\mathfrak{H}^S(A/K_\infty)$ ), we know that  $\mu=0$ . Additionally, if know that the growth of the Tate-Shafarevich group  $\text{III}(A/K_n)[p^\infty]$  is of the kind  $p^{\mu_1 \cdot p^n + c}$  (where  $c$  is a constant independent of  $n$ ), then we can conclude that  $\lambda = 0$  and hence the growth of  $\mathfrak{H}^S(A/K_n)$  stays bounded. In this case  $\mathfrak{H}^S(E/K_\infty)$  is finite.*

*An example arising via this technique in the number field case was given in [Wut07, page 11].*

**1.2. Part II: Function Field ( $\ell \neq p$  case).** Let  $\ell$  be a rational prime, with  $\ell \neq p$ . Consider  $K = \mathbb{F}(t)$ , where  $\mathbb{F}$  is a finite field of char  $p$ . The  $\ell^\infty$ -Selmer group (defined in 2.8) of an elliptic curve  $E$  coincides with the  $\ell^\infty$ -fine Selmer group (defined in 2.8). The main result here is to prove an analogue of Conjecture A (i.e. the vanishing of the  $\mu$ -invariant of the fine Selmer) over the non-commutative  $\ell$ -adic trivializing extension  $K_\infty$  of  $K$ . The Galois group of  $K_\infty$  over  $K$  is an open subgroup of  $GL_2(\mathbb{Z}_\ell)$  (cf. [Pal14, §4]). In particular we show the following result.

**Theorem 1.5.** *(Theorem 6.3) Let  $K_{\text{cyc}}$  be the unique  $\mathbb{Z}_\ell$ -extension of  $K$  and  $K_\infty$  be the trivialising extension of  $K$ , such that  $G = \text{Gal}(K_\infty/K)$  is pro- $p$ . Assume  $E$  to be a non-isotrivial elliptic curve over the function field  $K$ . Then,*

$$\mu_G(S(E/K_\infty)^\vee) = \mu_\Gamma(S(E/K_{\text{cyc}})^\vee) = 0.$$

**1.3. Isogeny and  $\mu$ -invariants ( $\ell = p$  case).** We again return to the setting of §1.1. Unlike the  $\ell \neq p$  case, here the Selmer and fine Selmer groups are distinct. Our goal in this section is to give a lower bound of  $\mu$ -invariant of

the Selmer group over the unramified  $\mathbb{Z}_p$ -extension  $K_\infty$  over  $K$ . The strategy that we adopt is to compare the  $\mu$ -invariants of isogenous elliptic curves. More precisely, we show the following result.

**Theorem 1.6.** *(Theorem 5.1) Let  $E_1$  and  $E_2$  be two non-isotrivial elliptic curves over the function field  $K$  and let  $\varphi : E_1 \rightarrow E_2$  be an isogeny of degree  $p^r$  for some  $r > 0$ . Also assume that  $H_{fl}^2(K_\infty, E_i[p^\infty]) = 0$ , for  $i = 1, 2$ .*

*Then,*

$$\mu(S(E_2/K_\infty)^\vee) - \mu(S(E_1/K_\infty)^\vee) = \text{ord}_p(\chi_{fl}(\text{Spec}(K), A)),$$

where  $A = \ker(\varphi)$  and  $\chi_{fl}(\text{Spec}(K), A)$  is the Euler characteristic for flat cohomology defined in §2.7.

It follows that if  $\text{ord}_p(\chi_{fl}(\text{Spec}(K), A))$  is positive, one can obtain examples of elliptic curves with positive  $\mu$ -invariant over  $K_\infty$ .

**Remark 1.7.** (i) In [LLS<sup>+</sup>21], the authors calculate the change in  $\mu$ -invariant with isogeny for Selmer groups of elliptic curves having semi-stable reduction everywhere, for the unramified  $\mathbb{Z}_p$ -extension  $K_\infty/K$ .

In Theorem 5.1 we do not make any assumption about the nature of reduction of the elliptic curve and use a different method to calculate the change in  $\mu$ -invariant with  $p^r$ -isogeny of elliptic curves, with the additional assumption that  $H_{fl}^2(K_\infty, E_i[p^\infty]) = 0$ . Infact our technique is much more general and can also be adapted in the  $\ell \neq p$  case both for the unramified cyclotomic  $\mathbb{Z}_\ell$ -extension and for a non-commutative  $GL_2(\mathbb{Z}_\ell)$  extension (see Appendix A). However we don't get any new result in the  $\ell \neq p$  case because of the stronger result that we showed earlier concerning the triviality of the  $\mu$ -invariant (see Theorem 1.5).

(ii) The assumption  $H_{fl}^2(K_\infty, E_i[p^\infty]) = 0$ , made in Theorem 5.1, is an analogue of the Weak-Leopoldt Conjecture for number fields. This assumption is similarly adopted in various other works (see for example [BV18, Remark 2.1]). We also rely on this assumption in our article; however, we are unable to provide a proof for it.

In the  $\ell = p$  case, the Selmer group and the  $S$ -fine Selmer group over the unramified  $\mathbb{Z}_p$ -extension of  $K$  are defined using flat-cohomology and hence the number field arguments doesn't carry over verbatim in this setup. This is also reciprocated in the formula of the characteristic ideal of the fine  $S$ -Selmer group. We note that the power of  $\Phi_n$  appearing in Theorem 1.1 is  $e_n$ , unlike the number field case where the power is given by  $e_n - 1$  by Greenberg (cf. [KP07, Problem 0.7]). Furthermore, this difficulty also appears while analyzing the local terms in proving Theorem 5.1 (cf. see (10)). We overcame this difficulty

by assuming that we are working over the unramified  $\mathbb{Z}_p$ -extension where the local terms doesn't contribute. The question of generalizing Theorem 5.1 for an arbitrary  $\mathbb{Z}_p$ -extension of  $K$  is much more involved and a work in progress.

In the  $\ell \neq p$ , setup, our contribution is to work with the non-commutative  $GL_2(\mathbb{Z}_\ell)$ -extension of  $K$  and to find a strategy to give instances of trivial  $\mu$ -invariant of (fine) Selmer group. The strategy is the following. Since we already know Conjecture A over the unramified  $\ell$ -adic extension  $K_{\text{cyc}}$  over  $K$ , using the analogous  $\mathfrak{M}_H(G)$  conjecture for  $E$  which is also known to hold from [Pal14, Theorem 4.1] (the conjecture states that  $S(E/K_\infty)^\vee$  is a finitely generated  $\mathbb{Z}_\ell[[G]]$ -module, such that  $S(E/K_\infty)^\vee/S(E/K_\infty)^\vee[p^\infty]$  is a finitely generated  $\mathbb{Z}_\ell[[H]]$ -module; here  $G = \text{Gal}(K_\infty/K)$  and  $H = \text{Gal}(K_\infty/K_{\text{cyc}})$ ) we show that  $\mu$ -invariant of the Selmer group  $S(E/K_\infty)$  is also trivial.

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#### 2. PRELIMINARIES

Fix an odd integer prime  $p$ . Let  $K$  be a function field in one variable over a finite field  $\mathbb{F}$  of characteristic  $p$ . Let  $A$  be an abelian variety defined over  $K$ . Consider an open dense subset  $U$  of  $C_K = \mathbb{P}_{\mathbb{F}}^1$ , where  $\mathbb{P}_{\mathbb{F}}^1$  denotes the projective space of dimension 1 over  $\mathbb{F}$ , such that  $A/K$  has good reductions at every place of  $U$ . Let  $\Sigma_K$  be the set of all the primes of  $K$  and  $S$  denote the set of primes of  $K$  outside  $U$  i.e., the places of  $C_K \setminus U$ . Therefore,  $S$  is a finite set of primes of  $K$  that contains the set of all primes of bad reduction of  $A/K$ . Let  $K_S$  denote the maximal algebraic extension of  $K$  unramified outside  $S$ . Consider a finite extension of  $K$ ,  $L \subset K_S$ .

**Definition 2.1.** *Let  $B$  be an abelian group.*

- (1) *Denote by  $B[p^n]$  the  $p^n$ -torsion points of  $B$  and let  $B[p^\infty] = \bigcup_n B[p^n]$ .*
- (2) *Let  $T_p B := \varprojlim_k B[p^k]$  the Tate module of  $B$ .*
- (3) *The  $p$ -adic completion of  $B$  is defined as  $B^* = \varprojlim_n B/p^n B$ .*
- (4) *Also, define*

$$B^\bullet := B^* \otimes \mathbb{Q}_p \text{ and } V_p B := T_p B \otimes \mathbb{Q}_p.$$

**2.1. The fine Selmer group.** Let  $v$  be any prime of  $K$  and  $w$  denote a prime of  $L$ . Define

$$J_v^1(A/L) := \prod_{w|v} \frac{H_{fl}^1(L_w, A[p^\infty])}{im(\kappa_w)} \text{ and } K_v^1(A/L) := \prod_{w|v} H_{fl}^1(L_w, A[p^\infty]).$$

Here  $H_{fl}^i(-, -)$  denotes the flat cohomology [Mil86, Chapters II, III] and  $\kappa_w : A(L_w) \otimes \mathbb{Q}_p/\mathbb{Z}_p \hookrightarrow H_{fl}^1(L_w, A[p^\infty])$  is induced by the Kummer map [BL09a, §2.1.2].

**Definition 2.2.** [KT03, Prop. 2.4] *Let  $\Sigma_K, S$  and  $K \subset L \subset K_S$  be as above. Then the Selmer group  $S(A/L)$  is defined as:*

$$S(A/L) := \ker (H_{fl}^1(L, A[p^\infty]) \longrightarrow \prod_{v \in \Sigma_K} J_v^1(A/L)). \quad (1)$$

We define the  $S$ -fine Selmer group as:

$$\begin{aligned} R^S(A/L) &:= \ker (H_{fl}^1(L, A[p^\infty]) \longrightarrow \bigoplus_{v \in S} K_v^1(A/L) \prod_{v \in \Sigma_K \setminus S} J_v^1(A/L)) \\ &\cong \ker (S(A/L) \longrightarrow \bigoplus_{w|v, v \in S} A(L_w) \otimes \mathbb{Q}_p/\mathbb{Z}_p). \end{aligned} \quad (2)$$

**Remark 2.3** (Dependence on  $S$ ). *Let  $A, S, K, U$  be as above and let  $\mathcal{A}$  be the Néron model of  $A$  over  $C_K$ . Recall the following equivalent definition [KT03] of the Selmer group:*

$$S(A/K) := \ker (H_{fl}^1(U, \mathcal{A}[p^\infty]) \longrightarrow \bigoplus_{v \in S} J_v^1(A/K)). \quad (3)$$

Using definitions 2 and 3, we have

$$R^S(A/K) := \ker (H_{fl}^1(U, \mathcal{A}[p^\infty]) \longrightarrow \bigoplus_{v \in S} K_v^1(A/K)). \quad (4)$$

In fact, in [KT03, Proposition 2.4], the authors showed that the two definitions (1 and 3) of  $S(A/K)$  are equivalent. The key ingredient in the proof is the following exact sequence [Mil86, Chapter 3, §7]:

$$0 \rightarrow H_{fl}^1(U, \mathcal{A}[p^\infty]) \longrightarrow H_{fl}^1(K, A[p^\infty]) \longrightarrow \bigoplus_{v \in U} H_{fl}^1(K_v, A[p^\infty]) / H_{fl}^1(O_v, A[p^\infty]), \quad (5)$$

where  $O_v$  is the valuation ring of  $K_v$  and  $H_{fl}^1(O_v, A[p^\infty]) \cong A(K_v) \otimes \mathbb{Q}_p/\mathbb{Z}_p$ .

