1) $f_\theta(x) = \frac{\lambda}{\theta}\left(\frac{x}{\theta}\right)^{\lambda-1} \exp\left\{-\left(\frac{x}{\theta}\right)^{\lambda}\right\}$, ~~x~~ $x>0$ e $\theta>0$.

$\lambda>0$ é conhecido.

a)

$$f_\theta(x) = \exp\left\{\log\left\{\frac{\lambda}{\theta}\left(\frac{x}{\theta}\right)^{\lambda-1}\exp\left\{-\left(\frac{x}{\theta}\right)^{\lambda}\right\}\right\}\right\}.$$

$$f_\theta(x_1,\dots,x_n) = \prod_{i=1}^{n} \theta^{-\lambda}\lambda\, x_i^{\lambda-1} \exp\left\{-\left(\frac{x_i}{\theta}\right)^{\lambda}\right\}$$

$$= \theta^{-n\lambda}\lambda^{n}\prod_{i=1}^{n} x_i^{\lambda-1}\exp\left\{-\sum_{i=1}^{n}\frac{x_i^{\lambda}}{\theta^{\lambda}}\right\}$$

$$= \exp\left\{\log\left\{\theta^{-n\lambda}\lambda^{n}\prod_{i=1}^{n} x_i^{\lambda-1}\exp\left\{-\sum_{i=1}^{n}\frac{x_i^{\lambda}}{\theta^{\lambda}}\right\}\right\}\right\}$$

$$= \exp\left\{\underbrace{-n\lambda\log\theta}_{d(\theta)} + \underbrace{n\log\lambda + \sum_{i=1}^{n}(\lambda-1)\log x_i}_{S(x)} \underbrace{-\sum_{i=1}^{n}\frac{x_i^{\lambda}}{\theta^{\lambda}}}_{c(\theta)T(x_1,\dots,x_n)}\right\}$$

$\Rightarrow$ É família exponencial

b) EMV

$$f_\theta(x_1,\dots,x_n) = \prod_{i=1}^{n}\theta^{-\lambda}\lambda\, x_i^{\lambda-1}\exp\left\{-\left(\frac{x_i}{\theta}\right)^{\lambda}\right\}$$

$$= \theta^{-n\lambda}\lambda^{n}\prod_{i=1}^{n} x_i^{\lambda-1}\exp\left\{-\sum_{i=1}^{n}\frac{x_i^{\lambda}}{\theta^{\lambda}}\right\}.$$

$$\log f_\theta(x_1,\dots,x_n) = -n\lambda\log\theta + n\log\lambda + (\lambda-1)\sum_{i=1}^{n}\log x_i - \sum_{i=1}^{n}\frac{x_i^{\lambda}}{\theta^{\lambda}}$$

$$\frac{\partial \log f_\theta(x_1,\dots,x_n)}{\partial\theta} = -\frac{n\lambda}{\hat\theta_{EMV}} + \lambda\sum_{i=1}^{n}\frac{x_i^{\lambda}}{\hat\theta_{EMV}^{\lambda+1}} = 0$$

$$\Rightarrow \quad \frac{-n\lambda}{\hat{\theta}_{EMV}} + \lambda \frac{\sum_{i=1}^{n} x_i^{\lambda}}{\hat{\theta}_{EMV}^{\lambda+1}} = 0$$

$$\Rightarrow \quad -n + \frac{\sum_{i=1}^{n} x_i^{\lambda}}{\hat{\theta}_{EMV}^{\lambda}} = 0 \Rightarrow \hat{\theta}_{EMV}^{\lambda+1} = \frac{\sum_{i=1}^{n} x_i^{\lambda}}{n} \Rightarrow \hat{\theta}_{EMV} = \sqrt[\lambda]{\frac{\sum_{i=1}^{n} x_i^{\lambda}}{n}}$$

$$\Rightarrow \quad E_{\theta}(x) = \theta\,\Gamma\left(1+\frac{1}{\lambda}\right) \Rightarrow \hat{\theta}_{EM}\,\Gamma\left(1+\frac{1}{\lambda}\right) = \overline{X}$$

