

FIELDS OF DEFINITION OF DYNAMICAL SYSTEMS ON \mathbb{P}^1 . IMPROVEMENTS ON A RESULT OF SILVERMAN

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ABSTRACT. J. Silverman proved that a dynamical system on \mathbb{P}^1 descends to the field of moduli if it is polynomial or it has even degree, but for non-polynomial ones of odd degree the picture is less clear. We give a complete characterization of which dynamical systems over \mathbb{P}^1 descend to the field of moduli.

We work over a perfect field k with algebraic closure K .

Consider $\phi : \mathbb{P}_K^1 \rightarrow \mathbb{P}_K^1$ a finite map of degree $d \geq 2$. The interplay between the number-theoretic and dynamical properties of ϕ has been the subject of a lot of research in the last decades, see e.g. [Sil07].

The dynamics of ϕ do not change if we conjugate it by a projective linear transformation $f \in \mathrm{PGL}_2(K)$. Define $\phi^f \stackrel{\mathrm{def}}{=} f^{-1} \circ \phi \circ f$; this gives an action of $\mathrm{PGL}_2(K)$ on the set of rational functions. A dynamical system ξ on \mathbb{P}_K^1 is an equivalence class for this action.

The Galois group $\mathrm{Gal}(K/k)$ acts on the set of dynamical systems by acting on the coefficients of the corresponding rational functions in $K(t)$. The *field of moduli* k_ξ is the subfield of K fixed by the elements of $\mathrm{Gal}(K/k)$ mapping ξ to itself. If ξ descends to a dynamical system on a subextension $k' \subset K$, then clearly $k_\xi \subset k'$. It is then natural to ask whether ξ descends to a dynamical system on k_ξ .

Recall that the stabilizer $\mathcal{A}_\phi \subset \mathrm{PGL}_2(K)$ of ϕ is the subgroup $\{f \in \mathrm{PGL}_2(K) \mid \phi^f = \phi\}$; its isomorphism class only depends on ξ , so that \mathcal{A}_ξ is well defined as an abstract group, and it is finite [Sil95, Proposition 4.1]. J. Silverman proved the following.

Theorem 1 ([Sil95, Theorem 5.1]). *Let ξ be a dynamical system on \mathbb{P}_K^1 , assume that $\mathrm{char} k$ does not divide $|\mathcal{A}_\xi|$. If $\deg \xi$ is even or if ξ contains a polynomial, then ξ descends to a dynamical system on $\mathbb{P}_{k_\xi}^1$.*

Furthermore, Silverman proved that ξ descends to a Brauer–Severi curve if the stabilizer \mathcal{A}_ξ is trivial [Sil95, Theorem 2.1]. We prove that every dynamical system with non-trivial stabilizer descends to $\mathbb{P}_{k_\xi}^1$. This is somewhat counter-intuitive, since the case of non-trivial stabilizer is more complex.

We thus get a complete characterization of descent properties for dynamical systems on \mathbb{P}_K^1 .

Theorem 2. *Every dynamical system $\xi = [\phi]$ on \mathbb{P}_K^1 descends to a dynamical system on some Brauer–Severi curve defined over the field of moduli k_ξ . Furthermore, the following are equivalent.*

- *The dynamical system ξ does not descend to a dynamical system over $\mathbb{P}_{k_\xi}^1$.*

- The stabilizer \mathcal{A}_ξ is trivial, and there is a commutative diagram

$$\begin{array}{ccc} \mathbb{P}_K^1 & \xrightarrow{\phi} & \mathbb{P}_K^1 \\ \downarrow j & & \downarrow j \\ B & \longrightarrow & B \end{array}$$

where B is a Brauer–Severi curve over a subextension $K/k'/k$ with $B(k') = \emptyset$, and $j_K : \mathbb{P}_K^1 \rightarrow B_K$ is an isomorphism.