For an infinite algebraic extension  $\mathcal{L}$  of  $K$ , the above definitions extend, as usual, by taking inductive limit over finite sub extensions of  $\mathcal{L}$  over  $K$ .

**Remark 2.4.** *Consider a  $p$ -adic Lie extension  $L_\infty/K$  where  $G = \text{Gal}(L_\infty/K)$  is a compact  $p$ -adic Lie group without any  $p$ -torsion. Then, one can show that*

$R^S(A/L_\infty)$  is independent of  $S$ , using the Noetherianess of  $\mathbb{Z}_p[[G]]$ . However, we cannot determine this set  $S$  explicitly (see [GJS22, Remark 3.2]).

**2.2.  $\mathbb{Z}_p^d$ -extensions.** Let  $\mathbb{F}^{(p)}$  be the unique subfield of  $\overline{\mathbb{F}}$  such that  $\text{Gal}(\mathbb{F}^{(p)}/\mathbb{F}) \cong \mathbb{Z}_p$ . Set  $K_\infty := K\mathbb{F}^{(p)}$ . Note that  $K_\infty/K$  is unramified everywhere. This follows from the fact that the prime ideals of  $K$  correspond to irreducible monic polynomials of  $K$  and  $K_\infty$  is obtained by extending the perfect field  $\mathbb{F}$ . This  $\mathbb{Z}_p$ -extension  $K_\infty$  is referred to as in the literature as the "arithmetic"  $\mathbb{Z}_p$ -extension.

The second type of  $\mathbb{Z}_p$ -extension that bears a close analogy with the cyclotomic  $\mathbb{Z}_p$ -extension of a number field, is the "cyclotomic extension at the prime ideal  $\mathfrak{P}$ ". We briefly outline its construction below:

Let  $P(t) = a_n t^n + \cdots + a_0 \in \mathbb{F}[t]$ . We define the Carlitz polynomial  $[P(t)](X)$  with coefficients in  $\mathbb{F}[t]$  recursively as follows:

$$\begin{aligned} [1](X) &= X, \\ [t](X) &= X^p + tX, \\ [t^n](X) &= [t]([t^{n-1}](X)) \text{ and} \\ [a_n t^n + \cdots + a_1 t + a_0](X) &= a_n [t^n](X) + \cdots + a_1 [t](X) + a_0(X). \end{aligned}$$

Consider a field extension  $F$  of  $K$ . Then  $F$  can be thought of as a  $\mathbb{F}[t]$ -module, where the action of  $\mathbb{F}[t]$  is given by the Carlitz polynomials.

Choose a prime  $\mathfrak{P}$  of  $\mathbb{F}[t]$ . For  $n > 0$ , let

$$\Lambda_{\mathfrak{P}^n} := \{\lambda \in \overline{\mathbb{F}(t)} \mid [\mathfrak{P}^n](\lambda) = 0\}.$$

Here  $K(\Lambda_{\mathfrak{P}^n})/K$  is Galois with  $\text{Gal}(K(\Lambda_{\mathfrak{P}^n})/K) \cong (\mathbb{F}[t]/\mathfrak{P}^n)^\times$ . Put  $\tilde{K} := \bigcup_{n \geq 1} K(\Lambda_{\mathfrak{P}^n})$ , then  $\text{Gal}(\tilde{K}/K) \cong \mathbb{Z}_p^\mathbb{N} \times (\mathbb{F}[t]/\mathfrak{P})^\times$ .

The  $\mathbb{Z}_p^d$ -extension obtained from  $\tilde{K}$ , for  $d \geq 1$  is ramified only at the prime  $\mathfrak{P}$  and it is totally ramified at that prime [Ros02, Proposition 12.7]. The  $\mathbb{Z}_p$ -extension, thus obtained, is referred to as the "geometric"  $\mathbb{Z}_p$ -extension.

**2.3. Fine Mordell-Weil groups.** We define the  $S$ -fine Mordell-Weil group similar to that of [Wut07] as follows:

**Definition 2.5.** Let  $k \geq 1$  and an extension  $L/K$ , the  $p^k$ - $S$ -fine Mordell-Weil group of  $A/L$ ,  $M_{p^k}^S(A/L)$  is defined as:

$$M_{p^k}^S(A/L) = \ker(A(L)/p^k \longrightarrow \bigoplus_{v|S} A(L_v)/p^k A(L_v))$$

The  $p$ -primary  $S$ -fine Mordell-Weil group of  $A/L$  is given by:

$$\mathcal{M}^S(A/L) = \varinjlim_k M_{p^k}^S(A/L) = \ker(A(L) \otimes \mathbb{Q}_p/\mathbb{Z}_p \longrightarrow \bigoplus_{v|S} A(L_v) \otimes \mathbb{Q}_p/\mathbb{Z}_p).$$



Again following [Wut07], we define the  $S$ -fine Tate-Shafarevich as follows:

**Definition 2.6.** *For an extension  $L/K$ ,  $S$ -fine Tate-Shafarevich is defined as:*

$$\mathfrak{K}^S(A/L) := \frac{R^S(A/L)}{\mathcal{M}^S(A/L)}.$$

Hence, we have the following short exact sequence:

$$0 \longrightarrow \mathcal{M}^S(A/L) \longrightarrow R^S(A/L) \longrightarrow \mathfrak{K}^S(A/L) \longrightarrow 0.$$

**Remark 2.7.** *Let  $k \geq 1$  and  $L$  be a finite extension of  $K$ . Note that  $\mathcal{M}^S(A/L)$  can also be defined as the intersection of  $R^S(A/L)$  with  $A(L) \otimes \mathbb{Q}_p/\mathbb{Z}_p$  inside  $S(A/L)$  [Wut07, §2]. Now, by using the exact sequence*

$$0 \longrightarrow A(L) \otimes \mathbb{Q}_p/\mathbb{Z}_p \longrightarrow S(A/L) \longrightarrow \text{III}(A/L)[p^\infty] \longrightarrow 0,$$

*it is easy to see that  $\mathfrak{K}^S(A/L) \subset \text{III}(A/L)[p^\infty]$ .*

**Remark 2.8.** *By Remark 2.7, we observe that for any separable extension  $\mathcal{L}$  of  $K$ , the finiteness of  $\mathfrak{K}^S(A/\mathcal{L})$  follows from finiteness of  $\text{III}(A/\mathcal{L})[p^\infty]$ .*

*The finiteness of  $\text{III}(A/\mathcal{L})$  (i.e., the BSD conjecture) has been established for a class of elliptic curves over function fields (see for example [HYZ23]).*

**2.4. The functor  $\mathfrak{G}$ .** Let  $K$  be a function field of char  $p$ . Let  $K_\infty$  be a  $\mathbb{Z}_p$ -extension of  $K$ , with Galois group  $\Gamma$ . Denote by  $\Lambda$  the Iwasawa algebra  $\mathbb{Z}_p[[\Gamma]]$ . We identify  $\mathbb{Z}_p[[\Gamma]]$  with the power series ring  $\mathbb{Z}_p[[T]]$  by identifying  $T$  with  $\gamma - 1$ , where  $\gamma$  is a topological generator of  $\Gamma$ . For  $n \geq 0$ , define  $\omega_n = (1 + T)^{p^n} - 1$  and  $\omega_{0,-1} := T$ .

For a finitely generated  $\Lambda$ -module  $X$ , we define

$$\mathfrak{G} := \varprojlim_n \left( \frac{X}{\omega_n X} [p^\infty] \right).$$

The properties of the functor  $\mathfrak{G}$  are discussed in [Lee20].

Let  $\Phi_n := \frac{\omega_n}{\omega_{n-1}}$  for  $n \geq 1$  and  $\Phi_0 = X$ . The functor  $\mathfrak{G}$  has the following properties [Lee20, Lemma A.2.9].

- (1)  $\mathfrak{G}(\Lambda) = 0$ . Hence,  $\mathfrak{G}(X)$  is a torsion  $\Lambda$ -module.
- (2)  $\mathfrak{G}(\Lambda/g^e) = \Lambda/g^e$  if  $g$  is coprime to  $\omega_n$  for all  $n$ .
- (3) For  $m \geq 0$ ,

$$\mathfrak{G}(\Lambda/\Phi_m^e) = \begin{cases} \Lambda/\Phi_m^{e-1}, & e \geq 2 \\ 0, & e = 1 \end{cases}$$

- (4)  $\mathfrak{G}$  is a covariant functor and preserves the pseudo-isomorphism.



Consider a finitely generated  $\mathbb{Z}_p$ -module  $M$ . Denote  $M_{\text{div}}$  as the maximal divisible subgroup of  $M$ . For an integral domain  $R$  and an  $R$ -module  $A$ , let  $A_{R\text{-tor}}$  represent the elements of  $A$  that are  $R$ -torsion. Let us recall the following lemma from [Lee20]:

**Lemma 2.9.** [Lee20, Lemma 2.1.4]

- (1) Let  $R$  be an integral domain and let  $Q(R)$  be the quotient field of  $R$ . Consider an exact sequence of  $R$ -modules  $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ , where  $A$  is an  $R$ -torsion module. Then we have a short exact sequence  $0 \rightarrow A_{R\text{-tor}} \rightarrow B_{R\text{-tor}} \rightarrow C_{R\text{-tor}} \rightarrow 0$ .
- (2) Let  $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$  be a short exact sequence of finitely generated  $\mathbb{Z}_p$ -modules. If  $A$  has finite cardinality, then we have a short exact sequence

$$0 \rightarrow A = A[p^\infty] \rightarrow B[p^\infty] \rightarrow C[p^\infty] \rightarrow 0.$$

- (3) If  $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow W \rightarrow 0$  is an exact sequence of cofinitely generated  $\mathbb{Z}_p$ -modules with finite  $W$ , then the sequence

$$X_{\text{div}} \rightarrow Y_{\text{div}} \rightarrow Z_{\text{div}} \rightarrow W \rightarrow 0$$

is exact.

We will use Lemma 2.9 and the properties of functor  $\mathfrak{G}$  later in the proof of Theorem 3.5.

**2.5. Elliptic curves over function fields.** Let  $E$  be an elliptic curve defined over the function field  $K$  of char  $p$ . Let  $E[p^\infty] := E(\bar{K})[p^\infty]$ , where  $\bar{K}$  is the separable closure of  $K$ .

We recall the Mordell-Weil-Lang-Néron theorem:

**Theorem 2.10.** [Ulm11, Lecture 1, Theorem 5.1] Assume that  $K = \mathbb{F}(C)$  is the function field of a curve  $C$  over a finite field  $\mathbb{F}$  and let  $E$  be an elliptic curve over  $K$ . Then  $E(K)$  is a finitely generated abelian group.

