ASYMMETRIC INFORMATION, LIQUIDITY CONSTRAINTS, AND EFFICIENT TRADE

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Abstract

This paper investigates trading mechanisms for efficiently (re)allocating a good to agents who face liquidity constraints and have private information about their valuation. More precisely, I derive a necessary and sufficient condition for the existence of ex post efficient, interim incentive compatible, interim individually rational, ex post budget balanced and ex post liquidity constrained trading mechanisms. The framework notably applies to buyers-seller relationships or partnership problems. In the latter case, I show that the optimal ownership structures are typically asymmetric and that agents with low liquidity resources should initially receive larger shares, and vice versa. I also show that a larger market size tends to increase the agents' minimal liquidity requirements necessary for existence. This is at odds with the standard property that a larger market size facilitates existence in asymmetric information environments. Finally, I propose a liquidity-constrained ex post efficient auction that implements the (re)allocation mechanism.

KEYWORDS: Mechanism design, efficient trade, liquidity constraints, partnerships.

1. INTRODUCTION

In markets with asymmetric information, liquidity constraints directly conflict with the design of incentive compatible trading mechanisms. On the one hand, such mechanisms must offer a price schedule from which agents with different valuations are incentivized to choose different prices so as to reveal their private information. On the other hand, liquidity constraints may prevent agents to choose the highest prices in the schedule, making it impossible to ensure full revelation of information. That is, trading mechanisms must create a spread in possible prices offered to each agent to allow them to reveal information while liquidity constraints restricts the size of this spread.

When designing centralized markets, this issue can be partially addressed by subsidizing the most liquidity-constrained agents with the resources of the least liquidity-constrained agents

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through lump-sum transfers. As long as those transfers are reasonably low not to affect the participation or liquidity constraints of the least liquidity-constrained agents, there might exist a feasible scheme that provide enough resources to liquidity-constrained agents to choose any price of the incentive compatible schedule. Although useful, this approach alone is insufficient to account for the most restrictive cases of constraints on liquidity as it takes the incentive compatible price schedule as given. Instead, the price schedule must be designed to minimize the need for liquidity resources – and consequently subsidies – while maintaining its incentive compatibility properties.

To illustrate this last point, consider the following example.

Example 1. Leave aside for a moment the usual participation and budget balance constraints and consider standard first and second-price auctions to allocate an indivisible good. When n agents with valuations v_i drawn i.i.d. from a uniform distribution over $[0,1]$ participate, the equilibrium bidding strategies induced by first and second-price auctions are respectively $\beta^F(v_i) = \frac{n-1}{n}v_i$ and $\beta^S(v_i) = v_i$.[1] In both cases, agents fully reveal their information through their choice of bid and the good is ex post efficiently allocated. Notice, however, that the second-price auction induces a price with range in $[0,1]$ while the first-price auction only requires it to be in $[0, \frac{n-1}{n}]$.

In the case $n = 2$, each agent i only needs an amount of liquidity $l_i \geq \frac{1}{2}$ to afford all possible prices of the first-price auction while they need twice as much in the second-price auction. Assume for instance that the two agents have initial liquidity resources $l_1 \in [0, \frac{1}{2})$ and $l_2 = 1$, and that the designer can enforce any liquidity redistribution between them. It is clear that there is no feasible redistribution of liquidity that ensures solvency in the second-price auction while one always exists in the first-price auction for any value of $l_1 \in [0, \frac{1}{2})$.

Coming back to the case $n \geq 2$, it is instructive to notice that the range of prices in the first-price auction is increasing in n and converges to $[0,1]$. That is, liquidity requirements in the first-price auction increase in the number of participants and eventually correspond to those in the second-price auction. Hence, when n is large both auctions pushes the individual liquidity requirements to the upper bound of the support of valuations.

Although this example deliberately ignores some fundamental constraints of the trading mechanisms studied in this paper, it captures two important intuitions that extend to the general setting: (i) two incentive compatible trading mechanisms can have rather different implications for liquidity requirements and/or for the amount of necessary cross-subsidy; and (ii) the range of the price schedule induced by an incentive compatible mechanism tends to increase in the number of participants – as illustrated by the first-price auction mechanism. From this last point, it follows that higher levels of aggregate and individual liquidity requirements are necessary when the number of market participants increases.

Previous and recent literature on this topic has mostly focused on the problem of designing optimal mechanisms in environments with one-sided private information such as the design of an

[1] In the general case of i.i.d. valuations with absolutely continuous cumulative distribution function $F(v_i)$ with support on $[\underline{v}, \overline{v}]$, the equilibrium bidding strategy in a first-price auction writes $\beta^F(v_i) = \frac{\int_{\underline{v}}^{v_i} y dF(y)^{n-1}}{F(v_i)^{n-1}}$.

auction for the sale of a good to privately informed buyers.[2] They generally assume that agents are ex ante identified buyers or sellers and do not allow for type-dependent outside options when agents refuse to participate in the trading mechanism. These assumptions rule out, for instance, buyers-seller relationships with private information on both sides in the spirit of Myerson and Satterthwaite (1983), partnership environments as in Cramton, Gibbons and Klemperer (1987), or the study ex post efficient auctions to allocate a good to privately informed agents with possibly type-dependent outside options.

In contrast, this paper deals with the design of ex post efficient trading mechanisms in an environment that encompasses the one-sided and two-sided private information cases. The specification of the agents' outside option is general and can depend on the valuations of all participants. Therefore the framework can be applied to a variety of environments, such as the ones mentioned above. More precisely, I investigate the existence of ex post efficient, interim incentive compatible, interim individually rational, ex post budget balanced, and ex post liquidity constrained trading mechanisms.

The three main contributions of this paper are as follows. First, I derive a necessary and sufficient condition for the existence of such trading mechanisms. I notably show how environments with liquidity constraints can be related to standard Groves mechanisms and to the expected externality mechanism.[3]

Second, I investigate the partnership dissolution problem as a special case of the framework so as to characterize the relationship between the initial allocation of property rights and liquidity constraints. In contrast to one of Cramton, Gibbons and Klemperer (1987)'s main results, equal-share partnerships do not always guarantee the existence of an efficient trading mechanism. Initial distributions of liquidity resources and property rights must be well-balanced in the sense that an agent's initial share of the asset must be (weakly) inversely related to their liquidity resources.

Third, the findings provide a cautionary message regarding the importance of liquidity constraints in asymmetric information environments. Most noticeably, a large number of participants in such markets tends to drastically increase their liquidity requirements as presented in Section 5. This finding is at odds with the standard argument that a larger market is unambiguously beneficial to the existence of trading mechanisms in asymmetric information environments. Section 6 presents a liquidity-constrained ex post efficient auction and show how it accounts for those features.

Several contributions related to liquidity constraints can be found in the auction literature. The early works of Laffont and Robert (1996) and Maskin (2000) respectively study the revenue-maximizing auction and a constrained-efficient auction with symmetrically liquidity-constrained bidders. Malakhov and Vohra (2008) derive the revenue-maximizing optimal mechanism with

[2]The optimal revenue maximizing and constrained-efficient auction with symmetric liquidity constraints have been respectively studied by Laffont and Robert (1996) and Maskin (2000). Boulatov and Severinov (2021) extend those results with asymmetric liquidity constraints while Pai and Vohra (2014) investigate the case of privately known liquidity constraints.

[3]See d'Aspremont and Gérard-Varet (1979) and Section 3.

discrete values and when only one bidder is liquidity constrained. Recently, Boulatov and Severinov (2021) greatly extended those results to asymmetrically liquidity-constrained bidders. Another strand of this literature investigates the case of bidders privately informed about their liquidity constraints. Pai and Vohra (2014) derive the optimal auction when both valuations and liquidity constraints are private information while Kotowski (2020) studies first-price auctions in a similar environment. Those frameworks, however, are limited to one-sided asymmetric information environments and study either revenue maximization or constrained efficiency at the ex ante stage. Therefore, their analysis is silent about conditions under which ex post efficiency can be achieved in the presence of liquidity constraints and cannot be extended to environments of bilateral trade or partnership problems.

By contrast, the present paper builds on the literature of ex post efficient trading mechanisms in two-sided asymmetric information environments, as introduced by the works of Myerson and Satterthwaite (1983) and Cramton, Gibbons and Klemperer (1987). Loertscher, Marx and Wilkening (2015) refer to these environments as *as secondary-market allocation problems*, in opposition to *primary-market allocation problems* such as the above mentioned auction settings.[4] Those models offer a general framework to analyze markets with privately informed traders who are not necessarily ex ante identified as buyers or sellers and can account for various initial ownership structures or type-dependent outside options. The framework has been extended to asymmetric and interdependent distributions (Figueroa and Skreta, 2012, Fieseler, Kittsteiner and Moldovanu, 2003, Jehiel and Pauzner, 2006), ex post individual rationality (Galavotti, Muto and Oyama, 2011), and second-best mechanisms (Lu and Robert, 2001, Loertscher and Wasser, 2019). However to the best of my knowledge, the present paper is the first to study ex post efficient trading mechanisms in two-sided asymmetric information environments with liquidity-constrained agents.

This paper is organized as follows. Section 2 introduces the theoretical framework. Section 3 presents the necessary and sufficient condition for existence and relate the result to Groves mechanisms and the expected externality mechanism. In Section 4, I apply the existence result to the special case of partnerships and characterize ownership and liquidity distributions compatible with existence. Section 5 investigates how the market size affects the agents' liquidity requirements. Section 6 proposes an auction to implement efficient trading mechanisms. Section 7 briefly concludes. All proofs are given in the Appendix.

2. THEORETICAL FRAMEWORK

There are n risk-neutral agents indexed by $i \in N := \{1, \dots, n\}$ and one good. Each agent $i \in N$ has private information about their valuation v_i for the entire good. It is, however, common knowledge that each valuation v_i is drawn independently from an absolutely continuous cumulative distribution function F_i with support $V_i := [\underline{v}_i, \overline{v}_i] \subseteq \mathbb{R}_+$ and positive continuous

[4] Loertscher, Marx and Wilkening (2015) also stress that the design of two-sided private information settings is fundamentally different from their one-sided counterparts. Notably, as the conflict between revenue and efficiency is more pronounced in two-sided settings, efficient mechanisms can merely fail to exist.

density f_i. Further assume that $V_i \cap V_j \neq \emptyset$ for any i, j and define $V := \times_{i\in N} V_i$.

The utility of agent i is assumed to be of the form $v_i x_i + m_i$, where x_i is the share of the good they receive and m_i is money. Each agent $i \in N$ is endowed with some liquidity resources, or budget, denoted by $l_i \in \mathbb{R}_+$. This amount corresponds to the maximal payment agent i can make given their current financial situation.

