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UNICAP - Ciência da Computação - 2º período

Disciplina: Elementos da integralização Computacional

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Lista #12 - Probabilidade: casos práticos

① $f_T(t) = \frac{1}{100} e^{-t/100}, \quad t \geq 0.$

$u = \frac{t}{100} \to \frac{du}{dt} = \frac{1}{100} \to dt = 100\,du$

a) $P(T > 200)$: $\int_{200}^{\infty} \frac{1}{100} e^{-t/100}\, dt$

$w = -u \to \frac{dw}{du} = -1 \to dw = -du \to -dw = du$

$\hookrightarrow \int_{2}^{\infty/100} \frac{1}{\cancel{100}} e^{-u} \cancel{100}\, du = \int_{2}^{\infty/100} e^{-u}\, du = \int_{2}^{\infty/100} e^{w}\, (-dw) = -\int_{2}^{\infty/100} e^{w}\, dw = -e^{w} \Big|_2^{\infty/100} = -e^{-u} \Big|_2^{\infty/100}$

$\hookrightarrow \lim_{a\to\infty} -e^{-u} \Big|_2^{a/100} = \lim_{a\to\infty} (-e^{-a/100} + e^{-2}) = \lim_{a\to\infty}\left(-\frac{1}{e^{a/100}}\right) + e^{-2}$

$= -\frac{1}{e^{\infty/100}} + e^{-2} = -\frac{1}{e^{\infty}} + e^{-2} = -\cancel{\frac{1}{\infty}}^{\,0} + e^{-2} = \boxed{e^{-2}}$

b) Esperança de uma variável aleatória discreta X é dada por:

$\hookrightarrow E(X) = \int_{-\infty}^{\infty} x \cdot f(x)\, dx \to$ multiplica a variável aleatória X pela sua função densidade de probabilidade $f(x)$ sobre todo o intervalo.

$t > 0$

$\therefore E(T) = \int_0^{\infty} t \cdot \frac{1}{100} e^{-t/100}\, dt = \int_0^{\infty} \frac{t}{100} e^{-t/100}\, dt$

$u = t/100 \to \frac{du}{dt} = \frac{1}{100} \to dt = 100\,du$

$= \int_0^{\infty/100} u \cdot e^{-u}\, 100\,du = 100 \int_0^{\infty/100} u \cdot e^{-u}\, du = 100 \cdot \lim_{a\to\infty} \int_0^{a/100} u \cdot e^{-u}\, du$

$v = u$; $dv = 1\,du$; $w = -e^{-u}$; $dw = e^{-u}\,du$

$= u \cdot -e^{-u} - \int -e^{-u}\, du = -u \cdot e^{-u} + \int e^{-u}\, du = -u \cdot e^{-u} - e^{-u} + C$

$= -e^{-u}(u+1) + C \to 100 \cdot \left[-e^{-u}(u+1)\right]_0^{\infty/100} = 100 \cdot \lim_{a\to\infty} \left[-e^{-u}(u+1)\right]_0^{a/100}$

$= 100 \cdot \lim_{a\to\infty} \left(-e^{-a/100}(a/100+1) + e^{0}(0/100+1)\right) = 100 \cdot \lim_{a\to\infty}\left(-e^{-a/100}(a/100+1) + 1\right)$

$= 100 \cdot \left(\frac{1}{-e^{\infty/100}}\left(\frac{\infty}{100}+1\right) + 1\right) = 100 \cdot \left(\cancel{\frac{1}{-e^{\infty}}}\,\cancel{(\infty+1)} + 1\right) = 100\,(0+1) = \boxed{100}$

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$0{,}8 \times 0{,}8 = 0{,}64$ $\quad 0{,}64 \times 0{,}8 = 0{,}512$ $\quad 1 - 0{,}512 = 0{,}488$

② $f_W(w) = 3w^2;\ 0 \leq w \leq 1$

a) $\int_0^1 3w^2 dw = 3\int_0^1 w^2 dw = \cancel{3}\cdot\frac{w^3}{\cancel{3}}\Big|_0^1 = \boxed{1} = P(0 \leq w \leq 1)$ ∴ é função densidade!

b) $P(W > 0{,}8)$: $\int_{0{,}8}^1 3w^2 dw = w^3\Big|_{0{,}8}^1 = 1^3 - 0{,}8^3 = 1 - 0{,}512 = \boxed{0{,}488}$

