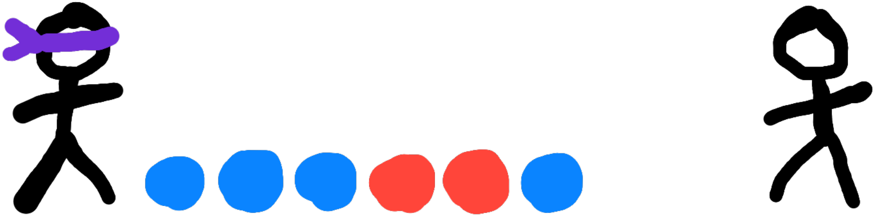
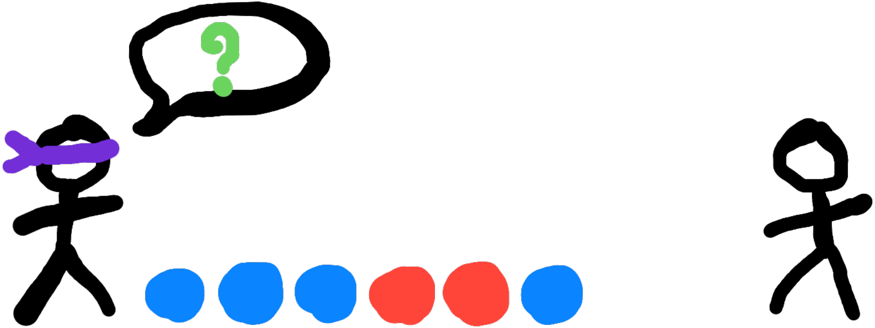
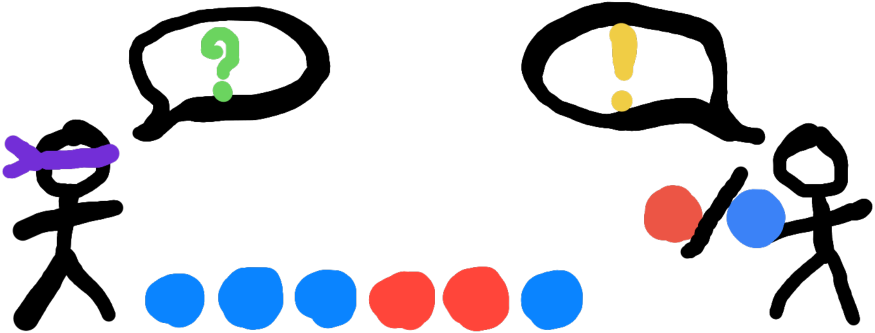
Characteristic Boolean algebras in classical realizability: a completeness result