For future reference, let $v := (v_1, \dots, v_n) \in V$ and $l := (l_1, \dots, l_n) \in \mathbb{R}^n_+$ define the vectors of agents' valuations and liquidity resources, respectively. Furthermore, let v_{-i} denote the vector of valuations of all agents except that of agent i. Finally, define $(v_i, v_{-i}) := v$ for any $i \in N$, where arguments are ordered differently only for readability.

By the Revelation Principle, there is no loss of generality in restricting mechanisms to direct revelation mechanisms. Hence, each agent $i \in N$ directly reports their valuation v_i, all reports are collected, and the mechanism determines each individual allocation $s_i : V \to [0,1]$ and each individual transfer $t_i : V \to \mathbb{R}$.[5] The resource constraint on the good further imposes that $\sum_{i\in N} s_i(v) \leq 1$. Let $s(v) := (s_1(v), \dots, s_n(v)) \in \Delta^{n-1}$ and $t(v) := (t_1(v), \dots, t_n(v)) \in \mathbb{R}^n$ be the collections of individual ex post allocation rules and transfers, respectively. The pair (s,t) is referred to as a *mechanism*.

The ex post net utility of agent $i \in N$ who participates in the mechanism (s,t) is given by $v_i s_i(v) + t_i(v) - u_i^0(v_i, v_{-i})$, where $u_i^0(v_i, v_{-i})$ is agent i's outside option if they refuse to participate.

Let $S_i(v_i) := \mathbb{E}_{-i} s_i(v_i, v_{-i})$, $T_i(v_i) := \mathbb{E}_{-i} t_i(v_i, v_{-i})$, and $U_i^0(v_i) := \mathbb{E}_{-i} u_i^0(v_i, v_{-i})$ denote the interim expected values of the allocation rule, the transfer rule, and the outside option, respectively. Further assume that $U_i^0(v_i)$ is continuously differentiable in v_i.

Assuming all agents $j \neq i$ report truthfully, the net interim expected utility of agent i with type v_i who decides to participate, and reports $\hat{v}_i$ is given by

$$U_i(v_i, \hat{v}_i) := v_i S_i(\hat{v}_i) + T_i(\hat{v}_i) - U_i^0(v_i).$$

The mechanism (s,t) is *interim incentive compatible* (IIC) if it is a Bayesian Nash equilibrium that each agent reports truthfully. Formally, (s,t) is IIC if for all $i \in N$, $v_i \in V_i$ and $\hat{v}_i \in V_i$, $U_i(v_i, v_i) \geq U_i(v_i, \hat{v}_i)$. For convenience, let $U_i(v_i) := U_i(v_i, v_i)$ denote the net interim expected utility of agent i in the associated IIC mechanism (s,t).

To ensure participation, a mechanism (s,t) must be *interim individually rational* (IIR). Given that all agents report truthfully, agent i is willing to participate in the mechanism at the interim stage if and only if $U_i(v_i) \geq 0$ for all $i \in N$, $v_i \in V_i$.

The mechanism must also be *ex post budget balanced* (EPBB) to avoid the need for external subsidy. Formally, the transfers among all agents must balance out, i.e. $\sum_{i\in N} t_i(v) = 0$ for all $v \in V$.

Liquidity constraints require that ex post transfers never exceed agents' liquidity resources, that is, $t_i(v) \geq -l_i$ for all $i \in N$ and all $v \in V$. Such mechanisms will be called *ex post liquidity*

[5] The allocation rule s_i can be interpreted either as the share of the good allocated to agent i or the probability of this agent to receive it.

constrained mechanisms (EPLC).

Finally, I restrict the analysis to *ex post efficient mechanisms* (EF), that is, mechanisms such that the good is ex post efficiently allocated to the agent with the highest valuation. Formally, (s,t) is EF if, for all $v \in V$, the ex post allocation rule satisfies:

$$s^*(v) \in \arg\max_{s \in \Delta^{n-1}} \sum_{i \in N} v_i s_i(v).$$

The ex post efficient allocation rule can be rewritten as follows. For each agent i and all $v \in V$,

$$s_i^*(v) = \begin{cases} 1, & \text{if } i = \rho(v) \\ 0, & \text{if } i \neq \rho(v) \end{cases},$$

where $\rho(v) := \min\left\{ j \in N \mid j \in \arg\max_i v_i \right\}$ breaks ties in favor of the agent with the lowest index.[6]

For later use, it is useful to define for each profile of valuations $v \in V$, the ex post gains from trade from implementing an ex post efficient mechanism (s^*, t) by

$$g(v) := \sum_{i \in N} v_i s_i^*(v).$$

3. THE EXISTENCE CONDITION

The statement of the theorem simply relies on four fundamental primitives of the environment: (i) liquidity resources, both at the individual and at the aggregate level; (ii) the shape and value of the outside options; (iii) the number of participants; and (iv) the distribution of agents' valuations.

The first two define an upper bound on how much each agent can contribute to the mechanism. The last two determine the deficit generated by the implementation an ex post efficient and interim incentive compatible trading mechanism. The existence of any ex post efficient trading mechanism simply relies on whether the sum of all agents' maximal feasible contributions is sufficiently large to cover the deficit generated by imposing incentive constraints.

To introduce those elements and expose the existence condition intuitively, I rely on the well-known result that any ex post efficient and interim incentive compatible mechanism is payoff-equivalent to a Groves mechanism at the interim stage. The formal proof of the existence theorem, however, does not rely on this methodology as I propose some results that go beyond ex post efficient mechanisms. The interested reader can find the proof in the Appendix.

Following Makowski and Mezzetti (1994) and Williams (1999), for any ex post efficient and interim incentive compatible mechanism (s^*, t) there exists a Groves mechanism (s^*, t^*) that is payoff-equivalent at the interim stage for each agent. Formally, a Groves mechanism (s^*, t^*) is

[6]This particular tie-breaking rule is without loss of generality. As valuations are drawn from absolutely continuous cumulative distribution functions, ties occur with zero probability so that agents' interim participation decision are unaffected by this choice.

such that the transfer to agent $i \in N$ writes as follows:

$$t_i^*(v) = g(v) - v_i s_i^*(v) - h_i(v), \tag{1}$$

where $h_i : V \to \mathbb{R}$ is referred to as a *non-distortionary charge* and is such that $\mathbb{E}_{-i}\, h_i(v_i, v_{-i}) = \mathbb{E}_{-i}\, h_i(\hat{v}_i, v_{-i}) =: H_i$ for all v_i, $\hat{v}_i \in V_i$.

In a nutshell, the first two terms, $g(v) - v_i s_i^*(v)$, constitute a basic Groves mechanism and ensure that truth telling is a Bayesian Nash equilibrium of the revelation game as each agent receives the entire gains from trade at the ex post stage. The last term, $h_i(v)$, serves as a means to redistribute money and ensure budget balance ex post without distorting incentives for truthful reporting as it appears as a constant in agent i's interim expected utility.

Using equation (1), the interim utility of agent i in the Groves mechanism (s^*, t^*) writes

$$U_i(v_i) = \mathbb{E}_{-i}\, g(v) - U_i^0(v_i) - H_i,$$

which is also the interim utility of agent i in any EF and IRR trading mechanism by payoff-equivalence at the interim stage. Hence, agent i's worst-off type receives utility $\inf_{v_i \in V_i} U_i(v_i) = \inf_{v_i \in V_i} \{\mathbb{E}_{-i}\, g(v) - U_i^0(v_i)\} - H_i$. Then, it is convenient to define

$$C_i := \inf_{v_i \in V_i} \{\mathbb{E}_{-i} g(v) - U_i^0(v_i)\}, \tag{2}$$

as the largest feasible interim non-distortionary charge that can be imposed on agent i due to interim individual rationality, or equivalently, $H_i \leq C_i$.

Agent i's maximal feasible contribution is ultimately defined by $\min\{C_i, l_i\}$, i.e. by whether it is more difficult to satisfy agent i's individual rationality or liquidity constraint.[7] It simply follows that $\sum_{i \in N} \min\{C_i, l_i\}$ determines the total amount of feasible contributions from all agents for a given distribution of liquidity resources and outside options.

Finally, notice that EPBB requires that $\sum_{i \in N} \big(g(v) - v_i s_i^*(v) - h_i(v)\big) = 0$, which is equivalent to $(n-1)g(v) = \sum_{i \in N} h_i(v)$ for all $v \in V$. The left-hand side, $(n-1)g(v)$, corresponds to the ex post deficit generated by the basic Groves mechanism. This last condition therefore implies $(n-1)\mathbb{E}g(v) = \sum_{i \in N} H_i$ must hold as necessary condition at the ex ante stage, that is, the sum of ex ante non-distortionary charges must be equal to the ex ante deficit generated by the basic Groves mechanism.

For convenience, let $G := \mathbb{E}g(v)$ denote the ex ante gains from trade. All together, the above conditions implies the following necessary condition: $\sum_{i \in N} \min\{C_i, l_i\} \geq \sum_{i \in N} H_i = (n-1)G$.

The existence theorem states that this condition is also sufficient for the existence of trading mechanisms.

Theorem 1. *Let $V_i = [\underline{v}, \overline{v}]$ for all $i \in N$. An EF, IIC, IIR, EPBB and EPLC trading mechanism*

[7] In the Groves mechanism (s^*, t^*), EPLC requires that $t_i^*(v) \geq -l_i$ for all $i \in N$ and all $v \in V$. EPLC implies the following necessary condition at the interim stage: $\mathbb{E}_{-i}\left[g(v) - v_i s_i^*(v)\right] - H_i \geq -l_i$. It is easy to show that $\min_{v_i \in V_i} \mathbb{E}_{-i}\left[g(v) - v_i s_i^*(v)\right] = 0$ at $v_i = \overline{v}_i$ so that the condition collapses to $H_i \leq l_i$.

exists ***if and only if***

$$\sum_{i\in N}\min\{C_i,l_i\}\geq(n-1)G. \tag{3}$$

Notice that when liquidity resources of all agents are *large enough*, condition (3) simply becomes $\sum_{i\in N}C_i\geq(n-1)G$ and thus corresponds to the existence condition of Makowski and Mezzetti (1994) and Williams (1999).[8] Condition (3) provides a natural extension of their existence condition to environments with liquidity-constrained agents.

