Denotational semantics of probabilistic programs, beyond the discrete case

Guillaume Geoffroy

DIAPASoN, Unibo

What do probabilistic programs denote?

Examples of probabilistic programs

```
Input: random integer r
Output: random integer
x ← sample r
y ← sample uniform {0,1}
return x + y
```

Examples of probabilistic programs

```
Input: random integer r
Output: random integer
x ← sample r
y ← sample uniform {0,1}
return x + y
```

```
Input: random integer r
Output: random integer
x ← sample r
y ← sample r
return x + y
```

Examples of probabilistic programs

```
Input: random integer r
Output: random integer
x ← sample r
y ← sample uniform {0,1}
return x + y
```

```
Input: random integer r
Output: random integer
x ← sample r
y ← sample r
return x + y
```

Denotation: function $\{\text{random integers}\} \to \{\text{random integers}\}$

Examples of probabilistic programs

```
Input: random integer r
Output: random integer
x ← sample r
y ← sample uniform {0,1}
return x + y
```

```
Input: random integer r
Output: random integer
x ← sample r
y ← sample r
return x + y
```

Denotation: function $\{\text{sub-proba measures on } \mathbb{Z}\} \to \{\text{s.p.m. on } \mathbb{Z}\}$

Examples of probabilistic programs

```
Input: random integer r
Output: random integer
x ← sample r
y ← sample uniform {0,1}
return x + y
```

```
Input: random integer r
Output: random integer
x ← sample r
y ← sample r
return x + y
```

Denotation: function $G(\mathbb{Z}) \to G(\mathbb{Z})$

Examples of probabilistic programs

Input: random integer r Output: random integer x ← sample r y ← sample uniform {0,1} return x + y	Input: random integer r Output: random integer x ← sample r y ← sample r return x + y
$(p_k)_{k\in\mathbb{Z}} \mapsto \left(\qquad\qquad \right)_{k\in\mathbb{Z}}$	

Denotation: function $G(\mathbb{Z}) \to G(\mathbb{Z})$

Examples of probabilistic programs

Input: random integer r Output: random integer x ← sample r y ← sample uniform {0,1} return x + y	Input: random integer r Output: random integer x ← sample r y ← sample r return x + y
$(p_k)_{k\in\mathbb{Z}} \mapsto \left(\frac{1}{2}p_k + \frac{1}{2}p_{k-1}\right)_{k\in\mathbb{Z}}$	

Denotation: function $G(\mathbb{Z}) \to G(\mathbb{Z})$

Examples of probabilistic programs

Input: random integer r **Output:** random integer `x` $\leftarrow$ `sample r` `y` $\leftarrow$ `sample uniform {0,1}` `return x + y`	**Input:** random integer r **Output:** random integer `x` $\leftarrow$ `sample r` `y` $\leftarrow$ `sample r` `return x + y`
$(p_k)_{k\in\mathbb{Z}} \mapsto \left(\frac{1}{2}p_k + \frac{1}{2}p_{k-1}\right)_{k\in\mathbb{Z}}$	$(p_k)_{k\in\mathbb{Z}} \mapsto \left(\qquad\qquad \right)_{k\in\mathbb{Z}}$

Denotation: function $G(\mathbb{Z}) \to G(\mathbb{Z})$

Examples of probabilistic programs

``Input: random integer r``` ```Output: random integer``` ```x ← sample r``` ```y ← sample uniform {0,1}``` ```return x + y```	```Input: random integer r``` ```Output: random integer``` ```x ← sample r``` ```y ← sample r``` ```return x + y```
$(p_k)_{k\in\mathbb{Z}} \mapsto \left(\frac{1}{2}p_k + \frac{1}{2}p_{k-1}\right)_{k\in\mathbb{Z}}$	$(p_k)_{k\in\mathbb{Z}} \mapsto \left(\sum_{i+j=k} p_i p_j\right)_{k\in\mathbb{Z}}$

