

LARGE BRICKS AND JOIN-IRREDUCIBLE TORSIONFREE CLASSES

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ABSTRACT. We show that every join-irreducible torsionfree class in the category of finitely generated modules over an artinian ring is cogenerated by a single (not necessarily finitely generated) brick.

This is a partial extension of the characterisation of completely join-irreducible torsionfree classes given by Barnard, Carroll and Zhu.

1. INTRODUCTION

Torsion pairs have been a central topic in the representation theory of artin algebras in the last few years.

In particular, one of the main aspects which was thoroughly investigated is the complete lattice structure of the set of torsion (or equivalently torsionfree) classes in the category of finitely generated modules over an artin algebra, see [6].

Recall that an object in some lattice is said to be *completely join-irreducible* if it cannot be written as an arbitrary join of strictly smaller elements.

The following result, holding in general in every abelian length category, gives a complete classification of completely join-irreducible torsion(free) classes in terms of bricks, i.e. objects whose endomorphism ring is a skew-field:

Theorem 1.1 ([3, Theorem 1.0.5]). *There is a bijection between the set of completely join-irreducible torsionfree classes in an abelian length category \mathcal{A} and the set of bricks of \mathcal{A} .*

However, in general we might have torsionfree classes which are join-irreducible, but not completely join-irreducible. Clearly, such classes can only occur when the lattice of torsionfree classes is infinite.

In fact, as a consequence of [11], we always have at least one such torsionfree class in the lattice of torsionfree classes of a τ -tilting infinite algebra.

The aim of this short note is to give a partial extension of Theorem 1.1, showing that every join-irreducible torsionfree class in the category of finitely generated modules over an artinian ring is determined by some brick.

Notation Throughout the paper, Λ will be a finite-dimensional k -algebra over some field k , $\Lambda\text{-Mod}$ (resp. $\Lambda\text{-mod}$) the category of (finite dimensional) left Λ -modules. Uniqueness of some object is always to be intended up to isomorphism. All the subcategories appearing in the work are assumed to be full and closed under isomorphism.

For a subcategory \mathcal{E} of some abelian category we denote by $\text{filt}(\mathcal{E})$ the closure of \mathcal{E} under extensions and by $\text{cogen}(\mathcal{E})$ the subcategory of objects which can be embedded in some finite coproduct of objects in \mathcal{E} .

2. MONOBRICKS AND JOIN-IRREDUCIBILITY

Monobricks were introduced by Enomoto in [7] to study left Schur subcategories of some abelian length category. This class of subcategories contains in particular the sets of torsionfree classes and wide subcategories.

They play a prominent role in the proof of our main result, thus we give a quick overview of the subject.

Definition 2.1. A collection of (isomorphism classes of) Λ -bricks $\{B_i\}_{i \in I}$ is a *monobrick* if for every $f \in \text{Hom}_\Lambda(B_i, B_j)$ we have $f = 0$ or f is a monomorphism.

Remark 2.2. We can see every monobrick as a poset with the following order relation $B_i \prec B_j$ if there exists a monomorphism $f : B_i \rightarrow B_j$. This relation is clearly reflexive and transitive. Antisymmetry is a consequence of the definition of brick.

Definition 2.3. Let (\mathbf{t}, \mathbf{f}) be a torsion pair in some abelian category \mathcal{A} . An object $0 \neq B$ in \mathbf{f} is *\mathbf{f} -simple* if all its proper quotients are in \mathbf{t} .

Examples 2.4. • Every semibrick is a monobrick, the corresponding poset is just a discrete set.

- The collection of \mathbf{f} -simple modules for a torsionfree class \mathbf{f} in $\Lambda - \text{mod}$ is a monobrick and the maximal elements of the corresponding poset are the minimal coextending modules for \mathbf{f} [7, Remark 4.6].

Notice that the poset corresponding to any monobrick in $\Lambda - \text{mod}$ has at most countable height (only countably many dimensions of finite-dimensional modules) but it might have uncountable width (consider the case of some torsionfree class with uncountably many minimal coextending modules).