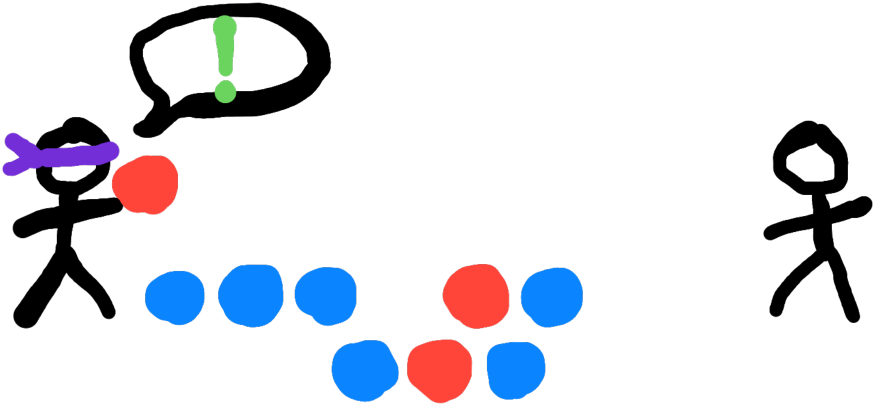
Guillaume Geoffroy

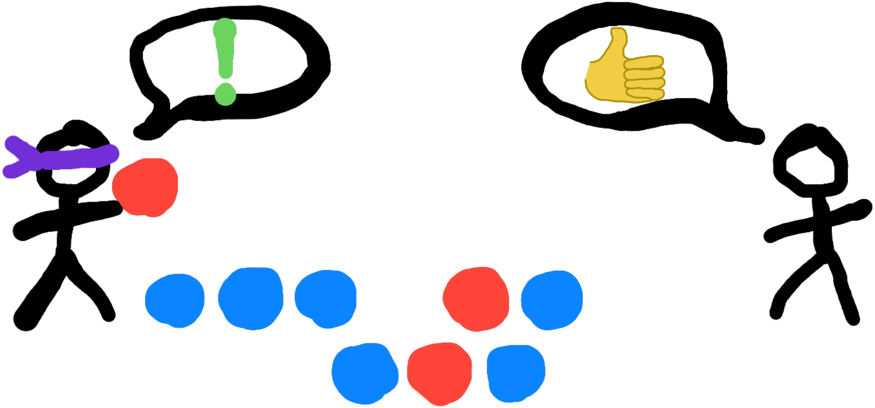
IRIF, Université Paris Cité

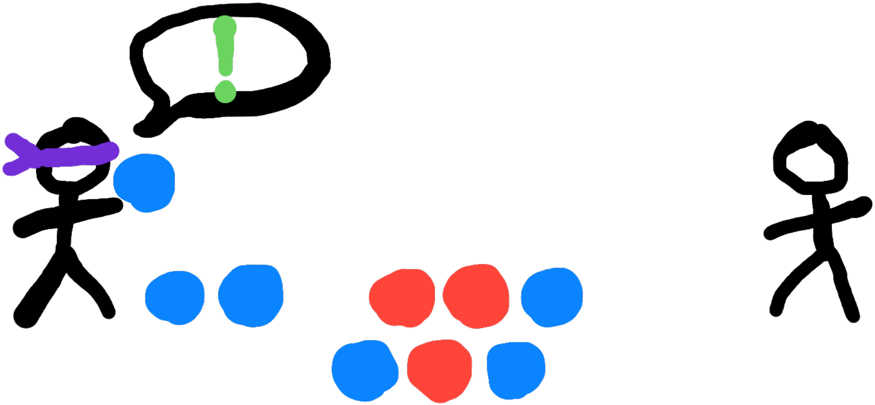


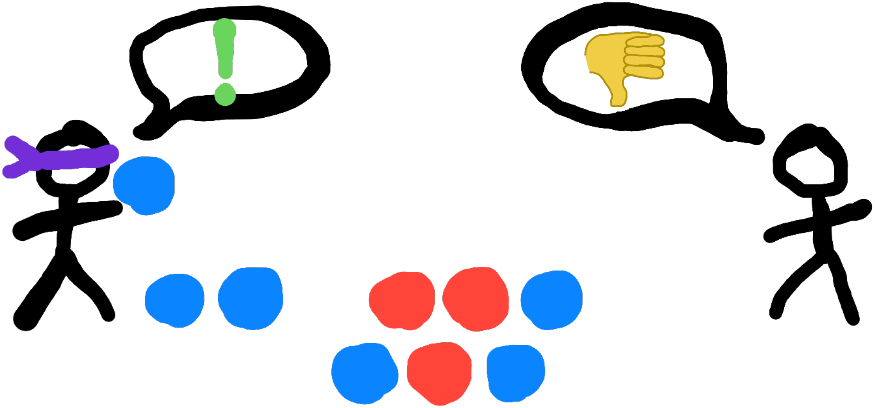