For any  $v \in \Sigma_K$ , choose a minimal Weierstrass equation for  $E$ . Let  $E_v$  be the reduction of  $E$  modulo  $v$ . For any point  $P \in E$ , let  $P_v$  be its image in  $E_v$ . Let  $E_{v,ns}(\mathbb{F}_v)$  be the set of non-singular points of  $E_v(\mathbb{F}_v)$  and define,

$$E_0(K_v) := \{P \in E(K_v) \mid P_v \in E_{v,ns}(\mathbb{F}_v)\}$$

**Remark 2.11.** (i) Let  $v$  be a prime of  $K$ , where the elliptic curve  $E$  has split multiplicative reduction. Then according to the theory of Tate curves,  $E_0(K_v)$  is isomorphic to  $\mathcal{O}_v^*$ , where  $\mathcal{O}_v$  represents the ring of integers of  $K_v$  [BL09a, §2.1.2]. Since  $E_0(K_v)$  has a finite index within  $E(K_v)$ , it follows that  $E(K_v)[p^\infty]$  is also finite.

- (ii) Consider  $v$ , a prime of  $K$ , where  $E$  has good reduction. According to [Tan10, Lemma 2.5.1], the formal group linked to  $E$  at  $v$ , also a subgroup of finite index of  $E(K_v)$ , is a torsion-free  $\mathbb{Z}_p$ -module. Consequently, this implies that  $E(K_v)[p^\infty]$  is finite.

Now, let us recall the lemma stated in [BL09a]:

**Lemma 2.12.** [BL09a, Lemma 4.1] *Let  $G \cong \mathbb{Z}_p^d$ , for some  $d \geq 1$  and  $B$  be a finite  $p$ -primary  $G$ -module. Then,*

$$|H^1(G, B)| \leq |B|^d \text{ and } |H^1(G, B)| \leq |B|^{\frac{d(d-1)}{2}}.$$

**Remark 2.13.** *Let  $\mathcal{K}_d$  be a  $\mathbb{Z}_p^d$ -extension of  $K$  for some  $d \geq 1$ . Then the group  $E[p^\infty](\mathcal{K}_d)$  is finite [BL09a, Lemma 4.3]. Therefore, by Lemma 2.12, we get that  $H^1(G, E[p^\infty](\mathcal{K}_d))$  is bounded by  $|E[p^\infty](\mathcal{K}_d)|^d$ .*

**2.6. Euler Characteristic and the  $\mu$ -invariant.** Let  $G$  be pro- $p$ ,  $p$ -adic Lie group without any elements of order  $p$ . Let  $\Lambda(G)$  represent the Iwasawa algebra of  $G$ . The completed group algebra, denoted as  $\Omega(G)$ , is defined as follows:

$$\Omega(G) = \varprojlim_U \mathbb{F}_p[G/U],$$

where  $U$  varies across the open normal subgroups of  $G$ .

**Definition 2.14.** [How02] *Let  $M$  be a finitely generated  $\Lambda(G)$ -module then we define*

$$\mu(M) := \sum_{i \geq 0} \text{rank}_{\Omega(G)}(p^i(M[p^\infty])/p^{i+1}).$$

**Remark 2.15.** *Note that the definition of  $\mu(M)$  does not require  $M$  to be a torsion  $\Lambda(G)$ -module.*

**Definition 2.16.** *Consider a compact  $p$ -adic Lie group  $G$  with no element of order  $p$ . Let  $M$  be a finitely generated  $\Lambda(G)$ -module. If  $H^i(G, M)$  is finite for every  $i \geq 0$ , then the  $G$ -Euler characteristic of  $M$  is defined and it is given by:*

$$\chi(G, M) := \prod_{i \geq 0} (\#H^i(G, M))^{(-1)^i}.$$

We recall the following results from [How02]:

**Corollary 2.17.** [How02, Corollary 1.7] *Assume that  $G$  contains no non-trivial element of finite order. If  $M$  is a finitely generated  $\Lambda(G)$ -module then*

$$\mu(M) = \text{ord}_p(\chi(G, M[p^\infty])),$$

where  $\text{ord}_p(\alpha)$  denotes the maximum power of  $p$  which divides  $\alpha$ .

**Proposition 2.18.** [How02, Proposition 1.8] *Assume that  $G$  contains no non-trivial element of finite order. In a short exact sequence of finitely generated  $\Lambda(G)$ -modules, one has*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0.$$

*we have  $\mu(B) \leq \mu(A) + \mu(C)$  with equality holding if  $B$ , and hence also  $A$  and  $C$ , is  $\Lambda(G)$ -torsion.*

## 2.7. Euler characteristic for flat cohomology.

**Definition 2.19.** [TW11, §5] *Let  $S$  be a scheme of characteristic  $p > 0$  and a finite flat group scheme  $N/S$  we define the Euler characteristic of  $N/S$  as*

$$\chi_{fl}(S, N) := \prod_i (\#H_{fl}^i(S, N))^{(-1)^i}.$$

*whenever the groups  $H_{fl}^i(S, N)$  are finite.*

**2.7.1. Relation with Hochschild-Serre spectral sequence.** Let  $K$  be a function field of char  $p$  and  $K_\infty$  be an unramified  $\mathbb{Z}_p$ -extension with  $\Gamma = \text{Gal}(K_\infty/K)$ . Let  $A$  be a sheaf for the flat topology on  $\text{Spec}(K)$ . Now, recall the Hochschild-Serre spectral sequence [Mil80, Chapter 3, Remark 2.21]

$$E_2^{a,b} := H^a(\Gamma, H_{fl}^b(K_\infty, A)) \implies H_{fl}^{a+b}(K, A), \quad (6)$$

where  $A$  is a sheaf for flat topology on  $\text{Spec}(K)$ .

**Remark 2.20.** *If  $A$  is a finite flat group scheme over  $\text{Spec}(K)$ , then by using (6), we obtain that*

$$\chi_{fl}(\text{Spec}(K), A) = \frac{\chi(\Gamma, H_{fl}^2(K_\infty, A)) \times \chi(\Gamma, H_{fl}^0(K_\infty, A))}{\chi(\Gamma, H_{fl}^1(K_\infty, A))}$$

**2.8. The  $\ell \neq p$  case.** Choose and fix a rational prime  $\ell \neq p$ . Consider  $K = \mathbb{F}(t)$ , where  $\mathbb{F}$  is a finite field of char  $p$  of cardinality  $p^r$ , for some  $r > 0$ . In this subsection, we will recall the properties of the  $\ell^\infty$ -Selmer groups of an elliptic curve  $E$  over certain compact  $\ell$ -adic Lie extensions over the function field  $K$  of characteristic  $p$ . Note that the image of the Kummer map, in this set up, is identically 0 [BL09b, Proposition 3.1], thus the  $\ell^\infty$ -fine Selmer group coincides with the  $\ell^\infty$ -Selmer group.

Let  $E/K$  be an elliptic curve. Let  $S$  be a finite set of places of  $K$  that contains the places which ramify over  $K_\infty$  and the places where  $E$  has a bad reduction. Let  $K_S$  be the maximal separable extension of  $K$ , which is unramified outside  $S$ . For each  $v \in S$ , put  $J_v^1(E/L) := \prod_{w|v} H^1(L_w, E[\ell^\infty])$ .

**Definition 2.21.** Let  $L \subset K_S$  be a finite extension of  $K$ . We define the  $\ell^\infty$ -Selmer group  $S(E/L)$  of  $E/L$  as:

$$S(E/L) := \ker(H^1(G_S(L), E[\ell^\infty]) \longrightarrow \bigoplus_{v \in S} J_v^1(E/L)), \quad (7)$$

where  $G_S(L) := \text{Gal}(K_S/L)$ .

The definition of  $S(E/\mathcal{L})$  extends to an infinite extension  $\mathcal{L}/K$  by taking direct limits over intermediate finite subextensions.

Denote by  $\mathbb{F}_p^{(\ell)}$  the unique  $\mathbb{Z}_\ell$ -extension of  $\mathbb{F}_p$  contained in  $\overline{\mathbb{F}_p}$ . Then  $K_{\text{cyc}} := \mathbb{F}_p^{(\ell)} K$  is the unique  $\mathbb{Z}_\ell$ -extension of  $K$  [BL09b, Proposition 4.3]. This is known in the literature as the “arithmetic”  $\mathbb{Z}_\ell$ -extension. The extension  $K_{\text{cyc}}$  of  $K$  is unramified everywhere.

Another  $p$ -adic Lie extension relevant to our discussion is the trivializing extension of  $K$ . It is defined as follows:

Let  $E/K$  be an elliptic curve such that  $j(E) \notin \mathbb{F}$ . Put  $K_\infty := K(E[\ell^\infty])$ . This is an  $\ell$ -adic Lie extension of the global field  $K$ . We know  $G_\infty := \text{Gal}(K_\infty/K)$  is an open subgroup of  $GL_2(\mathbb{Z}_\ell)$  [Pal14, §4].

**Remark 2.22.** The condition  $j(E) \notin \mathbb{F}$  ensures that the elliptic curve  $E$  has no complex multiplication [Ulm11, Lecture 1, §4]. Elliptic curves satisfying this property are called non-isotrivial elliptic curves.

### 3. STRUCTURE OF THE MORDELL-WEIL GROUP

Let  $K$  be a function field of char  $p$  and  $A/K$  be an abelian variety defined over  $K$ . Let  $K_\infty$  be a  $\mathbb{Z}_p$ -extension of  $K$ , with Galois group  $\Gamma$  (either  $K_\infty$  is the arithmetic  $\mathbb{Z}_p$ -extension or the geometric  $\mathbb{Z}_p$ -extension). From now on,  $S$  will be the finite set of primes of  $K$ , containing the primes of bad reduction of  $A$  and the primes of  $K$  ramified in  $K_\infty$ .

Let  $K_n$  be the unique sub-extension of  $K_\infty/K$  such that  $[K_n : K] = p^n$ . Denote by  $\Gamma_n := \text{Gal}(K_\infty/K_n)$ . Now consider the natural restriction map:

$$S_n^A : S(A/K_n) \longrightarrow S(A/K_\infty)^{\Gamma_n}.$$

**Remark 3.1.** The following theorem of [Tan10] shows that  $\ker(S_n^A)$  and  $\text{coker}(S_n^A)$  are both finite and bounded independently of  $n$ :

**Theorem 3.2.** [Tan10, Theorem 4] Let  $K_d$  be a  $\mathbb{Z}_p^d$ -extension of a global field  $K$  of characteristic  $p$  with Galois group  $\Gamma_d := \text{Gal}(K_d/K)$ . Assume that  $K_d/K$  is unramified outside a finite set  $S$  of places of  $K$ . Let  $A$  be an abelian variety over  $K$  with good ordinary reduction at every place in  $S$ . Then for every finite intermediate extension  $F$  of  $K_d/K$ , the kernel and the cokernel of the

restriction map  $\text{res } K_d/F : S(A/F) \longrightarrow S(A/K_d)^{\Gamma_F}$ , where  $\Gamma_F = \text{Gal}(K_d/F)$  are finite. Furthermore, if  $d = 1$ , then the orders of the kernel and the cokernel of  $\text{res } K_d/F$  are bounded as  $F$  varies.

