ECN 7059 Macroéconomie avancée

Lecture 8: Optimal monetary policy with commitment

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Outline of this lecture

1. Review of the New Keynesian model
2. Optimal labor and monetary policy in the NK model
3. Time inconsistency of optimal policy in the NK model with a distorted steady state (i.e., no labor policy)
4. Self-enforcement of a rule: suppose that the rule cannot be perfectly enforced. What is the best sustainable (or self-enforcing) rule ?
5. Tradeoff between commitment and discretion in the design of a rule: suppose that enforcement of the rule is perfect but the optimal state-contingent rule cannot be implemented. How to design a rule that balances the need for discretion and the need for discipline to address the time-inconsistency problem ?

1. Review of the New Keynesian model

"Three equations" model

IS dynamic:

$$y_t - y_t^n = E_t[y_{t+1} - y_{t+1}^n] - \frac{1}{\sigma}(i_t - E_t[\pi_{t+1}] - r_t^n)$$

New-keynesian Phillips curve:

$$\pi_t = \beta E_t[\pi_{t+1}] + \lambda(\sigma + \varphi)(y_t - y_t^n)$$

Monetary policy:

Example of interest rate rule:

$$i_t = \rho + \phi_\pi \pi_t + \phi_y(y_t - y_t^n) + v_t \qquad \text{(Taylor rule)}$$

in which $v_t = \rho_v v_{t-1} + \varepsilon_t^v$

IS dynamic:

$$y_t - y_t^n = E_t[y_{t+1} - y_{t+1}^n] - \frac{1}{\sigma}(i_t - E_t[\pi_{t+1}] - r_t^n)$$

- IS dynamic summarizes the choice of consumers in equilibrium
- IS dynamic is the log-linearized Euler condition in which consumption is replaced by production using the market clearing conditions
- In the neo-classical model (i.e., no nominal rigidities) output is at its natural level: $y_t = y_t^n$, $y_{t+1} = y_{t+1}^n$ and $r_t = r_t^n$.
- If the nominal interest rate i increases and the expected inflation $E_t[\pi_{t+1}]$ does not respond one for one, then the real interest rate $r_t = i_t - E_t[\pi_{t+1}]$ is higher that the neutral (natural) real interest rate r_t^n, which tends to lower the relative output gap between t and $t + 1$.

New-keynesian Phillips curve:

$$\pi_t = \beta E_t[\pi_{t+1}] + \lambda(\sigma + \varphi)(y_t - y_t^n)$$

- New-keynesian Phillips curve summarizes the choice of prices by monopolistically competitive firms in equilibrium
- New-keynesian Phillips curve is the firms' FOC log-linearized
- The choice of firms is forward looking (i.e. $\beta E_t[\pi_{t+1}]$) since the firm anticipates not being able to update its price due to nominal rigidities
- The choice of firms depends on the output gap because the output gap is a proxy for whether inflation is, on average, higher (negative output gap) or lower (positive output gap) than it would be without nominal rigidities.

New-keynesian Phillips curve:

$$\pi_t = \beta E_t[\pi_{t+1}] + \lambda\left(\sigma + \varphi\right)\left(y_t - y_t^n\right)$$

New-keynesian Phillips curve is different from the Phillips curve (important)

- New-keynesian Phillips curve is a *theoretical* relationship between inflation, expected inflation and output gaps. There is not unemployment in the model but if the output gap is negative, then hours worked are below their natural level. The New-keynesian Phillips curve is a *structural* equation.
- Phillips curve is an empirical relationship between inflation and unemployment.

2. Optimal labor and monetary policy in the NK model

Overview

1. Social welfare function
   - Exact: U
   - Second order Taylor approximation: Loss function $\mathbb{L}(\pi^2, (y_t - y_n)^2)$
2. Planner's problem: identify two sources of inefficiency:
   2.1 markup due to monopoly pricing: $\mathcal{M} > 1$
   2.2 relative price distortions due to nominal rigidities: $\mathcal{M}_t \neq \mathcal{M}$ and $P_t(j) \neq P_t(j')$
3. Implementation of efficient allocation with labor market and monetary policy
   3.1 labor market subsidy τ that correct the monopoly pricing $\mathcal{M} > 0$ of firms: $(1 - \tau)\mathcal{M} = 1$
   3.2 inflation targeting $\pi_t = 0$. If the central bank commits to 0 inflation, there is no need to update prices and nominal rigidities are not constraining

2. Optimal labor and monetary policy in the NK model

Social welfare function

- Social welfare:

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta_t U\left(C_t, N_t\right) Z_t$$

  Without capital, the dynamic economy is simply a repetition of the static economy so the planner can maximize utility period by period

- Second order Taylor expansion: to simplify the algebra, it is useful to use a second-order Taylor approximation

To simplify the algebra, it is useful to use a second-order Taylor approximation (see Gali Appendices 4.1 and 5.1)

- Let (C_t, N_t) be close to the steady state (C, N) so $U_t = U(C_t, N_t)$ is close to $U = U(C, N)$
-

$$\begin{aligned} U_t - U \approx & U_c C \frac{C_t - C}{C} + \frac{1}{2} U_{cc} C^2 \left(\frac{C_t - C}{C}\right)^2 + \\ & U_n N \frac{N_t - N}{N} + \frac{1}{2} U_{nn} N^2 \left(\frac{N_t - N}{N}\right)^2 + \\ & U_c C \frac{C_t - C}{C} \frac{Z_t - Z}{Z} + U_n N \frac{N_t - N}{N} \frac{Z_t - Z}{Z} \end{aligned}$$

where we used $U_{cn} = 0$.

