ECN 7059 Macroéconomie avancée

Lecture 3: Mirrleesian optimal taxation in a static environment

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Plan

So far

1.a) Is there a role for macroeconomic policy
1.b) Design of policy with exogenous instruments : Ramsey optimal taxation

This lecture :

1.c) Design of policy with endogenous instruments : Mirrleesian optimal taxation

Mirrleesian optimal taxation designs a mechanism

Readings :

- Mas-Colell, Winston and Greene, *Microeconomic Theory*, Chapter 23
- Fudenberg and Tirole *Game Theory*, Chapter 7
- Salanié, *The Economics of Contracts* Chapter 2 and 3
- Kocherlakota, *New Dynamic Public Finance* Chapter 3

Plan

1.c) Design of policy under information asymmetry : Mirrleesian approach

   i) Intro to mechanism design and the Revelation Principle

   ii) Static setting and the envelope/integral condition

   iii) (Lecture 4) Dynamic setting and the inverse Euler equation

- Lecture 3 (this lecture) studies optimal (Mirrleesian) non-linear taxation in a *static* environment. The focus is on intra-temporal tradeoffs such as the choice between consumption and labor. We focus on labor income tax while abstracting from investment.

- Lecture 4 studies optimal (Mirrleesian) non-linear taxation in a *dynamic* environment. The focus is on inter-temporal tradeoffs such as investments and savings and capital income tax.

1.c) Design of policy under information asymmetry : Mirrleesian approach

Recall that what should guide the choice of policy instruments in the Ramsey approach is what the government can observe.

Exercise : suppose that the government observes labor income $w_t \ell_t$ but does not observe hours worked ℓ_t and the wage rate w_t ?

- Can the government implement a linear ad-valorem tax ?
- Can the government implement a linear excise tax ?
- Can the government implement a non-linear tax as a function of labor income ?
- Can the government implement a non-linear tax as a function of the wage rate ?

1.c) Design of policy under information asymmetry : Mirrleesian approach

Mirrleesian approach :

- the government can use any non-linear tax schedule $T(\cdot)$ (*not* restricted to be linear as in the Ramsey approach)
- however, the tax schedule can only be a function of what the government can observe : T("observable variables")

Lack of information of the government determines the shape of the optimal tax schedule (a fully informed government would use lump-sum taxes.)

1.c) Mirrleesian approach on how to design optimal policy

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2. Define competitive equilibrium *given* a non-linear tax policy

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4. Characterize the constraint set of the Mirrleesian problem by characterizing the equilibrium allocation using constraints that depend only on the allocation. These constraints are called **Incentive Compatibility constraints** and **Participation constraints**. They are the analogue of the **implementability constraints**.

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5. Solve the **primal problem** : choose allocation to maximise social welfare subject to the constraint that the allocation is feasible and satisfies the implementability constraint
6. Decentralize : recover prices and optimal policy using the equivalence proved in Step 4.

1.c) Mirrleesian approach on how to design optimal policy

More details on the step by step approach to

5. Solve the **primal problem** : choose allocation to maximise social welfare subject to the constraint that the allocation is feasible and satisfies the implementability constraint

   5.1 Rewrite in primal form using the Revelation Principle

   5.2 Rewrite IC inequalities as IC equalities (integral/envelope conditions) and monotonicity

   5.3 Solve relaxed problem (ignore monotonicity)

      5.3.i substitute integral/envelope conditions into objective and market clearing

      5.3.ii integrate by parts to simplify nested integrals

      5.3.iii substitute market clearing into objective

   5.4 Maximize point-wise

1.c) Mirrleesian approach on how to design optimal policy

Remarks :

- Same overall methodology as the Ramsey approach, just a different environment and different tools
- Implementability constraints are called Incentive Compatibility and Participation constraints in the Mirrleesian approach
- To do Steps 4 and 5 we will use tools from mechanism design
  - In macroeconomics, usually, the *mechanism* through which ressources are allocated is a *competitive equilibrium given the government's policy*
  - Changing the government's policy changes the mechanism
  - Designing the optimal policy is a mechanism design problem
  - The Revelation Principle allows us to rewrite the government's problem in primal form (i.e. optimize over allocations instead of over policy)

1.c) i) Intro to mechanism design

First, let's put the macroeconomic context aside and study some tools to design mechanisms.