In the Appendix, I show that condition (3) can be generalized as a necessary condition of existence for any support of distributions of valuations $V=\times_{i\in N}[\underline{v}_i,\overline{v}_i]$, and any feasible and interim incentive compatible allocation rule $s_i:V\to[0,1]$ with $\sum_{i\in N}s_i(v)\leq 1$.[9]

Regarding the sufficiency of condition (3), the main challenge is to construct a transfer rule that accounts for the most restrictive cases of liquidity constraints. One of the most commonly used transfer rule in the literature is the *expected externality mechanism* introduced by d'Aspremont and Gérard-Varet (1979).[10] I now explain why this transfer rule fails in liquidity-constrained environments and show how it should be modified.

For the sake of exposition, let $F_i=F$ and $V_i=[\underline{v},\overline{v}]$ for all $i\in N$, the expected externality mechanism is such that the transfer to agent i writes:

$$\tilde{t}_i(v):=\tilde{\varphi}_i(v)-\frac{1}{n-1}\sum_{j\neq i}\tilde{\varphi}_j(v)+\phi_i, \tag{4}$$

where $\tilde{\varphi}_i(v):=\mathbb{E}_{-i}\sum_{j\neq i}v_js^*_j(v)$ and $\phi_i\in\mathbb{R}$ is a constant transfer to agent i such that $\sum_{i\in N}\phi_i=0$. The first term guarantees that truth-telling is Bayes-Nash equilibrium while the second term ensures ex post budget balance without distorting those incentives. The problem with the transfer rule defined by (4) is that its range of values is too wide. Simple computations show that this range is equal to $2\mathbb{E}[\max_{j\neq i}v_i]$, that is, two times the expected value of the maximum of $n-1$ valuations.[11] This ex post range imposes unnecessary large liquidity requirements on agents for transfers to be feasible in the same spirit as in Example 1.

It is possible to construct a transfer rule similar to (4) but inducing a lower range of values. Consider the following transfer rule for agent i:

$$t_i(v):=\varphi_i(v)-\frac{1}{n-1}\sum_{j\neq i}\varphi_j(v)+\phi_i, \tag{5}$$

[8] More precisely, Theorem 3.1 in Makowski and Mezzetti (1994) and condition (8) of Theorem 3 in Williams (1999).

[9] The generalized condition corresponds to equation (14). This generalization is notably used in Example 3 of Section 5 to account for asymmetric supports in a one seller and $n-1$ buyers trading problem. I believe that the generalized condition is also a sufficient condition for existence, but it is an open question.

[10] It can notably be found in Cramton, Gibbons and Klemperer (1987), Lu and Robert (2001), Fieseler, Kittsteiner and Moldovanu (2003), Ledyard and Palfrey (2007), or Segal and Whinston (2011), among others.

[11] It is easy to see that $\max_{v\in V}\tilde{t}_i(v)=\mathbb{E}[\max_{j\neq i}v_i]+\tilde{\phi}_i$ when $v_i=\underline{v}$ and $v_j=\overline{v}$ for all $j\neq i$, and that $\min_{v\in V}\tilde{t}_i(v)=-\mathbb{E}[\max_{j\neq i}v_i]+\tilde{\phi}_i$ when $v_i=\overline{v}$ and $v_j=\underline{v}$ for all $j\neq i$.

where $\varphi_i(v) := \sum_{j \neq i} \psi(v_j) s_j^*(v)$, and $\psi(y) := \frac{n-1}{n} \mathbb{E}[g(v) \mid g(v) \leq y]$.[12] It is useful to notice that $\psi(y)$ is increasing in $y \in [\underline{v}, \overline{v}]$, that $\psi(\underline{v}) = 0$ and $\psi(\overline{v}) = G$.

This transfer rule has a lower range than the one defined in (4) as it compresses the expected gains from trade of all agents except i in φ through ψ. Its effective range is equal to G, the ex ante expected gains from trade which is lower than $2\mathbb{E}[\max_{j \neq i} v_i]$, the range of the transfer rule of the expected externality mechanism.[13] It can be shown that it also induces truth-telling as a Bayes-Nash equilibrium (details can be found in the Appendix). Ex post budget balanced is obviously satisfied as long as $\sum_{i \in N} \phi_i = 0$.

4. PARTNERSHIPS

A special case of this framework is the partnership dissolution problem that was first introduced by Cramton, Gibbons and Klemperer (1987). A n-agent partnership is characterized by an initial ownership distribution over the asset to be traded and the problem consists in finding a dissolution mechanism such that ownership shares are reallocated to the partner who values them the most. I apply Theorem 1 to the partnership framework and characterize distributions of ownership shares and liquidity resources allowing for existence of a dissolution mechanism. For the sake of clarity, I rely once again on the approach introduced in Section 3.

Let $r := (r_1, \ldots, r_n) \in \Delta^{n-1}$ denote the initial distribution of ownership shares among the n agents. Each agent's share determines their outside option if they refuse to participate in the dissolution mechanism, that is, $u_i^0(v) = v_i r_i$. For notational convenience, define $F_{-i}(y) := \Pi_{j \neq i} F_j(y)$ and $f_{-i} := F'_{-i}$ with support on $[a_i, b_i] := [\max_{j \neq i} \underline{v}_j, \max_{j \neq i} \overline{v}_j]$. Using equation (2) and $u_i^0(v) = v_i r_i$, agent i's constraint on the non-distortionary charge due to individual rationality writes as follows:

$$C_i(r_i) := \inf_{v_i \in V_i} \left\{ \int_{a_i}^{v_i} v_i dF_{-i}(y) + \int_{v_i}^{b_i} y dF_{-i}(y) - v_i r_i \right\}, \tag{6}$$

where $C_i(\cdot)$ is explicitly defined as a function of agent i's ownership share r_i for later use. Agent i's worst-off type v_i^* is defined by the first-order condition of problem (6), that is,

$$F_{-i}(v_i^*) = r_i, \tag{7}$$

if there exists such a $v_i^* \in V_i$.[14] Otherwise, $v_i^* = \overline{v}_i$ if $F_{-i}(\overline{v}_i) < r_i$ and $v_i^* = \underline{v}_i$ if $F_{-i}(\underline{v}_i) > r_i$. This characterization of worst-off types in partnership problems is a generalization of Cramton,

[12] To avoid confusion with notations, the function ψ can also be written $\psi(y) = \frac{n-1}{n} \frac{\int_{\underline{v}}^{y} x dF(x)^n}{F(y)^n}$.

[13] It is useful to notice that $t_i(v) = -\psi(v_i)$ when $\rho(v) = i$ and $t_i(v) = \frac{1}{n-1}\psi(v_j)$ when $\rho(v) = j \neq i$. As $\psi(y)$ is increasing in y, it follows that $\max_{v \in V} t_i(v) = \frac{1}{n-1}\psi(\overline{v}) = \frac{1}{n} G$ and $\min_{v \in V} t_i(v) = -\psi(\overline{v}) = -\frac{n-1}{n} G$. The range of $t_i(v)$ immediately follows as $\max_{v \in V} t_i(v) - \min_{v \in V} t_i(v) = G$.

[14] The first-order condition is also a sufficient condition for a global minimum as $\frac{\partial^2}{\partial v_i^2}(\mathbb{E}_{-i} g(v_i, v_{-i}) - v_i r_i) = v_i f_{-i}(v_i) \geq 0$ for all $v_i \in V_i$. Notice that the solution to equation (7) might not exist or is not necessarily unique when the supports of valuation are asymmetric. When supports are symmetric, however, equation (7) fully and uniquely determines agent i's worst-off type.

Gibbons and Klemperer (1987) to the case of asymmetric distributions and supports of valuations.

Hence, $C_i(r_i)$ can be rewritten as

$$C_i(r_i) = \int_{v_i^*(r_i)}^{b_i} y dF_{-i}(y) + v_i^*(r_i)\big(F_{-i}(v_i^*(r_i)) - r_i\big), \tag{8}$$

where $v_i^*(r_i)$ denotes agent i's worst-off type and is defined as follows:

$$v_i^*(r_i) = \begin{cases} \underline{v}_i & \text{if} \quad F_{-i}(\underline{v}_i) > r_i \ \text{ or } \ r_i = 0 \\ \overline{v}_i & \text{if} \quad F_{-i}(\overline{v}_i) < r_i \ \text{ or } \ r_i = 1 \\ F_{-i}^{-1}(r_i) & \text{otherwise.} \end{cases} \tag{9}$$

The cases $v_i^*(0) = \underline{v}_i$ and $v_i^*(1) = \overline{v}_i$ are defined for convenience as agent i's worst-off type might not be unique when $r_i \in \{0,1\}$. Applying the envelope theorem to equation (6) gives that $C_i'(r_i) = -v_i^*(r_i)$ so that $C_i(r_i)$ is both decreasing and concave in r_i as $v_i^*(r_i)$ is increasing in r_i.

Under the symmetric supports assumption, $V_i = [\underline{v}, \overline{v}]$ for all $i \in N$, notice that $C_i(0) = \int_{\underline{v}}^{\overline{v}} y dF_{-i}(y)$ and $C_i(1) = 0$. In words, the largest non-distortionary charge that can be levied on agent i without any ownership share, $C_i(0)$, corresponds to the $(n-1)$th order statistics of the agents' valuations. This largest charge decreases as the ownership share of agent i increases and becomes null when agent i has full ownership of the asset. Ownership provides agents with a form of bargaining power in the dissolution mechanism.

The existence condition of a dissolution mechanism in partnership environments directly follows as a corollary of Theorem 1.

Corollary 1. *Let $V_i = [\underline{v}, \overline{v}]$ for all $i \in N$. A partnership with ownership rights $r \in \Delta^{n-1}$ and liquidity resources $l \in \mathbb{R}_+^n$ can be dissolved efficiently if and only if*

$$\sum_{i \in N} \min\{C_i(r_i), l_i\} \geq (n-1)G. \tag{10}$$

Equation (10) together with the characterization of $C_i(r_i)$ clearly highlight that the interplay between the distributions of ownership shares and liquidity resources among agents affects the sum of non-distortionary charges and thus the existence of a dissolution mechanism. As for Theorem 1, equation (3) also holds as a necessary condition of existence in the case of asymmetric supports of valuations.

I now present some characterization results relative to this issue. I further assume that agents' valuations are i.i.d. random variables, that is $F_i = F$ and $V_i = [\underline{v}, \overline{v}]$ for all $i \in N$, where F is an absolutely continuous cumulative distribution function. This assumption can be easily relaxed but I find it useful to present clear-cut characterization results about the interaction between ownership shares and liquidity resources.

