Simple Example for "Optimal multi-link intervention in transportation network"

Guilherme Junqueira Perticarari

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1 Problem

Let $(\mathcal{G}, a, b, \nu)$ be an affine routing game, where $\mathcal{G} = (\mathcal{N}, \mathcal{E})$ is a multigraph with $|\mathcal{N}| = N$, $|\mathcal{E}| = E$ and $a, b \in \mathbb{R}^E$. Also, let $u_e \in (-1, \infty)$ be an intervention on link $e \in \mathcal{E}$ with attributed cost $h_e(u)$. Intervention u_e changes the affine delay function on link e such that, given link flow f_e:

$$\tau_e(f_e, u_e) = b_e + \frac{a_e}{1 + u_e} f_e$$

The goal is to find $u^* \in (-1, \infty)^E$ that minimizes the total cost of the operation, given by:

$$u^* = \arg\min_u T(u) = \arg\min_u C(u) + \alpha h(u)$$

where:

$$C(u) = \sum_{e \in \mathcal{E}} f_e^* \tau(f_e^*, u_e)$$

$$h(u) = \sum_{e \in \mathcal{E}} h_e(u_e)$$

Here, $f_e^*(u)$ is the link flow in link e during the Wardrop equilibrium associated with intervention u.

2 Numerical Solution

We can use Gradient Descent to minimize the total cost, T, given a 'learning rate' μ and an initial guess for $u \xleftarrow{init.} u^{(0)}$:

$$u_e^{(i+1)} = u_e^{(i)} - \mu \frac{\partial T(u)}{\partial u_e}$$

We have that:

$$T(u) = C(u) + \alpha h(u) \implies \frac{\partial T(u)}{\partial u_e} = \frac{\partial C(u)}{\partial u_e} + \alpha \frac{\partial h(u)}{\partial u_e}$$

$$\frac{\partial C(u)}{\partial u_e} = -\frac{a_e}{1+u_e} f_e^*(u) y_e(u) = -\frac{a_e (f_e^*(u))^2}{1+u_e}$$

$$\frac{\partial h(u)}{\partial u_e} = \frac{dh_e}{du_e}$$

So, given an affine routing game $(\mathcal{G}, a, b, \nu)$, intervention cost $h_e(u_e)$, a cost coefficient α, an initial guess for the intervention $u^{(0)} \in (-1, \infty)^E$, learning rate μ, a $\nu = m(\delta^{(o)} - \delta^{(d)})$ for throughput $m > 0$, and some stopping criteria *condition*(), the algorithm would go as follows:

Algorithm 1 Gradient Descent to find u^*

$u \leftarrow u^{(0)}$
while *condition*() **do**
  $f^* = \arg\min_{f \in \mathbb{R}^+, Bf = \nu} \sum_{e \in \mathcal{E}} \int_0^{f_e} \tau_e^{u_e}(s) ds$
  for $e \in \mathcal{E}$ **do**
    $\Delta h_e \leftarrow h_e'(u_e)$
    $\Delta C_e \leftarrow -\frac{a_e (f_e^*)^2}{1+u_e}$
  end for
  $\Delta T = \Delta C + \alpha \Delta h$
  $\Delta u = -\mu \Delta T$
  $u \leftarrow u + \Delta u$
end while
$u^* \leftarrow u$

3 Simple Numerical Example

In order to test the algorithm, I created a simple graph with N=4 and E=6, as shown in Figure 1. All $b_e = 0$ for simplicity.

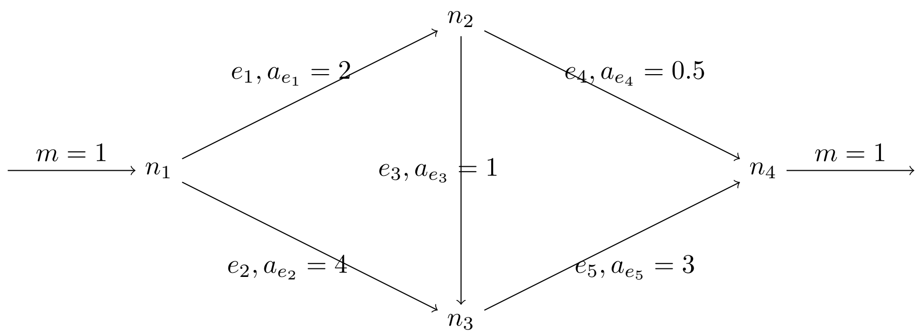

Figure 1: Simple graph with a single origin, n_1, and a single destination, n_4.

I'll be using the following assumptions:

1. $u^{(0)} \leftarrow [1, 0.5, 5, 0, 0]$
2. $\mu = 0.01$
3. $\alpha = 0.5$
4. $h_e(u_e) = u_e^2$
5. criteria: run for 300 iterations

The results can be seen below, and the graph of T (total cost of u) can be seen in Figure 2. The code can be found at github.com/guijp.

1. $u^* = [0.74, 0.15, 0.24, 0.24, 0.13]$
2. $f^* = [0.81, 0.19, 0.06, 0.75, 0.25]$
3. $C(u^*) = 1.27$
4. $h(u^*) = 0.3519$

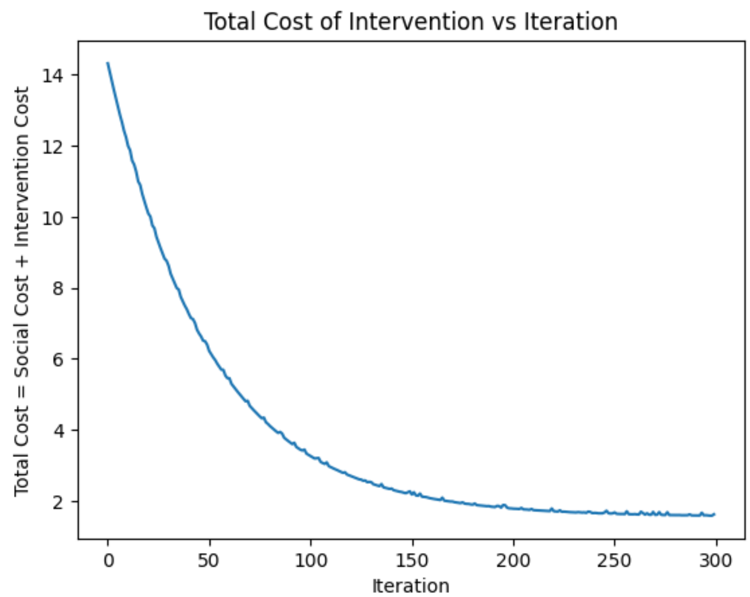

Figure 2: Total Cost of Intervention, as defined in Secion 1, for each iteration of the gradient descent algorithm.