2. Optimal labor and monetary policy in the NK model

Social welfare function

- Use market clearing conditions $c_t = y_t$
  Use technology to relate N_t to y_t and a measure of price dispersion (see Gali Appendices 4.1 and 5.1)

- Second order approximation to social welfare:

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\frac{1}{\epsilon}(y_t - y^n) - \frac{1}{2} \left(\frac{\epsilon}{\lambda} \pi_t^2 + (\sigma + \varphi)(y_t - y_n)^2 \right) \right]$$

  where

  - λ is an inverse measure of nominal rigidities
  - $\sigma \equiv -\frac{U_{cc}}{U_c} C$ and $\varphi \equiv \frac{U_{nn}}{U_n} N$ and $\lambda(\sigma + \varphi)$ gives the slope of the NKPC
  - ϵ is the elasticity of substitution (between different consumption goods in the same period) which is an inverse measure of the market power of the firms

2. Optimal labor and monetary policy in the NK model

Social welfare function

Second order approximation to social welfare:

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\frac{1}{\epsilon}(y_t - y^n) - \frac{1}{2} \left(\frac{\epsilon}{\lambda} \pi_t^2 + (\sigma + \varphi)(y_t - y_n)^2 \right) \right]$$

- $\frac{1}{\epsilon}(y_t - y^n)$ there is a first order effect of stimulating the economy above its steady state because the steady state is inefficient (monopoly distortion)
- $\frac{\epsilon}{\lambda}\pi_t^2$ there are inefficiencies due to fluctuations in prices due to nominal rigidities
- $(\sigma + \varphi)(y_t - y_n)^2$ there are inefficiencies due to output fluctuations because of concavity in the utility function of households

2. Optimal labor and monetary policy in the NK model

Planner's problem

For the planner's problem, we don't need to resort to the approximation of social welfare

$$\max U\left(C_t, N_t; Z_t\right)$$

subject to:

$$C_t(i) = A_t N_t(i)^{1-\alpha}, \ \ all \ \ i \in [0,1]$$

$$N_t = \int_0^1 N_t(i) di$$

$$C_t \equiv \left(\int_0^1 C_t(i)^{1-\frac{1}{\epsilon}} di\right)^{\frac{\epsilon}{\epsilon-1}}$$

Lagrangian:

$$\mathcal{L} \equiv U\left(C_t, N_t; Z_t\right) + \lambda_t \left[C_t - \left(\int_0^1 C_t(i)^{1-\frac{1}{\epsilon}} di\right)^{\frac{\epsilon}{\epsilon-1}}\right] + \mu_t(j)\left[C_t(i) - A_t N_t(i)^{1-\alpha}\right] + \xi\left[N_t - \int_0^1 N_t(i) di\right]$$

FOC of the planner's problem

$$C_t(i) = C_t, \text{ for all} i \in [0, 1]$$

$$N_t(i) = N_t, \text{ for all } i \in [0, 1]$$

$$-\frac{U_{n,t}}{U_{c,t}} = A_t(1 - \alpha)N_t^{-\alpha}$$

Symmetry in the model gives: $C_t(i) = C_t$ et $N_t(i) = N_t$

Equate MRS to marginal product of labor (we assume $\alpha = 0$ for simplicity)

2. Optimal labor and monetary policy in the NK model

Optimal policy to implement the planner's allocation

Suppose we design jointly the labor market policy and the monetary policy

If we set up policy instruments to make the equations characterizing an equilibrium with policy match the equations characterizing the planner's allocation, then we implement the first best and the policy is the optimal policy.

2. Optimal policy in the NK model

Comparing planner and equilibrium conditions

Planner's FOC:

$$-\frac{U_{n,t}^{plan.}}{U_{c,t}^{plan.}} = A_t(1-\alpha)(N_t^{plan.})^{-\alpha}$$

Equilibrium without policy:

$$-\frac{U_{n,t}^{eq}}{U_{c,t}^{eq}} = \frac{W_t}{P_t} \qquad \text{(Household FOC)}$$

$$\frac{W_t}{P_t} = \frac{A_t(1-\alpha)(N_t^{eq})^{-\alpha}}{\mathcal{M}}$$

(Firm FOC without nominal rigidities)

2. Optimal labor and monetary policy in the NK model

Monopoly distortion

Claim: The natural level of output is inefficiently low due to the monopoly distortion: $N_t^{eq} < N_t^{plan.}$.

