Hottest Summer school 2022 Lectures 10–12
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What you will learn:

- propositional truncations
- the univalence axiom
- univalent combinatorics
- How to express yourself in type theory
- How to think carefully about concepts and getting it right.
- How to spot univalence in your daily life.

Consider a map $f : A \to B$. If we want to say that f is surjective using the Curry-Howard interpretation, then we would express this in type theory as

$$\prod_{(b:B)} \sum_{(a:A)} f(a) = b$$

This asserts that "for all $b:B$ there is an $a:A$ equipped with an identification $f(a) = b$".

However, note that this captures something stronger than surjectivity: if f is surjective in this Curry-Howard sense, then we obtain

$$g : B \to A$$
$$H : f \circ g \sim id.$$

In order to properly express that f is surjective, we need a way to say that "for every $b:B$ there is an <u>unspecified</u> $a:A$ s.t. $f(a) = b$.

The Curry-Howard interpretation is <u>not</u> appropriate to express surjectivity.

Consider a type A. To count the elements of A is to obtain a number $n : \mathbb{N}$ and an equivalence

$$\mathrm{Fin}(n) \simeq A.$$

We define $\mathrm{count}(A) := \sum_{(n:\mathbb{N})} \mathrm{Fin}(n) \simeq A$

However, if A comes equipped with a counting $(n, e) : \mathrm{count}(A)$, then the type A inherits an ordering from the type $\mathrm{Fin}(n)$ via the equivalence e

So, types equipped with a counting are totally ordered finite types.

In order to express <u>only</u> that A is finite, we need a way to say that A has an unspecified counting.

Finally, we might want to say that a type A is inhabited, i.e. that it has an unspecified element. We would like that "A is inhabited" is a proposition in type theory.

We will introduce a new operation to type theory for this purpose:

The propositional truncation.

The propositional truncation operation is speified by its universal property.
Often in mathematics, when you introduce a new object, you first want to specify the characterizing features that you want your object to have. If you do this well, the specification will determine the object uniquely.
Universal properties are a common way to give such specifications.

The propositional truncation of a type A should be a proposition P s.t.

- If A is inhabited, then P is true (the propostion $\mathbf{1}$?)
- If Q is a proposition s.t. $A \to Q$, then $P \to Q$. (so probably not $\mathbf{1}$)

Definition. A map $f : A \to P$ into a proposition P is said to be a propositional truncation of A if for every proposition Q the map

$$(P \to Q) \to (A \to Q)$$

given by $h \mapsto h \circ f$ is an equivalence.

Rmk. Since Q is a proposition, it follows that both $(P \to Q)$ and $(A \to Q)$ are propositions. So the map

$$(P \to Q) \to (A \to Q)$$

is an equivalence iff.

$$(A \to Q) \to (P \to Q)$$

The universal property of propositional truncations says that every map $g : A \to Q$ extends uniquely along f

$$\begin{tikzcd} A \arrow[d, "f"'] \arrow[dr, "g"] & \\ P \arrow[r, dashed] & Q \end{tikzcd}$$

Proposition. Consider $f : A \to P$ and $f' : A \to P'$, where P and P' are propositions. Consider the following three conditions:

(i) f is a propositional truncation
(ii) f' is a propositional truncation
(iii) P and P' are equivalent.

Proof. Let Q be a proposition, and spse $P \simeq P'$. Then we have a triangle

$$\begin{tikzcd} & (A \to Q) \arrow[dl, "- \circ f"'] \arrow[dr, "- \circ f'"] & \\ (P \to Q) \arrow[rr, "\simeq"] & & (P' \to Q) \end{tikzcd}$$

By the 3-for-2 property of equivs, it follows that the left map is an equiv iff the right map is an equiv. This shows that (iii) implies (ii) $\Leftrightarrow$ (i)

If (i) and (ii) hold, then $(P \to P') \simeq (A \to P')$ gives $P \to P'$ and $(P' \to P) \simeq (A \to P)$ gives $P' \to P$ $\square$

From now on we will assume that for each type A there is a proposition $\|A\|$ equipped with a map $\eta : A \to \|A\|$ (the unit of the propositional truncation) satisfying the universal property that

$$(\|A\| \to Q) \to (A \to Q)$$

is an equivalence for each proposition Q.