Esperança de uma weibull,,

$$\Rightarrow \quad \hat{\theta}_{EM} = \overline{X}\left[\Gamma\left(1+\frac{1}{\lambda}\right)\right]^{-1}$$ //

c) $g(\theta) = \log\theta, \Rightarrow g(\hat{\theta}_{EMV}) = \log(\hat{\theta}_{EMV})$

$$= \log\left(\sqrt[\lambda]{\frac{\sum_{i=1}^{n} x_i^{\lambda}}{n}}\right) = \frac{1}{\lambda+1}\log\frac{\sum_{i=1}^{n} x_i^{\lambda}}{n},$$

$$\frac{g(\hat{\theta}_{EMV}) - g(\theta)}{\sqrt{\dfrac{g'(\hat{\theta}_{EMV})^2}{n I_F(\hat{\theta}_{EMV})}}} \sim N(0,1)$$

i) $g'(\hat{\theta}_{EMV}) = \dfrac{1}{\hat{\theta}_{EMV}} \Rightarrow g'(\hat{\theta}_{EMV}) = \dfrac{1}{\hat{\theta}_{EMV}} = \left(\dfrac{\sum_{i=1}^{n} x_i^{\lambda}}{n}\right)^{-\frac{1}{\lambda}}$ //

ii) $n I_F(\theta) = -E_{\theta}\left(\dfrac{\partial^2 \log f_{\theta}(x_1,\dots,x_n)}{\partial\theta^2}\right) =$

$$= -\mathcal{E}\left\{ \frac{\partial^2}{\partial\theta^2}\left\{ -n\lambda\log\theta + n\log\lambda + (\lambda-1)\sum_{i=1}^{n}\log X_i - \sum_{i=1}^{n}\frac{X_i^\lambda}{\theta^\lambda} \right\}\right\}$$

$$= -\mathcal{E}\left\{ \frac{\partial}{\partial\theta}\left\{ -\frac{n\lambda}{\theta} + \lambda\sum_{i=1}^{n}\frac{X_i^\lambda}{\theta^{\lambda+1}} \right\}\right\}$$

$$= -\mathcal{E}\left\{ \frac{n\lambda}{\theta^2} - \lambda(\lambda+1)\sum_{i=1}^{n}\frac{x_i^\lambda}{\theta^{\lambda+2}} \right\}$$

$$\mathcal{E}_\theta(X_i^\lambda) = \theta^\lambda$$

$$Y_i = X_i^\lambda \Rightarrow Y_i \sim Exp(1/\theta^\lambda)$$

$$= -\frac{n\lambda}{\theta^2} + \frac{\lambda(\lambda+1)}{\theta^{\lambda+2}}\sum_{i=1}^{n}E_\theta(X_i^\lambda) = -\frac{n\lambda}{\theta^2} + \frac{\lambda(\lambda+1)}{\theta^{\lambda+2}}\, n\,\theta^\lambda = -\frac{n\lambda}{\theta^2} + \frac{\lambda(\lambda+1)n}{\theta^2}$$

$$= \frac{n}{\theta^2}\left(\lambda^2 + \lambda - \lambda\right) = \frac{\lambda^2 n}{\theta^2} \;''$$

$$\Rightarrow n\, I_F(\hat\theta_{EMV}) = \frac{\lambda^2 n}{\hat\theta_{EMV}^2} = \frac{\lambda^2 n}{\left(\sum_{i=1}^{n}\frac{X_i^\lambda}{n}\right)^{\frac{2}{\lambda}}},$$

$$\frac{\log\left(\sum_{i=1}^{n}\frac{X_i^\lambda}{n}\right)^{\frac{1}{\lambda}} - \log\theta}{\dfrac{\left(\sum_{i=1}^{n}\frac{X_i^\lambda}{n}\right)^{-\frac{1}{\lambda}}\cdot\left(\sum_{i=1}^{n}\frac{X_i^\lambda}{n}\right)^{\frac{1}{\lambda}}}{\lambda\sqrt{n}}} = \frac{\frac{1}{\lambda}\log\sum_{i=1}^{n}\frac{X_i^\lambda}{n} - \log\theta}{\lambda\sqrt{n}} =$$

