Differential program semantics now with real bi-orthogonality pieces

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λ-terms – syntax

$$t, u \quad ::= \quad x \mid \lambda x.\, t \mid tu$$

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\begin{aligned}
t, u \quad ::= &\quad x \mid \lambda x.\, t \mid t u \\
\mid &\quad \langle t, u \rangle \mid \rho_L(t) \mid \rho_R(t)
\end{aligned}
$$

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& \mid & f\,(t_1, \ldots, t_n) \;\; (f : \mathbb{R}^n \to \mathbb{R}, n \in \mathbb{N})
\end{array}
$$

λ-terms – reduction

- $(\lambda x.\, t)u \to_\beta t\,[x := u]$

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  $(f \circ_j g)\,(t_1, \ldots, t_j, u_1, \ldots u_k, v_1, \ldots, v_l)$

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  $\mathsf{add}\,(1, \mathsf{add}\,(2, 3)) \to_\beta \mathsf{add}_3\,(1, 2, 3) \to_\beta^3 6$
- $\rho_L(\langle t, u\rangle) \to_\beta t$
- $\rho_R(\langle t, u\rangle) \to_\beta u$

λ-terms – types

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Expect: confluence + strong normalization

λ-terms – typing

$$\frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x.\ t : A \to B}$$

etc.

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$$\Lambda_A := \{\text{closed terms of type } A\}$$

Stacks (*i.e.* tests) – syntax

$$\pi \quad ::= \qquad I \text{ (closed interval)}$$

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Stacks (*i.e.* tests) – syntax

$$\begin{array}{rcl} \pi & ::= & I \text{ (closed interval)} \\ & | & t \cdot \pi \text{ (t closed term)} \\ & | & L \cdot \pi \mid R \cdot \pi \end{array}$$

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$$\frac{\vdash \pi : A}{\vdash L \cdot \pi : A \times B}$$

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$$\Pi_A := \{\text{stacks of type } A\}$$

Orthogonality : relation on $\Lambda_A \times \Pi_A$

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For $Y \subseteq \Pi_A$, $Y^{\perp\!\!\!\perp} := \{t \in \Lambda_A : \ \forall \pi \in Y \ \ t \perp\!\!\!\perp \pi\}$

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For $X \subseteq \Lambda_A$, $X^{\perp\!\!\!\perp} := \{\pi \in \Pi_A : \ \forall t \in X \ \ t \perp\!\!\!\perp \pi\}$

Approximate programs

$$[\![A]\!] \;:=\; \left\{X \subseteq \Lambda_A;\, X^{\perp\!\!\!\perp\perp\!\!\!\perp} = X\right\}$$

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Complete lattice

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- For all I, let $|I| := \{I\}^{\perp\!\!\!\perp}$

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- So $[\![\mathbb{R}]\!] = \{|I|\,;\ I \text{ closed interval}\}$

Approximate programs – examples

$$[\![A]\!]^* \ := \ [\![A]\!] \setminus \{\emptyset\}$$

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Example (Approximate functions)

- For all $F : \mathbb{R} \times \ldots \times \mathbb{R} \to \{I \neq \emptyset\}$, let $|F| := \{r_1 \cdot \ldots \cdot r_n \cdot F\,(r_1, \ldots, r_n)\,;\ r_1, \ldots, r_n \in \mathbb{R}\}^{\perp\!\!\!\perp}$

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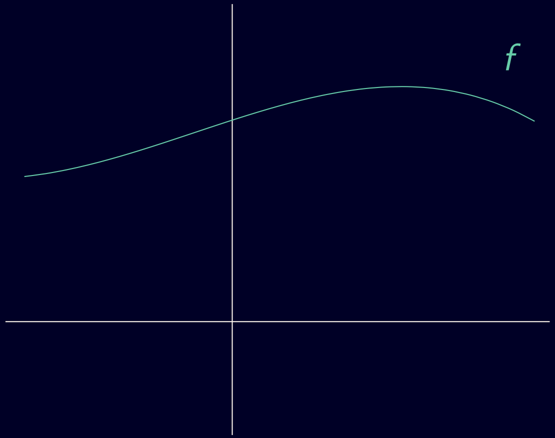
- Then $[\![\mathbb{R} \to \ldots \to \mathbb{R} \to \mathbb{R}]\!]^* = \{|F|\,;\, F \ldots\}$

