Master's thesis report: Entropy numbers of nonlinear systems

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Abstract

A nonlinear system is a function mapping input signals to output signals. An important problem in many fields, called *system identification*, is to recover the full system given its behaviour on only a finite set of inputs. In machine learning terms: learn the mapping between input and output signals, given a finite dataset of observations.

A crucial tool in learning theory is metric entropy, which essentially quantifies the difficulty of approximating a set by a finite number of "prototypical" objects. For classical regression or classification, learnability guarantees can be derived by estimating the metric entropy of a function set. Similarly, metric entropy estimates for sets of nonlinear systems would open the path to a theory of system identification with guarantees.

In this thesis, we take first steps towards that goal, by extending techniques for estimating metric entropy in function spaces to spaces of systems. We do this by leveraging explicit general parametric models such as Volterra or Wiener series (generalizing the convolution representation of LTI systems), or by viewing mappings between signals as generalizations of mappings between scalars. We focus on signals that are continuous time functions; this is well-suited for traditional system identification, e.g signal processing or automatic control.

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Contents

Notation

Generic notations

$\mathbb{S}, \mathbb{S}_+$	A vector space of systems $S : X \to \mathcal{Y}$ or $\mathcal{X} \to \mathcal{Y}$ / A set of systems of interest $\mathbb{S}_+ \subset \mathbb{S}$
X	A set of input signals x
$\mathcal{X}, \mathcal{X}(U)$	A vector space of input signals x / A vector space of input signals $x(u)$ over a domain U
$\mathcal{Y}, \mathcal{Y}(V)$	A normed vector space of output signals y / A normed vector space of output signals $y(v)$ over a domain V
$\mathcal{F}, \mathcal{F}+$	A vector space of (scalar-domain, scalar-valued) functions f / A set of functions of interest $\mathcal{F}_+ \subset \mathcal{F}$
$\mathcal{T}$	A domain for generic functions f, e.g $\mathcal{T} \subset \mathbb{R}$ or $\mathbb{R}^d$
$\mathcal{K}, \mathcal{K}_+$	A vector space of kernel functions k / A set of kernel functions of interest
Φ_k	The system deriving from the kernel function k
$\lVert \cdot \rVert_{\infty X}$	The worst-case error norm over X: $\lVert S \rVert_{\infty X} = \sup_X \lVert S[x] \rVert_{\mathcal{Y}}$

Volterra and Wiener series

Φ_{k_n}, Φ_k	Volterra monomial of order n with Volterra kernel k_n / Volterra series with Volterra kernels $k = (k_1, k_2, ...)$
$H_n[k_n; \cdot]$	Volterra monomial of order n with Volterra kernel k_n
$f_k(z)$	Gain bound function of the sequence of Volterra kernels k
$G_n[k_n; \cdot]$	Wiener G-functional of order n with leading Wiener kernel k_n

Entropy numbers

$N_\varepsilon(A; R), N_\varepsilon^R(A)$	ε-covering number of the set A in ambient metric space R
$\log N_\varepsilon(A; R)$	ε-entropy a.k.a metric entropy of the set A in ambient metric space R
$M_\varepsilon(A)$	ε-packing number of the metric space A
$D_\varepsilon(A; R), \mathcal{A}_\varepsilon(A; R)$	ε-dimension / ε-constructive-dimension of the set A in ambient Banach space R
$\varepsilon_n(A; R), e_n(A; R)$	Entropy number / Dyadic entropy number of the set A in ambient metric space R
$d_n(A; R), a_n(A; R)$	Kolmogorov number / Approximation number of the set A in ambient Banach space R
$\varepsilon_n(T), e_n(T)$	Entropy number / Dyadic entropy number of the linear operator T
$d_n(T), a_n(T)$	Kolmogorov number / Approximation number of the linear operator T

Band-limited square-integrable functions

$\hat{f}(\xi), \mathcal{F}[f](\xi)$	The Fourier transform of $f(t)$
$\mathbb{B}_B, \mathbb{B}_{\boldsymbol{B}}, \mathbb{B}_K$	The subset of $L^2(\mathbb{R})$ (resp. of $L^2(\mathbb{R}^N)$) consisting of functions f such that $\hat{f}$ is supported on $[\pm B]$ / (resp. supported on $[\pm \boldsymbol{B}]$ / supported on a compact K of $\mathbb{R}^N$)
$\Psi_0(t)$	Generic notation for a "(B, B')-synthesizer function" i.e such that $\forall f \in \mathbb{B}_B, f(t) = \sum_{n\in\mathbb{Z}} \frac{1}{2B'} f\left(\frac{n}{2B'}\right) \Psi_0\left(t - \frac{n}{2B'}\right)$

Multivariate shorthands

$t, \boldsymbol{t}$	A variable in $\mathbb{R}$ / A variable in $\mathbb{R}^n$ (or $\mathbb{R}^d$ or $\mathbb{R}^N$ according to context)
$\mathbf{1}_n$	The vector $(1, ..., 1)^T \in \mathbb{R}^n$
$\mathfrak{S}_n$	The set of permutations σ over $\{1, ..., n\}$
$\boldsymbol{\tau}_\sigma$	The vector obtained by permuting the entries of the vector $\boldsymbol{\tau} \in \mathbb{R}^n$: $\boldsymbol{\tau}_\sigma = (\tau_{\sigma(1)}, ..., \tau_{\sigma(n)})^T$
Sym	The symmetrization operator: $\mathrm{Sym}\, f(\boldsymbol{\tau}) = \frac{1}{n}\sum_{\sigma\in\mathfrak{S}_n} f(\boldsymbol{\tau}_\sigma)$
$f^\times(\boldsymbol{t}), f^{\times n}(\boldsymbol{t})$	For a univariate (scalar-valued) function $f(t)$, the product multivariate (scalar-valued) function $f^{\times n}(\boldsymbol{t}) = f(t_1)...f(t_n)$
$[\pm B], [\pm \boldsymbol{B}]$	The interval $[-B, B]$ / The hyperrectangle $[-B_1, B_1] \times ... \times [-B_n, B_n]$
$\boldsymbol{a} \odot \boldsymbol{b}, \boldsymbol{a}/\boldsymbol{b}$	Coordinate-wise multiplication / Coordinate-wise division
$L^p_{\mathrm{Sym}}(\mathbb{R}^n)$	The subset of $L^p(\mathbb{R}^n)$ consisting of permutation-invariant functions: $f(\boldsymbol{\tau}_\sigma) = f(\boldsymbol{\tau})$, also called symmetric functions in this thesis

Continuous functions and mappings

$C(\mathcal{T}; \mathbb{R}), C(\mathcal{T})$	The (Banach) space of real-valued continuous functions f over $\mathcal{T}$ (equipped with the sup norm, when $\mathcal{T}$ is compact)
$C_b(\mathbb{R}), C_0(\mathbb{R})$	The Banach space of real-valued continuous bounded functions over $\mathbb{R}$ / vanishing functions f i.e such that $\lim_{\pm\infty} f = 0$, equipped with the sup norm
$C_{b,w}(\mathbb{R}), C_{0,w}(\mathbb{R})$	For a valid weight function w (vanishing and $]0,1]$-valued), $C_{b,w}(\mathbb{R}) = w^{-1}C_b(\mathbb{R})$ and $C_{0,w}(\mathbb{R}) = w^{-1}C_0(\mathbb{R})$, equipped with the norm $\|x\|_w = \|wx\|_{L^\infty}$
$ba(\mathbb{R}), rca(\mathbb{R})$	The space of bounded finitely-additive measures over $\mathbb{R}$ / The space of regular bounded countably-additive measures over $\mathbb{R}$
$\omega_t(f;\delta), \omega_{\mathcal{T}}(f;\delta)$	Modulus of continuity at point t / Modulus of uniform continuity over $\mathcal{T}$ ($\mathcal{T}$ may be implicit)
$\omega_t(\mathcal{F}_+;\delta), \omega_{\mathcal{T}}(\mathcal{F}_+;\delta)$	Modulus of equicontinuity at point t / Modulus of uniform equicontinuity over $\mathcal{T}$ ($\mathcal{T}$ may be implicit)
$C(X; \mathcal{Y})$	The space of systems that are continuous from X to $\mathcal{Y}$, for some topology on X
δ_0, δ_t	The Dirac delta measure / The t-delayed Dirac delta measure: $\delta_t(f) = f(t)$

Banach spaces

$\mathcal{L}_b(E, F)$	The Banach space of bounded linear operators between Banachs E and F
$\|\|\|\cdot\|\|\|, \|\|\|\cdot\|\|\|_{E\to F}$	The operator norm / The operator norm over linear operators from E to F
E'	The topological dual space of the Banach E
$\langle \cdot, \cdot \rangle_E$	The duality bracket: $\langle x, X\rangle_E = X(x)$ for $x \in E$ and $X \in E'$
$B^{(E)}_{x,r}, B^{(E)}$	The ball of E of radius r centered at x / The centered unit ball of E
$L^p(\Omega; G)$	The p-th Bochner space from the measure space Ω to the Banach G
id, I	The identity mapping / The embedding operator of a Banach space into another

Chapter 0

Introduction

0.1 Informal motivation

Signal-to-signal tasks Theoretical machine learning research so far has focused on rather classical settings such as classification or regression of finite-dimensional vector inputs. By contrast, many applications are more naturally stated as signal-to-signal tasks, where we want to learn a function mapping signals to signals: in natural language processing, translation or summarization; in computer vision, super-resolution.

Thus, we are led to consider settings where the inputs and outputs are "*signals*", which we define informally as indexed collections of scalars: $(x_t) \in \mathbb{R}^{\mathcal{T}}$ for some arbitrary, possibly infinite or continuous, index set $\mathcal{T}$. Mappings from an input signal space to an output signal space are called *(nonlinear) systems*.

Admittedly, sentences and images (and indeed any other type of data) can be mapped into a vector representation, respectively via one-hot-encoding and sampling; the specificities of the input data then translate into properties of the vector representation (sparsity in some basis, smoothness, located on a manifold...). So since the inputs can be vectorized while preserving structure, it could be argued that assuming vector data is without loss of generality. However, for example, sampling a continuous signal may lead to aliasing issues; so one could imagine tasks where vectorization does lead to loss of generality. Moreover, settings involving unbounded domains, e.g infinite time-series, are not captured by analyses requiring vector inputs of fixed dimension.

Besides, working directly in a signal-to-signal framework may be fruitful from a theoretical perspective, as it enables directly applying tools from other fields such as signal processing. A similar situation arises with "discretize-optimize" analyses in optimization and scientific computing [Asc20].

Nonlinear system identification, inverse problems An important problem in any field studying systems (be they electrical, economic, mechanical, or environmental) is that of recovering the full behaviour of a system given its behaviour on only a small (e.g finite) set of inputs, usually referred to as the *system identification* problem. That is to say, in machine learning terminology: learning a mapping between input and output signals given only a finite dataset of observations.

The study of nonlinear systems is notoriously difficult, especially when given little prior knowledge. This stands in sharp contrast to linear systems, which are completely characterized by their impulse response function. Yet under certain conditions, one can approximately represent nonlinear systems by explicit formulas, e.g Volterra series. These explicit models provide a possible approach to nonlinear system identification [Sch81] [FS06]. To get quantitative theoretical guarantees, it remains to specify the conditions for such approximate formulas to hold, as well as approximation error bounds.

Metric entropy Two important topics in machine learning are that of *expressivity* of a given model class (e.g what functions can be represented by a L-layer, W-wide neural network), and PAC-*learnability* of a given instance class by a given hypothesis class (i.e how many i.i.d samples are required for the learned model to have a probability of error less than ε for a new sample).

In the context of nonlinear systems, both topics are subsumed under the term "system identification". Nonetheless we can distinguish the two corresponding questions: firstly, how well can one represent a set of systems using a given approximation scheme? secondly, how to obtain such representations in practice, with statistically low error when given a finite dataset of samples?

Consider the first topic – expressivity of machine learning models. A possible formalization is:

> Suppose given a class of functions $\mathcal{C}$ to approximate, a norm $\|\cdot\|$, and a set of models $\mathcal{M}_\ell$ of complexity bounded by ℓ. Estimate the worst-case oracle approximation error, $\sup_{f\in\mathcal{C}} \inf_{\hat{f}\in\mathcal{M}_\ell} \left\|f - \hat{f}\right\|$, as a function of ℓ.

The "complexity" bounded by ℓ may be, for example, the depth and maximal layer width of a neural network. An alternative formalization is:

> Suppose given $\mathcal{C}$, $\|\cdot\|$ and $\mathcal{M}$ (of unbounded complexity). Consider encoder-decoder schemes $E : \mathcal{C} \to \{0,1\}^\ell$, $D : \{0,1\}^\ell \to \mathcal{M}$. Estimate the best-obtainable worst-case error, $\inf_{E,D} \sup_{f\in\mathcal{C}} \|f - D(E(f))\|$, as a function of ℓ.

Here $\mathcal{M}$ does not incorporate a complexity bound; for example if $\mathcal{C} \subset L^2(\Omega)$, $\mathcal{M}$ may be taken as the entire space $L^2(\Omega)$. In a sense, the latter formalization is a special case of the former, when the complexity ℓ is measured as the minimum bitstring length required to store the model.

It turns out that these two formalizations are strongly related, and related to the notion of metric entropy [Elb+20]. Essentially, metric entropy quantifies the difficulty of approximating a set of objects $\mathcal{C}$ by a finite number of "prototypical" objects. In the above paragraph, the objects to be approximated were functions, but we may just as well take $\mathcal{C}$ to be a set of nonlinear systems.

Scope of this thesis Thus, a natural first step towards studying the approximability of sets of nonlinear systems is to estimate their metric entropy, which is what we set out to do in this thesis. We will restrict our attention to this first step, as it is sufficiently rich on its own. Furthermore, metric entropy is an essential building block in learning theory [CS01] [Wai19, chapter 5], so that its use allows results not only on expressivity but also on learnability.

We will focus on signals that are continuous time functions, e.g $x(t) \in C(\mathbb{R})$ the space of continuous functions over $\mathbb{R}$, or $L^p(\mathbb{R})$ the space of L^p-integrable functions over $\mathbb{R}$. This is well-suited for system identification in the traditional engineering sense, e.g for signal processing or automatic control. We expect that the insights developed here would be useful to tackle other settings as well (such as text-to-text tasks) – a key feature of our thesis is that the inputs and outputs typically live in some Banach space, rather than a finite-dimensional space.

0.2 Warning about ambiguity of the title

Entropy numbers refer to the same notion as metric entropy, only measured differently (the former are essentially the inverse function of the latter). According to the situation, one or the other may be more practical for calculations.

The term "entropy number" is often used in the context of linear operator approximation theory. The "entropy number of a linear operator" T between Banach spaces E and F is the entropy number of $T(B^{(E)})$, the image of the unit ball by T. We emphasize that estimating the entropy number of a given linear operator is a completely different question than estimating the entropy number of a given set of systems.

The risk of confusion is all the higher because linear systems can effectively be considered as linear operators, and because some papers refer to nonlinear systems as nonlinear operators.

0.3 Outline of this thesis

0.3.1 Two paths: "parametric" and "abstract"

On a high level, we explored two distinct paths towards our goal of estimating metric entropy of sets of nonlinear systems. Both of them rely on extending existing techniques for metric entropy estimates in function spaces.

The "parametric" path Leverage explicit general models, whereby a system is characterized by (a sequence of) so-called kernel functions. As a classical example, a linear time-invariant (LTI) system is characterized by its impulse response function, a.k.a convolution kernel. For nonlinear systems, a similar explicit model is given by Volterra series characterized by Volterra kernels, and by Wiener series characterized by Wiener kernels.

In machine learning terms, to *model the unknown system by an explicit LTI/Volterra/Wiener series model* is to *fit a parametric model to the unknown mapping*, where the parameters are the convolution/Volterra/Wiener kernel functions. (Of course those parameters are highly infinite-dimensional since they are themselves functions.)

Morally, to approximate a system, it suffices to model it by a Volterra series (for example) and to approximate the Volterra kernels. To follow this idea, it is necessary to check:

- What error is committed by the modeling step. As it turns out, for the formalism of metric entropy estimation, the modeling error is mostly irrelevant since Volterra series have good approximation properties. We still want to check for what class of systems that simplification holds, i.e what systems are arbitrarily-well approximated by that model;
- What normed space of functions the Volterra kernels lie in;
- How quantitative approximation bounds for the Volterra kernels translate into quantitative approximation bounds for the Volterra series.