R. Hidalgo proved that, in characteristic 0, every dynamical system ξ descends to a dynamical system on $\mathbb{P}_{k'}^1$, where k'/k_ξ is an extension of degree at most 2 [Hid14]. As a consequence of Theorem 2, Hidalgo’s theorem holds in any characteristic, since every dynamical system descends to a Brauer–Severi curve over k_ξ and every Brauer–Severi curve is split by an extension of degree at most 2.

Theorem 2 implies the following generalization of Silverman’s result.

Theorem 3. *A dynamical system $\xi = [\phi]$ on \mathbb{P}_K^1 descends to a dynamical system on $\mathbb{P}_{k_\xi}^1$ if any of the following conditions holds.*

- The stabilizer \mathcal{A}_ξ is non-trivial.
- The degree d is even.
- There exists an integer $e \geq 2$ such that the number of points with ramification index e is odd.

If ξ contains a polynomial, either ξ contains x^d (which is clearly defined over the field of moduli) or there is a unique K -point with ramification index d , so that ξ satisfies the third condition of Theorem 3.

We mention that analogous problems for dynamical systems on \mathbb{P}^n are studied in [HM14, DS19, Breb]. Furthermore, recently F. Veneziano and S. Vishkautsan have studied similar problems for iterates of a dynamical system on \mathbb{P}^1 [VV24].

A remark about Silverman’s results. Silverman does not assume the base field to be perfect, so our results seem to be less general in this regard. However, if we drop this hypothesis there is a technical flaw in Silverman’s paper. Let k be an arbitrary field, \bar{K} an algebraic closure, $K \subset \bar{K}$ the separable closure, $k^p \subset \bar{K}$ the perfect closure. There is a natural identification of Galois groups $\text{Gal}(K/k) = \text{Gal}(\bar{K}/k^p)$, call them G_k .

In [Sil95, Proposition 1.2], it is stated that $H^1(G_k, \text{PGL}_2(\bar{K}))$ classifies Brauer–Severi curves over k . However, the proof uses a reference [Sil09, §X, Theorem 2.2] which assumes the base field to be perfect. In general, Brauer–Severi curves over k are classified by $H^1(G_k, \text{PGL}_2(K))$ [GS17, Theorem 5.2.1], while $H^1(G_k, \text{PGL}_2(\bar{K}))$ classifies Brauer–Severi curves over k^p .

If $\text{char } k \neq 2$, the natural map

$$H^1(G_k, \text{PGL}_2(K)) = \text{Br}(k)[2] \rightarrow H^1(G_k, \text{PGL}_2(\bar{K})) = \text{Br}(k^p)[2]$$

is an isomorphism, since $\text{Br}(k) \rightarrow \text{Br}(k^p)$ is surjective [Jac96, Theorem 4.1.5] with $\text{char } k$ -primary kernel. Still, Silverman’s article remains flawed even if we assume $\text{char } k \neq 2$. In fact, in the proof of [Sil95, Theorem 2.1(b)], the morphism $\Phi = jg\phi g^{-1}j^{-1}$ is defined over \bar{K} (because g is defined over \bar{k}) as opposed to K : in order to verify that Φ is defined over k , Silverman checks that Φ is Galois-invariant, but since Φ is a priori defined over \bar{K} this only proves that Φ is defined over k^p .

Finally, let us mention that there is a way to study the problem even when the base field is not perfect. This is done by replacing Galois cohomology either with fppf cohomology or stack theory, such as in [BV24]. We don't know whether all the various results remain completely true in this generality. Still, most of their content remains true, see §4.

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1. PRELIMINARIES

A *model* of ϕ over a subfield k' is a dynamical system $b : B \rightarrow B$ on a Brauer–Severi curve B/k' with an isomorphism $\mathbb{P}_K^1 \xrightarrow{\sim} B_K$ such that the diagram

$$\begin{array}{ccc} \mathbb{P}_K^1 & \xrightarrow{\phi} & \mathbb{P}_K^1 \\ \downarrow & & \downarrow \\ B & \xrightarrow{b} & B \end{array}$$

commutes.