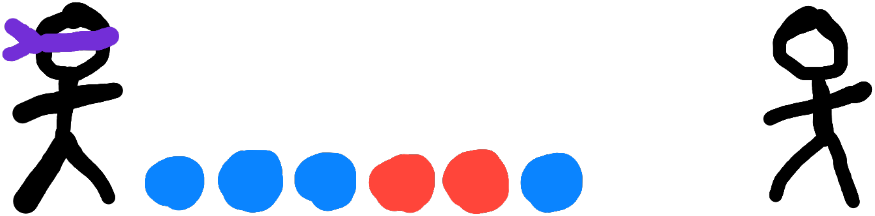




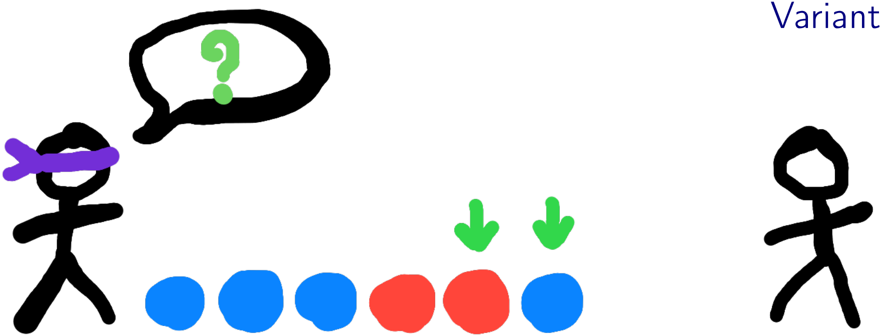




Variant 1/2



Variant $^1/_2$



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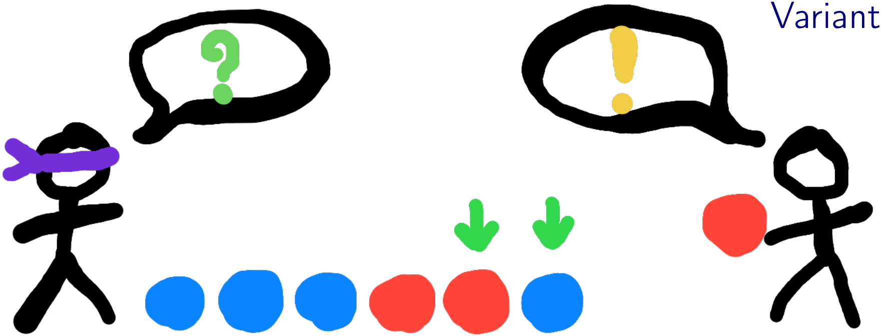
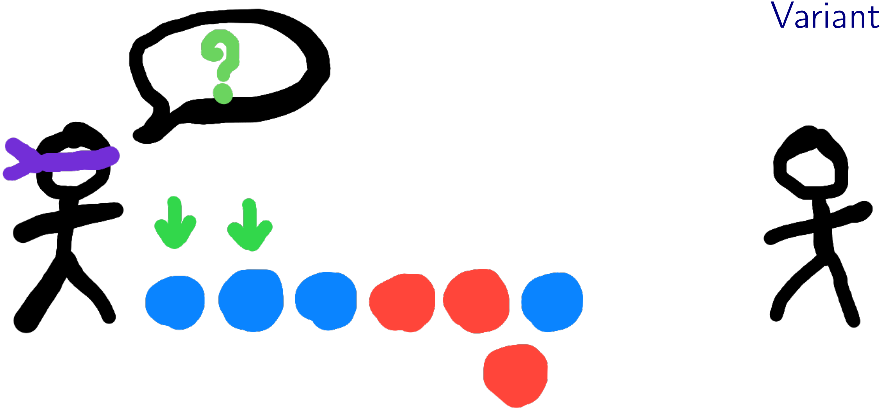