Thus upper-bounding metric entropy for sets of nonlinear systems can be reduced to upper-bounding metric entropy in the space of the Volterra kernels, which is a function space. The same idea would of course hold for other models than Volterra series.

chapter 3, 4 and 5 discuss the above items for the case of linear systems, Volterra series and Wiener series respectively. chapter 6 formalizes how to estimate metric entropy using explicit models, using the above idea.

The "abstract" path Nonlinear systems can be viewed simply as functions mapping input signals to output signals. As such, the basic techniques for estimating metric entropy in (scalar-domain, scalar-valued) function spaces can be expected to apply as well – provided that the input and output spaces are well-behaved, which is the case in the settings we considered (typically the input set X is a metric space and the output space $\mathcal{Y}$ is a Banach).

We review techniques and results on metric entropy in function spaces in chapter 2, and illustrate the technique of "sampling under smoothness assumptions" on continuous nonlinear systems in chapter 7.

0.3.2 Detailed outline by chapter; technical contributions

In chapter 1, we define the problem formally: we define metric entropy and entropy numbers as well as other related quantities, state their generic properties, and introduce a framework that will be used throughout this thesis. All of the subsequent chapters can, for the most part, be read independently (with the exception of section 4.4 which is also reused in several subsequent places).

In chapter 2 we review techniques for estimating the metric entropy of sets of (scalar-domain, scalar-valued) functions. For the purpose of estimating entropy numbers of nonlinear systems, this chapter is useful on three different levels. 1) For systems deriving from kernels, the problem reduces to estimating metric entropy in the space of the kernel functions. 2) For continuous systems over a compact set of signals X, results can be obtained that involve the metric entropy of X. 3) Nonlinear systems are simply mappings between signal spaces, so they can be seen as a generalization of functions from $\mathbb{R}$ to $\mathbb{R}$ (whereby the signals are actually scalars); so the proof techniques that apply for functions can be generalized to nonlinear systems.

In chapter 3, we study the case of linear systems, which are "well-known" to be characterized by their impulse responses; in particular we seek to make that statement mathematically rigorous under various settings. In chapter 4, we introduce the notion of Volterra series and thoroughly examine its definition, continuity and approximation properties, and relation to polynomial reproducing kernels. In chapter 5, we present the closely related notion of Wiener series, which have good approximation properties for a specific type of stochastic inputs.

All three of these explicit models have in common that they are characterized by kernel functions, allowing nice mathematical analyses, which we formalize in chapter 6. In chapter 7 we analyse the important case of systems that are continuous over a compact input signal set (for a metric to be specified).

Finally in chapter 8, we point out directions for future work and conclude.

Additionally, a summary of notations (roughly ordered by increasing commonness) can be found at the beginning of the report, before the introduction. Appendices on the sampling expansion of band-limited square-integrable functions (appendix A, B, C), as well as on reminders on Banach spaces (appendix D), are included at the end of the report.

Technical contributions Here we indicate what we find are this report's most important technical contributions. This paragraph is not meant to summarize the report's contents.

section 1.2 We show that the entropy number of a set can, under reasonable assumptions, be interpreted as the entropy number of an operator between Banach spaces. This bridge allows to directly translate results from operator approximation theory into metric entropy estimates.

section 2.2, 2.3 We illustrate how the proofs of fundamental theorems in functional analysis characterizing relatively compact sets, Arzela-Ascoli and Kolmogorov-Riesz theorems, can be refined into quantitative upper-bounds on approximation quantities and metric entropy.

section 3.3 We discuss the conditions under which a linear (resp. linear time-invariant) system can be represented as a kernel-integral (resp. convolution) operator, and the space in which the kernel function lies. We relate the operator norm of the linear system to the norm of the kernel function.

section 4.4 We clarify how studying time-invariant systems reduces to studying scalar-valued functionals, under natural assumptions on the input signal set and output signal space.

section 4.6 We show in what sense Volterra series are the elements of a polynomial reproducing kernel Banach space (RKBS), going beyond the discrete-time finite-memory setting of [FS06].

section 7.1 We notice that the Banach-valued Arzela-Ascoli theorem can be simplified: the "pointwise equicompactness" condition can be replaced by uniform equicompactness.

appendix B We explore how far the Nyquist-Shannon sampling expansion can be extended (without additional ideas). We characterize the functions that can be used for shift-invariant reconstruction from a multivariate band-limited function's samples.

Chapter 1

Problem statement

1.1 Metric entropy and entropy numbers

1.1.1 ε-coverings, ε-packings, metric entropy

Definition 1.1 ([KT59])**.** Let A a subset of a metric space (R, d), and $\varepsilon > 0$.

- A collection of points $(p_i) \subset R$ is called an ε*-covering of* A *with prototypes in* R, if any $a \in A$ is ε-close to one of the p_i's:
$$\forall a \in A, \exists i;\ d(a, p_i) \leq \varepsilon \tag{1.1.1}$$
Equivalently: $A \subset \bigcup_i B_{p_i,\varepsilon}$ the union of closed p_i-centered balls of radius ε. [1]

- An ε*-self-covering* of A is an ε-covering consisting of prototypes p_i in A itself. Equivalently, it is an ε-covering of A with prototypes in the induced metric space (A, d) (instead of R).

- A collection of points $(a_i) \subset A$ is called an ε*-packing* of A if they are pairwise distant by more than ε:
$$\forall i \neq j,\ d(a_i, a_j) > \varepsilon \tag{1.1.2}$$

- The minimum cardinality of an ε-covering of A (with prototypes in R) is denoted $N_\varepsilon(A; (R, d))$ or $N_\varepsilon^{(R,d)}(A)$, and called the ε*-covering number of* A *in* R.

- The maximum cardinality of an ε-packing of A is denoted $M_\varepsilon(A; d)$, and called the ε*-packing number* of A.

- When d is clear from context, we simply denote $N_\varepsilon^R(A) := N_\varepsilon^{(R,d)}(A)$, $M_\varepsilon(A) := M_\varepsilon(A; d)$.

  When the ambient space R is clear from context, we simply denote $N_\varepsilon(A) := N_\varepsilon^R(A)$.

We stress that the notion of ε-covering depends on the ambient space R. For example, if $R = \mathbb{R}^n$ and A is the sphere of radius r, then it clearly has a r-covering of cardinality 1: just take $p_1 = 0_{\mathbb{R}^n}$. By contrast, if we consider the ambient space of A to be A itself (with the induced metric), then clearly there does not exist a r-covering of cardinality 1 (Figure 1.1) In other words: $N_\varepsilon^R(A) \neq N_\varepsilon^A(A)$ in general.

A related caveat: for any $A \subset B$, clearly $N_\varepsilon^R(A) \leq N_\varepsilon^R(B)$. However, it is not the case that $N_\varepsilon^A(A) \leq N_\varepsilon^B(B)$, in general. The same situation as above provides a counter-example: the sphere of radius r (A) is contained in the ball of radius r (B), yet the latter has a r-self-covering of cardinality 1 while the former requires > 1 elements to cover with prototypes in the set itself.

[1] What we call "ε-covering" is sometimes called "ε-net", with the former term being reserved for a slightly more general notion: $(U_\gamma) \subset \mathcal{P}(R)$ such that $A \subset \bigcup_\gamma U_\gamma$. The two notions coincide when R is centralizable [KT59, theorem V].

In this thesis, it will almost always be clear from context what the ambient space is. It is also possible to define the minimal covering number of A as $\inf_{R; A\subset R} N_\varepsilon^R(A)$, and that quantity has nice characterizations. However for our purposes it is more natural to work with an explicit R.

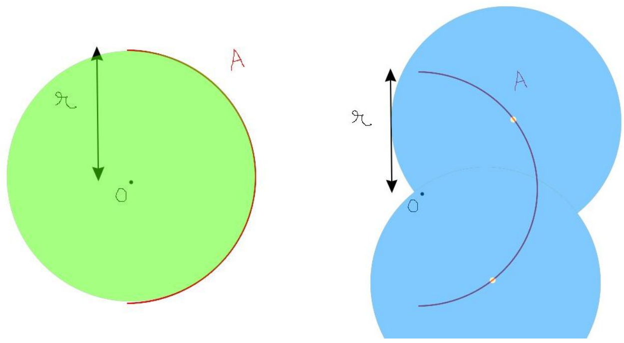


Figure 1.1: A is the curve drawn in red. Left: optimal r-covering of A with prototype in the ambient space. Right: optimal r-covering of A with prototypes in A itself.

Proposition 1.1 ([KT59, theorem IV])**.** For all $\varepsilon > 0$,

$$M_{2\varepsilon}(A) \le N_\varepsilon^R(A) \le N_\varepsilon^A(A) \le M_\varepsilon(A) \tag{1.1.3}$$

An important implication of this inequality is that to derive a lower-bound on the ε-covering number, is suffices to construct a 2ε-packing. Also, since $N_{2\varepsilon}^A(A) \le N_\varepsilon^R(A) \le N_\varepsilon^A(A)$, confusing covering- and self-covering-numbers is morally not too bad.

Note that so far, nothing guarantees that the covering and packing numbers are finite. The inequality above holds even when its terms are allowed to be infinite, which justifies the following definitions.

Definition 1.2. The metric space (A, d) is called a *totally bounded space* if it has a finite ε-(self)-covering for any $\varepsilon > 0$; equivalently, if it has a finite ε-packing for any $\varepsilon > 0$.

A is a totally bounded subset of (R, d) if it has a finite ε-covering with prototypes in R for any $\varepsilon > 0$; equivalently, if the induced metric space (A, d) is totally bounded.

Proposition 1.2. Suppose (R, d) is a complete metric space. Then $A \subset R$ is totally bounded if and only if it is *relatively compact*, i.e its closure in R is compact.

It turns out that it is often more convenient to work with the logarithm of the covering and packing numbers, which are given special names.

Definition 1.3. The *ε-entropy*, a.k.a *metric entropy*, of the set A with respect to R is $\log_2 N_\varepsilon^R(A)$.

The *ε-capacity* of A is $\log_2 M_\varepsilon(A)$.

The following characterization, which is easy to see from the definition, motivates the study of metric entropy in the context of approximation theory.

Proposition 1.3 ([Elb+20])**.** Let $\varepsilon > 0$. For any "bitstring length" $\ell \in \mathbb{N}$, consider encoders $E : A \to \{0,1\}^\ell$ and decoders $D : \{0,1\}^\ell \to R$. The metric entropy $\log_2 N_\varepsilon^R(A)$ is the minimum ℓ such that the best-obtainable worst-case error $\inf_{E,D} \sup_{a\in A} d(a, D(E(a)))$ is no more than ε.

1.1.2 Entropy numbers

In the context of operator approximation theory, it is common to study a quantity very closely related to metric entropy: entropy numbers.

We now define the entropy numbers of sets, and make explicit their relation to metric entropy. The connection to the notion of entropy numbers of operators will be discussed extensively in the next section.

Definition 1.4 ([CS90])**.** The n-th *entropy number of A in R* is

$$\varepsilon_n(A;R) = \inf\{\varepsilon > 0;\ \text{there exists an } \varepsilon\text{-covering of } A \text{ in } R \text{ of cardinality } n\} \tag{1.1.4}$$

The n-th *dyadic entropy number* of A in R is $e_n(A;R) = \varepsilon_{2^{n-1}}(A;R)$.
The n-th *self-entropy-number* of A is $\varepsilon_n^{\mathrm{self}}(A) = \varepsilon_n(A;A)$ (for the induced metric).

Morally, $n \mapsto \varepsilon_n(A;R)$ and $\varepsilon \mapsto N_\varepsilon(A;R)$ are just inverse functions of each other, and likewise $n \mapsto e_{n+1}(A;R)$ and $\varepsilon \mapsto \log_2 N_\varepsilon(A;R)$. (See Figure 1.2.) In other words, in the "bitstring length" characterization of metric entropy, $e_n(A;R)$ is the best worst-case error ε obtainable for fixed $\ell = n-1$.

Proposition 1.4. For all n_0, ε_0,

$$\varepsilon_{n_0}(A;R) \le \varepsilon_0 \iff N_{\varepsilon_0}(A;R) \le n_0 \tag{1.1.5}$$
$$\iff \quad \text{there exists an } (\varepsilon_0+\eta)\text{-covering of } A \text{ of cardinality } n_0, \text{ for any } \eta > 0 \tag{1.1.6}$$

For any strictly decreasing continuous $\phi : \mathbb{R}_+^* \to \mathbb{R}_+$,

$$\forall \varepsilon_0 > 0,\ N_{\varepsilon_0}(A;R) \le \phi(\varepsilon_0) \tag{1.1.7}$$
$$\iff \forall n_0 > \lim_{+\infty} \phi,\ \varepsilon_{n_0-1}(A;R) \le \phi^{-1}(n_0) \tag{1.1.8}$$

and

$$\forall \varepsilon_0 > 0,\ \log_2\left(N_{\varepsilon_0}(A;R) - 1\right) \le \phi(\varepsilon_0) \tag{1.1.9}$$
$$\iff \forall n_0 > \lim_{+\infty} \phi, \qquad e_{n_0+1}(A;R) \le \phi^{-1}(n_0) \tag{1.1.10}$$

In particular, a set $A \subset R$ is totally bounded, i.e $N_\varepsilon(A;R) < \infty$ for all $\varepsilon > 0$, if and only if $\lim_{n\to\infty} \varepsilon_n(A;R) = 0$.

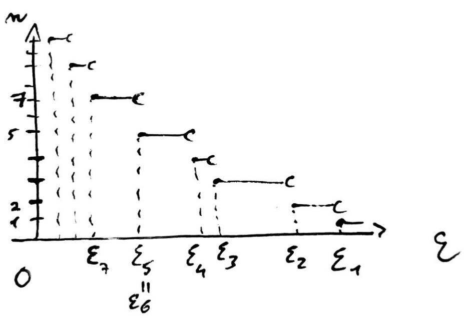


Figure 1.2: ε-covering number N_ε vs. entropy numbers ε_n. The curve represents $[\varepsilon \mapsto N_\varepsilon(A;R)]$ (for some A and R): integer-valued, decreasing, right-continuous.

Proof. Since the ε-covering number is monotonous in ε and integer-valued, there is no difficulty in defining the right-continuous variant of the covering number: $\widetilde{N}_\varepsilon(A;R) = \lim_{\rho\to\varepsilon^+} N_\rho(A;R)$. Then one can check that the equivalence $\varepsilon_{n_0}(A;R) \le \varepsilon_0 \iff \widetilde{N}_{\varepsilon_0}(A;R) \le n_0$ holds by definition.

Now, the proof of [KT59, theorem III] shows that $N_\varepsilon(A;R)$ is continuous from the right, [2] which proves the first part.

For the second part, dropping the dependency on A and R for concision,

$$\begin{aligned}
&\forall \varepsilon_0, && N_{\varepsilon_0} \leq \phi(\varepsilon_0) && (1.1.11)\\
\iff &\forall \varepsilon_0, && N_{\varepsilon_0} \leq \lfloor \phi(\varepsilon_0) \rfloor && (1.1.12)\\
\iff &\forall \varepsilon_0, && \varepsilon_{\lfloor \phi(\varepsilon_0) \rfloor} \leq \varepsilon_0 && (1.1.13)\\
\iff &\forall y > \lim_{+\infty} \phi, && \varepsilon_{\lfloor y \rfloor} \leq \phi^{-1}(y) && (1.1.14)\\
\iff &\forall n_0 > \lim_{+\infty} \phi, \forall \delta > 0, && \varepsilon_{\lfloor n_0 - \delta \rfloor} \leq \phi^{-1}(n_0 - \delta) && (1.1.15)\\
\iff &\forall n_0 > \lim_{+\infty} \phi, && \varepsilon_{n_0 - 1} \leq \phi^{-1}(n_0) && (1.1.16)
\end{aligned}$$

where we used successively that N_ε is integer-valued, the first part of the proposition, strict monotonicity of ϕ, and continuity of ϕ. For the reverse direction of the last equivalence we used that ϕ is decreasing.

For the equivalence of metric entropy and dyadic entropy numbers, simply check that

$$\begin{aligned}
&\forall \varepsilon_0, && \log_2(N_{\varepsilon_0} - 1) \leq \phi(\varepsilon_0) && (1.1.17)\\
\iff &\forall \varepsilon_0, && N_{\varepsilon_0} \leq 2^{\phi(\varepsilon_0)} + 1 =: \psi(\varepsilon_0) && (1.1.18)\\
\iff &\forall n_0, && \varepsilon_{n_0} \leq \psi^{-1}(n_0 + 1) && (1.1.19)\\
\iff &\forall n_0, && e_{n_0+1} = \varepsilon_{2^{n_0}} \leq \psi^{-1}(2^{n_0} + 1) && (1.1.20)
\end{aligned}$$

and that the right-hand-side of the last inequality is equal to $\phi^{-1}(n_0)$.