Approximate programs – examples



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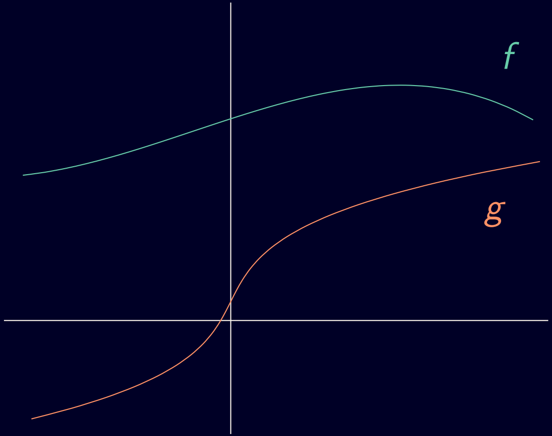

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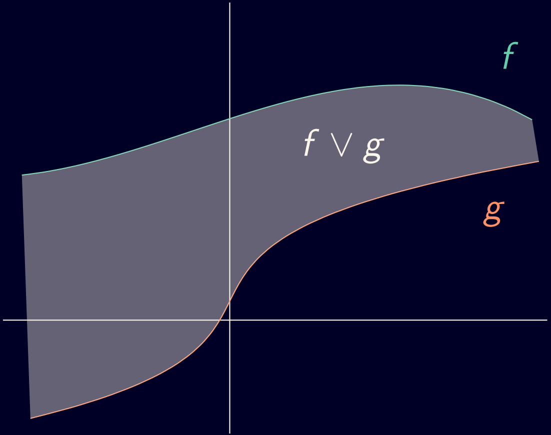

Approximate programs – examples

Example (Approximate pairs)

- For all $a \in [\![A]\!]$, $b \in [\![B]\!]$, let
  $|a \times b| := \left\{t;\ t \to_\beta^* \langle u, v\rangle, u \in a, v \in b\right\}$

Approximate programs – examples

Example (Approximate pairs)

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  $|a \times b| := \left\{t;\ t \rightarrow_\beta^* \langle u, v\rangle\,, u \in a, v \in b\right\}$
- Then $[\![A \times B]\!]^* = \{|a \times b|\,;\ a \in [\![A]\!]^*\,, b \in [\![B]\!]^*\}$

Substitution

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- $a_1 \in [\![A_1]\!], \ldots, a_n \in [\![A_n]\!]$

Substitution

- $t\,[x_1 : A_1, \ldots, x_n : A_n] : B$
- $a_1 \in [\![A_1]\!], \ldots, a_n \in [\![A_n]\!]$
- Then let

$$\begin{aligned} & t\,[x_1 := a_1, \ldots, x_n := a_n] \\ := & \left\{ \begin{array}{l} t\,[x_1 := u_1, \ldots, x_n := u_n]\,; \\ u_1 \in a_1, \ldots, u_n \in a_n \end{array} \right\}^{\perp\!\!\!\perp\perp\!\!\!\perp} \in [\![B]\!] \end{aligned}$$

Substitution – sub-functoriality

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- $a_1 \in [\![A_1]\!], \ldots, a_m \in [\![A_m]\!]$
- $\rightsquigarrow$ $$\begin{array}{rl} & t\,[u_1, \ldots, u_n]\,[a_1, \ldots, a_m] \\ \subseteq & t\,[u_1\,[a_1, \ldots, a_m], \ldots, u_n\,[a_1, \ldots, a_m]] \end{array}$$

