VaR using EVT approach with unknown shape parameter confidence interval width and threshold selection

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Abstract

Value at Risk (VaR_α) is an important metric of quantitative financial risk, it measures the value that a portfolio might lose at its α-quantile worse scenario. In recent years, Extreme Value Theory started being applied to estimate this number. This approach is dependent on the choice of the threshold u that defines the starting point of the tail for the returns distribution. There are some heuristics to choosing this limit but a case-by-case study is still required for this analysis.

We propose an assessment of the confidence interval of the VaR_α estimator by propagating the uncertainty of the shape parameter ξ from the tail distribution, approximated by a Generalized Pareto Distribution. The results show that choosing the threshold such that the proportion of points beyond is the complement of the level α is optimal when enough points are available. Otherwise, we find empirical evidence of a log-linear relationship between the width of the confidence interval of VaR_α with unknown ξ and the proportion of points above the threshold.

Keywords: value-at-risk (VaR); extreme value theory (EVT); confidence interval; threshold ; Generalized Pareto Distribution (GDP) ; quantitative risk management (QRM) ; shape parameter

1 Introduction

With quantitative risk management (QRM) rising in popularity among financial institutions, value-at-risk (VaR_α) has become a widely adopted risk metric Tsay (2010). It consists in estimating the loss of a portfolio in the α-quantile worst event. Given that this estimate lies on the tail of the returns distribution, the most appropriate approach to measure this value is by using extreme value theory (EVT) Taleb, Bar-Yam, and Cirillo (2020).

EVT provides two ways of estimating VaR_α, depending on the sampling method, points over the top (PoT) or block maxima Andersson (2020). In this work, we decided to study PoT since it is more efficient and more popular lately Forjan (2019).

To model VaR_α using this method, we look at the returns $X_1, X_2, ..., X_N$ and assume that they are i.i.d. coming from the same unknown distribution F.

From those returns, we are interested in modeling the values above a certain threshold u, such that:

$$F_u(x) = P(X - u < x | X > u) = \frac{F(u + x) - F(u)}{1 - F(u)}.$$

Using the Pickands-Balkema-de-Haan theorem we can approximate F_u by the Generalized Pareto distribution $G_{\xi,\sigma}$ Balkema and De Haan (1974); Bianchi, Stoyanov, Tassinari, Fabozzi, and Focardi (2019).

$$G_{\xi,\sigma}(x) = \begin{cases} 1 - (1 + \xi x/\sigma)^{-1/\xi} & \text{if } \xi \neq 0 \\ 1 - e^{-x/\sigma} & \text{if } \xi = 0 \end{cases}$$

where ξ is the shape parameter and σ is the scale parameter (not the standard deviation). Since return distributions are known for being heavy-tailed and this study is focused on the shape parameter, we assume that $\xi > 0$. Therefore, we have:

$$G_{\xi,\sigma}(x) = \frac{F(u + x) - F(u)}{1 - F(u)}. \tag{1}$$

To obtain F we need first to write $F(u)$ as:

$$F(u) = P(X \geq u) = 1 - \frac{N_u}{N},$$

where N is the total number of observations in the original sample and N_u is the number of observations above the threshold u.

Combining the equations, we have:

$$F(x) = 1 - \frac{N_u}{N}\left(1 + \xi\frac{x-u}{\sigma}\right)^{-1/\xi}$$

Having this approximation of F, we can define VaR_α as:

$$F(\mathrm{VaR}_\alpha) = \alpha$$
$$\mathrm{VaR}_\alpha = F^{-1}(\alpha).$$

Solving for F^{-1}, we have:

$$\mathrm{VaR}_\alpha = u + \frac{\sigma}{\xi}\left(\left(\frac{N}{N_u}(1-\alpha)\right)^{-\xi} - 1\right). \tag{2}$$

As we can observe from Equation (1) the ξ parameter depends on the choice of u and, from Equation (2), the VaR_α depends on ξ and u. Some previous studies have been made on the choice of u, for instance, A. J. McNeil and Frey (2000) and Coles, Bawa, Trenner, and Dorazio (2001) but they still require a heuristic approach when defining this value. Here we study the width of the confidence interval of the VaR_α estimator to define a new criteria for the choice of threshold.

2 Method

First, we collect data from 15 companies in the S&P 500 from different industries and from the index itself. Then, we fit the GPD on the log returns of these assets for different threshold levels. After that, for each ξ estimate, we compute the confidence interval by approximating the likelihood distribution using a grid sampling approach to avoid possible bias from random sampling methods. Once we have the confidence interval for ξ, we compute VaR_α and its confidence interval from ξ's confidence interval.