Lemma. Consider $f : A \to B$. Then we have $\|f\| : \|A\| \to \|B\|$.

Proof. We have $(\|A\| \to \|B\|) \xrightarrow{\simeq} (A \to \|B\|)$. Note that $\eta \circ f : A \to \|B\|$. This gives $\|f\| : \|A\| \to \|B\|$.

Cor. If $A \simeq B$ then we get $\|A\| \leftrightarrow \|B\|$, and hence $\|A\| \simeq \|B\|$.

Logic in type theory.

Recall that if P and Q are propositions, then $P+Q$ is a proposition iff $P \to \neg Q$.

Proof. Let $x, y : P+Q$. WTS $x=y$.

$\mathrm{inl}(x) = \mathrm{inl}(y)$ since $x=y$ in P.

$\mathrm{inl}(x) = \mathrm{inr}(y)$ since $x:P$ and $y:Q$ gives $\emptyset$.

$\mathrm{inr}(x) = \mathrm{inl}(y)$ same

$\mathrm{inr}(x) = \mathrm{inr}(y)$ since $x=y$ in Q. $\square$

We see that $P+Q$ is not always a proposition.

Defn. Given $P, Q : \mathrm{Prop}$, we define the disjunction $P \vee Q := \| P+Q \|$.

Proposition. For any $P, Q, R : \mathrm{Prop}$ we have

$$(P \vee Q \to R) \simeq (P \to R) \times (Q \to R)$$

Proof. $(P \vee Q \to R) \doteq (\|P+Q\| \to R) \simeq (P+Q \to R) \simeq$
$(P \to R) \times (Q \to R)$ (ex. 13.8).

$\square$

Consider a type A and $P : A \to \mathrm{Prop}$.
Then $\sum_{(a:A)} P(a)$ is a proposition iff
$$\prod_{(x,y:A)} P(x) \to P(y) \to x = y.$$
Indeed $\left(\sum_{(a:A)} P(a)\right) \to A$ is an embedding.
So for $x, y : A$, $p : P(x)$, $q : P(y)$ we have
$$(x,p) = (y,q)$$
This implies $\sum_{(x:A)} P(x)$ is a proposition. $\square$

By the above observation, $\sum_{(x:A)} P(x)$ is rarely a proposition. So, if we want to say that there exists an $x : A$ s.t $P(x)$ holds, then we must truncate.

Definition. Consider $P : A \to \mathrm{Prop}$. We define
$$\exists_{(x:A)} P(x) := \left\| \sum_{(x:A)} P(x) \right\|.$$

Proposition. Consider $P : A \to \mathrm{Prop}$, and $Q : \mathrm{Prop}$.
Then
$$\left(\exists_{(x:A)} P(x) \to Q\right) \simeq \left(\prod_{(x:A)} P(x) \to Q\right).$$

Proof. $(\exists_{(x:A)} P(x) \to Q)$

$$\doteq \left\| \sum_{(x:A)} P(x) \right\| \to Q$$

$$\simeq \left(\sum_{(x:A)} P(x) \right) \to Q$$

$$\simeq \prod_{(x:A)} (P(x) \to Q).$$

$\square$

Logical connective	interpretation
$\top$	$\mathbb{1}$
$\bot$	$\emptyset$
$P \Rightarrow Q$	$P \to Q$
$P \wedge Q$	$P \times Q$
$P \vee Q$	$\Vert P + Q \Vert$
$P \Leftrightarrow Q$	$(P \to Q) \times (Q \to P)$
$\exists_{(x:A)} P(x)$	$\Vert \sum_{(x:A)} P(x) \Vert$
$\forall_{(x:A)} P(x)$	$\prod_{(x:A)} P(x)$.

The image of a map.

