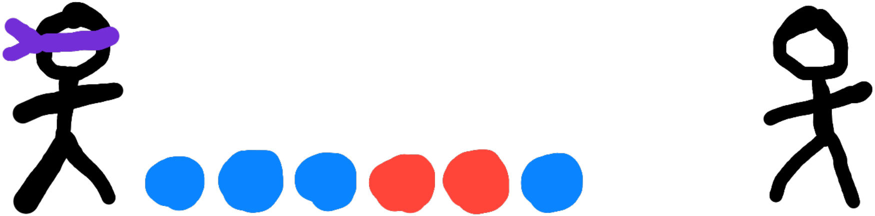
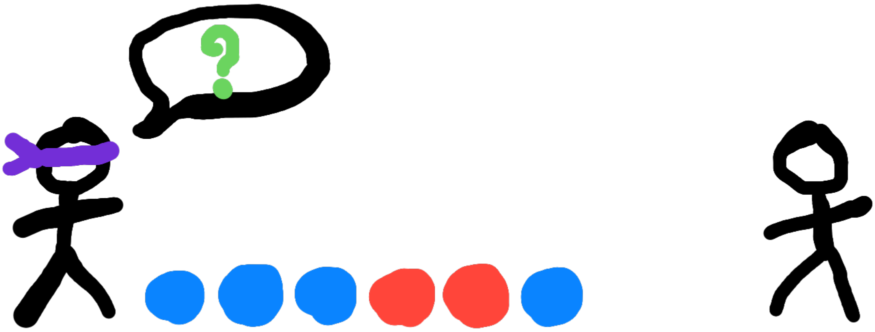
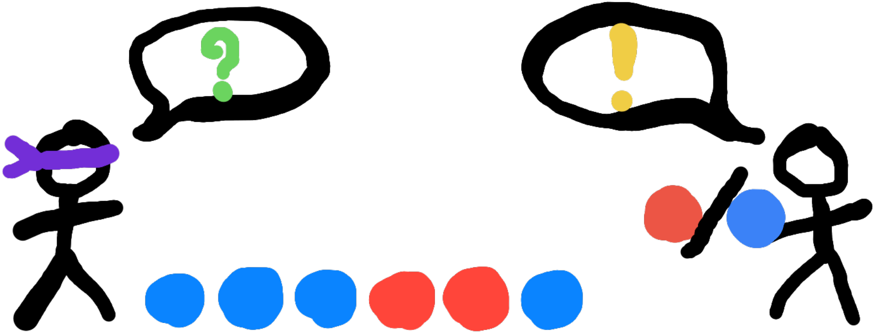
Characteristic Boolean algebras in classical realizability: a completeness result