Having VaR_α confidence interval, we measure the relative width by taking the log difference between the extremes of the interval, as in:

$$\mathrm{Width}_{CI} = \log(b) - \log(a) \;\Big|\; P(b > \mathrm{VaR}_\alpha > a) = \mathrm{level}_{CI}, \tag{3}$$

where a and b are the extremes of the VaR_α confidence interval, CI, having a confidence level, level_{CI}, set to 95% by default in our experiments.

3 Experiment

The data was taken from the Yahoo Finance API, using the yfinance python package Aroussi (2022). It contains the time series of the adjusted close price of each asset, indexed by date. We request the data, up to 1922, and apply the log transformation and the time differential operator to observe the daily log returns. A summary of the data can be observed in Table 1.

Then, we select the points where the return is beyond the threshold, see Figures 1 and 2. Note that, as we are studying the losses, i.e. the negative returns, we need to invert the signal to have our tail on the right side of the distribution.

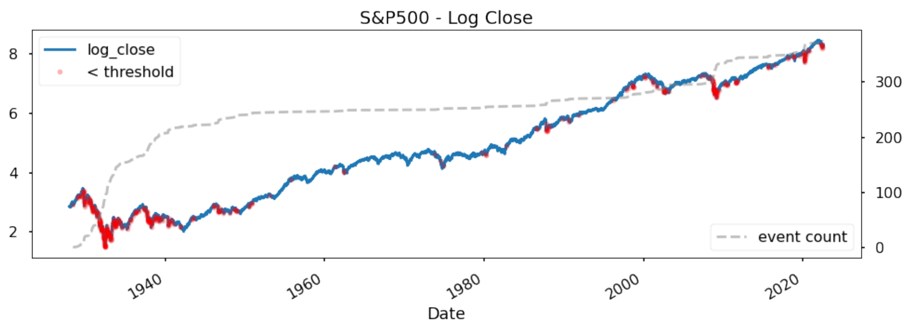

Figure 1: The blue curve shows S&P 500 log close price time series. On grey, we can observe the number of points selected across time at the predefined threshold.

Ticker	Security name	count	mean	std	min	25%	50%	75%	max
AAPL	Apple Inc.	10495	0.0007	0.0287	-0.7312	-0.0131	0.0000	0.0146	0.2868
AMZN	Amazon	6343	0.0011	0.0358	-0.2845	-0.0132	0.0004	0.0146	0.2961
DIS	Disney	15248	0.0004	0.0198	-0.3438	-0.0096	0.0000	0.0103	0.1747
GM	General Motors	2942	0.0001	0.0216	-0.1902	-0.0103	0.0002	0.0106	0.1818
IBM	IBM	15248	0.0002	0.0159	-0.2681	-0.0077	0.0000	0.0081	0.1236
JNJ	Johnson & Johnson	15248	0.0005	0.0145	-0.2027	-0.0069	0.0000	0.0076	0.1475
JPM	JP Morgan	10683	0.0004	0.0225	-0.3246	-0.0097	0.0000	0.0104	0.2239
KO	Coca Cola	15248	0.0004	0.0146	-0.2835	-0.0068	0.0000	0.0075	0.1795
MCD	Mc Donalds	14113	0.0005	0.0194	-0.7076	-0.0082	0.0000	0.0091	0.1665
META	Meta	2565	0.0005	0.0245	-0.3063	-0.0098	0.0009	0.0124	0.2593
PFE	Pfizer	12649	0.0004	0.0173	-0.1899	-0.0090	0.0000	0.0099	0.1030
PG	Procter & Gamble	15248	0.0004	0.0136	-0.3600	-0.0062	0.0000	0.0068	0.2004
TSLA	Tesla	3042	0.0017	0.0357	-0.2365	-0.0153	0.0012	0.0190	0.2182
V	Visa	3616	0.0007	0.0190	-0.1466	-0.0078	0.0012	0.0094	0.1397
XOM	Exxon Mobil	15248	0.0004	0.0144	-0.2669	-0.0069	0.0000	0.0079	0.1647
ˆGSPC	S&P 500	23757	0.0002	0.0120	-0.2289	-0.0045	0.0004	0.0054	0.1536

Table 1: Data description on the daily log return by asset.