Let \mathcal{N}_ϕ be the group of k -linear isomorphisms $f : \mathbb{P}_K^1 \rightarrow \mathbb{P}_K^1$ such that $f \circ \phi = \phi \circ f$. An element of \mathcal{N}_ϕ induces an homomorphism $\Gamma(\mathcal{O}_{\mathbb{P}_K^1}) = K \rightarrow \Gamma(\mathcal{O}_{\mathbb{P}_K^1}) = K$, hence we get a short exact sequence

$$1 \rightarrow \mathcal{A}_\phi \rightarrow \mathcal{N}_\phi \rightarrow \mathrm{Gal}(K/k).$$

Lemma 4. *The image of $\mathcal{N}_\phi \rightarrow \mathrm{Gal}(K/k)$ is $\mathrm{Gal}(K/k_\xi)$, hence we have a short exact sequence*

$$1 \rightarrow \mathcal{A}_\phi \rightarrow \mathcal{N}_\phi \rightarrow \mathrm{Gal}(K/k_\xi) \rightarrow 1.$$

Proof. Consider $\phi \in K(t)$ as a rational function. An element $\sigma \in \mathrm{Gal}(K/k)$ acts on the coefficients of ϕ , write $\phi^\sigma \in K(t)$ for the resulting rational function. If $\tau_\sigma : \mathbb{P}_K^1 \rightarrow \mathbb{P}_K^1$ is the standard Galois action of σ on \mathbb{P}_K^1 , then we have $\phi^\sigma = \tau_\sigma \circ \phi \circ \tau_\sigma^{-1}$.

If σ is the image of $f \in \mathcal{N}_\phi$, then

$$\phi^\sigma = \tau_\sigma \circ \phi \circ \tau_\sigma^{-1} = (\tau_\sigma \circ f^{-1}) \circ \phi \circ (\tau_\sigma \circ f^{-1})^{-1}.$$

Since $\tau_\sigma \circ f^{-1} \in \mathrm{PGL}_2(K)$ is K -linear, this implies that $\sigma \in \mathrm{Gal}(K/k_\xi)$.

On the other hand, if $\sigma \in \mathrm{Gal}(K/k_\xi)$, then there exists $f \in \mathrm{PGL}_2(K)$ such that $\phi^\sigma = f \circ \phi \circ f^{-1}$. Since $\phi^\sigma = \tau_\sigma \circ \phi \circ \tau_\sigma^{-1}$, this implies that $\tau_\sigma \circ f^{-1}$ is an element of \mathcal{N}_ϕ mapping to σ . \square

Recall that, since $d \geq 2$, the stabilizer \mathcal{A}_ϕ is finite [Sil95, Proposition 4.1], hence \mathcal{N}_ϕ is pro-finite. There are natural actions of \mathcal{N}_ϕ on \mathbb{P}_K^1 and $\mathrm{Spec} K$; denote by $\mathcal{P}_\xi^D, \mathcal{G}_\xi^D$ the quotient stacks $[\mathbb{P}_K^1/\mathcal{N}_\phi], [\mathrm{Spec} K/\mathcal{N}_\phi]$.

If the reader finds quotients by pro-finite groups uncomfortable, notice that we can find a large finite Galois extension k'/k where everything is defined: we can then proceed to repeat the above with k' instead of K , so that all the groups involved are finite. In particular, this shows that \mathcal{P}_ξ^D and \mathcal{G}_ξ^D are Deligne–Mumford stacks over k_ξ . Furthermore, \mathcal{G}_ξ^D is gerbe over k_ξ , since k_ξ is perfect and \mathcal{N}_ϕ maps surjectively on $\mathrm{Gal}(K/k_\xi)$ by Lemma 4.