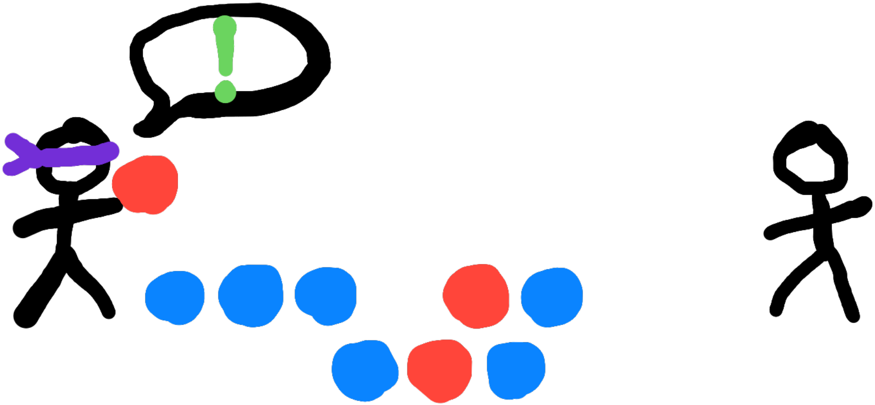
Guillaume Geoffroy

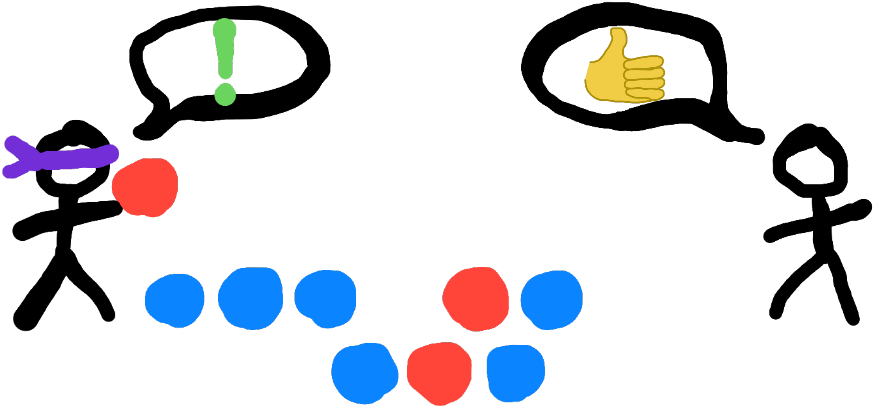
IRIF, Université Paris Cité

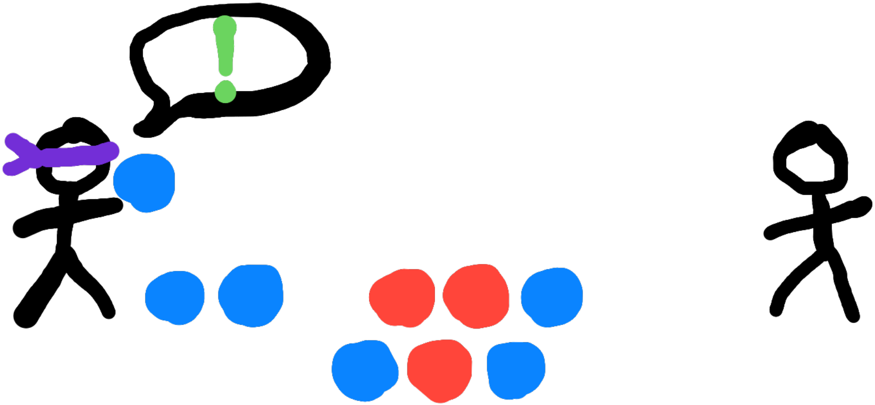


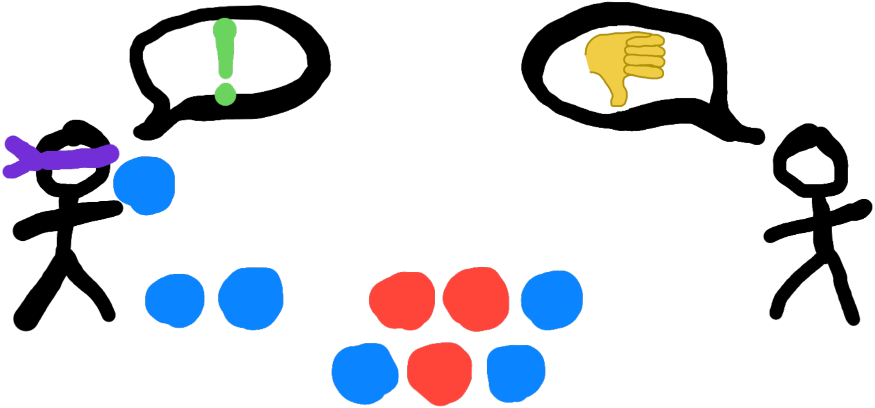




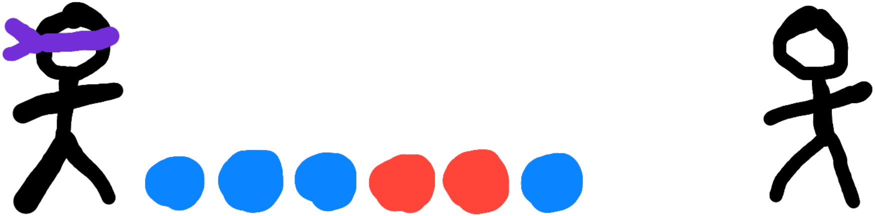




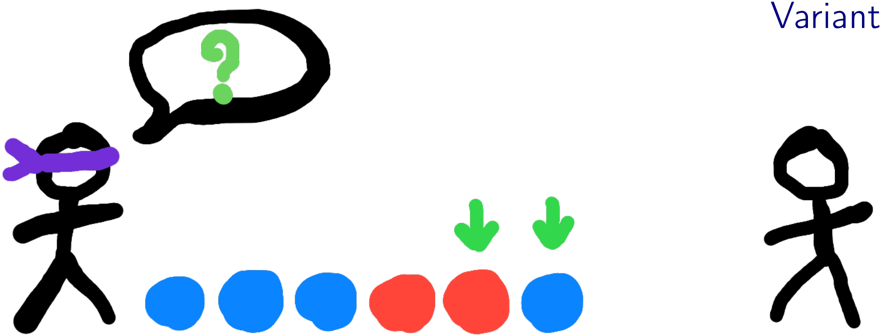




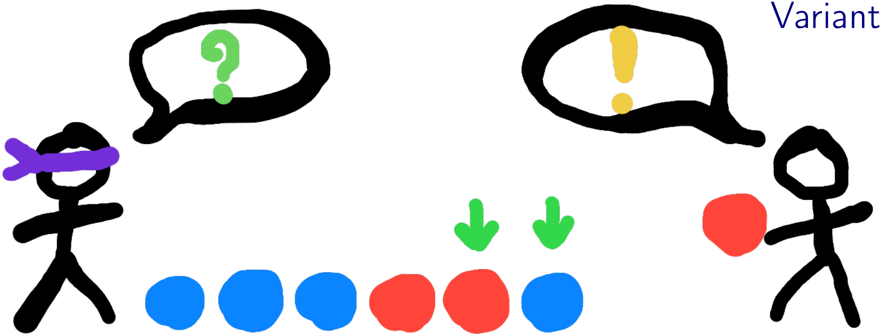
Variant $^1/_2$



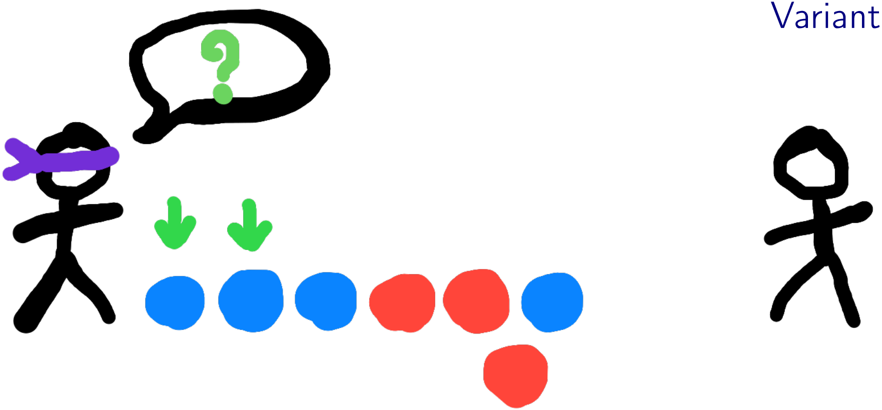
Variant $^1/_2$



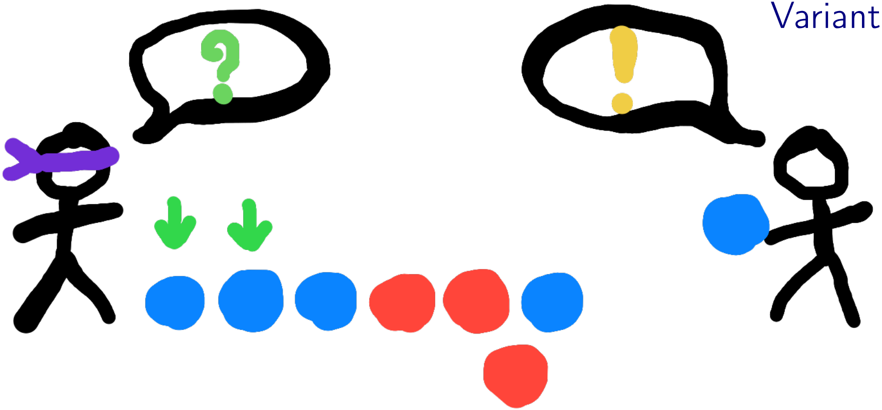
Variant 1/2



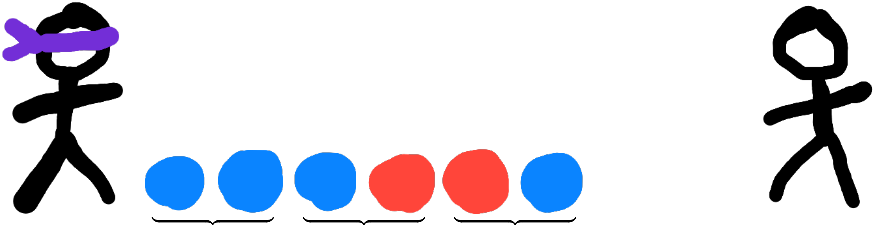
Variant $^1/_2$



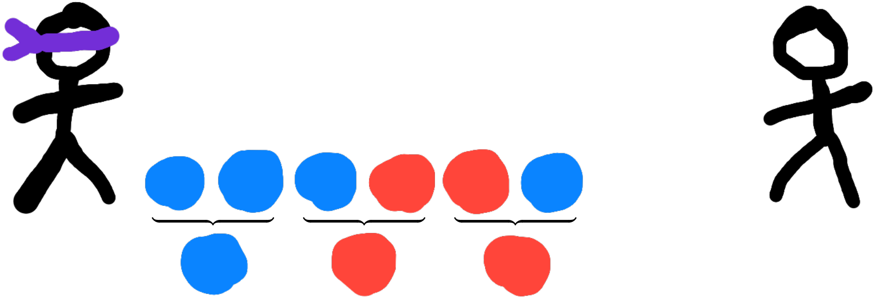
Variant $^{1}/_{2}$



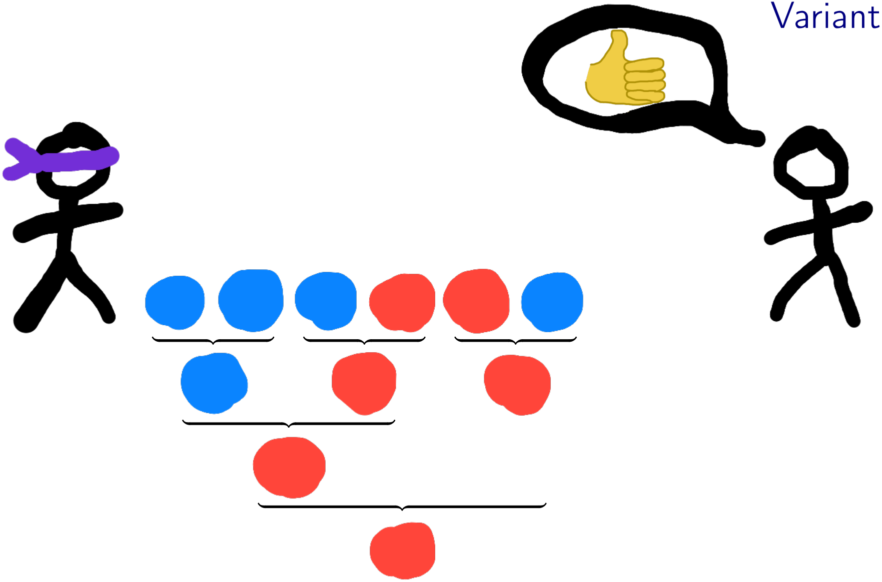
Variant $^{1}/_{2}$



Variant $^{1}/_{2}$