The last statement characterizing totally boundedness follows from the definitions and from the rest of the proposition. □

Thus, the problems of estimating metric entropy and estimating entropy numbers are completely equivalent. In the sequel we will use both points of view, whichever one is most convenient for each specific setting.

1.1.3 ε-(constructive)-dimension and approximation quantities

A finite ε-covering is essentially an approximation scheme using a finite number of prototypes. When the ambient space is a normed vector space, it is natural to consider approximation schemes using prototypes in a finite-dimensional set. This leads us to define the following notions.

Definition 1.5. Consider an ambient normed vector space $(R, \|\cdot\|)$, let $A \subset R$ and $\varepsilon > 0$.

The *ε-dimension of A in R*, denoted by $D_\varepsilon(A;R)$, is the least $d \in \mathbb{N}$ such that there exists a d-dimensional subspace $R_d \subset R$ satisfying

$$\forall a \in A, \exists x \in R_d; \|a - x\| \leq \varepsilon \tag{1.1.21}$$

In words, it is the least dimensionality of a finite-dimensional ε-covering of A.

The *ε-constructive-dimension of A in R*, denoted by $\mathcal{A}_\varepsilon(A;R)$, is the least $d \in \mathbb{N}$ such that there exists a bounded linear operator $S : R \to R$ of rank d satisfying

$$\forall a \in A, \|a - Sa\| \leq \varepsilon \tag{1.1.22}$$

In words, it is the least rank of a finite-rank approximation of identity up to an error of ε over A. [3]

[2] The proof in [KT59] is for ε-coverings in the sense of sets $A \subset \bigcup_\gamma U_\gamma$ ($\forall \gamma, U_\gamma \subset R$), but the same arguments apply for ε-coverings in the sense of points.

[3] The terminology "ε-dimension" does not seem well-established; we borrowed it from [Zam79]. The notion of "ε-constructive-dimension" has not been defined as such before, as far as we are aware, but it is equivalent to the well-studied notion of approximation number. Our choices of notation are equally non-standard.

Note that, clearly, $D_\varepsilon(A;R) \leq \mathcal{A}_\varepsilon(A;R)$. It turns out that equality holds if R is a Hilbert space, but not for general Banach R [CS90, proposition 2.4.4].

Proposition 1.5 ([CS90, proposition 2.2.1])**.** Let R a normed vector space. A bounded set $A \subset R$ has finite ε-dimension for all $\varepsilon > 0$ if and only if A is totally bounded. (But an unbounded set may also have finite ε-dimension for all ε.)

The conditions for the ε-constructive-dimension to be finite are less clear. In the sequel, when we write "$D_\varepsilon(A;R)$" and "$\mathcal{A}_\varepsilon(A;R)$", we implicitly make the assumption that ε is large enough so that they are finite.

In this thesis we define the Kolmogorov numbers of sets as (morally) the inverse function of the ε-dimension, and the approximation numbers of sets as the inverse function of the ε-constructive-dimension. This terminology comes from operator approximation theory, as will be discussed extensively in the next section.

Definition 1.6. Consider an ambient normed vector space $(R, \|\cdot\|)$, let $A \subset R$ and $\varepsilon > 0$.

The n-th *Kolmogorov number of A in R* is

$$\begin{aligned} d_n(A;R) &= \inf\left\{\varepsilon > 0;\ \exists R_n \subset R, \dim(R_n) \leq n-1,\ A \subset R_n + \varepsilon B^{(R)}\right\} && (1.1.23)\\ &= \inf\left\{\varepsilon > 0;\ \exists R_n \subset R, \dim(R_n) \leq n-1,\ \forall a \in A, \exists x \in R_n, \|a - x\| \leq \varepsilon\right\} && (1.1.24)\end{aligned}$$

Likewise, the n-th *approximation number of A in R* is

$$a_n(A;R) = \inf\left\{\varepsilon > 0;\ \exists S_n, \operatorname{rank}(S_n) \leq n-1,\ \forall a \in A, \|a - S_n a\| \leq \varepsilon\right\} \tag{1.1.25}$$

The formal correspondence between d_{n+1} and D_ε, respectively between a_{n+1} and $\mathcal{A}_\varepsilon$, is analogous to the one between ε_n and N_ε.

Proposition 1.6. For all n_0, ε_0,

$$d_{n_0+1}(A;R) \leq \varepsilon_0 \iff \widetilde{D}_{\varepsilon_0}(A;R) \leq n_0 \tag{1.1.26}$$

$$\iff \text{there exists a } n_0\text{-dimensional subspace of } R \text{ which is an } (\varepsilon_0+\eta)\text{-covering of } A, \text{ for any } \eta > 0 \tag{1.1.27}$$

where $\widetilde{D}_\varepsilon$ is the right-continuous variant of the ε-dimension: $\widetilde{D}_\varepsilon(A;R) = \lim_{\rho\to\varepsilon^+} D_\rho(A;R)$.

Similarly, for all n_0, ε_0,

$$a_{n_0+1}(A;R) \leq \varepsilon_0 \iff \widetilde{\mathcal{A}}_{\varepsilon_0}(A;R) \leq n_0 \tag{1.1.28}$$

$$\iff \text{there exists a rank-}n_0 \text{ operator } S \text{ such that } \sup_{a\in A} \|a - Sa\|_R \leq \varepsilon_0 + \eta, \text{ for any } \eta > 0 \tag{1.1.29}$$

where $\widetilde{\mathcal{A}}_\varepsilon$ is the right-continuous variant of the ε-constructive-dimension.

Proof. The claimed characterization of $d_{n_0+1}(A;R)$ follows from the definition.

To check the claimed characterization of $\widetilde{D}_{\varepsilon_0}(A;R)$, use that $\varepsilon \mapsto D_\varepsilon(A;R)$ is decreasing and integer-valued (which, by the way, justifies that $\widetilde{D}_\varepsilon(A;R)$ is well-defined), and write

$$\begin{aligned} \widetilde{D}_{\varepsilon_0}(A;R) = \lim_{\eta\to 0^+} D_{\varepsilon_0+\eta}(A;R) \leq n_0 &\iff \forall \eta > 0, D_{\varepsilon_0+\eta}(A;R) \leq n_0 && (1.1.30)\\ &\iff \forall \eta > 0, \exists R_{n_0} \subset R; \dim(R_{n_0}) \leq n_0,\ R_{n_0} \text{ is an } (\varepsilon_0+\eta)\text{-covering of } A && (1.1.31)\end{aligned}$$

The corresponding claims for the ε-constructive-dimension and the approximation numbers can be proved similarly. □

Relation to metric entropy The notions of metric entropy and ε-(constructive)-dimension are obviously related. However it is not obvious to state the relation in the general case:

- A finite-cardinality ε-covering of A clearly induces a finite-dimensional ε -covering of A, simply by taking the linear span, but the resulting dimensionality can be expected to be much larger than optimal.
- From a finite-dimensional ε-covering of A, one can morally extract a $(\varepsilon + \eta)$-covering of A, by decomposing the prototypes on some basis of the finite-dimensional subspace, and quantizing the coefficients in the decomposition. However, without additional assumptions, it is not clear how to choose the quantization level to achieve an $(\varepsilon + \eta)$-covering for a given η. We invite the curious reader to try it to see where it fails and what additional information are needed.

On the other hand, the relation between entropy-, Kolmogorov- and approximation-numbers (which is equally not obvious) is well-studied in [CS90, chapters 2 and 3]. Since that book takes the operator approximation theory view on those quantities, we defer the discussion of those relations to the next section.

1.1.4 Some simple useful facts

Let us state some simple facts on metric entropy (and ε-dimension and related quantities) that simplify how we can think about those objects.

Switching the ambient space We mentioned at the beginning of this chapter that, since $N_{2\varepsilon}^A(A) \leq N_\varepsilon^R(A) \leq N_\varepsilon^A(A)$, confusing covering- and self-covering-numbers is morally not too bad. For ease of reference, let us write down that observation formally.

Proposition 1.7. Let (R, d) an ambient metric space, $A \subset R$, and $\underline{R}$ an intermediate space: $A \subset \underline{R} \subset R$. All three sets are metric spaces equipped with d. Then

$$N_\varepsilon^R(A) \leq N_\varepsilon^{\underline{R}}(A) \leq N_\varepsilon^A(A) \leq M_\varepsilon(A) \leq N_{\varepsilon/2}^R(A) \tag{1.1.32}$$

$$\varepsilon_n(A; R) \leq \varepsilon_n(A; \underline{R}) \leq \varepsilon_n^{\mathrm{self}}(A) \qquad \leq 2\varepsilon_n(A; R) \tag{1.1.33}$$

Proof. The inequalities for the covering and packing numbers follow from the relation $M_{2\varepsilon}(A) \leq N_\varepsilon^R(A) \leq N_\varepsilon^A(A) \leq M_\varepsilon(A)$ [KT59, theorem IV], already seen at the beginning of this chapter.

The inequalities involving $N_\varepsilon^{\underline{R}}(A)$ follow from the definition, since any ε-covering of A with prototypes in A is a fortiori an ε-covering of A with prototypes in $\underline{R}$, and any ε-covering of A with prototypes in $\underline{R}$ is a fortiori an ε-covering of A with prototypes in R.

Taking inverse functions yields the announced inequalities for the entropy numbers. □

Dense ambient space If we are allowed to choose the prototypes of an ε-covering from some non-closed intermediary space $\underline{R} \subset (R, d)$, then morally there is no loss of generality in choosing the prototypes from the closure of $\underline{R}$. The following result formalizes that fact. It is stated here for ε-covering numbers and metric spaces, but analogues for ε-dimension and normed vector spaces hold.

Proposition 1.8. Let (R, d) a metric space and $\underline{R}$ a dense subset of R. Then for any $A \subset \underline{R}$,

$$N_\varepsilon(A; \underline{R}) = N_\varepsilon(A; R) \tag{1.1.34}$$

Proof. A covering of A with prototypes in $\underline{R}$ is also a covering of A with prototypes in R, so the inequality $N_\varepsilon(A; \underline{R}) \geq N_\varepsilon(A; R)$ clearly holds.

To show the other inequality $N_\varepsilon(A; \underline{R}) \leq N_\varepsilon(A; R)$, we first show that $N_{\varepsilon+\eta}(A; \underline{R}) \leq N_\varepsilon(A; R)$ for any $\eta > 0$. Let $(r_1, ..., r_n) \subset R$ an optimal ε-covering of A in R. For any $\eta > 0$, there exist $(\underline{r}_1, ..., \underline{r}_n) \subset \underline{R}$ such that $d(\underline{r}_i, r_i) \leq \eta$. So this family constitutes an $(\varepsilon + \eta)$-covering of A in $\underline{R}$.

Thus, $N_{\varepsilon+\eta}(A; \underline{R}) \leq N_\varepsilon(A; R)$ for any $\eta > 0$. We conclude by using the fact that $\varepsilon \mapsto N_\varepsilon(A; \underline{R})$ is right-continuous, as shown by the proof of [KT59, theorem III]. □

Metric entropy of the closure Similarly to the previous paragraph, if we can find an ε-covering of a non-closed set A, we might just as well cover its closure. In other words we have the following equalities, which are easy to check from the definitions.

Proposition 1.9. Let (R, d) a metric space. Let $A \subset R$ and denote $\overline{A}$ its closure.

$$\varepsilon_n(A; R) = \varepsilon_n(\overline{A}; R) \tag{1.1.35}$$

and if R is a normed vector space,

$$d_n(A; R) = d_n(\overline{A}; R) \qquad\qquad a_n(A; R) = a_n(\overline{A}; R) \tag{1.1.36}$$

Transformation by a (partial) isometry Since transformations by isometric isomorphisms preserve all the topological information, in particular they leave the metric entropy invariant.

Proposition 1.10. Let E and F be normed vector spaces and $\Phi : E \to F$ a linear operator. Let $E_+ \subset E$ a totally bounded subset, and $F_+ = \Phi(E_+)$.

- If the linear operator Φ is bounded, i.e $\forall x \in E, \|\Phi x\|_F \leq |||\Phi|||\, \|x\|_E$ with $|||\Phi||| < \infty$, then

$$\varepsilon_n(F_+) \leq |||\Phi|||\ \varepsilon_n(E_+) \tag{1.1.37}$$
$$N_{|||\Phi|||\delta}(F_+) \leq N_\delta(E_+) \tag{1.1.38}$$

  (This can be seen as a special case of the multiplicativity property of entropy numbers of operators, see infra.)

- If Φ is "bounded away from zero", i.e $\forall x \in E,\ \|\Phi x\|_F \geq a\, \|x\|_E$ for some $a > 0$ (restricted lower isometry property), then

$$M_{a\delta}(F_+) \geq M_\delta(E_+) \tag{1.1.39}$$

- If Φ is an isometric isomorphism, then

$$\varepsilon_n(F_+) = \varepsilon_n(E_+) \tag{1.1.40}$$
$$N_\varepsilon(F_+) = N_\varepsilon(E_+) \tag{1.1.41}$$
$$M_\varepsilon(F_+) = M_\varepsilon(E_+) \tag{1.1.42}$$

Relatively compact sets and continuity Recall that totally boundedness is equivalent to relative compactness when the ambient space is a complete metric space, and that continuous mappings send compact sets to compact sets. It turns out that the same holds for relatively compact sets.

Proposition 1.11. Let $f : E \to F$ a continuous mapping between complete metric spaces E and F. Then the image of a relatively compact set (in E) by f, is relatively compact (in F).

Proof. Let E_+ a relatively compact subset of E. For clarity, let $\overline{\cdot}^E$ and $\overline{\cdot}^F$ denote closure in E and in F respectively.

$$\overline{f(E_+)}^F \subset \overline{f(\overline{E_+}^E)}^F = f(\overline{E_+}^E) \tag{1.1.43}$$

Indeed, $\overline{E_+}^E$ is compact, so its image by f is compact and so closed.

So, $\overline{f(E_+)}^F$ is compact as a closed subset of the compact $f(\overline{E_+}^E)$. So by definition, $f(E_+)$ is relatively compact in F. $\square$

1.2 Bridge to operator approximation theory

We defined metric entropy and entropy numbers, and saw that they are two sides of the same coin. In the literature, metric entropy seems to be more widely used, for example in learning theory and related domains [Wai19] [Elb+20], but also studied in its own right in functional analysis [KT59] [CDN19]. The use of entropy numbers originates from, and seems mostly restricted to, the topic of linear operator approximation in Banach spaces [CS90].

To be clear, in this report, the term "operator" without qualifier will refer to bounded linear operators.

1.2.1 Definitions

Entropy numbers of operators

Definition 1.7. Let T a bounded linear operator between Banach spaces E and F. The n-th *entropy number of* T is the n-th entropy number of the image of the unit ball by T:

$$\varepsilon_n(T) = \varepsilon_n(T(B^{(E)})) \tag{1.2.1}$$

An operator T is called compact if $T(B^{(E)})$ is relatively compact in F. Such operators are of particular interest in operator approximation theory. By the previous proposition, and since totally boundedness is equivalent to relative compactness in Banach spaces (complete normed vector spaces), T is compact if and only if $\lim_n \varepsilon_n(T) = 0$. Thus the rate of decrease of an operator's entropy numbers is a measure of its degree of compactness.

Proposition 1.12 (verbatim from [CS90, section 1.3])**.** For bounded linear operators T between Banach spaces, denoting $|||\cdot|||$ the operator norm, the entropy numbers of operators $\varepsilon_n(T)$ satisfy:

Monotonicity $\varepsilon_n(T) \geq \varepsilon_{n+1}(T)$

Additivity $\varepsilon_{kn}(T_1 + T_2) \leq \varepsilon_k(T_1) + \varepsilon_n(T_2)$

Multiplicativity $\varepsilon_{kn}(RS) \leq \varepsilon_k(R)\varepsilon_n(S)$
In particular, $\varepsilon_n(RS) \leq |||R|||\varepsilon_n(S)$ and $\varepsilon_k(RS) \leq \varepsilon_k(R)|||S|||$.