Denotation: function $G(\mathbb{Z}) \to G(\mathbb{Z})$

Key idea

- Linear program $\rightsquigarrow$ linear function

- Non-linear program $\rightsquigarrow$ analytic function (power series)

Key idea

- Linear program $\rightsquigarrow$ linear function

$$\left\{ \begin{array}{ccc} G(X) & \to & G(Y) \\ \mu & \mapsto & \int_{x \in X} \varphi(x)\, \mu(\mathrm{d}x) \end{array} \right. \qquad \varphi : X \to G(Y)$$

- Non-linear program $\rightsquigarrow$ analytic function (power series)

Key idea

- Linear program $\rightsquigarrow$ linear function

$$\left\{ \begin{array}{ccc} G(X) & \to & G(Y) \\ \mu & \mapsto & \int_{x \in X} \varphi(x)\, \mu(\mathrm{d}x) \end{array} \right. \qquad \varphi : X \to G(Y)$$

- Non-linear program $\rightsquigarrow$ analytic function (power series)

$$\left\{ \begin{array}{ccc} G(X) & \to & G(Y) \\ \mu & \mapsto & \sum_{n \in \mathbb{N}} \int_{\vec{x} \in X^n} \varphi_n(\vec{x})\, \mu^n(\mathrm{d}\vec{x}) \end{array} \right. \qquad \forall n, \varphi_n : X^n \to G(Y)$$

But: what about higher-order?

But: what about higher-order?

Define a structure *s.t.*

But: what about higher-order?

Define a structure *s.t.*

- for all measurable spaces A, the set $G(A)$ can be equipped with this structure,

But: what about higher-order?

Define a structure *s.t.*

- for all measurable spaces A, the set $G(A)$ can be equipped with this structure,
- for any instances X, Y and any function $f : X \to Y$, we have notions of f being *linear* and f being *analytic*,

But: what about higher-order?

Define a structure *s.t.*

- for all measurable spaces A, the set $G(A)$ can be equipped with this structure,
- for any instances X, Y and any function $f : X \to Y$, we have notions of f being *linear* and f being *analytic*,
- for any instance X, Y, $\{f : X \to Y;\ f \text{ linear}\}$ and $\{f : X \to Y;\ f \text{ measurable}\}$ can be equipped with this structure.

Bonus: linear functions commute with integrals

Bonus: linear functions commute with integrals

We would like:

Bonus: linear functions commute with integrals

We would like:

- for any instance X and any sub-probability measure μ on X, we have a sensible definition for $\mathbb{E}(\mu) = \int_{x \in X} x \, \mu(\mathrm{d}x) \in X$

Bonus: linear functions commute with integrals

We would like:

- for any instance X and any sub-probability measure μ on X, we have a sensible definition for $\mathbb{E}(\mu) = \int_{x \in X} x \, \mu(\mathrm{d}x) \in X$ ($\rightsquigarrow \mathbb{E}$ G-algebra),

Bonus: linear functions commute with integrals

We would like:

- for any instance X and any sub-probability measure μ on X, we have a sensible definition for $\mathbb{E}(\mu) = \int_{x \in X} x \, \mu(\mathrm{d}x) \in X$ ($\rightsquigarrow \mathbb{E}$ G-algebra),
- for all $f : X \to Y$ linear,

$$f\left(\int_{x \in X} x \, \mu(\mathrm{d}x)\right) = \int_{x \in X} f(x) \, \mu(\mathrm{d}x)$$

Bonus: linear functions commute with integrals

We would like:

- for any instance X and any sub-probability measure μ on X, we have a sensible definition for $\mathbb{E}(\mu) = \int_{x \in X} x \, \mu(\mathrm{d}x) \in X$ ($\rightsquigarrow \mathbb{E}$ G-algebra),
- for all $f : X \to Y$ linear,

$$f\left(\int_{x \in X} x \, \mu(\mathrm{d}x)\right) = \int_{x \in X} f(x) \, \mu(\mathrm{d}x)$$