For convenience, let $\tilde{r}_i \in [0,1]$ be defined by $\tilde{r}_i = 0$ when $l_i > C_i(0)$ and by $C_i(\tilde{r}_i) = l_i$ when $l_i \leq C_i(0)$. This threshold characterizes which is the most restrictive constraint between IIR and EPLC as $\min\{C_i(r_i), l_i\} = l_i$ when $r_i \leq \tilde{r}_i$, and $\min\{C_i(r_i), l_i\} = C_i(r_i)$ when $r_i > \tilde{r}_i$. Notice also that $\tilde{r}_i$ is decreasing in l_i and can thus be seen as a measure of the severity of liquidity constraints on agent i's collectible charge. Then, $\sum_{i\in N}\tilde{r}_i$ is an aggregate measure of the severity of liquidity constraints in the partnership.

The first set of results assumes the distribution of liquidity resources fixed and characterizes the corresponding ownership structures that maximize the sum of agents' collectible charges. I refer to this ownership structure as the *optimal distribution of property rights*. For ease of exposition assume, without loss of generality, that $l_1 \geq \cdots \geq l_n$ so that $\tilde{r}_1 \leq \cdots \leq \tilde{r}_n$.

Proposition 1. *Let $F_i = F$ and $V_i = [\underline{v}, \overline{v}]$ for all $i \in N$. For any $l \in \mathbb{R}^n_+$ such that $\sum_{i\in N}\tilde{r}_i \leq 1$, the optimal distribution of property rights $r^* \in \Delta^{n-1}$ is as follows:*

a. *If $\tilde{r}_i \leq \frac{1}{n}$ for all $i \in N$, then $r^* = (\frac{1}{n}, \ldots, \frac{1}{n})$;*

b. *If $\tilde{r}_i > \frac{1}{n}$ for some $i \in N$, then $r^* = (\hat{r}, \hat{r}, \ldots, \hat{r},\ \tilde{r}_p, \tilde{r}_{p+1}, \ldots, \tilde{r}_n)$ where $\hat{r} = \frac{1-\sum_{j\geq p}\tilde{r}_j}{p-1}$ for some $p \in N$ such that $\max_{i<p}\tilde{r}_i < \hat{r} \leq \min_{j\geq p}\tilde{r}_j$.*

When liquidity constraints are not too severe at the aggregate level, i.e. $\sum_{i\in N}\tilde{r}_i \leq 1$, the optimal distribution of property rights r^* can take two forms. Proposition 1.a states that if liquidity constraints are also mild at the agent level for all agents, then equal sharing of ownership maximizes agents' collectible charges. This case corresponds to one of the main results of Cramton, Gibbons and Klemperer (1987) who also show that this ownership structure always ensures existence of a dissolution mechanism.

On the contrary, Proposition 1.b states that if liquidity constraints are too severe for some agents, then the optimal distribution of property rights allocates (weakly) more initial ownership shares to more liquidity-constrained agents. It should be noted that the sum of collectible charges in 1.b is always lower than in 1.a as the departure from equal sharing ownership is the result of a trade-off due to the severity of liquidity constraints at the individual level for some agents. Hence, distributing initial ownership shares as in 1.b does not necessarily guarantee that equation (10) holds, i.e, that a dissolution mechanism exists.

Example 2. To illustrate Proposition 1.b, consider a two-agent partnership in which agent 1 is not liquidity constrained, i.e. $\tilde{r}_1 = 0$, whereas agent 2 is heavily liquidity constrained so that $\tilde{r}_2 \in [\frac{1}{2}, 1)$.

It is clear that starting from any $r_2 < \tilde{r}_2$, and in particular $r_2 = \frac{1}{2}$, it is possible to strictly increase the sum of feasible contributions $\sum_{i=1,2}\min\{C_i(r_i), l_i\} = C_1(r_1) + l_2$ by increasing r_2 up to $\tilde{r}_2$ as $C_i(\cdot)$ is a decreasing function and $\min\{C_2(r_2), l_2\} = l_2$ is unchanged for all $r_2 \leq \tilde{r}_2$. In other words, it is innocuous to give more initial ownership rights to heavily liquidity-constrained agents as their feasible contribution is already limited by their liquidity resources. On the other hand, it allows to give less initial ownership rights to less liquidity-constrained agents and collect more from them.

It is easy to construct an example in which the equal-share partnership does not allow for a dissolution mechanism to exist while an asymmetric initial allocation or property rights does. For instance, take $F_i(v_i) = v_i$ and $V_i = [0, 1]$ for $i = 1, 2$, so that $G = \frac{2}{3}$ and $C_i(r_i) = \frac{1}{2}[1 - r_i^2]$. Further assume that $\tilde{r}_2 = 0.7$, which corresponds to $l_2 = 0.255$. In that case, it clear that existence fails under equal-share ownership as $\sum_{i=1,2} \min\{C_i(\frac{1}{2}), l_i\} = C_1(\frac{1}{2}) + l_2 = \frac{3}{8} + 0.255 = 0.63 < G$. On the contrary, let for instance $r_1 = 0.3$ and $r_2 = \tilde{r}_2 = 0.7$. Then the sum of feasible contributions becomes $C_1(0.3) + l_2 = 0.455 + 0.255 = 0.71 > G$ so that a dissolution mechanism exists.

While Proposition 1 shows that the distribution of property rights exhibits some structure when liquidity constraints are mild enough at the aggregate level, the next result shows that this not the case when there are more severe, that is, when $\sum_{i \in N} \tilde{r}_i > 1$ holds.

Proposition 2. *Let $F_i = F$ and $V_i = [\underline{v}, \overline{v}]$ for all $i \in N$. For any $l \in \mathbb{R}^n_+$ such that $\sum_{i \in N} \tilde{r}_i > 1$, an optimal distribution of property rights $r^* \in \Delta^{n-1}$ is such that $r^*_i \leq \tilde{r}_i$ for all $i \in N$, $\sum_{i \in N} r^*_i = 1$, and $\sum_{i \in N} \min\{C_i(r^*_i), l_i\} = \sum_{i \in N} \min\{C_i(0), l_i\}$.*

The only structure that property rights should satisfy is that agents' shares must be such that for all $i \in N$, $r^*_i \leq \tilde{r}_i$ and $\sum_{i \in N} r^*_i = 1$. This last condition can always be satisfied for some $r^* \in \Delta^{n-1}$ as $\sum_{i \in N} \tilde{r}_i > 1$ by assumption. It is worth noticing that if some agent j has $\tilde{r}_j = 0$, that is $l_j > C_j(0)$, the optimal distribution of property rights allocates no initial ownership share to this agent. The resulting optimal distribution of property rights do not have much structure on the side of heavily liquidity-constrained agents. It does, however, exhibit the particular feature that agents with large liquidity resources should receive no initial property rights.

The second set of characterization results only assumes that the total amount of liquidity resources is fixed but allow for any distribution of liquidity and ownership among agents. Let $L \in \mathbb{R}_+$ denote the total amount of available liquidity resources in the n-agent partnership. A couple (r^*, l^*) is said to be an *optimal organization of the partnership* if it solves

$$(r^*, l^*) \in \arg\max_{(r,l)} \sum_{i \in N} \min\{C_i(r_i), l_i\},$$

subject to $(r, l) \in \Delta^{n-1} \times \mathbb{R}^n_+$ and $\sum_{i \in N} l_i = L$.

As it is now possible to choose the distribution of liquidity resources and ownership shares simultaneously, the optimal organization of the partnership will rely crucially on the total amount of liquidity resources L. If it is large enough, it is possible to allocate enough liquidity resources to each agent so as to achieve the highest collectible charge in a partnership, i.e, the one corresponding to equal sharing of ownership in the absence of liquidity constraints (Cramton, Gibbons and Klemperer, 1987). Otherwise, the best that can be done is to arrange the distributions so as to collect the total amount of liquidity resources among agents. The next proposition formalizes those intuitions.

Proposition 3. *For any $L \in \mathbb{R}_+$, where $L := \sum_{i \in N} l_i$, an optimal organization of the partnership (r^*, l^*) achieves the following maximal collectible charge*

$$\sum_{i \in N} \min\{C_i(r^*_i), l^*_i\} = \min\{\sum_{i \in N} C_i(1/n), L\}.$$

From Proposition 3 it is straightforward that a dissolution mechanism exists if and only if $L \geq (n-1)G$, assuming one can choose an optimal organization for the partnership.[15] That is, as long as the total amount of liquidity resources is enough to cover the budget deficit generated by a Groves mechanism, there always exists a way to allocate liquidity resources and ownership shares for a dissolution mechanism to exist.

5. MARKET SIZE

The existence condition of Theorem 1 crucially relies on the ex ante expected deficit generated by a Groves mechanism $(n-1)G$. This deficit directly stems from the incentive compatibility requirement as from equation (1), the interim expected payoff of agent i must be equal to $\mathbb{E}_{-i}[g(v) - v_i s_i^*(v)] - H_i$ in any ex post efficient and interim incentive compatible trading mechanism. As the number of agents participating in the trading mechanism increases, the deficit $(n-1)G$ increases as well. Not only it increases through $(n-1)$ but also through G, the ex ante expected gains from trade.[16] Hence, as the number of agents increases, the volume of transfers among participants must increase to cover the deficit generated by the incentive compatibility requirement.

At first glance, it seems difficult to assess the impact of the number of agents on the existence of efficient trading mechanisms in the general case described by Theorem 1 as both sides of equation (3) may vary with the number of agents. It is clear, however, that a necessary condition for equation (3) to hold is that $\sum_{i\in N} l_i \geq (n-1)G$. The next result illustrates that liquidity constraints become quite restrictive when the number of agents increases.

Proposition 4. *Assume $l_i = \tilde{l} \in \mathbb{R}_+$ for all $i \in N$. An ex post efficient trading mechanism exists only if*

$$\tilde{l} \geq \frac{n-1}{n} G.$$

Moreover, assume $V_i = [\underline{v}, \overline{v}]$ and $F_i = F$ for all $i \in N$, then $\tilde{l}$ increases in n and converges to $\overline{v}$ when n goes to infinity.

The most important feature of Proposition 4 is the fact that $\tilde{l}$ increases in the number of agents and eventually reaches the upper bound of the support of valuations in the limit case. It means that having more participants in the trading mechanism increases the pressure on each agent with respect to their minimal liquidity resources requirement.

In my opinion, the main message of Proposition 4 is that ignoring the presence of liquidity constraints in trading environments with multi-sided asymmetric information seems unjustified if the market size is assumed to be large. In other words, one should be careful when studying the

[15]This stems directly from Cramton, Gibbons and Klemperer (1987) who show that equal sharing of ownership always allow for ex post efficient dissolution in the absence of liquidity constraints, that is, $\sum_{i\in N} C_i(1/n) \geq (n-1)G$.