$$\Rightarrow \mathbb{P}_\theta\left(c_1 \leq \frac{\frac{1}{\lambda}\log \frac{\sum_{i=1}^{n} X_i^\lambda}{n} - \log\theta}{\lambda\sqrt{n}} \leq c_2 \right) = \gamma$$

$$= \mathbb{P}_\theta\left(c_1(\lambda\sqrt{n}) \leq \frac{1}{\lambda}\log \frac{\sum_{i=1}^{n} x_i^\lambda}{n} - \log\theta \leq c_2\lambda\sqrt{n} \right) = \gamma$$

$$= \mathbb{P}_\theta\left(c_1(\lambda\sqrt{n}) - \frac{1}{\lambda}\log \frac{\sum_{i=1}^{n} x_i^\lambda}{n} \leq -\log\theta \leq c_2\lambda\sqrt{n} - \frac{1}{\lambda}\log \frac{\sum_{i=1}^{n} x_i^\lambda}{n} \right) = \gamma$$

$$= \mathbb{P}_\theta\left(\frac{1}{\lambda}\log \frac{\sum_{i=1}^{n} x_i^\lambda}{n} - c_2\lambda\sqrt{n} \leq \log\theta \leq \frac{1}{\lambda}\log \frac{\sum_{i=1}^{n} X_i^\lambda}{n} - c_1(\lambda\sqrt{n}) \right) = \gamma$$

d) pela (a) $f_\theta(x)$ é família exponencial,

$$\Rightarrow T(x) = \sum_{i=1}^{n} x_i^\lambda \Rightarrow Y_i = X_i^\lambda \Rightarrow Y_i \sim Exp(1/\theta^\lambda)$$

$$\Rightarrow \sum_{i=1}^{n} x_i^\lambda \sim gama(n, 1/\theta^\lambda) \Rightarrow \frac{1}{\theta^\lambda}\sum_{i=1}^{n} x_i^\lambda \sim gama(n, 1)$$

$$\Rightarrow \frac{2}{\theta^\lambda}\sum_{i=1}^{n} x_i^\lambda \sim \chi^2_{2n}.$$

i) $Y_i = X_i^{\lambda} \Rightarrow Y_i = h(X_i) \Rightarrow h^{-1}(Y_i) = Y_i^{1/\lambda}$,

$$\Rightarrow \frac{\partial h^{-1}(Y_i)}{\partial Y_i} = \frac{1}{\lambda} Y_i^{\frac{1}{\lambda}-1},$$

$$\Rightarrow f_Y(Y_i) = f_X(h^{-1}(Y_i)) \left| \frac{\partial h^{-1}(Y_i)}{\partial Y_i} \right|$$

$$= \theta^{-\lambda} \cancel{\lambda} \left(Y_i^{\frac{1}{\lambda}} \right)^{\lambda-1} \exp\left\{ - \frac{\left(Y_i^{\frac{1}{\lambda}}\right)^{\lambda}}{\theta^{\lambda}} \right\} \frac{1}{\cancel{\lambda}} Y_i^{\frac{1}{\lambda}-1}$$

$$= \theta^{-\lambda} Y_i^{\cancel{1-\frac{1}{\lambda}}} \exp\left\{ -\frac{Y_i}{\theta^{\lambda}} \right\} Y_i^{\cancel{\frac{1}{\lambda}-1}} = \frac{1}{\theta^{\lambda}} \exp\left\{ -\frac{Y_i}{\theta^{\lambda}} \right\}$$

$$\Rightarrow Y_i \sim Exp(1/\theta^{\lambda}).$$

e)

$$\frac{L_{\theta_1}(\underset{\sim}{x})}{L_{\theta_0}(\underset{\sim}{x})} = \frac{\theta_1^{-\lambda n} \lambda^n \prod_{i=1}^{n} x_i^{\lambda-1} \exp\left\{ -\left(\frac{x_i}{\theta_1}\right)^{\lambda} \right\}}{\theta_0^{-\lambda n} \cancel{\lambda^n} \cancel{\prod_{i=1}^{n} x_i^{\lambda-1}} \exp\left\{ -\left(\frac{x_i}{\theta_0}\right)^{\lambda} \right\}} = \frac{\theta_1^{-\lambda n} \exp\left\{ -\left(\frac{x_i}{\theta}\right)^{\lambda} \right\}}{\theta_0^{-\lambda n} \exp\left\{ -\left(\frac{x_i}{\theta}\right)^{\lambda} \right\}}$$

