

A MISSING THEOREM ON DUAL SPACES

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ABSTRACT. We answer in the affirmative the surprisingly difficult questions: If a complex Banach space possesses a real predual X , then is X a complex Banach space? If a complex Banach space possesses a real predual, does it have a complex predual? We also answer the analogous questions for operator spaces, that is, spaces of operators on a Hilbert space, up to complete isometry. Indeed, we use operator space methods to solve these Banach space questions.

1. INTRODUCTION

In 1932 Banach's famous monograph [2] appeared, in which Banach spaces and their dual (or 'conjugate' or 'adjoint') space play a central role. Shortly thereafter F. J. Murray of 'Murray and von Neumann' fame provided the well known $\varphi(x) - i\varphi(ix)$ trick (see the second page of [25]) to deal with complex dual spaces X^* . It then becomes clear that the real Banach space dual X^\dagger of a complex Banach space X is itself a complex Banach space. That is, this real Banach space has *isometric complex structure*. Indeed it is easy to see that $X^\dagger \cong X^*$ real linearly isometrically and weak* homeomorphically (see e.g. [23, Proposition 1.1.6]). However there remain a couple of obvious questions:

Problem 1. If Y is a complex Banach space possessing a real predual X , then is X (isometrically) a complex Banach space?

That is, does X have (isometric) complex structure? Rephrasing, Problem 1 asks if X is a real Banach space such that its real Banach space dual X^\dagger has (isometric) complex structure, then does X have (isometric) complex structure?

In Problem 1, one would also desire that the sought-for complex structure on X is a complex predual of Y , if mercy prevails. Indeed we have:

Problem 2. If a complex Banach space possesses a real predual X , then does it have a complex predual? Is X a complex predual?

We note that by 'predual' in Problems 1 and 2 we mean an isometric predual. We will discuss the isomorphic question later.

Problems 1 and 2 do not appear explicitly in any textbook. However related questions have been discussed by Dixmier, Dieudonné, Bourgain, Szarek, Kalton, Godefroy, Ferenczi, and many others (see e.g. [15, 12, 36, 17, 18] and references therein). Some of these papers are connected to the question above of whether mercy and desire prevail. Indeed if a complex Banach space X is a real predual of complex Banach space Y then the complex dual X^* is real isometric to Y , as in the first paragraph above. We claim that this need not imply in general that X is a complex predual of Y . So mercy does not prevail in this generality, although it

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does for us, as we shall see. Note that if the claim were false then Problem 2 would follow immediately from an affirmative solution to Problem 1. In fact the claim (and its isomorphic variant) is related to nasty examples in some of the works just cited. For example, let Y be a complex dual Banach space which is not complex isometric to its complex conjugate \bar{Y} (e.g. Bourgain's reflexive example [12]). If E is a complex predual of \bar{Y} then E is a real predual of Y and \bar{E} is a complex predual of Y . These complex preduals of Y and \bar{Y} are real isometrically isomorphic to each other, but not complex isometrically so, and E is not a complex predual of Y . This establishes the claim.

Problems 1 and 2 turn out to be quite difficult and subtle, as any direct attack eventually reveals. Duality is often a slippery business, particularly when higher duals are involved, as they turn out to be. We are able to answer Problems 1 and 2 in the affirmative. Indeed the complex dual of this complex structure on X agrees with Y complex isometrically and weak* homeomorphically, as one would wish. This seems a useful contribution to Banach space theory, clarifying an aspect of the relationship between real and complex Banach spaces. Thus it should be helpful in certain problems in the theory of both of these classes of Banach spaces, in the sense that it should sometimes allow each of them to better access the other.

As a corollary we have the equivalent reformulation of Problem 1:

Theorem 1.1. *If X is a real Banach space such that its real Banach space bidual $X^{\dagger\dagger}$ has complex structure (isometrically), then X has complex structure (isometrically).*

To see that this is equivalent to an affirmative solution to Problem 1: Indeed if X^\dagger has (isometric) complex structure then its real dual is isometrically a complex Banach space by the facts in the first paragraph of our paper (but with X replaced by X^\dagger). Thus X is isometrically a complex Banach space by the theorem. The converse is trivial.

Our techniques use the theory of operator spaces and C^* -algebras, although we would guess that at some later point this will be removed, i.e. there will be a later solution only involving Banach spaces. Indeed at the end of our paper we make some remarks on how our methods suggest the direction in which this might be done. The reader unfamiliar with operator spaces should browse early chapters of e.g. [6, 16, 27] or [29]. While our proofs may often appear petite, this is usually because they include only the modifications to be made in the real case of selected points in existing rather technical and lengthy arguments in the complex case. One would have to master the latter to completely understand some of our 'short proofs'.

Operator spaces may be viewed as Banach spaces X with a specified norm on the $n \times n$ matrices over X satisfying (Ruan's) conditions. The category of operator spaces is very similar to the category of Banach spaces, indeed there are canonical functors between these categories, e.g. Min , which turns every Banach space into a operator space in a 'minimal way' (see e.g. 1.2.21 and 1.4.12 in [6], and Section 2 below). Real operator spaces have been studied in e.g. [33, 34, 35, 9, 4], and recent progress here led to the present paper.