The Tate-Shafarevich group of an abelian variety  $A/K$  is defined as:

$$\text{III}(A/K) := \ker(H^1(K, A) \longrightarrow \prod_{v \in \Sigma_K} H^1(K_v, A))$$

**Proposition 3.3.** *Let  $A/K$  be an abelian variety and  $S$  be as above. Assume that  $\ker(S_n^A)$  and  $\text{coker}(S_n^A)$  are finite and bounded independently of  $n$ .*

*Further, if the groups  $A(K_n)$  and  $\text{III}(A/K_n)[p^\infty]$  are finite for each  $n > 0$ , then there exists  $\mu, \lambda, \nu \geq 0$ , such that*

$$\#S(A/K_n) = \#\text{III}(A/K_n)[p^\infty] = p^{e_n}$$

*for  $n \gg 0$ , where  $e_n = p^n \mu + n\lambda + \nu$ .*

*Proof.* The proof follows easily from the structure of finitely generated  $\Lambda$ -modules (see [Gre01, Corollary 4.11]).  $\square$

The subsequent lemma extends the result of [Lee20, Lemma 2.0.1] to function fields of characteristic  $p$ . We provide the proof below:

**Lemma 3.4.** *Let  $L$  be a finite extension of  $K$  or  $K_v$  for some prime  $v$  and let  $A/L$  be an abelian variety. Let  $L_\infty/L$  be a  $\mathbb{Z}_p$ -extension and  $X = (A(L_\infty)[p^\infty])^\vee$ . Then,*

- (i)  *$X$  is a finitely generated torsion  $\Lambda$ -module with  $\mu = 0$  and  $\text{char}_\Lambda(X)$  is coprime to  $\omega_n$  for all  $n$ .*
- (ii) *The modules  $\frac{A(L_\infty)[p^\infty]}{p^n A(L_\infty)[p^\infty]}$  and  $\frac{A(L_\infty)[p^\infty]}{\omega_n A(L_\infty)[p^\infty]}$  are finite and bounded independent of  $n$ .*
- (iii) *For the natural maps*

$$MW_n^A : A(L_n) \otimes \mathbb{Q}_p/\mathbb{Z}_p \longrightarrow (A(L_\infty) \otimes \mathbb{Q}_p/\mathbb{Z}_p)^{\Gamma_n}$$

*and*

$$S_n^A : S(A/L_n) \longrightarrow S(A/L_\infty)^{\Gamma_n}$$

*the groups  $\ker(MW_n^A)$  and  $\ker(S_n^A)$  are finite and bounded independently of  $n$*

*Proof.* Note that (ii) follows from (i), via the structure theorem of finitely generated  $\Lambda$ -modules. For (i), observe that

$$\frac{X}{pX} \cong (A(L_\infty)[p])^\vee, \frac{X}{\omega_n X} \cong (A(L_n)[p^\infty])^\vee$$

$A(L_\infty)[p]$  and  $A(L_n)[p^\infty]$  are finite (see proof of [GJS22, Proposition 3.3]). This proves (i).

Now, notice that by the definition of Selmer groups, we have injections:

$$\ker(MW_n^A) \hookrightarrow \ker(S_n^A) \hookrightarrow \frac{A(L_\infty)[p^\infty]}{\omega_n A(L_\infty)[p^\infty]}$$

The second injectivity follows from the fact that  $\ker(S_n^A) \subset H^1(\Gamma_n, A(F_\infty)[p^\infty])$ . Now, (iii) follows from (ii).  $\square$

The next proposition is a generalisation of [Lee20, Theorem 3.3] in the setting of function fields.

**Proposition 3.5.** *Let  $L$  be a finite extension of  $K$  or  $K_v$  for some prime  $v$  and let  $A/L$  be an abelian variety. Let  $L_\infty/L$  be a  $\mathbb{Z}_p$ -extension. Then, we have:*

- (1)  $\mathfrak{S}((A(L_\infty) \otimes \mathbb{Q}_p/\mathbb{Z}_p)^\vee) = 0$ .
- (2) *There is a  $\Lambda$ -linear injection*

$$(A(L_\infty) \otimes \mathbb{Q}_p/\mathbb{Z}_p)^\vee \hookrightarrow \Lambda^r \oplus \left( \bigoplus_{i=1}^u \frac{\Lambda}{\Phi_{c_i}} \right) \quad (8)$$

*with finite cokernel for some  $r, u, c_i \geq 0$ .*

The proof essentially follows from the arguments of [Lee20, Theorem 3.3]. We present a brief sketch below:

*Proof.* We first identify  $\mathbb{Z}_p[[\text{Gal}(L_\infty/L)]]$  with  $\mathbb{Z}_p[[X]]$ . Let  $C_n$  be the cokernel of the natural map

$$MW_n^A : A(L_n) \otimes \mathbb{Q}_p/\mathbb{Z}_p \longrightarrow (A(L_\infty) \otimes \mathbb{Q}_p/\mathbb{Z}_p)^{\Gamma_n}$$

As  $\varinjlim_n A(L_n) \otimes \mathbb{Q}_p/\mathbb{Z}_p = A(L_\infty) \otimes \mathbb{Q}_p/\mathbb{Z}_p$  by definition, it follows that  $\varinjlim_n C_n = 0$  and  $\varinjlim_n (C_n)_{\text{div}} = 0$ .

Now, from the short exact sequence

$$A(L_n) \otimes \mathbb{Q}_p/\mathbb{Z}_p \longrightarrow (A(L_\infty) \otimes \mathbb{Q}_p/\mathbb{Z}_p)^{\Gamma_n} \longrightarrow C_n \longrightarrow 0$$

we obtain the following short exact sequence

$$0 = (A(L_n) \otimes \mathbb{Q}_p/\mathbb{Z}_p)_{\text{div}} \longrightarrow ((A(L_\infty) \otimes \mathbb{Q}_p/\mathbb{Z}_p)^{\Gamma_n})_{\text{div}} \longrightarrow (C_n)_{\text{div}} \longrightarrow 0.$$

by applying Lemma 2.9(iii). Taking direct limits to the above sequence, we get that

$$\varinjlim_n ((A(L_\infty) \otimes \mathbb{Q}_p/\mathbb{Z}_p)^{\Gamma_n})_{\text{div}} = 0.$$

Taking Pontryagin dual, we obtain that

$$\mathfrak{S}((A(L_\infty) \otimes \mathbb{Q}_p/\mathbb{Z}_p)^\vee) = \varprojlim_n \frac{(A(L_\infty) \otimes \mathbb{Q}_p/\mathbb{Z}_p)^\vee}{\omega_n (A(L_\infty) \otimes \mathbb{Q}_p/\mathbb{Z}_p)^\vee} [p^\infty] = 0.$$

Recall from the description of functor  $\mathfrak{G}$ , given before the statement of Lemma 2.9, that  $\mathfrak{G}(\Lambda) = 0$  and  $(\frac{\Lambda}{\omega_{m+1,m}^e}) = 0$ , if  $e = 1$  and  $= \frac{\Lambda}{\omega_{m+1,m}^{e-1}}$ , if  $e \geq 2$ . Using the fact that  $(A(L_\infty) \otimes \mathbb{Q}_p/\mathbb{Z}_p)^\vee$  is a finitely generated  $\Lambda$  module and  $\mathfrak{G}((A(L_\infty) \otimes \mathbb{Q}_p/\mathbb{Z}_p)^\vee) = 0$ , we get the following  $\Lambda$ -linear pseudo-isomorphism:

$$(A(L_\infty) \otimes \mathbb{Q}_p/\mathbb{Z}_p)^\vee \longrightarrow \Lambda^r \oplus \left( \bigoplus_{i=1}^u \frac{\Lambda}{\Phi_{c_i}} \right)$$

for some  $r, u, c_i \geq 0$ .

The injectivity follows from the fact that  $(A(L_\infty) \otimes \mathbb{Q}_p/\mathbb{Z}_p)^\vee$  has no non-trivial finite  $\Lambda$ -module since it is  $\mathbb{Z}_p$ -torsion free.  $\square$

**Remark 3.6.** Let  $L$  be a finite extension of  $K$ . Let  $L_\infty$  be a  $\mathbb{Z}_p$ -extension of the function field  $L$ . Assuming that  $\text{III}(E/L_n)[p^\infty]$  is finite for all  $n$ , we can use the finiteness of the kernel of the map  $S_n^A$  and apply the snake lemma to deduce that the natural map

$$E(L_n) \otimes \mathbb{Q}_p/\mathbb{Z}_p \longrightarrow (E(L_\infty) \otimes \mathbb{Q}_p/\mathbb{Z}_p)^{\Gamma_n}$$

has both finite kernel and cokernel. Moreover, by comparing the  $\mathbb{Z}_p$ -ranks of the two modules as in [Lee20, Page-2409], we get that

$$r = \lim_{n \rightarrow \infty} \frac{\text{rank}_{\mathbb{Z}} A(L_n)}{p^n} \text{ and } e_n = \frac{\text{rank}_{\mathbb{Z}} E(L_n) - \text{rank}_{\mathbb{Z}} E(L_{n-1})}{\varphi(p^n)} - r;$$

for  $n \geq 1$  and  $e_n$  is defined as the number between  $1 \leq i \leq t$  satisfying  $i = n$  in (8).

#### 4. STRUCTURE OF FINE MORDELL WEIL GROUPS

**Corollary 4.1.** Assume that  $r = 0$  in equation (8). Then there is a pseudo-isomorphism of  $\Lambda$ -modules

$$\mathcal{M}^S(A/L_\infty)^\vee \sim \bigoplus_{i=1}^u \frac{\Lambda}{\Phi_{c_i}},$$

for some  $u, c_i \geq 0$ .

We now prove a control theorem for the Fine Mordell-Weil group:

**Proposition 4.2.** Let  $L$  be a finite extension of  $K$ . Let  $E/L$  be an elliptic curve and let  $m_n$  denote the natural morphism

$$m_n : \mathcal{M}^S(E/L_n) \longrightarrow \mathcal{M}^S(E/L_\infty)^{\Gamma_n}$$

induced by the inclusion  $E(L_n) \hookrightarrow E(L_\infty)$ .

(a) The kernel of  $m_n$  is finite, with order bounded independently of  $n$ .



(b) Suppose that  $\mathfrak{K}^S(E/L_n)$  is finite, then the cokernel of  $m_n$  is finite

*Proof.* Consider the following commutative diagram:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & R^S(E/L_\infty)^{\Gamma_n} & \longrightarrow & H^1(L_\infty, E[p^\infty])^{\Gamma_n} & \xrightarrow{\lambda_{L_\infty}} & (\prod_{v \in S} K_v^1(E/L_\infty) \prod_{v \in \Sigma_K \setminus S} J_v^1(E/L_\infty))^{\Gamma_n} \\
 & & \uparrow f_n & & \uparrow g_n & & \uparrow h_n \\
 0 & \longrightarrow & R^S(E/L_n) & \longrightarrow & H^1(L_n, E[p^\infty]) & \xrightarrow{\lambda_{L_n}} & \prod_{v \in S} K_v^1(E/L_n) \prod_{v \in \Sigma_K \setminus S} J_v^1(E/L_n)
 \end{array}$$

First, we show that the kernel of  $f_n$  is finite and bounded independently of  $n$  and that the cokernel of  $f_n$  is also finite.