Proof.

$$-\frac{U_{n,t}^{eq}}{U_{c,t}^{eq}} = \frac{W_t}{P_t} = \frac{A_t(1-\alpha)(N_t^{eq})^{-\alpha}}{\mathcal{M}} < A_t(1-\alpha)(N_t^{plan.})^{-\alpha}$$

since the markup $\mathcal{M} > 1$

- Left-hand side $-\frac{U_{n,t}^{plan.}}{U_{c,t}^{plan.}}$ is increasing N_t
- Right-hand side $A_t(1-\alpha)(N_t^{plan.})^{-\alpha}$ is decreasing in N_t
- Hence $N_t^{eq} < N_t^{plan.}$

$\square$

2. Optimal labor and monetary policy in the NK model

Optimal labor market policy

Labor subsidy τ correct the monopoly distortion

- the CPO of the firm becomes

$$P_t = \mathcal{M} \times \frac{(1-\tau)W_t}{A_t(1-\alpha)N_t^{-\alpha}}$$

Hence

$$-\frac{U_{n,t}^{eq}}{U_{c,t}^{eq}} = \frac{W_t}{P_t} = \frac{A_t(1-\alpha)(N_t^{eq})^{-\alpha}}{(1-\tau)\ \mathcal{M}}$$

match the planner's FOC

$$-\frac{U_{n,t}^{plan.}}{U_{c,t}^{plan.}} = A_t(1-\alpha)(N_t^{plan.})^{-\alpha}$$

if we set

$$(1-\tau)\ \mathcal{M} = 1 \qquad \text{which is } \tau = \frac{1}{\epsilon}$$

2. Optimal labor and monetary policy in the NK model

This is intuitive: the larger is the degree of market power (i.e., lower ϵ), the larger is the corrective subsidy on labor $\tau = \frac{1}{\epsilon}$.

We found the optimal labor market policy conditional on not having distortions due to nominal rigidities (i.e., we determined the optimal τ on the basis of markups $\mathcal{M}$ not $\mathcal{M}_t$).

Let's see if monetary policy can achieve $\mathcal{M}_t = \mathcal{M}$.

2. Optimal labor and monetary policy in the NK model

Two types of distortions due to nominal rigidities:

1. firms may not have their preferred markup $\mathcal{M}_t \neq \mathcal{M}$

$$\mathcal{M}_t = \frac{P_t}{(1-\tau)(W_t/(A_t(1-\alpha)(N_t^{eq})^{-\alpha})}$$
$$= \mathcal{M}\frac{P_t}{W_t/(A_t(1-\alpha)(N_t^{eq})^{-\alpha})}$$

   and so

$$-\frac{U_{n,t}}{U_{c,t}} = \frac{W_t}{P_t} = A_t(1-\alpha)(N_t^{eq})^{-\alpha}\frac{\mathcal{M}}{\mathcal{M}_t} \neq A_t(1-\alpha)(N_t^{eq})^{-\alpha}$$

2. relative prices between commodities may not be equal to 1:
   - if $P_t(j) \neq P_t(j')$, then $C_t^{eq}(j) \neq C_t^{eq}(j')$
   - planner: $C_t^{plan.}(j) \neq C_t^{plan.}(j')$

2. Optimal labor and monetary policy in the NK model

- The monetary policy can control the price level and inflation with a Taylor rule that satisfies the Taylor principle.
- If the monetary policy is set so as to perfectly stabilize the nominal marginal cost, then firms do not need to change their prices.
- If firms do not need to change their prices, then nominal rigidities are not constraining

2. Optimal labor and monetary policy in the NK model

To summarize, if we jointly design labor and monetary policy, it is optimal to have:

- $\tau = \frac{1}{\epsilon}$: subsidy on labor to correct the monopoly distortion
- $\pi = 0$: target 0 inflation to make the nominal rigidity non-constraining. This can be done with a Taylor rule.

2. Optimal labor and monetary policy in the NK model

Implementation

- We determined that monetary policy should achieve (given $\tau = 1/\epsilon$) an inflation that is null: $\pi_t = 0$.
- However the central bank does not choose inflation. Inflation is the result of a market equilibrium (NKPC and Dynamic IS) given a monetary policy.
- How can the central bank *implement* the 0 inflation?

How can the central bank *implement* the 0 inflation?

- IS dynamic

$$y_t - y_t^n = E_t[y_{t+1} - y_{t+1}^n] - \frac{1}{\sigma}(i_t - E_t[\pi_{t+1}] - r_t^n)$$

  tells us that if the output gap and inflation are both null, then

$$i_t = r_t^n$$

  on equilibrium path.

- The problem with a rule $i_t = r_t^n$ is that there are multiple equilibria and only one of which features output gaps and inflation that are both null. The equilibrium is *indeterminate*.

How can the central bank *implement* the 0 inflation? (continued)

- **Optimal monetary policy:** Determinacy of equilibrium can

  be achieved with a Taylor rule with target $\pi = 0$:

$$i_t = r_t^n + \pi + \phi_\pi(\pi_t - \pi) + \phi_y(y_t - y_t^n)$$

  and set $\phi_\pi > 1$ to satisfy the *Taylor principle*.