Theorem 1.2. *Let X be a real operator space such that the real operator space bidual $X^{\dagger\dagger}$ has complex operator space structure. Then X has complex operator space structure, such that if G is X with this complex operator space structure then $G^{**} \cong X^{\dagger\dagger}$ real completely isometrically and weak* homeomorphically.*

(This result was advertised in [4, Section 3] without proof, as a future application of Corollary 3.2 from that paper.) Due to the nature of operator space duality one cannot expect that if the real operator space dual of X has complex operator space structure (completely isometrically) then so does X (see Example 2 below). However the correct result matching our solution to Problem 1 is a small modification of the last statement (namely, Theorem 3.3 (3) below). We also note that a real operator space can have complex Banach space structure isometrically but not complex operator space structure completely isometrically (e.g. the quaternions [4, Remark 1 after Theorem 3.1]).

Valentin Ferenczi has shown us that, interestingly, the isomorphic version of Problem 1 has a quick negative answer. It is well known that a bounded map $J : X \rightarrow X$ on a real Banach space corresponds isomorphically to a complex Banach space structure on X (with $ix = Jx$) if and only if $J^2 = -I$ and J is an isomorphism. Thus for example, by this criterion (real) ℓ_1 has isomorphic complex structure using $J(\alpha) = (-\alpha_2, \alpha_1, -\alpha_4, \alpha_3, \dots)$. From this it is easy to see for example that a real Banach space which is real isomorphic to a complex Banach space, need not have isometric complex structure. There are various formulations of the isomorphic version of Problem 1, e.g.:

Problem 3. If Y is a real Banach space with real predual X , and Y is isomorphic to a complex Banach space, then is X isomorphic to a complex Banach space?

Example 1. (Ferenczi) In the last fifteen years many authors have studied real Banach spaces X on which every operator is a scalar multiple of the identity plus a compact. Considering the operator J above Problem 3, one sees immediately that such X cannot be isomorphic to a complex space. (For $J^2 = (aI + k)^2 = -I$, for k compact and a real, gives the contradiction $a^2 = -1$.)

One such space X is the Argyros-Haydon space [1], whose dual X^* is isomorphic to ℓ_1 , so is isomorphic to a complex Banach space as we stated above. This answers Problem 3 in the negative, and it also shows that the issue (and the various reformulations of Problem 3) cannot be fixed by any kind of renorming trick, etc. If Problem 1 has an affirmative solution then it follows that no renorming of the real dual of X to have complex structure can be the real dual space of a space isomorphic to X .

It is easy to see using real Min-Max duality (see [35, Proposition 2.6]), that $\text{Max}(X)$ is a counterexample to the operator space (completely isomorphic) version of Problem 3.

Ferenczi and his coauthors have many very striking results about complexifications, e.g. an example of complex Banach spaces that are isometric as real spaces but ‘totally incomparable’ as complex spaces (so no infinite-dimensional subspace of the one is complex isomorphic to a subspace of the other) [17].

We will give our solution to Problems 1 and 2 in Section 4, and give some immediate corollaries or sample applications there. Since the solution to Problems 1 and 2 is complicated we suggest that the reader might warm up to it, and start to get a feeling for the technique, by first mastering the proof of the relatively easy Theorem 3.3 (1). Indeed Section 3 is mostly devoted to the operator space version of Problems 1 and 2, but also contains some applications and some preparatory material for Section 4. In the rest of Section 1 and in Section 2 we provide necessary background and lemmata.

Turning to definitions, we write i_X for the canonical map X into its bidual. Because there are many 'i's around we sometimes write ι for the complex number i . It is well known and obvious that a map $J : X \rightarrow X$ on a real Banach space corresponds isometrically to a complex Banach space structure on X (with $ix = Jx$) if and only if $J^2 = -I$ and $x \mapsto sx + tJ(x)$ is an isometry whenever $s = \cos \theta, t = \sin \theta$, that is whenever $s^2 + t^2 = 1$.

We say that an algebra is *unital* if it has an identity 1 of norm 1, and a map T is unital if $T(1) = 1$. We say that a map T is selfadjoint if $T(x^*) = T(x)^*$. For us, a contraction is a linear map with norm ≤ 1 . For some theory of real or complex C^* -algebras we refer the reader to e.g. [29, 23]. We reserve the letters H and K for Hilbert spaces. A *TRO* (or ternary system) is a closed linear subspace $Z \subset B(K, H)$, for Hilbert spaces K and H , satisfying $ZZ^*Z \subset Z$.

We refer the reader to [24] for a survey of relationships between real and complex Banach spaces. If X is a complex Banach space we will sometimes write X_r for X viewed as a real Banach space. We usually will write X^\dagger for the real dual and X^* for the complex dual. The canonical map $X^\dagger \rightarrow X^*$ described at the start of our paper is given by the canonical duality pairing

$$\langle \varphi, x \rangle = \varphi(x) - i\varphi(ix), \quad \varphi \in X^\dagger, x \in X.$$

Lemma 1.3. *If Y is a complex Banach space then $Y^{**} \cong Y^{\dagger\dagger}$. Writing Y_r as E , and E^\dagger for the real dual and Y^* for the complex dual, we have*

$$Y^{**} \cong E^{\dagger\dagger}$$

real linear isometrically via an isomorphism which is a homeomorphism for the two weak topologies, and takes $i_Y(y)$ to $i_E(y)$ for $y \in Y$.*

Proof. By [23, Proposition 1.1.6] we have $Y^* \cong E^\dagger$ real linear isometrically via the map $j_Y : \varphi \mapsto \operatorname{Re} \varphi$. The inverse map is $\varphi \mapsto \tilde{\varphi}$ where $\tilde{\varphi}(x) = \varphi(x) - i\varphi(ix)$. Moreover this map is clearly a homeomorphism for the two weak* topologies. Similarly

$$Y^{**} \cong ((Y^*)_r)^\dagger \cong (E^\dagger)^\dagger = E^{\dagger\dagger}$$

isometrically, via a homeomorphism μ for the two weak* topologies. Indeed $\mu = ((j_Y)^\dagger)^{-1} \circ j_{Y^*}$, where $(j_Y)^\dagger : E^{\dagger\dagger} \rightarrow (Y^*)^\dagger$, and $j_{Y^*} : Y^{**} \rightarrow (Y^*)^\dagger$. Note that $i_Y(y)$ maps to $i_E(y)$ here (exercise). \square