[16]Recall that $G := \mathbb{E}g(v)$ corresponds to the expectation of the maximum of the random variables $(v_1,\ldots,v_n)$ which is naturally increasing in n for any distributions of valuations.

properties of such environments as liquidity requirements seem to be at odds with the standard intuition that a larger market size is unambiguously beneficial.

Indeed, it is standard that in the absence of liquidity constraints, a larger number of participants favors the existence of efficient trading mechanisms through (i) an increase in *competition* among them and thus lowers each agent's informational rent; and (ii) larger expected gains from trade as the expected highest valuation increases when more agents participate. Proposition 4, however, suggests that having too many participants can compromise the existence of efficient trading mechanisms: the benefits on gains from trade mentioned in (ii) directly lead to a larger ex ante deficit induced by the incentive compatibility constraints which translates into larger liquidity requirements.

Notice also that adding new agents with large or unlimited liquidity resources to the trading mechanis is of no help in the symmetric liquidity resources case. Worse still, their arrival weighs on the liquidity requirements of the original set of participants. The existence of an efficient trading mechanism therefore relies critically upon the liquidity constraint of the most liquidity-constrained agent – that is less likely to hold as the number of participants increases.

To illustrate those ideas, I now present a simple example.

Example 3. Consider the case of a single seller, $i=1$, facing $n-1$ potential buyers denoted by $i=2,\dots,n$. The seller has valuation $v_1 \in [0,c]$ for the good they own and buyers have valuations $v_i \in [0,1]$ for all $i \in N \setminus \{1\}$. Valuations are uniformly and independently distributed on their respective support and $c \in [0,1]$. To account for the seller's full ownership of the good, the trading environment can be seen as a n-agent partnership (see Section 4) with initial ownership shares $r_1 = 1$ and $r_i = 0$ for all $i \in N \setminus \{1\}$. This framework is an extension of Myerson and Satterthwaite (1983) with multiple buyers as first introduced by Makowski and Mezzetti (1993). I further assume that each agent has limited liquidity resources $l_i \in \mathbb{R}_+$.

As previously mentioned, the existence condition (10) of Corollary 1 applies as a necessary condition also in the case of asymmetric supports for distributions of valuations. A necessary condition for existence of a trading mechanism in this environment therefore writes $\min\{C_1(1), l_1\} + \sum_{i=2}^{n} \min\{C_i(0), l_i\} \geq (n-1)G$.

It is straightforward to compute $G = \frac{n-1}{n} + \frac{c^n}{n(n+1)}$.[17] Then, equation (9) defines the worst-off types of the seller and the buyers, that is, $v_1^*(1) = c$ and $v_i^*(0) = 0$ for all $i \in N \setminus \{1\}$. The maximal non-distortionary charges due to individual rationality constraints follow from equation (8):

$$C_1(1) = \frac{n-1}{n} - c + \frac{c^n}{n},$$
$$C_i(0) = \frac{n-2}{n-1} + \frac{c^{n-1}}{n(n-1)} \text{ for any } i \in N \setminus \{1\}.$$

First, ignore liquidity constraints so that a necessary condition for existence simply writes

[17] The joint cumulative distribution functions write $F_{-1}(y) = y^{n-1}$ and $F_{-i}(y) = y^{n-2}\left(\frac{y}{c}\right)^{\mathbb{1}\{y<c\}}$ for all $i \in N \setminus \{1\}$ with support on $[a_j, b_j] = [0,1]$ for all $j \in N$.

$C_1(1)+\sum_{i=2}^{n} C_i(0) \geq (n-1)G$. The case $n=2$ corresponds to Myerson and Satterthwaite (1983)'s impossibility result as the condition requires $c \geq 3/2$, contradicting the assumption that $c \leq 1$. For $n \geq 3$, it is possible to show that there exists a $n^*(c)$ such that for any $c \in [0,1]$, the necessary condition without liquidity constraints is satisfied for all $n \geq n^*(c)$.[18] The threshold $n^*(c)$ is increasing in c, that is, if the seller *values the good more*, there must be more potential buyers for the condition to be satisfied.

Consider now that the potential buyers have limited liquidity resources, $l_i = \tilde{l} \in \mathbb{R}_+$ for all $i \in N \setminus \{1\}$ such that $\tilde{l} \leq C_i(0)$.[19] The seller, however, is assumed to have unlimited liquidity resources, i.e, $l_1 = +\infty$. In this environment, a necessary condition for the existence of a efficient trading mechanism is $C_1(1)+(n-1)\tilde{l} \geq (n-1)G$, or, after straightforward computations:

$$\tilde{l} \geq \frac{n-2}{n} + \frac{c}{n-1} - \frac{2c^n}{n(n-1)(n+1)}. \tag{11}$$

To focus on one of the most favorable environment for the existence of trading mechanisms, assume that c is arbitrarily close to zero.[20] The following table reports the minimal value of $\tilde{l}$ depending on the number of potential buyers for equation (11) to be satisfied.

# of buyers	1	2	3	4	5	6	7	8	9
$\tilde{l}$	0	.33	.50	.60	.66	.71	.75	.77	.80

In line with Proposition 4, the minimal liquidity requirement is increasing in the number of agents participating in the trading mechanism. More importantly, recall that buyers' valuations have support on $[0,1]$ meaning that with three buyers the minimal liquidity requirement must equal half the maximal possible valuation, about two-thirds with five buyers, and three-quarters with seven buyers.

This example immediately contradicts the intuition that having more buyers is unambiguously beneficial. Additional buyers increase the likelihood of high realizations of valuations which in turn increases the ex ante expected cost of a Groves mechanism to reveal agents' private information. As this cost must be somehow financed by the agents through transfers, the largest payments that agents may have to make increase as well and so are their minimal liquidity requirements.

6. LIQUIDITY-CONSTRAINED EFFICIENT AUCTION

Whenever condition (3) is satisfied, ex post efficient trading mechanisms can be implemented by an auction. Let $b := (b_1, \ldots, b_n) \in \mathbb{R}_+^n$ denote the vector of bids submitted by the n participants

[18] Makowski and Mezzetti (1993) provide a treatment of the case without liquidity constraints in the same environment and characterize a threshold $c^*(n)$. I instead rely on the converse threshold $n^* := (c^*)^{-1}$ for the purposes of the analysis.

[19] Notice that when c is close to zero, $C_i(0) \approx \frac{3}{4}, \frac{5}{6}, \frac{6}{7}, \frac{7}{8}$ for $n = 5,6,7,8$, respectively. Recall that $\overline{v} = 1$ in this framework so that the condition $\tilde{l} \leq C_i(0)$ is not extremely restrictive.

[20] It is easy to show that the right-hand side of (11) is increasing in c so that the case $c \to 0$ corresponds to the less demanding scenario in terms of liquidity resources.

and $U^0 := (U_1^0, \dots, U_n^0)$ be the vector of agents' interim outside options.[21]

Proposition 5. *Let $F_i = F$, $V_i = [\underline{v}, \overline{v}]$ for all $i \in N$ and assume condition (3) holds. The bidding game in which agents bid $b \in \mathbb{R}_+^n$, the highest bidder receives the good, agent i pays a price*

$$p_i(b) := \begin{cases} (n-1)\left[b_i + \frac{1}{n}\underline{v}\right] & \text{if } b_i \geq \max_k b_k \\ -\left[b_j + \frac{1}{n}\underline{v}\right] & \text{if } b_j \geq \max_k b_k, \end{cases}$$

and receives a side payment

$$\phi_i(U^0, l) := \frac{1}{n} \sum_{j \in N} \min\{C_j(U_j^0), l_j\} - \min\{C_i(U_i^0), l_i\},$$

ensures an ex post efficient allocation (EF), and satisfies IIR, EPBB, and EPLC.

In this auction, the highest bidder simply pays every other agents her bid (modulo the term $\frac{1}{n}\underline{v}$). Side payments ensure IIRR and EPLC. It is easy to see that the auction is ex post budget balanced. Finally, ex post efficiency follows from the fact that $p_i(b)$ induces increasing equilibrium bidding strategies $b_i(v_i)$ for all $i \in N$.

The important feature of the pricing rule is that, at equilibrium, it makes agents bid over a lower range of values than with other pricing rules therefore minimizing the need for too large side payments. Concretely, equilibrium prices under pricing rule $p_i(b)$ have a range of size G like the transfer rule (5) proposed in Section 3. As a comparison, the pricing rule proposed by Cramton, Gibbons and Klemperer (1987), $\tilde{p}_i(b) := b_i - \frac{1}{n-1}\sum_{j \neq i} b_j$, induces equilibrium prices with a range of size $2\mathbb{E}[\max_{j \neq i} v_j]$.[22] This last pricing rule may fail to satisfy EPLC even when condition (3) holds. The *performance* of each of those pricing rules with respect to liquidity constraints is also tied to the number of participants in the auction.

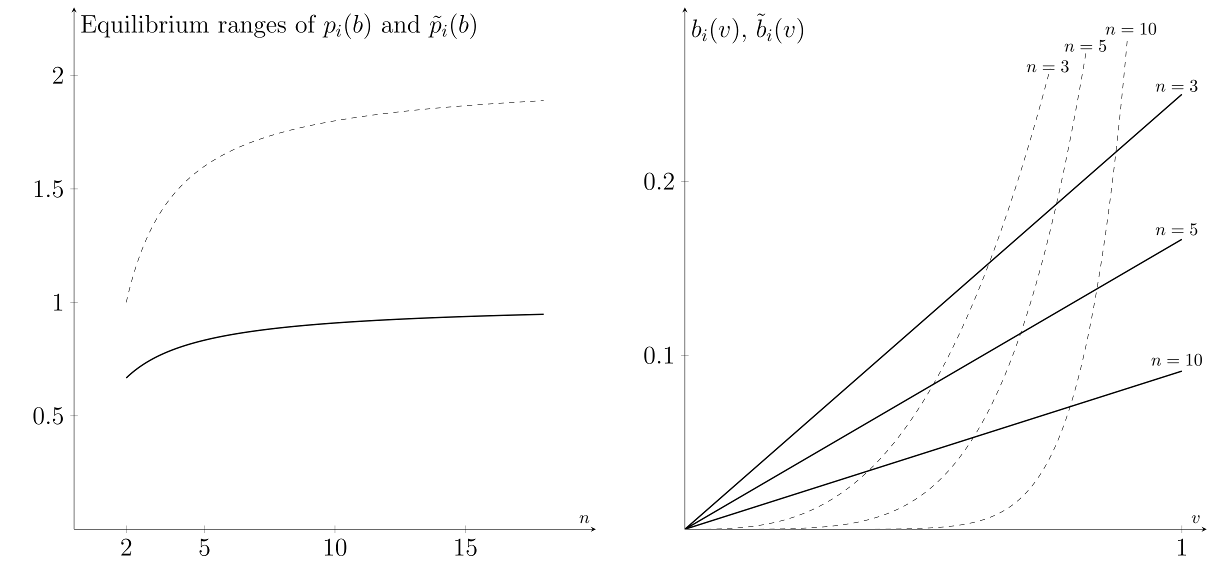
To illustrate those results, suppose that valuations are uniformly distributed on the unit interval, i.e., $F_i(v_i) = v_i$ and $V_i = [0, 1]$ for all $i \in N$. Figure 1a represents the equilibrium ranges of the two pricing rules, $p_i(b)$ and $\tilde{p}_i(b)$. Figure 1b shows their associated equilibrium bidding strategies.[23] As mentioned above, the equilibrium range of prices of the liquidity-constrained auction is lower than in the auction with pricing rule $\tilde{p}_i(b)$. Both equilibrium ranges of prices increase in the number of participants, but that of the liquidity-constrained auction is less affected by this phenomenon. This suggests that an improper design of the pricing rule can also lead to very different performances depending on the size the market.