We use the superscript \bullet^D to distinguish the Deligne–Mumford stacks \mathcal{P}_ξ^D and \mathcal{G}_ξ^D from their non-Deligne–Mumford siblings \mathcal{P}_ξ and \mathcal{G}_ξ that will appear in §4, and which will be used to study the problem over non-perfect fields.

The morphisms $\mathbb{P}_K^1 \rightarrow \operatorname{Spec} K$, $\phi : \mathbb{P}_K^1 \rightarrow \mathbb{P}_K^1$ induce morphisms $\pi : \mathcal{P}_\xi^D \rightarrow \mathcal{G}_\xi^D$, $\varphi : \mathcal{P}_\xi^D \rightarrow \mathcal{P}_\xi^D$ with a commutative diagram

$$\begin{array}{ccc} \mathcal{P}_\xi^D & \xrightarrow{\varphi} & \mathcal{P}_\xi^D \\ & \searrow \pi & \swarrow \pi \\ & \mathcal{G}_\xi^D & \end{array}$$

The restriction of $\varphi : \mathcal{P}_\xi^D \rightarrow \mathcal{P}_\xi^D$ to the tautological section $\operatorname{Spec} K \rightarrow \mathcal{G}_\xi^D = [\operatorname{Spec} K / \mathcal{N}_\phi]$ identifies canonically with $\phi : \mathbb{P}_K^1 \rightarrow \mathbb{P}_K^1$. Since \mathcal{G}_ξ^D is a gerbe over k_ξ , given any section $s : \operatorname{Spec} k_\xi \rightarrow \mathcal{G}_\xi^D$ the base change s_K is isomorphic to the tautological section and hence the restriction $\varphi|_s : \mathcal{P}_{\xi,s}^D \rightarrow \mathcal{P}_{\xi,s}^D$ is a model of ϕ over k_ξ .

On the other hand, if $b : B \rightarrow B$ is a model of ϕ over k_ξ , the natural Galois action on $B_K \simeq \mathbb{P}_K^1$ defines a section $\operatorname{Gal}(K/k_\xi) \rightarrow \mathcal{N}_\xi$, which in turn induces a section $s : \operatorname{Spec} k \rightarrow \mathcal{G}_\xi^D$ by [Brea, Lemma 5]. It is straightforward to check that these constructions are inverses to each other; we have thus proved the following.

Lemma 5. *The dynamical system ξ descends to a Brauer–Severi curve over k_ξ if and only if $\mathcal{G}_\xi^D(k_\xi) \neq \emptyset$. Furthermore, it descends to a dynamical system on $\mathbb{P}_{k_\xi}^1$ if and only if $\mathcal{P}_\xi^D(k_\xi) \neq \emptyset$. \square*

If k_ξ is finite, then $\mathcal{G}_\xi^D(k_\xi) \neq \emptyset$ by [Bre24, Lemma 13]. Since Brauer–Severi curves over finite fields always have rational points [GS17, Remark 5.1.6], then ξ is defined over $\mathbb{P}_{k_\xi}^1$ by Lemma 5. From now on, we assume that k_ξ is infinite.

Let $\mathcal{P}_\xi^D \rightarrow \mathbf{P}_\xi^D$ the coarse moduli space of \mathcal{P}_ξ^D , it is a Brauer–Severi curve over k_ξ with a natural identification $\mathbf{P}_{\xi,K}^D \simeq \mathbb{P}_K^1 / \mathcal{A}_\phi \simeq \mathbb{P}_K^1$. Since the action of \mathcal{A}_ϕ on \mathbb{P}_K^1 is faithful, the stack \mathcal{P}_ξ^D is generically a scheme, thus we have a birational inverse $\mathbf{P}_\xi^D \dashrightarrow \mathcal{P}_\xi^D$. Since we are assuming that k_ξ is infinite, we get the following corollary of Lemma 5.