Remarks. 1) Since $Y^{**} \cong ((Y^*)_r)^\dagger$, the canonical real linear isometry $E \rightarrow Y \rightarrow Y^{**}$ taking $x \mapsto i_Y(x)$ extends uniquely to a weak* continuous real linear contraction $\rho : E^{\dagger\dagger} \rightarrow Y^{**}$ with $i_E(x)$ mapping to $i_Y(x)$. By uniqueness this is the isometry μ in the proof, and we will need this fact later.

2) Any complex Banach space Y with an (isometric) complex predual X clearly has an (isometric) real predual. Indeed $(X_r)^\dagger \cong X^* \cong Y$.

2. SOME RESULTS ON REAL OPERATOR SPACES

If $T : X \rightarrow Y$ is a linear map between operator spaces we write T_n for the canonical 'entrywise' amplification taking $M_n(X)$ to $M_n(Y)$. We say that T is *completely bounded* if $\|T\|_{\text{cb}} = \sup_n \|T_n\| < \infty$, and write $\text{CB}(X, Y)$ for the space of such completely bounded maps. Then T is *completely contractive* (resp. *completely isometric*, *completely positive*) if T_n is a contraction (resp. is an isometry, is positive) for all n . We write UCP for 'unital completely positive'. A real *operator system* is a unital selfadjoint subspace of a unital real C^* -algebra (or of $B(H)$).

The following result improves on Proposition 4.1 in [33], and is in [9] (but not stated in this form).

Proposition 2.1. *Let X be a real operator system and $T : X \rightarrow B(H)$ a real linear unital map. Then T is completely positive if and only if T is completely contractive, and in this case T is selfadjoint.*

Proof. If T is completely positive then it is selfadjoint and completely contractive by [9, Lemma 2.3]. The other direction formally follows from [9, Corollary 2.5 and Proposition 2.4], but in fact as indicated there is immediate by going to the complexifications (i.e. by the complex result applied to the complexified map T_c). \square

Many well known results about completely positive maps generalize to the real case. In [9, 4] we verified the real case of the results 1.3.4–1.3.8 and 1.3.11–1.3.13 in [6] (that $\text{Ran}(u) \subset \Delta(Y)$ in 1.3.8 follows since u is selfadjoint by [9, Lemma 5.16]). Indeed items 1.3.4, 1.3.5 and 1.3.9 there had previously been verified by Ruan in [33] for selfadjoint unital completely positive maps, and he also checked 1.3.10. However [9, Lemma 2.3] shows that any completely positive map is selfadjoint, so that in fact 1.3.9, 1.3.11, 1.3.12 and 1.3.13 in [6] hold verbatim in the real case for UCP maps or equivalently for unital completely contractive maps. In particular we will need the powerful ‘multiplicative domain’ trick:

Theorem 2.2. *Suppose that $u : A \rightarrow B$ is a real linear UCP map (or equivalently is a unital completely contractive map) between unital real C^* -algebras, and that there is a C^* -subalgebra C of A with $1_A \in C$, such that $\pi = u|_C$ is a $*$ -homomorphism. Then u is a ‘bimodule map’:*

$$u(ac) = u(a)\pi(c) \quad \text{and} \quad u(ca) = \pi(c)u(a), \quad a \in A, c \in C.$$

We remind the reader of the minimal operator space structure Min (see e.g. 1.2.21 and 1.4.12 in [6]). Here (in both the real and complex case) the matrix norm $\|[y_{ij}]\|$ on $M_n(Y)$ is the supremum of $\|\psi(y_{ij})\|$ over all contractive functionals ψ on Y . Min has the universal property that any contraction $T : X \rightarrow Y$ on an operator space T is a complete contraction into $\text{Min}(Y)$.

Proposition 2.3. *Let $T : X \rightarrow Y$ be a continuous real linear map between complex Banach spaces, and let T_m be T regarded as a real linear map from $\text{Min}_{\mathbb{C}}(X)$ to $\text{Min}_{\mathbb{C}}(Y)$. Then $\|T_m\|_{\text{cb}} = \|T\|$.*

Proof. Certainly $\|T\| \leq \|T_m\|_{\text{cb}}$. Conversely, assume that $\|T\| \leq 1$. Since $\varphi \circ T$ is a contractive real functional for all $\varphi \in \text{Ball}(Y^\dagger)$, it is clear that $T : \text{Min}_{\mathbb{R}}(X) \rightarrow \text{Min}_{\mathbb{C}}(Y)$ is a complete contraction. Call it R . On the other hand the identity contraction $\text{Min}_{\mathbb{C}}(X) \rightarrow X$ induces a complete contraction $\text{Min}_{\mathbb{C}}(X) \rightarrow \text{Min}_{\mathbb{R}}(X)$. Composing this with R we obtain T_m . So T_m is a complete contraction. \square

We believe that the correct tool for Problems 1 and 2 (and their operator space version) is the largely forgotten tool of ‘multipliers of linear spaces’. Indeed the main technical part of our proof uses the *operator space multipliers* which we introduced in 2001. An operator $T : X \rightarrow X$ on an operator space is a *left operator space multiplier*, or is in the left multiplier operator algebra $\mathcal{M}_\ell(X)$, if there is a complete isometry $\sigma : X \rightarrow B$ for a C^* -algebra B , and an operator $a \in B$, with $\sigma(Tx) = a\sigma(x)$ for all $x \in X$. Of course there are two definitions hidden here, corresponding

to the real and complex case. We will not review the theory of multipliers in detail here: the reader is referred to [10] for a very short introduction that does a far better job than we could do here. In fact we will only need to know that one particular map, multiplication by i , is an operator space multiplier. However we will then need to feed this one map into the rather complicated operator space multiplier machinery. See the second half of [6, Chapter 4] for a more detailed account of this theory, and see [35] and [4, Section 4] for the real case. These multipliers may be defined in terms of the *injective envelope*, which we shall mention in more detail above Lemma 3.5.