- Households indexed by i.
- Heterogeneity : $\theta_i \in \Theta$ is the type of household i
- Social choice function $f((\theta_i)_i)$ maps a profile of types into a feasible allocation

1.c) i) Intro to mechanism design

Some definitions

A *mechanism* is a collection of choice sets C_i and an allocation rule g that maps choices into a feasible allocation.

- An *equilibrium* of the mechanism $((C_i)_i, g)$ is a collection of choices $(c(\theta_i))_i$ such that all agents are privately optimizing in their respective choice sets C_i given the allocation rule g
- A mechanism $((C_i)_i, g)$ *implements* the social choice function f if there exists an equilibrium $(c(\theta_i))$ of the mechanism such that

$$f((\theta_i)_i) = g((c(\theta_i))_i)$$

- A *direct mechanism* is a mechanism where the choice set is the type space : $C_i = \Theta$ for all i.

1.c) i) Revelation Principle

Revelation principle : If a mechanism $((C_i)_i, g)$ implements a social function f, then there exists a direct mechanism $((\Theta)_i, g^d)$ that also implements the social choice function f.

Proof : Since the mechanism $((C_i)_i, g)$ implements a social function f, there exists an equilibrium of the mechanism $(c(\theta_i))$ such that

$$f((\theta_i)_i) = g((c(\theta_i))_i).$$

Define $g^d(\theta) = g(c(\theta))$.

Exercise : verify that the direct mechanism so defined implements the social choice function f.

1.c) i) Revelation Principle

The Revelation Principle allows us to rewrite the government's problem in **primal form**

- instead of designing the choice set C_i by choosing the optimal tax schedule $T(\cdot)$

$$\max_{T(\cdot)} \int_{\Theta} u(c(\theta); \theta) f(\theta) d\theta$$

$$c(\theta) \in \arg\max_{c} u(c, \theta) : \ c \in C_i(T(\cdot)) \quad \text{for all } \theta \in \Theta$$

$$(c(\theta))_{\theta \in \Theta} \text{ satisfies the resource constraint}$$

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- choose directly the allocation subject to incentive compatibility constraints

$$\max_{c(\cdot)} \int_{\Theta} u(c(\theta);\theta) f(\theta) d\theta$$

$$\theta \in \arg\max_{\tilde{\theta}} u(c(\tilde{\theta}),\theta) \qquad \text{for all } \theta \in \Theta$$

$$(c(\theta))_{\theta \in \Theta} \text{ satisfies the resource constraint}$$

1.c) i) Revelation Principle

Note the equivalence between :

- the condition that individuals are maximizing

$$\theta \in \arg\max_{\tilde{\theta}} u(c(\tilde{\theta}), \theta) \qquad \text{for all } \theta \in \Theta$$

- and the incentive compatibility constraints :

$$u(c(\theta), \theta) \geq u(c(\hat{\theta}), \theta) \quad \text{for all } \theta, \hat{\theta} \in \Theta \qquad \text{(IC)}$$

There are infinitely many incentive compatibility constraints, one for each pair $(\theta, \hat{\theta})$.

1.c) ii) Static : characterization of Incentive Compatibility

We now show that we can replace the set of infinitely many incentive compatibility constraints by two conditions :

i) monotonicity condition

ii) integral condition

Characterization of Incentive Compatibility

Utility linear in types : $u(q, h; \theta) = \theta q + h$

Theorem

A social choice function $f(\theta) = (q(\theta), h(\theta))$, on a domain restricted to preferences parametrized by types θ, which are linear in θ in the sense that the utility of type θ at an allocation for type θ' is $\tilde{U}(\theta'; \theta) \equiv u(q(\theta'), h(\theta'); \theta) = \theta q(\theta') + h(\theta')$, is Incentive Compatible if and only if

i) *q is monotonic increasing ;*

ii) *$U(\theta) = U(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} q(x)dx$ for every θ*
where[1] *$U(\theta) \equiv \tilde{U}(\theta, \theta)$ and $\tilde{U}(\theta, \hat{\theta}) = \theta q(\hat{\theta}) + h(\hat{\theta})$.*

Condition i) is called monotonicity and condition ii) is often called the **integral/envelope condition**

1. U is the utility derived from telling the truth, while $\tilde{U}$ is the utility from reporting type $\hat{\theta}$ when of type θ.

