

→ Begin with directed graph (edges are links).

→ Remove directions (may get multigraph). Work with matchings in resulting undirected G .

Notation: E = set of edges of G .

\mathcal{M} = set of all matchings of G .

\mathcal{M}_{ok} = set of all matchings of G satisfying sink constraints.

$\mathcal{M}_{\text{ok}}^c = \mathcal{M} \setminus \mathcal{M}_{\text{ok}}$.

Primal LP: variable y_M for each $M \in \mathcal{M}$.

$$\text{minimize } \sum_{M \in \mathcal{M}_{\text{ok}}} y_M + \mu \sum_{M \in \mathcal{M}_{\text{ok}}^c} y_M \quad \mu \geq 1$$

$$\text{s.t. } y_M \geq 0 \quad \forall M \in \mathcal{M}$$

$$\sum_{\substack{M \in \mathcal{M} \\ e \in M}} y_M = 1 \quad \forall e \in E$$

(Note: $\lim_{\mu \rightarrow \infty} \text{objective function} = 1 / \text{capacity}$.)

Dual LP: variable x_e for each $e \in E$.

$$\text{maximize } \sum_{e \in E} x_e$$

$$\text{s.t. } \sum_{e \in M} x_e \leq 1 \quad \forall M \in \mathcal{M}_{\text{ok}} \quad (\text{c})$$

$$\sum_{e \in M} x_e \leq \mu \quad \forall M \in \mathcal{M}_{\text{ok}}^c \quad (\text{d})$$

Algorithm :

Initialize set of constraints C ;

$\mu := 1$;

repeat

$D := \emptyset$;

$ok := \underline{\text{false}}$;

repeat

solve the dual LP given $C \cup D$ and let
 $x_1^*, x_2^* \dots x_{|E|}^*$ be the optimal solution;

find a maximum-weight matching m^* on G
using weights $x_1^*, x_2^* \dots x_{|E|}^*$;

if $w(m^*, x^*) \leq 1$

$ok := \underline{\text{true}}$;

if $w(m^*, x^*) > \mu$ and $m^* \in M_{ok}$

add $\sum_{e \in m^*} x_e \leq \mu$ to D ;

if $w(m^*, x^*) > 1$ and $m^* \in M_{ok}$

add $\sum_{e \in m^*} x_e \leq 1$ to C ;

if $1 < w(m^*, x^*) \leq \mu$ and $m^* \in M_{ok}$

search (exhaustively) for $M \in M_{ok}$ such that

$1 < w(M, x^*) \leq w(m^*, x^*)$;

if none found

$ok := \underline{\text{true}}$;

else add $\sum_{e \in m} x_e \leq 1$ to C ;

until ok ;

if $D \neq \emptyset$

$\mu := \mu + \Delta\mu$; $(\Delta\mu > 0)$

until $D = \emptyset$;

notation : $w(M, x) = \sum_{e \in M} x_e$.