Now, using snake lemma, we get the following exact sequence:

$$0 \longrightarrow \ker(f_n) \longrightarrow \ker(g_n) \longrightarrow \ker(h_n) \cap \text{image}(\lambda_{L_n}) \longrightarrow \text{coker}(f_n) \longrightarrow 0.$$

Therefore, it is enough to show that  $\ker(g_n)$  is finite and bounded independently of  $n$  and  $\ker(h_n)$  is finite. Notice, that  $\ker(g_n) = H^1(\Gamma_n, E(L_\infty)[p^\infty])$ , which is finite and bounded independently of  $n$  by Remark 2.13.

For  $v \notin S$ , the kernel of the map from  $J_v^1(E/L_n) \longrightarrow J_v^1(E/L_\infty)^{\Gamma_{n,v}}$ , where  $\Gamma_{n,v}$  is the decomposition subgroup of  $\Gamma_n$  at the prime  $v$ , is 0 [BL09a, §4.2.4]. Therefore, by Shapiro's lemma, we get that  $\ker(h_n) \cong \bigoplus_{v \in S} H^1(\Gamma_{n,v}, E(L_{\infty,v})[p^\infty])$ .

Now, the finiteness of  $\ker(h_n)$  follows from the fact that  $H^0(\Gamma_{n,v}, E(L_{\infty,v})[p^\infty]) = E(L_{n,v})[p^\infty]$  is finite (see Remark 2.11).

To prove (a) and (b), we consider the commutative diagram:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \mathcal{M}^S(E/L_\infty)^{\Gamma_n} & \longrightarrow & R^S(E/L_\infty)^{\Gamma_n} & \longrightarrow & \mathfrak{K}^S(E/L_\infty)^{\Gamma_n} \\
 & & \uparrow m_n & & \uparrow f_n & & \uparrow \gamma_n \\
 0 & \longrightarrow & \mathcal{M}^S(E/L_n) & \longrightarrow & R^S(E/L_n) & \longrightarrow & \mathfrak{K}^S(E/L_n) \longrightarrow 0
 \end{array}$$

Applying snake lemma, we get that  $\ker(m_n)$  is finite and bounded independently of  $n$  as  $\ker(f_n)$  is finite and bounded independently of  $n$ . This proves (a).

Now, if we assume that  $\mathfrak{K}^S(E/L_n)$  is finite. Then, (b) follows from the fact that  $\text{coker}(f_n)$  is finite. □

The proof of the next Corollary is similar to that of [Lei23, Corollary 3.8]. We briefly sketch it below:

**Corollary 4.3.** *Assume that  $r = 0$  in equation (8) and  $\mathfrak{H}^S(E/L_n)$  is finite for each  $n$ . Then there is a  $\Lambda$ -isomorphism*

$$T_p \mathcal{M}^S(E/L_n) \longrightarrow \bigoplus_{c_i \leq n} \frac{\Lambda}{\Phi_{c_i}},$$

where the integers  $c_i$  are as in Corollary 4.1.

*Proof.* First, note that  $(\Lambda/\Phi_{c_i})_{\Gamma_n}$  has finite cardinality for all  $c_i > n$ . Now, from Proposition 4.2, it follows that  $\mathcal{M}^S(E/L_n)^\vee$  is given by, up to finite modules,  $\bigoplus_{c_i \leq n} \Lambda/\Phi_{c_i}$ . The proof now follows from [Lei23, Remark 2.4].  $\square$

**Remark 4.4.** *If  $L_\infty$  is the arithmetic  $\mathbb{Z}_p$ -extension of  $L$ , then  $S(E/L_\infty)^\vee$  is a torsion  $\Lambda$ -module [OT09, Theorem 1.7]. Now, recall the following short exact sequence:*

$$0 \longrightarrow E(L_\infty) \otimes \mathbb{Q}_p/\mathbb{Z}_p \longrightarrow S(E/L_\infty) \longrightarrow \text{III}(E/L_\infty)[p^\infty] \longrightarrow 0.$$

By additivity of co-rank of  $\Lambda$ -modules, we get that  $r = 0$  in equation (8).

**Example 4.5.** *Let  $p > 3$  and  $K = \mathbb{F}_p(t)$ . Consider the family of elliptic curves  $E_q$  defined over  $K$  given by the Weierstrass equation of the form*

$$y^2 = x^3 + t^q - t,$$

where  $q = p^f$  for some  $f > 0$ . From, [GU20, Theorem 8.2] it follows that  $\text{III}(E_q/L_n)[p^\infty]$  is finite.

Now, considering the observations mentioned in Remark 4.4, we can conclude that all the conditions specified in Corollary 4.3 are satisfied.

**Proposition 4.6.** *Let  $L$  be a finite extension of  $K$ . Let  $L_\infty/L$  be the arithmetic  $\mathbb{Z}_p$ -extension. Assume that  $E/L$  has split multiplicative reduction or good reduction at all primes of  $L$  and  $\text{III}(E/L_n)[p^\infty]$  is finite for all  $n \geq 0$ .*

*Also, suppose that  $E$  has split multiplicative reduction at all primes of  $S$ . Then,*

$$\text{Char}_\Lambda(\mathcal{M}^S(E/L_\infty))^\vee = \left( \prod_{e_n \geq 1, n \geq 0} \Phi_n^{e_n} \right),$$

$$\text{where } e_n = \frac{\text{rank } E(L_n) - \text{rank } E(L_{n-1})}{\varphi(p^n)}.$$

*Proof.* By using Remark 3.6 and the definition of  $E(L_n)^\bullet$  as in [Lei23, §4], we have the following isomorphism of  $\Lambda$ -modules:

$$E(L_n)^\bullet \cong \bigoplus_{m=0}^n (\mathbb{Q}_p[X]/\Phi_m)^{e_m}.$$

Let  $v \in S$  and  $f_n : E(L_n)^\bullet \rightarrow E(L_{n,v})^\bullet$  denote the natural map induced by inclusion  $E(L_n) \hookrightarrow E(L_{n,v})$ . Now, note that  $E_0(L_{n,v})$ , the points of  $E(L_{n,v})$  that do not reduce to a singular point, have finite index in  $E(L_{n,v})$ . Hence,  $E_0(L_{n,v})^\bullet \cong E(L_{n,v})^\bullet$ . Note that since  $v \in S$ ,  $E_0(L_{n,v}) \cong \mathcal{O}_{n,v}^\times$ , the units in the ring of integers  $\mathcal{O}_{n,v}$  (Remark 2.11). By the structure of  $\mathcal{O}_{n,v}$ , we get that  $E_0(L_{n,v})^\bullet$  is isomorphic to countable copies of  $\mathbb{Q}_p$ . Hence, we conclude that  $(\text{Image } f_n)[\Phi_n] = 0$ .

Now, consider the following short exact sequences,

$$0 \rightarrow M_{p^k}^S(E/L_n) \rightarrow E(L_n)/p^k \rightarrow \bigoplus_{v \in S} E(L_{n,v})/p^k$$

Taking inverse limit as  $k$  varies and tensoring with  $\mathbb{Q}_p$ , we get

$$0 \rightarrow V_p \mathcal{M}_{p^k}^S(E/L_n) \rightarrow E(L_n)^\bullet \xrightarrow{f_n} \bigoplus_{v \in S} E(L_{n,v})^\bullet$$

Using Corollary 4.3, we get that

$$V_p \mathcal{M}_{p^k}^S(E/L_n) \cong \bigoplus_{m=0}^n (\mathbb{Q}_p[X]/\Phi_m)^{s_m},$$

where  $s_m$  is the number of times  $\Lambda/\Phi_n$  appears on the right-hand side of the pseudo-isomorphism given by Corollary 4.1.

As  $(\text{Image } f_n)[\Phi_n] = 0$ , we get that  $e_m = s_m$ . This completes the proof.  $\square$

**Remark 4.7.** Note that the characteristic ideal of  $\mathcal{M}^S(E/L_\infty)$  depends on the choice of  $S$ . In particular, as seen in the proof of Proposition 4.6, we obtained  $s_m = e_m$  by showing that  $(\text{Image } f_n)[\Phi_n] = 0$ . This is obtained by observing the behavior of the elliptic curve  $E$  at the primes in  $S$ .

**Remark 4.8.** The condition that  $E/K$  has split multiplicative reduction or good reduction at all primes of  $K$  is trivially satisfied by constant elliptic curves (i.e., elliptic curves defined over a finite field).

Also note that for any elliptic curve  $E$  over  $K$ , there exists a finite extension  $L/K$  such that  $E$  has good or split multiplicative reduction at all places of  $L$  [Sil09, Proposition 5.4].

The following result is an obvious consequence of Proposition 4.6. Therefore, we state it without proof:

**Theorem 4.9.** Let  $L$  be a finite extension of  $K$ . Let  $L_\infty/L$  be the arithmetic  $\mathbb{Z}_p$ -extension. Assume that  $E/L$  has split multiplicative reduction or good reduction at all primes of  $L$  and  $\text{III}(E/L_n)[p^\infty]$  is finite for all  $n \geq 0$ .

Also, suppose that  $E$  has split multiplicative reduction at all primes of  $S$  and that  $\mathcal{H}^S(E/L_\infty)$  is finite. Then,

$$\text{Char}_\Lambda(R^S(E/L_\infty)^\vee) = \left( \prod_{e_n \geq 1, n \geq 0} \Phi_n^{e_n} \right)$$

### 5. BEHAVIOUR OF $\mu$ -INVARIANTS UNDER ISOGENY ( $\ell = p$ CASE).

Let  $K = \mathbb{F}(t)$  be the function field over the finite field of char  $p$ . Assume that  $E_1/K$  and  $E_2/K$  are two non-isotrivial elliptic curves (i.e.,  $j(E_i) \notin \mathbb{F}$  for  $i = 1, 2$ ). Consider  $\varphi : E_1 \rightarrow E_2$  as an isogeny of degree  $p^r$  for some  $r > 0$ .

In this section, our goal is to consider the change in the  $\mu$ -invariant of  $p^\infty$ -Selmer groups for  $E_1$  and  $E_2$  over  $K_\infty$ , the unramified  $\mathbb{Z}_p$ -extension.

**Theorem 5.1.** *Let  $E_1$  and  $E_2$  be two non-isotrivial elliptic curves over the function field  $K$  and let  $\varphi : E_1 \rightarrow E_2$  be an isogeny of degree  $p^r$  for some  $r > 0$ . Also assume that  $H_{fl}^2(K_\infty, E_i[p^\infty]) = 0$ , for  $i = 1, 2$ .*

Then,

$$\mu(S(E_2/K_\infty)^\vee) - \mu(S(E_1/K_\infty)^\vee) = \text{ord}_p(\chi_{fl}(\text{Spec}(K), A)),$$

where  $A = \ker(\varphi)$ .