$$= \left(\frac{\theta_0}{\theta_1}\right)^{\lambda n} \exp\left\{ \sum_{i=1}^{n} x_i^{\lambda} \left(\frac{1}{\theta_0^{\lambda}} - \frac{1}{\theta_1^{\lambda}} \right) \right\} > k;$$

$$\mathbb{P}_{\theta}\left(\sum_{i=1}^{n} x_i^{\lambda} > \underbrace{\frac{\log\left\{ k \left(\frac{\theta_0}{\theta_1}\right)^{-\lambda n} \right\}}{\left(\frac{1}{\theta_0^{\lambda}} - \frac{1}{\theta_1^{\lambda}} \right)}}_{k'} \right) \geq \alpha$$

f) $p(H_0; \underset{\sim}{x}) = \sup_{\theta \in \Theta_0} \mathbb{P}_\theta(T \geq t)$.

$$\underbrace{\sum_{i=1}^{10} x_i = 10,1}_{\parallel}, \quad \mathbb{P}_\theta\left(\frac{1}{\theta^\lambda}\sum_{i=1}^{n} x_i^\lambda \geq t\right) =$$

$$= \mathbb{P}_\theta\left(\frac{2}{\theta^\lambda}\sum_{i=1}^{n} x_i^\lambda \geq 2t\right) = \mathbb{P}_0\left(2\sum_{i=1}^{n} x_i \geq 2t\right), \quad \theta_0 = 1, \lambda = 1$$

$$\Rightarrow \mathbb{P}_0\left(\chi^2_{20} \geq 20{,}8\right) = 1 - \underbrace{\mathbb{P}_0\left(\chi^2_{20} < 20{,}8\right)}$$

$$\cong 1 - 0{,}62 \cong 0{,}38$$

$\Rightarrow$ não rejeitamos H_0, à um ~~um~~ nível de significância de $\alpha = 0{,}05$.

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g) Temos que.

$$\lambda(\underset{\sim}{x}) = \frac{\sup_{\theta \in \Theta_0} h_\theta(\underset{\sim}{x})}{\sup_{\theta \in \Theta} h_\theta(\underset{\sim}{x})} = \frac{\theta_0^{-\lambda n} \cancel{\lambda^n \prod_{i=1}^{n} x_i} \exp\left\{-\frac{\sum_{i=1}^{n} x_i^\lambda}{\theta_0^\lambda}\right\}}{\hat{\theta}_{EMV}^{-\lambda n} \cancel{\lambda^n \prod_{i=1}^{n} x_i} \exp\left\{-\sum_{i=1}^{n} \frac{x_i^\lambda}{\hat{\theta}_{EMV}^\lambda}\right\}} =$$

$$= \left(\frac{\hat{\theta}_{EMV}}{\theta_0}\right)^{\lambda n} \exp\left\{-\left(\frac{n}{\theta_0^\lambda} - \frac{n}{\hat{\theta}_{EMV}^\lambda}\right)\frac{\sum_{i=1}^{n} x_i^\lambda}{n}\right\} =$$

(4)

$$= \exp\left\{-\left(\frac{n}{\theta_0^\lambda}-\frac{n}{\hat{\theta}_{EMV}^\lambda}\right)\sum_{i=1}^{n}\frac{x_i^\lambda}{n} + n\log\left(\frac{\hat{\theta}_{EMV}}{\theta_0}\right)^\lambda\right\}$$

$$= \exp\left\{-\left(\frac{1}{\theta_0^\lambda}-\frac{1}{\hat{\theta}_{EMV}^\lambda}\right)n\,\hat{\theta}_{EMV}^\lambda + n\log\left(\frac{\hat{\theta}_{EMV}}{\theta_0}\right)^\lambda\right\} < K$$

$\Rightarrow$ rejeitamos H_0 se $\lambda(\underset{\sim}{x}) < K$

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2)

$$P_\theta(X=x)=\begin{cases}\theta/2, & \text{se } x=0,\\ 1/2, & \text{se } x=1,\\ (1-\theta)/2, & \text{se } x=2,\\ 0, & \text{c.c.}\end{cases}$$