The dyadic entropy numbers $e_n(T) = \varepsilon_{2^{n-1}}(T)$ satisfy:

Additivity $e_{k+n-1}(T_1 + T_2) \leq e_k(T_1) + e_n(T_2)$

Multiplicativity $e_{k+n-1}(RS) \leq e_k(R)e_n(S)$
In particular, $e_n(RS) \leq |||R|||e_n(S)$ and $e_k(RS) \leq e_k(R)|||S|||$.

All of those properties naturally extend to the case of entropy numbers of sets rather than operators, and transpose to covering numbers and metric entropy.

Approximation quantities (Kolmogorov and approximation numbers) of operators The quantities from operator approximation theory corresponding to ε-dimension and ε-constructive-dimension, are respectively the Kolmogorov numbers and the approximation numbers of operators. [4] Their definition, given below, was the basis for our definition (and choice of terminology) for the Kolmogorov and approximation numbers of sets.

[4] Since Kolmogorov's name is given to so many objects, a more useful keyword may be "s-number". Approximation, Kolmogorov and dyadic entropy numbers are special cases of s-numbers.

Definition 1.8 ([CS90, sections 2.1, 2.2])**.** Let T a bounded linear operator between Banachs E and F. The n-th *Kolmogorov number of* T is

$$d_n(T) = \inf\left\{\varepsilon > 0;\ \exists F_n \subset F, \dim(F_n) \leq n-1,\ T(B^{(E)}) \subset F_n + \varepsilon B^{(F)}\right\} \tag{1.2.2}$$

The n-th *approximation number of* T is

$$a_n(T) = \inf\left\{\varepsilon > 0;\ \exists S_n, \operatorname{rank}(S_n) \leq n-1,\ |||T - S_n||| \leq \varepsilon\right\} \tag{1.2.3}$$

1.2.2 Relation between entropy numbers and approximation quantities

The relation between entropy-, Kolmogorov- and approximation-numbers is made explicit in [CS90, chapters 2 and 3]. Here we list only simple results, and refer to that book for a detailed discussion, and for results leveraging additional assumptions on the spaces E and F.

Proposition 1.13 ([CS90, propositions 2.2.3, 2.4.4])**.** For any T bounded linear operator between arbitrary Banachs E and F,

$$d_n(T) \leq a_n(T) \leq (1 + \sqrt{n-1})d_n(T) \tag{1.2.4}$$

Moreover, if F is a Hilbert space, or if E satisfies the "metric lifting property", then $d_n(T) = a_n(T)$.

Proposition 1.14 ([CS90, theorem 3.1.1])**.** Let T bounded linear operator between arbitrary Banachs E and F. For any $0 < p < \infty$ and any $m \in \mathbb{N}^*$,

$$\sup_{1\leq k\leq m} k^{1/p} e_k(T) \leq c_p \sup_{1\leq k\leq m} k^{1/p} a_k(T) \tag{1.2.5}$$

where $c_p = 2^7(16(2 + 1/p))^{1/p}$ for Banachs over the field $\mathbb{R}$ and $c_p = 2^7(32(2 + 1/p))^{1/p}$ for Banachs over the field $\mathbb{C}$. In particular,

$$e_m(T) \leq c_p m^{-1/p} \sup_{1\leq k\leq m} k^{1/p} a_k(T) \leq c_p \left(\frac{\sum_{i=1}^m a_i(T)^p}{m}\right)^{1/p} \tag{1.2.6}$$

Proposition 1.15. Let T a bounded linear operator between Banachs E and F.

[CS90, lemma 2.5.2] $\lim_n \varepsilon_n(T) \leq \lim_n a_n(T)$ and equality does not hold in general.

[CS90, prop. 2.2.3] If $\lim_n a_n(T) = 0$ then T is a compact operator, but the converse is not true, unless F has the approximation property.

[CS90, prop. 2.2.1] T is compact if and only if $\lim_n d_n(T) = 0$.

The next subsection will show that similar statements on relative compactness and approximation quantities hold for subsets $A \subset R$ instead of operators $T : E \to F$.

1.2.3 Entropy numbers and approximation quantities of sets vs. of operators

So far we introduced the Kolmogorov and approximation numbers of sets by analogy with the corresponding quantities for operators. Let us show how the two settings relate precisely. Namely, we will show that if R is a Banach and A satisfies some reasonable assumptions, then $d_n(A; R)$ (resp. $a_n(A; R)$) is equal to $d_n(I)$ (resp. $a_n(I)$) for I a bounded linear operator between Banach spaces to be defined. This implies that, under those assumptions, the relations between entropy numbers and approximation quantities summarized in the previous subsection for operators, also hold for sets.

First we recall the Minkowski functional construction, which establishes a sort of equivalence between subsets and semi-norms.

Lemma 1.16. Let a normed vector space $(R, \|\cdot\|)$ and $A \subset R$. Suppose A is convex, balanced and absorbing. Denote $\overline{A}$ its closure in R.

- *Balanced* means that $\forall |\lambda| \leq 1, \lambda A \subset A$. Equivalently since A is convex, $\forall |\lambda| = 1, \lambda A = A$. We assume for simplicity that the ground field is $\mathbb{R}$, so yet equivalently, $-A = A$.
- *Absorbing* and balanced means that $\forall x \in R, \exists b > 0; x/b \in A$.

The following formula defines a semi-norm (called the gauge or Minkowski functional of A):

$$\forall x \in R, \ \ \|x\|_A = \inf \{b > 0; \ x/b \in A\} \tag{1.2.7}$$

Moreover the closed unit ball of $(R, \|\cdot\|_A)$ satisfies $A \subset B^{(R,\|\cdot\|_A)} \subset \overline{A}$.

Note that, conversely, the unit ball of R for any semi-norm is convex, balanced and absorbing.

Furthermore, if A is also bounded in $(R, \|\cdot\|)$, then $\|\cdot\|_A$ is positive definite, i.e a norm.

Proof. To check that the gauge $\|\cdot\|_A$ is a semi-norm,

Homogeneity For all $\lambda \in \mathbb{R}$,

$$\begin{aligned} \|\lambda x\|_A &= \inf \{b > 0; \ \lambda x \in bA\} & (1.2.8)\\ &= \inf \left\{ b > 0; \ x \in \frac{b}{\lambda} A = \frac{b}{|\lambda|} A \right\} & (1.2.9)\\ &= \inf \{|\lambda|\, b > 0; \ x \in bA\} \quad = |\lambda| \, \|x\|_A & (1.2.10)\end{aligned}$$

because A is balanced.

Triangle inequality For all $x, y \in R$, let $b_x, b_y > 0$ and $a_x, a_y \in A$ such that $x = b_x a_x, y = b_y a_y$. Then

$$x + y = b_x a_x + b_y a_y = (b_x + b_y) \left(\frac{b_x}{b_x + b_y} a_x + \frac{b_y}{b_x + b_y} a_y \right) \in (b_x + b_y) A \tag{1.2.11}$$

since A is convex. So by definition $\|x + y\|_A \leq b_x + b_y$. Taking the inf over b_x and b_y, we conclude that $\|x + y\|_A \leq \|x\|_A + \|y\|_A$.

To check that $A \subset B^{(R,\|\cdot\|_A)} \subset \overline{A}$, simply note that for all $x \in R$,

$$\|x\|_A = \inf \{b > 0; x \in bA\} \leq 1 \iff \forall \varepsilon > 0, x \in (1 + \varepsilon) A \tag{1.2.12}$$

Conversely, let $\|\cdot\|'$ a semi-norm on R and B its unit ball. Then B is clearly convex and balanced. For any $x \in R$, setting $b = 2 \|x\|'$ yields $\|x/b\|' < 1$ i.e $x/b \in B$, so B is absorbing.

Finally, assume A is also bounded in $(R, \|\cdot\|)$, and denote $M = \sup_A \|a\|$. Now for any $x \in R$,

$$\begin{aligned} \frac{x}{\|x\|_A} &\in B^{(R,\|\cdot\|_A)} = \overline{A} & (1.2.13)\\ \left\| \frac{x}{\|x\|_A} \right\| &\leq M & (1.2.14)\\ \|x\| &\leq M \|x\|_A & (1.2.15)\end{aligned}$$

Thus $\|x\|_A = 0 \implies \|x\| = 0 \implies x = 0$, so $\|\cdot\|_A$ is positive definite, as claimed. □

It turns out that if R is a Banach space (or more generally a barrelled space), then A convex, balanced and absorbing in R implies that $\overline{A}$ contains a neighborhood of zero: $\overline{A} \supset r B^{(R)}$ for some $r > 0$. [5] Now by Riesz theorem, as soon as R is infinite-dimensional, $B^{(R)}$ is not totally bounded. So the above lemma applies only if A is not totally bounded in R or if R is not complete.

Let us continue our construction, by removing the very strong requirement that A is absorbing in R. Instead we look at the largest intermediary space in which A is absorbing.

[5] `https://en.wikipedia.org/wiki/Barrelled_space`

Lemma 1.17. Let a Banach space $(R, \|\cdot\|)$ and $A \subset R$. Suppose A is convex, balanced and bounded. Denote $\overline{A}$ its closure in R.

Let $\mathbb{R}A = \{\lambda a;\ \lambda \in \mathbb{R}, a \in A\}$. Then $\mathbb{R}A$ is equal to the linear span of A:

$$\mathbb{R}A = \left\{ \sum_{i=1}^{N} \lambda_i a_i;\ \lambda_i \in \mathbb{R}, a_i \in A, N \in \mathbb{N} \right\} \tag{1.2.16}$$

In particular $\mathbb{R}A$ is a vector subspace which contains A. Clearly A is absorbing in $\mathbb{R}A$.

Thus A is a convex, balanced, absorbing and bounded subset of the normed vector space $(\mathbb{R}A, \|\cdot\|)$, so we may define the norm $\|\cdot\|_A$ over $\mathbb{R}A$ as in the previous lemma.

Denote $Q = \overline{\mathbb{R}A}^{\|\cdot\|_A}$ the Cauchy completion of $\mathbb{R}A$ with respect to $\|\cdot\|_A$. Then, $Q \subset R$.

Moreover, the closed unit ball of Q satisfies

$$A \subset B^{(Q, \|\cdot\|_A)} \subset \overline{A} \tag{1.2.17}$$

Proof. Denote span(A) the linear span of A. Clearly $\mathbb{R}A \subset \text{span}(A)$. Conversely for any $x \in \text{span}(A)$, we can write

$$x = \sum_{i=1}^{N} \lambda_i a_i = \left(\sum_j |\lambda_j| \right) \left(\sum_{i=1}^{N} \underbrace{\frac{|\lambda_i|}{\sum_j |\lambda_j|}}_{\text{sums to 1}} \cdot \underbrace{\frac{\lambda_i}{|\lambda_i|} a_i}_{\in A} \right) \in \mathbb{R}A \tag{1.2.18}$$

since A is balanced and convex.

To show that $Q \subset R$, denote $M = \sup_A \|a\|$ and recall from the proof of the previous lemma that $\forall x \in \mathbb{R}A,\ \|x\| \leq M \|x\|_A$. So any Cauchy sequence in $(\mathbb{R}A, \|\cdot\|_A)$ is also a Cauchy sequence in $(R, \|\cdot\|)$. Thus since the latter is a Banach, it contains the Cauchy completion of the former.

The unit ball of Q is by definition $B^{(Q,\|\cdot\|_A)} = \{x \in Q;\ \forall \varepsilon > 0, x \in (1+\varepsilon)A\}$. The inclusion $A \subset B^{(Q,\|\cdot\|_A)}$ is obvious. To check the inclusion $B^{(Q,\|\cdot\|_A)} \subset \overline{A}$, simply note that for all $x \in Q$,

$$[\forall \varepsilon > 0, x \in (1+\varepsilon)A = A + \varepsilon A] \implies \left[\forall \varepsilon > 0, x \in A + \varepsilon M B^{(R)}\right] \iff x \in \overline{A} \tag{1.2.19}$$

□

Remark 1.1. Our construction simplifies considerably if A is closed in R i.e $A = \overline{A}$, as $B^{(Q)}$ is then simply equal to A and $Q = \mathbb{R}B^{(Q)} = \mathbb{R}A$.

The following proposition shows that, thanks to this construction, the entropy numbers and approximation quantities of a bounded subset A of a Banach ambient space R can be interpreted as the corresponding quantities for an operator between Banachs (under the reasonable assumption that A is convex and balanced).

Proposition 1.18. Let a Banach space $(R, \|\cdot\|)$ and $A \subset R$ convex, balanced and bounded. Let $Q = \overline{\mathbb{R}A}^{\|\cdot\|_A}$ the Banach space defined as in the lemma just above.

Denote $I : (Q, \|\cdot\|_A) \to (R, \|\cdot\|)$ the identity map. Then

$$\varepsilon_n(A; R) = \varepsilon_n(I) \qquad d_n(A; R) = d_n(I) \qquad a_n(A; R) = a_n(I) \tag{1.2.20}$$

Proof. Let $\overline{A}$ the closure of A in R, B^Q the unit ball of $(Q, \|\cdot\|_A)$, and recall that $A \subset B^Q \subset \overline{A}$. In particular, the closure of B^Q in R is $\overline{A}$.

For entropy numbers: $\varepsilon_n(I) = \varepsilon_n(I(B^Q); R) = \varepsilon_n(B^Q; R)$. Now the entropy numbers of a set are equal to the entropy numbers of its closure, so $\varepsilon_n(B^Q; R) = \varepsilon_n(\overline{A}; R)$, which in turn is nothing else than $\varepsilon_n(A; R)$.

For Kolmogorov numbers: recall the definitions

$$d_n(A;R) = \inf\left\{\varepsilon > 0;\ \exists R_n \subset R, \dim(R_n) \le n-1,\ A \subset R_n + \varepsilon B^{(R)}\right\} \tag{1.2.21}$$

$$d_n(I) = \inf\left\{\varepsilon > 0;\ \exists R_n \subset R, \dim(R_n) \le n-1,\ I(B^Q) \subset R_n + \varepsilon B^{(R)}\right\} \tag{1.2.22}$$

So we can write that $d_n(I) = d_n(B^Q;R)$. Now the Kolmogorov numbers of a set are equal to the Kolmogorov numbers of its closure, so $d_n(B^Q;R) = d_n(\overline{A};R) = d_n(A;R)$.

For approximation numbers: recall the definitions

$$a_n(A;R) = \inf\left\{\varepsilon > 0;\ \exists S_n, \mathrm{rank}(S_n) \le n-1,\ \forall a \in A, \|a - S_n a\| \le \varepsilon\right\} \tag{1.2.23}$$

$$a_n(I) = \inf\left\{\varepsilon > 0;\ \exists S_n, \mathrm{rank}(S_n) \le n-1,\ \|\!|I - S_n|\!\|_{Q\to R} \le \varepsilon\right\} \tag{1.2.24}$$

Note that $\|\!|I - S_n|\!\|_{(Q,\|\cdot\|_A)\to(R,\|\cdot\|)} = \sup_{a\in B^Q} \|a - S_n a\|$. So we can write $a_n(I) = a_n(B^Q;R)$. Now the approximation numbers of a set are equal to the approximation numbers of its closure, so $a_n(B^Q;R) = a_n(\overline{A};R) = a_n(A;R)$. ☐

Remark 1.2. With slight adaptations in the definition of the semi-norm $\|\cdot\|_A$, the assumption that A is balanced can probably be removed for Kolmogorov and approximation numbers. Indeed, it is not hard to check that those quantities are unchanged by "balanced-ification": denoting $\mathrm{bal}(A) := \bigcup_{|\lambda|=1} A$ the balanced hull,

$$d_n(\mathrm{bal}(A);R) = d_n(A;R) \qquad a_n(\mathrm{bal}(A);R) = a_n(A;R) \tag{1.2.25}$$

However, since $\mathrm{bal}(A)$ may not be convex even for convex A, it is not obvious what the adaptations to the definition of $\|\cdot\|_A$ should be. [6]

1.3 Framework for this thesis

In this section we formally pose the general problem; in the rest of this report we will discuss several instantiations of this framework, with notations consistent with the ones presented here. We distinguish three slight variants of the framework.

1.3.1 Systems defined over a set X, with worst-case error norm

Suppose given

- X a set of input signals;
- $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$ a normed vector space of output signals;
- $\mathbb{S}$ a vector space of systems $S : \begin{bmatrix} X \to \mathcal{Y} \\ x \mapsto y = S[x] \end{bmatrix}$; [7]
- $\mathbb{S}_+ \subset \mathbb{S}$ a set of systems of interest.