In other words: linear map $\Rightarrow$ morphism of G-algebras

Bonus: linear functions commute with integrals

We would like:

- for any instance X and any sub-probability measure μ on X, we have a sensible definition for $\mathbb{E}(\mu) = \int_{x \in X} x \, \mu(\mathrm{d}x) \in X$ ($\rightsquigarrow \mathbb{E}$ G-algebra),
- for all $f : X \to Y$ linear,

$$f\left(\int_{x \in X} x \, \mu(\mathrm{d}x)\right) = \int_{x \in X} f(x) \, \mu(\mathrm{d}x)$$

In other words: linear map $\Rightarrow$ morphism of G-algebras
$\Leftrightarrow$ (for X, Y data types) sub-Markov kernel.

Bonus 2: arbitrary fixed points

Bonus 2: arbitrary fixed points

We would like:

Bonus 2: arbitrary fixed points

We would like:

- any analytic function $f : X \to X$ has a least fixed point $\mathrm{fix}\, f \in X$,

Bonus 2: arbitrary fixed points

We would like:

- any analytic function $f : X \to X$ has a least fixed point $\operatorname{fix} f \in X$,
- the map $\left\{ \begin{array}{ccc} \{f : X \to X \text{ analytic}\} & \to & X \\ f & \mapsto & \operatorname{fix} f \end{array} \right.$ is analytic.

Bonus 3: linear logic

Bonus 3: linear logic

We would like:

Bonus 3: linear logic

We would like:

- a model of linear logic,

Bonus 3: linear logic

We would like:

- a model of linear logic,
- in which each construction of linear logic has an intuitive interpretation in "probabilistic" terms.

Things to avoid

Things to avoid

- sigma-algebras

Things to avoid

- sigma-algebras
- limits, sups (topology in general)

Things to avoid

- sigma-algebras $\rightsquigarrow$ quasi-Borel spaces
- limits, sups (topology in general)

Things to avoid

- sigma-algebras $\rightsquigarrow$ quasi-Borel spaces
- limits, sups (topology in general) $\rightsquigarrow$ infinite sums

Things to avoid

- sigma-algebras $\rightsquigarrow$ quasi-Borel spaces
- limits, sups (topology in general) $\rightsquigarrow$ infinite sums (of positive reals, with sum ≤ 1)

Linear quasi-Borel spaces

Linear QBSs

A linear QBS X is:

Linear QBSs

A linear QBS X is:

- two linear QBSs $|X|$ and $|X^{\perp}|$,

Linear QBSs

A linear QBS X is:

- two linear QBSs $|X|$ and $|X^{\perp}|$,
- a measurable map (:⇔ QBS morph.) $\left\{ \begin{array}{ccc} |X| \times |X^{\perp}| & \to & [0,1] \\ (\eta, x) & \mapsto & \eta x \end{array} \right.$

Linear QBSs

A linear QBS X is:

- two linear QBSs $|X|$ and $|X^{\perp}|$,
- a measurable map (:⇔ QBS morph.) $\left\{ \begin{array}{rcl} |X| \times |X^{\perp}| & \to & [0,1] \\ (\eta, x) & \mapsto & \eta x \end{array} \right.$

such that

Linear QBSs

A linear QBS X is:

- two linear QBSs $|X|$ and $|X^{\perp}|$,
- a measurable map (:$\Leftrightarrow$ QBS morph.) $\begin{cases} |X| \times |X^{\perp}| & \to & [0,1] \\ (\eta, x) & \mapsto & \eta x \end{cases}$

such that

- $\forall \mu \in G\left(|X|\right), \exists! \int_{x \in X} x\mu(\mathrm{d}x) \in |X|, \forall \eta \in |X^{\perp}|,$

$$\eta \int_{x \in X} x\mu(\mathrm{d}x) = \int_{x \in X} \eta x \mu(\mathrm{d}x)$$