Variant $\frac{1}{2}$



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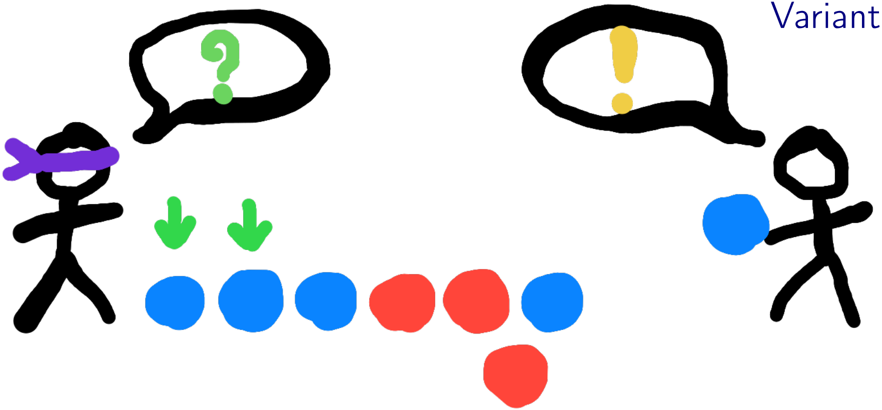

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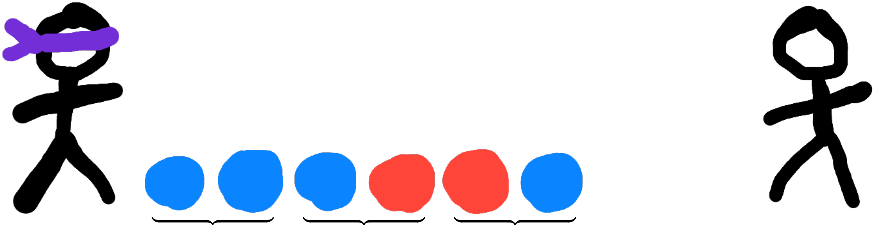

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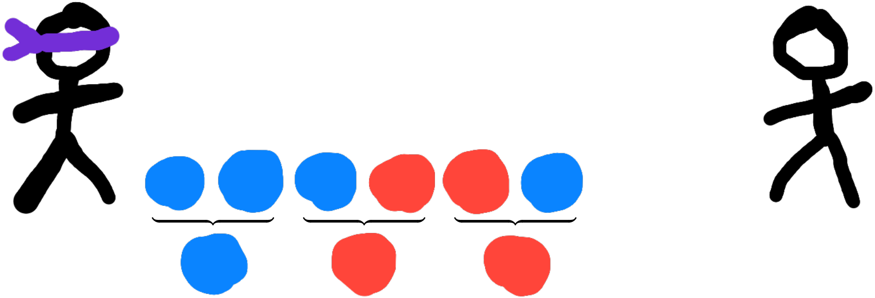