The goal will be to estimate the metric entropy $\log N_\varepsilon(\mathbb{S}_+; (\mathbb{S}, d))$. Several choices are possible for the error metric d; we will specifically look at the case of worst-case error over inputs in X. For $S, T \in \mathbb{S}$, let

$$d(S,T)_{\infty X} = \sup_{x\in X} \|S[x] - T[x]\|_{\mathcal{Y}} \tag{1.3.1}$$

[6] `https://en.wikipedia.org/wiki/Absolutely_convex_set`

[7] Requiring that $\mathbb{S}$ is stable by linear combinations makes sense from a physical point of view – think of block diagram representations.

the *worst-case error (as measured in* $\|\cdot\|_{\mathcal{Y}}$*) of approximating the system* $S \in \mathbb{S}$ *by the system* T*, over input signals* $x \in X$. In fact one quickly realizes that this metric corresponds to a norm: $\forall S, T, d(S,T)_{\infty X} = \|S - T\|_{\infty X}$ where

$$\|S\|_{\infty X} = \sup_{x \in X} \|S[x]\|_{\mathcal{Y}} \tag{1.3.2}$$

This formula does not define a norm, strictly speaking, as the "sup" may be infinite a priori. But this must be forbidden for our goal to make sense, as we now argue.

Suppose without loss of generality that $0 \in \mathbb{S}_+$ (otherwise, take instead $\widetilde{\mathbb{S}}_+ = \mathbb{S}_+ + (-S_0)$ for any $S_0 \in \mathbb{S}_+$, since $N_\varepsilon(\widetilde{\mathbb{S}}_+) = N_\varepsilon(\mathbb{S}_+)$), and make the very natural assumption that $\mathbb{S}_+$ is star-shaped around 0 i.e $\forall S \in \mathbb{S}_+, \forall \theta \in [0,1], \theta S \in \mathbb{S}_+$. Suppose by contradiction that there exists $S_1 \in \mathbb{S}_+$ such that $\|S_1\|_{\infty X} = \infty$, and denote

$$\forall \theta \in [0,1],\ S_\theta := \theta S_1 \in \mathbb{S}_+ \tag{1.3.3}$$

The goal is to estimate $\log N_\varepsilon(\mathbb{S}_+; (\mathbb{S}, \|\cdot\|_{\infty X}))$ as $\varepsilon \to 0$. But the packing number $M_{2\varepsilon}(\mathbb{S}_+; \|\cdot\|_{\infty X})$ is infinite for any $\varepsilon > 0$. Indeed for any finite collection $S_{\theta_1}, ..., S_{\theta_m}$ with $\theta_i \in [0,1]$ pairwise distinct,

$$\forall i \neq j, \left\|S_{\theta_i} - S_{\theta_j}\right\|_{\infty X} = |\theta_i - \theta_j| \, \|S_1\|_{\infty X} = \infty \tag{1.3.4}$$

So the ε-packing number is infinite for any ε, and so it doesn't make sense to estimate the metric entropy, since it is also always infinite. Thus for our purpose we should assume that $\|S\|_{\infty X}$ is finite for all $S \in \mathbb{S}_+$.

Further, we may therefore assume without loss of generality that $\|S\|_{\infty X}$ is finite also for all $S \in \mathbb{S}$, since any covering of $\mathbb{S}_+$ by prototypes in $\mathbb{S}$ will necessarily consist of prototypes of finite norm. More formally, denote $\mathbb{S}_b = \{S \in \mathbb{S}; \|S\|_{\infty X} < \infty\}$. One can check that $\mathbb{S}_b$ is a linear subspace that contains $\mathbb{S}_+$, and that $N_\varepsilon(\mathbb{S}_+; (\mathbb{S}_b; \|\cdot\|_{\infty X})) = N_\varepsilon(\mathbb{S}_+; (\mathbb{S}; \|\cdot\|_{\infty X}))$; so that we may just as well take $\mathbb{S}_b$ instead of $\mathbb{S}$.

To summarize, we may assume without loss of generality that $\|S\|_{\infty X} = \sup_X \|S[x]\|_{\mathcal{Y}}$ defines a proper norm over $\mathbb{S}$. The goal is to estimate the ε-covering number $N_\varepsilon(\mathbb{S}_+; \|\cdot\|_{\infty X})$, i.e the smallest $m \in \mathbb{N}$ such that

$$\exists S_1, ..., S_m \in \mathbb{S}, \forall S \in \mathbb{S}_+, \exists i \leq m, \forall x \in X, \|S[x] - S_i[x]\|_{\mathcal{Y}} \leq \varepsilon \tag{1.3.5}$$

In words, it is *the minimum number of "prototypical systems" needed to represent* $\mathbb{S}_+$*, such that the worst-case error (as measured in* $\|\cdot\|_{\mathcal{Y}}$*) of approximating any* $S \in \mathbb{S}_+$ *by its closest prototype is at most* ε*, over inputs* $x \in X$.

1.3.2 Systems over a space of inputs, with worst-case error norm over a set X

In many cases it is natural to consider systems S defined over a vector space of input signals, but unreasonable to expect S to be uniformly bounded over the entire space. If we restrict our ambition to approximating the systems over a subset of input signals, we can still formulate our framework as follows.

Suppose given

- $\mathcal{X}$ a vector space of input signals;
- $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$ a normed vector space of output signals;
- $\mathbb{S}$ a vector space of systems $S : \begin{bmatrix} \mathcal{X} \to \mathcal{Y} \\ x \mapsto y = S[x] \end{bmatrix}$;
- $X \subset \mathcal{X}$ a set of input signals of interest;

- $\mathbb{S}_+ \subset \mathbb{S}$ a set of systems of interest.

We again define

$$\|S\|_{\infty X} = \sup_{x \in X} \|S[x]\|_{\mathcal{Y}} \tag{1.3.6}$$

For the same reason as previously, we may assume, without loss of generality for our purpose, that $\|S\|_{\infty X} < \infty$ for all $S \in \mathbb{S}$.

Note that this formula defines a priori only a semi-norm over $\mathbb{S}$, since two systems may coincide over X while having a different behaviour in other regions of the space $\mathcal{X}$. However the notions of coverings and packings can still be defined for semi-norms or semi-metrics. Besides, in applications we may expect that perfect (not approximate!) knowledge of a system's behaviour over X allows to completely recover the system, provided that X is sufficiently diverse; in which case this semi-norm is a norm.

Tradeoff between restricting X and restricting $\mathbb{S}_+$ A common theme in system identification and learning theory is that it can be relatively easy to infer a system's behaviour "near" observed samples, but it becomes difficult when test signals are drawn from a large region of the signal space [Sch81]. Our framework reflects this by the simple fact that $X \mapsto \|S\|_{\infty X}$ is increasing.

Conversely, the task may again become tractable if prior knowledge of the system is given, i.e if the concept class to learn is small. For our goal, this corresponds to the fact that $\mathbb{S}_+ \mapsto N_\varepsilon(\mathbb{S}_+; (\mathbb{S}, \|\cdot\|_{\infty X}))$ is increasing.

So to hope to have estimatable metric entropy, or indeed to have $\mathbb{S}_+$ totally bounded for $\|\cdot\|_{\infty X}$, we must assume either X or $\mathbb{S}_+$ sufficiently small.

1.3.3 Systems over a space of inputs, with $L^p_{\mathbb{P}}$-average error norm

In the two first variants of the framework, morally, the system approximation scheme is tested against input signals drawn from X, and we measure the worst-case error. A natural extension is to consider test input signals drawn from a certain distribution, and to measure the average error or squared-error.

This leads us to formulate the third variant of the framework: suppose given

- $(\mathcal{X}, \Sigma_{\mathcal{X}})$ a measurable space of input signals;
- $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$ a normed vector space of output signals;
- $\mathbb{S}$ a vector space of systems $S : \begin{bmatrix} \mathcal{X} \to \mathcal{Y} \\ x \mapsto y = S[x] \end{bmatrix}$;
- $\mathbb{P}$ a finite measure over $(\mathcal{X}, \Sigma_{\mathcal{X}})$, and $1 \leq p < \infty$ an exponent;
- $\mathbb{S}_+ \subset \mathbb{S}$ a set of systems of interest.

The $L^p_{\mathbb{P}}$-average error metric is defined as

$$d(S,T)_{p\mathbb{P}} = \left(\mathbb{E}_{x\sim\mathbb{P}} \|S[x] - T[x]\|^p_{\mathcal{Y}}\right)^{1/p} \tag{1.3.7}$$

Similarly to the worst-case error, one quickly realizes that it arises from the semi-norm

$$\|S\|_{p\mathbb{P}} = \left(\mathbb{E}_{x\sim\mathbb{P}} \|S[x]\|^p_{\mathcal{Y}}\right)^{1/p} \tag{1.3.8}$$

For the same reason as in the first framework variant, we may assume, without loss of generality for our purpose, that $\|S\|_{p\mathbb{P}} < \infty$ for all $S \in \mathbb{S}$. So the above formula indeed defines a semi-norm.

When the support of $\mathbb{P}$ spans all of $\mathcal{X}$, the degeneracy of the semi-norm is morally negligible, and we may treat it as a norm, with the same abuses of notation as for L^p function spaces.

Chapter 2

Metric entropy estimates in function spaces

This chapter reviews and illustrates proof techniques for bounding metric entropy in (real-domain, real- or complex-valued) function spaces. For the purpose of estimating entropy numbers of nonlinear systems, this chapter is useful on three different levels.

1. In the case of systems deriving from kernels, the problem reduces to estimating metric entropy in the space of the kernel functions (chapter 6).
2. In the case of continuous systems over a set of signals $x(t) \in X$, results can be obtained that involve the metric entropy of X (chapter 7).
3. Nonlinear systems can be seen simply as mappings between signal spaces, from a metric space X to a Banach $\mathcal{Y}$ for example. Now this can be seen as a generalization of the real-domain real-valued functions case, whereby the signals are actually scalars. So the general ideas that appear here will also apply for spaces of systems.

2.1 Overview of basic proof techniques

We start by a brief review of generic proof techniques. The techniques presented here are extracted from [KT59], [Zam79], [Pro66], and [Elb+20]. It is remarkable that almost all of these papers' results are underlain by only a small number of ideas. Since we aimed to present those ideas generically, the presentation in this section is necessarily relatively informal.

The setting In this section we will use the following generic notations.

- $\mathcal{F}$ a set of functions $f : \mathcal{T} \to \mathbb{R}$ or $\mathbb{C}$ with some nice assumptions, equipped with a norm $\|\cdot\|_{\mathcal{F}}$. For example, we may consider the case $\mathcal{T} = \mathbb{R}^n$ and $(\mathcal{F}, \|\cdot\|_{\mathcal{F}}) = (L^2(\mathbb{R}^n), \|\cdot\|_{L^2(\mathbb{R}^n)})$.
- $\mathcal{F}_+$ a subset of $\mathcal{F}$.
- The goal is to upper-bound $N_\varepsilon(\mathcal{F}_+; \mathcal{F})$ – in other words, to construct an ε-covering of $\mathcal{F}_+$ with smallest possible cardinality.

Remark 2.1. We only reviewed proof techniques for the problem of constructing coverings, and not packings. In other words we look only at upper-bounds on the metric entropy, and not lower-bounds. Indeed it seems that there are no really general reusable techniques, and that packings are constructed on an ad-hoc basis, although often building on the same ideas as for coverings.

2.1.1 Orthonormal decomposition: reduction to $\ell^2(\mathbb{N})$

Suppose that $\mathcal{F}$ is a separable Hilbert space. Then $\mathcal{F}$ is isometrically isomorphic to $\ell^2(\mathbb{N})$ and, by considering $\Sigma \subset \ell^2(\mathbb{N})$ such that $\mathcal{F}_+ \simeq \Sigma$, we can simply consider coverings in $\ell^2(\mathbb{N})$.

In typical concrete settings, the isometric isomorphism corresponds to the expansion in a naturally-arising orthonormal basis (ONB) $(\phi_n)_n$:

$$f = \sum_{n \in \mathbb{N}} c_n \phi_n \;\simeq\; (c_n)_{n \in \mathbb{N}} \in \ell^2(\mathbb{N}) \tag{2.1.1}$$

$$\mathcal{F}_+ = \left\{ \sum_{n \in \mathbb{N}} c_n \phi_n ;\; (c_n)_n \in \Sigma \right\} \;\simeq\; \Sigma \tag{2.1.2}$$

As a typical example [KT59, section 6], Σ is an ellipsoid in $\ell^2(\mathbb{N})$. That is, $\Sigma = \left\{ (c_n)_n ;\; \sum_n \frac{c_n^2}{a_n^2} < \infty \right\}$ for some sequence $a_n \downarrow 0$. A rather complete treatment of the ellipsoid case is contained in [Pro66], Theorem 3 and Corollary 4.

2.1.2 Frame-like decomposition: truncating the expansion and quantizing the coefficients

See [Böl20, sections 1.3, 1.4] for background on frame theory.

Suppose that $\mathcal{F}$ is a separable Hilbert space and denote $\langle \cdot, \cdot \rangle$ its inner product. In many cases, it is natural to express the functions $f \in \mathcal{F}$ using a certain frame $(\phi_n)_{n \in \mathbb{N}}$. That is,

$$\forall f \in \mathcal{F},\; f = \sum_{n \in \mathbb{N}} \langle f, \phi_n \rangle \, \psi_n \tag{2.1.3}$$

where $(\psi_n)_n$ is any dual frame of $(\phi_n)_n$.

Let us describe the main steps of a general strategy for constructing coverings based on this [KT59, sections 7.2, 8] [Zam79].

1. Suppose that we know a frame expansion as above: $\forall f \in \mathcal{F}, f = \sum_{n \in \mathbb{N}} \langle f, \phi_n \rangle \, \psi_n$.

   Remark 2.2. Here we assume that the frame expansion holds for all functions $f \in \mathcal{F}$, and not just $f \in \mathcal{F}_+$. The technique presented here would still apply without that assumption – it is just much easier to present this way.

2. Suppose also that we know a bound on the N-term truncation error for functions in $\mathcal{F}_+$, so that

$$\forall f \in \mathcal{F}_+, \left\| f - \sum_{n \leq N} \langle f, \phi_n \rangle \, \psi_n \right\| = \left\| \sum_{n=N+1}^{\infty} \langle f, \phi_n \rangle \, \psi_n \right\| \leq \varepsilon_{\text{trunc}}(N) \tag{2.1.4}$$

This already yields an upper-bound on the ε-constructive-dimension of $\mathcal{F}_+$. Indeed, by definition, $f \mapsto \sum_{n \leq N} \langle f, \phi_n \rangle \, \psi_n$ is an $\varepsilon_{\text{trunc}}(N)$-approximation of identity over $\mathcal{F}_+$.

To obtain an upper-bound on metric entropy, a possible path is thus to construct an ε-covering of

$$\{ (\langle f, \phi_n \rangle)_{n \leq N},\; f \in \mathcal{F}_+ \} \tag{2.1.5}$$

for some corresponding norm on $\mathbb{R}^{N+1}$. As an example:

3. Suppose furthermore that the coefficients are uniformly bounded, for functions in $\mathcal{F}_+$:

$$\forall f \in \mathcal{F}_+, \forall n \leq N,\;\; |c_n| = |\langle f, \phi_n \rangle| \leq C \tag{2.1.6}$$

   If entry-wise bounds (of the form $|c_n| \leq C_n$) are available, then they can be used to get a more economical covering. The uniform bound assumption is just for illustration.