Linear QBSs

- $\forall \mu \in G\left(\left|X\right|\right), \exists! \int_{x \in X} x\mu\left(\mathrm{d}x\right) \in \left|X\right|, \forall \eta \in \left|X^{\perp}\right|,$

$$\eta \int_{x \in X} x\mu\left(\mathrm{d}x\right) = \int_{x \in X} \eta x\mu\left(\mathrm{d}x\right)$$

- $\forall (x_n)_{n \in \mathbb{N}}$ *s.t.* $\forall \eta, \sum_{n \in \mathbb{N}} \eta x_n \leq 1$
  $\exists! \sum_{n \in \mathbb{N}} x_n \in \left|X\right|, \forall \eta \in \left|X^{\perp}\right|,$

$$\eta \sum_{n \in \mathbb{N}} x_n = \sum_{n \in \mathbb{N}} \eta x_n$$

Linear QBSs

- $\forall \mu \in G\left(\left|X\right|\right), \exists! \int_{x \in X} x\mu\left(\mathrm{d}x\right) \in \left|X\right|, \forall \eta \in \left|X^{\perp}\right|,$
$$\eta \int_{x \in X} x\mu\left(\mathrm{d}x\right) = \int_{x \in X} \eta x\mu\left(\mathrm{d}x\right)$$
- $\forall (x_n)_{n \in \mathbb{N}}$ *s.t.* $\forall \eta, \sum_{n \in \mathbb{N}} \eta x_n \leq 1$
$\exists! \sum_{n \in \mathbb{N}} x_n \in \left|X\right|, \forall \eta \in \left|X^{\perp}\right|,$
$$\eta \sum_{n \in \mathbb{N}} x_n = \sum_{n \in \mathbb{N}} \eta x_n$$
- $\mu \mapsto \int_{x \in X} x\mu\left(\mathrm{d}x\right)$ and $(x_n)_{n \in \mathbb{N}} \mapsto \sum_{n \in \mathbb{N}} x_n$ are measurable,

Linear QBSs

- $\forall \mu \in G\left(\left|X\right|\right), \exists! \int_{x \in X} x\mu\left(\mathrm{d}x\right) \in \left|X\right|, \forall \eta \in \left|X^{\perp}\right|,$

$$\eta \int_{x \in X} x\mu\left(\mathrm{d}x\right) = \int_{x \in X} \eta x\mu\left(\mathrm{d}x\right)$$

- $\forall (x_n)_{n \in \mathbb{N}}$ $s.t.$ $\forall \eta, \sum_{n \in \mathbb{N}} \eta x_n \leq 1$
  $\exists! \sum_{n \in \mathbb{N}} x_n \in \left|X\right|, \forall \eta \in \left|X^{\perp}\right|,$

$$\eta \sum_{n \in \mathbb{N}} x_n = \sum_{n \in \mathbb{N}} \eta x_n$$

- $\mu \mapsto \int_{x \in X} x\mu\left(\mathrm{d}x\right)$ and $(x_n)_{n \in \mathbb{N}} \mapsto \sum_{n \in \mathbb{N}} x_n$ are measurable,
- and symmetrically (exchanging $\left|X\right|$ and $\left|X^{\perp}\right|$).

Canonical example: data types

For all QBSs A, we define a linear QBS $\underline{A}$:

Canonical example: data types

For all QBSs A, we define a linear QBS $\underline{A}$:

- $\left|\underline{A}\right| = G\left(A\right)$

Canonical example: data types

For all QBSs A, we define a linear QBS $\underline{A}$:

- $\left|\underline{A}\right| = G\left(A\right)$
- $\left|\underline{A}^{\perp}\right| = \mathrm{Qbs}\left(A, [0,1]\right)$

Canonical example: data types

For all QBSs A, we define a linear QBS $\underline{A}$:

- $\left|\underline{A}\right| = G\left(A\right)$
- $\left|\underline{A}^{\perp}\right| = \mathrm{Qbs}\left(A, [0,1]\right)$
- $\forall f \in \left|\underline{A}^{\perp}\right|, \forall \mu \in \left|\underline{A}\right|,$

$$f\,\mu = \int_{a \in A} f(a)\,\mu(\mathrm{d}a)$$

Dual space

Given a linear QBS $X = \left(|X|, |X^\perp|, (\eta, x) \mapsto \eta x \right)$, we define its dual $X^\perp = \left(|X^\perp|, |X|, (x, \eta) \mapsto \eta x \right)$.