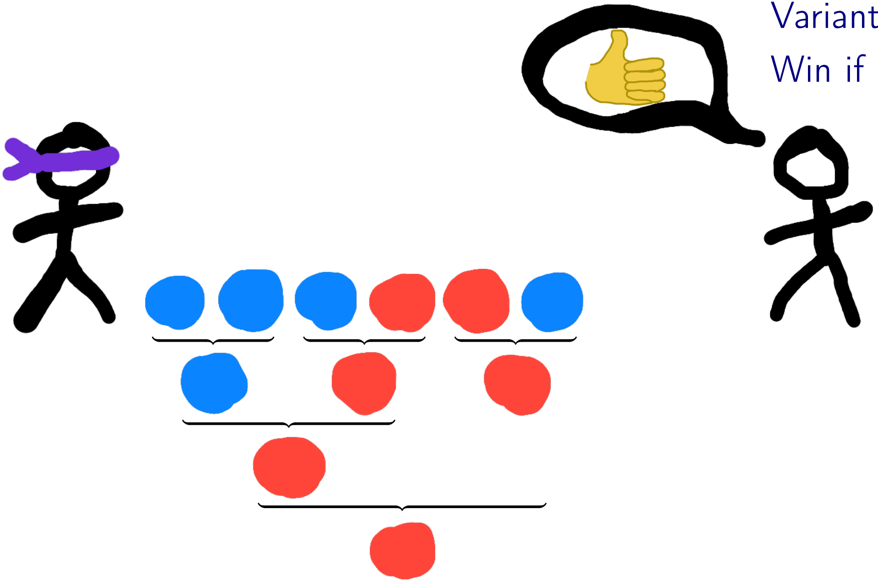


Variant $1/2$

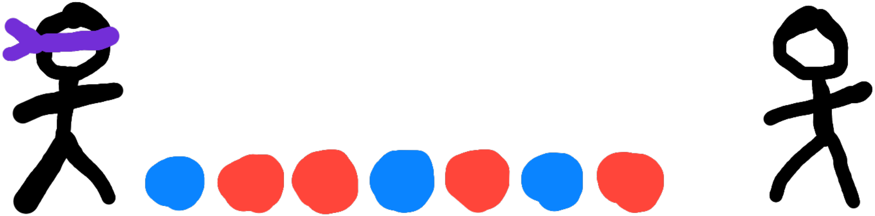


Variant $^{1}/_{2}$

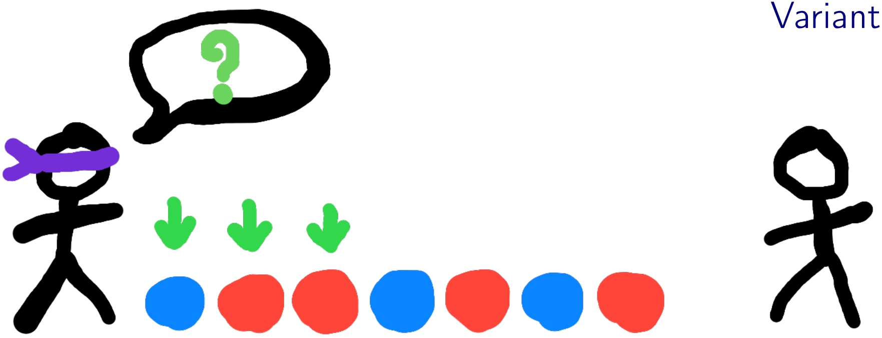
Win if $r > 0$



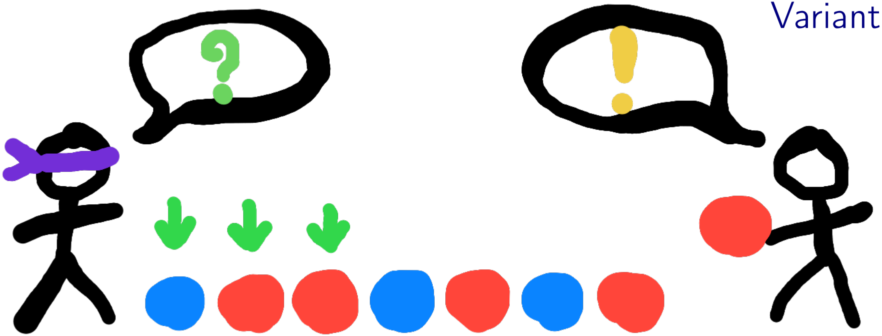
Variant $^{2}/_{3}$



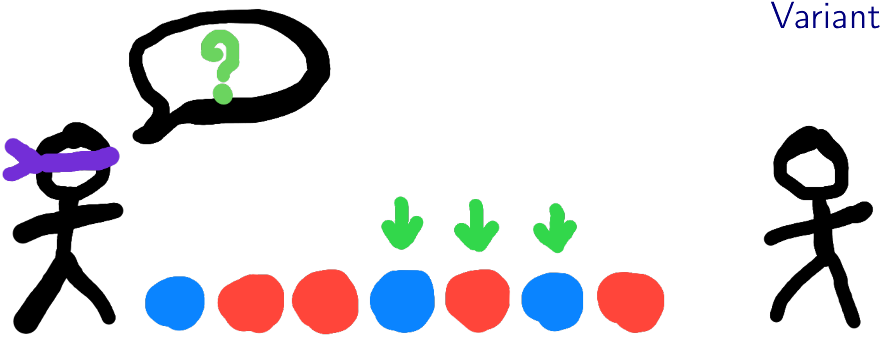
Variant $^{2}/_{3}$



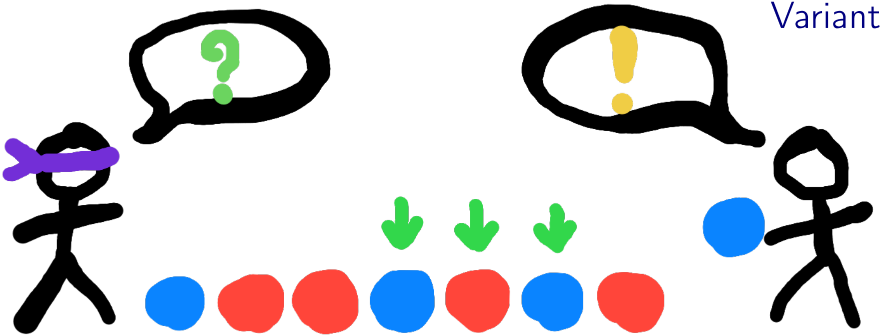
Variant $^{2}/_{3}$

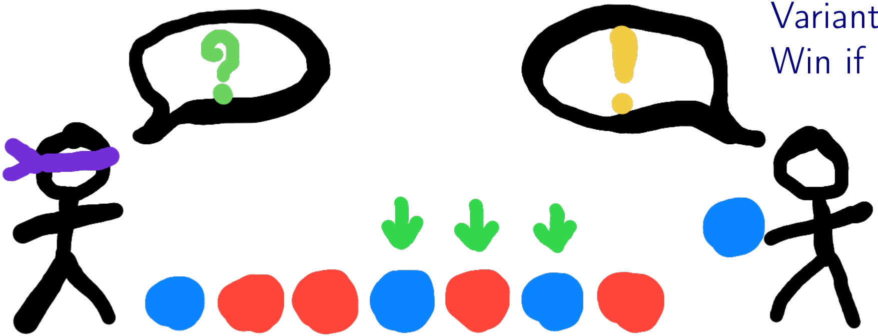


Variant $^{2}/_{3}$



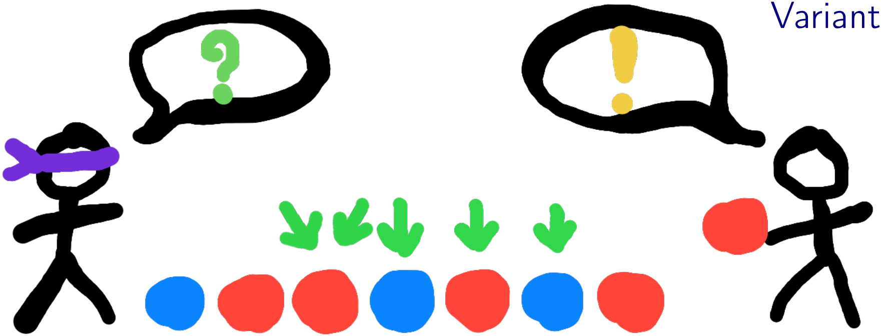
Variant $^{2}/_{3}$

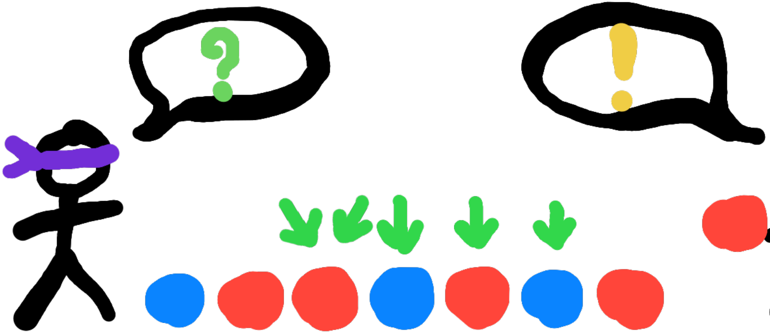




Variant $\frac{2}{3}$
Win if $r > \frac{1}{2}$

Variant $^3/_5$



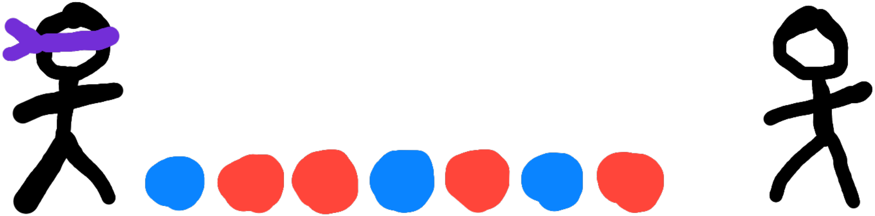


Variant $^{3}/_{5}$

Win if $r > {}^{1}/_{2}$