Corollary 6. *The dynamical system ξ descends to a dynamical system on $\mathbb{P}_{k_\xi}^1$ if and only if $\mathbf{P}_\xi^D(k_\xi) \neq \emptyset$.*

2. PROOF OF THEOREM 2

By [Sil95, Theorem 2.1], it is enough to prove that ξ descends to $\mathbb{P}_{k_\xi}^1$ if the stabilizer \mathcal{A}_ξ is non-trivial.

So let us assume that \mathcal{A}_ξ is non-trivial; equivalently, $\mathcal{G}_\xi^D \neq \operatorname{Spec} k_\xi$. Assume by contradiction that ξ does not descend to $\mathbb{P}_{k_\xi}^1$; by Corollary 6, this is equivalent to assuming that $\mathbf{P}_\xi^D(k_\xi) = \emptyset$.

The composition $\mathbf{P}_\xi^D \dashrightarrow \mathcal{P}_\xi^D \rightarrow \mathcal{G}_\xi^D$ defines a morphism whose geometric fibers are irreducible curves of genus 0. In fact, if $\operatorname{Spec} K \rightarrow \mathcal{G}_\xi^D = [\operatorname{Spec} K / \mathcal{N}_\xi]$ is the preferred section and we base change to K , we have a commutative diagram

$$\begin{array}{ccc} \mathbb{P}_K^1 & \xrightarrow{\quad} & \operatorname{Spec} K \\ \downarrow & & \downarrow \\ \mathbb{P}_K^1 / \mathcal{A}_\phi = \mathbf{P}_{\xi,K}^D & \dashrightarrow & \mathcal{G}_{\xi,K}^D = [\operatorname{Spec} K / \mathcal{A}_\phi] \end{array}$$

which is cartesian on the locus of points where the action is free. Since $\mathbf{P}_\xi^D(k_\xi) = \emptyset$, by [Bre24, Proposition 28] we get that \mathcal{A}_ϕ is cyclic of order prime with $\text{char } k$.

Choose coordinates on \mathbb{P}_K^1 so that $0, \infty$ are the two fixed points for the action of \mathcal{A}_ξ . Let n be the order of \mathcal{A}_ϕ , then \mathcal{A}_ξ acts on \mathbb{P}_K^1 by $x \mapsto \zeta_n x$, where ζ_n is a primitive n -th root of 1. The fact that \mathcal{A}_ϕ stabilizes ϕ implies that ϕ has the form

$$\phi(x) = \lambda x^a \prod_{i=0}^{n-1} \frac{q_1(\zeta_n^i x)}{q_2(\zeta_n^i x)}$$

for some $\lambda \in K$, $a \in \mathbb{Z}$ and $q_1, q_2 \in K[x]$.

Notice that, since $x \mapsto \zeta_n x$ stabilizes ϕ , we get the equation $\phi(\zeta_n x) = \zeta_n \phi(x)$. Applying this to the formula above, we get

$$\zeta_n = \zeta_n^a,$$

i.e. $a \equiv 1 \pmod{n}$.

Let us now prove that

$$2a + n \deg q_1 - n \deg q_2 = 0.$$

Equivalently, we want to prove that the orders of vanishing of ϕ on 0 and ∞ are equal.

Notice that the subset $\{0, \infty\} \subset \mathbb{P}_K^1$ is distinguished in the sense of [Bre23, §7]: if $h \in \mathcal{N}_\phi$ and $g \in \mathcal{A}_\phi$, then $g(h(0)) = h(h^{-1}gh(0)) = h(0)$ and hence $h(0)$ is a fixed point for \mathcal{A}_ϕ . If the orders of vanishing of ϕ on 0 and ∞ are different, then $\{0\}$ is a distinguished subset as well, hence by [Bre23, Lemma 17] the image of 0 in $\mathbb{P}_K^1/\mathcal{A}_\xi$ descends to a k_ξ -rational point of \mathbf{P}_ξ^D , which is absurd.