3. OPERATOR SPACE DUALITY

The space $\text{CB}(X, Y)$ defined at the start of the last section has canonical matrix norms with respect to which it is an operator space. In particular for $\mathbb{F} = \mathbb{R}$ or \mathbb{C} the dual Banach space $B_{\mathbb{F}}(Y, \mathbb{F}) = CB_{\mathbb{F}}(Y, \mathbb{F})$ of an operator space X has a canonical operator space structure. This is called the *operator space dual*. It turns out that the canonical map i_X of X into its operator space bidual is completely isometric. This and other very basics of the duality theory of operator spaces are explained in e.g. [6, Section 1.4]. The real case of these results were checked in [33, 34, 35, 9, 4]. Thus a *dual operator space* is an operator space completely isometric to the operator space dual of another operator space.

A key fact for us will be the following result from [7] concerning the multipliers mentioned at the end of the last section:

Theorem 3.1. *Every operator space left multiplier on a dual operator space is weak* continuous.*

Lemma 3.2. *If E is a complex operator space and $F = E_r$ then $E^{**} \cong F^{\dagger\dagger}$ real linear completely isometrically, where E^{**} here is the complex operator space bidual, and $F^{\dagger\dagger}$ is the real operator space bidual. This isomorphism is a homeomorphism for the two weak* topologies, and takes $i_E(x)$ to $i_F(x)$ for $x \in X$.*

Proof. By Lemma 1.3 we have $E^{**} \cong F^{\dagger\dagger}$ isometrically, via a homeomorphism μ for the two weak* topologies, with $i_E(x)$ mapping to $i_F(x)$.

By the Remark 1 after Lemma 1.3 we have that μ^{-1} is the unique weak* continuous real linear extension ρ of the canonical complete isometry $F \rightarrow E \rightarrow E^{**}$ taking $i_F(x)$ to $i_E(x)$. Now $M_n(E)_r = M_n(F)$ isometrically, so taking the complex bidual we obtain a unique real linear isometric weak* homeomorphism $(M_n(E))^{**} \cong M_n(F)^{\dagger\dagger}$ taking $i_{M_n(E)}([x_{ij}])$ to $i_{M_n(F)}([x_{ij}])$. So by [6, Theorem 1.4.11] and its real version there is a unique real linear isometry $M_n(E^{**}) \cong M_n(F^{\dagger\dagger})$ taking $[i_E(x_{ij})]$ to $[i_F(x_{ij})]$. By uniqueness this must be the inverse of $\rho_n : M_n(F^{\dagger\dagger}) \rightarrow M_n(E^{**})$. Thus ρ is a real linear complete isometry. \square

As mentioned in the introduction, due to the nature of operator space duality one cannot expect that: if the real operator space dual of X has complex operator space structure (completely isometrically) then so does X . See the simple Example 2 below. To correct this we need to change the matrix norms to match those in the bidual. The correct statement is the following modification of the expectation mentioned above:

Theorem 3.3. *Let X be a real operator space such that the real operator space dual X^\dagger has complex operator space structure (completely isometrically). Let $Y = X^\dagger$ with this complex operator space structure.*

- (1) *X has complex Banach space structure isometrically, whose complex dual is weak* homeomorphically complex linearly isometric to Y via the canonical duality pairing*

$$\langle \varphi, x \rangle = \varphi(x) - i\varphi(ix), \quad \varphi \in Y, x \in X.$$

- (2) *$M_n(X)$ has complex Banach space structure isometrically for each $n \in \mathbb{N}$.*
 (3) *The canonical complex linear embedding of X in its bidual $j : X \rightarrow B_{\mathbb{C}}(Y, \mathbb{C}) = CB_{\mathbb{C}}(Y, \mathbb{C})$, the latter with its canonical dual operator space structure mentioned at the start of Section 3, endows X with a canonical complex operator space structure such that j is a complete isometry. With this structure its complex operator space dual is completely isometric to Y via a complex linear weak* homeomorphism. Moreover this structure is real isometric and completely isomorphic to the original operator space structure on X .*

Proof. (1) Let $u = L_i$, multiplication by i on Y , which is almost trivially a (complex) operator space multiplier of Y . (We do not need this here but since it will be important in future work we mention that clearly u is a (complex) operator space centralizer of Y in the sense of e.g. [11, 4].) Since $\mathcal{M}_{\ell}^{\mathbb{R}}(Y) = \mathcal{M}_{\ell}^{\mathbb{C}}(Y)$ (this is Theorem 4.5 in [4]), the operator L_i is a real operator space multiplier on $(X^*)_r$. So it is $\sigma(Y, X)$ -continuous by Theorem 3.1. Let $J : X \rightarrow X$ be the predual map of L_i . It is easy to see by duality theory that $J^2 = I$ (since $(J^2)^* = -I$). So X has (isometric) complex structure by the criterion mentioned after Example 1. Indeed if s, t are real with $s^2 + t^2 = 1$ then $(sI + tJ)^* = sI + tL_i$ is an isometric isomorphism, so $sI + tJ$ is isometric. We remark that $sI + tJ$ is clearly seen to have inverse $sI - tJ$. Write W for X viewed as a complex Banach space. Then by [23, Proposition 1.1.6] the complex dual $W^* \cong Y$, weak* homeomorphically and real isometrically via the duality $\langle \varphi, x \rangle = \varphi(x) - i\varphi(ix)$ for $\varphi \in Y, x \in W = X$. However because $J^* = L_i$ it is easy to check that this isomorphism is a complex linear isometry.