Variant $^{k}/_{n}$

Win if $r >$???



Puzzle for you!

Classical realizability

Formulas: (A,B,...)

- first-order
- language of Boolean algebras

Programs: $t, u \quad ::= \quad x \mid \lambda x.\, t \mid tu$

Realizability relation: "t realizes A"

Classical realizability

Formulas: (A,B,...)

- first-order
- language of Boolean algebras

Programs: $t, u \;::=\; x \mid \lambda x.\, t \mid tu$

Realizability relation: "t realizes A"

$$\left\{\begin{array}{l}\text{e.g. } \lambda x.\lambda y.\, xyy \\ \text{realizes } (A \to A \to B) \to A \to B \\ \text{for all } A, B\end{array}\right.$$

A completeness result

Adequacy (standard): if A is true in all Boolean algebras,
then there exists a λ-term t that realizes A

A completeness result

Adequacy (standard): if A is true in all Boolean algebras,
then there exists a λ-term t that realizes A

Theorem: A is true in all Boolean algebras
<u>iif</u> there exists a λ-term t that realizes A

Back to the game

Allowed questions (variant k/n)	$\leftrightsquigarrow$	Formula $A_{k,n}$
Initial situation $(N_{\text{red}}, N_{\text{balls}})$	$\leftrightsquigarrow$	Formula $A_{N_{\text{red}},N_{\text{balls}}}$

Back to the game

Allowed questions (variant ${}^{k}/_{n}$)	$\leftrightsquigarrow$	Formula $A_{k,n}$
Initial situation $(N_{\text{red}}, N_{\text{balls}})$	$\leftrightsquigarrow$	Formula $A_{N_{\text{red}},N_{\text{balls}}}$
