Asymmetric Information, Liquidity Constraints, and Efficient Trade

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This paper investigates trading mechanisms for efficiently (re)allocating a good to agents who face liquidity constraints and have private information about their valuation. I derive a necessary and sufficient condition for the existence of ex post efficient, interim incentive compatible, interim individually rational, ex post budget balanced and ex post liquidity constrained trading mechanisms. The framework notably applies to partnership problems for which I show that the optimal ownership structures are typically asymmetric and that agents with low liquidity resources should initially receive larger shares, and vice versa. I also show that a larger market size tends to increase the agents' minimal liquidity requirements necessary for existence. This is at odds with the standard property that a larger market size facilitates existence in asymmetric information environments. Finally, I propose a liquidity constrained ex post efficient auction that implements the (re)allocation mechanism.

KEYWORDS. Mechanism design, efficient trade, liquidity constraints, partnerships.

1. INTRODUCTION

In markets with asymmetric information, liquidity constraints directly conflict with the design of incentive compatible trading mechanisms. On the one hand, such mechanisms must offer a price schedule from which agents with different valuations are incentivized to choose different prices so as to reveal their private information. On the other hand, liquidity constraints may prevent agents from choosing the highest prices in the schedule, making it impossible to ensure full revelation of information. That is, trading mechanisms must create a spread in possible prices offered to each agent to allow them to reveal information while liquidity constraints restrict the size of this spread.

When designing centralized markets, this issue can be partially addressed through ex ante redistribution of resources from richer to poorer agents. Although useful, this

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approach alone is insufficient to account for the most restrictive cases of liquidity constraints as it takes the incentive compatible price schedule as given. Instead, the price schedule must be designed to minimize the need for liquidity resources – and consequently subsidies – while maintaining its incentive compatibility properties.

To illustrate these two points, consider the following example.

EXAMPLE 1. Two siblings, A and B, have inherited a property for which they have independent and private valuations v_A and v_B both uniformly distributed on the unit interval. Additionally, assume sibling A has limited financial resources $l_A = 0.5$ and B is at least twice as rich, i.e., $l_B \geq 1$. The notary has been instructed to ensure that the sibling with the highest valuation obtains full ownership of the property and is considering using either a first-price or a second-price auction to that end. While the two auction formats implement the same allocation and yield the same interim expected payments from each sibling, it is straightforward to show that the actual final price paid by the winner ranges from 0 to 0.5 in the first-price auction whereas it ranges from 0 to 1 in the second-price auction.[1] Hence, only the price range induced by the first-price auction is feasible for sibling A. Notice that the first-price auction would still remain (weakly) more desirable if the notary had the power to enforce any ex ante redistribution of money between the siblings. While using the first-price auction requires no such transfers, the second-price auction would require sibling B to be at least three times as wealthy as sibling A, that is $l_B \geq 1.5$, to allow for the appropriate transfers. $\Diamond$

Although this example deliberately ignores some fundamental constraints of the trading mechanisms studied in this paper, it captures the main intuition behind the general setting. That is, two incentive compatible trading mechanisms implementing the same allocation can have rather different implications for liquidity requirements and/or for the amount of necessary cross-subsidy. This naturally raises the question of determining how to construct a trading mechanism that minimizes the need for liquidity from the market participants. Even without strict liquidity constraints such a mechanism would be preferable, as it achieves the same allocation while preserving agents' financial resources for other needs.

In this paper, I investigate this question in a general trading environment. A policymaker seeks to design a market to ex post efficiently (re)allocate a good among n agents privately informed about their valuations while simultaneously minimizing the mechanism's largest monetary transfers so as to reduce the financial burden on the market participants. To encompass a variety of environments, such as buyers-seller relationships or reallocation problems with initial ownership shares, I allow agents to have a rich type-dependent outside option. Imposing the usual participation and budget balance constraints, I provide an explicit construction of the ex post efficient and liquidity constrained mechanism and characterize necessary and sufficient conditions for its existence. To the best of my knowledge, this is the first characterization of existence of

[1]The equilibrium bidding strategy in the first-price auction is $\beta^F(v_i) = \frac{1}{2}v_i$ and that in the second-price auction is $\beta^S(v_i) = v_i$, for $i = A, B$. It is therefore immediate that β^F ranges from 0 to 0.5, while β^S ranges from 0 to 1.

ex post efficient mechanisms under liquidity constraints in two-sided asymmetric environments which can be notably be applied to generalized Myerson and Satterthwaite (1983)'s environments as well as partnership problems.

A building block in constructing efficient, incentive compatible and budget balanced mechanisms is the expected externality mechanism (EEM) introduced by d'Aspremont and Gérard-Varet (1979). While aligning each agent's incentives with social surplus maximization, the EEM also ensures budget balance through the help of non-distortionary fees. Unfortunately, this widely used construction[2] is too permissive with ex post monetary transfers and does not account for the most restrictive case of liquidity constraints. In other words, the EEM induces unnecessarily large ex post transfers that a better design could reduce without compromising on any of the other properties of the trading mechanism. I propose a construction – the liquidity constrained mechanism – that minimizes the range of individual ex post transfers and show that it is almost twice as small as the range induced by the EEM. Unlike the EEM, the liquidity constrained mechanism does not rely directly on the expected externalities to incentivize agents. Instead, it offers each of them the entire expected social surplus, provided it does not exceed their ex post externality. This novel construction allows me to maintain the interim incentive properties and balance transfers while *compressing* ex post transfers to their minimal possible range.

Despite its non-trivial construction, the liquidity constrained mechanism can be easily implemented through an auction resembling a first-price auction together with some entry bonus or fee. The highest bidder receives the good and pays the entire amount of their bid to each of the other $(n-1)$ agents. Perhaps surprisingly, this auction initially appears to induce very large payments, as the winner's bid is multiplied by the number of competitors. However, this threat of large payments actually pushes each agent's equilibrium bidding strategy to the lowest necessary range to ensure information revelation. Another notable property of the liquidity constrained auction is that, with ex ante symmetric agents, only the winner pays and no entry bonus or fee is required. As a comparison, the auction proposed by Cramton et al. (1987), in the specific case of partnership dissolution, may require losers to pay at equilibrium.

Importantly, liquidity requirements to ensure existence of an efficient trading mechanism are closely tied to the features of the trading environment. The expected social surplus achievable in the market crucially determines the minimum required levels of individual and aggregate liquidity. Essentially, higher expected social surplus leads to greater minimum liquidity requirements. A direct consequence of that property is that individual liquidity requirements are typically increasing in the number of agents participating in the market. This finding contrasts with the standard argument that a larger market is unambiguously beneficial to the existence of trading mechanisms in asymmetric information environments. The construction of liquidity constrained mechanisms also reveals the relationship between the distribution of individual and aggregate level of liquidity resources and the pursuit of ex post efficiency. From a redistributive

[2]It is notably used in Cramton et al. (1987), Lu and Robert (2001), Fieseler et al. (2003), Ledyard and Palfrey (2007), or Segal and Whinston (2011), among others.

point of view, a more equal distribution of initial liquidity resources tends to be weakly preferable to ensure the existence of ex post efficient trading mechanisms. This result naturally echoes the finding that equal-share ownership structures are preferable to more asymmetric ones in the special case of partnership dissolution problems. To shed light on the relationship between the initial distribution of property rights and liquidity constraints, I also investigate the partnership dissolution problem together with liquidity constraints. In contrast to one of Cramton et al. (1987)'s main results, equal-share partnerships do not necessarily guarantee the existence of an efficient trading mechanism. Initial distributions of liquidity resources and property rights must be well-balanced in the sense that an agent's initial ownership share must be (weakly) inversely related to their liquidity resources.

Existing literature has primarily focused on designing optimal mechanisms in one-sided private information settings, such as auctions for selling goods to privately informed buyers.[3] These models typically assume that agents are ex ante designated as buyers or sellers and do not account for type-dependent outside options when they opt out of the mechanism.[4] Furthermore, the focus is often on revenue maximization rather than efficiency, or on constrained efficiency while assuming that liquidity constraints are already preventing an ex post efficient allocation without providing nonexistence conditions.

In contrast, this paper studies the design of ex post efficient trading mechanisms in an environment that encompasses both one-sided and two-sided private information cases and provides a necessary and sufficient condition for existence with liquidity constrained agents. The reason to focus on ex post efficient allocation mechanism is twofold. First, while much of the literature examines how allocation distortions accommodate severe liquidity constraints, this paper highlights that even when the allocation remains fixed, the design of the ex post transfer structure plays a crucial role in ensuring existence. Second, understanding the existence of such mechanisms requires explicitly constructing the liquidity constrained mechanism, making it essential to refining possibility results in asymmetric information settings.

Several contributions related to liquidity constraints can be found in the auction literature. The early works of Laffont and Robert (1996) and Maskin (2000) respectively study the revenue-maximizing auction and a constrained-efficient auction with symmetrically liquidity-constrained bidders. Malakhov and Vohra (2008) derive the revenue-maximizing optimal mechanism with discrete values and when only one bidder is liquidity constrained. Recently, Boulatov and Severinov (2021) significantly extended these results to asymmetrically liquidity-constrained bidders. Another strand of this literature investigates the case of bidders privately informed about their liquidity

[3]The optimal revenue maximizing and constrained-efficient auction with symmetric liquidity constraints have been respectively studied by Laffont and Robert (1996) and Maskin (2000). Boulatov and Severinov (2021) extend these results with asymmetric liquidity constraints while Pai and Vohra (2014) investigate the case of privately known liquidity constraints.

[4]As mentioned earlier, these assumptions notably rule out buyers-seller relationships with private information on both sides in the spirit of Myerson and Satterthwaite (1983), partnership environments as in Cramton et al. (1987), or the study of ex post efficient auctions to allocate a good to privately informed agents with type-dependent outside options.

constraints. Che and Gale (1998, 2006) compare the performances of standard auctions in that case and Che et al. (2013b,a) study alternative assignment mechanisms. Pai and Vohra (2014) derive the optimal auction when both valuations and liquidity constraints are private information while Kotowski (2020) studies first-price auctions in a similar environment. These frameworks, however, are limited to one-sided asymmetric information environments and study either revenue maximization or constrained efficiency at the ex ante stage. Therefore, their analysis is silent about conditions under which ex post efficiency can be achieved in the presence of liquidity constraints and does not cover environments of bilateral trade or partnership problems.

By contrast, the present paper builds on the literature of ex post efficient trading mechanisms in two-sided asymmetric information environments, pioneered by Myerson and Satterthwaite (1983) and Cramton et al. (1987). Loertscher et al. (2015) refer to these environments as *as secondary-market allocation problems*, in contrast to *primary-market allocation problems* such as the above mentioned auction settings.[5] These models offer a general framework to analyze markets with privately informed traders who are not necessarily ex ante identified as buyers or sellers and can account for various initial ownership structures or type-dependent outside options. The framework has been extended to asymmetric and interdependent distributions (Figueroa and Skreta, 2012, Fieseler et al., 2003, Jehiel and Pauzner, 2006), ex post individual rationality (Galavotti et al., 2011), and second-best mechanisms (Lu and Robert, 2001, Loertscher and Wasser, 2019). However to the best of my knowledge, the present paper is the first to study ex post efficient trading mechanisms in two-sided asymmetric information environments with liquidity-constrained agents.

The paper is organized as follows. Section 2 introduces the theoretical framework. Section 3 presents the construction of the liquidity constrained mechanism and provides the necessary and sufficient condition for existence. Section 4 proposes an auction design to implement the liquidity constrained mechanism in a simple way. Section 5 investigates how the market size affects the agents' liquidity requirements. In Section 6, I apply the existence result to the special case of partnerships and characterize ownership and liquidity distributions that ensure existence. Section 7 briefly concludes. All proofs are given in the Appendix.