(2) As above, if s, t are real with $s^2 + t^2 = 1$ then $(sI + tJ)^* = sI + tL_i$ is a surjective complete isometry, so $sI + tJ$ is a complete isometry. Hence $sI_n + tJ_n$ is an isometry for each n . Thus $M_n(X)$ has complex Banach space structure isometrically.

(3) We endow W with possibly new matrix norms for $n \geq 2$ so that W is a complex operator space with complex operator space dual Y , via the duality

$$\langle \varphi, x \rangle = \varphi(x) - i\varphi(ix), \quad \varphi \in Y, x \in W = X.$$

To do this we view $W \subset Y^*$ as usual via $i_W : W \rightarrow Y^*$, and W is given the operator space structure making i_W a complete isometry. We use the second criterion in 1.6.4 in [6] to check that W has complex operator space dual Y . Suppose that $[y_{ij}^\lambda]$ is a net in $\text{Ball}(M_n(Y))$ with $y_{ij}^\lambda \rightarrow y_{ij}$ in $\sigma(Y, W)$. Thus

$$y_{ij}^\lambda(x) - \iota y_{ij}^\lambda(\iota x) \rightarrow y_{ij}(x) - \iota y_{ij}(\iota x), \quad x \in X.$$

Taking real parts, we see that $y_{ij}^\lambda \rightarrow y_{ij}$ in $\sigma(X^*, X)$. So since Y is the real operator space dual of X , by the second criterion in 1.6.4 in [6] in the real case, $[y_{ij}]$ is in $\text{Ball}(M_n(Y))$. Thus by the same criterion in the complex case, $W^* \cong Y$ completely

isometrically as complex operator spaces. We checked in (1) that this is a weak* homeomorphism.

We claim that the map $Id : W \rightarrow X$ is a real linear isometry and complete contraction with completely bounded inverse. For $n, m \in \mathbb{N}$ and $[\varphi_{ij}] \in \text{Ball}(M_m(Y))$, $[x_{kl}] \in M_n(X)$ we have

$$\|[\varphi_{ij}(x_{kl})]\| \leq \|[\varphi_{ij}(x_{kl}) - \iota\varphi_{ij}(\iota x_{kl})]\| \leq 2\|[\varphi_{ij}]\| \| [x_{kl}] \|_{M_n(X)},$$

and $\|[\varphi_{ij}]\|_{M_m(Y)} = \|[\varphi_{ij}]\|_{M_m(X^\dagger)} \leq 1$. Here we used the fact that $Jx = \iota x$ is completely isometric on X (see e.g. the proof of (2) above). We also used the fact that for real matrices α, β we have $\|\alpha\| \leq \|\alpha + i\beta\|$. Taking the supremum over such $[\varphi_{ij}]$, and using that $X \subset X^{\dagger\dagger}$ completely isometrically we obtain

$$\| [x_{kl}] \|_{M_n(X)} \leq \| [x_{kl}] \|_{M_n(W)} \leq 2 \| [x_{kl}] \|_{M_n(X)}$$

for all n and $x_{kl} \in X$. Thus W is a real isometric copy of X but with an equivalent operator space renorming. \square

Example 2. Let X be the real operator space dual of the complex numbers \mathbb{C} . Then the real operator space dual X^\dagger , which is \mathbb{C} , certainly has complex operator space structure. However X does not have complex operator space structure completely isometrically. Indeed if it did it would be a one dimensional complex operator space, hence is complex linearly completely isometric to \mathbb{C} . However the real operator space dual of the complex numbers \mathbb{C} is known (see [35, Proposition 2.8]) to be not completely isometric to \mathbb{C} . Thus the real operator space dual (or predual) of a real operator space E having complex operator space structure (completely isometrically does not imply that E does.

The following complement matches the Remark at the end of Section 2:

Corollary 3.4. *A complex operator space with a (completely isometric) complex operator space predual X has a (completely isometric) real operator space predual. Indeed we may find a real operator space predual that is real isometric and completely boundedly isomorphic to X .*

Proof. This proof is similar to that of Theorem 3.3 (3). Suppose that $X^* = Y$ as complex operator spaces. We view $X \subset Y^\dagger$ as usual via the duality pairing

$$\langle \varphi, x \rangle = \text{Re } \varphi(x), \quad \varphi \in Y, x \in W = X.$$

and let W be X but with the operator space structure making $i_X : W \rightarrow Y^\dagger$ a complete isometry. We use the second criterion in 1.6.4 in [6] to check that W has real operator space dual Y . Suppose that $[y_{ij}^\lambda]$ is a net in $\text{Ball}(M_n(Y))$ with $y_{ij}^\lambda \rightarrow y_{ij}$ in $\sigma(Y, W)$. Thus $\text{Re } y_{ij}^\lambda(x) \rightarrow \text{Re } y_{ij}(x)$ for all $x \in X$. Replacing x by ix we see that in fact $y_{ij}^\lambda(x) \rightarrow y_{ij}(x)$. That is $y_{ij}^\lambda \rightarrow y_{ij}$ in $\sigma(Y, X)$. So since Y is the operator space dual of X , by the second criterion in 1.6.4 in [6], $[y_{ij}]$ is in $\text{Ball}(M_n(Y))$. Thus by the same criterion in the real case, $W^\dagger \cong Y$ completely isometrically as real operator spaces. It is easy to see that this is a weak* homeomorphism.