Two particular cases

- The multiplicative unit $\mathbb{W} \approx \underline{\{*\}}$
  - $\left|\mathbb{W}\right| = \left|\mathbb{W}^{\perp}\right| = [0,1]$
  - observation is given by standard multiplication

Two particular cases

- The multiplicative unit $\mathbb{W} \approx \underline{\{*\}}$
  - $|\mathbb{W}| = |\mathbb{W}^{\perp}| = [0,1]$
  - observation is given by standard multiplication
- The additive unit $\mathbb{0} \approx \underline{\emptyset}$
  - $|\mathbb{0}| = |\mathbb{0}^{\perp}| = \{0\}$

Linear maps

Linear maps

Let X, Y be linear QBSs. A map $f : |X| \to |Y|$ is *linear* if:

Linear maps

Let X, Y be linear QBSs. A map $f : |X| \to |Y|$ is *linear* if:

- f is measurable

Linear maps

Let X, Y be linear QBSs. A map $f : |X| \to |Y|$ is *linear* if:

- f is measurable (meaning: *morphism of QBSs*),

Linear maps

Let X, Y be linear QBSs. A map $f : |X| \to |Y|$ is *linear* if:

- f is measurable (meaning: *morphism of QBSs*),
- there exits a (necessarily unique) measurable map $f : |X^\perp| \to |Y^\perp|$ *s.t.* $\forall x \in |X|, \forall \eta \in |Y^\perp|$,

$$\eta\left(f\left(x\right)\right) = \left(f^\perp\left(\eta\right)\right) x$$

Linear maps

Let X, Y be linear QBSs. A map $f : |X| \to |Y|$ is *linear* if:

- f is measurable (meaning: *morphism of QBSs*),
- there exits a (necessarily unique) measurable map $f : |X^{\perp}| \to |Y^{\perp}|$ *s.t.* $\forall x \in |X|, \forall \eta \in |Y^{\perp}|$,

$$\eta\left(f\left(x\right)\right) = \left(f^{\perp}\left(\eta\right)\right)x =: \eta f x$$

Linear maps commute with integrals and sums

Linear maps commute with integrals and sums

For all $f : X \to Y$ linear,

- for all $\mu \in G\left(|X|\right)$,

$$f\left(\int_{x\in|X|} x\,\mu\left(\mathrm{d}x\right)\right) = \int_{x\in|X|} f\left(x\right)\,\mu\left(\mathrm{d}x\right)$$

- for all $(x_n)_{n\in\mathbb{N}} \in |X|^{\mathbb{N}}$, if $\sum_{n\in\mathbb{N}} x_n$ is defined, then

$$f\left(\sum_{n\in\mathbb{N}} x_n\right) = \sum_{n\in\mathbb{N}} f\left(x_n\right)$$

Spaces of linear functions

For all linear QBSs X, Y, we define a linear QBS $X \multimap Y$

Spaces of linear functions

For all linear QBSs X, Y, we define a linear QBS $X \multimap Y$

- $|X \multimap Y|$ is the set of linear maps from X to Y

Spaces of linear functions

For all linear QBSs X, Y, we define a linear QBS $X \multimap Y$

- $|X \multimap Y|$ is the set of linear maps from X to Y
- $\left|(X \multimap Y)^{\perp}\right| = G\left(|X| \times |Y^{\perp}|\right)$