Theorem 2.5 ([7, Theorem 2.11]). Let \mathbf{f} be a torsionfree class in $\Lambda - \text{mod}$ and $\text{simp}(\mathbf{f})$ be the collection of \mathbf{f} -simple modules. Then $\mathbf{f} = \text{filt}(\text{simp}(\mathbf{f}))$.

The result above tells us that every torsionfree object admits a filtration in terms of \mathbf{f} -simples.

Corollary 2.6. Let \mathbf{f} be a torsionfree class. Assume there exist torsionfree classes \mathbf{v}, \mathbf{w} with $\mathbf{f} = \mathbf{v} \vee \mathbf{w}$. Then $\text{simp}(\mathbf{f}) \subseteq \text{simp}(\mathbf{v}) \cup \text{simp}(\mathbf{w})$.

Proof. Any \mathbf{f} -simple object B is contained in the join, which can be expressed as $\text{filtcogen}(\mathbf{v} \cup \mathbf{w})$. However, \mathbf{f} -simples are not filtered by other torsionfree modules, thus it must be that $B \in \text{cogen}(\mathbf{v} \cup \mathbf{w})$ that is $B \in \mathbf{v}$ or $B \in \mathbf{w}$. This shows the inclusion, as an \mathbf{f} -simple contained in \mathbf{v} is automatically \mathbf{v} -simple (the torsion class corresponding to \mathbf{v} contains the torsion class corresponding to \mathbf{f}). The same for \mathbf{w} . \square

This result is enough to give a characterisation of join-irreducibility of a torsionfree class in terms of properties of the corresponding monobrick of simples.

Definition 2.7. Let L be a lattice, $l \in L$ is *join-irreducible* if l is not the 0 of L and whenever $l = x \vee y$ we have $l = x$ or $l = y$.

Definition 2.8. A nonempty poset P is *directed* if every finite subset of P has an upper bound.

Proposition 2.9. *Let \mathbf{f} be a torsionfree class. Then \mathbf{f} is join irreducible if and only if the monobrick of \mathbf{f} -simple objects is directed.*

Proof. “ \implies ” We prove the contrapositive: assume that the monobrick is not directed.

If the monobrick is empty, then $\mathbf{f} = 0$ thus it is not join irreducible. So assume it is non-empty.

Then we can find \mathbf{f} -simples B_1, B_2 such that $B_1^\uparrow := \{B \in \mathbf{simp}(\mathbf{f}) \mid B_1 \prec B\}$ and B_2^\uparrow are disjoint.

Consider the two torsionfree classes $\mathbf{v} := \text{filtcogen}(B_1^\uparrow)$ and $\mathbf{w} := \text{filtcogen}(\mathcal{C})$ where $\mathcal{C} := \mathbf{simp}(\mathbf{f}) \setminus B_1^\uparrow$. Clearly, we have that $\mathbf{f} = \mathbf{v} \vee \mathbf{w}$ as every \mathbf{f} -simple is contained in \mathbf{v} or in \mathbf{w} .

Moreover $\mathbf{v} \neq \mathbf{f}$. In fact \mathbf{f} contains B_2 . If $B_2 \in \mathbf{v}$ i.e. if B_2 is in $\text{filtcogen}(B_1^\uparrow)$ using \mathbf{f} -simplicity we would have B_2 is in $\text{cogen}(B_1^\uparrow)$ but this would contradict our hypothesis on the disjointness of B_1^\uparrow and B_2^\uparrow .

In a similar way, $\mathbf{w} \neq \mathbf{f}$ as it does not contain B_1 : \mathcal{C} does not contain, by definition, any module B such that B_1 embeds in B , so using \mathbf{f} -simplicity we can conclude that $B_1 \notin \text{filtcogen}(\mathcal{C})$. This shows that \mathbf{f} is not join-irreducible.

“ \impliedby ” Assume that the monobrick of simples is directed. Let \mathbf{v}, \mathbf{w} be torsionfree classes such that $\mathbf{f} = \mathbf{v} \vee \mathbf{w}$.