- Although this Taylor rule satisfying the Taylor principle has the same interest rate *on equilibrium path* as the rule $i_t = r_t^n$, the Taylor rule specifies threats that ensure that the efficient allocation is the only equilibrium.

  Are these threats credible though?

2. Optimal labor and monetary policy in the NK model

Time consistency

Exercise:

1. Is this optimal policy time-consistent?

2. Optimal labor and monetary policy in the NK model

Time consistency

Exercise:

1. Is this optimal policy time-consistent?

   Yes, we solved the problem period by period just like a recursive problem.

2. Could we do better by designing policy from time 0 (i.e. with commitment)?

2. Optimal labor and monetary policy in the NK model

Time consistency

Exercise:

1. Is this optimal policy time-consistent?

   Yes, we solved the problem period by period just like a recursive problem.

2. Could we do better by designing policy from time 0 (i.e. with commitment)?

   No, the optimal time-consistent policy achieves the first best.

3. Time inconsistency of optimal policy in the NK model

Suppose that there is no labor market policy (or it is set such that $\tau < \frac{1}{\epsilon}$). Then the labor market is inefficient and so we say that the steady state is distorted.

The design of monetary policy aims to achieve two goals:

1. correct for the inefficiency of the labor market
2. limit the distortions due to nominal rigidities

This is exactly what the approximation to social welfare captures

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\frac{1}{\epsilon}(y_t - y^n) - \frac{1}{2}\left(\frac{\epsilon}{\lambda}\pi_t^2 + (\sigma + \varphi)(y_t - y_n)^2 \right) \right]$$

With $\tau = \frac{1}{\epsilon}$, the linear term disappears and the central bank can focus on the quadratic terms. We saw that inflation targeting achieves both $\pi = 0$ and $y_t = y_n$ (the so-called "divine coincidence")

3. Time inconsistency of optimal policy in the NK model

Ramsey plan: optimal monetary policy with commitment

$$\max_{((y_t - y_n), \pi_t, i_t)_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\frac{1}{\epsilon}(y_t - y^n) - \frac{1}{2}\left(\frac{\epsilon}{\lambda}\pi_t^2 + (\sigma + \varphi)(y_t - y_n)^2 \right) \right]$$

s.t. $((y_t - y_n), \pi_t)_{t=0}^{\infty}$ is an equilibrium given the policy $(i_t)_{t=0}^{\infty}$

We now rewrite the constraint:

$((y_t - y_n), \pi_t)_{t=0}^{\infty}$ is an equilibrium given the policy $(i_t)_{t=0}^{\infty}$

if and only if:

$$\pi_t = \beta E_t[\pi_{t+1}] + \lambda\left(\sigma + \varphi\right)\left(y_t - y_t^n\right) \qquad \text{(NKPC)}$$

$$y_t - y_t^n = E_t[y_{t+1} - y_{t+1}^n] - \frac{1}{\sigma}(i_t - E_t[\pi_{t+1}] - r_t^n) \qquad \text{(Dynamic IS)}$$

3. Time inconsistency of optimal policy in the NK model

Ramsey plan: optimal monetary policy with commitment

Exercise: We saw that an equilibrium is characterized by three equations: (1. NKPC, 2. Dynamic IS, 3. Monetary policy). Explain why we can take the NKPC as the only implementability constraint and rewrite the problem as follows:

$$\max_{(y_t - y_n, \pi_t)_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\frac{1}{\epsilon}(y_t - y_n) - \frac{1}{2} \left(\frac{\epsilon}{\lambda} \pi_t^2 + (\sigma + \varphi)(y_t - y_n)^2 \right) \right]$$

s.t.

$$\pi_t = \beta E_t[\pi_{t+1}] + \lambda (\sigma + \varphi) (y_t - y_n) \qquad \text{(NKPC)}$$

3. Time inconsistency of optimal policy in the NK model

Ramsey plan: optimal monetary policy with commitment

$$\max_{(y_t - y_n, \pi_t)_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\frac{1}{\epsilon}(y_t - y_n) - \frac{1}{2}\left(\frac{\epsilon}{\lambda}\pi_t^2 + (\sigma + \varphi)(y_t - y_n)^2 \right) \right]$$

s.t.

$$\pi_t = \beta E_t[\pi_{t+1}] + \lambda (\sigma + \varphi)(y_t - y_n) \qquad \text{(NKPC)}$$

3. Time inconsistency of optimal policy in the NK model

Ramsey plan: optimal monetary policy with commitment

Let $\beta^t \xi_t$ denote the Lagrange multiplier on the implementability constraint (i.e. the NKPC).

FOCs of the planner's problem:

$$\frac{1}{\epsilon} - (\sigma + \varphi)(y_t - y_n) = \xi_t \lambda (\sigma + \varphi) \qquad \text{(FOC w.r.t. } (y_t - y^n)\text{)}$$

$$-\frac{\epsilon}{\lambda}\pi_t = \xi_{t-1} - \xi_t \qquad \text{(FOC w.r.t. } \pi_t\text{)}$$

and $\xi_{-1} = 0$.