Proof : Necessity of i) and ii) for IC.

Assume w.l.o.g. that $\theta \geq \theta'$. Incentive compatibility $\tilde{U}(\theta,\theta) \geq \tilde{U}(\theta,\theta')\ \forall \theta, \theta'$ can be rewritten :

$$U(\theta) \geq U(\theta') + (\theta - \theta')q(\theta') \quad \forall \theta, \theta'.$$

Switching the roles of θ and θ' gives another similar inequality which, combined with the above gives :

$$q(\theta) \ \geq \frac{U(\theta) - U(\theta')}{\theta - \theta'} \geq \ q(\theta') \quad \forall \theta, \theta'. \tag{1}$$

It follows that q is monotonic, and hence i) is proved.
A monotonic function on an interval is almost everywhere differentiable so it is almost everywhere continuous. Taking limits ($\theta' \to \theta$) in (1) gives :

$$U'(\theta) = q(\theta) \quad a.e.,$$

which, once integrated gives

$$U(\theta) = U(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} q(x)dx.$$

Proof : Sufficiency of *i*) and *ii*) for IC.

From above proof, we want to show that $U(\theta) \geq U(\theta') + (\theta - \theta')q(\theta') \quad \forall \theta, \theta'$, since it is equivalent to Incentive Compatibility. From *ii*), it follows that

$$\begin{aligned} U(\theta) - U(\theta') &= \int_{\theta'}^{\theta} q(x)dx; \\ &\geq \int_{\theta'}^{\theta} q(\theta')dx \qquad \text{by i);} \\ &= q(\theta')(\theta - \theta'). \end{aligned}$$

Q.E.D.

Characterization of Incentive Compatibility

Utility linear in transfers : $U(q, t; \theta) = u(q, \theta) + t$

Spence-Mirrlees, Single-crossing, Sorting condition

$$\frac{\partial^2 u}{\partial q \partial \theta}(q, \theta) > 0 \quad \text{for all} \quad q, \theta$$

This condition implies that :

- indifference curves of two different types can only cross once
- higher types are willing to pay more for a given increase in q than lower types
- allows to characterize *global* incentive compatible allocations using only *local* Incentive Compatibility constraints that come from First Order Conditions and Second Order Conditions

Intuition : the government can design the policy so that higher types get higher allocations and pay something that a low type would not want to pay.

Characterization of Incentive Compatibility

Utility linear in transfers : $U(q, t; \theta) = u(q, \theta) + t$

Theorem

Suppose that the utility satisfies $U(q, t; \theta) = u(q, \theta) - t$ and u satisfies the Spence-Mirrlees condition. An allocation $(q(\theta), t(\theta))$ is **globally incentive compatible**

$$u(q(\theta), \theta) - t(\theta) \geq u(q(\hat{\theta}), \theta) - t(\hat{\theta}) \qquad \textit{for all } \theta, \hat{\theta} \quad \text{(global IC)}$$

if and only if it is **locally incentive compatible**

$$\frac{d\ t}{d\theta}(\theta) = \frac{\partial u}{\partial q}(q(\theta), \theta)\ \frac{d\ q}{d\theta}(\theta) \qquad \text{(local IC 1)}$$

$$q \textit{ is monotonic increasing.} \qquad \text{(local IC 2)}$$

We can replace the set of infinitely many incentive compatibility constraints by two conditions. Local IC 1 is often call the **Integral condition**

Problem Set 1, Exercise 3

Utility linear in transfers : $U(q, t; \theta) = u(q, \theta) + t$

Prove that the local IC are necessary for global IC in the theorem above.

Hint : write the problem of the individual as

$$\max_{\hat{\theta}} U(q(\hat{\theta}), t(\hat{\theta}); \theta),$$

Derive the First Order and Second Order conditions and use that $U(q, t; \theta) = u(q, \theta) + t$ *and the Spence-Mirrlees condition.*

Problem Set 1, Exercise 4

Show that if $u(q,\theta) = \theta\ q$ the local IC 1 implies that :

$$\theta\ q(\theta) - t(\theta) = \underline{\theta}\ q(\underline{\theta}) - t(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} q(x)dx$$

for every θ.

Hint : integrate local IC 1 from $\underline{\theta}$ to θ to the $t(\theta)$

Mirrlees : static optimal non-linear taxation

We apply these techniques to solve for the optimal non-linear tax.