4. Suppose finally that we have a known bound over the coefficient-quantization error. More precisely, for a $\delta > 0$ to be chosen,

$$[\forall n, |c_n - \hat{c}_n| \leq \delta] \implies \left\| \sum_{n \leq N} (c_n - \hat{c}_n)\psi_n \right\| \leq \varepsilon_{\text{quant}}(\delta, N) \tag{2.1.7}$$

For example if the frame $(\psi_n)_n$ is an ONB,

$$\left\| \sum_{n \leq N} (c_n - \hat{c}_n)\psi_n \right\|^2 = \sum_{n \leq N} |c_n - \hat{c}_n|^2 \leq N\delta^2 \tag{2.1.8}$$

(This also holds with an extra constant factor if $(\psi_n)_n$ is a tight frame [Böl20, theorem 1.3.20].) Alternatively if the norms of the vectors are known,

$$\left\| \sum_{n \leq N} (c_n - \hat{c}_n)\psi_n \right\| \leq \sum_{n \leq N} |c_n - \hat{c}_n| \|\psi_n\| \leq \left[\sum_{n \leq N} \|\psi_n\| \right] \delta \tag{2.1.9}$$

5. With the assumptions above, by choosing N and δ so that $\varepsilon_{\text{trunc}} + \varepsilon_{\text{quant}} \leq \varepsilon$, we have constructed an ε-covering, since

$$g := \sum_{n \leq N} \hat{c}_n \psi_n \tag{2.1.10}$$

$$\|f - g\| = \left\| \sum_{n \leq N} (c_n - \hat{c}_n)\psi_n + \sum_{n=N+1}^{\infty} c_n \psi_n \right\| \leq \varepsilon_{\text{quant}}(\delta, N) + \varepsilon_{\text{trunc}}(N) \leq \varepsilon \tag{2.1.11}$$

6. All that remains is to count the number of elements required for the covering. If the $|c_n|$ are known to be uniformly bounded by C for example, we can quantize each coefficient by its closest point in $\left\{ -\left[\frac{C}{\delta}\right]\delta, ..., +\left[\frac{C}{\delta}\right]\delta \right\}$, so that the covering uses (approximately) $\left(\frac{2C}{\delta}\right)^N$ prototypes g, and so

$$N_{\varepsilon_{\text{quant}}(\delta,N) + \varepsilon_{\text{trunc}}(N)}(\mathcal{F}_+; \|\cdot\|) \leq O\left(\left(\frac{2C}{\delta}\right)^N\right) \tag{2.1.12}$$

Note that it was necessary to have a bound on the coefficients $|c_n|$, in order to quantize the c_n's using a finite number of points.

Remark 2.3. In our presentation above, we implicitly assumed that the norm of interest $\|\cdot\|$ is the one associated to the inner product $\langle \cdot, \cdot \rangle$ of the separable Hilbert space $\mathcal{F}$. Although it does greatly simplify things, this assumption is not necessary for the general strategy to make sense. For example in section C.3, "$\|\cdot\|$" corresponds to $\|\cdot\|_{L^\infty(\mathbb{R})}$ whereas "$\langle \cdot, \cdot \rangle$" corresponds to $\langle \cdot, \cdot \rangle_{L^2(\mathbb{R})}$.

2.1.3 Sampling under smoothness assumptions: quantizing the samples on a δ-covering of the input set

With a good choice of sample points and with appropriate smoothness assumptions, $f : \mathcal{T} \to \mathbb{R}$ can be approximately reconstructed from its samples $(f(t_i))_{i \leq m}$.

1. Suppose that the input set $\mathcal{T}$ is a relatively compact subset of a Banach $(\mathbb{T}, \|\cdot\|_{\mathbb{T}})$. A typical example would be $\mathcal{T} = [0,1]^d \subset \mathbb{R}^d = \mathbb{T}$. $\mathcal{T}$ is totally bounded so $N_\delta^{\mathcal{T}}(\mathcal{T}) < \infty$ for all $\delta > 0$.

   For a fixed δ to be chosen, let a smallest δ-self-covering of $\mathcal{T}$, $(t_1, ..., t_m) \subset \mathcal{T}$ where $m = N_\delta^{\mathcal{T}}(\mathcal{T})$.

Remark 2.4. If $\mathcal{T}$ is not itself totally bounded but contains a totally bounded subset $\mathcal{T}'$ such that $\forall f \in \mathcal{F}_+, \|f - f|_{\mathcal{T}'}\|_{\mathcal{F}} \leq \varepsilon_{restr}$, then we may instead construct coverings of $\mathcal{F}'_+ = \{ f|_{\mathcal{T}'} : \mathcal{T}' \to \mathbb{R},\ f \in \mathcal{F}\}$. Coverings of $\mathcal{F}'_+$ then induce coverings of $\mathcal{F}_+$ with an additional term ε_{restr} in the error.

This is the case for example if f is exponentially decaying [Zam79]: $\mathcal{T} = \mathbb{R}$ and $|f(t)| \leq e^{-|t|}$, then taking $\mathcal{T}' = [-t_0, t_0]$ we have

- For $\|\cdot\|_{\mathcal{F}} = \|\cdot\|_{L^\infty}$: $\left\| f - f|_{[-t_0,t_0]} \right\|_{L^\infty} \leq e^{-t_0} =: \varepsilon_{restr}$
- For $\|\cdot\|_{\mathcal{F}} = \|\cdot\|_{L^1}$: $\left\| f - f|_{[-t_0,t_0]} \right\|_{L^1} \leq 2\int_{t_0}^{\infty} e^{-t} dt = 2e^{-t_0} =: \varepsilon_{restr}$

Similar things can be said for other norms (e.g $\|\cdot\|_{\mathcal{F}} = \|\cdot\|_{L^p}$) and for different kinds of decay ($|f(t)| \leq \phi(t)$).

2. Suppose also that $\mathcal{F}_+$ is such that we have a bound of the form

$$\forall f \in \mathcal{F}_+, \forall t, t' \in \mathcal{T}, |f(t) - f(t')| \leq \omega(\|t - t'\|_{\mathbb{T}}) \tag{2.1.13}$$

for some known function ω (corresponding to the modulus of equicontinuity of $\mathcal{F}_+$ as discussed later in this chapter).

These two conditions guarantee that the samples $(f(t_i))_{i \leq m}$ approximately characterize f, which gives a bound on the ε-dimension of $\mathcal{F}_+$. More precisely, if the metric is $\|\cdot\|_{\mathcal{F}} = \|\cdot\|_{L^\infty(\mathcal{T})}$, the $(\omega(\delta))$-dimension is bounded by $m = N_\delta^{\mathcal{T}}(\mathcal{T})$. If we are actually interested in the metric $\|\cdot\|_{L^p_\mu(\mathcal{T})}$ for some $1 \leq p < \infty$ and some finite measure μ over $\mathcal{T}$, we can always use that $\|\cdot\|_{L^p} \leq \mu(\mathcal{T})^{1/p} \ \|\cdot\|_{L^\infty}$.

To obtain an upper-bound on metric entropy, a possible path is thus to construct an ε-covering of

$$\{(f(t_i))_{i \leq m},\ f \in \mathcal{F}_+\} \tag{2.1.14}$$

which is a subset of a finite-dimensional space. As an example:

3. Suppose furthermore that the $f \in \mathcal{F}_+$ are uniformly bounded: $\forall f \in \mathcal{F}_+, \forall t \in \mathcal{T}, |f(t)| \leq M$. If a point-wise bound is available, then it can be used to get a more economical covering.
4. Suppose finally that the space is $\mathcal{F} = L^\infty(\mathcal{T})$.
5. With these assumptions, quantize the sequence of samples $(f(t_i))_{i \leq m}$ using a finite number of sequences $(y_i)_{i \leq m}$, with precision $\varepsilon_{\text{quant}}$ to be chosen.

   More precisely, fix $f \in \mathcal{F}_+$, and

   - For each i, denote by $\hat{y}_i$ the point of $\varepsilon_{\text{quant}}\mathbb{Z} \cap [-M, M]$ that is closest to $f(t_i)$.
   - Define g by: $\forall t, g(t) = \hat{y}_{i(t)}$ where $i(t)$ is any i such that $\|t - t_i\|_{\mathbb{T}} \leq \delta$. Note that the sequence $(\hat{y}_i)_i$ can only take a finite number of values, so that we require only a finite number of prototypes g.
   - Then the approximation error is bounded by:

$$\begin{aligned} \forall t \in \mathcal{T}, |f(t) - g(t)| &= |f(t) - \hat{y}_{i(t)}| && (2.1.15)\\ &\leq |f(t) - f(t_{i(t)})| + |f(t_{i(t)}) - \hat{y}_i| && (2.1.16)\\ \|f - g\|_{L^\infty} &\leq \omega(\delta) + \varepsilon_{\text{quant}} && (2.1.17) \end{aligned}$$

   Hence, by choosing δ and $\varepsilon_{\text{quant}}$ so that $\omega(\delta) + \varepsilon_{\text{quant}} \leq \varepsilon$, we have constructed an ε-covering of $\mathcal{F}_+$.

6. All that remains is to count the number of elements required for the covering, i.e the number of prototypes g required, i.e the number of values that can be taken by the sequence of quantized values $(\hat{y}_i)_i$.

   Clearly each $\hat{y}_i \in \varepsilon_{\text{quant}}\mathbb{Z} \cap [-M, M]$ can take at most $O\left(\frac{2M}{\varepsilon_{\text{quant}}}\right)$ values. Consequently the sequence $(\hat{y}_i)_{i \leq m}$ can take at most $O\left(\left(\frac{2M}{\varepsilon_{\text{quant}}}\right)^m\right)$ values, and so

$$N_{\omega(\delta)+\varepsilon_{\text{quant}}}(\mathcal{F}_+, \|\cdot\|_{L^\infty}) \leq \left(\frac{2M}{\varepsilon_{\text{quant}}}\right)^m \tag{2.1.18}$$

Note that here, we quantized the sample values $f(t_i)$ *separately* for each i, so our covering uses all of the sequences $(y_i)_{i \leq m}$ in a cartesian product $Y_1 \times ... \times Y_m$. This already yields an upper-bound and is sometimes sufficient, e.g [KT59, section 9.1]. However, in many cases we can do much better by jointly quantizing the sequence of sampled values, as this allows us to better exploit regularity properties of $f \in \mathcal{F}_+$ [KT59, sections 2.2, 9.2].

Remark 2.5. As made obvious by our presentation, there is a strong resemblance between the strategy based on truncating a frame decomposition, and the one based on sampling at well-chosen points. Morally, this is because our sampling-based construction can be interpreted as expanding f on the "frame" of piece-wise continuous functions, with pieces corresponding to a δ-covering of the input set. The natural framework to formalize this idea is that of reproducing kernel spaces.

2.1.4 Approximate sparse representation (representation systems)

The idea of "truncating and quantizing an expansion" is applicable more generally when we have assumptions of the following form. They arise naturally in the context of approximation by (ordered) representation systems, a.k.a (ordered) dictionaries.

As an introduction, suppose that there exist functions $(\phi_i)_{i \in I}$ such that

$$\forall f \in \mathcal{F}_+, \forall M \in \mathbb{N}, \exists (c_j)_{j \in J} \text{ where } J \subset I \text{ and } |J| = M; \quad \left\| f - \sum_{j \in J} c_j \phi_j \right\|_{\mathcal{F}} \leq \varepsilon_{repr}(M) \tag{2.1.19}$$

Note that J may depend on f and M, and I is not assumed ordered yet.

The above includes the assumptions of the frame-decomposition-based strategy as a special case. There we approximated f by quantizing the coefficients of its N-first-terms approximation. Morally in the dictionary-representation case, we may approximate f by quantizing the coefficients that appear in its M-term approximation. Following the same steps as in the two previous constructions, suppose $\mathcal{F}_+$ is such that:

1.,2. The representation error is bounded as in the above equation.

3. The coefficients are bounded e.g uniformly: $\forall i, |c_i| \leq C$.

4. The coefficient-quantization error can be bounded as a function of the quantization interval δ:

$$[\forall j, |c_j - \hat{c}_j| \leq \delta] \implies \left\| \sum_{j \in J} (c_j - \hat{c}_j)\phi_j \right\|_{\mathcal{F}} \leq \varepsilon_{\text{quant}}(\delta, M) \tag{2.1.20}$$

However, we are still missing a piece. To see this, try to construct an ε-covering of $\mathcal{F}_+$ using our current ingredients.

- Contrary to the frame-decomposition-based strategy, we cannot simply take as prototypes $g_{(\hat{c}_j)_{j \in J}} = \sum_{j \in J} \hat{c}_j \phi_j$, because the set J may depend on f.

- We cannot either simply take the union of all such prototypes $\bigcup_{J\subset I,|J|=M}\left\{g_{(\hat{c}_j)_{j\in J}};\ (\hat{c}_j)_j\in(\delta\mathbb{Z}\cap[-C,C])^J\right\}$, because there are infinitely many such J.

Hence, we also need to restrict which J may be chosen in the representation (2.1.19). This leads to the concept of "best M-term approximation subject to polynomial-depth search", where "depth" is with respect to a prespecified ordering of I. These considerations are presented in detail in [Elb+20].

In particular, [Elb+20, table 1] lists a number of known metric entropy estimates, for the $\|\cdot\|_{L^2}$ norm. The corresponding $\mathcal{F}_+$ are unit balls in some normed space that embeds compactly into $L^2(\mathbb{R})$ (leftmost column). So the results of the table can also be seen as quantifying the degree of compactness of those embeddings (see the last section of this chapter).

2.2 Continuous functions

In this section we illustrate the proof technique based on sampling under smoothness assumptions, by spaces of continuous functions.

Throughout this section we assume $\mathcal{T}\subset\mathbb{R}$, but similar results hold for $\mathcal{T}\subset\mathbb{R}^d$, and for $\mathcal{T}\subset\mathbb{T}$ for a well-behaved metric space $\mathbb{T}$.

2.2.1 Definitions

Definition 2.1. Let $\mathcal{T}\subset\mathbb{R}$. Let F a set of functions $f:\mathcal{T}\to\mathbb{R}$.

- F is *equibounded at point* $t\in\mathcal{T}$ if $\exists M_t;\ \forall f\in F,\ \ |f(t)|\leq M_t$. It is *pointwise equibounded* if bounded at every point.

  F is *uniformly equibounded* if $\exists M;\ \forall t\in\mathcal{T},\forall f\in F,\ \ |f(t)|\leq M$.

- F is *equicontinuous at point* $t\in\mathcal{T}$ if $\forall\varepsilon>0,\exists\delta_t>0;\ \forall f\in F,\forall t'\in\mathcal{T},\ \ |t-t'|\leq\delta_t\implies|f(t)-f(t')|\leq\varepsilon$. It is *pointwise equicontinuous* if equicontinuous every point.

  F is *uniformly equicontinuous* if $\forall\varepsilon>0,\exists\delta>0;\ \forall f\in F,\forall t,t'\in\mathcal{T},\ \ |t-t'|\leq\delta\implies|f(t)-f(t')|\leq\varepsilon$.

Definition 2.2. Let $f:\mathcal{T}\to\mathbb{R}$.

- The *modulus of continuity of f at point* $t\in\mathcal{T}$ is $\omega_t(f;\delta)=\sup_{t'\in\mathcal{T};|t-t'|\leq\delta}|f(t)-f(t')|$
- The *modulus of (uniform) continuity* of f is $\omega(f;\delta)=\sup_{t,t'\in\mathcal{T};|t-t'|\leq\delta}|f(t)-f(t')|$

Note that f is continuous at $t\in\mathcal{T}$ if and only if $\lim_{\delta\to0}\omega_t(f;\delta)=0$, and that f is uniformly continuous if and only if $\lim_{\delta\to0}\omega(f;\delta)=0$.

Let $F\subset C(\mathcal{T})$.

- The *modulus of equicontinuity of F at point* $t\in\mathcal{T}$ is $\omega_t(F;\delta)=\sup_{f\in F}\omega_t(f;\delta)$
- The *modulus of (uniform) equicontinuity* of F is $\omega(F;\delta)=\sup_{f\in F}\omega(f;\delta)$

Note that F is equicontinuous at $t\in\mathcal{T}$ if and only if $\lim_{\delta\to0}\omega_t(F;\delta)=0$, and that F is uniformly equicontinuous if and only if $\lim_{\delta\to0}\omega(F;\delta)=0$.

Denote $C(\mathcal{T})$ the space of real-valued continuous functions over $\mathcal{T}$. Note that a subset $F\subset C(\mathcal{T})$ is a priori not equicontinuous at any point, even if uniformly bounded (just take as F a centered ball in $C(\mathcal{T})$).

The following proposition illustrates that, by focusing on the modulus of continuity, this section's discussion covers some natural notions of smoothness. It does not cover other notions, however, such as Hoelderness of the derivatives [KT59, section 5]; nor global regularity notions such as Sobolev spaces.

Proposition 2.1. Let $L > 0$. $f \in C(\mathcal{T})$ is called *L-lipschitz* if $\forall t, t' \in \mathcal{T}, |f(t) - f(t')| \leq L\,|t - t'|$.
f is L-lipschitz if and only if

$$\forall \delta > 0,\ \omega(f;\delta) \leq L\delta \tag{2.2.1}$$

Let $C > 0, 0 < \alpha < 1$. $f \in C(\mathcal{T})$ is called (C,α)*-Hoelder* if $\forall t, t' \in \mathcal{T}, |f(t) - f(t')| \leq C\,|t - t'|^\alpha$.
f is (C,α)-Hoelder if and only if

$$\forall \delta > 0,\ \omega(f;\delta) \leq C\delta^\alpha \tag{2.2.2}$$

2.2.2 Over a compact domain

Let us start by recalling the Arzela-Ascoli theorem, which characterizes totally bounded (i.e relatively compact) sets of continuous functions over compact domains.