Recall that we are assuming that \mathcal{A}_ϕ is non-trivial, or equivalently $n \neq 1$. Since $a \equiv 1 \pmod{n}$ and $2a + n \deg q_1 - n \deg q_2 = 0$, the only possibility is $n = 2$. Hence, we get the equality $a = \deg q_2 - \deg q_1$ and the formula

$$\phi(x) = \lambda x^{\deg q_2 - \deg q_1} \frac{q_1(x)q_1(-x)}{q_2(x)q_2(-x)}.$$

This implies that ϕ has degree equal to $2 \max(\deg q_1, \deg q_2)$. By Silverman's theorem, this implies that ϕ descends to $\mathbb{P}_{k_\xi}^1$, giving the desired contradiction.

3. PROOF OF THEOREM 3

The fact that ξ descends to \mathbb{P}_K^1 under the first condition is a particular case of Theorem 2. If the second condition holds, then this follows by Theorem 2 plus the well-known fact that a finite morphism $B \rightarrow B$ has odd degree if B is a non-trivial Brauer–Severi curve; Silverman gives a proof of this fact in [Sil95, Corollary 2.2].

Assume that there exists an integer $e > 1$ such that the number of geometric points with ramification index e is odd. By Theorem 2, we may assume that \mathcal{A}_ξ is trivial, hence $\mathcal{P}_\xi^D = \mathbf{P}_\xi^D$ is a Brauer–Severi curve over k_ξ and we get a commutative diagram

$$\begin{array}{ccc} \mathbb{P}_K^1 & \xrightarrow{\phi} & \mathbb{P}_K^1 \\ \downarrow & & \downarrow \\ \mathbf{P}_\xi^D & \longrightarrow & \mathbf{P}_\xi^D. \end{array}$$

with $\mathbb{P}_K^1 \rightarrow \mathbf{P}_{\xi, K}^D$ an isomorphism.

Let $S \subset \mathbf{P}_\xi^D$ be the image of the points $p \in \mathbb{P}_K^1(K)$ with ramification index e , considered as a closed scheme with the reduced structure. We have that k_ξ is perfect since the same holds for k ; this implies that the degree of S coincides with the number of geometric points with ramification index e , and hence it is odd. We conclude that $\mathbf{P}_\xi^D(k_\xi) \neq \emptyset$ by [Bre24, Lemma 27].

4. NON-PERFECT BASE FIELD

Let k be any field with algebraic closure \bar{K} , separable closure $K \subset \bar{K}$ and perfect closure $k^p \subset \bar{K}$. A dynamical system is an equivalence class for the action of $\mathrm{PGL}_2(\bar{K})$ on the set of finite morphisms $\mathbb{P}_K^1 \rightarrow \mathbb{P}_K^1$. Fix ξ a dynamical system of degree $d \geq 2$. We have already seen in §1 that if the base field is finite then dynamical systems always descend to the field of moduli. We may then assume that k is infinite.

Definition 7. Define categories fibered in groupoids \mathcal{G}_ξ , \mathcal{P}_ξ as follows.

If S is a scheme over k , then $\mathcal{G}_\xi(S)$ is the groupoid of commutative diagrams

$$\begin{array}{ccc} P & \xrightarrow{\psi} & P \\ & \searrow \pi & \swarrow \pi \\ & S & \end{array}$$

where $\pi : P \rightarrow S$ is a projective bundle of relative dimension 1 such that, for every geometric point $s \in S$, the restriction $\psi|_s : P_s \rightarrow P_s$ is an element of ξ .

We define $\mathcal{P}_\xi(S)$ to be the groupoid of commutative diagrams as above plus a section $S \rightarrow P$. There is an obvious induced commutative diagram

$$\begin{array}{ccc} \mathcal{P}_\xi & \longrightarrow & \mathcal{P}_\xi \\ & \searrow & \swarrow \\ & \mathcal{G}_\xi & \end{array}$$

Arrows in $\mathcal{G}_\xi(S)$ are isomorphisms $P' \simeq P$ over S commuting with the diagrams. Arrows in $\mathcal{P}_\xi(S)$ are isomorphisms $P' \simeq P$ over S commuting with the diagrams and mapping the section $S \rightarrow P'$ to the section $S \rightarrow P$.