We claim that the map $Id : X \rightarrow W$ is a real linear isometry and complete contraction with completely bounded inverse. For $n, m \in \mathbb{N}$ and $[\varphi_{ij}] \in \text{Ball}(M_m(Y))$, $[x_{kl}] \in M_n(X)$ we have

$$\|[\text{Re } \varphi_{ij}(x_{kl})]\| \leq \|[\varphi_{ij}(x_{kl})]\| \leq \|[\varphi_{ij}]\| \| [x_{kl}] \|_{M_n(X)}.$$

So $Id : X \rightarrow W$ is completely contractive. That its inverse is completely bounded follows from

$$\|[\varphi_{ij}(x_{kl})]\| \leq \|[\operatorname{Re} \varphi_{ij}(x_{kl})]\| + \|[\operatorname{Re} \varphi_{ij}(\iota x_{kl})]\| \leq 2\|x_{kl}\|_{M_n(W)}.$$

and taking the supremum over such $[\varphi_{ij}] \in \operatorname{Ball}(M_m(Y))$. \square

We end this section with some results which are needed in Section 4.

Let B be a real unital C^* -algebra, such as $B(H)$. The real Paulsen system of a real operator space $X \subset B$, is the real operator system

$$\mathcal{S}_{\mathbb{R}}(X) = \left[\begin{array}{cc} \mathbb{R}I & X \\ X^* & \mathbb{R}I \end{array} \right] = \left\{ \left[\begin{array}{cc} \lambda I & x \\ y^* & \mu I \end{array} \right] : x, y \in X, \lambda, \mu \in \mathbb{R} \right\}$$

in $M_2(B)$, where $I = I_B$. The well known ‘Paulsen trick’ (see e.g. [6, Lemma 1.3.15] or [27, Chapter 8]) holds in the real case (see e.g. the proof of [33, Theorem 4.3]).

We will also need the theory of injective envelopes $I(\cdot)$ of operator spaces and operator systems. For the uninitiated, probably an efficient way to get hold of the main details and facts here is to read e.g. Section 4.2 up to and including 4.2.7, 4.4.2, and 4.4.7, in [6].

If B is a C^* -algebra then the multiplier algebra $M(B)$ contains a copy of every C^* -algebra containing B as an essential ideal (2.6.10 in [6]). We checked this in the real case in [4]. This is one ingredient for the real case of [8, Corollary 1.8], a result for nonzero operator spaces X . Indeed the latter has the same proof, using the real versions from e.g. [4] of the complex facts and results used in [8, Corollary 1.8]. We shall need the first assertion of [8, Corollary 1.8 (iii)] in the real case in the next proof. In both the real and complex case we remark that 1) this first assertion also easily follows from the second, and 2) the interested reader can find a proof of the second in the real case in our Appendix to a recently published paper of Cecco (again $X \neq (0)$). We sketch a proof of 1) in the complex case (the real case is the same): Let W be the complex TRO $I(X)$, and $\mathcal{L}(W)$ its usual ‘TRO linking C^* -algebra’, with WW^*, W, W^* , and W^*W as its four corners. Let $\mathcal{L} = \mathcal{L}(W)$. Since \mathcal{L} is an ideal in $I(\mathcal{S}(X))$, by that second assertion there is a canonical completely contractive unital homomorphism

$$I(\mathcal{L}) \cong I(\mathcal{S}(X)) \rightarrow \operatorname{LM}(\mathcal{L}) \subset I(\mathcal{L}),$$

that compose to the identity, and factors through a $*$ -homomorphism $\pi : I(\mathcal{S}(X)) \rightarrow \operatorname{M}(\mathcal{L})$. Alternatively, π is faithful by the ‘essential property’ of the injective envelope. So $I(\mathcal{L}) \cong \operatorname{M}(\mathcal{L})$, the desired first assertion of [8, Corollary 1.8 (iii)].

Lemma 3.5. *Let E be a nonzero complex operator space, and $X = E_r$, the underlying real operator space. The complex ‘injective linking algebra’ $I(\mathcal{S}(E))$ (see [6, 4.4.2 ff.]) is a real injective envelope of the real Paulsen system of X . That is (loosely speaking), $I(\mathcal{S}(E)) = I_{\mathbb{R}}(\mathcal{S}_{\mathbb{R}}(X))$.*

Proof. We will use notation from [6, 4.4.2 ff.] and [8]. Let $B = I(\mathcal{S}(E))$. Let W be the complex TRO $I_{\mathbb{C}}(E)$, and $\mathcal{L}(W)$ its usual ‘TRO linking C^* -algebra’ as discussed above the lemma. By [5, Corollary 4.14] we have that W is a real injective envelope of X : that is $W = I_{\mathbb{C}}(E) = I_{\mathbb{R}}(X)$. By [8, Corollary 1.8 (iii)] and its ‘real case’ discussed above we have

$$I_{\mathbb{R}}(\mathcal{S}_{\mathbb{R}}(X)) = M_{\mathbb{R}}(\mathcal{L}(W)) = M_{\mathbb{C}}(\mathcal{L}(W)) = B.$$