Defn. Let $f : A \to B$. We define

$$\mathrm{im}(f) := \Sigma_{(b:B)} \| \mathrm{fib}_f(b) \|.$$

Question. What would happen if we defined the image of f to be

$$\Sigma_{(b:B)} \mathrm{fib}_f(b) \ ?$$

$$\begin{aligned}
\Sigma_{(b:B)} \mathrm{fib}_f(b) &\simeq \Sigma_{(b:B)} \Sigma_{(x:A)} f(x) = b \\
&\simeq \Sigma_{(x:A)} \Sigma_{(y:B)} f(x) = y \\
&\simeq A.
\end{aligned}$$

We also define

$$g_f : A \to \mathrm{im}(f) \qquad g_f(a) := (f(a), \eta(a, \mathrm{refl}))$$
$$i_f : \mathrm{im}(f) \to B \qquad i_f(x) := \mathrm{pr}_1(x).$$

The map i_f is an embedding, because $\|\mathrm{fib}_f(y)\|$ is a proposition for each $y : B$.
The map g_f is surjective.
Defn. A map $f : A \to B$ is said to be surjective if it has an element

$$\mathrm{is\text{-}surj}(f) := \prod_{b:B} \|\mathrm{fib}_f(b)\|.$$

g_f is surjective. Let $(b, x) : \mathrm{im}(f)$
$b : B$, $x : \|\mathrm{fib}_f(b)\|$.
We have $(\|\mathrm{fib}_f(b)\| \to \|\mathrm{fib}_{g_f}(b,x)\|) \simeq$

$$\mathrm{fib}_f(b) \to \|\mathrm{fib}_{g_f}(b,x)\|.$$

so by the universal property we may assume $a : A$, $p : f(a) = b$.

STS $\mathrm{fib}_{q_f}(f(a), \eta(a, \mathrm{refl}))$

i.e. $\mathrm{fib}_{q_f}(q_f(a))$ $\square$

Theorem. Let $f : A \to B$ be surjective.
Let $P : B \to \mathrm{Prop}$. Then

$$\left(\prod_{(b:B)} P(b)\right) \xrightarrow{\simeq} \left(\prod_{(x:A)} P(f(x))\right)$$

Proof. It suffices to construct

$$\left(\prod_{(x:A)} P(f(x))\right) \to \prod_{(b:B)} P(b)).$$

Let $h : \prod_{(x:A)} P(f(x))$, $b : B$.

then $\|\mathrm{fib}_f(b)\|$ holds. We're proving $P(b)$.
Note: $(\|\mathrm{fib}_f(b)\| \to P(b)) \xrightarrow{\simeq} (\mathrm{fib}_f(b) \to P(b))$

Let $a : A$, $p : f(a) = b$. Then we have $h(a) : P(f(a))$
and $\mathrm{tr}_P(p, h(a)) : P(b)$. $\square$

Defn. We say that a type A is finite if it comes equipped with an elt of type
$\text{is-finite}(A) := \| \sum_{(n:\mathbb{N})} \text{Fin}(n) \simeq A \|$.

Rmk. (Thm. 16.3.3)
$\text{is-finite}(A) \simeq \sum_{(n:\mathbb{N})} \| \text{Fin}(n) \simeq A \|$

Defn. $\mathbb{F} := \sum_{(X:\mathcal{U}_0)} \text{is-finite}(X)$.

Also define $BS_n := \sum_{(X:\mathcal{U}_0)} \| \text{Fin}(n) \simeq X \|$.

(book writes $\mathbb{F}_n$ for BS_n).

Eg. Let A, B be finite. Then $A+B$ is finite.
Assume $H : \| \sum_{n:\mathbb{N}} \text{Fin}(n) \simeq A \|$, $K : \| \sum_{(n:\mathbb{N})} \text{Fin}(n) \simeq B \|$
We're proving a proposition, so assume
$(n, e) : \sum_{(n:\mathbb{N})} \text{Fin}(n) \simeq A$, $(m, f) \simeq \sum_{(n:\mathbb{N})} \text{Fin}(n) \simeq B$.

Claim. $\text{Fin}_{n+m} \simeq A + B$.

$\text{Fin}_{n+m} \simeq \text{Fin}_n + \text{Fin}_m \simeq A + B$ $\square$