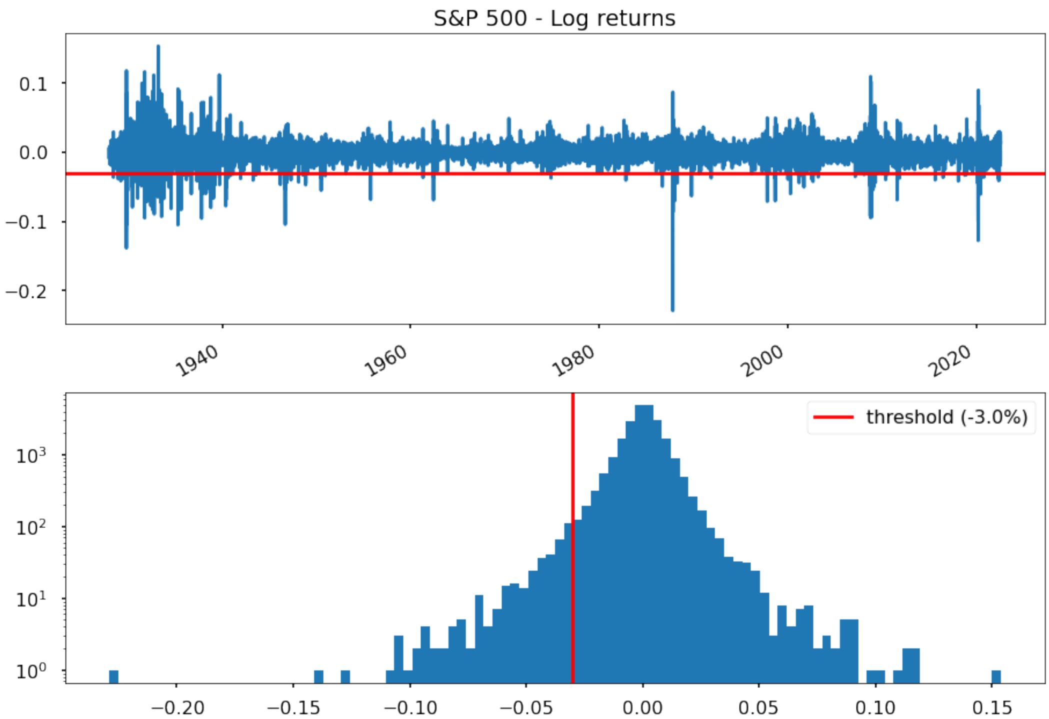

Figure 2: The blue curve shows S&P 500 log return time series. The red line is the threshold that defines the extreme returns, i.e. the tail distribution.

Then, we use the Pickands-Balkema-de-Haan theorem, discussed in Section 1, to fit our sample to the GDP, see Figure 3. The confidence interval of each VaR_α estimate is given by propagating the uncertainty of the ξ parameter from the fit distribution. To compute this uncertainty we approximate the likelihood distribution of ξ using a grid sampling method, see Figure 4.

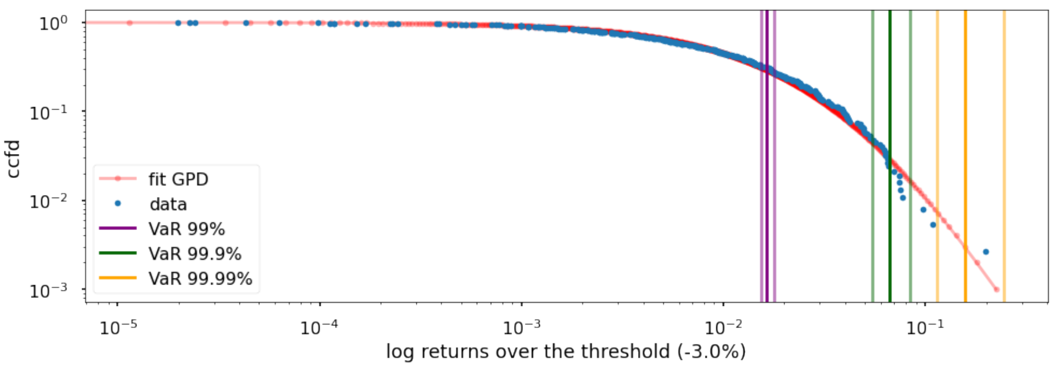

Figure 3: The blue curve shows the complementary cumulative distribution function of the S&P 500 extreme log return. The red line is the parametric GDP counterpart. On purple, green and orange, we have the VaR_α at different levels and the faded lines represent the 95% confidence interval. The confidence interval is given by propagating the uncertainty observed on the shape parameter, ξ, from the GDP.