Theorem 2.2 (Arzela-Ascoli)**.** Suppose $\mathcal{T}$ is a compact, i.e closed bounded subset of $\mathbb{R}$. Then $C(\mathcal{T})$ equipped with the sup norm $\|\cdot\|_{L^\infty}$ is a Banach, and

1. A subset F of $C(\mathcal{T})$ is pointwise equicontinuous if and only if uniformly equicontinuous.
2. If F is pointwise equibounded and equicontinuous, then it is uniformly equibounded over $\mathcal{T}$.
3. F is a relatively compact subset of $C(\mathcal{T})$ if and only if it is (pointwise or uniformly) equibounded and equicontinuous.

Proof. 1. Straightforward variant of Heine theorem. [1]

2. Take $\varepsilon = 1$, a corresponding δ as in the definition of uniform equicontinuity, and $(t_1, ..., t_m)$ a finite δ-covering of $\mathcal{T}$. Then $\forall f \in F, \forall t \in \mathcal{T}, |f(t)| \leq \left|f(t) - f(t_{i(t)})\right| + \left|f(t_{i(t)})\right| \leq 1 + \max_{j \leq m} M_{t_j}$.
3. Pointwise-equibounded version: Arzela-Ascoli theorem [HH10]. Uniformly-equibounded version: the three previous points.

□

Henceforth in this subsection, let the domain $\mathcal{T}$ be a compact subset of $\mathbb{R}$. Note that with minimal effort we could generalize to $\mathcal{T}$ relatively compact, since uniformly continuous functions then have a unique continuous extension to $\overline{\mathcal{T}}$ by Cauchy-continuity. [2]

Let $F \subset (C(\mathcal{T}), \|\cdot\|_{L^\infty})$. Since we want to estimate the metric entropy of F, we must assume F totally bounded, i.e relatively compact. Equivalently we must assume F equibounded, i.e $\sup_{f \in F} \|f\|_{L^\infty} < \infty$, and equicontinuous, i.e $\lim_{\delta \to 0} \omega(F;\delta) = 0$.

Is it possible to estimate $N_\varepsilon(F; C(\mathcal{T}))$ given only its uniform bound and its modulus of equicontinuity? It turns out that upper estimates can be found, provided some knowledge of $\mathcal{T}$ is also given.

Proposition 2.3 ([CS90, theorem 5.6.1])**.** Let F a relatively compact subset of $C(\mathcal{T})$. The approximation number of F is bounded by

$$\forall n \in \mathbb{N}^*,\ a_{n+1}(F; C(\mathcal{T})) \leq \omega(F; \varepsilon_n^{\text{self}}(\mathcal{T})) \tag{2.2.3}$$

where $\varepsilon_n^{\text{self}}(\mathcal{T})$ is the inverse function of the ε-self-covering number.

Proof. Let $n \in \mathbb{N}^*$, $\delta > \varepsilon_n^{\text{self}}(\mathcal{T})$ and $(t_1, ..., t_n) \subset \mathcal{T}$ a δ-self-covering of $\mathcal{T}$.

Let $\{\varphi_1, ..., \varphi_n\}$ an associated partition of unity:

$$\operatorname{dist}(t, (B_{t_i,\delta})^c) = \max(0, \delta - \|t - t_i\|_{\mathbb{T}}) \tag{2.2.4}$$

$$\forall i,\ \varphi_i(t) = \frac{\operatorname{dist}(t, (B_{t_i,\delta})^c)}{\sum_{j=1}^n \operatorname{dist}(t, (B_{t_j,\delta})^c)} \tag{2.2.5}$$

Other choices of $\{\varphi_i\}$ would work, as long as they satisfy

[1] `https://math.stackexchange.com/questions/3947351/uniform-equicontinuity-from-pointwise-equicontinuity-over-compact-sey`
[2] `https://en.wikipedia.org/wiki/Cauchy-continuous_function`

- $\varphi_i \in C(\mathcal{T})$
- $0 \le \varphi_i(t) \le 1$
- $\sum_i \varphi_i(t) = 1$
- $\varphi_i(t) \neq 0 \implies t \in B_{t_i,\delta}$

For each $f \in F$, denote $\hat{f}$ the function

$$\hat{f}(t) = \sum_{i=1}^{n} f(t_i)\varphi_i(t) \tag{2.2.6}$$

Clearly $f \mapsto \hat{f}$ is a linear operator from $C(\mathcal{T})$ to itself of rank at most n. To show that this operator approximates identity over F up to an error of $\omega(F;\delta)$, write for each $f \in F$

$$f(t) = f(t)\sum_{i=1}^{n} \varphi_i(t) \tag{2.2.7}$$

$$f(t) - \hat{f}(t) = \sum_{i=1}^{n} \left(f(t) - f(t_i)\right)\varphi_i(t) \tag{2.2.8}$$

$$\left|f(t) - \hat{f}(t)\right| \le \sum_{i=1}^{n} |f(t) - f(t_i)|\,\varphi_i(t) \tag{2.2.9}$$

For a fixed t, denote $i_1, ..., i_k$ the indices such that $t \in B_{t_i,\delta}$,

$$\begin{aligned} \left|f(t) - \hat{f}(t)\right| &\le \sum_{i\in\{i_1,\dots,i_k\}} |f(t) - f(t_i)|\,\varphi_i(t) + 0 && (2.2.10)\\ &\le \omega(f;\delta) \sum_{i\in\{i_1,\dots,i_k\}} \varphi_i(t) && (2.2.11)\\ &= \omega(f;\delta) && (2.2.12)\\ \forall f \in F,\ \left\|f - \hat{f}\right\|_{L^\infty} &\le \omega(F;\delta) && (2.2.13) \end{aligned}$$

by the properties of the partition of unity, listed above.

Thus we showed that $f \mapsto \hat{f}$ provides a rank-n approximation of identity over F up to error of $\omega(F;\delta)$, so

$$\forall n \in \mathbb{N}^*, \forall \delta > \varepsilon_n^{\mathrm{self}}(\mathcal{T}),\ a_{n+1}(F; C(\mathcal{T})) \le \omega(F;\delta) \tag{2.2.14}$$

Now by the following lemma, $\delta \mapsto \omega(F;\delta)$ is right-continuous, which concludes the proof. □

Lemma 2.4 ([CS90, propositions 5.4.1 and 5.5.2])**.** Let $f \in C(\mathcal{T})$. The modulus of continuity $\delta \mapsto \omega(f;\delta)$ is non-decreasing and continuous from the right at each $\delta \ge 0$.

Let F a relatively compact subset of $C(\mathcal{T})$. The modulus of equicontinuity $\delta \mapsto \omega(F;\delta)$ is non-decreasing and continuous from the right at each $\delta \ge 0$.

2.2.3 Over an unbounded domain

The previous subsection requires $\mathcal{T}$ to be compact, e.g limited to a finite interval if $\mathcal{T} \subset \mathbb{R}$, whereas we would also like to consider signals defined over all of $\mathbb{R}$.

Definition 2.3. [3] $C(\mathbb{R})$ denotes the set of real-valued continuous functions over $\mathbb{R}$.

[3] https://regularize.wordpress.com/2011/11/11/dual-spaces-of-continuous-functions/

- Denote $C_b(\mathbb{R})$ the set of bounded continuous functions. Then $(C_b(\mathbb{R}), \|\cdot\|_{L^\infty})$ is a Banach space (as a closed subset of $L^\infty(\mathbb{R})$).
- Denote $C_0(\mathbb{R})$ the set of continuous functions $f(t)$ such that $\lim_{t\to\pm\infty} f(t) = 0$. Then $(C_0(\mathbb{R}), \|\cdot\|_{L^\infty})$ is a Banach space (as a closed subset of $C_b(\mathbb{R})$).

The following proposition characterizes the relatively compact subsets of $C_0(\mathbb{R})$, analogously to the Arzela-Ascoli theorem. Essentially, uniform boundedness must be replaced by domination by some vanishing function. Recall that since the ambient space is Banach, F is relatively compact if and only if it is totally bounded i.e has finite ε-coverings for any $\varepsilon > 0$.

Proposition 2.5. *$F \subset C_0(\mathbb{R})$ is relatively compact if and only if it is pointwise equicontinuous and there exists $\phi : \mathbb{R} \to \mathbb{R}_+$ such that $\lim_{t\to\pm\infty} \phi(t) = 0$ and $\forall t, \sup_{f\in F} |f(t)| \le \phi(t)$.*
(This is also a sufficient condition for $F \subset C_b(\mathbb{R})$ to be relatively compact.)

Proof. This is a straightforward application of the proof method of [HH10].

Necessity To show equicontinuity at $t_0 \in \mathbb{R}$, let $\mathcal{T}$ any compact neighborhood of t_0. Then $\{f|_{\mathcal{T}}, f \in F\}$ is totally bounded with respect to $C(\mathcal{T})$ (since F is with respect to $C(\mathbb{R})$ and ε-coverings of the latter induce coverings of the former), so by Arzela-Ascoli theorem F is equicontinuous at t_0.

To show $\lim_{\pm\infty} \phi = 0$, let $\varepsilon > 0$. Since F is relatively compact i.e totally bounded, there exist $f^{(1)}, ..., f^{(m)} \in C_0(\mathbb{R})$ $(m < \infty)$ such that $\forall f \in F,\ \inf_i \left\|f - f^{(i)}\right\|_{L^\infty} \le \varepsilon/2$. Let t_0 such that $\forall |t| > t_0,\ \sup_i \left|f^{(i)}(t)\right| \le \varepsilon/2$. Then $\forall |t| > t_0,\ \phi(t) \le \varepsilon$.

Sufficiency To show that F is totally bounded, let arbitrary $\varepsilon > 0$ and let us show that F has a finite ε-covering.

Let t_0 such that $\forall |t| > t_0, \phi(t) \le \varepsilon$, and let $\mathcal{T} = [-t_0, t_0]$. Then $\{f|_{\mathcal{T}}, f \in F\}$ satisfies the conditions of Arzela-Ascoli theorem, so is totally bounded, so has a finite ε-covering $f^{(1)}, ..., f^{(m)} \in C(\mathcal{T})$. Extend those functions to elements of $C_0(\mathbb{R})$ (e.g by linear interpolation such that $f^{(i)}(t) = 0$ for $|t| > 2t_0$). This gives an ε-covering of F, since the norm is the sup norm.

□

To make a slightly more precise characterization, let us look at the least dominating function ϕ.

Proposition 2.6. *Let $F \subset C_0(\mathbb{R})$ and denote $\phi(t) = \sup_{f\in F} |f(t)|$. Then ϕ is lower semi-continuous.*
If furthermore F is pointwise equicontinuous, then ϕ is continuous.
In particular F is relatively compact if and only if it is pointwise equicontinuous and $\phi \in C_0(\mathbb{R})$.

Proof. Suppose $F \subset C_0(\mathbb{R})$. Let $t \in \mathbb{R}$ and $\varepsilon > 0$, and $f_0 \in F$ such that $|f_0(t)| \ge \phi(t) - \varepsilon/2$. There exists a neighborhood of t on which f_0 stays $(\varepsilon/2)$-close to $f_0(t)$. For all t' in that neighborhood,

$$\phi(t') \ge |f_0(t')| \ge |f_0(t)| - \varepsilon/2 \ge \phi(t) - \varepsilon \tag{2.2.15}$$

Which shows that ϕ is lower semi-continuous at t.

Now suppose F pointwise equicontinuous, i.e $\forall t, \omega_t(F;\delta) \xrightarrow{\delta\to 0} 0$. Let $t \in \mathbb{R}$ and $\varepsilon > 0$, and δ such that $\omega_t(F;\delta) \le \varepsilon$. For all t' such that $|t' - t| \le \delta$,

$$\begin{aligned}
&\forall f \in F,\ |f(t') - f(t)| \le \varepsilon && (2.2.16)\\
\iff &\forall f \in F,\ |f(t)| - \varepsilon \le |f(t')| \le |f(t)| + \varepsilon && (2.2.17)\\
\implies &\forall f \in F,\ |f(t)| - \varepsilon \le \phi(t') \text{ and } |f(t')| \le \phi(t) + \varepsilon && (2.2.18)\\
\iff &\phi(t) - \varepsilon \le \phi(t') \le \phi(t) + \varepsilon && (2.2.19)
\end{aligned}$$

Which shows that ϕ is continuous at t. □

Let $F \subset (C_0(\mathbb{R}), \|\cdot\|_{L^\infty})$. Again assume F totally bounded, i.e relatively compact, and we want to estimate the metric entropy of F.

In view of the above characterization, is it possible to estimate $N_\varepsilon(F; C_0(\mathbb{R}))$ given only its pointwise bound and its pointwise modulus of equicontinuity? It turns out that the same idea as in the above proof allows us to transfer the upper-estimate for the compact-domain case to the $C_0(\mathbb{R})$ case.

Proposition 2.7. Let F relatively compact subset of $C_0(\mathbb{R})$. Denote $\phi(t) = \sup_{f\in F} |f(t)|$ and recall the definition of $\omega_t(F; \delta) = \sup_{f\in F} \sup_{t';|t-t'|\le\delta} |f(t) - f(t')|$. By assumption, $\phi \in C_0(\mathbb{R})$, and $\lim_{\delta\to 0} \omega_t(F; \delta) = 0$ for each t.

The approximation number of F in $L^\infty(\mathbb{R})$ is bounded by

$$\forall n \in \mathbb{N}^*,\ a_{n+1}(F; L^\infty(\mathbb{R})) \le \inf_{\eta>0}\ \max\left\{\eta,\ \omega_{\mathcal{T}_\eta}(F; \varepsilon_n(\mathcal{T}_\eta))\right\} \tag{2.2.20}$$

where $\mathcal{T}_\eta = \phi^{-1}([\eta, +\infty[)$ and $\omega_{\mathcal{T}}(F; \delta) = \sup_{t\in\mathcal{T}} \omega_t(F; \delta)$.

The approximation number of F in $C_0(\mathbb{R})$, i.e when the prototypes are required to be continuous and vanishing, is bounded by

$$\forall n \in \mathbb{N}^*,\ a_{n+1}(F; C_0(\mathbb{R})) \le \inf_{\eta>0}\ \eta + \omega_{\mathcal{T}_\eta}(F; \varepsilon_n^{\text{self}}(\mathcal{T}_\eta)) \tag{2.2.21}$$

Proof. Let $n \in \mathbb{N}^*$ and $\eta > 0$. Since $\phi \in C_0(\mathbb{R})$, $\mathcal{T}_\eta = \phi^{-1}([\eta, +\infty[)$ is a closed bounded i.e compact subset of $\mathbb{R}$. So we may apply a similar construction as in the compact-domain case, except that the functions are known to be continuous on a larger space than just the compact $\mathcal{T}_\eta$.