Lemma 8. *The fibered categories \mathcal{G}_ξ , \mathcal{P}_ξ are fppf stacks locally of finite presentation.*

Proof. This follows from [Vis05, Theorem 4.38, Example 4.39]. \square

Notice that the Galois groups $\mathrm{Gal}(\bar{K}/k^p)$ and $\mathrm{Gal}(K/k)$ naturally coincide, and that they act on the set of dynamical systems by acting on the coefficients of the corresponding functions.

Definition 9. Denote by $H \subset \mathrm{Gal}(\bar{K}/k^p) = \mathrm{Gal}(K/k)$ the subgroup of elements σ such that $\xi^\sigma = \xi$; it is an open subgroup since ξ contains a function defined over a finite extension of k .

Assume that ξ contains a morphism defined over K . The *field of moduli* of ξ is the fixed field $k_\xi = K^H \subset K$.

Proposition 10. *Assume that ξ contains a morphism defined over K . The stacks \mathcal{G}_ξ , \mathcal{P}_ξ are algebraic with finite inertia, and \mathcal{G}_ξ is a finite gerbe over the field of moduli k_ξ .*

Proof. Let $\phi \in \xi$ be a morphism defined over K . Notice that $\underline{\text{Aut}}_K(\phi) \subset \text{PGL}_2$ is a finite group scheme over K : in fact, it is clearly of finite type, and it has a finite number of geometric points by [Sil95, Proposition 4.1]. The statement then follows from [BV24, Proposition 3.9, Proposition 3.13]. \square

The gerbe \mathcal{G}_ξ is called *residual gerbe* of ξ , and $\mathcal{P}_\xi \rightarrow \mathcal{G}_\xi$ is the *universal family*. Denote by \mathbf{P}_ξ the coarse moduli space of \mathcal{P}_ξ , it is called the *compression* of ξ .

Lemma 11. *The dynamical system ξ descends to $\mathbb{P}_{k_\xi}^1$ if and only if $\mathbf{P}_\xi(k_\xi) \neq \emptyset$. It descends to a Brauer–Severi curve over k_ξ if and only if $\mathcal{G}_\xi(k_\xi) \neq \emptyset$.*

Proof. The second statement follows from the definition of \mathcal{G}_ξ .

Notice that \mathcal{P}_ξ is generically a scheme, since its base change to \bar{K} is the quotient stack $[\mathbb{P}_{\bar{K}}^1 / \underline{\text{Aut}}_K(\phi)]$. Furthermore, \mathbf{P}_ξ is a Brauer–Severi curve over k_ξ , since its base change to \bar{K} is $\mathbb{P}_{\bar{K}}^1 / \underline{\text{Aut}}_K(\phi)$. Since we are assuming that k is infinite, then $\mathcal{P}_\xi(k_\xi)$ is non-empty if and only if $\mathbf{P}_\xi(k_\xi)$ is non-empty. The first statement then follows from the definition of \mathcal{P}_ξ . \square

Denote by $\underline{\mathcal{A}}_\xi$ the automorphism group scheme of any representative of ξ , its isomorphism class is well defined as an abstract finite group scheme over \bar{K} (as opposed to the automorphism group of a fixed representative, which has a natural embedding in PGL_2).