$E(W) = \int_0^1 w \cdot 3w^2 dw = \int_0^1 3w^3 dw = 3\cdot\frac{w^4}{4}\Big|_0^1 = \frac{3}{4}\cdot 1^4 - \cancel{\frac{3}{4}\cdot 0^4} = \boxed{\frac{3}{4}}$

③ $f_R(r) = \frac{1}{2\sqrt{r}};\ 0 < r < 1.$

a) $\int_0^1 \frac{1}{2\sqrt{r}} dr = \frac{1}{2}\int_0^1 r^{-1/2} dr = \frac{1}{2}\cdot\frac{r^{1/2}}{1/2}\Big|_0^1 = \frac{1}{\cancel{2}}\cdot\frac{\cancel{2}}{1}\cdot r^{1/2}\Big|_0^1 = \sqrt{1} - \sqrt{0} = \boxed{1}$

∴ é função densidade!

b) $P(0{,}25 < R < 0{,}81)$: $\int_{0{,}25}^{0{,}81} \frac{1}{2\sqrt{r}} dr = r^{1/2}\Big|_{0{,}25}^{0{,}81} = \sqrt{0{,}81} - \sqrt{0{,}25} = 0{,}9 - 0{,}5 = \boxed{0{,}4}$

$E(R)$: $\int_0^1 r\cdot\frac{1}{2\sqrt{r}} dr = \frac{1}{2}\int_0^1 r\cdot r^{-1/2} dr = \frac{1}{2}\int_0^1 r^{1/2} dr = \frac{1}{2}\cdot\frac{r^{3/2}}{3/2}\Big|_0^1 = \frac{1}{\cancel{2}}\cdot\frac{\cancel{2}}{3}\cdot r^{3/2}\Big|_0^1$

$= \frac{1}{3}\cdot r^{3/2}\Big|_0^1 = \frac{1}{3}\cdot 1^{3/2} - \cancel{\frac{1}{3}\cdot 0^{3/2}} = \boxed{\frac{1}{3}}$

④ $f_X(x) = e^{-x};\ x > 0.$ $E(X)$: $\int_0^\infty x\cdot e^{-x} dx$ $\left\{\begin{array}{l} u = x \\ du = 1dx \\ v = -e^{-x} \\ dv = e^{-x}dx \end{array}\right.$ $\to x\cdot -e^{-x} - \int -e^{-x} dx$

$= -xe^{-x} + \int e^{-x} dx = -xe^{-x} - e^{-x} + C = -e^{-x}(x+1) + C$

$\therefore -e^{-x}(x+1)\Big|_0^\infty = \lim_{a\to\infty}\left[-e^{-x}(x+1)\Big|_0^a\right] = \lim_{a\to\infty}\left[-e^{-a}(a+1) + \cancel{e^{0}}(0+1)\right]$

$= \lim_{a\to\infty}\left[-\frac{1}{e^a}\cancel{(a+1)} + 1\right] = \boxed{1}$

$E(X^2)$: $\int_0^\infty x^2 e^{-x} dx$ $\left\{\begin{array}{l} u = x^2 \\ du = 2x dx \\ v = -e^{-x} \\ dv = e^{-x}dx \end{array}\right.$ $\to x^2\cdot -e^{-x} - \int -e^{-x}\cdot 2x\cdot dx = -x^2e^{-x} + 2\int x\cdot e^{-x} dx$ $\left\{\begin{array}{l} u = x \\ du = 1dx \\ v = -e^{-x} \\ dv = e^{-x}dx \end{array}\right.$

$= -x^2e^{-x} + 2\left(-xe^{-x} + \int e^{-x} dx\right) = -x^2e^{-x} - 2xe^{-x} - 2e^{-x} + C$

$= -e^{-x}(x^2 + 2x + 2) + C$ $\therefore -e^{-x}(x^2+2x+2)\Big|_0^\infty = \lim_{a\to\infty}\left[-e^{-a}(a^2+2a+2) + \cancel{e^{0}}(\cancel{0^2+2\cdot 0}+2)\right]$

$= \lim_{a\to\infty}\left[-e^{-a}(a^2+2a+2) + 2\right] = \lim_{a\to\infty}\left[-\frac{1}{e^a}\cancel{(a^2+2a+2)} + 2\right] = \boxed{2}$

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⑤ $f_{X,Y}(x,y) = k(x+y)$; $0 \leq x \leq 1$ e $0 \leq y \leq 1$

a) $\int_0^1 \int_0^1 k(x+y)\,dx\,dy = 1 \therefore k\left(\int_0^1\int_0^1 x\,dx\,dy + \int_0^1\int_0^1 y\,dx\,dy\right)$