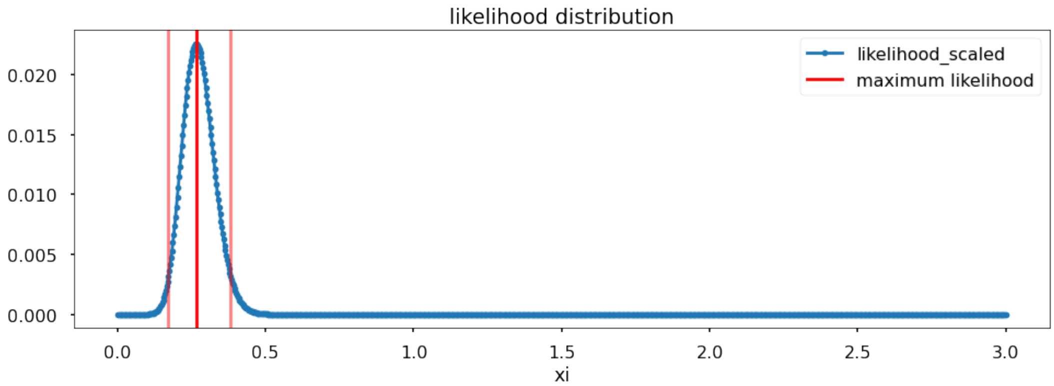

Figure 4: ξ (xi) likelihood distribution approximated by a grid sampling method. The red line shows the ξ best estimate by maximum likelihood and the faded lines show the 95% confidence interval.

Finally, we repeat the same process for each asset varying the threshold level u, having at least ten points and at most 50% of the points beyond the threshold. After collecting the results we clip the points where the grid sampler fails to estimate the ξ CI, plot the VaR_α's CI width against the threshold at different levels and fit the curve of $\mathrm{VaR}_{99.99\%}$'s CI width.

4 Results

After running the experiment over different assets, we observe that the confidence interval for ξ grows wider as we increase the threshold as expected A. McNeil, Frey, Embrechts, and Hofert (2022), see Figure 5 for the results on a subset of assets and see Figure 9 in the Annex for the full set result. However, as opposed to what we might think by looking at Equation (2), the confidence interval for VaR_α does not necessarily increases with ξ but rather decreases until the complement of the VaR_α level, i.e. $1-\alpha$, see Figure 6 for the results on a subset of assets and see Figure 10 for the full set result.

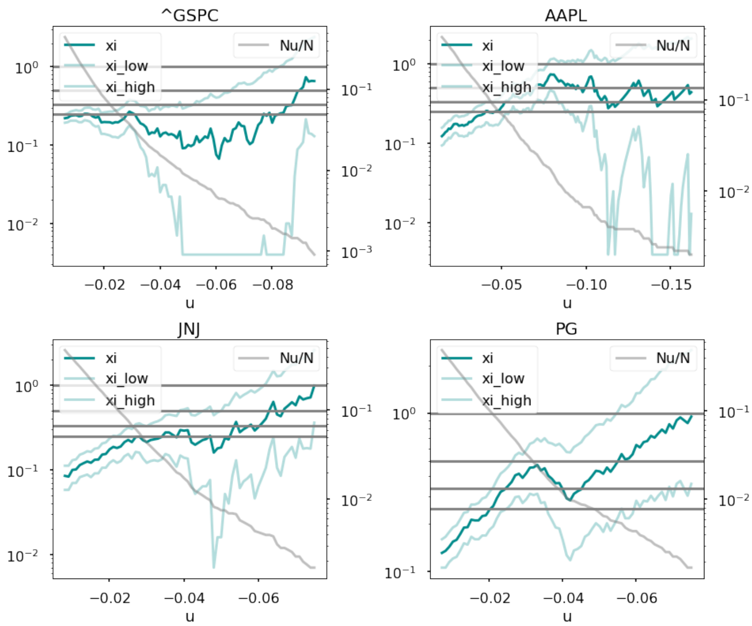

Figure 5: The four graphs represent the change in the estimated ξ (xi) parameter as a function of the threshold of the GPD (u). The cyan curve represents the estimate at the maximum likelihood and the adjacent faded curves represent the 95% confidence interval. The grey curve represents the proportion of points above the threshold ($\frac{N_u}{N}$). The grey horizontal lines represent the levels of ξ: 1, $\frac{1}{2}$, $\frac{1}{3}$ and $\frac{1}{4}$ which delimits respectively the existence of the first, second, third and fourth order moments of the underlying GDP distribution Cirillo and Taleb (2020).