2. THEORETICAL FRAMEWORK

There are n risk-neutral agents indexed by $i \in N := \{1, \dots, n\}$ and one good. Each agent $i \in N$ has private information about their valuation v_i for the entire good. It is common knowledge that each valuation v_i is independently drawn from an absolutely continuous cumulative distribution function F_i with support $V_i := [\underline{v}_i, \overline{v}_i] \subseteq \mathbb{R}_+$ and positive continuous density f_i. Further assume that $V_i \cap V_j \neq \emptyset$ for any i, j and define $V := \times_{i \in N} V_i$. The utility of agent i is assumed to be of the form $v_i x_i + m_i$, where x_i is the share of the good they receive and m_i is money. For convenience, let $v := (v_1, \dots, v_n) \in V$ denote the

[5] Loertscher et al. (2015) also stress that the design of two-sided private information settings is fundamentally different from their one-sided counterparts. Notably, as the conflict between revenue and efficiency is more pronounced in two-sided settings, efficient mechanisms can merely fail to exist.

vectors of agents' valuations, v_{-i} denote the vector of valuations of all agents except that of agent i. Define $(v_i, v_{-i}) := v$ for any $i \in N$, where arguments are ordered differently only for readability.

By the Revelation Principle, there is no loss of generality in restricting mechanisms to direct revelation mechanisms. Each agent $i \in N$ directly reports their valuation v_i, all reports are collected, and the mechanism determines each individual allocation $s_i : V \to [0,1]$ and each individual transfer $t_i : V \to \mathbb{R}$, where $\sum_{i \in N} s_i(v) \leq 1$ for all $v \in V$ is the resource constraint on the good. Let $s(v) := (s_1(v), \ldots, s_n(v)) \in \Delta^{n-1}$ and $t(v) := (t_1(v), \ldots, t_n(v)) \in \mathbb{R}^n$ be the collections of individual ex post allocation rules and transfers, respectively. The pair (s,t) is referred to as a *mechanism*.

The ex post net utility of agent $i \in N$ who participates in the mechanism (s,t) is given by $v_i s_i(v) + t_i(v) - u_i^0(v)$, where $u_i^0(v)$ is agent i's outside option if they refuse to participate. Let $S_i(v_i) := \mathbb{E}_{-i} s_i(v)$, $T_i(v_i) := \mathbb{E}_{-i} t_i(v)$, and $U_i^0(v_i) := \mathbb{E}_{-i} u_i^0(v)$ denote the interim expected values of the allocation rule, the transfer rule, and the outside option, respectively.[6] Further assume that $U_i^0(v_i)$ is continuously differentiable in v_i.

The market designer seeks to achieve an ex post efficient (EF) allocation of the good. Formally, (s,t) is EF if, for all $v \in V$, the ex post allocation rule satisfies:

$$s^*(v) \in \operatorname*{arg\,max}_{s \in \Delta^{n-1}} \sum_{i \in N} v_i s_i(v).$$

The ex post efficient allocation rule can be rewritten as follows. For each agent i and all $v \in V$, $s_i^*(v) = \mathbb{1}\{i = \rho(v)\}$ where $\rho(v) := \min\left\{j \in N \mid j \in \arg\max_i v_i\right\}$ breaks ties in favor of the agent with the lowest index.[7]

Within the set of EF mechanisms (s^*, t) the designer wants to ensure that each agent i's ex post transfer never exceeds some lower bound $l_i \in \mathbb{R}_+$. That is, for all $v \in V$ the transfer of agent i must satisfy $t_i(v) \geq -l_i$. The individual bound l_i can be equivalently interpreted either as agent i's liquidity constraint or as some choice of the designer on the maximal possible contribution of agent i. It is assumed that the vector $l := (l_1, \ldots, l_n) \in \mathbb{R}_+$ is common knowledge. Mechanisms satisfying this boundary condition will be called *ex post liquidity constrained mechanisms* (EPLC).

In order to elicit the private information held by the agents, a mechanism (s,t) is said to *interim incentive compatible* (IIC) if it is a Bayesian Nash equilibrium that each agent reports truthfully. Formally, (s,t) is IIC if for all $i \in N$, $v_i \in V_i$ and $\hat{v}_i \in V_i$, $U_i(v_i, v_i) \geq U_i(v_i, \hat{v}_i)$, where $U_i(v_i, \hat{v}_i) := v_i S_i(\hat{v}_i) + T_i(\hat{v}_i) - U_i^0(v_i)$ is agent i's net interim expected utility when their type is v_i, they report $\hat{v}_i$, and all agents $j \neq i$ report truthfully. For convenience, let $U_i(v_i) := U_i(v_i, v_i)$ denote the net interim expected utility of agent i in the associated IIC mechanism (s,t).

As participation in the mechanism must be voluntary, a mechanism (s,t) must be *interim individually rational* (IIR). Given that all agents report truthfully, agent i is willing to participate in the mechanism at the interim stage if and only if $U_i(v_i) \geq 0$ for all $i \in N$, $v_i \in V_i$.

[6] The operator $\mathbb{E}_{-i}$ denotes the expectation over $v_{-i} \in V \setminus V_i$.

[7] This particular tie-breaking rule is without loss of generality. As valuations are drawn from absolutely continuous cumulative distribution functions, ties occur with zero probability so that agents' interim participation decision are unaffected by this choice.

Finally, the designer must maintain budget balance to account for the fact that all monetary transfers in the market occurs between its participants. Formally mechanism (s,t) is said to be *ex post budget balanced* (EPBB) whenever $\sum_{i\in N} t_i(v)=0$ for all $v\in V$.

3. Liquidity constrained mechanisms

Even in the absence of liquidity constraints, effectively constructing a transfer rule to implement a mechanism satisfying all the other desired properties (EF, IIC, IIR, and EPBB) can be a challenging task. The main issue usually comes from the interplay between incentive compatibility and budget balance: aligning each agent's incentive to reveal information typically generates a deficit that must be covered by report-contingent transfers in a non-distortionary way. One of the most well-known and widely used constructions is the *expected externality mechanism* (EEM) introduced by d'Aspremont and Gérard-Varet (1979). Unfortunately, the EEM heavily relies on liquidity, as it requires a broad range of individual ex post transfers. The analysis of liquidity-constrained mechanisms shows that ex post transfers can be significantly reduced while preserving all other desired properties.

Of course, due to incentive compatibility the size of ex post transfers cannot be restricted without limit. Interim incentive compatibility constraints precisely determine the range of interim transfers: In ex post efficient mechanisms, the range of each agent's interim transfer must equal the expected maximum valuation among all other agents. This constraint implies that ex post transfers must span at least the same range as interim transfers, though they may be larger. For example, in the EEM, the range of ex post transfers is precisely twice that of interim transfers. In contrast, the liquidity constrained mechanism nearly halves the range of ex post transfers in the EEM, bringing it as close as possible to that of interim transfers.

The first step in designing liquidity-constrained mechanisms is to assess the liquidity requirements at both the individual and aggregate levels. To derive the necessary condition for the existence of a liquidity constrained mechanism, I leverage the well-known result that any ex post efficient and interim incentive-compatible mechanism is payoff-equivalent to a Groves mechanism at the interim stage. This approach provides an intuitive exposition of the necessary condition and explicitly links liquidity requirements to fundamental primitives of the environment, such as the social surplus from trade. The formal proof of the main theorem of existence, however, does not rely on this methodology as I propose some results that go beyond ex post efficient mechanisms. The full proof is provided in the Appendix.

Following Makowski and Mezzetti (1994) and Williams (1999), for any ex post efficient and interim incentive compatible mechanism (s^*,t) there exists a Groves mechanism (s^*,t^*) that is payoff-equivalent at the interim stage for each agent. Formally, a Groves mechanism (s^*,t^*) is such that the transfer to agent $i\in N$ is given by:

$$t_i^*(v)=g(v)-v_i s_i^*(v)-h_i(v), \tag{1}$$

where for all $v \in V$

$$g(v) = \sum_{i \in N} v_i s_i^*(v),$$

denotes the social surplus generated by trade when v is realized and where the function $h_i : V \to \mathbb{R}$ is referred to as a *non-distortionary charge* and is such that $\mathbb{E}_{-i}\, h_i(v_i, v_{-i}) = \mathbb{E}_{-i}\, h_i(\hat{v}_i, v_{-i}) =: H_i$ for all v_i, $\hat{v}_i \in V_i$.

Starting from a Groves mechanism provides an easily interpretable characterization of interim transfers and payoffs in any EF and IIC mechanism. Using equation (1), these immediately write as

$$\begin{aligned} T_i^*(v_i) &= \mathbb{E}_{-i}\left[g(v) - v_i s_i^*(v)\right] - H_i \\ U_i^*(v_i) &= \mathbb{E}_{-i}\, g(v) - U_i^0(v_i) - H_i. \end{aligned}$$

While using the restricted class of Groves mechanisms is not fruitful to understand liquidity restrictions on ex post transfers, it nonetheless provides a useful characterization at the interim stage. Imposing that $t_i^*(v) \geq -l_i$ for all $i \in V$ and all $v \in V$ immediately implies that $T_i^*(v_i) \geq -l_i$ for all i and v_i in any EF and IIC mechanism. It is also easy to see that $\min_{v_i \in V_i} T_i^*(v_i) = -H_i$ so that a necessary condition implied by EPLC is that $H_i \leq l_i$ for all $i \in N$. Intuitively, this condition states that the non-distortionary charge on agent i can never exceed their liquidity resources.

These non-distortionary charges are subject to two additional conditions, arising from IIR and EPBB, respectively. At the interim stage, notice that agent i's worst-off type receives utility $\inf_{v_i \in V_i} U_i^*(v_i) = \inf_{v_i \in V_i}\{\mathbb{E}_{-i}\, g(v) - U_i^0(v_i)\} - H_i$. Then, it is convenient to define

$$C_i := \inf_{v_i \in V_i} \{\mathbb{E}_{-i} g(v) - U_i^0(v_i)\}, \tag{2}$$

as the largest feasible interim non-distortionary charge that can be imposed on agent i due to interim individual rationality. It immediately follows that $H_i \leq C_i$.

Finally, EPBB requires that for all $v \in V$, $\sum_{i \in N} t_i^*(v) = 0$ which is equivalent to $(n-1)g(v) = \sum_{i \in N} h_i(v)$. Notice that $(n-1)g(v)$ can be interpreted as the ex post deficit generated by a Groves mechanism absent the non-distortionary charges. In other words, it is the deficit created by eliciting the agents' private information. As this condition holds for all $v \in V$, it must also hold a necessary condition at the ex ante stage, i.e., $(n-1)\mathbb{E}g(v) = \sum_{i \in N} H_i$. The sum of interim non-distortionary charges must therefore exactly cover the ex ante budget deficit due to revelation of information.

Altogether, these necessary conditions on interim non-distortionary charges yield that

$$\sum_{i \in N} \min\{C_i, l_i\} \geq (n-1)G,$$

where $G := \mathbb{E}g(v)$ defines the ex ante social surplus.