By the corollary above, every \mathbf{f} -simple is either a \mathbf{v} -simple or a \mathbf{w} -simple.

So assuming that $\mathbf{v} \subsetneq \mathbf{f}$ and $\mathbf{w} \subsetneq \mathbf{f}$ we can find an \mathbf{f} -simple module B_v which is in $\mathbf{v} \setminus \mathbf{w}$ and an \mathbf{f} -simple B_w which is in $\mathbf{w} \setminus \mathbf{v}$. However, using the directness of the poset, we can find an \mathbf{f} -simple B_{vw} such that $B_w \prec B_{vw}$ and $B_v \prec B_{vw}$. Now it must be the case that this new simple is an element of \mathbf{v} or of \mathbf{w} and in both cases this yields a contradiction, as these classes are closed under submodules.

It follows that either $\mathbf{f} = \mathbf{v}$ or $\mathbf{f} = \mathbf{w}$, proving join-irreducibility. \square

We can also restate the description of completely join irreducible elements in terms of monobricks:

Corollary 2.10 ([3], [6]). *Let \mathbf{f} be a torsionfree class in $\Lambda - \text{mod}$. Then \mathbf{f} is completely join irreducible if and only if the monobrick of \mathbf{f} -simple modules is directed and has finite height (i.e. it has a maximum).*

Proof. Being completely join irreducible is equivalent to the existence of a (finitely-generated) brick B such that $\mathbf{f} = \text{filtcogen}(B)$, moreover a standard computation shows that the brick B is \mathbf{f} -simple.

Using this characterisation, it is immediate to see that the equivalence holds as the monobrick being directed and of finite height means that there exists a top element. \square

So by exclusion we have:

Corollary 2.11. *Let \mathbf{f} be a torsionfree class. Then \mathbf{f} is join irreducible, but not completely join irreducible if and only if the monobrick of \mathbf{f} -simple modules is directed and has infinite height.*

3. LARGE BRICKS AND JOIN-IRREDUCIBILITY

To obtain the connection with large bricks we need to use some tools from the theory of cosilting modules. We recall a couple of fundamental results and refer the reader to the survey [1] for the basic notions of silting/cosilting theory.

The first result associates with every torsion pair in the small module category a nice torsion pair in the large one. We use $\mathbf{Cosilt}(R)$ for the set of torsion pairs cogenerated by a cosilting module.

Theorem 3.1. *When R is a left noetherian ring, there is a bijection*

$$\mathbf{tors}(R) \leftrightarrow \mathbf{Cosilt}(R).$$

It associates to a torsion pair (\mathbf{t}, \mathbf{f}) in $R - \text{mod}$ the limit closure $(\mathcal{T}, \mathcal{F}) := (\varinjlim \mathbf{t}, \varinjlim \mathbf{f})$, which coincides with the torsion pair $(\text{Gen } \mathbf{t}, \mathbf{t}^{\perp_0})$ generated by \mathbf{t} . The inverse of this map sends a cosilting torsion pair $(\mathcal{T}, \mathcal{F})$ to its restriction $(\mathcal{T} \cap R - \text{mod}, \mathcal{F} \cap R - \text{mod})$.

The second result relates cosilting torsion pairs with t -structures in the derived category :

Theorem 3.2 ([8], [9, Theorem 1.2], [10, Theorem 7.1]). *Let R be a left noetherian ring, (\mathbf{t}, \mathbf{f}) a torsion pair in $R - \text{mod}$ and $(\mathcal{T}, \mathcal{F}) = (\varinjlim \mathbf{t}, \varinjlim \mathbf{f})$ the corresponding cosilting torsion pair in $R - \text{Mod}$.*

Then $(\mathcal{T}, \mathcal{F})$ determines, via the HRS-construction, a t -structure $(\mathcal{U}, \mathcal{V})$ in the unbounded derived category.

The heart \mathcal{H} of this t -structure is a locally coherent Grothendieck category and $(\mathcal{F}, \mathcal{T}[-1])$ is a torsion pair in it.