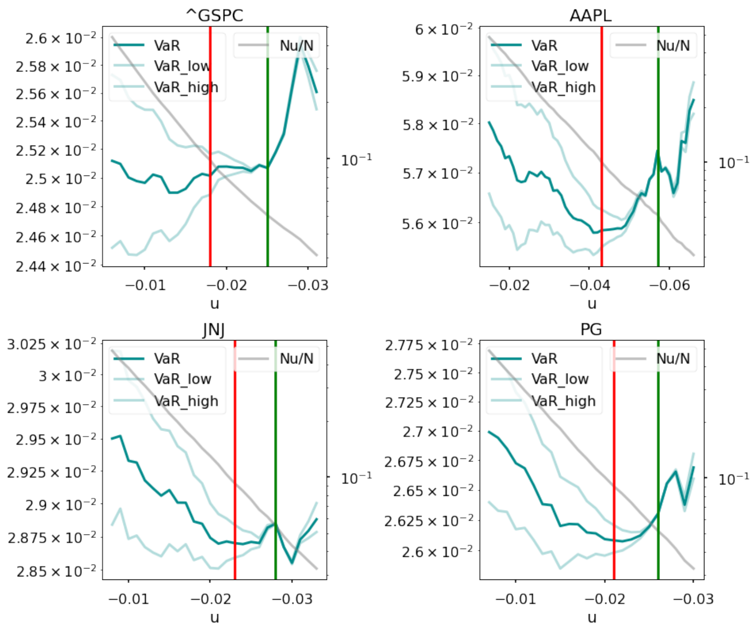

Figure 6: The four graphs represent the change in the estimated $\text{VaR}_{95\%}$ as a function of the threshold of the GPD (u). The cyan curve represents the estimate at the maximum likelihood and the adjacent faded curves represent the 95% confidence interval due to the propagated uncertainty from ξ. The grey curve represents the proportion of points above the threshold ($\frac{N_u}{N}$). The green vertical line is placed at the threshold that splits the proportion of points beyond it at the complement of of the $\text{VaR}_{95\%}$ level, i.e. 5%. The red vertical line is placed at the threshold that sets 10% of the points as the tail distribution, which is often used as the default threshold criteria.

These results can be replicated for different VaR_α levels. However, at levels far beyond the minimum proportion of points available at the sample, $inf_u(\frac{N_u}{N}) \gg \alpha$, the width of the confidence interval increases as initially expected, see Figure 7. For such cases, we hypothesize a linear relationship:

$$\log(\text{Width}_{CI}) = c_0 + c_1 \log\left(\frac{N_u}{N}\right) + c_2\beta + \epsilon \;\Big|\; \{c_0, c_1, c_2\} \subset \mathbb{R}$$

where the c_0, c_1 and c_2 the coefficients of the linear equation, ϵ is the idiosyncratic error term and β is the market exposure term computed as the covariance between the index's and the asset's log returns divided by the variance of the index log return:

$$\beta_{\text{security}} = \frac{\text{cov}(\text{index}, \text{security})}{\text{var}(\text{index})} \tag{4}$$

where index and security represent respectively the market's log returns and the underlying security's log returns.