*Proof.* Consider the following commutative diagrams:

$$\begin{array}{ccccccc} 0 & \longrightarrow & S(E_2/K_\infty) & \longrightarrow & H^1(K_\infty, E_2[p^\infty]) & \xrightarrow{\lambda_2} & \text{Im}(\lambda_2) \longrightarrow 0 \\ & & \uparrow f_1 & & \uparrow f_2 & & \uparrow f_3 \\ 0 & \longrightarrow & S(E_1/K_\infty) & \longrightarrow & H^1(K_\infty, E_1[p^\infty]) & \xrightarrow{\lambda_1} & \text{Im}(\lambda_1) \longrightarrow 0 \end{array} \quad (9)$$

Here,  $\lambda_i$  denotes the maps:

$$\lambda_i : H^1(K_\infty, E_i[p^\infty]) \longrightarrow \prod_{v \in \Sigma_K} J_v^1(E_i/K_\infty).$$

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Im}(\lambda_2) & \longrightarrow & \prod_{v \in \Sigma_K} J_v^1(E_2/K_\infty) & \longrightarrow & \text{coker}(\lambda_2) \longrightarrow 0 \\ & & \uparrow f_3 & & \uparrow f_4 & & \uparrow f_5 \\ 0 & \longrightarrow & \text{Im}(\lambda_1) & \longrightarrow & \prod_{v \in \Sigma_K} J_v^1(E_1/K_\infty) & \longrightarrow & \text{coker}(\lambda_1) \longrightarrow 0 \end{array} \quad (10)$$

Let  $\Gamma = \text{Gal}(K_\infty/K)$ . As  $S(E_i/K_\infty)^\vee$  is torsion  $\Lambda$ -module for  $i = 1, 2$ , we get by Proposition 2.18 that

$$\mu(\ker(f_1)^\vee) - \mu(\text{coker}(f_1)^\vee) = \mu(S(E_1/K_\infty)^\vee) - \mu(S(E_2/K_\infty)^\vee)$$

We start by showing that  $\ker(f_1)$  and  $\operatorname{coker}(f_1)$  are both finite and therefore annihilated by a power of  $p$ . By Corollary 2.17, we know that for a finitely generated  $\Lambda$  module  $M$ ,  $\mu(M) = \operatorname{ord}_p(\chi(\Gamma, M[p^\infty]^\vee))$ . Therefore it suffices to calculate the logarithm to the  $p$ -base of  $\frac{\chi(\Gamma, \operatorname{coker} f_1)}{\chi(\Gamma, \ker f_1)}$ , once we establish the finiteness of  $\ker(f_1)$  and  $\operatorname{coker}(f_1)$ .

By applying snake lemma to equation (9), we obtain the following exact sequence:

$$0 \longrightarrow \ker(f_1) \longrightarrow \ker(f_2) \longrightarrow \ker(f_3) \longrightarrow \operatorname{coker}(f_1) \longrightarrow \operatorname{coker}(f_2) \longrightarrow \operatorname{coker}(f_3) \longrightarrow 0.$$

Now, we claim that for  $i = 2, 3$ ,  $\ker(f_i)$  and  $\operatorname{coker}(f_i)$  are finite. This in turn will show that  $\ker(f_1)$  and  $\operatorname{coker}(f_1)$  are finite. Following this, we will conclude our calculation of  $\frac{\chi(\Gamma, \operatorname{coker} f_1)}{\chi(\Gamma, \ker f_1)}$ , by using the relation

$$\frac{\chi(\Gamma, \operatorname{coker} f_1)}{\chi(\Gamma, \ker f_1)} = \frac{\chi(\Gamma, \operatorname{coker} f_2)}{\chi(\Gamma, \ker f_2)} \times \frac{\chi(\Gamma, \ker f_3)}{\chi(\Gamma, \operatorname{coker} f_3)},$$

which can be easily derived from equation (5).

Recall that the kernel of an isogeny is finite and its order divides the degree of the isogeny. Hence,  $A = \ker(\varphi(p))$ , where  $\varphi(p)$  is the map between  $E_1[p^\infty]$  and  $E_2[p^\infty]$ , which is induced by  $\varphi$ .

We claim that  $\chi(\Gamma, \ker(f_3)) = \chi(\Gamma, \operatorname{coker}(f_3)) = 1$ . Consider the long-exact sequence:

$$\begin{aligned} 0 \longrightarrow S(E_1/K_\infty)^\Gamma &\longrightarrow H^1(K_\infty, E_1[p^\infty])^\Gamma \longrightarrow (\operatorname{im}(\lambda_1))^\Gamma \longrightarrow H^1(\Gamma, S(E_1/K_\infty)) \\ &\longrightarrow H^1(\Gamma, H^1(K_\infty, E_1[p^\infty])) \longrightarrow H^1(\Gamma, \operatorname{im}(\lambda_1)) \longrightarrow 0. \end{aligned}$$

The assumption  $H_{fl}^2(K_\infty, E_1[p^\infty]) = 0$  implies that  $H^1(\Gamma, H^1(K_\infty, E_1[p^\infty])) = 0$  ([BV18, Proposition 2.2]). Hence,  $H^1(\Gamma, \operatorname{Im}(\lambda_1)) = 0$ .

Again, consider the exact sequence

$$\begin{aligned} 0 \longrightarrow (\operatorname{Im}(\lambda_1))^\Gamma &\longrightarrow \left( \prod_{v \in \Sigma_K} J_v^1(E_1/K_\infty) \right)^\Gamma \longrightarrow (\operatorname{coker}(\lambda_1))^\Gamma \longrightarrow H^1(\Gamma, \operatorname{Im}(\lambda_1)) \\ &\longrightarrow H^1(\Gamma, \prod_{v \in \Sigma_K} J_v^1(E_1/K_\infty)) \longrightarrow H^1(\Gamma, \operatorname{coker}(\lambda_1)) \longrightarrow 0 \end{aligned}$$

By [BV18, Remark 2.7(3)], we know that

$$\operatorname{coker} \left( (\operatorname{Im}(\lambda_1))^\Gamma \longrightarrow \left( \prod_{v \in \Sigma_K} J_v^1(E_1/K_\infty) \right)^\Gamma \right) = 0.$$

This along with the fact that  $H^1(\Gamma, \operatorname{Im}(\lambda_1)) = 0$  implies that  $(\operatorname{coker}(\lambda_1))^\Gamma = 0$ . Hence,  $(\ker(f_3))^\Gamma = 0$ .

As  $K_\infty$  is unramified at every prime  $v$ , we obtain that for any prime  $v$  and  $w \mid v$ ,  $\text{Gal}(\overline{K_{\infty,w}}/K_{\infty,w}) \cong \prod_{\ell \neq p} \mathbb{Z}_\ell$ . Therefore,  $\ker f_4 = \text{coker } f_4 = 0$ .

By applying a snake lemma on equation (10), we obtain that  $\ker(f_3) = 0$  and  $\ker(f_5) \cong \text{coker}(f_3)$ . Since,  $(\text{coker}(\lambda_1))^\Gamma = 0$ , hence  $(\ker(f_5))^\Gamma = (\text{coker}(f_3))^\Gamma = 0$ .

By imitating the arguments above, the assumption  $H_{fl}^2(K_\infty, E_2[p^\infty])$  implies that  $H^1(\Gamma, \text{Im}(\lambda_2)) = 0$  as well. Now, from the long exact sequence corresponding to the short exact sequence

$$0 \longrightarrow \ker(f_3) \longrightarrow \text{Im}(\lambda_1) \longrightarrow \text{Im}(\lambda_2) \longrightarrow \text{coker}(f_3) \longrightarrow 0,$$

we obtain that  $H^1(\Gamma, \text{coker}(f_3)) = 0$ . This proves over claim that  $\chi(\Gamma, \ker(f_3)) = \chi(\Gamma, \text{coker}(f_3)) = 1$ .

Now, considering the long exact sequence of flat cohomology corresponding to the short exact sequence  $0 \longrightarrow A \longrightarrow E_1[p^\infty] \longrightarrow E_2[p^\infty] \longrightarrow 0$  and using the fact  $H_{fl}^2(K_\infty, E_2[p^\infty]) = 0$ , we get

$$\frac{\chi(\Gamma, \text{coker}(f_1))}{\chi(\Gamma, \ker(f_1))} = \frac{\chi(\Gamma, \text{coker}(f_2))}{\chi(\Gamma, \ker(f_2))} = \frac{\chi(\Gamma, H_{fl}^2(K_\infty, A)) \times \chi(\Gamma, E_1[p^\infty](K_\infty)) \times \chi(\Gamma, H_{fl}^0(K_\infty, A))}{\chi(\Gamma, H_{fl}^1(K_\infty, A)) \times \chi(\Gamma, E_2[p^\infty])} \quad (11)$$

Since,  $E_1$  and  $E_2$  are non-isotrivial, hence we get that  $E_1(K_\infty)[p^\infty]$  and  $E_2(K_\infty)[p^\infty]$  are finite (Remark 2.13). Also note that  $A$  is finite. Therefore,  $\chi(\Gamma, E_1[p^\infty](K_\infty)) = \chi(\Gamma, E_2[p^\infty](K_\infty)) = \chi(\Gamma, H_{fl}^0(K_\infty, A)) = 1$ .

Using the Hochschild-Serre spectral sequence ([Mil80]) and its connection with the Euler characteristic, as mentioned in Remark 2.20, in equation (11), we derive our required result.  $\square$

**Remark 5.2.** Let  $E/K$  be a non-isotrivial elliptic curve and  $q = p^r$ , for some  $r > 0$ . Consider the elliptic curve  $E^{(q)}$ , which is obtained by raising the coefficients in Weierstrass equation of  $E$  by a power of  $q$ . Clearly,  $E^{(q)}$  is also non-isotrivial.

Let us call  $E$  as  $E_1$  and  $E^{(q)}$  as  $E_2$ . The obvious isogeny map  $\varphi$  between  $E_1$  and  $E_2$  is the Frobenius isogeny,  $\text{Fr}_q$ . This is a purely inseparable isogeny. As we know that the order of the kernel of isogeny equals its separable degree, we get that  $\mu(S(E_2/K_\infty)^\vee) = \mu(S(E_1/K_\infty)^\vee)$ .

## 6. VANISHING OF $\mu_G(S(E/K_\infty)^\vee)$ IN THE $\ell \neq p$ CASE

Let  $p$  be an odd prime and  $\ell \geq 5$  be a prime distinct from  $p$ . Set  $K = \mathbb{F}(t)$ , where  $\mathbb{F}$  is a field of char  $p$  as defined in §2.8. Let  $E$  be a non-isotrivial elliptic curve over the function field  $K$ . Let  $K_\infty$  be the trivialising extension of  $K$  (defined in §2.8). As a consequence of Weil pairing, we get that  $K_{\text{cyc}} \subset K_\infty$ . Denote by  $G = \text{Gal}(K_\infty/K)$  and  $H = \text{Gal}(K_\infty/K_{\text{cyc}})$ .