$H_0: \theta = 0{,}1,$

$H_1: \theta \in \{0{,}6,\ 0{,}9\},$

temos que,

$X_1 \backslash X_2$	0	1	2
0	(0,0)	(0,1)	(0,2)
1	(1,0)	(1,1)	(1,2)
2	(2,0)	(2,1)	(2,2)

$X_1 \backslash X_2$	0	1	2
0	$\theta^2/4$	$\theta/4$	$\theta(1-\theta)/4$
1	$\theta/4$	$1/4$	$(1-\theta)/4$
2	$\theta(1-\theta)/4$	$(1-\theta)/4$	$(1-\theta)^2/4$
	$\theta/2$	$1/2$	$(1-\theta)/2$

$P_\theta(X_1=2, X_2=2) = P_\theta(X_1=2)\cdot P_\theta(X_2=2) = (1-\theta)^2/4$

$P_\theta(X_1=0, X_1=0) = P_\theta(X_1=0)P_\theta(X_1=0) = \theta^2/4$

$P_\theta(X_1=0, X_2=1) = P_\theta(X_1=0)P_\theta(X_2=1) = \theta/4$

$P_\theta(X_1=1, X_2=1) = P_\theta(X_1=1)P_\theta(X_2=1) = 1/4$

$P_\theta(X_1=0, X_2=2) = P_\theta(X_1=0)\cdot P_\theta(X_2=2) = \theta(1-\theta)/4$

$P_\theta(X_1=1, X_2=2) = P_\theta(X_1=1)\,P_\theta(X_2=2) = \frac{1}{2}\cdot(1-\theta)/2 = (1-\theta)/4$

a) Obtenha o tamanho do teste.

$$\pi_0 = \sup_{\theta \in \Theta_0} \mathbb{P}_\theta(X_1 + X_2 \geq 2)$$

$$= \sup_{\theta \in \Theta_0} \left\{ \frac{\theta(1-\theta)}{2} + \frac{(1-\theta)}{2} + \frac{(1-\theta)^2}{4} \right\}$$

$$= \frac{0.1(0.9)}{2} + \frac{0.9}{2} + \frac{(0.9)^2}{4}$$

$$= 0{,}045 + 0{,}45 + 0.2025 = 0{,}6975,$$

b)

$$\beta_\delta(\theta) = \begin{cases} \frac{\theta(1-\theta)}{2} + \frac{(1-\theta)}{2} + \frac{(1-\theta)^2}{4} = 0{,}6975, & \theta = 0{,}1 \\ 0.12 + 0{,}2 + 0{,}04 = 0{,}18, & \theta = 0{,}6 \\ 0.045 + 0{,}05 + 0{,}0025 = 0{,}0975, & \theta = 0{,}9 \end{cases}$$

3) $X_1, \dots, X_n$, $X_i \sim Exp(\lambda^i \theta)$, $\lambda^i > 0, \theta > 0$. (J)

I) a) Encontre EMV p/ θ;

$$f_\theta(x_1, \dots, x_n) = \prod_{i=1}^{n} \lambda^i \theta \exp\{-x_i \lambda^i \theta\}$$

$$= \lambda^{\sum_{i=1}^{n} i} \theta^n \exp\{-\sum_{i=1}^{n} x_i \lambda^i \theta\}, \quad \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\log f_\theta(x_1, \dots, x_n) = \frac{n(n+1)}{2} \log \lambda + n \log \theta - \theta \sum_{i=1}^{n} x_i \lambda^i,$$

$$\frac{\partial \log f_\theta(x_1, \dots, x_n)}{\partial \theta} = \frac{n}{\hat{\theta}_{EMV}} - \sum_{i=1}^{n} x_i \lambda^i = 0 \Rightarrow \hat{\theta}_{EMV} = \frac{1}{\sum_{i=1}^{n} \frac{x_i \lambda^i}{n}},$$