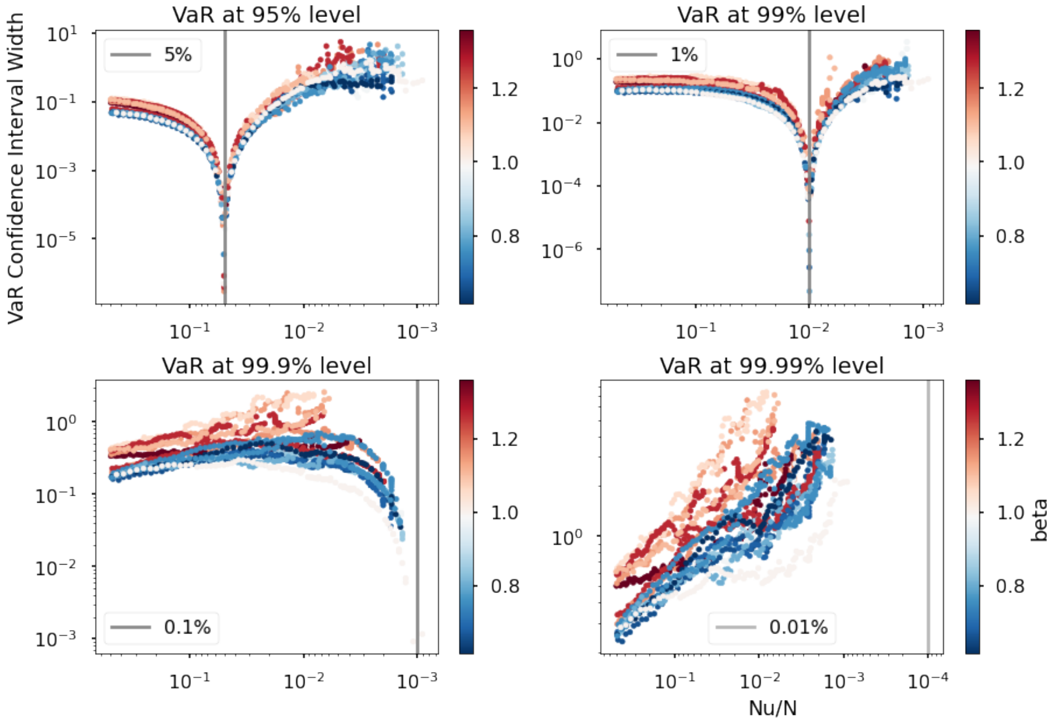

Figure 7: The four graph show the VaR_α confidence interval width as a function of the proportion of points beyond the threshold $\frac{N_u}{N}$, see Equation (3) for the width definition. Each graph projects the relationship for a different α. The color scale represents the β (beta), a measure of market exposure of the underlying security, see Equation (4). The grey vertical is set at the point where the proportion of points beyond the threshold is the complement of the value-at-risk level, i.e. $\frac{N_u}{N} = 1 - \alpha$.

After running a linear regression on the $\log(\text{Width}_{CI})$ we can observe the results summary in Table 2. The R^2 metric is above 0.6 and all the coefficients have a high T-stat with a low P-value. For a visual representation of the results, see Figure 8. Note that the error is not normally distributed, which might indicate that there are more hidden factors to justify the dependent variable.

Dep. Variable:	log(Width)	**R-squared:**	0.608
Model:	OLS	**Adj. R-squared:**	0.607
Method:	Least Squares	**F-statistic:**	1178.
Date:	Sat, 30 Jul 2022	**Prob (F-statistic):**	7.13e-310
Time:	23:18:21	**Log-Likelihood:**	-981.04
No. Observations:	1525	**AIC:**	1968.
Df Residuals:	1522	**BIC:**	1984.
Df Model:	2		

	coef	**std err**	**t**	**P> |t|**	**[0.025**	**0.975]**
const	-1.8835	0.063	-29.823	0.000	-2.007	-1.760
log(N_u/N)	-0.3493	0.007	-48.003	0.000	-0.364	-0.335
β	0.7467	0.051	14.697	0.000	0.647	0.846

Omnibus:	39.654	**Durbin-Watson:**	0.055
Prob(Omnibus):	0.000	**Jarque-Bera (JB):**	48.318
Skew:	0.322	**Prob(JB):**	3.22e-11
Kurtosis:	3.589	**Cond. No.**	29.5

Table 2: OLS Regression Results

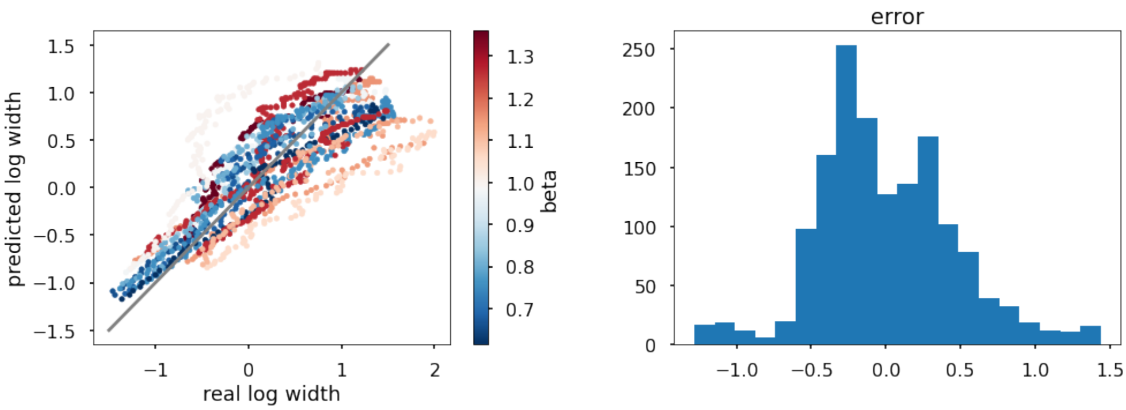

Figure 8: The graph on the left shows the real Width_{CI} against the predicted Width_{CI} with the market exposure (beta) of the underlying security on a color scale. The graph on the right shows the distribution of the model error ϵ.