Distances – distance spaces

- $(\!|\mathbb{R}|\!) := \mathbb{R}_+^\infty$
- $(\!|A \times B|\!) := (\!|A|\!) \times (\!|B|\!)$
- $(\!|A \to B|\!) := [\![A]\!] \to (\!|B|\!)$

Distances – diameter function

$\delta_A : [\![A]\!] \to (\!|A|\!)$:

- $\delta_{\mathbb{R}}\left(|I|\right) := \text{length}\left(I\right)$
- $\delta_{A\times B}\left(p\right) := \left(\delta_A\left(\rho_L\left(p\right)\right), \delta_B\left(\rho_R\left(p\right)\right)\right)$
- $\delta_{A\to B}\left(f\right)\left(a\right) := \delta_B\left(fa\right)$

Distances – diameter function

$\delta_A : [\![A]\!] \to (\!|A|\!)$:

- $\delta_{\mathbb{R}}(|I|) := \mathrm{length}(I)$
- $\delta_{A\times B}(p) := (\delta_A(\rho_L(p)), \delta_B(\rho_R(p)))$
- $\delta_{A\to B}(f)(a) := \delta_B(fa)$

$d_A(a, b) := \delta_A(a \vee b)$

Distances – sub-modularity

Proposition

If $a \wedge b \neq \emptyset$ *then*

$$\delta_A(a \vee b) + \delta_A(a \wedge b) \leq \delta_A(a) + \delta_A(b)$$

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If $a \wedge b \neq \emptyset$ *then*

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Corollary

For all $a, b, c \in [\![A]\!]^*$,

$$\delta_A(a \vee c) + \delta_A(b) \leq \delta_A(a \vee b) + \delta_A(b \vee c)$$

Distances – sub-modularity

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If $a \wedge b \neq \emptyset$ *then*

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Corollary

For all $a, b, c \in [\![A]\!]^*$,

$$d_A(a, c) + d_A(b, b) \leq d_A(a, b) + d_A(b, c)$$

Distances – partial metric ?

Proposition?

$[\![A]\!]^*$ is a partial metric space:

- $d_A(a, a) \leq d_A(a, b)$
- $d_A(a, b) = d_A(b, a)$
- $d_A(a, c) + d_A(b, b) \leq d_A(a, b) + d_A(b, c)$
- *if* $d_A(a, a) = d_A(a, b) = d_A(b, b)$, *then* $a = b$

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$\llbracket A \rrbracket^*$ *is a partial metric space:*

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- $\cancel{\textit{if } d_A(a, a) = d_A(a, b) = d_A(b, b), \textit{ then } a = b}$

Distances – locally finite distances

- $\llparenthesis \mathbb{R} \rrparenthesis_f := \mathbb{R}_+$

Distances – locally finite distances

- $(\!|\mathbb{R}|\!)_f := \mathbb{R}_+$
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  $\{\varepsilon : [\![A]\!] \to (\!|B|\!);\ \forall u : A,\ \varepsilon(\{u\}) \in (\!|B|\!)_f\}$

Distances – locally finite distances

- $(\!|\mathbb{R}|\!)_f := \mathbb{R}_+$
- $(\!|A \times B|\!)_f := (\!|A|\!)_f \times (\!|B|\!)_f$
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$$[\![A]\!]_f = \{a \in [\![A]\!]\,;\ \delta_A(a) \in (\!|A|\!)_f\}$$

Distances – partial metric

Proposition

$[\![A]\!]^*$ is almost a partial metric space:

- $d_A(a, a) \leq d_A(a, b)$
- $d_A(a, b) = d_A(b, a)$
- $d_A(a, c) + d_A(b, b) \leq d_A(a, b) + d_A(b, c)$
- *if* $d_A(a, a) = d_A(a, b) = d_A(b, b) \in (\!|A|\!)_f$, *then* $a = b$