Winning strategy	$\leftrightsquigarrow$	λ-term that realizes $A_{k,n} \to A_{N_{\text{red}},N_{\text{balls}}}$

Back to the game

Allowed questions (variant k/n)	$\leftrightsquigarrow$	Formula $A_{k,n}$
Initial situation $(N_{\text{red}}, N_{\text{balls}})$	$\leftrightsquigarrow$	Formula $A_{N_{\text{red}},N_{\text{balls}}}$
$\exists$ Winning strategy	$\leftrightsquigarrow$	In all Boolean algebras, $A_{k,n} \to A_{N_{\text{red}},N_{\text{balls}}}$ holds

Back to the game

Allowed questions (variant k/n)	$\leftrightsquigarrow$	Formula $A_{k,n}$ "$\exists$ at most $f(k/n)$ elements"
Initial situation $(N_{\text{red}}, N_{\text{balls}})$	$\leftrightsquigarrow$	Formula $A_{N_{\text{red}},N_{\text{balls}}}$ "$\exists$ at most $f(N_{\text{red}}/N_{\text{balls}})$ elements"
$\exists$ Winning strategy	$\leftrightsquigarrow$	In all Boolean algebras, $A_{k,n} \to A_{N_{\text{red}},N_{\text{balls}}}$ holds

Back to the game

Allowed questions (variant ${}^{k}/_{n}$)	$\rightsquigarrow\!\!\!\!\leftarrow$ ⟷	Formula $A_{k,n}$ "$\exists$ at most $f({}^{k}/_{n})$ elements"
Initial situation $(N_{\text{red}}, N_{\text{balls}})$	⟷	Formula $A_{N_{\text{red}},N_{\text{balls}}}$ "$\exists$ at most $f({}^{N_{\text{red}}}/_{N_{\text{balls}}})$ elements"
$\exists$ Winning strategy	⟷	In all Boolean algebras, $A_{k,n} \to A_{N_{\text{red}},N_{\text{balls}}}$ holds $f({}^{k}/_{n}) \leq f({}^{N_{\text{red}}}/_{N_{\text{balls}}})$