5 Conclusion

Setting the threshold u is a critical choice for quantitative risk analysts when measuring value-at-risk. In terms of the VaR_α confidence interval with an unknown shape parameter ξ, the best choice of threshold is the one such that the proportion of points above u is the complement of α, i.e. $\frac{N_u}{N} = 1 - \alpha$.

At extreme levels of α, where it is not possible to select the threshold as proposed, we model the width of the confidence interval. The model shows empirical evidence that, for traditional companies, the proportion of points above the threshold u, $\frac{N_u}{N}$, and the market exposure β can justify 60% of the variance of the log confidence interval width of the VaR_α estimator. This result can hopefully help improve financial risk reporting by bringing insight into the choice of level α and tail threshold u.

Acknowledgement

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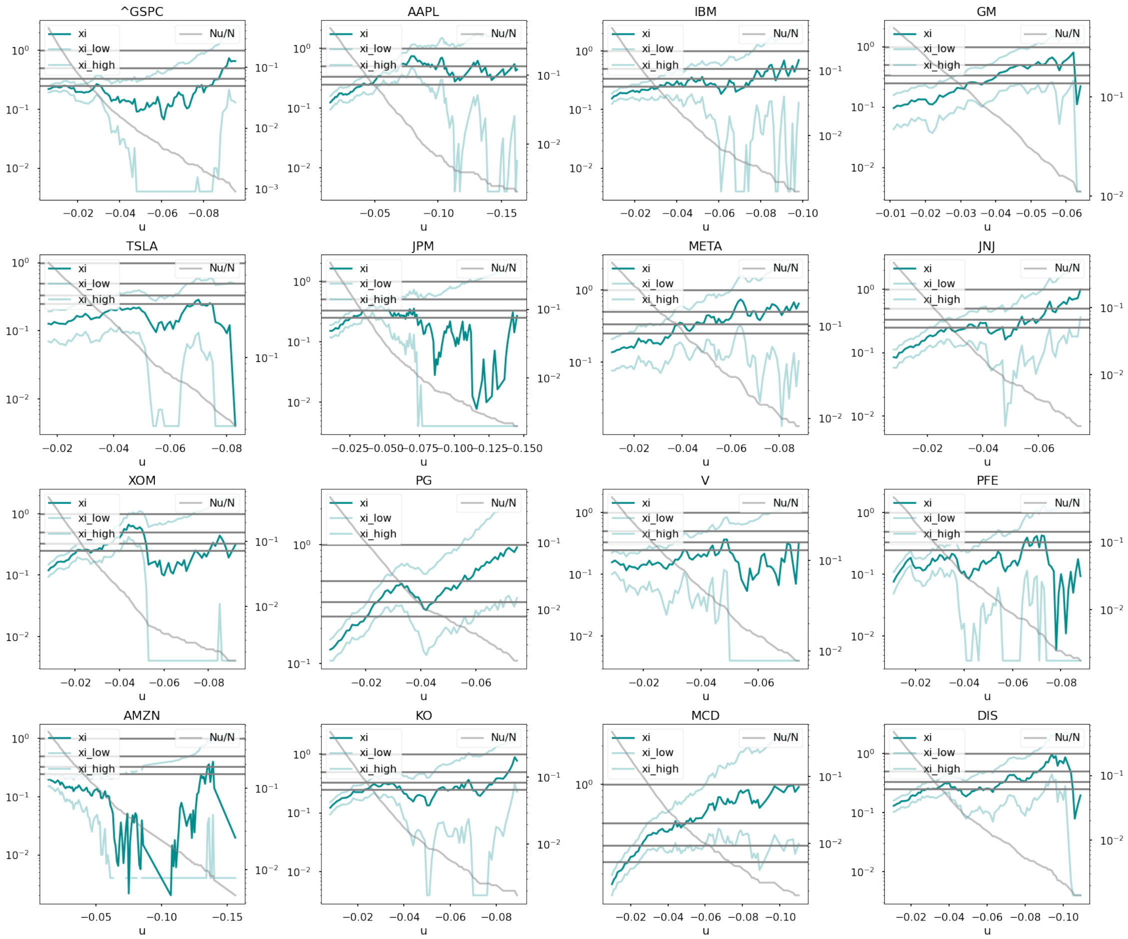
Annex