b) Seja $Y_i = X_i \lambda^i \Rightarrow Y_i \sim Exp(\theta)$

$$Y_i = h(X_i) \Rightarrow h^{-1}(Y_i) = Y_i / \lambda^i \Rightarrow \frac{\partial h^{-1}(Y_i)}{\partial Y_i} = \frac{1}{\lambda^i}$$

$$f_Y(Y_i) = f_X(h^{-1}(Y_i)) \left| \frac{\partial h^{-1}(Y_i)}{\partial Y_i} \right|$$

$$= \theta \lambda^i \exp\{-\frac{Y_i}{\lambda^i} \lambda^i \theta\} \cdot \frac{1}{\lambda^i} = \theta \exp\{-Y_i \theta\}$$

$$\Rightarrow Y_i \sim Exp(\theta). \Rightarrow \sum_{i=1}^{n} Y_i \sim gama(n, \theta) \Rightarrow \theta \sum_{i=1}^{n} Y_i \sim ga(n, 1)$$

$$\Rightarrow 2\theta \sum_{i=1}^{n} X_i \lambda^i \sim \chi^2_{2n}.$$

c) $\mathbb{P}_\theta\left(c_1 \leq \theta \sum_{i=1}^{n} x_i \lambda^i \leq c_2 \right) = 1-\alpha,$

$= \mathbb{P}_\theta\left(2c_1 \leq 2\theta \sum_{i=1}^{n} x_i \lambda^i \leq 2c_2 \right) = 1-\alpha, \quad \Rightarrow \chi^2_{2n}$

$= \mathbb{P}_\theta\left(\frac{2c_1}{\theta \sum_{i=1}^{n} x_i \lambda^i} \leq \theta \leq \frac{2c_2}{\sum_{i=1}^{n} x_i \lambda^i} \right) = 1-\alpha,$

II. modelo Bayesiano,

$$\pi_0(\theta) = \frac{\theta^{a-1}}{b^a \Gamma(a)} \exp\left\{-\frac{\theta}{b}\right\}, \quad \theta > 0, \quad a > 0 \text{ e } b > 0,$$

$$\pi(\theta | x_1, \dots, x_n) = \frac{\pi(\theta) f_\theta(x_1, \dots, x_n)}{\int \pi(\theta) f_\theta(x_1, \dots, x_n) d\theta} \propto \pi(\theta) f_\theta(x_1, \dots, x_n)$$

$$\pi(\theta) f(x_1, \dots, x_n) = \frac{\theta^{a-1}}{b^a \Gamma(a)} \exp\left\{-\frac{\theta}{b}\right\} \prod_{i=1}^{n} \lambda^i \theta \exp\left\{-x_i \lambda^i \theta\right\}$$

$$= \frac{\theta^{a-1}}{b^a \Gamma(a)} \exp\left\{-\frac{\theta}{b}\right\} \lambda^{\frac{n(n+1)}{2}} \theta^n \exp\left\{-\sum_{i=1}^{n} x_i \lambda^i \theta\right\}$$

$$\propto \theta^{a+n-1} \exp\left\{-\left(\sum_{i=1}^{n} x_i \lambda^i + \frac{1}{b}\right)\theta\right\}$$

$$\Rightarrow \pi(\theta \mid x_1,\dots,x_n) \sim \text{Gamma}\left(a+n, \left(\sum_{i=1}^{n} x_i \lambda^i + \frac{1}{b}\right)\right)$$

c) $\mathbb{P}_{\theta}\left(\theta \in (\tilde{\theta}, \infty)\right) = 1-\alpha$

$\mathbb{P}_{\theta}\left(\theta \geq \tilde{\theta} \mid x_1,\dots,x_n\right) = 1-\alpha \quad , \quad \theta \mid x_1,\dots,x_n \sim \text{Gam}\left(a+n, \sum_{i=1}^{n} x_i \lambda^i + \frac{1}{b}\right)$

i) $$\mathbb{P}_{\theta}\left(2\theta\left(\sum_{i=1}^{n} x_i \lambda^i + \frac{1}{b}\right) \geq \underbrace{2\tilde{\theta}\left(\sum_{i=1}^{n} x_i \lambda^i + b\right)}_{C_1 \Rightarrow \chi^2_{2n, \alpha}}\right) = 1-\alpha$$