It is natural to interpret $\sum_{i\in N}\min\{C_i,l_i\}$ as the highest collectible interim charge on agent i, determined by whether the individual rationality or the liquidity constraint is binding. Notice that when liquidity resources of all agents are *large enough*, this necessary condition collapses to $\sum_{i\in N} C_i \geq (n-1)G$ and corresponds to the necessary and sufficient condition of existence of trading mechanisms without liquidity requirements.[8]

The following theorem claims that the necessary condition $\sum_{i\in N}\min\{C_i,l_i\} \geq (n-1)G$ is also sufficient for the existence of liquidity constrained mechanisms and therefore provides a natural extension of existence to environments with liquidity-constrained agents.[9]

THEOREM 1. *Let $V_i = [\underline{v},\overline{v}]$ for all $i \in N$. An EF, IIC, IIR, EPBB and EPLC trading mechanism exists* ***if and only if***

$$\sum_{i\in N}\min\{C_i,l_i\} \geq (n-1)G. \tag{3}$$

At first glance, condition (3) may suggest that liquidity constraints act in a similar fashion as individual rationality constraints and therefore that a similar strategy can be applied to prove sufficiency. Without liquidity constraints, this strategy simply consists of constructing any ex post transfer rule t that satisfies EF, IIC and EPBB – such as the EEM – and redistribute interim utilities among agents through budget-balanced lump-sum transfers. In that case, the existence condition ($\sum_{i\in N} C_i \geq (n-1)G$) simply ensures that there exists a distribution of lump-sum transfers such that IIR is satisfied. The particular design of the ex post rule t is irrelevant, as any distribution of interim utilities induced by t can be replicated by another EF, IIC and EPBB transfer rule $\tilde{t}$ and well-chosen individual constants $(\phi_1,\ldots,\phi_n)$ such that each agent i receives $\tilde{t}_i(v)+\phi_i$, and where $\sum_{i\in N}\phi_i = 0$. In other words, satisfying IIR is only a matter of redistributing interim utilities.

With liquidity constraints, however, this strategy alone produces only limited results. The use of budget-balanced lump-sum transfers between agents is ultimately part of the construction of the liquidity constrained mechanism but its efficacy heavily relies on the precise design of the ex post transfer rule. The larger the range of ex post transfers for each agent, the less flexibility there is in adjusting the lump-sum transfers. Designing a liquidity constrained mechanism involves the additional step of minimizing the need for large transfers directly at the ex post stage while maintaining the incentive compatibility properties of the mechanism at the interim stage.

[8]More precisely, see Theorem 3.1 in Makowski and Mezzetti (1994) and condition (8) of Theorem 3 in Williams (1999).

[9]Although the sufficiency result is proven only in the case of symmetric supports and EF mechanisms, I show in the appendix that condition (3) can be generalized as a necessary condition of existence for asymmetric support of distributions of valuations, and any feasible and interim incentive compatible allocation rule. The generalized condition corresponds to equation (14). This generalization is notably used in Example 3 of Section 5 to account for asymmetric supports in a one seller and $n-1$ buyers trading problem. I conjecture that the generalized condition is also a sufficient condition for existence, but this remains an open question.

For the sake of exposition, I assume here symmetric distributions of valuations and $V_i = [0, \overline{v}]$ for all $i \in N$.[10] As previously introduced through Groves mechanisms, the most basic way of ensuring incentive compatibility consists in offering to each agent their ex post externality $g(v) - v_i s_i^*(v)$. Of course, this alone directly produces unbalanced transfers. The clever construction of the EEM results from the observation that giving each agent their expected externality $\varphi_i(v_i) := \mathbb{E}_{-i}[g(v) - v_i s_i^*(v)]$ generates the same interim incentives but has the advantage of depending only on agent i's valuation. The construction of the EEM relies on this slight change to satisfy ex post budget balance in the following way:

$$t_i^{EEM}(v) := \varphi_i(v_i) - \frac{1}{n-1} \sum_{j \neq i} \varphi_j(v_j) - \hat{\phi}_i, \tag{4}$$

where $\hat{\phi}_i$ is a constant such that $\sum_{i \in N} \hat{\phi}_i = 0$. Interim incentive compatibility is obviously directly satisfied through $\varphi_i(v_i)$. The second term ensures ex post budget balance in a non-distortionary fashion as it is independent of v_i for agent i. This construction, however, creates a large range of ex post payments which in turn require agents to have important liquidity resources. Formally, the transfer rule defined by (4) has a range of $2\mathbb{E}[\max_{j \neq i} v_j]$, that is, two times the expected value of the maximum of $n-1$ valuations.[11] This range of ex post payments is *excessive*, as the EEM may fail due to liquidity constraints even when condition (3) is satisfied.

The construction of the liquidity constrained mechanism also relies on the agents' expected externality but not as a direct payment to provide incentives. The individual transfer rule of the liquidity constrained mechanism writes as follows:

$$t_i^{LCM}(v) = \delta_i(v) - \frac{1}{n} \sum_{j \in N} \delta_j(v) - \phi_i, \tag{5}$$

where $\delta_i(v) := \mathbb{E}_{\tilde{v}}[g(\tilde{v}) \mid g(\tilde{v}) \leq g(v) - v_i s_i^*(v)]$. To avoid confusion, $\mathbb{E}_{\tilde{v}}$ denotes the expectation with respect to $\tilde{v} \in V$ while $v \in V$ is a vector of realizations of the agents' valuations. That is, each agent i's is offered the entire expected social surplus provided that it is lower than their ex post externality minus the average value of this truncated expected social surplus over all agents in the market. While the transfer rule defined by (5) straightforwardly satisfies budget balance when $\sum_{i \in N} \phi_i = 0$, the case of incentive compatibility is not as immediate as in the EEM. Indeed, both terms, $\delta_i(v)$ and $\frac{1}{n} \sum_{j \in N} \delta_j(v)$, depend on all valuations such that the incentive part of the rule cannot be separated from its budget balance part as clearly as in the EEM. In the appendix, I show that $t_i^{LCM}(v)$ is indeed IIC.

[10] The symmetry assumption greatly simplifies the interpretation and setting $\underline{v} = 0$ removes some additional constants in the transfer rule of the liquidity constrained mechanism.

[11] It is easy to see that $\max_{v \in V} t_i^{EEM}(v) = \mathbb{E}[\max_{j \neq i} v_j] - \hat{\phi}_i$ when $v_i = \underline{v}$ and $v_j = \overline{v}$ for all $j \neq i$, and that $\min_{v \in V} t_i^{EEM}(v) = -\mathbb{E}[\max_{j \neq i} v_j] - \hat{\phi}_i$ when $v_i = \overline{v}$ and $v_j = \underline{v}$ for all $j \neq i$.

Finally, the liquidity constrained mechanism has a range equal to G, the expected social surplus, which is unambiguously lower than the range of the EEM.[12] This construction therefore allows for a greater degree of freedom in choosing the budget balanced lump-sum transfers ϕ_i for each agent. Simple computation (see the appendix for details) show that IIR and EPLC are satisfied provided that $\phi_i \leq \min\{C_i, l_i\} - \frac{n-1}{n}G$ for all $i \in N$. One natural way to choose the constant terms to jointly satisfy IIR, EPLC and EPBB is as follows:

$$\phi_i^* = \min\{C_i, l_i\} - \frac{n-1}{n}G - \frac{1}{n}\left[\sum_{j\in N}\min\{C_j, l_j\} - (n-1)G\right],$$

which is obviously budget balanced and satisfies IIR and EPLC as long as condition (3) holds.[13] Notice that in the simple case in which,for all agents, liquidity constraints are more stringent than the individual rationality ones the constants can be written as $\phi_i^* = l_i - \frac{1}{n}\sum_{j\in N}\min\{C_j, l_j\}$. That is, the mechanism redistributes money to agents whose individual liquidity resources are lower than the average ones by taxing agents above the average. Although this redistribution scheme is not unique, it suggests that achieving an equitable distribution of liquidity resources is desirable for achieving efficiency. I further explore the question of the initial distribution of liquidities in Section 6 by considering a reallocation problem in which agents have explicit initial ownership rights of the traded good.

I conclude this section with an illustrative comparison of the EEM and the liquidity constrained mechanism.

EXAMPLE 2. Assume $n = 2$ and $F_i(v_i) = v_i$ for $i = 1, 2$ with support $V_i = [0, 1]$. The expected externality of agent i writes as $\varphi(v_i) = \mathbb{E}_{-i}[g(v) - v_i s_i^*(v)] = \frac{1}{2} - \frac{v_i^2}{2}$. It immediately follows that the EEM writes as follows:

$$t_i^{EEM}(v) = \frac{1}{2}(v_j^2 - v_i^2) - \hat{\phi}_i.$$

Instead, the basic payment $\delta_i(v) = \mathbb{E}[g(\tilde{v}) \mid g(\tilde{v}) \leq g(v) - v_i s_i^*(v)]$ is such that $\delta_i(v) = \frac{1}{3}v_j s_j(v)$ so that the liquidity constrained mechanism writes as:

$$t_i^{LCM}(v) = \frac{1}{3}(v_j s_j(v) - v_i s_i(v)) - \phi_i.$$

At the interim stage we have that $\mathbb{E}_{-i}t_i^{EEM}(v) = \hat{\phi}_i - \frac{1}{2}v_i^2$ and $\mathbb{E}_{-i}t_i^{LCM}(v) = \phi_i - \frac{1}{2}v_i^2$, i.e., both mechanisms induce the same interim payments, up to a constant. Interim

[12] First, it is useful to notice that $t_i^{LCM}(v) = -\frac{n-1}{n}\mathbb{E}[g(\tilde{v}) \mid g(\tilde{v}) \leq v_i] - \phi_i$ when $\rho(v) = i$ and $t_i^{LCM}(v) = \frac{1}{n}\mathbb{E}[g(\tilde{v}) \mid g(\tilde{v}) \leq v_j] - \phi_i$ when $\rho(v) = j \neq i$. As $\mathbb{E}[g(\tilde{v}) \mid g(\tilde{v}) \leq y]$ is clearly increasing in y, it is immediate that $\max_{v\in V} t_i^{LCM}(v) = \frac{1}{n}\mathbb{E}[g(\tilde{v}) \mid g(\tilde{v}) \leq \overline{v}] - \phi_i = \frac{1}{n}G + \phi_i$ and $\min_{v\in V} t_i^{LCM}(v) = -\frac{n-1}{n}\mathbb{E}[g(\tilde{v}) \mid g(\tilde{v}) \leq \overline{v}] - \phi_i = -\frac{n-1}{n}G - \phi_i$. It follows that the range of $t_i^{LCM}(v)$ is G.

[13] The choice of the constants is of course nonunique when condition (3) holds with strict inequality and there is still some freedom in choosing the distribution of surplus among agents.

incentive compatibility of the liquidity constrained mechanism is therefore satisfied as we know it is in the EEM.

It is easy to see that $t_i^{EEM}(v) \in [-\frac{1}{2}, \frac{1}{2}]$ while $t_i^{LCM}(v) \in [-\frac{1}{3}, \frac{1}{3}]$. Hence, the range of the liquidity constrained mechanism is clearly lower than that of the EEM. Non-distortionary trading charges $\hat{\phi}_i$ and ϕ_i are clearly subject to the same constraints at the interim stage (for IIC) but not at the ex post stage: ϕ_i is less constrained than $\hat{\phi}_i$ due to the smaller range of the liquidity constrained mechanism. $\diamondsuit$

4. A LIQUIDITY CONSTRAINED AUCTION

At first glance, the construction of the liquidity constrained mechanism seems complicated when compared to the EEM that relies on a slight but ingenious modification of basic Groves mechanisms. However, the practical implementation of the liquidity constrained mechanism is arguably simple and resembles a first-price auction. Additionally, it has desirable properties, such as the fact that only the winner pays, regardless of the valuation profile.

Let $b := (b_1, \dots, b_n) \in \mathbb{R}_+^n$ denote the vector of bids submitted by the n participants, and let $(C_1, \dots, C_n)$ be the vector of largest feasible interim non-distortionary charges (as defined by equation (2)).