Code repository github.com/guilhermessc/VaR-threshold-and-confidence-interval



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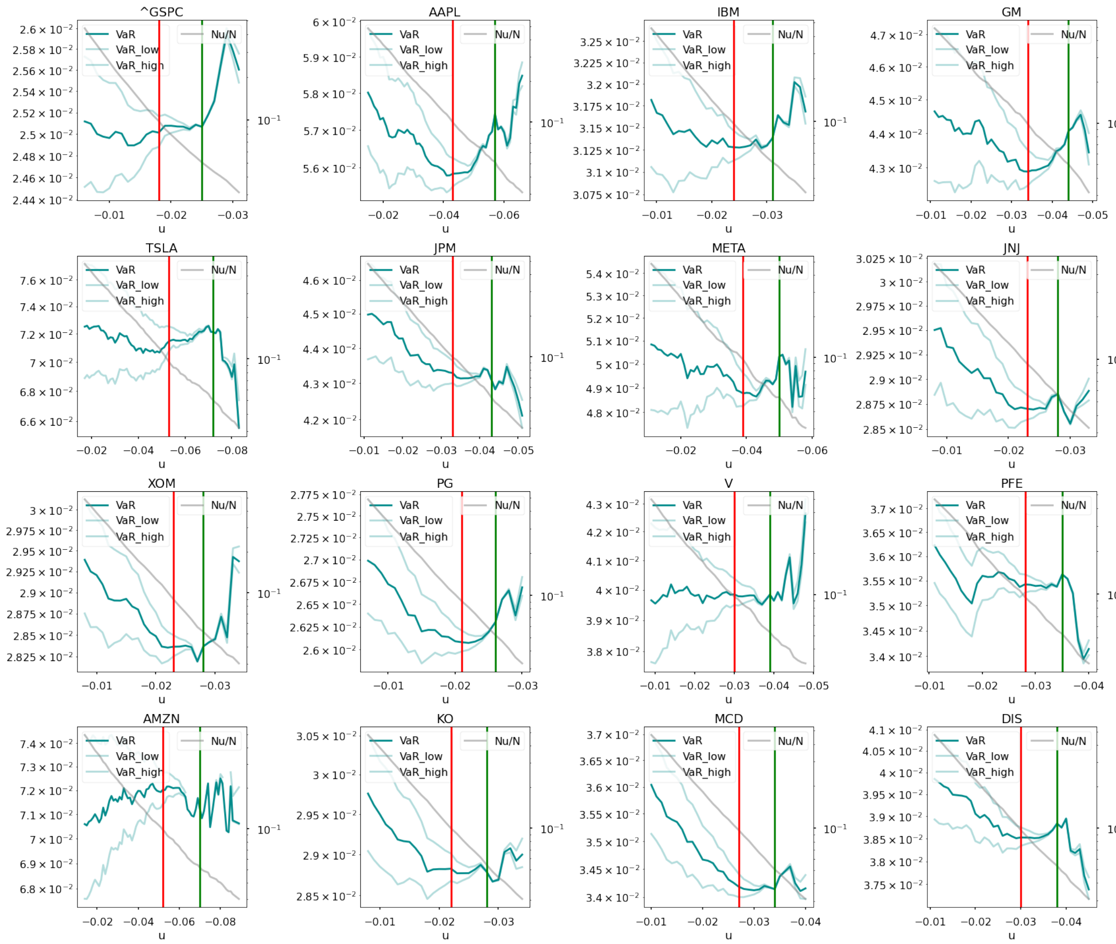

Figure 10: The graphs represent the change in the estimated $VaR_{95\%}$ as a function of the threshold of the GPD (u). The cyan curve represent the estimate at the maximum likelihood and the adjacent faded curves represent the 95% confidence interval due to the propagated uncertainty from ξ. The grey curve represents the proportion of points above the threshold ($\frac{N_u}{N}$). The green vertical line is placed at the threshold that splits the proportion of points beyond it at the complement of of the $VaR_{95\%}$ level, i.e. 5%. The red vertical line is placed at the threshold that sets 10% of the points as the tail distribution, which is often used as the default threshold criteria.