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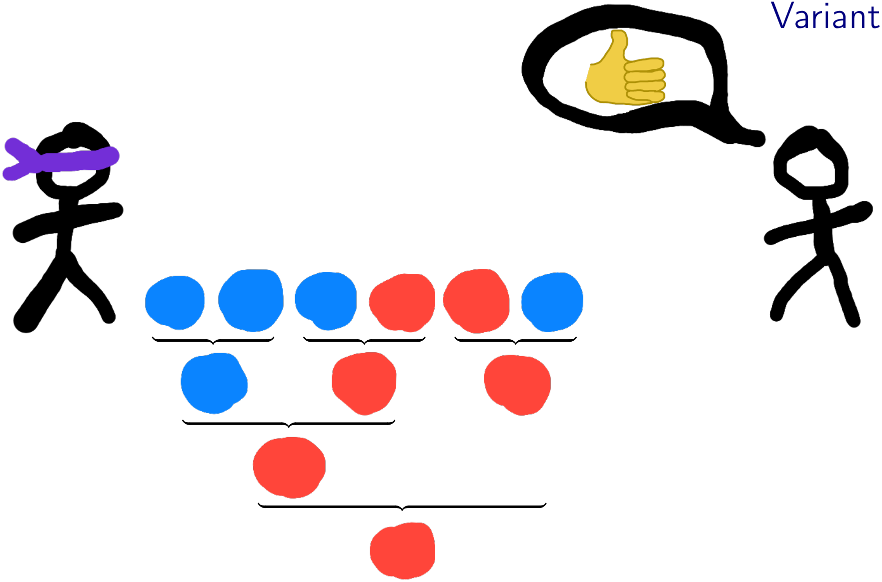
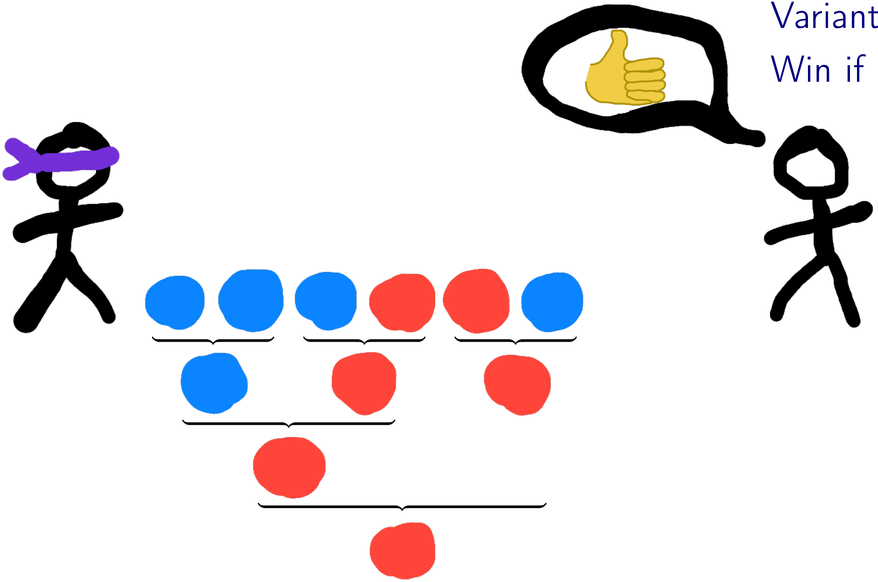
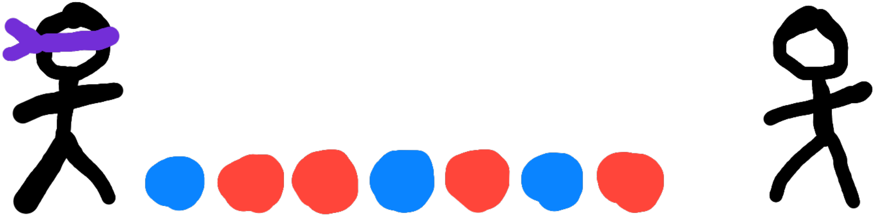




Variant $^1/_2$

Win if $r > 0$

Variant $^{2}/_{3}$



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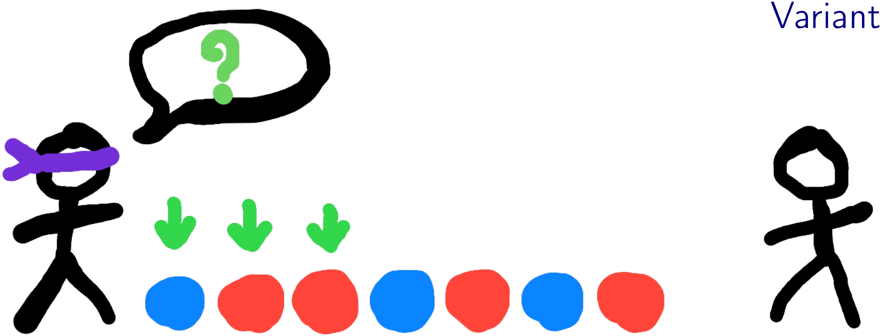
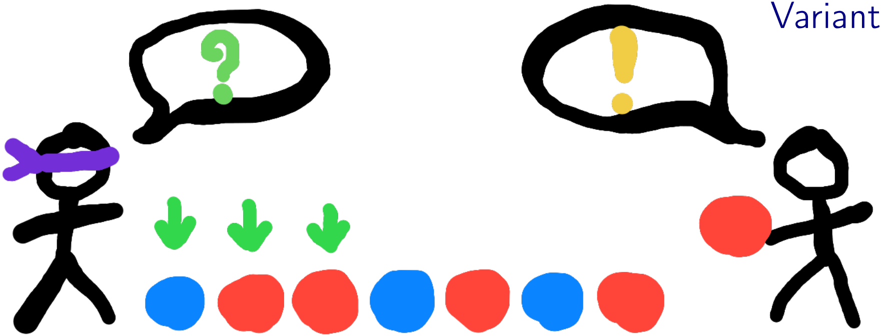