$= k\cdot\left(\int_0^1 \frac{x^2}{2}\Big|_0^1 dy + \int_0^1 x\Big|_0^1 y\,dy\right) = k\left(\frac{1}{2}\cdot y\Big|_0^1 + \frac{y^2}{2}\Big|_0^1\right) = k\left(\frac{1}{2}+\frac{1}{2}\right) = k\cdot 1 = k$

$= \boxed{k = 1}$

b) $P(X > 0{,}5,\ Y < 0{,}25)$: ($x: [0,1]$ e $y: [0,1]$, $k=1$) $\int_0^{0,25}\int_{0,5}^1 1(x+y)\,dx\,dy = \int_0^{0,25}\int_{0,5}^1 x\,dx\,dy + \int_0^{0,25}\int_{0,5}^1 y\,dx\,dy$

$= \int_0^{0,25} \frac{x^2}{2}\Big|_{0,5}^1 dy + \int_0^{0,25} x\Big|_{0,5}^1 \cdot y\,dy$ → $x\Big|_{1/2}^1 = 1 - \frac{1}{2} = \boxed{\frac{1}{2}}$

↳ $\frac{x^2}{2}\Big|_{1/2}^1 = \frac{1}{2} - \frac{\left(\frac{1}{2}\right)^2}{2} = \frac{1}{2} - \frac{\frac{1}{4}}{2} = \frac{1}{2} - \frac{1}{8} = \frac{4-1}{8} = \boxed{\frac{3}{8}}$ → $\frac{\left(\frac{1}{4}\right)^2}{2} - \frac{0^2}{2} = \frac{\frac{1}{16}}{2} = \boxed{\frac{1}{32}}$

$\therefore \int_0^{0,25} \frac{3}{8}\,dy + \int_0^{0,25} \frac{1}{2}y\,dy = \frac{3}{8}\cdot y\Big|_0^{1/4} + \frac{1}{2}\cdot\frac{y^2}{2}\Big|_0^{1/4} = \frac{3}{8}\cdot\frac{1}{4} + \frac{1}{2}\cdot\frac{1}{32} = \frac{3}{32} + \frac{1}{64} = \frac{6+1}{64}$

$= \boxed{\frac{7}{64}}$

c) $F_X(x)$: $\int_0^1 (x+y)\,dy = \int_0^1 x\,dy + \int_0^1 y\,dy = x\cdot y\Big|_0^1 + \frac{y^2}{2}\Big|_0^1$

$= \boxed{x + \frac{1}{2}}$

⑥ $f_L(l) = \frac{3}{8}l^2$; $0 \leq l \leq 2$ → $Z = 2L+1$; $P(3 < Z < 5) = ?$

$Z = 2L+1 \to 2L = Z-1 \to L = \frac{Z-1}{2} \to f_Z(z) = \frac{3}{8}\cdot\left(\frac{z-1}{2}\right)^2 = \frac{3}{32}\cdot(z-1)^2$

↳ multiplica-se por $\frac{d}{dz}\left(\frac{z-1}{2}\right)$ para manter área sob $f_Z(z)$ em 1 (fator correção):

$\frac{3}{32}\cdot(z-1)^2\cdot\frac{d}{dz}\left(\frac{z-1}{2}\right) = \frac{3}{32}\cdot(z-1)^2\cdot\frac{1}{2} = \boxed{\frac{3}{64}(z-1)^2}$

$\therefore P(3 < Z < 5)$: $\int_3^5 \frac{3}{64}(z-1)^2\,dz = \frac{3}{64}\int_3^5 (z-1)^2\,dz = \frac{3}{64}\int_2^4 u^2\,du = \frac{\cancel{3}}{64}\cdot\frac{u^3}{\cancel{3}}\Big|_2^4$

↳ $u = z-1$; $\frac{du}{dz} = 1$; $du = dz$

$= \frac{1}{64}\cdot u^3\Big|_2^4 = \frac{1}{64}(64-8) = \frac{56}{64} = \frac{14}{16} = \boxed{\frac{7}{8}}$

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(7) $f_{X,Y}(x,y) = 4xy \; ; \; 0 \leq x \leq 1 \text{ e } 0 \leq y \leq 1$.