Spaces of linear functions

For all linear QBSs X, Y, we define a linear QBS $X \multimap Y$

- $|X \multimap Y|$ is the set of linear maps from X to Y
- $\left|(X \multimap Y)^{\perp}\right| = G\left(|X| \times |Y^{\perp}|\right)$
- $\mu f = \int \eta f x \, \mu\left(\mathrm{d}(x, \eta)\right)$

Spaces of linear functions

For all linear QBSs X, Y, we define a linear QBS $X \multimap Y$

- $\left|X \multimap Y\right|$ is the set of linear maps from X to Y
- $\left|(X \multimap Y)^{\perp}\right| = G\left(|X| \times \left|Y^{\perp}\right|\right) / \sim$, where $\mu \sim \nu$ if $\mu f = \nu f$ for all $f : X \multimap Y$
- $\mu f = \int \eta f x \, \mu\left(\mathrm{d}(x, \eta)\right)$

Spaces of linear functions

For all linear QBSs X, Y, we define a linear QBS $X \multimap Y$

- $|X \multimap Y|$ is the set of linear maps from X to Y
- $\left|(X \multimap Y)^\perp\right| = G\left(|X| \times |Y^\perp|\right) / \sim$, where $\mu \sim \nu$ if $\mu f = \nu f$ for all $f : X \multimap Y$
- $\mu f = \int \eta f x \, \mu \left(\mathrm{d}(x, \eta)\right)$

Spaces of linear functions

For all linear QBSs X, Y, we define a linear QBS $X \multimap Y$

- $|X \multimap Y|$ is the set of linear maps from X to Y
- $$\left|(X \multimap Y)^\perp\right| = \left\{ \begin{array}{l} (\mu_n)_{n\in\mathbb{N}} \in G\left(|X| \times |Y^\perp|\right)^{\mathbb{N}}; \\ \forall f, (\mu_n)_{n\in\mathbb{N}}\, f \leq 1 \end{array} \right\} / \sim$$
- $(\mu_n)_{n\in\mathbb{N}}\, f = \sum_{n\in\mathbb{N}} \int \eta f x\, \mu_n\left(\mathrm{d}(x, \eta)\right)$

Multiplicatives

- $X \multimap \mathbb{W} \simeq X^{\perp}$
- $X \parr Y := X^{\perp} \multimap Y \simeq Y^{\perp} \multimap X$
- $X \otimes Y :\simeq (X \multimap Y \multimap \mathbb{W}) \multimap \mathbb{W}$
- $\underline{A} \otimes \underline{B} \simeq \underline{A \times B}$
- $(\mathrm{QbsLin}, \mathbb{W}, \otimes, \multimap)$ monoidal closed

The cartesian product $\&$

For all linear QBSs $(X_n)_{n\in\mathbb{N}}$, we define $\&_{n\in\mathbb{N}} X_n$

- $\left|\&_{n\in\mathbb{N}} X_n\right| = \prod_{n\in\mathbb{N}} |X_n|$
- $\left|\left(\&_{n\in\mathbb{N}} X_n\right)^{\perp}\right| = \left\{ \begin{array}{l} (\eta_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} \left|X_n^{\perp}\right| ; \\ \forall\, (x_n)_{n\in\mathbb{N}} , \sum_{n\in\mathbb{N}} \eta_n\, x_n \leq 1 \end{array} \right\}$
- $(\eta_n)_{n\in\mathbb{N}}\, (x_n)_{n\in\mathbb{N}} = \sum_{n\in\mathbb{N}} \eta_n\, x_n$

The coproduct $\bigoplus$

For all linear QBSs $(X_n)_{n\in\mathbb{N}}$, we define $\bigoplus_{n\in\mathbb{N}} X_n$