The proof of the following result is similar to that of [CSS03a, Lemma 2.5]. We recall the key steps below:

**Proposition 6.1.** *Let  $G, H$  be as above, and  $E/K$  be a non-isotrivial elliptic curve. Then  $H^i(H, S(E/K_\infty)) = 0$ , for  $i \geq 1$ .*

*Proof.* Recall that the maps

$$\lambda(\mathcal{K}) : H^1(G_S(\mathcal{K}), E[\ell^\infty]) \longrightarrow \prod_{v \in S} J_v^1(E/\mathcal{K})$$

are surjective for  $\mathcal{K} = K_{\text{cyc}}$  or  $\mathcal{K} = K_\infty$  (see [Pal14, proof of Theorem 4.4]). Now, by arguments similar to the proof of [CSS03a, Lemma 2.3], we obtain the following exact sequence:

$$0 \longrightarrow S(E/K_\infty)^H \longrightarrow H^1(G_S(K_\infty), E[\ell^\infty])^H \longrightarrow \prod_{v \in S} J_v^1(E/K_\infty)^H \longrightarrow 0 \quad (12)$$

Then, by using Hochschild-Serre spectral sequence and the finiteness of  $H^3(H, E[\ell^\infty])$ , which follows from [Sec07, Chapter-2, Remark 3.14], we obtain as in [CSS03a, Lemma 2.4] that  $H^i(H, H^1(G_S(F_\infty), E[\ell^\infty])) = 0$  for all  $i \geq 1$ . By [Sec07, Chapter-1, Lemma III.7], we obtain that  $H^i(H, J_v^1(E/K_\infty)) = 0$ . The result now easily follows by taking the  $H$ -cohomology of the short exact sequence

$$0 \longrightarrow S(E/K_\infty) \longrightarrow H^1(G_S(K_\infty), E[\ell^\infty]) \longrightarrow \prod_{v \in S} J_v^1(E/K_\infty) \longrightarrow 0,$$

and considering the corresponding long exact sequence.  $\square$

**Proposition 6.2.** *Let  $E/K$  be a non-isotrivial elliptic curve and  $G = \text{Gal}(K_\infty/K)$  a pro- $p$  group. Then,  $\mu_G(S(E/K_\infty)^\vee) = \mu_\Gamma(S(E/K_{\text{cyc}})^\vee)$ .*

*Proof.* By [Pal14, Theorem 4.4], we obtain that the kernel of the map  $S(E/K_{\text{cyc}}) \xrightarrow{f} S(E/K_\infty)^H$  is finite and the cokernel of  $f$  is a co-finitely generated  $\mathbb{Z}_\ell$ -module. Furthermore, we know from [Wit20, Corollary 4.38] that  $S(E/K_{\text{cyc}})^\vee$  is a finitely generated  $\mathbb{Z}_\ell$ -module. Then, it follows from Nakayama lemma that  $S(E/K_\infty)^\vee$  is finitely generated over  $\Lambda(H)$ . The proof now follows by using arguments similar to the proof of [CSS03b, Proposition 2.13].  $\square$

As discussed in [GJS22, §2.1], we know that  $S(E/K_{\text{cyc}})^\vee$  is a finitely generated  $\mathbb{Z}_\ell$ -module. Therefore, as an immediate consequence of Proposition 6.2, we obtain the following theorem:

**Theorem 6.3.** *Let  $K_{\text{cyc}}$  be the unique  $\mathbb{Z}_\ell$ -extension of  $K$  and  $K_\infty$  be the trivialising extension of  $K$ , such that  $G = \text{Gal}(K_\infty/K)$  is pro- $p$ . Assume  $E$  to*



be a non-isotrivial elliptic curve over the function field  $K$ . Then,

$$\mu_G(S(E/K_\infty)^\vee) = \mu_\Gamma(S(E/K_{\text{cyc}})^\vee) = 0.$$

#### APPENDIX A. BEHAVIOUR OF $\mu$ -INVARIANTS UNDER ISOGENY ( $\ell \neq p$ CASE)

We choose and fix a rational prime  $\ell$  distinct from  $p$ . Set  $K = \mathbb{F}(t)$ , where  $\mathbb{F}$  is a field of char  $p$  as defined in §2.8. Now, let  $E_1$  and  $E_2$  are two non-isotrivial elliptic curves over the function field  $K$ . Let  $\varphi : E_1 \rightarrow E_2$  be an isogeny of any degree. Let  $\mathcal{K}$  be either the unramified  $\mathbb{Z}_\ell$ -extension  $K_{\text{cyc}}$  or the trivialising extension  $K_\infty$ .

In this appendix, although we don't get any new result, we show that the arguments given in the proof of Theorem 5.1 in the  $\ell = p$  case also carries over in the  $\ell \neq p$  case. Infact in this case we don't get any "extra" factor of the Euler-characteristic and obtain an equality of the  $\mu$ -invariants of the (fine) Selmer groups under isogeny, and thus confirming the validity of Theorem 1.5.

Consider the following commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & S(E_2/\mathcal{K}) & \longrightarrow & H^1(G_S(\mathcal{K}), E_2[\ell^\infty]) & \longrightarrow & \prod_{v \in S} J_v^1(E_2/\mathcal{K}) \longrightarrow 0 \\ & & \uparrow g_1 & & \uparrow g_2 & & \uparrow g_3 = \prod_{v \in S} g_{3,v} \\ 0 & \longrightarrow & S(E_1/\mathcal{K}) & \longrightarrow & H^1(G_S(\mathcal{K}), E_1[\ell^\infty]) & \longrightarrow & \prod_{v \in S} J_v^1(E_1/\mathcal{K}) \longrightarrow 0 \end{array} \quad (13)$$

Again, our goal is to consider the change in the  $\mu$ -invariant of  $\ell^\infty$ -Selmer groups of  $E_1$  and  $E_2$  over  $K_{\text{cyc}}$  and  $K_\infty$ .

**Theorem A.1.** *Consider  $\mathcal{K} = K_{\text{cyc}}$  or  $K_\infty$ . Assume  $E_1$  and  $E_2$  are two non-isotrivial elliptic curves over the function field  $K$ , and  $\varphi : E_1 \rightarrow E_2$  is an isogeny. Also, suppose that  $S(E_i/\mathcal{K})^\vee$  is a torsion  $\Lambda(G)$ -module for  $i = 1, 2$ . Then,*

$$\mu(S(E_1/\mathcal{K})^\vee) = \mu(S(E_2/\mathcal{K})^\vee).$$

*Proof.* Let  $G = \text{Gal}(\mathcal{K}/K)$ . We start by calculating  $\mu(\ker(f_1)^\vee) - \mu(\text{coker}(f_1)^\vee)$ . Note that by our assumption that  $S(E_i/\mathcal{K})^\vee$  is torsion  $\Lambda(G)$ -module for  $i = 1, 2$ , we get by Proposition 2.18 that

$$\mu(\ker(f_1)^\vee) - \mu(\text{coker}(f_1)^\vee) = \mu(S(E_1/\mathcal{K})^\vee) - \mu(S(E_2/\mathcal{K})^\vee)$$

Recall that for a finitely generated  $\Lambda$  module  $M$ ,  $\mu(M) = \text{ord}_p(\chi(\Gamma, M[p^\infty]^\vee))$  (Corollary 2.17). Now, suppose  $\ker(g_1)$  and  $\text{coker}(g_1)$  are both finite and therefore annihilated by a power of  $p$ . Then, it suffices to calculate the logarithm to the  $p$ -base of  $\frac{\chi(G, \text{coker } g_1)}{\chi(G, \ker g_1)}$ .

By applying snake lemma to equation (13), we obtain the following exact sequence:

$$0 \longrightarrow \ker(g_1) \longrightarrow \ker(g_2) \longrightarrow \ker(g_3) \longrightarrow \text{coker}(g_1) \longrightarrow \text{coker}(g_2) \longrightarrow \text{coker}(g_3) \longrightarrow 0.$$

Now, we claim that for  $i = 2, 3$ ,  $\ker(g_i)$  and  $\text{coker}(g_i)$  are finite, which shows that  $\ker(g_1)$  and  $\text{coker}(g_1)$  are finite. We establish the finiteness of  $\ker(g_1)$  and  $\text{coker}(g_1)$  and calculate  $\frac{\chi(G, \text{coker } g_1)}{\chi(G, \ker g_1)}$  separately for  $K_{\text{cyc}}$  and  $K_\infty$ . Note that the equation (A) also implies that  $\frac{\chi(G, \text{coker } g_1)}{\chi(G, \ker g_1)} = \frac{\chi(G, \text{coker } g_2)}{\chi(G, \ker g_2)} \times \frac{\chi(G, \ker g_3)}{\chi(G, \text{coker } g_3)}$ .

Case 1: Take  $\mathcal{K} = K_{\text{cyc}}$ . Then,  $G = \Gamma = \text{Gal}(K_{\text{cyc}}/K)$ . Let  $A = \ker(\varphi(\ell))$ , the  $\ell$ -primary part of  $\ker(\varphi)$ . From the long exact sequence corresponding to the short exact sequence  $0 \longrightarrow A \longrightarrow E_1[\ell^\infty] \longrightarrow E_2[\ell^\infty] \longrightarrow 0$  and using the fact  $H^2(K_{\text{cyc}}, E_2[\ell^\infty]) = 0$  ([GJS22, Proposition 2.10]), we get that

$$\text{coker}(g_2) = H^1(G_S(K), A) \text{ and } \ker(g_2) = \frac{H^1(G_S(K), A)}{\left( \frac{E_2[\ell^\infty](K_{\text{cyc}})}{(E_1[\ell^\infty](K_{\text{cyc}})/H^0(G_S(K), A))} \right)}.$$

Since,  $E_1$  and  $E_2$  are non-trivial, hence we get that  $E_1[\ell^\infty](K_{\text{cyc}})$  and  $E_2[\ell^\infty](K_{\text{cyc}})$  are finite ([BL09b, Lemma 3.2]). Hence,  $\ker(g_2)$  and  $\text{coker}(g_2)$  are finite.

As  $K_{\text{cyc}}$  is unramified at every prime  $v$ , we obtain that for any prime  $v$  and  $w \mid v$ ,  $\text{Gal}(\overline{K}_{\text{cyc},w}/K_{\text{cyc},w}) \cong \prod_{r \neq \ell} \mathbb{Z}_r$ . Therefore,  $\ker g_3$  and  $\text{coker } g_3$  are trivial.

As  $\ker g_i$  and  $\text{coker } g_i$  are finite for  $i = 1, 2$ , we get that  $\chi(\Gamma, \ker g_j)$  and  $\chi(\Gamma, \text{coker } g_j)$  exists for  $j = 1, 2, 3$ . Therefore,

$$\frac{\chi(\Gamma, \text{coker}(g_2))}{\chi(\Gamma, \ker(g_2))} = \frac{\chi(\Gamma, H^2(G_S(K_\infty), A)) \times \chi(\Gamma, E_1[\ell^\infty](K_\infty)) \times \chi(\Gamma, H^0(G_S(K_\infty), A))}{\chi(\Gamma, H^1(G_S(K_\infty), A)) \times \chi(\Gamma, E_2[\ell^\infty])} \quad (14)$$

As  $E_1[\ell^\infty](K_\infty)$  and  $E_2[\ell^\infty](K_\infty)$  and  $A$  are all finite. Therefore,  $\chi(\Gamma, E_1[\ell^\infty](K_{\text{cyc}})) = \chi(\Gamma, E_2[\ell^\infty](K_{\text{cyc}})) = \chi(\Gamma, H^0(G_S(K_{\text{cyc}}), A)) = 1$ .