The term in red reflect the intertemporal considerations that a central bank with the ability to commit to its policy can afford. It's a telltale sign of the time-inconsistency of the optimal policy with commitment.

3. Time inconsistency of optimal policy in the NK model

Ramsey plan: optimal monetary policy with commitment

Combining the FOCs of the planner's problem, we get:

$$y_0 - y_0^n = \frac{1}{\epsilon(\sigma + \varphi)} - \epsilon\pi_0 \qquad \text{(for } t = 0)$$

$$y_t - y_t^n = y_{t-1} - y_{t-1}^n - \epsilon\pi_t \qquad \text{(for } t \geq 1)$$

Again, the difference between the FOC at $t = 0$ and $t \geq 1$ is a telltale sign of time-inconsistency.

Solving it backwards, we get:

$$\begin{aligned} y_t - y_t^n &= \frac{1}{\epsilon(\sigma + \varphi)} - \epsilon \sum_{j=0}^{t} \pi_j \\ &= \frac{1}{\epsilon(\sigma + \varphi)} - \epsilon(p_t - p_{-1}) \end{aligned}$$

since $\pi_t = p_t - p_{t-1}$.

3. Time inconsistency of optimal policy in the NK model

Ramsey plan: optimal monetary policy with commitment

The NKPC can be rewritten:

$$p_t - p_{t-1} = \beta E_t[p_{t+1} - p_t] + \lambda\left(\sigma + \varphi\right)\left(y_t - y_t^n\right) \quad \text{(for } t \geq 0\text{)}$$

Combining the NKPC (which is the implementability constraint) and the FOC of the planner, we get:

$$\frac{p_t - p_{t-1}}{\lambda\left(\sigma + \varphi\right)} - \beta \frac{E_t[p_{t+1} - p_t]}{\lambda\left(\sigma + \varphi\right)} = \frac{1}{\epsilon(\sigma + \varphi)} - \epsilon(p_t - p_{-1})$$

This is a second order difference equation. It can be solved as in QuantEcon: Linear Rational Expectations Models (link). Solving it gives a feedback and a feedforward part. Alternatively, one can use the method of undetermined coefficients as in Gali.

Qualitative features of the solution with commitment:

- The price level p_t increases over time to converge to a level that is strictly positive and bounded.
- The output gap is strictly positive and converges to the (inefficient) natural level.

3. Time inconsistency of optimal policy in the NK model

Ramsey plan: optimal monetary policy with commitment

Comparison of

2. optimal monetary policy when the natural equilibrium is not distorted (e.g., optimal labor subsidy is in place) to
3. optimal monetary policy when the steady state is distorted

We found

2. optimal monetary policy when the labor subsidy makes the natural equilibrium efficient
   - The price level p_t is constant.
   - The output gap is 0.
3. optimal monetary policy with commitment in the absence of a labor subsidy (i.e., inefficient natural equilibrium)
   - The price level p_t increases over time to converge to a level that is strictly positive and bounded.
   - The output gap is strictly positive and converges to the (inefficient) natural level.

Comparison (continued):

- *Short run:* in the absence of a labor subsidy and an inefficient low natural level, there is some inflation and a positive output gap. This is the **inflation bias** that result from the first order gain from using monetary policy to correct for the inefficiently low natural level of output.
- *Long run:* the optimal monetary policy with commitment implements an equilibrium with the same properties in the long run as the policy with an efficient natural equilibrium.

Exercise: (Recursive formulation of the Ramsey problem)
Suppose that there is no uncertainty.

1. Rewrite the problem of finding the Ramsey plan in recursive form.
2. Explain how you would solve it.

(Hint: Chapter 19 “Dynamic Stackelberg Problem” from Ljungqvist and Sargent’s textbook provides the theory. The QuantEcon lecture 40. “Ramsey Plans, Time Inconsistency, Sustainable Plans” puts the theory in practice for a classical monetary model in which there is a cost to changing the money supply. (The Friedman rule is the optimal policy if there is not cost of changing the money supply.)

3. Time inconsistency of optimal policy in the NK model

Exercise: Recursive formulation of the Ramsey problem. Answer to 1.