As it can be seen on Figure 1b, the liquidity-constrained auction is inducing a lower spread of equilibrium bids while the other auction makes this spread larger as the number of agents increases. In both cases, agents with different types can bid differently so as to fully reveal their information but the liquidity-constrained auction ensures that this spread is as low as possible.

[21] Recall that $U_i^0 : V_i \to \mathbb{R}_+$ is defined as $U_i^0(v_i) := \mathbb{E}_{-i} u_i^0(v_i, v_{-i})$ for all $i \in N$.

[22] It is also naturally reminiscent of the transfer rule of the expected externality mechanism discussed in Section 3.

[23] The equilibrium range of $p_i(b)$ is defined as $\max_{v \in V} p_i(b(v)) - \min_{v \in V} p_i(b(v))$, where $b(v) := (b_1(v_1), \dots, b_n(v_n))$ is the vector of equilibrium bidding strategy induced by $p_i(b)$, and similarly for $\tilde{p}_i(b)$.



(a) (b)

Figure 1: (a) Ranges of pricing rules $p_i(b)$ and $\tilde{p}_i(b)$ evaluated at their respective equilibrium bidding strategy as n varies and (b) Equilibrium bidding strategies for the two pricing rules for some values of n. Solid curves correspond to pricing rule $p_i(b)$ and dashed curves to $\tilde{p}_i(b)$. Valuations are supposed to be uniformly distributed on the unit interval.

7. CONCLUSION

This paper studied a trading model with asymmetric information and liquidity constraints. It analyzed the conditions under which efficient trade is possible, and showed that the details of the design of incentive compatibility constraints are crucial to accommodate the most severe cases of liquidity constraints. The framework also raises the issue of the benefits of large markets – or with highly valued goods – in asymmetric information environments as it shows that an increase in the number of participants tend to increase the need for liquidity resources. As the assumption that agents have no liquidity constraints is often made in those environments, the interaction between asymmetric information, liquidity resources and market size is worth further investigation.

APPENDIX

For convenience, I define the following notations. Let $F_{-i}(y) := \Pi_{j \neq i} F_j(y)$ and $f_{-i} := F'_{-i}$ with support on $[a_i, b_i] := [\max_{j \neq i} \underline{v}_j, \max_{j \neq i} \overline{v}_j]$. Similarly, let $\boldsymbol{F}(y) := \Pi_{i \in N} F_i(y)$, and $\boldsymbol{f} := \boldsymbol{F}'$.

Proof of Theorem 1 (Necessity). The proof of necessity of condition (3) extends the result of Theorem 1 to asymmetric supports of valuations and to any feasible and interim incentive compatible allocation rule $s_i : V \to [0, 1]$ with $\sum_{i \in N} s_i(v) \leq 1$. Consider any interim incentive compatible mechanism (s,t), then the following inequalities must hold for any two valuations v_i, $\hat{v}_i \in V_i$,

$$U_i(v_i) \geq v_i S_i(\hat{v}_i) + T_i(\hat{v}_i) - U_i^0(v_i),$$
$$U_i(\hat{v}_i) \geq \hat{v}_i S_i(v_i) + T_i(v_i) - U_i^0(\hat{v}_i),$$

which in turn implies that

$$(v_i - \hat{v}_i)S_i(v_i) \geq U_i(v_i) - U_i(\hat{v}_i) + \big(U_i^0(v_i) - U_i^0(\hat{v}_i)\big) \geq (v_i - \hat{v}_i)S_i(\hat{v}_i).$$

This inequality implies that $S_i(v_i)$ must be nonincreasing and that $U_i(v_i)$ is absolutely continuous and almost everywhere differentiable with derivative $\partial U_i/\partial v_i(v_i) = S_i(v_i) - \partial U_i^0/\partial v_i(v_i)$. Integrating this last equation over $[\hat{v}_i, v_i]$ gives that

$$U_i(v_i) = U_i(\hat{v}_i) + \int_{\hat{v}_i}^{v_i} \Big(S_i(y) - \frac{\partial U_i^0}{\partial v_i}(y)\Big)dy.$$

Standard computations show that the expected interim transfer rule writes as follows:

$$T_i(v_i) = T_i(\hat{v}_i) - \int_{\hat{v}_i}^{v_i} y dS_i(y). \tag{12}$$

Imposing IIR requires $U_i(v_i) \geq 0$ for all $i \in N$ and $v_i \in V_i$. It is equivalent to impose $U_i(v_i^*) \geq 0$ for all $i \in N$ where $v_i^* \in \arg\min_{v_i \in V_i} U_i(v_i)$ represents agent i's worst-off type. Expressed in terms of interim transfers this yields:

$$T_i(v_i^*) \geq U_i^0(v_i^*) - v_i^* S_i(v_i^*).$$

EPLC requires that $t_i(v) \geq -l_i$ for all $i \in N$ and $v \in V$ which implies that $T_i(v_i) \geq -l_i$ for all $i \in N$ and $v_i \in V_i$ must hold. Clearly, $T_i(v_i)$ is a decreasing in v_i so that it is sufficient to satisfy $T_i(\overline{v}_i) \geq -l_i$ for all $i \in N$, or equivalently,

$$T_i(v_i^*) \geq \int_{v_i^*}^{\overline{v}_i} y dS_i(y) - l_i \tag{13}$$

Hence, IIR and EPLC implies that interim transfers must satisfy

$$T_i(v_i^*) \geq \max\Big\{U_i^0(v_i^*) - v_i^* S_i(v_i^*), \int_{v_i^*}^{\overline{v}_i} y dS_i(y) - l_i\Big\}.$$

As EPBB requires that $\sum_{i\in N} t_i(v) = 0$ for all $i \in N$ and $v \in V$, it must be true that $\sum_{i\in N} \mathbb{E}_i T_i(v_i) = 0$ for all $v_i \in V_i$. Using this last equality and equation (12) for $\hat{v}_i = v_i^*$ yields

$$\begin{aligned}\sum_{i\in N} T_i(v_i^*) &= \sum_{i\in N} \mathbb{E}_i \int_{v_i^*}^{v_i} y dS_i(y) \\ &= \sum_{i\in N} \left\{ \int_{v_i^*}^{\overline{v}_i} (1-F_i(y)) y dS_i(y) - \int_{\underline{v}_i}^{v_i^*} y F_i(y) dS_i(y) \right\},\end{aligned}$$

where the second line is obtained by changing the order of integration.

Summing equation (13) over all $i \in N$ and using the above result immediately gives that

$$\sum_{i\in N} \left\{ \int_{v_i^*}^{\overline{v}_i} (1-F_i(y)) y dS_i(y) - \int_{\underline{v}_i}^{v_i^*} y F_i(y) dS_i(y) \right\} \geq \sum_{i\in N} \max \left\{ U_i^0(v_i^*) - v_i^* S_i(v_i^*), \int_{v_i^*}^{\overline{v}_i} y dS_i(y) - l_i \right\},$$

which can be rewritten as follows:

$$\sum_{i\in N} \min \left\{ v_i^* S_i(v_i^*) - U_i^0(v_i^*), l_i - \int_{v_i^*}^{\overline{v}_i} y dS_i(y) \right\} \geq \sum_{i\in N} \left\{ \int_{\underline{v}_i}^{v_i^*} y F_i(y) dS_i(y) - \int_{v_i^*}^{\overline{v}_i} (1-F_i(y)) y dS_i(y) \right\}.$$

Now it is convenient to add the term $\int_{v_i^*}^{\overline{v}_i} y dS_i(y)$ on both sides and define $\mathscr{C}_i := v_i^* S_i(v_i^*) + \int_{v_i^*}^{\overline{v}_i} y dS_i(y) - U_i^0(v_i^*)$, so that

$$\sum_{i\in N} \min \left\{ \mathscr{C}_i, l_i \right\} \geq \sum_{i\in N} \int_{\underline{v}_i}^{\overline{v}_i} y F_i(y) dS_i(y), \tag{14}$$

which corresponds to the generalized version of the necessary condition (3).

To conclude the proof of the "only if" part, it only remains to prove that equation (14) corresponds to condition (3) under the ex post efficient allocation rule and symmetric support assumption $V_i = [\underline{v}, \overline{v}]$ for $i \in N$.

First, notice that in that case $S_i(v_i) = F_{-i}(v_i)$ so that $\mathscr{C}_i = v_i^* F_{-i}(v_i^*) + \int_{v_i^*}^{\overline{v}_i} y dF_{-i}(y) - U_i^0(v_i^*)$. Then, notice that equation (2) can be rewritten as

$$C_i := \inf_{v_i \in V_i} \{ v_i F_{-i}(v_i) + \int_{v_i}^{\overline{v}_i} y dF_{-i}(y) - U_i^0(v_i) \}.$$

By payoff equivalence at the interim stage and as $v_i^* \in \arg\min_{v_i \in V_i} U_i(v_i)$, it follows that $\mathscr{C}_i = C_i$.

Second, the right-hand side of (14) rewrites as follows:

$$\begin{aligned}
\sum_{i\in N}\int_{\underline{v}}^{\overline{v}} yF_i(y)dS_i(y) &= \sum_{i\in N}\int_{\underline{v}}^{\overline{v}} yF_i(y)\sum_{k\neq i} f_k(y)\frac{F_{-i}(y)}{F_k(y)}dy \\
&= \sum_{i\in N}\sum_{k\neq i}\int_{\underline{v}}^{\overline{v}} yf_k(y)F_{-k}(y)dy \\
&= (n-1)\sum_{i\in N}\int_{\underline{v}}^{\overline{v}} yf_i(y)F_{-i}(y)dy \\
&= (n-1)\int_{\underline{v}}^{\overline{v}} yd\Pi_{i\in N}F_i(y) \\
&= (n-1)G,
\end{aligned}$$

which concludes the proof of the "only if" part. ■

Proof of Theorem 1 (Sufficiency). The sufficiency of condition (3) is only proven in the symmetric support case and under the ex post efficient allocation rule.