a) $\int_0^1 \int_0^1 4xy\,dx\,dy = 4y \iint_0^1 x\,dx\,dy = 4y \int_0^1 \cancel{\frac{x^2}{2}}\Big|_0^1 dy = \cancel{4} \cdot \frac{1}{\cancel{2}} \int_0^1 y\,dy = \cancel{2} \cdot \frac{y^2}{\cancel{2}}\Big|_0^1$

$= \boxed{1}$ ✓ ∴ é função densidade conjunta.

b) $P(Y > 0{,}5 \mid X = 0{,}6)$: $f(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} \to f_X(x) = \int_0^1 4xy\,dy$

$= 4x \cdot \frac{y^2}{2}\Big|_0^1 = \boxed{2x}$

∴ $f(y|x) = \frac{\cancel{4}^2 \cancel{x} y}{\cancel{2x}} = \boxed{2y} \; ; \; 0 \leq y \leq 1$

↳ p/ $x = 0{,}6$: $f(y|0{,}6) = 2y \; ; \; 0 \leq y \leq 1 \to$ ñ depende de x!

↳ $\int_{0,5}^1 2y\,dy = 2 \cdot \int_{0,5}^1 y\,dy = \cancel{2} \cdot \frac{y^2}{\cancel{2}}\Big|_{0,5}^1 = 1 - \left(\frac{1}{2}\right)^2 = 1 - \frac{1}{4} = \boxed{\frac{3}{4}}$

$P(Y > 0{,}5 \mid X > 0{,}5) = \frac{P(Y > 0{,}5 \text{ e } X > 0{,}5)}{P(X > 0{,}5)}$

$\to \frac{1}{2} - \frac{\left(\frac{1}{2}\right)^2}{2} = \frac{1}{2} - \frac{\frac{1}{4}}{2} = \frac{1}{2} - \frac{1}{8} = \frac{3}{8}$

↳ numerador: $\int_{0,5}^1 \int_{0,5}^1 4xy\,dx\,dy = 4 \int_{1/2}^1 \frac{x^2}{2}\Big|_{1/2}^1 y\,dy = \cancel{4} \cdot \frac{3}{2\cancel{8}} \int_{1/2}^1 y\,dy = \frac{3}{2} \cdot \frac{y^2}{2}\Big|_{1/2}^1$

$= \frac{3}{4} \cdot y^2\Big|_{1/2}^1 = \frac{3}{4} \cdot \left(1 - \left(\frac{1}{2}\right)^2\right) = \frac{3}{4}\left(1 - \frac{1}{4}\right) = \frac{3}{4} \cdot \frac{3}{4} = \boxed{\frac{9}{16}}$ //

↳ Denominador: $\int_{0,5}^1 f_X(x)\,dx = \int_{0,5}^1 2x\,dx = 2 \int_{1/2}^1 x\,dx = \cancel{2} \cdot \frac{x^2}{\cancel{2}}\Big|_{1/2}^1 = 1 - \frac{1}{4} = \boxed{\frac{3}{4}}$ //

∴ $P(Y > 0{,}5 \mid X > 0{,}5) = \frac{\frac{9}{16}}{\frac{3}{4}} = \frac{\cancel{9}^3}{\cancel{16}_4} \cdot \frac{\cancel{4}}{\cancel{3}} = \boxed{\frac{3}{4}}$

(8) $f_T(t) = \begin{cases} \frac{t}{2}, & 0 \leq t \leq 2 \\ 0, & \text{caso contrário} \end{cases}$

a) $\int_0^2 \frac{t}{2}\,dt = \frac{1}{2} \int_0^2 t\,dt = \frac{1}{2} \frac{t^2}{2}\Big|_0^2 = \frac{1}{4} \cdot t^2\Big|_0^2 = \boxed{1}$ ✓ ∴ é função densidade!

b) $P(1 < t < 1{,}5)$: $\int_1^{1,5} \frac{t}{2}\,dt = \frac{1}{4} t^2\Big|_1^{3/2} = \frac{1}{4} \cdot \left[\left(\frac{3}{2}\right)^2 - 1\right] = \frac{1}{4}\left(\frac{9}{4} - 1\right) = \frac{9}{16} - \frac{1}{4}$

$= \frac{9-4}{16} = \boxed{\frac{5}{16}}$

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$E(T)$: $\int_0^2 t \cdot \frac{t}{2}\,dt = \frac{1}{2}\int_0^2 t^2\,dt = \frac{1}{2}\frac{t^3}{3}\Big|_0^2 = \frac{1}{6}t^3\Big|_0^2 = \frac{1}{6}\cdot 8 = \frac{8}{6} = \boxed{\frac{4}{3}}$