Variant 2/3



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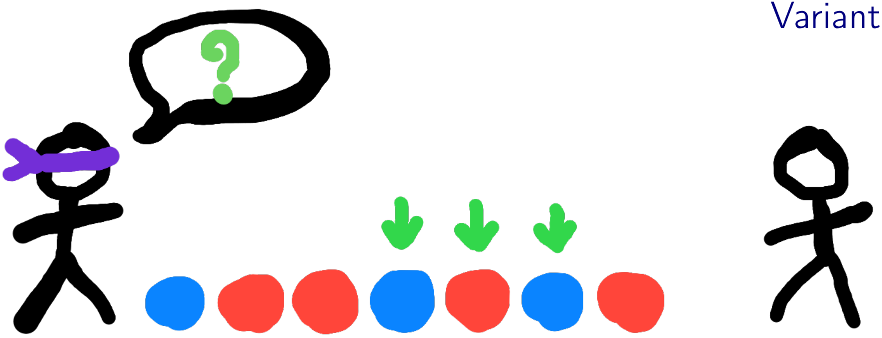

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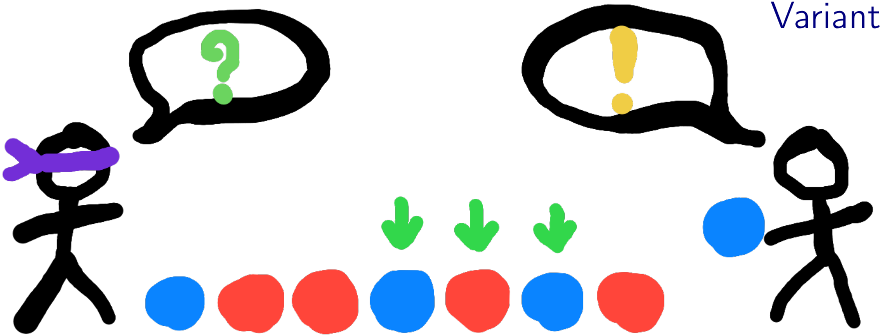
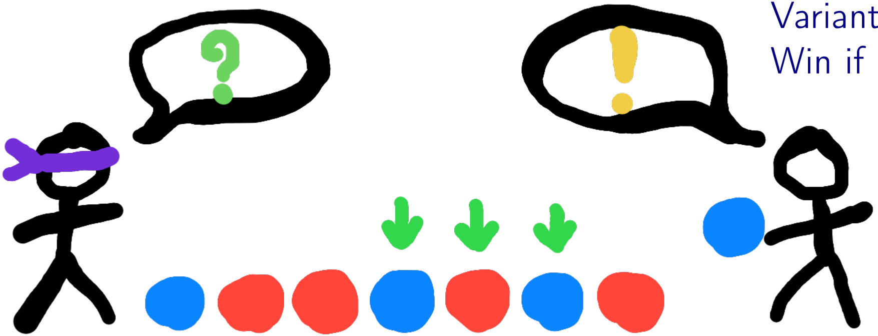
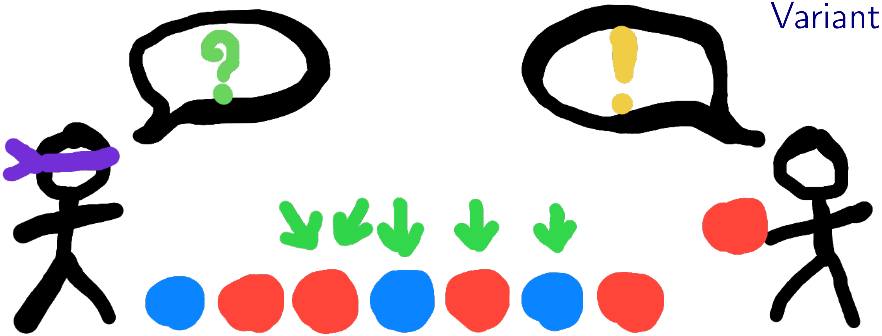