PROPOSITION 1. *Let $F_i = F$, $V_i = [\underline{v}, \overline{v}]$ for all $i \in N$ and assume condition (3) holds. The bidding game in which agents bid $b \in \mathbb{R}_+^n$, the highest bidder receives the good, agent i pays a price*

$$p_i(b) := \begin{cases} (n-1)\left[b_i + \frac{1}{n}\underline{v}\right] & \textit{if } b_i \geq \max_{k\in N} b_k \\ -\left[b_j + \frac{1}{n}\underline{v}\right] & \textit{if } b_j \geq \max_{k\in N} b_k, \end{cases}$$

and receives a side payment

$$k_i := \frac{1}{n}\sum_{j\in N} \min\{C_j, l_j\} - \min\{C_i, l_i\},$$

ensures an ex post efficient allocation (EF), and satisfies IIR, EPBB, and EPLC.

In this auction, the winner pays their bid not just once, but to each of the other $n-1$ agents (modulo the term $\frac{1}{n}\underline{v}$). In that sense, it resembles a first-price auction, but with payments inflated by the number of losing bidders. At first, it may seem that the pricing rule will imply large ex post payments from the winner. But it is precisely this feature that generates very low equilibrium bidding strategies. At the Bayes-Nash equilibrium of this game, each agent bids according to:

$$b(v_i) := \int_{\underline{v}}^{v_i} \frac{\int_{\underline{v}}^{y} F(x)^n dx}{F(y)^{n+1}} dF(y).$$

This strategy results in bids ranging from 0 to $\frac{1}{n}[G - \underline{v}]$ and induces a range of equilibrium payments of G as in the liquidity constrained mechanism presented in Section 3. As a comparison, the the pricing rule proposed by Cramton et al. (1987),

$\tilde{p}_i(b) := b_i - \frac{1}{n-1}\sum_{j\neq i} b_j$, induces equilibrium bids $\tilde{b}(v_i) = \int_{\underline{v}}^{v_i} y dF(y)^{n-1}$ ranging from 0 to $\mathbb{E}[\max_{j\neq i} v_j]$. Equilibrium payments in that case exhibits a range of $2\mathbb{E}[\max_{j\neq i} v_j]$ as the EEM mechanism. Of course, these results directly stem from the construction of the liquidity constrained mechanism in Section 3 but Proposition 1 demonstrates that its practical implementation is quite simple.

Side payments in the auction serve to ensure IIR and EPLC. Ex post budget balance follows immediately from the fact that $\sum_{i\in N} p_i(b) = 0$ for all b and $\sum_{i\in N} k_i = 0$. Finally, ex post efficiency follows from the fact that the equilibrium bidding strategy $b(v_i)$ is increasing in v_i for all $i \in N$. Even if side payments are not possible – for legal or practical reasons – or if the agents' liquidity resources are unknown to the designer, the liquidity constrained auction's payment rule minimizes the use of large payments, thereby *maximizing* the likelihood that all agents meet the liquidity requirements to participate. In that sense, it is always weakly preferable to other types of mechanisms as it does not involve any compromise on efficiency, incentive compatibility, or budget balance.

It is also worth noting that the *performance* of each of these pricing rules with respect to liquidity constraints depends on the number of participants in the auction. To illustrate this property, suppose that valuations are uniformly distributed on the unit interval, i.e., $F_i(v_i) = v_i$ and $V_i = [0, 1]$ for all $i \in N$. Figure 1a shows the equilibrium bidding strategies associated with the pricing rules $p_i(b)$ (solid curve) and $\tilde{p}_i(b)$ (dashed curve). Figure 1b represents the equilibrium ranges of the two pricing rules evaluated at their respective equilibrium bidding strategies.

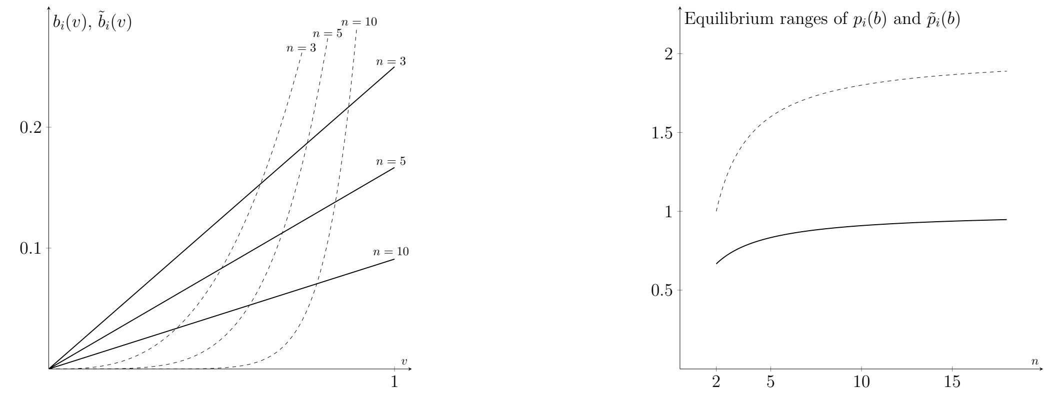
As it can be seen on Figure 1a, the liquidity constrained auction is inducing a decreasing spread of equilibrium bids while the other auction makes this spread larger as the number of agents increases. As a result, the effective price paid by bidders has a lower range under the pricing rule $p_i(b)$ than under the pricing rule $\tilde{p}_i(b)$, and this range is much less affected by an increase in the number of participants.

This property is not specific to the assumptions made in this example.The next section highlights that the number of participants in general trading mechanisms is crucial for determining the minimum liquidity each agent must possess.

5. Market size

The existence condition of Theorem 1 crucially relies on the ex ante expected deficit generated by a Groves mechanism $(n-1)G$. This deficit directly stems from the incentive compatibility requirement as from equation (1), the interim expected transfer of agent i must be equal to $\mathbb{E}_{-i}[g(v) - v_i s_i^*(v)] - H_i$ in any ex post efficient and interim incentive compatible trading mechanism. As the number of agents participating in the trading mechanism increases, the deficit $(n-1)G$ increases as well. Not only does it increase through $(n-1)$ but also through G, the ex ante expected social surplus.[14] Hence, as the number of agents increases, the volume of transfers among participants must increase to cover the deficit generated by the incentive compatibility requirement.

[14]Recall that $G := \mathbb{E}g(v)$ corresponds to the expectation of the maximum of the random variables $(v_1, \ldots, v_n)$ which is naturally increasing in n for any distributions of valuations.



(a) (b)

FIGURE 1. (a) Equilibrium bidding strategies of pricing rules $p_i(b)$ and $\tilde{p}_i(b)$ for some values of n; (b) Ranges of pricing rules $p_i(b)$ and $\tilde{p}_i(b)$ evaluated at their respective equilibrium bidding strategy as n varies. Solid curves correspond to pricing rule $p_i(b)$ and dashed curves to $\tilde{p}_i(b)$. Valuations are supposed to be uniformly distributed on the unit interval.

At first glance, it seems difficult to assess the impact of the number of agents on the existence of efficient trading mechanisms in the general case described by Theorem 1 as both sides of equation (3) may vary with the number of agents. It is clear, however, that a necessary condition for equation (3) to hold is that $\sum_{i\in N} l_i \geq (n-1)G$. The next result illustrates that liquidity constraints become quite restrictive when the number of agents increases.

PROPOSITION 2. *Assume $l_i = \tilde{l} \in \mathbb{R}_+$ for all $i \in N$. An ex post efficient trading mechanism exists only if*

$$\tilde{l} \geq \frac{n-1}{n} G.$$

Moreover, assume $V_i = [\underline{v}, \overline{v}]$ and $F_i = F$ for all $i \in N$, then the minimum individual resource requirement $\underline{l}(n) := \frac{n-1}{n} G$ increases in n and converges to $\overline{v}$ when n goes to infinity.

The most important feature of Proposition 2 is the fact that $\underline{l}(n)$ increases in the number of agents and eventually reaches the upper bound of the support of valuations in the limit case. It means that having more participants in the trading mechanism increases the pressure on each agent with respect to their minimal liquidity resources requirement.

In my opinion, the main message of Proposition 2 is that ignoring the presence of liquidity constraints in trading environments with multi-sided asymmetric information seems unjustified if the market size is assumed to be large. In other words, one should be careful when studying the properties of such environments as liquidity requirements seem to be at odds with the standard intuition that a larger market size is unambiguously beneficial.

Indeed, it is standard that in the absence of liquidity constraints, a larger number of participants favors the existence of efficient trading mechanisms through (i) an increase in *competition* among them and thus lowers each agent's informational rent; and (ii) larger expected social surplus as the expected highest valuation increases when more agents participate. Proposition 2, however, suggests that having too many participants can compromise the existence of efficient trading mechanisms: the benefits on the social surplus mentioned in (ii) directly lead to a larger ex ante deficit induced by the incentive compatibility constraints which translates into larger liquidity requirements.

Notice also that adding new agents with large or unlimited liquidity resources to the trading mechanism is of no help in the symmetric liquidity resources case. Worse still, their arrival weighs on the liquidity requirements of the original set of participants. The existence of an efficient trading mechanism therefore relies critically upon the liquidity constraint of the most liquidity-constrained agent – that is less likely to hold as the number of participants increases.

To illustrate these ideas, I now present a simple example.

EXAMPLE 3. Consider the case of a single seller, $i=1$, facing $n-1$ potential buyers denoted by $i=2,\ldots,n$. The seller has valuation $v_1 \in [0,c]$ for the good they own and buyers have valuations $v_i \in [0,1]$ for all $i \in N \setminus \{1\}$. Valuations are uniformly and independently distributed on their respective support and $c \in [0,1]$. To account for the seller's full ownership of the good, the trading environment can be seen as a n-agent partnership with initial ownership shares $r_1 = 1$ and $r_i = 0$ for all $i \in N \setminus \{1\}$, and ex post outside option $u_i^0(v) = v_i r_i$ (see Section 6 for details). This framework is an extension of Myerson and Satterthwaite (1983) with multiple buyers as first introduced by Makowski and Mezzetti (1993). I further assume that each agent has limited liquidity resources $l_i \in \mathbb{R}_+$.

A necessary condition for existence of a trading mechanism in this environment writes $\min\{C_1(1), l_1\} + \sum_{i=2}^{n} \min\{C_i(0), l_i\} \geq (n-1)G$, where $C_i(r_i)$ denotes the maximal interim non-distortionary charge that can be taken on agent i with ownership share r_i.[15]

It is straightforward to compute $G = \frac{n-1}{n} + \frac{c^n}{n(n+1)}$.[16] Then, equation (10) defines the worst-off types of the seller and the buyers, that is, $v_1^*(1) = c$ and $v_i^*(0) = 0$ for all $i \in N \setminus \{1\}$. The maximal non-distortionary charges due to individual rationality constraints follow from equation (9):

$$C_1(1) = \frac{n-1}{n} - c + \frac{c^n}{n},$$
$$C_i(0) = \frac{n-2}{n-1} + \frac{c^{n-1}}{n(n-1)} \text{ for any } i \in N \setminus \{1\}.$$

First, ignore liquidity constraints so that a necessary condition for existence simply writes $C_1(1) + \sum_{i=2}^{n} C_i(0) \geq (n-1)G$. The case $n=2$ corresponds to Myerson and Satterthwaite (1983)'s impossibility result as the condition requires $c \geq 3/2$, contradicting

[15]The computations are a direct application of Corollary 1 presented in Section 6 and are therefore omitted here.