Theorem 12. *Assume that ξ contains a morphism defined over K . Then ξ descends to $\mathbb{P}_{k_\xi}^1$ if any of the following conditions holds.*

- (1) *The group scheme $\underline{\mathcal{A}}_\xi$ is not connected, i.e. $\underline{\mathcal{A}}_\xi(\bar{K})$ is non-trivial.*
- (2) *The degree d is even.*
- (3) *$\text{char } k \neq 2$ and ξ contains a polynomial.*
- (4) *More generally, $\text{char } k \neq 2$ and there exists an integer $e \geq 2$ such that the number of points with ramification index e is odd.*

Furthermore, ξ descends to a unique Brauer–Severi curve over k_ξ if the group scheme $\underline{\mathcal{A}}_\xi$ is trivial. Finally, ξ always descends to $\mathbb{P}_{k'}^1$, where k' is a finite extension of k_ξ of degree at most 2.

Proof. In general, there exists an extension k'/k_ξ of degree at most 2 with $\mathbf{P}_\xi(k_\xi) \neq \emptyset$, hence ξ descends to $\mathbb{P}_{k'}^1$ by Lemma 11.

If $\underline{\mathcal{A}}_\xi$ is trivial, then $\mathcal{G}_\xi = \text{Spec } k_\xi$ and $\mathcal{P}_\xi = \mathbf{P}_\xi$ is a Brauer–Severi curve. By construction, \mathcal{P}_ξ is equipped with a model of ξ over k_ξ , since $\mathcal{G}_\xi = \text{Spec } k_\xi$ this is the only model over k_ξ .

Under condition (1), the proof of Theorem 2 still works by replacing \mathcal{G}_ξ with the maximal étale quotient of \mathcal{G}_ξ , i.e. the rigidification of \mathcal{G}_ξ [AGV08, Appendix C] by the connected component of the inertia stack; condition (1) ensures that this maximal quotient is non-trivial.

Assume condition (2). The morphism $\mathbf{P}_\xi \rightarrow \mathbf{P}_\xi$ has degree d as well, since its base change to \bar{K} is the morphism $\mathbb{P}_{\bar{K}}^1 / \underline{\text{Aut}}_K(\phi) \rightarrow \mathbb{P}_{\bar{K}}^1 / \underline{\text{Aut}}_K(\phi)$ induced by ϕ . If d is even, then $\mathbf{P}_\xi(k_\xi) \neq \emptyset$ (this is shown in the proof of [Sil95, Corollary 2.2]), hence we conclude by Lemma 11.

Under condition (3), i.e. ξ contains a polynomial, either ξ contains x^d (which is clearly defined over k_ξ) or ξ satisfies condition (4).

Assume condition (4), i.e. there exists an integer $e \geq 2$ such that the number of points with ramification index e is odd, and let $S \subset \mathbf{P}$ be the image of said points

with the reduced structure. The assumption is equivalent to saying that $|S(\bar{K})|$ is odd; if $\text{char } k \neq 2$, this is equivalent to saying that S has odd degree. We conclude that $\mathbf{P}(k_\xi) \neq \emptyset$ by [Bre24, Lemma 27]. \square

Remark 13. The gerbe \mathcal{G}_ξ^D constructed in §1 is the rigidification of \mathcal{G}_ξ [AGV08, Appendix C] by the connected component of the inertia stack. It follows that the induced morphism $\mathcal{G}_\xi \rightarrow \mathcal{G}_\xi^D$ is a relative gerbe with connected relative inertia. If k is perfect, it can be proved that such a morphism induces a bijection on isomorphism classes of rational points: this is the reason why the simpler formalism used in the previous sections works when k is perfect.

Remark 14. It is natural to ask if we can drop the assumption that ξ contains a morphism defined over K . We only use this assumption to define the field of moduli and to prove that \mathcal{G}_ξ is an algebraic stack. Even if ξ does not contain a morphism defined over K , if we know that \mathcal{G}_ξ is an algebraic stack then the field of moduli can be defined as the coarse moduli space of \mathcal{G}_ξ [BV24, Proposition 3.10] and Theorem 12 holds in this generality. See [BV24, Proposition 3.8] for a criterion to check whether \mathcal{G}_ξ is an algebraic stack.

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