Moreover, the category of finitely presented objects of \mathcal{H} is precisely the intersection of \mathcal{H} with the bounded derived category $\mathbf{D}^b(R - \text{mod})$.

We will make use of the following computational lemma:

Lemma 3.3 ([2, Lemma 2.3]). *Let F, V be two modules in a (cosilting) torsionfree class \mathcal{F} . Then a morphism $f : F \rightarrow V$ is an epimorphism in the corresponding heart \mathcal{H} if and only if the cokernel of f is a torsion module.*

Finally, we recall the definition of a torsionfree, almost torsion module and a characterisation of those

Definition 3.4. Let R be a ring, let $(\mathcal{T}, \mathcal{F})$ be a torsion pair in $R - \text{Mod}$. Then a non-zero module B is *torsionfree, almost torsion* if:

- (i) $B \in \mathcal{F}$
- (ii) Every proper quotient of B is in \mathcal{T} .
- (iii) For every short exact sequence

$$0 \rightarrow B \rightarrow F \rightarrow C \rightarrow 0$$

with $F \in \mathcal{F}$, we have that $C \in \mathcal{F}$.

Proposition 3.5 ([2, Theorem 3.6]). *A module B is torsionfree, almost torsion with respect to a torsion pair $(\mathcal{T}, \mathcal{F})$ if and only if it is a simple object in the heart \mathcal{H} which is torsion with respect to $(\mathcal{F}, \mathcal{T}[-1])$.*

To obtain the desired connection we will make use of the following:

Lemma 3.6. *Let (\mathbf{t}, \mathbf{f}) be a torsion pair in $\Lambda\text{-mod}$ and let $\{F_i, (\alpha_{ij})\}$ be a direct system of non-zero maps between \mathbf{f} -simple modules. Then $F = \varinjlim \{F_i, (\alpha_{ij})\}$ is $\varinjlim \mathbf{f}$ -simple.*

Proof. First, we notice that the direct limit is non-zero, as any non-zero map between \mathbf{f} -simple objects is a monomorphism. In particular, each of the canonical maps $F_i \rightarrow F$ must be a monomorphism.

Now we have to show that every proper quotient of F is contained in the torsion class $\varinjlim \mathbf{t}$.

Using the fact that torsion classes are closed under quotients, this task can be reduced to proving that every maximal quotient of F is a torsion module.

So let S be a simple module with some non-zero map $f : S \rightarrow F$. As S is a finite-dimensional module, any map from S to F can be factored through the direct system $\{F_i, (\alpha_{ij})\}$. This yields a direct system of short exact sequences:

$$0 \longrightarrow S \xrightarrow{f_i} F_i \longrightarrow \text{coker}(f_i) \longrightarrow 0$$

where each $\text{coker}(f_i)$ is contained in \mathbf{t} as F_i is \mathbf{f} -simple.

The direct limit of this system of sequences is

$$0 \longrightarrow S \xrightarrow{f} F \longrightarrow \text{coker}(f) \longrightarrow 0$$

so that $\text{coker}(f)$ is in $\varinjlim \mathbf{t}$. This shows that every maximal quotient of F is torsion. \square

While $\varinjlim \mathbf{f}$ -simple might be infinite-dimensional, it is easy to observe that they are finitely generated as objects of the heart. So we can obtain a second useful lemma:

Lemma 3.7. *Let F be a $\varinjlim \mathbf{f}$ -simple module, then there exists a torsionfree, almost torsion module B such that F is isomorphic to a submodule of B .*

Proof. Let F be $\varinjlim \mathbf{f}$ -simple and consider any non-trivial finite-dimensional submodule M of F . This results in a short exact sequence

$$0 \rightarrow M \rightarrow F \rightarrow N \rightarrow 0$$

where N must be a torsion module, by simplicity. But then by Lemma 3.3, the inclusion $M \rightarrow F$ is an epimorphism, when considered as a map in the heart of the HRS-tilt at $(\varinjlim \mathbf{t}, \varinjlim \mathbf{f})$. So F is the epimorphic image of a finitely presented object, thus it is finitely generated.