Usual method :

1. Describe the environment
2. Define an equilibrium given a tax policy
3. Write government's problem
4. Use the Revelation Principle to rewrite the problem in **primal form**
5. Use the theorems that characterize IC to solve the problem
6. Decentralize : recover the optimal non-linear tax

Reference : Salanié, *The Economics of Contracts, 2nd edition* Section 3.1.2

Mirrlees : static optimal non-linear taxation

1. Describe the environment

- Unit mass of workers
- Heterogeneity in their skills-productivity
- Heterogeneity parametrized by $\theta \sim F[\underline{\theta}, \bar{\theta}]$ with pdf f
- utility depends on consumption and labor :

  $$C - v(\ell)$$

  where v is increasing and convex
- Constant Returns to Scale firm produces using labor only
- workers paid their marginal product (firm observes productivity) :

  $$w = \theta$$
- total income of a worker is the production :

  $$Q = \theta\ell$$

Mirrlees : static optimal non-linear taxation

1. Describe the environment

- Social welfare function :

$$\int_{\underline{\theta}}^{\bar{\theta}} (C(\theta) - v(\ell(\theta))\ g(\theta)d\theta$$

- $F \leq G$ gives the motive for redistribution : F First Order Stochastically Dominate G implies that the social planner puts more weight on lower types.
- Government observes total labor income : $Q = \theta\ell$
- Government does not observe : wage rate θ, hours worked ℓ
  Exercise : Discuss the interpretation of the wage rate as productivity ? What model justifies this ? Is this a good model for the U.S. ?
- Government has access to non-linear taxes $T(Q)$

Mirrlees : static optimal non-linear taxation

2. Define an equilibrium given a tax policy

An equilibrium, given government policy $T(\cdot)$, is an allocation $(C(\theta), Q(\theta), \ell(\theta))$ and a price system $w(\theta) = \theta$ such that

- for each θ, the agent optimizes : $C(\theta), Q(\theta)$ solves

$$\max_{C,Q} \; C - v\left(\frac{Q}{\theta}\right) \quad \text{s.t.} \quad C \leq Q - T(Q)$$

  where $Q(\theta) = \theta\ell(\theta)$

- market clears :

$$\int_{\underline{\theta}}^{\bar{\theta}} C(\theta)\; f(\theta) d\theta = \int_{\underline{\theta}}^{\bar{\theta}} Q(\theta)\; f(\theta) d\theta$$

- Government budget balances :

$$\int_{\underline{\theta}}^{\bar{\theta}} \; T(\theta\ell(\theta))\; f(\theta) d\theta = 0$$

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- Government budget balances :

$$\int_{\underline{\theta}}^{\bar{\theta}} T(\theta\ell(\theta))\; f(\theta)d\theta = 0$$

Exercise : show that the government budget balance condition is redundant.

Mirrlees : static optimal non-linear taxation

2. Define an equilibrium given a tax policy

The equilibrium features :

$$v'(\ell(\theta)) = \theta \left(1 - T'(\theta\ \ell(\theta))\right)$$

Exercise :

- Suppose that there is no tax $T(x) = 0$. Does the first welfare theorem apply ?
- What is the role of policy in this economy ?
- If the government wanted to lump-sum tax type $\bar{\theta}$ to redistribute to type $\underline{\theta}$, could it do it ? Why not ?

Mirrlees : static optimal non-linear taxation

3. Write government's problem

$$\max_{T(\cdot)} \int_{\underline{\theta}}^{\bar{\theta}} \left(C(\theta) - v\left(\frac{Q(\theta)}{\theta}\right)\right) g(\theta)d\theta$$

s.t. $(C(\theta), Q(\theta))$ is an equilibrium allocation given $T(\cdot)$

Mirrlees : static optimal non-linear taxation

3. Write government's problem

$$\max_{T(\cdot)} \int_{\underline{\theta}}^{\bar{\theta}} \left(C(\theta) - v\left(\frac{Q(\theta)}{\theta}\right)\right) \; g(\theta) d\theta$$

s.t.

$$C(\theta), Q(\theta) \in \arg\max_{C,Q} \quad C - v\left(\frac{Q}{\theta}\right) \quad \text{s.t.} \quad C \leq Q - T(Q)$$

$$\int_{\underline{\theta}}^{\bar{\theta}} C(\theta) \; f(\theta) d\theta = \int_{\underline{\theta}}^{\bar{\theta}} Q(\theta) \; f(\theta) d\theta$$