Translating variant $^{2}/_{3}$

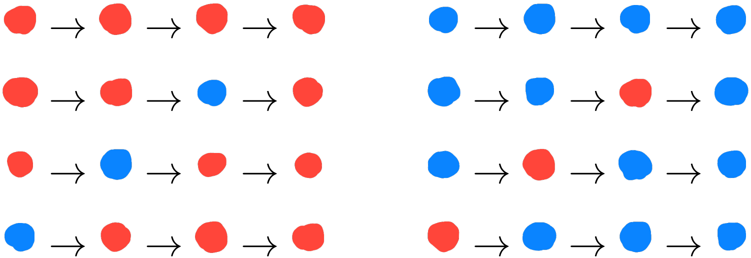
- The key: •  $\Leftrightarrow \bot$  $\Leftrightarrow \top$

Translating variant $^{2}/_{3}$

- The key:
  -  $\Leftrightarrow \bot$  $\Leftrightarrow \top$
  - universal quantification $\Leftrightarrow$ intersection type

Translating variant $^2/_3$

- The key:
  - $\Leftrightarrow \bot$ $\Leftrightarrow \top$
  - universal quantification $\Leftrightarrow$ intersection type
- Variant $^2/_3$ is the intersection of:



Translating variant ${}^{2}/_{3}$

- The key:
  -  $\Leftrightarrow \bot$  $\Leftrightarrow \top$
  - universal quantification $\Leftrightarrow$ intersection type
- Variant ${}^{2}/_{3}$ is the intersection of:

$$
\begin{array}{lcl}
\bot \to \bot \to \bot \to \bot & \qquad & \top \to \top \to \top \to \top \\
\bot \to \bot \to \top \to \bot & & \top \to \top \to \bot \to \top \\
\bot \to \top \to \bot \to \bot & & \top \to \bot \to \top \to \top \\
\top \to \bot \to \bot \to \bot & & \bot \to \top \to \top \to \top
\end{array}
$$

Translating variant $^{2}/_{3}$

- The key:
  -  $\Leftrightarrow \bot$  $\Leftrightarrow \top$
  - universal quantification $\Leftrightarrow$ intersection type
- Variant $^{2}/_{3}$ is the intersection of:

$$
\begin{array}{ccc}
\bot \to \bot \to \bot \to \bot & \qquad & \top \to \top \to \top \to \top \\
\bot \to \bot \to \top \to \bot & & \top \to \top \to \bot \to \top \\
\bot \to \top \to \bot \to \bot & & \top \to \bot \to \top \to \top \\
\top \to \bot \to \bot \to \bot & & \bot \to \top \to \top \to \top
\end{array}
$$

- $A_{2,3}$ is the formula:

$$\forall x \forall y \forall z\ (x{\neq}0 \to y{\neq}0 \to z{\neq}0 \to (x{\wedge}y \vee y{\wedge}z \vee z{\wedge}x) \neq 0)$$

Translating variant $^{2}/_{3}$

- The key:
  - $\Leftrightarrow \bot$ $\qquad$ $\Leftrightarrow \top$
  - universal quantification $\Leftrightarrow$ intersection type
- Variant $^{2}/_{3}$ is the intersection of:

$$
\begin{array}{ccc}
\bot \to \bot \to \bot \to \bot & \qquad & \top \to \top \to \top \to \top \\
\bot \to \bot \to \top \to \bot & \qquad & \top \to \top \to \bot \to \top \\
\bot \to \top \to \bot \to \bot & \qquad & \top \to \bot \to \top \to \top \\
\top \to \bot \to \bot \to \bot & \qquad & \bot \to \top \to \top \to \top
\end{array}
$$

- $A_{2,3}$ is the formula:

$$\forall x \forall y \forall z\ (x \neq 0 \to y \neq 0 \to z \neq 0 \to (x \wedge y \vee y \wedge z \vee z \wedge x) \neq 0)$$