As the heart is a locally coherent Grothendieck category, we know that every finitely generated object has a simple quotient, thus we have an epimorphism $F \rightarrow S$. In particular, as F is an element of the torsion class $\varinjlim \mathbf{f}$, S will be a torsion simple.

Now by Proposition 3.5, this means that S as a module is torsionfree, almost torsion for the pair $(\varinjlim \mathbf{t}, \varinjlim \mathbf{f})$.

We therefore have a non-zero map $F \rightarrow S$ which must be a monomorphism, by simplicity of F . \square

We need a last technical lemma:

Lemma 3.8. *Let G be a non-empty connected graph, then G contains a spanning tree (i.e. an acyclic connected subgraph containing all the vertices).*

Proof. This is a well-known consequence of Zorn's lemma. We give the proof for completeness. Let \mathcal{T} be the collection of all the subtrees of G which we can order by inclusion. Notice that \mathcal{T} is non-empty as every singleton gives a subtree of G and G itself is non-empty. It is easy to notice that every ascending chain of elements of \mathcal{T} has an upper bound, this is just the union of the elements of the chain.

Then by Zorn's lemma the set \mathcal{T} contains some maximal element T . I claim that T must be a spanning tree.

It is a tree by definition, so assume there is some vertex v of G such that $v \notin T$. If T were empty, then clearly $\{v\}$ would be a tree containing T contradicting maximality. Thus we may assume that T contains some element t . As G is connected there exists a path starting at $t = v_0 \rightarrow v_1 \rightarrow \cdots \rightarrow v_n = v$, we may assume that this is the shortest path connecting an element of T to v . In particular, v_1 is not in T and we can enlarge T to a new subtree T' containing the vertex v_1 and the edge $v_0 \rightarrow v_1$.

Then $T \subsetneq T'$ giving a contradiction. Thus T must be a spanning tree. \square

We can now prove our main result:

Proposition 3.9. *Let \mathbf{f} be a join-irreducible torsionfree class in $\Lambda - \text{mod}$. Then there exists a brick B in $\Lambda - \text{Mod}$ which is torsionfree, almost torsion for $\varinjlim \mathbf{f}$ such that $\mathbf{f} = \mathbf{F}(B) \cap \Lambda - \text{mod}$.*

Proof. By Proposition 2.9 the poset of \mathbf{f} -simples is directed. We show that the graph G underlying the Hasse quiver of this poset is connected.

Let F_1 and F_2 be two isomorphism classes of \mathbf{f} -simples. Then by directedness there exists a third isoclass F_3 with $F_1 \leq F_3$ and $F_2 \leq F_3$. As all the isoclasses involved correspond to finite-dimensional modules, the poset of objects smaller than F_3 has finite height, therefore we have finite paths in the Hasse quiver connecting F_1 and F_2 to F_3 .

This proves that the graph G is connected.

Now by Lemma 3.8 we can find a spanning tree T in G , and consequently a directed acyclic subquiver T' of the Hasse quiver containing all the vertices. Using the axiom of choice we assign to each vertex v_i of T' an \mathbf{f} -simple module V_i in the isoclass v_i and successively to each edge $v_i \rightarrow v_j$ a non-zero map $V_i \rightarrow V_j$ (such a map exists by the definition of the order on \mathbf{f} -simples).

This gives a directed system of \mathbf{f} -simples consisting of monomorphisms (we don't have any compatibility condition to check on the maps as the underlying graph is a tree), therefore we can use Lemma 3.6 to obtain a $\varinjlim \mathbf{f}$ -simple F with the property that every \mathbf{f} -simple is isomorphic to a submodule of F .

But now, Lemma 3.7 tells us that F is a submodule of some torsionfree, almost torsion module B of the class $\varinjlim \mathbf{f}$.

So we obtain that $\mathbf{f} = \mathbf{F}(B) \cap \Lambda - \text{mod}$ by Theorem 2.5. \square

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