Before we start, it is worth solving the NKPC forward to see how future output gaps matter for current inflation:

$$\pi_t = \beta\pi_{t+1} + \lambda\left(\sigma+\varphi\right)\left(y_t - y_t^n\right) \qquad \text{(NKPC)}$$

Suppose that $(\pi_t)_{t=0}^{\infty}$ is square summable, then we can solve the NKPC forward to get:

$$\pi_t = \beta(\beta[\pi_{t+2}] + \lambda\left(\sigma+\varphi\right)\left(y_{t+1} - y_{t+1}^n\right)]) + \lambda\left(\sigma+\varphi\right)\left(y_t - y_t^n\right)$$
$$\pi_t = \beta^2\pi_{t+2} + \beta\lambda\left(\sigma+\varphi\right)\left(y_{t+1} - y_{t+1}^n\right)] + \lambda\left(\sigma+\varphi\right)\left(y_t - y_t^n\right)$$

...

$$\pi_t = \lambda\left(\sigma+\varphi\right)\sum_{j=0}^{\infty}\beta^j(y_{t+j+1} - y_{t+j+1}^n) + \lim_{j\to\infty}\beta^j\pi_{t+j}$$

So π_{t+1} in the NKPC summarizes the effect of all the future output gaps on π_t. This is a hint that we need a forward looking state variable to define the state in the recursive formulation.

Finding the state variable

Since π_{t+1} in the NKPC summarizes the effect of all the future output gaps on π_t, we use it as a state variable.

- It is forward looking because it summarizes the effect of future output gaps on current inflation.
- It is backward looking because it is a promise made at time 0 by the central bank.

Note that π_t is not a *natural* state variable (i.e., it is not a stock and it is not predetermined at time t). We use it is as a state variable to be able to formulate the Ramsey problem (i.e., with commitment) recursively.

3. Time inconsistency of optimal policy in the NK model

Exercise: Recursive formulation of the Ramsey problem. Answer to 1.

Implementability constraint: rewriting the NKPC in state space form

$$\begin{bmatrix} 1 \\ \pi_t \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \beta \end{bmatrix} \begin{bmatrix} 1 \\ \pi_{t+1} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \lambda(\sigma+\varphi) \end{bmatrix} \begin{bmatrix} 1 \\ y_t - y_t^n \end{bmatrix}$$

$$\tilde{x}_t \equiv \begin{bmatrix} z_t \\ x_t \end{bmatrix} \equiv \begin{bmatrix} 1 \\ \pi_t \end{bmatrix} \quad , \qquad A = \begin{bmatrix} 1 & 0 \\ 0 & \beta \end{bmatrix}^{-1}$$

and

$$u_t \equiv \begin{bmatrix} 1 \\ y_t - y_t^n \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & \beta \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \lambda(\sigma+\varphi) \end{bmatrix}.$$

Hence, the implementability constraint reads:

$$\tilde{x}_{t+1} = A\tilde{x}_t + Bu_t$$

3. Time inconsistency of optimal policy in the NK model

Exercise: Recursive formulation of the Ramsey problem. Answer to 1.

Objective of the Ramsey planner is Linear-Quadratic:

$$-\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t r(\tilde{x}_t, u_t) = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\frac{1}{\epsilon}(y_t - y_n) - \frac{1}{2}\left(\frac{\epsilon}{\lambda}\pi_t^2 + (\sigma + \varphi)(y_t - y_n)^2 \right) \right]$$

$$r(\tilde{x}, u) = y' R y + u' Q u$$

$$\tilde{x}' R \tilde{x} = \frac{1}{2}\frac{\epsilon}{\lambda}\pi_t^2, \quad \text{so} \quad R = \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{2}\frac{\epsilon}{\lambda} \end{bmatrix}$$

$$u' Q u = -\frac{1}{\epsilon}(y_t - y_n) + \frac{1}{2}(\sigma + \varphi)(y_t - y_n)^2, \quad \text{so} \quad Q = \begin{bmatrix} 0 & -\frac{1}{\epsilon} \\ 0 & \frac{1}{2}(\sigma + \varphi)\frac{\epsilon}{\lambda} \end{bmatrix}$$

So far, we have only introduced notation and rewritten the Ramsey problem:

$$\max_{(y_t - y_n, \pi_t)_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\frac{1}{\epsilon}(y_t - y_n) - \frac{1}{2}\left(\frac{\epsilon}{\lambda}\pi_t^2 + (\sigma + \varphi)(y_t - y_n)^2 \right) \right]$$

$$\text{s.t.} \qquad \pi_t = \beta E_t[\pi_{t+1}] + \lambda(\sigma + \varphi)(y_t - y_n) \qquad \text{(NKPC)}$$

in matrix notation:

at time 0, choose a sequence $(u_t, \tilde{x}_t)_{t=0}^{\infty}$ to solve

$$\max_{(u_t, \tilde{x}_t)_{t=0}^{\infty}} -\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t r(\tilde{x}_t, u_t)$$

$$\text{s.t.} \qquad \tilde{x}_{t+1} = A\tilde{x}_t + Bu_t$$

3. Time inconsistency of optimal policy in the NK model

Exercise: Recursive formulation of the Ramsey problem. Answer to 1.

The recursive formulation is split into two sub-problems which reflects the special role of period 0 when policy is chosen without prior commitment but with the power to commit futur decisions:

1. Sub-problem of continuation Ramsey planners at time $t \geq 1$.
   The Ramsey planner delegates the implementation of the plan to a sequence of continuation Ramsey planners. Each continuation Ramsey planner inherits the state variable $\tilde{x}_t$ which constrains the continuation Ramsey planner to implement the Ramsey plan.