Mirrlees : static optimal non-linear taxation

4. Use the Revelation Principle to rewrite the problem in **primal form**

$$\max_{C(\cdot),Q(\cdot)} \int_{\underline{\theta}}^{\bar{\theta}} \left(C(\theta) - v\left(\frac{Q(\theta)}{\theta}\right)\right) \, g(\theta) d\theta$$

s.t.

$$C(\theta) - v\left(\frac{Q(\theta)}{\theta}\right) \geq C(\hat{\theta}) - v\left(\frac{Q(\hat{\theta})}{\theta}\right) \qquad \text{(global IC)}$$

$$\int_{\underline{\theta}}^{\bar{\theta}} C(\theta) \, f(\theta) d\theta = \int_{\underline{\theta}}^{\bar{\theta}} Q(\theta) \, f(\theta) d\theta \quad \text{(market clearing)}$$

There are infinitely many Incentive Constraints. We use the theorem above to replace global IC with local IC.

Mirrlees : static optimal non-linear taxation

5. Use the theorems that characterize IC to solve the problem

Check that the Spence-Mirrlees condition is satisfied.
From the theorem above :

$$C(\theta) - v\left(\frac{Q(\theta)}{\theta}\right) \geq C(\hat{\theta}) - v\left(\frac{Q(\hat{\theta})}{\theta}\right) \qquad \text{(global IC)}$$

if and only if

$$C'(\theta) = v'\left(\frac{Q(\theta)}{\theta}\right) \frac{Q'(\theta)}{\theta} \qquad \text{(local IC 1)}$$

$$Q'(\theta) \geq 0 \qquad \text{(local IC 2)}$$

Mirrlees : static optimal non-linear taxation

5. Use the theorems that characterize IC to solve the problem

$$\max_{C(\cdot),Q(\cdot)} \int_{\underline{\theta}}^{\bar{\theta}} \left(C(\theta) - v\left(\frac{Q(\theta)}{\theta}\right)\right) g(\theta) d\theta$$

s.t.

$$C'(\theta) = v'\left(\frac{Q(\theta)}{\theta}\right) \frac{Q'(\theta)}{\theta} \qquad \text{(local IC 1)}$$

$$Q'(\theta) \geq 0 \qquad \text{(local IC 2)}$$

$$\int_{\underline{\theta}}^{\bar{\theta}} C(\theta)\, f(\theta) d\theta = \int_{\underline{\theta}}^{\bar{\theta}} Q(\theta)\, f(\theta) d\theta \qquad \text{(market clearing)}$$

Mirrlees : static optimal non-linear taxation

5. Use the theorems that characterize IC to solve the problem

To solve the problem in primal form :

i) Ignore local IC 2 and solve a **relaxed primal problem**

ii) Integrate local IC 1 by parts so that $C(\theta)$ can be substituted in the objective and in the market clearing constraint

iii) Substitute $C(\theta)$ in the market clearing constraint and integrate by parts

iv) Substitute $C(\theta)$ in the objective and integrate by parts

v) Substitute market clearing in the objective

vi) Maximize point-wise

vii) Decentralize by rewriting FOC as a tax formula

viii) Verify that IC 2 is not violated although we ignored it : the solution to the relaxed primal problem also solves the primal problem

Mirrlees : static optimal non-linear taxation

5. Use the theorems that characterize IC to solve the problem

i) Ignore local IC 2

Relaxed primal problem :

$$\max_{C(\cdot),Q(\cdot)} \int_{\underline{\theta}}^{\bar{\theta}} \left(C(\theta) - v\left(\frac{Q(\theta)}{\theta}\right)\right) g(\theta)d\theta$$

s.t.

$$C'(\theta) = v'\left(\frac{Q(\theta)}{\theta}\right) \frac{Q'(\theta)}{\theta} \qquad \text{(local IC 1)}$$

$$\int_{\underline{\theta}}^{\bar{\theta}} C(\theta)\ f(\theta)d\theta = \int_{\underline{\theta}}^{\bar{\theta}} Q(\theta)\ f(\theta)d\theta \qquad \text{(market clearing)}$$