We are also using the fact that the complex multiplier algebra $M_{\mathbb{C}}(\cdot)$ of a nonzero complex C^* -algebra is its real multiplier algebra $M_{\mathbb{R}}(\cdot)$. The latter is easily seen since every complex Hilbert space H is a real Hilbert space, and a faithful nondegenerate complex $*$ -representation on H may be viewed as a real $*$ -representation. \square

4. BANACH SPACE DUALITY

Theorem 4.1. *If the real dual Y of a real Banach space X has complex Banach space structure (isometrically) then X has complex Banach space structure (isometrically). Moreover, there exists a canonical complex structure on X whose complex dual is Y (with its given complex structure) complex isometrically and weak* homeomorphically.*

Proof. We adapt the difficult proof of Theorem 3.1, as it is presented in [6, Theorem 4.7.1]. Let Y be a complex Banach space having an isometric real predual X via an isometry $\nu : Y \rightarrow X^\dagger$. There is a canonical weak* continuous real linear contractive projection $Y^{\dagger\dagger} \rightarrow Y$, the dual of the inclusion $i_X : X \rightarrow Y^\dagger$. Via the isomorphism $Y^{**} \cong Y^{\dagger\dagger}$ in Lemma 1.3 we obtain a canonical weak* continuous real linear contraction $q : Y^{**} \rightarrow Y$, which is a projection (taking $i_Y(y)$ to y for $y \in Y$). We endow Y with its (complex) minimal operator space structure (see e.g. 1.2.21 and 1.4.12 in [6]). Then Y is a complex operator space, and q is a weak* continuous completely contractive real linear projection by Proposition 2.3.

Let $u = L_i$, multiplication by i on Y . As in the first line of the proof of Theorem 3.3 (1), this is a (complex) multiplier (indeed centralizer, but we do not need this) of Y . To prove the first assertion of the theorem it suffices to prove that u is weak* continuous, by the ideas in the proof of Theorem 3.3 (1). We will not repeat that argument here. Of course u^{**} is in $B(Y^{**})$ and is weak* continuous. By the argument in the four lines below equation (4.14) in [6, Theorem 4.7.1] it suffices to prove that equation, namely that

$$q(u^{**}(\eta)) = u(q(\eta)), \quad \eta \in Y^{**}.$$

Let $A = I(\mathcal{S}(Y))$ (see 4.4.2 ff. in [6]), the complex ‘injective linking algebra’ of Y . We now follow the proof in [6, Theorem 4.7.1] verbatim (but with X there being our Y), up to the point where we view $\mathcal{S}(Y^{**}) \subset A^{**}$. We then adapt the rest of the proof of [6, Theorem 4.7.1] as follows:

By the ‘Paulsen trick’ in the remark below Corollary 4.5, q induces a canonical real linear unital complete contraction $Q : \mathcal{S}_{\mathbb{R}}(Y^{**}) \rightarrow \mathcal{S}_{\mathbb{R}}(Y) \subset A$. It is evident that $Q|_{\mathcal{S}(Y)} = I|_{\mathcal{S}(Y)}$. Since A is real injective, we can extend Q to a real linear unital complete contraction $R : A^{**} \rightarrow A$. Since A is a real injective envelope $I_{\mathbb{R}}(\mathcal{S}_{\mathbb{R}}(Y))$ by Lemma 3.5, and $R|_{\mathcal{S}_{\mathbb{R}}(Y)} = I|_{\mathcal{S}_{\mathbb{R}}(Y)}$, we see that $R|_A = I_A$ by the rigidity property of the real injective envelope. So R is a ‘conditional expectation’ from A^{**} onto its C^* -subalgebra A (it is UCP by Proposition 2.1). By Theorem 2.2 R is an A -bimodule map (and in particular R is complex linear). The rest of the proof of the displayed equation above, and hence of the weak* continuity of u and the first assertion of our theorem, is as in [6, Theorem 4.7.1].

The last assertion follows as in the proof of the last assertion of Theorem 3.3 (1). \square

Remarks. 1) The complex subspace of Y^* corresponding to the $\sigma(Y, X)$ -continuous complex valued functionals is real linearly isometric to X of course,

and is complex linearly isometric to X with the complex structure that we found. This uses the last assertion of the theorem. Thus all is as one would desire.

2) Special cases of Theorem 4.1 are sometimes valid with a simpler proof. Gilles Godefroy reminded us that if Y has (strongly) unique real predual X then every invertible isometry of Y is weak*-continuous. In particular, multiplication by i is weak* continuous, so Theorem 4.1 holds. As he kindly communicated, this applies to a score of spaces, e.g. when X has RNP, or Y does not contain l_1 , or when X is isomorphic to a subspace of a wsc Banach lattice, spaces which are isomorphic to subspaces of L -embedded spaces in the sense of [22, Chapter 4], etc. See [19] and e.g. [30, Corollary 4]. He also mentioned that Sections 2, 6, and 9 of [20, Chapter VII] have particular relevance to our paper. This book should be forthcoming in English in 2025. We thank him warmly for these and other interesting historical comments.