Now, using the Hochschild-Serre spectral sequence,

$$H^i(\Gamma, H^j(G_S(K_{\text{cyc}}), A)) \implies H^{i+j}(G_S(K), A),$$

in equation (14). Therefore,

$$\frac{\chi(\Gamma, \text{coker}(f_2))}{\chi(\Gamma, \ker(f_2))} = \chi(\text{Gal}(K_S/K), A).$$

By [Mil86, Remark I.5.2], we get that  $\chi(\text{Gal}(K_S/K), A) = 1$ . This concludes Case 1.

Case 2: Take  $\mathcal{K} = K_\infty$  and  $G = G_\infty = \text{Gal}(K_\infty/K)$ . Again, let  $A = \ker(\varphi(\ell))$ , the  $\ell$ -primary part of  $\ker(\varphi)$ . Similar to Case 1, we obtain from the long exact sequence corresponding to the short exact sequence  $0 \rightarrow A \rightarrow E_1[\ell^\infty] \rightarrow E_2[\ell^\infty] \rightarrow 0$  and  $H^2(K_\infty, E_2[\ell^\infty]) = 0$  ([BV14, Proposition 4.5]) that

$$\text{coker}(g_2) = H^1(G_S(K), A) \text{ and } \ker(g_2) = H^0(G_S(K), A).$$

Let  $S_{\text{un}} \subset S$  be the set of primes of  $K$  is  $S$  that are unramified in  $K_\infty$ . For  $v \in S_{\text{un}}$ ,  $\ker(g_{3,v}) = 0$ , by arguments similar to Case 1. And for  $v \in S \setminus S_{\text{un}}$ ,  $\ker(g_{3,v}) = \prod_{w|v} H^1(K_{\infty,w}, A)$  and  $\text{coker}(g_{3,v}) \subset \prod_{w|v} H^2(K_{\infty,w}, A)$ . As  $cd_\ell(K_{\infty,w}) = 1$  ([NSW13, Theorem 7.1.8]), we get that  $\text{coker}(g_{3,v}) = 0$ . Therefore,  $\ker(g_3) = \prod_{v \in S \setminus S_{\text{un}}} \ker(g_{3,v})$

and  $\text{coker}(g_3) = 0$ .

For each prime  $v \in S$  we fix a choice of prime  $K_\infty$  lying above, which we will also denote by  $v$ . It follows from Shapiro's lemma that

$$H^j(G_\infty, \ker(g_{3,v})) \cong H^j(G_{\infty,v}, H^i(K_{\infty,v}, A)).$$

Thus,  $\chi(G_\infty, \ker(\varphi_3)) = \prod_{v \in S \setminus S_{\text{un}}} \chi(G_{\infty,v}, H^1(K_{\infty,v}, A))$ .

Again, using the Hochschild-Serre spectral sequence, we obtain

$$H^i(G_{\infty,v}, H^j(K_{\infty,v}, A)) \implies H^{i+j}(K_v, A)$$

Thus,

$$\frac{\chi(G_\infty, \ker(\varphi_3))}{\chi(G_\infty, \text{coker}(\varphi_3))} = \frac{1}{\prod_{v \in S \setminus S_{\text{un}}} \chi(\text{Gal}(\overline{K}_v/K_v), A)}.$$

By [Mil86, Chapter 1, Lemma 2.9], we know that  $\chi(\text{Gal}(\overline{K}_v/K_v), A) = 1$ . Moreover, using calculations similar to Case 1, we obtain that  $\frac{\chi(G_\infty, \text{coker}(g_2))}{\chi(G_\infty, \ker(g_2))} = 1$ .

Hence, in both cases, we obtain that  $\frac{\chi(G, \text{coker } g_1)}{\chi(G, \ker g_1)} = 1$ , for  $G = G_\infty$  or  $\Gamma$ . This concludes the proof of the theorem.  $\square$

**Remark A.2.** (i) For an arbitrary abelian variety  $A$ ,  $S(A/K_{\text{cyc}})^\vee$  is a finitely generated torsion  $\Lambda$ -module [Pal14, Theorem 3.12].

(ii) By [Wit20, Corollary 4.38],  $S(E/K_{\text{cyc}})^\vee$  is a finitely generated  $\mathbb{Z}_\ell$ -modules, then by [Pal14, Theorem 4.4]  $S(E/K_\infty)^\vee$  is a torsion  $\Lambda(G)$ -module.

## REFERENCES

- [BL09a] A. Bandini and I. Longhi. Control theorems for elliptic curves over function fields. *Int. J. Number Theory*, 5(2):229–256, 2009.
- [BL09b] Andrea Bandini and Ignazio Longhi. Selmer groups for elliptic curves in  $\mathbb{Z}_\ell^d$ -extensions of function fields of characteristic  $p$ . *Ann. Inst. Fourier (Grenoble)*, 59(6):2301–2327, 2009.

- [BV14] Andrea Bandini and Maria Valentino. On Selmer groups of abelian varieties over  $\ell$ -adic Lie extensions of global function fields. *Bull. Braz. Math. Soc. (N.S.)*, 45(3):575–595, 2014.
- [BV18] Andrea Bandini and Maria Valentino. Euler characteristic and Akashi series for Selmer groups over global function fields. *J. Number Theory*, 193:213–234, 2018.
- [CSS03a] John Coates, Peter Schneider, and Ramdorai Sujatha. Links between cyclotomic and  $\mathrm{GL}_2$  Iwasawa theory. *Doc. Math.*, (Extra Vol.):187–215, 2003. Kazuya Kato’s fiftieth birthday.
- [CSS03b] John Coates, Peter Schneider, and Ramdorai Sujatha. Links between cyclotomic and  $\mathrm{GL}_2$  Iwasawa theory. *Doc. Math.*, (Extra Vol.):187–215 (electronic), 2003. Kazuya Kato’s fiftieth birthday.
- [GJS22] Sohan Ghosh, Somnath Jha, and Sudhanshu Shekhar. Iwasawa theory of fine selmer groups over global fields, 2022.
- [Gre01] Ralph Greenberg. Introduction to Iwasawa theory for elliptic curves. In *Arithmetic algebraic geometry (Park City, UT, 1999)*, volume 9 of *IAS/Park City Math. Ser.*, pages 407–464. Amer. Math. Soc., Providence, RI, 2001.
- [GU20] Richard Griffon and Douglas Ulmer. On the arithmetic of a family of twisted constant elliptic curves. *Pacific J. Math.*, 305(2):597–640, 2020.
- [HKLR23] Jeffrey Hatley, Debanjana Kundu, Antonio Lei, and Jishnu Ray. Control theorems for fine Selmer groups, and duality of fine Selmer groups attached to modular forms. *Ramanujan J.*, 60(1):237–258, 2023.
- [How02] Susan Howson. Euler characteristics as invariants of Iwasawa modules. *Proc. London Math. Soc. (3)*, 85(3):634–658, 2002.
- [HYZ23] Paul Hamacher, Ziquan Yang, and Xiaolei Zhao. Finiteness of the tate-shafarevich group for some elliptic curves of analytic rank  $> 1$ , 2023.
- [KP07] Masato Kurihara and Robert Pollack. Two  $p$ -adic  $L$ -functions and rational points on elliptic curves with supersingular reduction. *London Mathematical Society Lecture Note Series*, 320:300, 2007.
- [KT03] Kazuya Kato and Fabien Trihan. On the conjectures of Birch and Swinnerton-Dyer in characteristic  $p > 0$ . *Invent. Math.*, 153(3):537–592, 2003.
- [Lee20] Jaehoon Lee. Structure of the Mordell-Weil group over the  $\mathbb{Z}_p$ -extensions. *Trans. Amer. Math. Soc.*, 373(4):2399–2425, 2020.
- [Lei23] Antonio Lei. Algebraic structure and characteristic ideals of fine Mordell-Weil groups and plus/minus Mordell-Weil groups. *Math. Z.*, 303(1):Paper No. 14, 17, 2023.
- [LLS<sup>+</sup>21] King-Fai Lai, Ignazio Longhi, Takashi Suzuki, Ki-Seng Tan, and Fabien Trihan. On the  $\mu$ -invariants of abelian varieties over function fields of positive characteristic. *Algebra Number Theory*, 15(4):863–907, 2021.
- [LLTT16] King Fai Lai, Ignazio Longhi, Ki-Seng Tan, and Fabien Trihan. The Iwasawa main conjecture for semistable abelian varieties over function fields. *Math. Z.*, 282(1-2):485–510, 2016.
- [Mil80] James S. Milne. *Étale cohomology*, volume No. 33 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1980.
- [Mil86] James S. Milne. *Arithmetic duality theorems*, volume 1 of *Perspectives in Mathematics*. Academic Press, Inc., Boston, MA, 1986.

- [NSW13] Jürgen Neukirch, Alexander Schmidt, and Kay Wingberg. *Cohomology of number fields*, volume 323. Springer Science & Business Media, 2013.
- [OT09] Tadashi Ochiai and Fabien Trihan. On the Selmer groups of abelian varieties over function fields of characteristic  $p > 0$ . *Math. Proc. Cambridge Philos. Soc.*, 146(1):23–43, 2009.
- [Pal14] Aprameyo Pal. Functional equation of characteristic elements of abelian varieties over function fields ( $\ell \neq p$ ). *Int. J. Number Theory*, 10(3):705–735, 2014.
- [Ros02] Michael Rosen. *Number theory in function fields*, volume 210 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2002.
- [Sec07] Gianluigi Sechi. *Iwasawa theory over function field*. PhD thesis, Ph. D. thesis, University of Cambridge, 2007.
- [Sil09] Joseph H Silverman. *The arithmetic of elliptic curves*, volume 106. Springer, 2009.
- [Tan10] Ki-Seng Tan. A generalized Mazur’s theorem and its applications. *Trans. Amer. Math. Soc.*, 362(8):4433–4450, 2010.
- [TW11] Fabien Trihan and Christian Wuthrich. Parity conjectures for elliptic curves over global fields of positive characteristic. *Compos. Math.*, 147(4):1105–1128, 2011.
- [Ulm11] Douglas Ulmer. Elliptic curves over function fields. In *Arithmetic of L-functions*, volume 18 of *IAS/Park City Math. Ser.*, pages 211–280. Amer. Math. Soc., Providence, RI, 2011.
- [Wit20] Malte Witte. Non-commutative Iwasawa main conjecture. *Int. J. Number Theory*, 16(9):2041–2094, 2020.
- [Wut07] Christian Wuthrich. The fine Tate–Shafarevich group. In *Mathematical Proceedings of the Cambridge Philosophical Society*, volume 142, pages 1–12. Cambridge University Press, 2007.

(Ghosh) HARISH CHANDRA RESEARCH INSTITUTE, A CI OF HOMI BHABHA NATIONAL INSTITUTE, CHHATNAG ROAD, JHUNSI, PRAYAGRAJ (ALLAHABAD) 211 019 INDIA  
*Email address:* ghoshsohan4@gmail.com

(Ray) HARISH CHANDRA RESEARCH INSTITUTE, A CI OF HOMI BHABHA NATIONAL INSTITUTE, CHHATNAG ROAD, JHUNSI, PRAYAGRAJ (ALLAHABAD) 211 019 INDIA  
*Email address:* jishnuray@hri.res.in; jishnuray1992@gmail.com