Mirrlees : static optimal non-linear taxation

5. Use the theorems that characterize IC to solve the problem

ii) Integrate local IC 1 by parts so that $C(\theta)$ can be substituted in the objective and in the market clearing constraint

$$
\begin{aligned}
C'(\theta) &= \left.\frac{dv\left(\frac{Q(\theta)}{\hat{\theta}}\right)}{d\theta}\right|_{\hat{\theta}=\theta} = v'\left(\frac{Q(\theta)}{\theta}\right)\ \frac{Q'(\theta)}{\theta} \\
&= \theta v'\left(\frac{Q(\theta)}{\theta}\right)\ \frac{Q'(\theta)}{\theta^2} \\
&= \frac{dv\left(\frac{Q(\theta)}{\theta}\right)}{d\theta} + v'\left(\frac{Q(\theta)}{\theta}\right)\frac{Q(\theta)}{\theta^2}
\end{aligned}
$$

After integrating from $\underline{\theta}$ to θ, we get :

$$
C(\theta) - v\left(\frac{Q(\theta)}{\theta}\right) = A + \int_{\underline{\theta}}^{\theta} v'\left(\frac{Q(t)}{t}\right)\frac{Q(t)}{t^2}\ dt
$$

where $A = C(\underline{\theta}) - v\left(\frac{Q(\underline{\theta})}{\underline{\theta}}\right)$

Mirrlees : static optimal non-linear taxation

5. Use the theorems that characterize IC to solve the problem

iii) substitute $C(\theta)$ in the market clearing constraint

$$\int_{\underline{\theta}}^{\bar{\theta}} \left(A + v\left(\frac{Q(\theta)}{\theta}\right) + \int_{\underline{\theta}}^{\theta} v'\left(\frac{Q(t)}{t}\right) \frac{Q(t)}{t^2}\, dt \right) f(\theta)d\theta = \int_{\underline{\theta}}^{\bar{\theta}} Q(\theta)\; f(\theta)d\theta$$

which we rewrite

$$A + \int_{\underline{\theta}}^{\bar{\theta}} \left(v\left(\ell(\theta)\right) + \int_{\underline{\theta}}^{\theta} v'\left(\ell(t)\right) \frac{\ell(t)}{t}\, dt \right) f(\theta)d\theta = \int_{\underline{\theta}}^{\bar{\theta}} \theta\; \ell(\theta)\; f(\theta)d\theta$$

iii) integrate the market clearing constraint by parts to get rid of the nested integrals

$$A = \int_{\underline{\theta}}^{\bar{\theta}} \left(\left(\theta \ell(\theta) - v\left(\ell(\theta)\right)\right) f(\theta) - \frac{\ell(\theta)}{\theta} v'\left(\ell(\theta)\right)\left(1 - F(\theta)\right) \right) d\theta$$

Mirrlees : static optimal non-linear taxation

5. Use the theorems that characterize IC to solve the problem

iv) substitute $C(\theta)$ in the government objective and integrate by parts

$$\int_{\underline{\theta}}^{\bar{\theta}} \left(C(\theta) - v\left(\frac{Q(\theta)}{\theta}\right)\right) \, g(\theta)d\theta$$
$$= \int_{\underline{\theta}}^{\bar{\theta}} \left(A + \int_{\underline{\theta}}^{\theta} v'\left(\ell(t)\right) \frac{\ell(t)}{t} \, dt\right) \, g(\theta)d\theta$$

Again, integrating by parts, we get :

$$A + \int_{\underline{\theta}}^{\bar{\theta}} (1 - G(\theta)) \frac{\ell(\theta)v'(\ell(\theta))}{\theta} d\theta$$

Mirrlees : static optimal non-linear taxation

5. Use the theorems that characterize IC to solve the problem

iv) the relaxed primal problem now is :

$$\max_{\ell(\cdot)} A + \int_{\underline{\theta}}^{\bar{\theta}} (1 - G(\theta)) \frac{\ell(\theta) v'(\ell(\theta))}{\theta} d\theta$$

s.t.

$$A = \int_{\underline{\theta}}^{\bar{\theta}} \left((\theta \ell(\theta) - v\left(\ell(\theta)\right)) \; f(\theta) - \frac{\ell(\theta)}{\theta} v'\left(\ell(\theta)\right)(1 - F(\theta)) \right) d\theta$$