[16]The joint cumulative distribution functions write $F_{-1}(y) = y^{n-1}$ and $F_{-i}(y) = y^{n-2}\left(\frac{y}{c}\right)^{\mathbb{1}\{y<c\}}$ for all $i \in N \setminus \{1\}$ with support on $[a_j, b_j] = [0,1]$ for all $j \in N$.

the assumption that $c \leq 1$. For $n \geq 3$, it is possible to show that there exists a $n^*(c)$ such that for any $c \in [0,1]$, the necessary condition without liquidity constraints is satisfied for all $n \geq n^*(c)$.[17] The threshold $n^*(c)$ is increasing in c, that is, if the seller *values the good more*, there must be more potential buyers for the condition to be satisfied.

Consider now that the potential buyers have limited liquidity resources, $l_i = \tilde{l} \in \mathbb{R}_+$ for all $i \in N \setminus \{1\}$ such that $\tilde{l} \leq C_i(0)$.[18] The seller, however, is assumed to have unlimited liquidity resources, i.e, $l_1 = +\infty$. In this environment, a necessary condition for the existence of an efficient trading mechanism is $C_1(1) + (n-1)\tilde{l} \geq (n-1)G$, or, after straightforward computations:

$$\tilde{l} \geq \frac{n-2}{n} + \frac{c}{n-1} - \frac{2c^n}{n(n-1)(n+1)}. \tag{6}$$

To focus on one of the most favorable environment for the existence of trading mechanisms, assume that c is arbitrarily close to zero.[19] The following table reports the minimal value of $\tilde{l}$ depending on the number of potential buyers for equation (6) to be satisfied.

# of buyers	1	2	3	4	5	6	7	8	9
$\tilde{l}$	0	.33	.50	.60	.66	.71	.75	.77	.80

In line with Proposition 2, the minimal liquidity requirement is increasing in the number of agents participating in the trading mechanism. More importantly, recall that buyers' valuations have support on $[0,1]$ meaning that with three buyers the minimal liquidity requirement must equal half the maximal possible valuation, about two-thirds with five buyers, and three-quarters with seven buyers. $\diamondsuit$

This example immediately contradicts the intuition that having more buyers is unambiguously beneficial. Additional buyers increase the likelihood of high realizations of valuations which in turn increases the ex ante expected cost of a Groves mechanism to reveal agents' private information. As this cost must be somehow financed by the agents through transfers, the largest payments that agents may have to make increase as well and so are their minimal liquidity requirements.

6. Partnerships

A special case of this framework is the partnership dissolution problem that was first introduced by Cramton et al. (1987). A n-agent partnership is characterized by an initial ownership distribution over the asset to be traded and the problem consists in finding

[17] Makowski and Mezzetti (1993) provide a treatment of the case without liquidity constraints in the same environment and characterize a threshold $c^*(n)$. I instead rely on the converse threshold $n^* := (c^*)^{-1}$ for the purposes of the analysis.

[18] Notice that when c is close to zero, $C_i(0) \approx \frac{3}{4}, \frac{5}{6}, \frac{6}{7}, \frac{7}{8}$ for $n = 5, 6, 7, 8$, respectively. Recall that $\overline{v} = 1$ in this framework so that the condition $\tilde{l} \leq C_i(0)$ is not extremely restrictive.

[19] It is easy to show that the right-hand side of (6) is increasing in c so that the case $c \to 0$ corresponds to the less demanding scenario in terms of liquidity resources.

a dissolution mechanism such that ownership shares are reallocated to the partner who values them the most. I apply Theorem 1 to the partnership framework and characterize distributions of ownership shares and liquidity resources allowing for existence of a dissolution mechanism. For the sake of clarity, I rely once again on the approach introduced in Section 3.

Let $r := (r_1, \ldots, r_n) \in \Delta^{n-1}$ denote the initial distribution of ownership shares among the n agents. Each agent's share determines their outside option if they refuse to participate in the dissolution mechanism, that is, $u_i^0(v) = v_i r_i$. For notational convenience, define $F_{-i}(y) := \prod_{j \neq i} F_j(y)$ and $f_{-i} := F'_{-i}$ with support on $[a_i, b_i] := [\max_{j \neq i} \underline{v}_j, \max_{j \neq i} \overline{v}_j]$. Using equation (2) and $u_i^0(v) = v_i r_i$, agent i's constraint on the non-distortionary charge due to individual rationality writes as follows:

$$C_i(r_i) := \inf_{v_i \in V_i} \left\{ \int_{a_i}^{v_i} v_i dF_{-i}(y) + \int_{v_i}^{b_i} y dF_{-i}(y) - v_i r_i \right\}, \tag{7}$$

where $C_i(\cdot)$ is explicitly defined as a function of agent i's ownership share r_i for later use. Agent i's worst-off type v_i^* is defined by the first-order condition of problem (7), that is,

$$F_{-i}(v_i^*) = r_i, \tag{8}$$

if there exists such a $v_i^* \in V_i$.[20] Otherwise, $v_i^* = \overline{v}_i$ if $F_{-i}(\overline{v}_i) < r_i$ and $v_i^* = \underline{v}_i$ if $F_{-i}(\underline{v}_i) > r_i$. This characterization of worst-off types in partnership problems is a generalization of Cramton et al. (1987) to the case of asymmetric distributions and supports of valuations.

Hence, $C_i(r_i)$ can be rewritten as

$$C_i(r_i) = \int_{v_i^*(r_i)}^{b_i} y dF_{-i}(y) + v_i^*(r_i)\big(F_{-i}(v_i^*(r_i)) - r_i\big), \tag{9}$$

where $v_i^*(r_i)$ denotes agent i's worst-off type and is defined as follows:

$$v_i^*(r_i) = \begin{cases} \underline{v}_i & \text{if} \quad F_{-i}(\underline{v}_i) > r_i \ \text{ or } \ r_i = 0 \\ \overline{v}_i & \text{if} \quad F_{-i}(\overline{v}_i) < r_i \ \text{ or } \ r_i = 1 \\ F_{-i}^{-1}(r_i) & \text{otherwise.} \end{cases} \tag{10}$$

The cases $v_i^*(0) = \underline{v}_i$ and $v_i^*(1) = \overline{v}_i$ are defined for convenience as agent i's worst-off type might not be unique when $r_i \in \{0, 1\}$. Applying the envelope theorem to equation (7) gives that $C_i'(r_i) = -v_i^*(r_i)$ so that $C_i(r_i)$ is both decreasing and concave in r_i as $v_i^*(r_i)$ is increasing in r_i.

Under the symmetric supports assumption, $V_i = [\underline{v}, \overline{v}]$ for all $i \in N$, notice that $C_i(0) = \int_{\underline{v}}^{\overline{v}} y dF_{-i}(y)$ and $C_i(1) = 0$. In words, the largest non-distortionary charge that can be levied on agent i without any ownership share, $C_i(0)$, corresponds to the $(n-1)$th

[20] The first-order condition is also a sufficient condition for a global minimum as $\frac{\partial^2}{\partial v_i^2}(\mathbb{E}_{-i} g(v_i, v_{-i}) - v_i r_i) = v_i f_{-i}(v_i) \geq 0$ for all $v_i \in V_i$. Notice that the solution to equation (8) might not exist or is not necessarily unique when the supports of valuation are asymmetric. When supports are symmetric, however, equation (8) fully and uniquely determines agent i's worst-off type.

order statistics of the agents' valuations. This largest charge decreases as the ownership share of agent i increases and becomes null when agent i has full ownership of the asset. Ownership provides agents with a form of bargaining power in the dissolution mechanism.

The existence condition of a dissolution mechanism in partnership environments directly follows as a corollary of Theorem 1.

COROLLARY 1. *Let $V_i = [\underline{v}, \overline{v}]$ for all $i \in N$. A partnership with ownership rights $r \in \Delta^{n-1}$ and liquidity resources $l \in \mathbb{R}^n_+$ can be dissolved efficiently if and only if*

$$\sum_{i \in N} \min\{C_i(r_i), l_i\} \geq (n-1)G. \tag{11}$$

Equation (11) together with the characterization of $C_i(r_i)$ clearly highlight that the interplay between the distributions of ownership shares and liquidity resources among agents affects the sum of non-distortionary charges and thus the existence of a dissolution mechanism. As for Theorem 1, equation (3) also holds as a necessary condition of existence in the case of asymmetric supports of valuations.

I now present some characterization results relative to this issue. I further assume that agents' valuations are i.i.d. random variables, that is $F_i = F$ and $V_i = [\underline{v}, \overline{v}]$ for all $i \in N$, where F is an absolutely continuous cumulative distribution function. This assumption can be easily relaxed but I find it useful to present clear-cut characterization results about the interaction between ownership shares and liquidity resources.

For convenience, let $\tilde{r}_i \in [0,1]$ be defined by $\tilde{r}_i = 0$ when $l_i > C_i(0)$ and by $C_i(\tilde{r}_i) = l_i$ when $l_i \leq C_i(0)$. This threshold characterizes which is the most restrictive constraint between IIR and EPLC as $\min\{C_i(r_i), l_i\} = l_i$ when $r_i \leq \tilde{r}_i$, and $\min\{C_i(r_i), l_i\} = C_i(r_i)$ when $r_i > \tilde{r}_i$. Notice also that $\tilde{r}_i$ is decreasing in l_i and can thus be seen as a measure of the severity of liquidity constraints on agent i's collectible charge. Then, $\sum_{i \in N} \tilde{r}_i$ is an aggregate measure of the severity of liquidity constraints in the partnership.

The first set of results assumes the distribution of liquidity resources is fixed and characterizes the corresponding ownership structures that maximize the sum of agents' collectible charges. I refer to this ownership structure as the *optimal distribution of property rights*. For ease of exposition assume, without loss of generality, that $l_1 \geq \cdots \geq l_n$ so that $\tilde{r}_1 \leq \cdots \leq \tilde{r}_n$.

PROPOSITION 3. *Let $F_i = F$ and $V_i = [\underline{v}, \overline{v}]$ for all $i \in N$. For any $l \in \mathbb{R}^n_+$ such that $\sum_{i \in N} \tilde{r}_i \leq 1$, the optimal distribution of property rights $r^* \in \Delta^{n-1}$ is as follows:*

a. *If $\tilde{r}_i \leq \frac{1}{n}$ for all $i \in N$, then $r^* = (\frac{1}{n}, \ldots, \frac{1}{n})$;*

b. *If $\tilde{r}_i > \frac{1}{n}$ for some $i \in N$, then $r^* = (\hat{r}, \hat{r}, \ldots, \hat{r}, \ \tilde{r}_p, \tilde{r}_{p+1}, \ldots, \tilde{r}_n)$ where $\hat{r} = \frac{1 - \sum_{j \geq p} \tilde{r}_j}{p-1}$ for some $p \in N$ such that $\max_{i<p} \tilde{r}_i < \hat{r} \leq \min_{j \geq p} \tilde{r}_j$.*

When liquidity constraints are not too severe at the aggregate level, i.e. $\sum_{i \in N} \tilde{r}_i \leq 1$, the optimal distribution of property rights r^* can take two forms. Proposition 3.a states that if liquidity constraints are also mild at the agent level for all agents, then equal sharing of ownership maximizes agents' collectible charges. This case corresponds to one of the main results of Cramton et al. (1987) who also show that this ownership structure always ensures existence of a dissolution mechanism.

On the contrary, Proposition 3.b states that if liquidity constraints are too severe for some agents, then the optimal distribution of property rights allocates (weakly) more initial ownership shares to more liquidity-constrained agents. It should be noted that the sum of collectible charges in 3.b is always lower than in 3.a as the departure from equal sharing ownership is the result of a trade-off due to the severity of liquidity constraints at the individual level for some agents. Hence, distributing initial ownership shares as in 3.b does not necessarily guarantee that equation (11) holds, i.e, that a dissolution mechanism exists.