Consider the following ex post transfer rule:

$$t_i(v) := \begin{cases} -\sum_{k\neq i}\psi_k(v_i) - \dfrac{n-1}{n}\underline{v} - \phi_i & \text{if } \rho(v) = i \\ \psi_i(v_j) + \dfrac{1}{n}\underline{v} - \phi_i & \text{if } \rho(v) = j \neq i, \end{cases} \tag{15}$$

where $\psi_k(v_p) := \int_{\underline{v}}^{v_p} \frac{\int_{\underline{v}}^{x}\boldsymbol{F}(y)dy}{\boldsymbol{F}(x)}\frac{f_k(x)}{F_k(x)}dx$ and $\phi_i \in \mathbb{R}$ is a constant.

Before proceeding with the proof is it useful to show that

$$\begin{aligned}
\sum_{i\in N}\psi_i(v_p) &= \int_{\underline{v}}^{v_p}\frac{\int_{\underline{v}}^{x}\boldsymbol{F}(y)dy}{\boldsymbol{F}(x)}\sum_{i\in N}\frac{f_i(x)}{F_i(x)}dx \\
&= -\left[\frac{\int_{\underline{v}}^{x}\boldsymbol{F}(y)dy}{\boldsymbol{F}(x)}\right]_{\underline{v}}^{v_p} + \int_{\underline{v}}^{v_p} dx \\
&= v_p - \frac{\int_{\underline{v}}^{v_p}\boldsymbol{F}(y)dy}{\boldsymbol{F}(v_p)} - \underline{v} \\
&= \frac{\int_{\underline{v}}^{v_p} yd\boldsymbol{F}(y)}{\boldsymbol{F}(v_p)} - \underline{v},
\end{aligned}$$

where the second line follows from integration by parts and L'Hôpital's rule. In particular, notice that $\sum_{i\in N}\psi_i(\overline{v}) = G - \underline{v}$.

Step 1 (Budget Balance). EPBB only requires that $\sum_{i\in N}\phi_i = 0$ as all other terms cancel out for any $v \in V$. This condition will be used in the last step of the proof.

Step 2 (Liquidity constraints). EPLC requires $\min_{v\in V} t_i(v) \geq -l_i$ for all $i \in N$. It is easy to see that $t_i(v)$ as defined by equation (15) is always lower when $\rho(v) = i$, and in this case it is minimized when $v_i = \overline{v}$ as $\sum_{k\neq i} \psi_k(v_i)$ is increasing in v_i. It follows that

$$\begin{aligned}\min_{v\in V} t_i(v) &= -\sum_{k\neq i} \psi_k(\overline{v}) - \frac{n-1}{n}\underline{v} - \phi_i \\ &= \psi_i(\overline{v}) - G + \frac{1}{n}\underline{v} - \phi_i,\end{aligned}$$

which implies that EPLC is satisfied whenever,

$$\phi_i \leq l_i + \psi_i(\overline{v}) - G + \frac{1}{n}\underline{v}.$$

Step 3 (Individual rationality). The interim expected transfer, $T_i(v_i) = \mathbb{E}_{-i} t_i(v)$, writes as follows:

$$T_i(v_i) = \int_{\underline{v}}^{\overline{v}} \Big[\big(-\sum_{k\neq i} \psi_k(v_i) - \frac{n-1}{n}\underline{v}\big)\mathbb{1}\{v_i > y\} + \big(\psi_i(y) + \frac{1}{n}\underline{v}\big)\mathbb{1}\{v_i < y\}\Big] dF_{-i}(y) - \phi_i.$$

Using the above result on $\sum_{k\in N} \psi_k(v_i)$ and integrating by part the term $\int_{\underline{v}}^{\overline{v}} \psi_i(y) dF_{-i}(y)$ yields

$$\begin{aligned}T_i(v_i) = F_{-i}(v_i)\Big(\psi_i(v_i) - \frac{\int_{\underline{v}}^{v_i} y d\boldsymbol{F}(y)}{\boldsymbol{F}(v_i)} + \frac{1}{n}\underline{v}\Big) + \psi_i(\overline{v}) - \psi_i(v_i)F_{-i}(v_i) \\ - \int_{v_i}^{\overline{v}}\int_{\underline{v}}^{y} F(x)dx \frac{f_i(y)}{F_i(y)^2} dy + (1 - F_{-i}(y))\frac{1}{n}\underline{v} - \phi_i,\end{aligned}$$

which simplifies to

$$\begin{aligned}T_i(v_i) &= -\frac{\int_{\underline{v}}^{v_i} y d\boldsymbol{F}(y)}{F_i(v_i)} + \psi_i(\overline{v}) + \int_{\underline{v}}^{\overline{v}} F(x)dx - \frac{\int_{\underline{v}}^{v_i} \boldsymbol{F}(y)dy}{F_i(v_i)} - \int_{v_i}^{\overline{v}} F_{-i}(y)dy + \frac{1}{n}\underline{v} - \phi_i \\ &= \int_{v_i}^{\overline{v}} y dF_{-i}(y) + \psi_i(\overline{v}) - G + \frac{1}{n}\underline{v} - \phi_i.\end{aligned}$$

IIR requires that $T_i(v_i^*) \geq U_i^0(v_i^*) - v_i^* F_{-i}(v_i^*)$ where $v_i^* \in \arg\min_{v_i \in V_i} U_i(v_i)$. With this interim transfer rule it follows that

$$\phi_i \leq C_i + \psi_i(\overline{v}) - G + \frac{1}{n}\underline{v},$$

where $C_i := v_i^* F_{-i}(v_i^*) + \int_{v_i^*}^{\overline{v}} y dF_{-i}(y) - U_i^0(v_i^*)$.

Step 4 (Incentive compatibility). IIC is immediate as $T_i(v_i) = \int_{v_i}^{\overline{v}} y dF_{-i}(y) + \psi_i(\overline{v}) - G + \frac{1}{n}\underline{v} - \phi_i$ directly satisfies equation (12).

Step 5 (Sufficiency). Putting together the EPLC and IRR conditions on ϕ_i yields

$$\phi_i \leq \min\{C_i, l_i\} + \psi_i(\overline{v}) - G + \frac{1}{n}\underline{v}, \tag{16}$$

and EPBB requires that $\sum_{i\in N} \phi_i = 0$.

Simply let

$$\phi_i := \min\{C_i, l_i\} + \psi_i(\overline{v}) - G + \frac{1}{n}\underline{v} - \frac{1}{n}\left[\sum_{j\in N}\min\{C_j, l_j\} - (n-1)G\right]. \tag{17}$$

As condition (3) holds, it is clear that the term in square brackets is nonnegative so that ϕ_i satisfies condition (16), i.e., both EPLC and IIR. Finally, it is straightforward to see that $\sum_{i\in N}\phi_i = 0$ given the earlier result that $\sum_{i\in N}\psi_i(\overline{v}) = G - \underline{v}$. The transfer rule is therefore also EPBB which concludes the proof of sufficiency. ■

Proof of Proposition 1. Starting with Proposition 1.a., assume that $\tilde{r}_i \leq \frac{1}{n}$ for all $i \in N$. Notice that $\max_{r\in\Delta^{n-1}}\sum_{i\in N} C_i(r_i) = \sum_{i\in N} C_i(\frac{1}{n})$ and thus for all $r \in \Delta^{n-1}$, $\sum_{i\in N}\min\{C_i(r_i), l_i\} \leq \sum_{i\in N} C_i(\frac{1}{n})$. It is then clear that choosing $r_i^* = \frac{1}{n}$ for all $i \in N$ is such that for each $i \in N$, $\min\{C_i(r_i^*), l_i\} = C_i(r_i^*) = C_i(\frac{1}{n})$ provided that $r_i^* \geq \tilde{r}_i$ for all $i \in N$. Hence, $\sum_{i\in N}\min\{C_i(r_i), l_i\} = \sum_{i\in N} C_i(\frac{1}{n})$ which is the upper bound.

Consider now Proposition 1.b., *i.e.* assume that $\tilde{r}_i > \frac{1}{n}$ for some $i \in N$. Define $\mathscr{L}(r,\lambda) = \sum_{i\in N}\min\{C_i(r_i), l_i\} + \lambda(\sum_{i\in N} r_i - 1)$ where $\lambda \in \mathbb{R}$ is the Lagrange multiplier associated with the constraint $\sum_{i\in N} r_i = 1$. Notice that $\sum_{i\in N}\min\{C_i(r_i), l_i\}$ is concave as $C_i(r_i)$ is concave for each $i \in N$ and differentiable everywhere except at $r_i = \tilde{r}_i$. Let $\delta_{r_i}\mathscr{L}(r,\lambda)$ denote the superdifferential of the Lagrangian in r_i, then

$$\delta_{r_i}\mathscr{L}(r,\lambda) = \lambda + \begin{cases} 0 & \text{if } r_i < \tilde{r}_i \\ [C_i'(\tilde{r}_i), 0] & \text{if } r_i = \tilde{r}_i \\ C_i'(r_i) & \text{if } r_i > \tilde{r}_i. \end{cases}$$

The necessary optimality condition writes $0 \in \delta_{r_i}\mathscr{L}(r,\lambda)$ for all $i \in N$. First, assume that there is at least one $r_j^* < \tilde{r}_j$. Then $\lambda = 0$ and $r_i > \tilde{r}_i$ is impossible as it is impossible to have $C_i'(r_i) = 0$ with $r_i > \tilde{r}_i$ (indeed $C_i'(r_i) = 0$ only occurs when $\underline{v} = 0$ and $r_i = 0$). But then, if all $r_i^* \leq \tilde{r}_i$ with one strict inequality at least, it follows that $\sum_{i\in N} r_i^* < \sum_{i\in N}\tilde{r}_i \leq 1$ which is also impossible. Therefore, it is necessary that $r_i \geq \tilde{r}_i$ for all $i \in N$. Assume now that $r_i > \tilde{r}_i$ for all $i \in N$. Then, the necessary optimality condition implies that $\lambda + C_i'(r_i^*) = 0$ for all $i \in N$. But then it follows that $r_i^* = \frac{1}{n}$ for all $i \in N$ which is impossible as some $\tilde{r}_i > \frac{1}{n}$ contradicting that $r_i^* > \tilde{r}_i$ for all $i \in N$.