- $\left|\bigoplus_{n\in\mathbb{N}} X_n\right| = \left\{ \begin{array}{l} (x_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} |X_n|\,; \\ \forall \left(\eta_n^{\perp}\right)_{n\in\mathbb{N}}, \sum_{n\in\mathbb{N}} \eta_n\, x_n \leq 1 \end{array} \right\}$
- $\left|\left(\bigoplus_{n\in\mathbb{N}} X_n\right)\right| = \prod_{n\in\mathbb{N}} \left|X_n^{\perp}\right|$
- $(\eta_n)_{n\in\mathbb{N}}\,(x_n)_{n\in\mathbb{N}} = \sum_{n\in\mathbb{N}} \eta_n\, x_n$

The coproduct $\bigoplus$

For all linear QBSs $(X_n)_{n\in\mathbb{N}}$, we define $\bigoplus_{n\in\mathbb{N}} X_n$

- $\left|\bigoplus_{n\in\mathbb{N}} X_n\right| = \left\{ \begin{array}{l} (x_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} \left|X_n\right| ; \\ \forall \left(\eta_n^\perp\right)_{n\in\mathbb{N}}, \sum_{n\in\mathbb{N}} \eta_n x_n \leq 1 \end{array} \right\}$
- $\left|\left(\bigoplus_{n\in\mathbb{N}} X_n\right)\right| = \prod_{n\in\mathbb{N}} \left|X_n^\perp\right|$
- $(\eta_n)_{n\in\mathbb{N}} (x_n)_{n\in\mathbb{N}} = \sum_{n\in\mathbb{N}} \eta_n x_n$

For all QBSs A, B, $\underline{A} \oplus \underline{B} \simeq \underline{A + B}$

The exponential modality "!"

The exponential modality "!"

- $!X := \varprojlim (X \mathbin{\&} \mathbb{W})^{\otimes_s n}$

The exponential modality "!"

- $!X := \varprojlim (X \,\&\, \mathbb{W})^{\otimes_s n}$(Melliès, Tabareau, Tasson. *An explicit formula for the free exponential modality of linear logic.*)

The exponential modality "!"

- $!X := \varprojlim (X \,\&\, \mathbb{W})^{\otimes_s n}$(Melliès, Tabareau, Tasson. *An explicit formula for the free exponential modality of linear logic.*)
- $\nabla_X := \left\{ \begin{array}{ccc} X & \to & !X \\ x & \mapsto & \sup_{n \in \mathbb{N}} (x, 1)^{\otimes_s n} \end{array} \right.$

The exponential modality "!"

- $!X := \varprojlim (X \,\&\, \mathbb{W})^{\otimes_s n}$(Melliès, Tabareau, Tasson. *An explicit formula for the free exponential modality of linear logic.*)
- $\nabla_X := \begin{cases} X & \to & !X \\ x & \mapsto & \sup_{n \in \mathbb{N}} (x, 1)^{\otimes_s n} \end{cases}$
- we call $f : X \to Y$ *analytic* if $\exists f_! : !X \multimap Y$ *s.t.* $f = f_! \circ \nabla_X$

The exponential modality "!"

- $!X := \varprojlim (X \,\&\, \mathbb{W})^{\otimes_s n}$(Melliès, Tabareau, Tasson. *An explicit formula for the free exponential modality of linear logic.*)
- $\nabla_X := \begin{cases} X & \to & !X \\ x & \mapsto & \sup_{n \in \mathbb{N}} (x, 1)^{\otimes_s n} \end{cases}$
- we call $f : X \to Y$ *analytic* if $\exists f_! : !X \multimap Y$ *s.t.* $f = f_! \circ \nabla_X$

Proposition

- $f_!$ is unique
- $g \circ f = \left(g_! \circ_{\mathrm{Kleisli}_!} f_!\right) \circ \nabla$

Arbitrary fixed points

Proposition

- any analytic function $f : X \to X$ has a least fixed point $\operatorname{fix} f \in X$,
- the map $\left\{ \begin{array}{ccc} \{f : X \to X \text{ analytic}\} & \to & X \\ f & \mapsto & \operatorname{fix} f \end{array} \right.$ is analytic.