EXAMPLE 4. To illustrate Proposition 3.b, consider a two-agent partnership in which agent 1 is not liquidity constrained, i.e. $\tilde{r}_1 = 0$, whereas agent 2 is heavily liquidity constrained so that $\tilde{r}_2 \in [\frac{1}{2}, 1)$.

It is clear that starting from any $r_2 < \tilde{r}_2$, and in particular $r_2 = \frac{1}{2}$, it is possible to strictly increase the sum of feasible contributions $\sum_{i=1,2} \min\{C_i(r_i), l_i\} = C_1(r_1) + l_2$ by increasing r_2 up to $\tilde{r}_2$ as $C_i(\cdot)$ is a decreasing function and $\min\{C_2(r_2), l_2\} = l_2$ is unchanged for all $r_2 \leq \tilde{r}_2$. In other words, it is innocuous to give more initial ownership rights to heavily liquidity-constrained agents as their feasible contribution is already limited by their liquidity resources. On the other hand, it allows to give less initial ownership rights to less liquidity-constrained agents and collect more from them.

It is easy to construct an example in which the equal-share partnership does not allow for a dissolution mechanism to exist while an asymmetric initial allocation or property rights does. For instance, take $F_i(v_i) = v_i$ and $V_i = [0, 1]$ for $i = 1, 2$, so that $G = \frac{2}{3}$ and $C_i(r_i) = \frac{1}{2}[1 - r_i^2]$. Further assume that $\tilde{r}_2 = 0.7$, which corresponds to $l_2 = 0.255$. In that case, it is clear that existence fails under equal-share ownership as $\sum_{i=1,2} \min\{C_i(\frac{1}{2}), l_i\} = C_1(\frac{1}{2}) + l_2 = \frac{3}{8} + 0.255 = 0.63 < G$. On the contrary, let for instance $r_1 = 0.3$ and $r_2 = \tilde{r}_2 = 0.7$. Then the sum of feasible contributions becomes $C_1(0.3) + l_2 = 0.455 + 0.255 = 0.71 > G$ so that a dissolution mechanism exists. $\Diamond$

While Proposition 3 shows that the distribution of property rights exhibits some structure when liquidity constraints are mild enough at the aggregate level, the next result shows that this is not the case when there are more severe, that is, when $\sum_{i \in N} \tilde{r}_i > 1$ holds.

PROPOSITION 4. *Let $F_i = F$ and $V_i = [\underline{v}, \overline{v}]$ for all $i \in N$. For any $l \in \mathbb{R}^n_+$ such that $\sum_{i \in N} \tilde{r}_i > 1$, an optimal distribution of property rights $r^* \in \Delta^{n-1}$ is such that $r_i^* \leq \tilde{r}_i$ for all $i \in N$, $\sum_{i \in N} r_i^* = 1$, and $\sum_{i \in N} \min\{C_i(r_i^*), l_i\} = \sum_{i \in N} \min\{C_i(0), l_i\}$.*

The only structure that property rights should satisfy is that agents' shares must be such that for all $i \in N$, $r_i^* \leq \tilde{r}_i$ and $\sum_{i \in N} r_i^* = 1$. This last condition can always be satisfied for some $r^* \in \Delta^{n-1}$ as $\sum_{i \in N} \tilde{r}_i > 1$ by assumption. It is worth noticing that if some agent j has $\tilde{r}_j = 0$, that is $l_j > C_j(0)$, the optimal distribution of property rights allocates no initial ownership share to this agent. The resulting optimal distribution of property rights does not have much structure on the side of heavily liquidity-constrained agents. It does, however, exhibit the particular feature that agents with large liquidity resources should receive no initial property rights.

The second set of characterization results only assumes that the total amount of liquidity resources is fixed but allows for any distribution of liquidity and ownership among agents. Let $L \in \mathbb{R}_+$ denote the total amount of available liquidity resources in the n-agent partnership. A couple (r^*, l^*) is said to be an *optimal organization of the partnership* if it solves

$$(r^*, l^*) \in \arg\max_{(r,l)} \sum_{i \in N} \min\{C_i(r_i), l_i\},$$

subject to $(r, l) \in \Delta^{n-1} \times \mathbb{R}^n_+$ and $\sum_{i \in N} l_i = L$.

As it is now possible to choose the distribution of liquidity resources and ownership shares simultaneously, the optimal organization of the partnership will rely crucially on the total amount of liquidity resources L. If it is large enough, it is possible to allocate enough liquidity resources to each agent so as to achieve the highest collectible charge in a partnership, i.e, the one corresponding to equal sharing of ownership in the absence of liquidity constraints (Cramton et al., 1987). Otherwise, the best that can be done is to arrange the distributions so as to collect the total amount of liquidity resources among agents. The next proposition formalizes these intuitions.

PROPOSITION 5. *For any $L \in \mathbb{R}_+$, where $L := \sum_{i \in N} l_i$, an optimal organization of the partnership (r^*, l^*) achieves the following maximal collectible charge*

$$\sum_{i \in N} \min\{C_i(r_i^*), l_i^*\} = \min\{\sum_{i \in N} C_i(1/n), L\}.$$

From Proposition 5 it is straightforward that a dissolution mechanism exists if and only if $L \geq (n-1)G$, assuming one can choose an optimal organization for the partnership.[21] That is, as long as the total amount of liquidity resources is enough to cover the budget deficit generated by a Groves mechanism, it is always possible to allocate liquidity resources and ownership shares in a way that a dissolution mechanism exists.

7. CONCLUSION

This paper studied a trading model with asymmetric information and liquidity constraints. It analyzed the conditions under which efficient trade is possible, and showed that the details of the design of incentive compatibility constraints are crucial to accommodate the most severe cases of liquidity constraints. The framework also raises the issue of the benefits of large markets in asymmetric information environments as

[21] This stems directly from Cramton et al. (1987) who show that equal sharing of ownership always allows for ex post efficient dissolution in the absence of liquidity constraints, that is, $\sum_{i \in N} C_i(1/n) \geq (n-1)G$.

it shows that an increase in the number of participants tends to increase the need for liquidity resources. As the assumption that agents have no liquidity constraints is often made in these environments, the interaction between asymmetric information, liquidity resources and market size is worth further investigation.

APPENDIX

For convenience, I define the following notations. Let $F_{-i}(y) := \prod_{j\neq i} F_j(y)$ and $f_{-i} := F'_{-i}$ with support on $[a_i, b_i] := [\max_{j\neq i} \underline{v}_j, \max_{j\neq i} \overline{v}_j]$. Similarly, let $\boldsymbol{F}(y) := \prod_{i\in N} F_i(y)$, and $\boldsymbol{f} := \boldsymbol{F}'$.

PROOF OF THEOREM 1 (NECESSITY). The proof of necessity of condition (3) extends the result of Theorem 1 to asymmetric supports of valuations and to any feasible and interim incentive compatible allocation rule $s_i : V \to [0,1]$ with $\sum_{i\in N} s_i(v) \leq 1$. Consider any interim incentive compatible mechanism (s,t), then the following inequalities must hold for any two valuations v_i, $\hat{v}_i \in V_i$,

$$U_i(v_i) \geq v_i S_i(\hat{v}_i) + T_i(\hat{v}_i) - U_i^0(v_i),$$
$$U_i(\hat{v}_i) \geq \hat{v}_i S_i(v_i) + T_i(v_i) - U_i^0(\hat{v}_i),$$

which in turn implies that

$$(v_i - \hat{v}_i)S_i(v_i) \geq U_i(v_i) - U_i(\hat{v}_i) + \big(U_i^0(v_i) - U_i^0(\hat{v}_i)\big) \geq (v_i - \hat{v}_i)S_i(\hat{v}_i).$$

This inequality implies that $S_i(v_i)$ must be nonincreasing and that $U_i(v_i)$ is absolutely continuous and almost everywhere differentiable with derivative $\partial U_i/\partial v_i(v_i) = S_i(v_i) - \partial U_i^0/\partial v_i(v_i)$. Integrating this last equation over $[\hat{v}_i, v_i]$ gives that

$$U_i(v_i) = U_i(\hat{v}_i) + \int_{\hat{v}_i}^{v_i} \Big(S_i(y) - \frac{\partial U_i^0}{\partial v_i}(y)\Big)dy.$$

Standard computations show that the expected interim transfer rule writes as follows:

$$T_i(v_i) = T_i(\hat{v}_i) - \int_{\hat{v}_i}^{v_i} y dS_i(y). \tag{12}$$

Imposing IIR requires $U_i(v_i) \geq 0$ for all $i \in N$ and $v_i \in V_i$. It is equivalent to impose $U_i(v_i^*) \geq 0$ for all $i \in N$ where $v_i^* \in \arg\min_{v_i\in V_i} U_i(v_i)$ represents agent i's worst-off type. Expressed in terms of interim transfers this yields:

$$T_i(v_i^*) \geq U_i^0(v_i^*) - v_i^* S_i(v_i^*).$$

EPLC requires that $t_i(v) \geq -l_i$ for all $i \in N$ and $v \in V$ which implies that $T_i(v_i) \geq -l_i$ for all $i \in N$ and $v_i \in V_i$ must hold. Clearly, $T_i(v_i)$ is decreasing in v_i so that it is sufficient to satisfy $T_i(\overline{v}_i) \geq -l_i$ for all $i \in N$, or equivalently,

$$T_i(v_i^*) \geq \int_{v_i^*}^{\overline{v}_i} y dS_i(y) - l_i. \tag{13}$$

Hence, IIR and EPLC implies that interim transfers must satisfy

$$T_i(v_i^*) \geq \max\left\{U_i^0(v_i^*) - v_i^* S_i(v_i^*), \int_{v_i^*}^{\overline{v}_i} y dS_i(y) - l_i\right\}.$$

As EPBB requires that $\sum_{i\in N} t_i(v) = 0$ for all $i \in N$ and $v \in V$, it must be true that $\sum_{i\in N} \mathbb{E}_i T_i(v_i) = 0$ for all $v_i \in V_i$. Using this last equality and equation (12) for $\hat{v}_i = v_i^*$ yields

$$\begin{aligned}
\sum_{i\in N} T_i(v_i^*) &= \sum_{i\in N} \mathbb{E}_i \int_{v_i^*}^{v_i} y dS_i(y) \\
&= \sum_{i\in N} \left\{\int_{v_i^*}^{\overline{v}_i} (1 - F_i(y)) y dS_i(y) - \int_{\underline{v}_i}^{v_i^*} y F_i(y) dS_i(y)\right\},
\end{aligned}$$

where the second line is obtained by changing the order of integration.

Summing equation (13) over all $i \in N$ and using the above result immediately gives that

$$\begin{aligned}
&\sum_{i\in N} \left\{\int_{v_i^*}^{\overline{v}_i} (1 - F_i(y)) y dS_i(y) - \int_{\underline{v}_i}^{v_i^*} y F_i(y) dS_i(y)\right\} \\
&\qquad\qquad \geq \sum_{i\in N} \max\left\{U_i^0(v_i^*) - v_i^* S_i(v_i^*), \int_{v_i^*}^{\overline{v}_i} y dS_i(y) - l_i\right\},
\end{aligned}$$

which can be rewritten as follows:

$$\begin{aligned}
&\sum_{i\in N} \min\left\{v_i^* S_i(v_i^*) - U_i^0(v_i^*), l_i - \int_{v_i^*}^{\overline{v}_i} y dS_i(y)\right\} \\
&\qquad\qquad \geq \sum_{i\in N} \left\{\int_{\underline{v}_i}^{v_i^*} y F_i(y) dS_i(y) - \int_{v_i^*}^{\overline{v}_i} (1 - F_i(y)) y dS_i(y)\right\}.
\end{aligned}$$

Now it is convenient to add the term $\int_{v_i^*}^{\overline{v}_i} y dS_i(y)$ on both sides and define $\mathcal{C}_i := v_i^* S_i(v_i^*) + \int_{v_i^*}^{\overline{v}_i} y dS_i(y) - U_i^0(v_i^*)$, so that

$$\sum_{i\in N} \min\{\mathcal{C}_i, l_i\} \geq \sum_{i\in N} \int_{\underline{v}_i}^{\overline{v}_i} y F_i(y) dS_i(y), \tag{14}$$

which corresponds to the generalized version of the necessary condition (3).