3) Continuing this theme, we give an interesting reduction in the case that the bidual of a real commutative C^* -algebra A has complex Banach space structure. Indeed the bidual is a real commutative W^* -algebra so by [5, Theorem 3.1] it is a C^* -algebra real $*$ -isomorphic to $M = L^\infty(\Omega, \mathbb{C}) \oplus^\infty L^\infty(X, \mathbb{R})$, where we allow the two summands to possibly be (0). If M has complex Banach space structure then L_i is a complex Banach space centralizer, and the latter commute with every M -projection (see e.g. [22, Section I.3]). In particular if the second summand is not (0) then L_i commutes with $(0, 1) \in M$, which forces $L^\infty(X, \mathbb{R})$ to have complex Banach space structure. However a $C(K, \mathbb{R})$ cannot have complex structure isometrically. Indeed if $J : C(K, \mathbb{R}) \rightarrow C(K, \mathbb{R})$ is a surjective isometric antisymmetry then by the Banach-Stone theorem it is of form $u \cdot (f \circ \tau)$ for a unimodular u . We also need $|\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}u(\cdot)| \leq 1$, which forces $u = -1$. Since $J^2(1) = -1$ we have $u \cdot (u \circ \tau) = -1$, a contradiction. Thus $A^{\dagger\dagger}$ is real $*$ -isomorphic (and real completely isometric by e.g. [9, Theorem 2.6]) to $L^\infty(\Omega, \mathbb{C})$.

We end with some quick ‘typical applications’. By the well known Dixmier-Ng characterization of dual spaces [26, 15] we deduce:

Corollary 4.2. *Let X be a complex normed space. Suppose that there exists a real Hausdorff locally convex topology on X with respect to which $\text{Ball}(X)$ is compact. Then X is (isometrically) a complex dual Banach space.*

Corollary 4.3. *Let X be a complex normed space. Suppose that there exists a total subspace V of the real dual X^\dagger of X such that $\text{Ball}(X)$ is $\sigma(X, V)$ -compact. Then X is (isometrically) a complex dual Banach space.*

It is tempting to think that the Dixmier-Ng conditions could be used to prove Theorem 4.1. However there appears to be a subtle but serious problem with this.

Corollary 4.4. *Let A be a complex C^* -algebra. Then A is a complex von Neumann algebra if and only if it has a real Banach space predual.*

Proof. By Remark 2 after Lemma 1.3 any complex von Neumann algebra has a real predual. Conversely, if A has a real predual then it has a complex predual by Theorem 4.1 (or by Remark 2 above), and hence it is a von Neumann algebra by Sakai’s theorem (characterizing von Neumann algebras). \square

Remark. A similar result holds for TRO’s, or indeed in any category that has a ‘Sakai theorem’ of the type used above, There is a similar result for unital operator

algebras in the operator space category using our characterization of dual operator algebras in [6, Theorem 2.7.9].

We now give a quick proof of **Theorem 1.2**:

Proof. By Theorem 4.1 applied twice, X is a complex Banach space, and $(L_i)^{\dagger\dagger} = L_i$. It is easy to see that $\iota_X(X) \subset i_X(X)$ in $X^{\dagger\dagger}$. Indeed this is a Banach space statement which we leave as a pleasant exercise for the reader. So X is a complex operator subspace of X^{**} . The last assertion is clear by Lemma 3.2. \square

Remark. It is shown in [4, Section 3] that a real operator space has at most one complex operator space structure up to complete isometry.

In the spirit of deducing complex structure from complex structure in the bidual, we offer the following result:

Corollary 4.5. *Let A be a real C^* -algebra. The following are equivalent:*

- (i) *A is a complex C^* -algebra.*
- (ii) *The operator space bidual $A^{\dagger\dagger}$ has (completely isometrically) complex operator space structure.*
- (iii) *$A^{\dagger\dagger}$ is (real $*$ -isomorphic to) a complex von Neumann algebra.*

Proof. (iii) \Rightarrow (ii) Use e.g. [9, Theorem 2.6].

(ii) \Rightarrow (i) If $A^{\dagger\dagger}$ has complex operator space structure then A has complex operator space structure by Theorem 1.2. So A is a complex C^* -algebra by [5, Theorem 4.14].

(i) \Rightarrow (iii) This can be seen in several ways. Of course A^{**} is a complex von Neumann algebra. There is a weak* continuous complete isometry $A^{**} \cong A^{\dagger\dagger}$ from Lemma 3.2, and it is easy to see that it is unital. Hence by an operator space version of the Kadison-Banach-Stone theorem (see e.g. [34, Theorem 4.4]) the map is also a $*$ -isomorphism. \square

Remark. A similar result holds for TRO's, or for approximately unital operator algebras (using again [5, Theorem 4.14]). We leave these to the interested reader to state. We also remark that one may not drop the words 'completely' and 'operator space' in (ii). A counterexample is the quaternions (see [4, Remark 1 after Theorem 3.1]).

Closing Remark. It is natural to ask for a proof of Theorem 4.1 only involving Banach spaces. We believe that 'multipliers' of linear spaces, and of their bidual, are essentially the correct tool for any proof. Indeed since the key to our proof involves centralizers, and Banach space centralizers correspond to certain maps on the extreme points of the ball of the dual space (see [22, Section I.3] or [3]), we expect that there should be an ingenious extreme point/multiplier argument. Unfortunately multiplication L_i by i is *not* a real Banach space centralizer or multiplier of a complex space X in the sense of the just cited literature. That is, $L_i \notin \text{Cent}_{\mathbb{R}}(X)$, unlike in the operator space case where $L_i \in Z_{\mathbb{R}}(X) = Z_{\mathbb{C}}(X)$ for operator space centralizers [4, Theorem 4.5]. Thus it appears that one cannot access the classical Banach space centralizer/multiplier theory. We would guess that the main idea should be a clever and technical manipulation of φ and restrictions of its real part,

for extreme points of the ball of the dual space, and the third dual space. However if such a proof is found it would only effect a small part of our paper, namely providing a simpler (but not typographically shorter) proof of Theorem 4.1.

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