We can substitute the constraint in the objective

Mirrlees : static optimal non-linear taxation

5. Use the theorems that characterize IC to solve the problem

v) substitute market clearing in the objective

$$\max_{\ell(\cdot)} \int_{\underline{\theta}}^{\bar{\theta}} \left(\left(\theta\ell(\theta) - v\left(\ell(\theta)\right)\right) \ f(\theta) - \left(G(\theta) - F(\theta)\right) \frac{\ell(\theta) v'(\ell(\theta))}{\theta} \right) d\theta$$

Define the **virtual surplus** as :

$$\underbrace{\left(\theta\ell(\theta) - v\left(\ell(\theta)\right)\right)}_{\text{surplus}} \underbrace{- \frac{\left(G(\theta) - F(\theta)\right)}{f(\theta)} \frac{\ell(\theta) v'(\ell(\theta))}{\theta}}_{\text{effect of incentive constraints}}$$

vi) maximize the virtual surplus point-wise **virtual surplus** as :

$$(\theta\ell(\theta) - v(\ell(\theta))) \underbrace{- \frac{(G(\theta) - F(\theta))}{f(\theta)} \frac{\ell(\theta)v'(\ell(\theta))}{\theta}}_{\text{effect of incentive constraints}}$$

FOC from maximizing point-wise gives :

$$\theta - v'(\ell(\theta)) = \frac{(G(\theta) - F(\theta))}{f(\theta)\ \theta} (v''(\ell(\theta))\ \ell(\theta) + v'(\ell(\theta)))$$

Mirrlees : static optimal non-linear taxation

6. Decentralize : recover the optimal non-linear tax

Let $\epsilon \equiv \frac{v'(\ell)}{\ell v''(\ell)}$ denote the wage[2]-elasticity of labor supply since

$$\frac{\ell'(\theta)}{\ell(\theta)}\theta = \frac{v'(\ell(\theta))}{\ell(\theta)v''(\ell(\theta))} \ .$$

Use FOC of consumer problem in an equilibrium with non-linear taxes to recover optimal tax formula from the equation that characterizes the optimal allocation

$$v'(\ell(\theta)) = \theta \ \left(1 - T'(\theta \ \ell(\theta))\right)$$

2. Recall that each worker is paid its marginal product so the wage rate equals the type. Stantcheva (2014) ReStud studies the case where firms do not observe workers productivities.

$$\frac{T'}{1-T'} = \left(1+\frac{1}{\epsilon}\right) \frac{1-F}{\theta f} \frac{G-F}{1-F}$$

Exercise : what have we learnt from this optimal tax formula ?

1. Identify each of the three terms on the right hand side
2. How does the marginal tax rate depend on the elasticity of labor supply ?
3. How does the marginal tax rate depend on the elasticity of the tail of the distribution of skills ?
4. How does the marginal tax rate depend on the redistribution motive ?

$$\frac{T'}{1-T'} = \left(1+\frac{1}{\epsilon}\right)\ \frac{1-F}{\theta f}\ \frac{G-F}{1-F}$$

1. Identify each of the three terms on the right hand side

- Higher $\frac{T'}{1-T'}$ is equivalent to a higher *marginal* tax rate T'. The tax paid for a wage $\theta\ell(\theta)$ is $T(\theta\ell(\theta)) = \int_{\underline{\theta}\ell(\underline{\theta})}^{\theta\ell(\theta)} T'(w)dw$.
- $\left(1+\frac{1}{\epsilon}\right)$ is an inverse measure of the wage-elasticity of labor supply.
- $\frac{1-F}{\theta f}$ is a measure of the thickness of the distribution of skills, also a measure of inequality. It also shows how private information alters the design of optimal taxation. Since the exogenous source of heterogeneity/inequality, namely skills, is private information, taxation is a function of income, not a function of skills.
- $\frac{G-F}{1-F}$ is a measure of the strength of the redistribution motive modelled in the social welfare function.