2. Sub-problem of Ramsey planner at time $t = 0$.
   The time 0 Ramsey planner is not constrained by $\tilde{x}_0$

3. Time inconsistency of optimal policy in the NK model

Exercise: Recursive formulation of the Ramsey problem. Answer to 1.

1. Sub-problem of continuation Ramsey planners at time $t \geq 1$. Bellman equation of the Ramsey planner:

$$v(\tilde{x}) = \max_{u,\tilde{x}'} -r(\tilde{x}, u) + \beta v(\tilde{x}')$$
$$\text{s.t.} \quad \tilde{x}' = A\tilde{x} + Bu$$

   Note that $\tilde{x}$ constrains the choice of the continuation Ramsey planner.

2. Sub-problem of Ramsey planner at time $t = 0$.

$$W = \max_{\tilde{x}_0} v(\tilde{x}_0)$$

3. Time inconsistency of optimal policy in the NK model

Exercise: Recursive formulation of the Ramsey problem. Answer to 2.

To solve the problem in recursive form, note that the Linear Quadratic environment implies that we only need to solve algebraic Riccati equations which we can solve in closed form

See Ljungqvist Sargent Chapter 5 for more on algebraic Riccati equations.

3. Time inconsistency of optimal policy in the NK model

Exercise: Recursive formulation of the Ramsey problem. Answer to 2.

We can use the solution in closed form of the continuation Ramsey planner to solve the Ramsey planner's problem at time $t = 0$ in closed form.

2. Sub-problem of Ramsey planner at time $t = 0$.

$$
\begin{aligned}
W &= \max_{\tilde{x}_0} v(\tilde{x}_0) \\
&= \max_{\tilde{x}_0} -\tilde{x}_0' P \tilde{x}_0
\end{aligned}
$$

Taking FOC w.r.t. to x_0 where $\tilde{x}_0 = [1 x_0]^T$ gives:

$$
-2P_{21} - 2P_{22}x_0 = 0
$$

and so

$$
x_0 = -P_{22}^{-1} P_{21}.
$$

3. Time inconsistency of optimal policy in the NK model

Exercise: Recursive formulation of the Ramsey problem. Answer to 2.

Now, we formalize the sense in which the time 0 Ramsey planner delegates the implementation of the plan to a sequence of continuation Ramsey planners. Each continuation Ramsey planner inherits the state variable $\tilde{x}_t$ which constrains the continuation Ramsey planner to implement the Ramsey plan.

Chapter 5 of Ljungqvist Sargent shows that $P\tilde{x}_t$ can be interpreted at the Lagrange multiplier on the contraint

$$\tilde{x}_{t+1} = A\tilde{x}_t + Bu_t.$$

The Lagrange multiplier on the NKPC is thus:

$$P_{21} + P_{22}x_t$$

which is 0 for the time 0 Ramsey planner but not zero for the continuation Ramsey planners. This is another telltale sign of time inconsistency of the Ramsey plan.

Exercise: Recursive formulation of the Ramsey problem. Answer to 2.

1. Sub-problem of continuation Ramsey planners at time $t \geq 1$. With the Linear Quadratic environment, we guess that

$$v(\tilde{x}) = -\tilde{x}P\tilde{x}$$

for an undetermined matrix P. We determine the matrix P by solving Bellman equation of the Ramsey planner:

$$v(\tilde{x}) = -\tilde{x}P\tilde{x} = \max_{u,\tilde{x}'} -r(\tilde{x}, u) + \beta = -\tilde{x}'P\tilde{x}'$$

which is the following algebraic Riccati equation:

$$P = R + \beta A'PA - \beta^2 A'PB(Q + \beta B'PB)^{-1}B'PA.$$

The continuation Ramsey planner's decision rule reads:

$$u(\tilde{x}) = F\tilde{x}, \quad \text{with} \quad F = \beta(Q + \beta B'PB)^{-1}B'PA.$$

3. Time inconsistency of optimal policy in the NK model

Exercise (ZLB):

1. Augment the Ramsey problem with an Effective Lower Bound on the interest rate at zero (also known as a Zero Lower Bound $i_t \geq 0$).
2. How would you modify the implementability constraints to impose a Zero Lower Bound on the nominal interest rate ?

3. Time inconsistency of optimal policy in the NK model

Ramsey plan: optimal monetary policy with commitment

Exercise (ZLB):

1. Augment the Ramsey problem with an Effective Lower Bound on the interest rate at zero (also known as a Zero Lower Bound $i_t \geq 0$).
2. How would you modify the implementability constraints to impose a Zero Lower Bound on the nominal interest rate ?

1. Answer

$$\max_{((y_t - y_n), \pi_t, i_t)_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\frac{1}{\epsilon}(y_t - y^n) - \frac{1}{2} \left(\frac{\epsilon}{\lambda} \pi_t^2 + (\sigma + \varphi)(y_t - y_n)^2 \right) \right]$$

s.t. $((y_t - y_n), \pi_t)_{t=0}^{\infty}$ is an equilibrium given the policy $(i_t)_{t=0}^{\infty}$ and $i_t \geq 0$.