To conclude the proof of the "only if" part, it only remains to prove that equation (14) corresponds to condition (3) under the ex post efficient allocation rule and symmetric support assumption $V_i = [\underline{v}, \overline{v}]$ for $i \in N$.

First, notice that in that case $S_i(v_i) = F_{-i}(v_i)$ so that $\mathcal{C}_i = v_i^* F_{-i}(v_i^*) + \int_{v_i^*}^{\overline{v}_i} y dF_{-i}(y) - U_i^0(v_i^*)$. Then, notice that equation (2) can be rewritten as

$$C_i := \inf_{v_i \in V_i} \{v_i F_{-i}(v_i) + \int_{v_i}^{\overline{v}_i} y dF_{-i}(y) - U_i^0(v_i)\}.$$

By payoff equivalence at the interim stage and as $v_i^* \in \arg\min_{v_i \in V_i} U_i(v_i)$, it follows that $\mathcal{C}_i = C_i$.

Second, the right-hand side of (14) rewrites as follows:

$$
\begin{aligned}
\sum_{i\in N}\int_{\underline{v}}^{\overline{v}} yF_i(y)dS_i(y) &= \sum_{i\in N}\int_{\underline{v}}^{\overline{v}} yF_i(y)\sum_{k\neq i} f_k(y)\frac{F_{-i}(y)}{F_k(y)}dy \\
&= \sum_{i\in N}\sum_{k\neq i}\int_{\underline{v}}^{\overline{v}} yf_k(y)F_{-k}(y)dy \\
&= (n-1)\sum_{i\in N}\int_{\underline{v}}^{\overline{v}} yf_i(y)F_{-i}(y)dy \\
&= (n-1)\int_{\underline{v}}^{\overline{v}} yd\prod_{i\in N}F_i(y) \\
&= (n-1)G,
\end{aligned}
$$

which concludes the proof of the “only if” part. □

PROOF OF THEOREM 1 (SUFFICIENCY). The sufficiency of condition (3) is only proven in the symmetric support case and under the ex post efficient allocation rule.

Consider the following ex post transfer rule:

$$
t_i(v) := \begin{cases} -\sum_{k\neq i}\psi_k(v_i) - \frac{n-1}{n}\underline{v} - \phi_i & \text{if } \rho(v) = i \\ \psi_i(v_j) + \frac{1}{n}\underline{v} - \phi_i & \text{if } \rho(v) = j \neq i, \end{cases} \tag{15}
$$

where $\psi_k(v_p) := \int_{\underline{v}}^{v_p} \frac{\int_{\underline{v}}^{x}\boldsymbol{F}(y)dy}{\boldsymbol{F}(x)}\frac{f_k(x)}{F_k(x)}dx$ and $\phi_i \in \mathbb{R}$ is a constant.

Before proceeding with the proof is it useful to show that

$$
\begin{aligned}
\sum_{i\in N}\psi_i(v_p) &= \int_{\underline{v}}^{v_p}\frac{\int_{\underline{v}}^{x}\boldsymbol{F}(y)dy}{\boldsymbol{F}(x)}\sum_{i\in N}\frac{f_i(x)}{F_i(x)}dx \\
&= -\left[\frac{\int_{\underline{v}}^{x}\boldsymbol{F}(y)dy}{\boldsymbol{F}(x)}\right]_{\underline{v}}^{v_p} + \int_{\underline{v}}^{v_p}dx \\
&= v_p - \frac{\int_{\underline{v}}^{v_p}\boldsymbol{F}(y)dy}{\boldsymbol{F}(v_p)} - \underline{v} \\
&= \frac{\int_{\underline{v}}^{v_p} yd\boldsymbol{F}(y)}{\boldsymbol{F}(v_p)} - \underline{v},
\end{aligned}
$$

where the second line follows from integration by parts and L’Hôpital’s rule. In particular, notice that $\sum_{i\in N}\psi_i(\overline{v}) = G - \underline{v}$.

Step 1 (Budget Balance). EPBB only requires that $\sum_{i\in N}\phi_i = 0$ as all other terms cancel out for any $v\in V$. This condition will be used in the last step of the proof.

Step 2 (Liquidity constraints). EPLC requires $\min_{v\in V} t_i(v) \geq -l_i$ for all $i\in N$. It is easy to see that $t_i(v)$ as defined by equation (15) is always lower when $\rho(v)=i$, and in this case it is minimized when $v_i=\overline{v}$ as $\sum_{k\neq i}\psi_k(v_i)$ is increasing in v_i. It follows that

$$\begin{aligned}\min_{v\in V} t_i(v) &= -\sum_{k\neq i}\psi_k(\overline{v}) - \frac{n-1}{n}\underline{v} - \phi_i \\ &= \psi_i(\overline{v}) - G + \frac{1}{n}\underline{v} - \phi_i,\end{aligned}$$

which implies that EPLC is satisfied whenever,

$$\phi_i \leq l_i + \psi_i(\overline{v}) - G + \frac{1}{n}\underline{v}.$$

Step 3 (Individual rationality). The interim expected transfer, $T_i(v_i) = \mathbb{E}_{-i}t_i(v)$, writes as follows:

$$\begin{aligned}T_i(v_i) = \int_{\underline{v}}^{\overline{v}} \Big[\big(-\sum_{k\neq i}\psi_k(v_i) - \frac{n-1}{n}\underline{v}\big)\mathbb{1}\{v_i>y\} \\ + \big(\psi_i(y)+\frac{1}{n}\underline{v}\big)\mathbb{1}\{v_i<y\}\Big] dF_{-i}(y) - \phi_i.\end{aligned}$$

Using the above result on $\sum_{k\in N}\psi_k(v_i)$ and integrating by part the term $\int_{\underline{v}}^{\overline{v}}\psi_i(y)dF_{-i}(y)$ yields

$$\begin{aligned}T_i(v_i) = F_{-i}(v_i)\Big(\psi_i(v_i) - \frac{\int_{\underline{v}}^{v_i} yd\boldsymbol{F}(y)}{\boldsymbol{F}(v_i)} + \frac{1}{n}\underline{v}\Big) + \psi_i(\overline{v}) - \psi_i(v_i)F_{-i}(v_i) \\ - \int_{v_i}^{\overline{v}}\int_{\underline{v}}^{y} F(x)dx\frac{f_i(y)}{F_i(y)^2}dy + (1-F_{-i}(y))\frac{1}{n}\underline{v} - \phi_i,\end{aligned}$$

which simplifies to

$$\begin{aligned}T_i(v_i) &= -\frac{\int_{\underline{v}}^{v_i} yd\boldsymbol{F}(y)}{F_i(v_i)} + \psi_i(\overline{v}) + \int_{\underline{v}}^{\overline{v}} F(x)dx - \frac{\int_{\underline{v}}^{v_i}\boldsymbol{F}(y)dy}{F_i(v_i)} - \int_{v_i}^{\overline{v}} F_{-i}(y)dy + \frac{1}{n}\underline{v} - \phi_i \\ &= \int_{v_i}^{\overline{v}} ydF_{-i}(y) + \psi_i(\overline{v}) - G + \frac{1}{n}\underline{v} - \phi_i.\end{aligned}$$

IIR requires that $T_i(v_i^*) \geq U_i^0(v_i^*) - v_i^*F_{-i}(v_i^*)$ where $v_i^*\in\arg\min_{v_i\in V_i} U_i(v_i)$. With this interim transfer rule it follows that

$$\phi_i \leq C_i + \psi_i(\overline{v}) - G + \frac{1}{n}\underline{v},$$

where $C_i := v_i^* F_{-i}(v_i^*) + \int_{v_i^*}^{\overline{v}} y dF_{-i}(y) - U_i^0(v_i^*)$.

Step 4 (Incentive compatibility). IIC is immediate as $T_i(v_i) = \int_{v_i}^{\overline{v}} y dF_{-i}(y) + \psi_i(\overline{v}) - G + \frac{1}{n}\underline{v} - \phi_i$ directly satisfies equation (12).

Step 5 (Sufficiency). Putting together the EPLC and IRR conditions on ϕ_i yields

$$\phi_i \le \min\{C_i, l_i\} + \psi_i(\overline{v}) - G + \frac{1}{n}\underline{v}, \tag{16}$$

and EPBB requires that $\sum_{i\in N} \phi_i = 0$.

Simply let

$$\phi_i := \min\{C_i, l_i\} + \psi_i(\overline{v}) - G + \frac{1}{n}\underline{v} - \frac{1}{n}\left[\sum_{j\in N} \min\{C_j, l_j\} - (n-1)G\right]. \tag{17}$$

As condition (3) holds, it is clear that the term in square brackets is nonnegative so that ϕ_i satisfies condition (16), i.e., both EPLC and IIR. Finally, it is straightforward to see that $\sum_{i\in N} \phi_i = 0$ given the earlier result that $\sum_{i\in N} \psi_i(\overline{v}) = G - \underline{v}$. The transfer rule is therefore also EPBB which concludes the proof of sufficiency. □

PROOF OF PROPOSITION 1. Let b_i be agent i's bidding strategy and $b(v_j)$ be the bidding strategy of agent $j \neq i$ with valuation v_j. Agent i's interim expected utility (omitting side payments) when bidding b_i writes

$$\begin{aligned} U_i(b_i; v_i) := &\left[v_i - (n-1)(b_i + \frac{1}{n}\underline{v})\right] \mathbb{E}_{-i} \mathbb{1}\{b_i > \max_{k\neq i} b(v_k)\} \\ &+ \sum_{j\neq i} \mathbb{E}_{-i}\left[\mathbb{1}\{b(v_j) > b_i\}\mathbb{1}\{b(v_j) > \max_{k\neq i,j} b(v_k)\}\left[b(v_j) + \frac{1}{n}\underline{v}\right]\right]. \end{aligned}$$

Solving for a strictly increasing symmetric Bayesian equilibrium, the bidding strategy of agent $j \neq i$ players, $b(v_j)$, is strictly increasing and therefore invertible. Notice that $\mathbb{1}\{b_i > \max_{k\neq i} b(v_k)\} = \mathbb{1}\{b^{-1}(b_i) > \max_{k\neq i} v_k\}$ and $\mathbb{1}\{b(v_j) > \max_{k\neq i} b(v_k)\} = \mathbb{1}\{v_j > \max_{k\neq i} v_k\}$. It follows that agent i's interim expected utility rewrites

$$U_i(b_i; v_i) = \left[v_i - (n-1)(b_i + \frac{1}{n}\underline{v})\right] Z(b^{-1}(b_i)) + \int_{b^{-1}(b_i)}^{\overline{v}} \left[b(v_j) + \frac{1}{n}\underline{v}\right] dZ(v_j),$$

where $Z := F^{n-1}$. Let $z = Z'$, differentiating $U(b_i; v_i)$ with respect to b_i and simplifying using $\frac{\partial b^{-1}}{\partial b_i}(b_i) = \frac{1}{b'(b^{-1}(b_i))}$ gives

$$\frac{\partial U_i}{\partial b_i}(b_i; v_i) = -(n-1)Z(b^{-1}(b_i)) + \frac{z(b^{-1}(b_i))}{b'(b^{-1}(b_i))}\left[v_i - nb_i - \underline{v}\right].$$

At equilibrium, $b(v_i)$ must be such that $\frac{\partial U_i}{\partial b_i}(b(v_i); v_i) = 0$. Therefore, $b(v_i)$ must solve

$$-(n-1)Z(v_i) + \frac{z(v_i)}{b'(v_i)}\left[v_i - nb(v_i) - \underline{v}\right] = 0.$$

It is easy to show that $b(v_i) := \int_{\underline{v}}^{v_i} \frac{\int_{\underline{v}}^{t} F(s)^n ds}{F(t)^{n+1}} f(t)dt$ solves this first-order differential equation and is strictly increasing in v_i. This first-order condition is also sufficient. First, notice from the first-order condition that $b'(v_i) = \frac{f(v_i)}{F(v_i)}(v_i - nb(v_i) - \underline{v})$. Assume that instead of $b(v_i)$, agent i of type v_i bids $b(x)$ where $x \in V_i$, then

$$\begin{aligned}\frac{\partial U_i}{\partial b_i}(b(x); v_i) &= (n-1)Z(x)\left[-1 + \frac{f(x)}{b'(x)F(x)}\left[v_i - nb(x) - \underline{v}\right]\right] \\ &= (n-1)Z(x)\left[-1 + \frac{v_i - nb(x) - \underline{v}}{x - nb(x) - \underline{v}}\right].\end{aligned}$$

Hence, as $b(x)$ is increasing in x, it follows that $\frac{\partial U_i}{\partial b_i}(b(x); v_i) > 0$ (resp. < 0) when $x < v_i$ (resp. $x > v_i$) for any $v_i \in V_i$ and $x \neq v_i$.