Hence, the solution must be such $r_i^* \geq \tilde{r}_i$ for all $i \in N$ with at least one equality. Let $\mathscr{A} := \{i \in N \mid r_i^* > \tilde{r}_i\}$ and $\mathscr{B} := \{j \in N \mid r_j^* = \tilde{r}_j\}$. Then, for all $i \in \mathscr{A}$, $\lambda + C_i'(r_i^*) = 0$ implies that $\lambda > 0$ and $r_i^* = r_k^*$ for any two $i,k \in \mathscr{A}$. For any $i \in \mathscr{A}$, and let $r_i^* = \hat{r}$ with $\hat{r} := \frac{1-\sum_{j\in\mathscr{B}}\tilde{r}_j}{|\mathscr{A}|}$. As by assumption $\tilde{r}_1 \leq \cdots \leq \tilde{r}_n$ and for all $i \in \mathscr{A}$ it is necessary that $\hat{r} > \tilde{r}_i$, it is possible to rewrite $\mathscr{A} := \{i \in N \mid i < p\}$ and $\mathscr{B} := \{j \in N \mid j \geq p\}$ for some $p \in N \setminus \{1\}$ and $\hat{r} = \frac{1-\sum j\geq p\tilde{r}_j}{p-1}$. It is also necessary that $\hat{r} \leq \tilde{r}_j$ for all $j \in \mathscr{B}$. The solution therefore writes $r^* = (\hat{r}, \hat{r}, \ldots, \hat{r}, \tilde{r}_p, \tilde{r}_{p+1}, \ldots, \tilde{r}_n)$ and $\max_{i<p}\tilde{r}_i < \hat{r} \leq \min_{j\geq p}\tilde{r}_j$. ■

Proof of Proposition 2. First, notice the following upper bound on the sum of collectible charges: $\sum_{i\in N}\min\{C_i(r_i),l_i\} \leq \sum_{i\in N}\min\{C_i(0),l_i\}$ for all $r\in\Delta^{n-1}$. For every $i\in N$, let $r_i \leq \tilde{r}_i$ which is always possible as $\sum_{i\in N} r_i = 1 < \sum_{i\in N}\tilde{r}_i$. Then $\min\{C_i(r_i),l_i\} = \min\{C_i(0),l_i\}$ for all $i\in N$ and $\sum_{i\in N}\min\{C_i(r_i),l_i\} = \sum_{i\in N}\min\{C_i(0),l_i\}$. To conclude, it is clear that choosing any $r_i > \tilde{r}_i$ would decrease $\sum_{i\in N}\min\{C_i(r_i),l_i\}$. ■

Proof of Proposition 3. Notice that $\sum_{i\in N}\min\{C_i(r_i),l_i\} \leq \sum_{i\in N} C_i(r_i) \leq \sum_{i\in N} C_i(1/n)$ where the second inequality follows from the optimality of equal-share ownership in the absence of liquidity constraints. Similarly $\sum_{i\in N}\min\{C_i(r_i),l_i\} \leq L$. Hence, it is clear that $\sum_{i\in N}\min\{C_i(r_i),l_i\} \leq \sum_{i\in N}\min\{C_i(1/n),L\}$ is an upper bound of the maximal collectible charges. Which of these bound is attained depends on the aggregate level of liquidity.

First, consider the case $L \geq \sum_{i\in N} C_i(1/n)$. It is always possible to construct $l^*\in\mathbb{R}^n_+$ such that $l_i^* \geq C_i(1/n)$ for all $i\in N$ and $\sum_{i\in N} l_i^* = L$. Similarly, let $r_i^* = 1/n$ for all $i\in N$. It immediately follows that $\sum_{i\in N}\min\{C_i(r_i^*),l_i^*\} = \sum_{i\in N} C_i(1/n)$.

Second, consider the case $L < \sum_{i\in N} C_i(1/n)$. It is clear that now $\sum_{i\in N} C_i(1/n)$ is not attainable as the previous distribution of liquidity is not feasible. Simply let $r_i^* = 1/n$ for all $i\in N$ and it is possible to let $l_i^* < C_i(1/n)$ for all $i\in N$ such that $\sum_{i\in N} l_i^* = L$. Hence, $\sum_{i\in N}\min\{C_i(r_i^*),l_i^*\} = \sum_{i\in N} l_i^* = L$.

Proof of Proposition 4. From condition (3) it is immediate that an ex post trading mechanism exists only if $\sum_{i\in N} l_i \geq (n-1)G$. From the assumption that $l_i = \tilde{l}$ for all $i\in N$ this condition rewrites $\tilde{l} \geq \frac{n-1}{n}G$.

Now, assume that $V_i = [\underline{v},\overline{v}]$ and $F_i = F$. For convenience let $G(n) == \int_{\underline{v}}^{\overline{v}} y dF(y)^n$ be the expected gains from trade when n agents participate. As $F(y)^{n+1}$ first-order stochastically dominates $F(y)^n$, it is clear that $G(n+1)-G(n)\geq 0$. It immediately follows that $\tilde{l}$ is increasing in n as $\frac{n-1}{n}G(n)$ is also increasing in n.

Finally, notice that $G(n) = \overline{v} - \int_{\underline{v}}^{\overline{v}} F(y)^n dy$ after integration by parts. By the monotone convergence theorem, $G(n)$ converges to $\overline{v}$ when n converges to $+\infty$. It follows that $\frac{n-1}{n}G(n)$ converges to $\overline{v}$ as well. ■

Proof of Proposition 5. Let b_i be agent i's bidding strategy and $b(v_j)$ be the bidding strategy of agent $j\neq i$ with valuation v_j. Agent i's interim expected utility (omitting side payments) when bidding b_i writes

$$\begin{aligned} U_i(b_i;v_i) := &\left[v_i-(n-1)(b_i+\frac{1}{n}\underline{v})\right]\mathbb{E}_{-i}\mathbb{1}\{b_i > \max_{k\neq i} b(v_k)\} \\ &+\sum_{j\neq i}\mathbb{E}_{-i}\left[\mathbb{1}\{b(v_j)>b_i\}\mathbb{1}\{b(v_j)>\max_{k\neq i,j} b(v_k)\}\left[b(v_j)+\frac{1}{n}\underline{v}\right]\right]. \end{aligned}$$

Solving for a strictly increasing symmetric Bayesian equilibrium, the bidding strategy of agent $j\neq i$ players, $b(v_j)$, is strictly increasing and therefore invertible. Notice that $\mathbb{1}\{b_i >$

$\max_{k\neq i} b(v_k)\} = \mathbb{1}\{b^{-1}(b_i) > \max_{k\neq i} v_k\}$ and $\mathbb{1}\{b(v_j) > \max_{k\neq i} b(v_k)\} = \mathbb{1}\{v_j > \max_{k\neq i} v_k\}$. It follows that agent i's interim expected utility rewrites

$$U_i(b_i;v_i) = \Big[v_i - (n-1)(b_i + \frac{1}{n}\underline{v})\Big]Z(b^{-1}(b_i)) + \int_{b^{-1}(b_i)}^{\overline{v}} \Big[b(v_j) + \frac{1}{n}\underline{v}\Big]dZ(v_j),$$

where $Z := F^{n-1}$. Let $z = Z'$, differentiating $U(b_i;v_i)$ with respect to b_i and simplifying using $\frac{\partial b^{-1}}{\partial b_i}(b_i) = \frac{1}{b'(b^{-1}(b_i))}$ gives

$$\frac{\partial U_i}{\partial b_i}(b_i;v_i) = -(n-1)Z(b^{-1}(b_i)) + \frac{z(b^{-1}(b_i))}{b'(b^-1(b_i))}\Big[v_i - nb_i - \underline{v}\Big].$$

At equilibrium, $b(v_i)$ must be such that $\frac{\partial U_i}{\partial b_i}(b(v_i);v_i) = 0$. Therefore, $b(v_i)$ must solve

$$-(n-1)Z(v_i) + \frac{z(v_i)}{b'(v_i)}\Big[v_i - nb(v_i) - \underline{v}\Big] = 0.$$

It is easy to show that $b(v_i) := \int_{\underline{v}}^{v_i} \frac{\int_{\underline{v}}^{t} F(s)^n ds}{F(t)^{n+1}} f(t)dt$ solves this first-order differential equation and is strictly increasing in v_i. This first-order condition is also sufficient. First, notice from the first-order condition that $b'(v_i) = \frac{f(v_i)}{F(v_i)}(v_i - nb(v_i) - \underline{v})$. Assume that instead of $b(v_i)$, agent i of type v_i bids $b(x)$ where $x \in V_i$, then

$$\begin{aligned}\frac{\partial U_i}{\partial b_i}(b(x);v_i) &= (n-1)Z(x)\left[-1 + \frac{f(x)}{b'(x)F(x)}[v_i - nb(x) - \underline{v}]\right] \\ &= (n-1)Z(x)\left[-1 + \frac{v_i - nb(x) - \underline{v}}{x - nb(x) - \underline{v}}\right].\end{aligned}$$

Hence, as $b(x)$ is increasing in x, it follows that $\frac{\partial U_i}{\partial b_i}(b(x);v_i) > 0$ (resp. < 0) when $x < v_i$ (resp. $x > v_i$) for any $v_i \in V_i$ and $x \neq v_i$.

At the Bayesian equilibrium, agent i pays a price

$$p_i(b(v_1),\ldots,b(v_n)) = \begin{cases} (n-1)\left[\int_{\underline{v}}^{v_i} \frac{\int_{\underline{v}}^{t} F(s)^n ds}{F(t)^{n+1}} f(t)dt + \frac{1}{n}\underline{v}\right] & \text{if } b_i \geq \max_k b_k \\ -\left[\int_{\underline{v}}^{v_j} \frac{\int_{\underline{v}}^{t} F(s)^n ds}{F(t)^{n+1}} f(t)dt + \frac{1}{n}\underline{v}\right] & \text{if } b_j \geq \max_k b_k. \end{cases}$$

It is immediate to see that this price rule corresponds to the ex post transfer rule defined by equation (15) in the symmetric distribution case (and omitting the constant term). It can easily be proven that $\phi_i(U^0,l)$ replicates the constant term defined by equation (17) after noticing that $\psi_i(\overline{v}) = \frac{1}{n}[G - \underline{v}]$ in the symmetric distribution case.

The bidding game is thus EF as $b(\cdot)$ is increasing, i.e., the bidder with the highest valuation obtains the good. It is also IIR, EPBB and EPCC as it reproduces the transfer rule defined in equation (17). ■

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