3. Time inconsistency of optimal policy in the NK model

Exercise: Ramsey problem with ZLB

Exercise:

2. How would you modify the implementability constraints to impose a Zero Lower Bound on the nominal interest rate ?

2. Answer: use the dynamic IS to translate the constraint $i_t \geq 0$ into a constraint on the output gap and inflation:

$$0 \leq \frac{1}{\sigma} i_t = E_t[y_{t+1} - y^n_{t+1}] - (y_t - y^n_t) - \frac{1}{\sigma}(-E_t[\pi_{t+1}] - r^n_t)$$

and the Ramsey problem then reads:

$$\max_{(y_t - y_n, \pi_t)_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\frac{1}{\epsilon}(y_t - y_n) - \frac{1}{2} \left(\frac{\epsilon}{\lambda} \pi_t^2 + (\sigma + \varphi)(y_t - y_n)^2 \right) \right]$$

s.t.

$$\pi_t = \beta E_t[\pi_{t+1}] + \lambda (\sigma + \varphi) (y_t - y_n) \qquad \text{(NKPC)}$$

$$0 \leq E_t[y_{t+1} - y^n_{t+1}] - (y_t - y^n_t) + \frac{1}{\sigma}(E_t[\pi_{t+1}] + r^n_t) \qquad \text{(ZLB)}$$

3. Time inconsistency of optimal policy in the NK model

Optimal policy without commitment

The economy is a repetition of a static environment. The optimal policy for a given period in isolation solves :

$$\max_{(y_t - y_n), \pi_t} \frac{1}{\epsilon}(y_t - y_n) - \frac{1}{2}\left(\frac{\epsilon}{\lambda}\pi_t^2 + (\sigma + \varphi)(y_t - y_n)^2\right)$$

s.t.

$$\pi_t = \beta E_t[\pi_{t+1}] + \lambda\left(\sigma + \varphi\right)\left(y_t - y_n\right) \qquad \text{(NKPC)}$$

3. Time inconsistency of optimal policy in the NK model

Optimal policy without commitment

$$\max_{(y_t - y_n), \pi_t} \frac{1}{\epsilon}(y_t - y^n) - \frac{1}{2}\left(\frac{\epsilon}{\lambda}\pi_t^2 + (\sigma + \varphi)(y_t - y_n)^2\right)$$

s.t.

$$\pi_t = \beta E_t[\pi_{t+1}] + \lambda\left(\sigma + \varphi\right)\left(y_t - y_t^n\right) \qquad \text{(NKPC)}$$

FOCs of the planner's problem:

$$\frac{1}{\epsilon} - (\sigma + \varphi)(y_t - y_t^n) = \xi_t \lambda(\sigma + \varphi) \qquad \text{(FOC w.r.t. } (y_t - y_t^n))$$

$$-\frac{\epsilon}{\lambda}\pi_t = -\xi_t \qquad \text{(FOC w.r.t. } \pi_t)$$

FOCs of the planner's problem:

$$\frac{1}{\epsilon} - (\sigma + \varphi)(y_t - y_t^n) = \xi_t \lambda (\sigma + \varphi) \qquad \text{(FOC w.r.t. } (y_t - y_t^n))$$

$$\frac{\epsilon}{\lambda} \pi_t = \xi_t \qquad \text{(FOC w.r.t. } \pi_t)$$

Combining the FOCs of the planner's problem, we get:

$$y_t - y_t^n = \frac{1}{\epsilon(\sigma + \varphi)} - \epsilon \pi_t$$

Using the NKPC to substitute $y_t - y_t^n$, we get

$$\frac{\pi_t - \beta E_t[\pi_{t+1}]}{\lambda\,(\sigma + \varphi)} = \frac{1}{\epsilon(\sigma + \varphi)} - \epsilon \pi_t.$$

This is a second order difference equation. It can be solved as in QuantEcon: Linear Rational Expectations Models (link).

Qualitative features of the solution without commitment:

- Inflation is strictly positive and constant over time.
- The output gap is strictly positive and constant over time.

The first-order gain from raising output above its natural level gives rise to an *inflation bias*. The central bank cannot commit to have inflation and the output gap converge to zero. The optimal policy with commitment is time-inconsistent.

3. Time inconsistency of optimal policy in the NK model

Further study

- Other sources of time-inconsistency: Gali Sections 5.2 (undistorted steady state but inefficient fluctuations of the natural output) and Section 5.3 (zero lower bound on the nominal interest rate)

4. Self-enforcement of a rule

Further readings

- Barro and Gordon (1983)
- Chari and Kehoe (1990)
- QuantEcon https://python-advanced.quantecon.org/calvo.html Ramsey Plans, Time Inconsistency, Sustainable Plans
- Abreu Pearce Stachetti (1990)

5. Design of a rule

Further readings

- Athey Atkeson Kehoe (2005)
- Halac and Yared (2020, 2021)
- Clayton and Schaab (2021)