At the Bayesian equilibrium, agent i pays a price

$$p_i(b(v_1), \ldots, b(v_n)) = \begin{cases} (n-1)\left[\int_{\underline{v}}^{v_i} \frac{\int_{\underline{v}}^{t} F(s)^n ds}{F(t)^{n+1}} f(t)dt + \frac{1}{n}\underline{v}\right] & \text{if } b_i \geq \max_k b_k \\ -\left[\int_{\underline{v}}^{v_j} \frac{\int_{\underline{v}}^{t} F(s)^n ds}{F(t)^{n+1}} f(t)dt + \frac{1}{n}\underline{v}\right] & \text{if } b_j \geq \max_k b_k. \end{cases}$$

It is immediate to see that this price rule corresponds to the ex post transfer rule defined by equation (15) in the symmetric distribution case (and omitting the constant term). It can easily be proven that k_i replicates the constant term defined by equation (17) after noticing that $\psi_i(\overline{v}) = \frac{1}{n}[G - \underline{v}]$ in the symmetric distribution case.

The bidding game is thus EF as $b(\cdot)$ is increasing, i.e., the bidder with the highest valuation obtains the good. It is also IIR, EPBB and EPCC as it reproduces the transfer rule defined in equation (17). □

PROOF OF PROPOSITION 2. From condition (3) it is immediate that an ex post trading mechanism exists only if $\sum_{i \in N} l_i \geq (n-1)G$. From the assumption that $l_i = \tilde{l}$ for all $i \in N$ this condition rewrites $\tilde{l} \geq \frac{n-1}{n}G$.

Now, assume that $V_i = [\underline{v}, \overline{v}]$ and $F_i = F$. For convenience let $G(n) = \int_{\underline{v}}^{\overline{v}} y dF(y)^n$ be the expected social surplus when n agents participate. As $F(y)^{n+1}$ first-order stochastically dominates $F(y)^n$, it is clear that $G(n+1) - G(n) \geq 0$. It immediately follows that $\tilde{l}$ is increasing in n as $\frac{n-1}{n}G(n)$ is also increasing in n.

Finally, notice that $G(n) = \overline{v} - \int_{\underline{v}}^{\overline{v}} F(y)^n dy$ after integration by parts. By the monotone convergence theorem, $G(n)$ converges to $\overline{v}$ when n converges to $+\infty$. It follows that $\frac{n-1}{n}G(n)$ converges to $\overline{v}$ as well. □

PROOF OF PROPOSITION 3. Starting with Proposition 3.a., assume that $\tilde{r}_i \leq \frac{1}{n}$ for all $i \in N$. Notice that $\max_{r\in\Delta^{n-1}} \sum_{i\in N} C_i(r_i) = \sum_{i\in N} C_i(\frac{1}{n})$ and thus for all $r \in \Delta^{n-1}$, $\sum_{i\in N} \min\{C_i(r_i), l_i\} \leq \sum_{i\in N} C_i(\frac{1}{n})$. It is then clear that choosing $r_i^* = \frac{1}{n}$ for all $i \in N$ is such that for each $i \in N$, $\min\{C_i(r_i^*), l_i\} = C_i(r_i^*) = C_i(\frac{1}{n})$ provided that $r_i^* \geq \tilde{r}_i$ for all $i \in N$. Hence, $\sum_{i\in N} \min\{C_i(r_i), l_i\} = \sum_{i\in N} C_i(\frac{1}{n})$ which is the upper bound.

Consider now Proposition 3.b., *i.e.* assume that $\tilde{r}_i > \frac{1}{n}$ for some $i \in N$. Define $\mathcal{L}(r,\lambda) = \sum_{i\in N} \min\{C_i(r_i), l_i\} + \lambda(\sum_{i\in N} r_i - 1)$ where $\lambda \in \mathbb{R}$ is the Lagrange multiplier associated with the constraint $\sum_{i\in N} r_i = 1$. Notice that $\sum_{i\in N} \min\{C_i(r_i), l_i\}$ is concave as $C_i(r_i)$ is concave for each $i \in N$ and differentiable everywhere except at $r_i = \tilde{r}_i$. Let $\delta_{r_i}\mathcal{L}(r,\lambda)$ denote the superdifferential of the Lagrangian in r_i, then

$$\delta_{r_i}\mathcal{L}(r,\lambda) = \lambda + \begin{cases} 0 & \text{if } r_i < \tilde{r}_i \\ [C_i'(\tilde{r}_i), 0] & \text{if } r_i = \tilde{r}_i \\ C_i'(r_i) & \text{if } r_i > \tilde{r}_i. \end{cases}$$

The necessary optimality condition writes $0 \in \delta_{r_i}\mathcal{L}(r,\lambda)$ for all $i \in N$. First, assume that there is at least one $r_j^* < \tilde{r}_j$. Then $\lambda = 0$ and $r_i > \tilde{r}_i$ is impossible as it is impossible to have $C_i'(r_i) = 0$ with $r_i > \tilde{r}_i$ (indeed $C_i'(r_i) = 0$ only occurs when $\underline{v} = 0$ and $r_i = 0$). But then, if all $r_i^* \leq \tilde{r}_i$ with one strict inequality at least, it follows that $\sum_{i\in N} r_i^* < \sum_{i\in N} \tilde{r}_i \leq 1$ which is also impossible. Therefore, it is necessary that $r_i \geq \tilde{r}_i$ for all $i \in N$. Assume now that $r_i > \tilde{r}_i$ for all $i \in N$. Then, the necessary optimality condition implies that $\lambda + C_i'(r_i^*) = 0$ for all $i \in N$. But then it follows that $r_i^* = \frac{1}{n}$ for all $i \in N$ which is impossible as some $\tilde{r}_i > \frac{1}{n}$ contradicting that $r_i^* > \tilde{r}_i$ for all $i \in N$.

Hence, the solution must be such $r_i^* \geq \tilde{r}_i$ for all $i \in N$ with at least one equality. Let $\mathcal{A} := \{i \in N \mid r_i^* > \tilde{r}_i\}$ and $\mathcal{B} := \{j \in N \mid r_j^* = \tilde{r}_j\}$. Then, for all $i \in \mathcal{A}$, $\lambda + C_i'(r_i^*) = 0$ implies that $\lambda > 0$ and $r_i^* = r_k^*$ for any two $i,k \in \mathcal{A}$. For any $i \in \mathcal{A}$, and let $r_i^* = \hat{r}$ with $\hat{r} := \frac{1-\sum_{j\in\mathcal{B}} \tilde{r}_j}{|\mathcal{A}|}$. As by assumption $\tilde{r}_1 \leq \cdots \leq \tilde{r}_n$ and for all $i \in \mathcal{A}$ it is necessary that $\hat{r} > \tilde{r}_i$, it is possible to rewrite $\mathcal{A} := \{i \in N \mid i < p\}$ and $\mathcal{B} := \{j \in N \mid j \geq p\}$ for some $p \in N \setminus \{1\}$ and $\hat{r} = \frac{1-\sum_{j\geq p} \tilde{r}_j}{p-1}$. It is also necessary that $\hat{r} \leq \tilde{r}_j$ for all $j \in \mathcal{B}$. The solution therefore writes $r^* = (\hat{r}, \hat{r}, \ldots, \hat{r}, \tilde{r}_p, \tilde{r}_{p+1}, \ldots, \tilde{r}_n)$ and $\max_{i<p} \tilde{r}_i < \hat{r} \leq \min_{j\geq p} \tilde{r}_j$. □

PROOF OF PROPOSITION 4. First, notice the following upper bound on the sum of collectible charges: $\sum_{i\in N} \min\{C_i(r_i), l_i\} \leq \sum_{i\in N} \min\{C_i(0), l_i\}$ for all $r \in \Delta^{n-1}$. For every $i \in N$, let $r_i \leq \tilde{r}_i$ which is always possible as $\sum_{i\in N} r_i = 1 < \sum_{i\in N} \tilde{r}_i$. It follows that $\min\{C_i(r_i), l_i\} = \min\{C_i(0), l_i\}$ for all $i \in N$ and $\sum_{i\in N} \min\{C_i(r_i), l_i\} = \sum_{i\in N} \min\{C_i(0), l_i\}$. To conclude, it is clear that choosing any $r_i > \tilde{r}_i$ would decrease $\sum_{i\in N} \min\{C_i(r_i), l_i\}$. □

PROOF OF PROPOSITION 5. First notice that $\sum_{i\in N} \min\{C_i(r_i), l_i\} \leq \sum_{i\in N} C_i(r_i)$ and $\sum_{i\in N} C_i(r_i) \leq \sum_{i\in N} C_i(1/n)$ where the second inequality follows from the optimality of equal-share ownership in the absence of liquidity constraints (Proposition 3.a). Similarly $\sum_{i\in N} \min\{C_i(r_i), l_i\} \leq L$. Hence, it is clear that $\sum_{i\in N} \min\{C_i(r_i), l_i\} \leq \sum_{i\in N} \min\{C_i(1/n), L\}$ is an upper bound of the maximal collectible charges. Which of these bound is attained depends on the aggregate level of liquidity.

First, consider the case $L \geq \sum_{i \in N} C_i(1/n)$. It is always possible to construct $l^* \in \mathbb{R}^n_+$ such that $l_i^* \geq C_i(1/n)$ for all $i \in N$ and $\sum_{i \in N} l_i^* = L$. Similarly, let $r_i^* = 1/n$ for all $i \in N$. It immediately follows that $\sum_{i \in N} \min\{C_i(r_i^*), l_i^*\} = \sum_{i \in N} C_i(1/n)$.

Second, consider the case $L < \sum_{i \in N} C_i(1/n)$. It is clear that now $\sum_{i \in N} C_i(1/n)$ is not attainable as the previous distribution of liquidity is not feasible. Simply let $r_i^* = 1/n$ for all $i \in N$ and it is possible to let $l_i^* < C_i(1/n)$ for all $i \in N$ such that $\sum_{i \in N} l_i^* = L$. Hence, $\sum_{i \in N} \min\{C_i(r_i^*), l_i^*\} = \sum_{i \in N} l_i^* = L$. $\square$

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