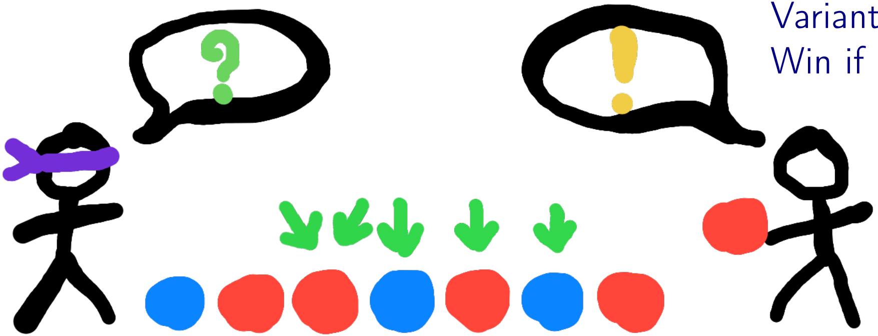
Variant $^{2}/_{3}$

Win if $r > {}^{1}/_{2}$



Variant $^{3}/_{5}$



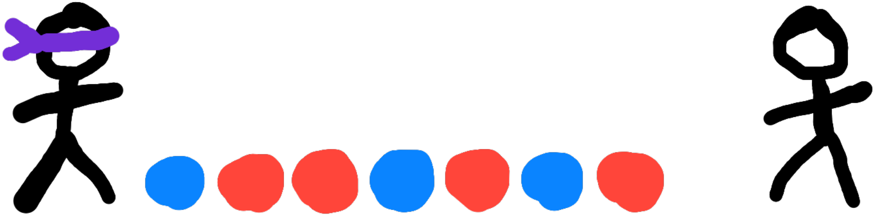


Variant $^{\color{red}3}/_{\color{green}5}$

Win if $\color{red}r$ $> {}^1/_2$

Variant ${}^{k}/_{n}$

Win if $r >$???



Puzzle for you!

Classical realizability

Formulas: (A,B,...)

- first-order
- language of Boolean algebras

Programs: $t, u \;::=\; x \mid \lambda x.\, t \mid tu$

Realizability relation: "t realizes A"

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Realizability relation: "t realizes A"

$$\left\{\begin{array}{l}\text{e.g. } \lambda x.\lambda y.\, xyy \\ \text{realizes } (A \to A \to B) \to A \to B \\ \text{for all } A, B\end{array}\right.$$

A completeness result

Adequacy (standard): if A is true in all Boolean algebras,
then there exists a λ-term t that realizes A

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Adequacy (standard): if A is true in all Boolean algebras,
then there exists a λ-term t that realizes A

Theorem: A is true in all Boolean algebras
<u>iif</u> there exists a λ-term t that realizes A

Back to the game

Allowed questions (variant $^{k}/_{n}$) $\leftrightsquigarrow$ Formula $A_{k,n}$

Initial situation $(N_{\text{red}}, N_{\text{balls}})$ $\leftrightsquigarrow$ Formula $A_{N_{\text{red}},N_{\text{balls}}}$

Back to the game

Allowed questions (variant $^{k}/_{n}$)	$\leftrightsquigarrow$	Formula $A_{k,n}$
Initial situation $(N_{\text{red}}, N_{\text{balls}})$	$\leftrightsquigarrow$	Formula $A_{N_{\text{red}},N_{\text{balls}}}$
Winning strategy	$\leftrightsquigarrow$	λ-term that realizes $A_{k,n} \to A_{N_{\text{red}},N_{\text{balls}}}$