Mirrlees : static optimal non-linear taxation

Lessons from the optimal non-linear tax

$$\frac{T'(\theta\ \ell(\theta))}{1 - T'(\theta\ \ell(\theta))} = \left(1 + \frac{1}{\epsilon(\ell(\theta))}\right) \frac{1 - F(\theta)}{\theta f(\theta)} \frac{G(\theta) - F(\theta)}{1 - F(\theta)}$$

2. The marginal tax depend on the elasticity of labor supply
   - If $v(\ell) = \frac{\ell^{1+\frac{1}{\bar{\epsilon}}}}{1+\frac{1}{\bar{\epsilon}}}$, the elasticity is constant : $\epsilon(\ell) \equiv \frac{v'(\ell)}{\ell v''(\ell)} = \bar{\epsilon}$
   - Take two countries i and j that differ only by their elasticity the labor supply $\epsilon_i > \epsilon_j$. The optimal marginal tax rate is lower in country i than in country j. The reason is that the loss in efficiency from taxation is higher in country i than in country j.
   - This is a standard economic force in optimal taxation. For instance, the optimal tax is higher on goods with a more inelastic demand because the deadweight loss from taxation is higher the more elastic is the demand.

Mirrlees : static optimal non-linear taxation

Lessons from the optimal non-linear tax

$$\frac{T'(\theta\ \ell(\theta))}{1-T'(\theta\ \ell(\theta))} = \left(1+\frac{1}{\epsilon(\ell(\theta))}\right)\ \frac{1-F(\theta)}{\theta f(\theta)}\ \frac{G(\theta)-F(\theta)}{1-F(\theta)}$$

3. How does the marginal tax rate depend on the elasticity of the tail of the distribution of skills ?

- $\frac{1-F(\theta)}{\theta f(\theta)}$ is the inverse of the elasticity of the tail $1-F$
- We solved a relaxed problem without the monotonicity condition IC2. This is a solution to the problem with IC2 if the allocation is monotonic.
- If T' is non-increasing, the allocation is monotonic as can be seen from the household FOC, where v' is increasing since v is convex : $v'(\ell(\theta)) = \theta\ \left(1-T'(\theta\ \ell(\theta))\right)$

$$\frac{T'(\theta\ \ell(\theta))}{1-T'(\theta\ \ell(\theta))} = \left(1+\frac{1}{\epsilon(\ell(\theta))}\right)\ \frac{1-F(\theta)}{\theta f(\theta)}\ \frac{G(\theta)-F(\theta)}{1-F(\theta)}$$

3. (continued)

- All else equal, T' is non-increasing if $\frac{1-F(\theta)}{\theta f(\theta)}$ is non-increasing. A sufficient condition for this is that the distribution is log-concave (cf Bagnoli and Bergstrom (2005)).
- $\frac{1-F(\theta)}{\theta f(\theta)}$ measures the thickness of the tail of the distribution of skills. The thicker is the tail, the more unequal is the society. The optimal marginal tax rate increases with the degree of inequality.

Mirrlees : static optimal non-linear taxation

Lessons from the optimal non-linear tax

$$\frac{T'(\theta\ \ell(\theta))}{1-T'(\theta\ \ell(\theta))} = \left(1+\frac{1}{\epsilon(\ell(\theta))}\right)\ \frac{1-F(\theta)}{\theta f(\theta)}\ \frac{G(\theta)-F(\theta)}{1-F(\theta)}$$

4. How does the marginal tax rate depend on the redistribution motive ?

- The marginal tax rate is null at the lowest realization of skills since $G(\underline{\theta}) = F(\underline{\theta}) = 0$
- For a utilitarian planner, the welfare weight is constant so $F = G$. The optimal (marginal) tax is zero.
- A formal way to say that a society puts more welfare weight on low realizations of skills than on high realization of skills is that F First Order Stochastically Dominates G. In this case, $G(\theta) \geq F(\theta)$ and the marginal tax rate is positive.

Sufficient Statistics approach

"The sufficient statistics approach provides formulas for welfare/revenue consequences of policies that are functions of high-level elasticities rather than deep primitives".

Chetty (2009) "Sufficient Statistics for Welfare Analysis : A Bridge Between Structural and Reduced-Form Methods" *Annual Review of Economics* (paper is on StudiUM).

Sufficient statistics approach : estimate ϵ and take it as a constant. In this model, it is like assuming that v is CRRA.

- what is ϵ in the data ?
- what is the distribution F ?
- what about G ?

Further readings

Further readings to get closer to the research frontier on this topic :

- Werning (2007) “Pareto Efficient Income Taxation”
- Heathcote Tsujiyama (2020) “Optimal Income Taxation : Mirrlees Meets Ramsey”