Back to the game

Allowed questions (variant ${}^{k}/_{n}$)	$\leftrightsquigarrow$	Formula $A_{k,n}$
Initial situation $(N_{\mathrm{red}}, N_{\mathrm{balls}})$	$\leftrightsquigarrow$	Formula $A_{N_{\mathrm{red}},N_{\mathrm{balls}}}$
$\exists$ Winning strategy	$\leftrightsquigarrow$	In all Boolean algebras, $A_{k,n} \to A_{N_{\mathrm{red}},N_{\mathrm{balls}}}$ holds

Back to the game

Allowed questions (variant k/n)	$\leftrightsquigarrow$	Formula $A_{k,n}$ "$\exists$ at most $f(k/n)$ elements"
Initial situation $(N_{\text{red}}, N_{\text{balls}})$	$\leftrightsquigarrow$	Formula $A_{N_{\text{red}},N_{\text{balls}}}$ "$\exists$ at most $f(N_{\text{red}}/N_{\text{balls}})$ elements"
$\exists$ Winning strategy	$\leftrightsquigarrow$	In all Boolean algebras, $A_{k,n} \to A_{N_{\text{red}},N_{\text{balls}}}$ holds

Back to the game

Allowed questions (variant $^{k}/_{n}$)	$\leftrightsquigarrow$	Formula $A_{k,n}$ "$\exists$ at most $f(^{k}/_{n})$ elements"
Initial situation $(N_{\text{red}}, N_{\text{balls}})$	$\leftrightsquigarrow$	Formula $A_{N_{\text{red}},N_{\text{balls}}}$ "$\exists$ at most $f(^{N_{\text{red}}}/_{N_{\text{balls}}})$ elements"
$\exists$ Winning strategy	$\leftrightsquigarrow$	In all Boolean algebras, $A_{k,n} \to A_{N_{\text{red}},N_{\text{balls}}}$ holds $f(^{k}/_{n}) \leq f(^{N_{\text{red}}}/_{N_{\text{balls}}})$

Translating variant $^{2}/_{3}$

- The key:
  -  $\Leftrightarrow \bot$  $\Leftrightarrow \top$

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  - universal quantification $\Leftrightarrow$ intersection type

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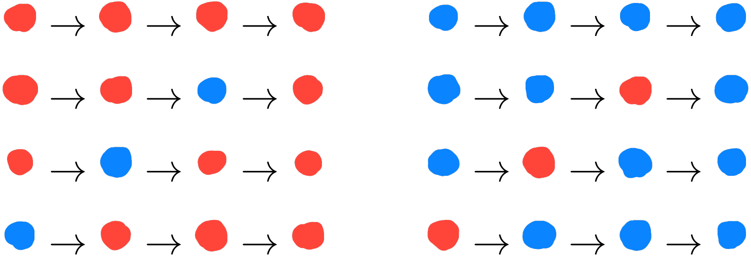

Translating variant $^2/_3$

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$$
\begin{array}{lll}
\bot \to \bot \to \bot \to \bot & \qquad & \top \to \top \to \top \to \top \\
\bot \to \bot \to \top \to \bot & & \top \to \top \to \bot \to \top \\
\bot \to \top \to \bot \to \bot & & \top \to \bot \to \top \to \top \\
\top \to \bot \to \bot \to \bot & & \bot \to \top \to \top \to \top
\end{array}
$$

Translating variant $\frac{2}{3}$

- The key:
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\bot \to \bot \to \bot \to \bot & \quad & \top \to \top \to \top \to \top \\
\bot \to \bot \to \top \to \bot & \quad & \top \to \top \to \bot \to \top \\
\bot \to \top \to \bot \to \bot & \quad & \top \to \bot \to \top \to \top \\
\top \to \bot \to \bot \to \bot & \quad & \bot \to \top \to \top \to \top
\end{array}$$

- $A_{2,3}$ is the formula:

$$\forall x \forall y \forall z \; (x{\neq}0 \to y{\neq}0 \to z{\neq}0 \to (x \wedge y \vee y \wedge z \vee z \wedge x) \neq 0)$$