Let us start by the first statement, with ambient space $L^\infty(\mathbb{R})$. Let $\delta > \varepsilon_n(\mathcal{T}_\eta)$ and $(t_1, ..., t_n) \subset \mathbb{R}$ a δ-covering of $\mathcal{T}_\eta$ in $\mathbb{R}$. Let $\{\varphi_1, ..., \varphi_n\}$ an associated (non-continuous!) partition of unity over $\mathcal{T}_\eta$:

$$\operatorname{dist}(t, (B_{t_i,\delta})^c) = \max(0, \delta - |t - t_i|) \tag{2.2.22}$$

$$\forall i,\ \varphi_i(t) = \mathbb{1}_{t\in\mathcal{T}_\eta} \frac{\operatorname{dist}(t, (B_{t_i,\delta})^c)}{\sum_{j=1}^n \operatorname{dist}(t, (B_{t_j,\delta})^c)} \tag{2.2.23}$$

Other choices of $\{\varphi_i\}$ would work, as long as

- $\varphi_i \in L^\infty(\mathbb{R})$
- $\forall t \in \mathbb{R}, 0 \le \varphi_i(t) \le 1$
- $\forall t \in \mathcal{T}_\eta, \sum_i \varphi_i(t) = 1$
- $\varphi_i(t) \ne 0 \implies t \in \mathcal{T}_\eta \cap B_{t_i,\delta}$

As for the compact-domain case, for each $f \in F$, denote $\hat{f}$ the function

$$\hat{f}(t) = \sum_{i=1}^n f(t_i)\varphi_i(t) \tag{2.2.24}$$

Note that $f \mapsto \hat{f}$ is a linear operator from $C_0(\mathbb{R})$ to $L^\infty(\mathbb{R})$ of rank at most n. By the same reasoning as before, we show that

$$\forall t \in \mathcal{T}_\eta,\ \left|f(t) - \hat{f}(t)\right| \le \omega_{\mathcal{T}_\eta}(F; \delta) \tag{2.2.25}$$

Now for all $t \notin \mathcal{T}_\eta$, $\hat{f}(t) = 0$ by definition, so

$$\forall t \notin \mathcal{T}_\eta,\ \left|f(t) - \hat{f}(t)\right| = |f(t)| \le \phi(t) \le \eta \tag{2.2.26}$$

Thus, $f \mapsto \hat{f}$ has rank at most n and approximates identity over F up to error of $\max\left\{\eta, \omega_{\mathcal{T}_\eta}(F; \delta)\right\}$, so

$$\forall n \in \mathbb{N}^*, \forall\eta > 0, \forall\delta > \varepsilon_n(\mathcal{T}_\eta),\ a_{n+1}(F; L^\infty(\mathbb{R})) \le \max\left\{\eta,\ \omega_{\mathcal{T}_\eta}(F; \delta)\right\} \tag{2.2.27}$$

The announced bound follows by letting $\delta \to \varepsilon_n(\mathcal{T}_\eta)$ from the right and taking the inf over η.

Let us now show the second statement, with ambient space $C_0(\mathbb{R})$. Let $\delta > \varepsilon_n^{\text{self}}(\mathcal{T}_\eta)$ and $(t_1, ..., t_n) \subset \mathcal{T}_\eta$ a δ-self-covering of $\mathcal{T}_\eta$. Let $\{\varphi_1, ..., \varphi_n\}$ an associated continuous partition of unity, i.e such that

- $\varphi_i \in C_0(\mathbb{R})$
- $\forall t \in \mathbb{R}, 0 \leq \varphi_i(t) \leq 1$
- $\forall t \in \mathcal{T}_\eta, \sum_i \varphi_i(t) = 1$
- $\varphi_i(t) \neq 0 \implies t \in B_{t_i,\delta}$
- $\forall t \in \mathbb{R}, 0 \leq \sum_i \varphi_i \leq 1$

Such a family can be obtained for example by starting from $\psi_i(t) = \frac{\text{dist}(t,(B_{t_i,\delta})^c)}{\sum_{j=1}^n \text{dist}(t,(B_{t_j,\delta})^c)}$, and taking $\varphi_i = (\psi_i * \zeta_\sigma)$ for some appropriately chosen mollifier ζ_σ [Gol18, theorem 1.4.3]. As above, denote $\hat{f}(t) = \sum_{i=1}^n f(t_i)\varphi_i(t)$. The linear operator $f \mapsto \hat{f}$ from $C_0(\mathbb{R})$ to itself has rank at most n. By the same reasoning as before, we show that

$$\forall t \in \mathcal{T}_\eta, \quad \left|f(t) - \hat{f}(t)\right| \leq \omega_{\mathcal{T}_\eta}(F;\delta) \tag{2.2.28}$$

Now for $t \notin \mathcal{T}_\eta$:

$$\forall t \notin \mathcal{T}_\eta, \; f(t) - \hat{f}(t) = \sum_{i=1}^n (f(t) - f(t_i))\,\varphi_i(t) + f(t)\left(1 - \sum_{i=1}^n \varphi_i(t)\right) \tag{2.2.29}$$

$$\left|f(t) - \hat{f}(t)\right| \leq \sum_{i=1}^n |f(t) - f(t_i)|\,\varphi_i(t) + |f(t)| \cdot 1 \tag{2.2.30}$$

$$\leq \sum_{i:|t-t_i|\leq\delta} |f(t) - f(t_i)|\,\varphi_i(t) + \phi(t) \tag{2.2.31}$$

$$\leq \omega_{\mathcal{T}_\eta}(F;\delta) + \eta \tag{2.2.32}$$

Thus, $f \mapsto \hat{f}$ has rank at most n and values in $C_0(\mathbb{R})$, and approximates identity over F up to error of $\max\left\{\eta, \omega_{\mathcal{T}_\eta}(F;\delta)\right\}$, so

$$\forall n \in \mathbb{N}^*, \forall \eta > 0, \forall \delta > \varepsilon_n(\mathcal{T}_\eta), \; a_{n+1}(F; C_0(\mathbb{R})) \leq \eta + \omega_{\mathcal{T}_\eta}(F;\delta) \tag{2.2.33}$$

The announced bound follows by letting $\delta \to \varepsilon_n^{\text{self}}(\mathcal{T}_\eta)$ from the right and taking the inf over η. □

2.2.4 Fading-memory norm

The above examples were all variations on standard functional analysis results. [BC85, lemma 1] proposes one that is less well-known, but more useful in practice.

Definition 2.4. A *weight function* is any $w : \mathbb{R} \to \;]0,1]$, such that $\lim_{t\to\pm\infty} w(t) = 0$.

For example, $w(t) = e^{-\lambda|t|}$ is a valid weight function.

Denote $\|f\|_w = \sup_{t\in\mathbb{R}} |w(t)f(t)|$. This defines a norm over $C_b(\mathbb{R})$.

Note that $(C_b(\mathbb{R}), \|\cdot\|_w)$ is not a complete space. To specify its completion, suppose for simplicity that $w(t)$ itself is continuous, and define

$$C_{b,w}(\mathbb{R}) = w^{-1}C_b(\mathbb{R}) = \left\{w^{-1}(t)\underline{f}(t), \; \underline{f} \in C_b(\mathbb{R})\right\} \tag{2.2.34}$$

$$= \{f(t), \; w(t)f(t) \in C_b(\mathbb{R})\} \tag{2.2.35}$$

$$= \left\{f(t) \in C(\mathbb{R}), \; f(t) = O_{t\to\pm\infty}(w^{-1}(t))\right\} \tag{2.2.36}$$

and

$$C_{0,w}(\mathbb{R}) = w^{-1}C_0(\mathbb{R}) = \left\{w^{-1}(t)\underline{f}(t),\ \underline{f} \in C_0(\mathbb{R})\right\} \tag{2.2.37}$$
$$= \{f(t),\ w(t)f(t) \in C_0(\mathbb{R})\} \tag{2.2.38}$$
$$= \left\{f(t) \in C(\mathbb{R}),\ f(t) = o_{t\to\pm\infty}(w^{-1}(t))\right\} \tag{2.2.39}$$

Note that $C_b(\mathbb{R}) \subsetneq C_{0,w}(\mathbb{R})$, e.g $\sqrt{w^{-1}(t)} \in C_{0,w}(\mathbb{R}) \setminus C_b(\mathbb{R})$.

Clearly, since $\|f\|_w = \|wf\|_{L^\infty}$, we have the isometries (see Figure 2.1)

$$(C_{b,w}(\mathbb{R}), \|\cdot\|_w) \simeq (C_b(\mathbb{R}), \|\cdot\|_{L^\infty}) \tag{2.2.40}$$
$$(C_{0,w}(\mathbb{R}), \|\cdot\|_w) \simeq (C_0(\mathbb{R}), \|\cdot\|_{L^\infty}) \tag{2.2.41}$$
$$\text{by } f \mapsto \underline{f} = wf \tag{2.2.42}$$

Proposition 2.8. Let w a continuous weight function. The completions of $C_0(\mathbb{R})$ and $C_b(\mathbb{R})$ with respect to the $\|\cdot\|_w$ norm are:

$$\overline{C_0(\mathbb{R})}^{\|\cdot\|_w} = \overline{C_b(\mathbb{R})}^{\|\cdot\|_w} = C_{0,w}(\mathbb{R}) \tag{2.2.43}$$

Proof. Recall $C_0(\mathbb{R}) \subset C_b(\mathbb{R}) \subset C_{0,w}(\mathbb{R})$. Since $(C_{0,w}(\mathbb{R}), \|\cdot\|_w)$ is complete (since isometric to $C_0(\mathbb{R})$), this shows the left-to-right inclusions.

Let us show the converse inclusion: $\overline{C_0(\mathbb{R})}^{\|\cdot\|_w} \supset C_{0,w}(\mathbb{R})$. By isometry, it suffices to show $\overline{C_{0,w^{-1}}(\mathbb{R})}^{\|\cdot\|_{L^\infty}} \supset C_0(\mathbb{R})$, i.e that any $f \in C_0(\mathbb{R})$ is arbitrarily close in $\|\cdot\|_{L^\infty}$ norm to some $f' \in C_{0,w^{-1}}(\mathbb{R})$. Note that $C_{0,w^{-1}}(\mathbb{R}) = \{f \in C(\mathbb{R});\ f(t) = o(w(t))\}$.

Let $\varepsilon > 0$ and let us show existence of $f' \in C(\mathbb{R})$ such that $f'(t) = o(w(t))$ and $\|f - f'\|_{L^\infty} \le \varepsilon$. Since $f \in C_0(\mathbb{R})$, let t_0 such that $\forall\, |t| > t_0, |f(t)| \le \varepsilon/2$, and let $\mathcal{T} = [-t_0, t_0]$. Simply let $f'(t) := f(t)$ for all $t \in \mathcal{T}$, and extend f' to all of $\mathbb{R}$ by zero (interpolating as needed so that f' is continuous). $\square$

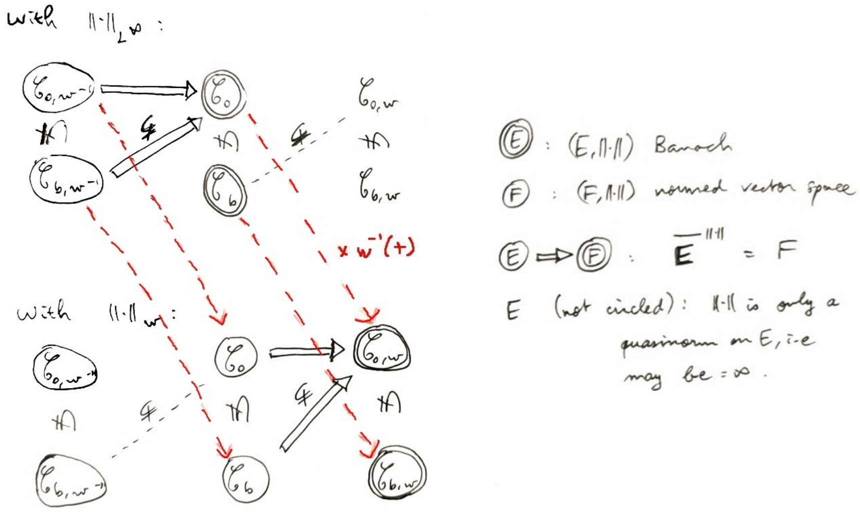


Figure 2.1: Summary of spaces associated with fading-memory norm

Let F a relatively compact subset of $(C_{0,w}(\mathbb{R}), \|\cdot\|_w)$ with w continuous. Equivalently, wF is relatively compact in $(C_0(\mathbb{R}), \|\cdot\|_{L^\infty})$. Furthermore,

$$N_\varepsilon(F; (C_{0,w}(\mathbb{R}), \|\cdot\|_w)) = N_\varepsilon(wF; (C_0(\mathbb{R}), \|\cdot\|_{L^\infty})) \tag{2.2.44}$$

The previous subsection gives an upper-bound on that quantity involving $(w\phi)(t) = \sup_{f\in F} |wf|(t)$ and $\omega_t(wF;\delta) = \sup_f \sup_{t';|t-t'|\leq\delta} |(wf)(t) - (wf)(t')|$. In concrete settings, we can expect to have F with $\phi(t) = \sup_{f\in F} |f(t)|$ and $\omega_t(F;\delta)$ given. This automatically gives us $(w\phi)(t)$; but what about $\omega_t(wF;\delta)$? One quickly notices that, morally, $\omega_t(wF;\delta) \approx w(t)\omega_t(F;\delta)$. More precisely:

Proposition 2.9. Let any $w : \mathbb{R} \to \mathbb{R}_+$ and any F set of functions from $\mathbb{R}$ to $\mathbb{R}$. For all $t \in \mathbb{R}$ and $\delta > 0$,

$$\omega_t(wF;\delta) \leq w(t)\ \omega_t(F;\delta) + \phi(t)\ \omega_t(w;\delta) + \omega_t(w;\delta)\ \omega_t(F;\delta) \tag{2.2.45}$$

Consequently, for w a continuous weight function and $F \subset C_{0,w}(\mathbb{R})$, F is pointwise equicontinuous if and only if wF is.

Proof. Let $f \in F$ and t' such that $|t - t'| \leq \delta$. Abbreviating $w = w(t)$, $w' = w(t')$, $f = f(t)$, $f' = f(t')$,

$$wf - w'f' = w(f - f') + f(w - w') - (w - w')(f - f') \tag{2.2.46}$$

$$|wf - w'f'| \leq w\,|f - f'| + |f|\,|w - w'| + |w - w'|\,|f - f'| \tag{2.2.47}$$

$$\leq w\ \omega_t(f;\delta) + \phi(t)\ \omega_t(w;\delta) + \omega_t(w;\delta)\ \omega_t(f;\delta) \tag{2.2.48}$$

$$|(wf)(t) - (wf)(t')| \leq w(t)\ \omega_t(f;\delta) + \phi(t)\ \omega_t(w;\delta) + \omega_t(w;\delta)\ \omega_t(f;\delta) \tag{2.2.49}$$

Taking the sup over f and t' yields the announced bound.

We turn to the second part of the proposition. Recall that F is pointwise equicontinuous if and only if $\forall t,\ \lim_{\delta\to 0} \omega_t(F;\delta) = 0$. Suppose F pointwise equicontinuous, then wF is too by the above inequality. For the converse, apply the result for w^{-1} instead of w (check that we didn't use that w has values in $]0,1]$, nor that it is vanishing, but only that it is continuous and has values in $\mathbb{R}_+$). ☐

Example 2.1. According to [BC85, lemma 1], the set of M-uniformly bounded and L-lipschitz functions

$$F = \left\{ f(t);\ \begin{array}{c} \forall t,\ |f(t)| \leq M \\ \forall s,t,\ |f(t) - f(s)| \leq L\,|t - s| \end{array} \right\} \tag{2.2.50}$$

is a compact subset of $(C_b(\mathbb{R}), \|\cdot\|_w)$, for any weight function w. Note that it is not a compact subset of $(C_b(\mathbb{R}), \|\cdot\|_{L^\infty})$.

In the case where w is continuous, the relative compactness can also be seen easily from our last two subsections, as we now show. Clearly $F \subset C_b(\mathbb{R})$, so let us show that F is relatively compact in the completion $C_{0,w}(\mathbb{R})$, or equivalently that wF is relatively compact in $C_0(\mathbb{R})$. Now wF is clearly dominated by a vanishing function, namely $Mw(t)$, and it is pointwise equicontinuous since F is and w is continuous. So wF satisfies our characterization of relative compactness from the previous subsection.

Showing closedness alone is also not difficult: check that the Dirac delta $\delta_t : [f \mapsto f(t)]$ is continuous for the $\|\cdot\|_w$ norm for each t (its operator norm is $|||\delta_t|||_{C_{0,w}(\mathbb{R})\to\mathbb{R}} = w(t)^{-1}$), and write F as an intersection of closed sets:

$$F = \bigcap_{t\in\mathbb{R}} \delta_t^{-1}([-M, M])\ \cap\ \bigcap_{s,t\in\mathbb{R}} \left(\frac{\delta_t - \delta_s}{|t - s|}\right)^{-1} ([-L, L]) \tag{2.2.51}$$

This proves that F is compact for the $\|\cdot\|_w$ norm using seemingly different (but ultimately equivalent) arguments than in [BCD84], which used the sequential characterization of compact sets instead.

Since F is compact, it is totally bounded. We stress that the quantity shown to be finite, and that we may wish to estimate, is $\log N_\varepsilon(F; \|\cdot\|_w)$ the metric entropy of F *for the* $\|\cdot\|_w$ *norm*.

2.3 L^p-integrable functions

Besides the sup norm $\|f\|_{L^\infty} = \sup_t |f(t)|$, other norms for which we may be interested in metric entropy estimates include the L^p-norms:

$$\|f\|_{L^p(\mathcal{T})} = \left(\int_{\mathcal{T}} dt \ |f(t)|^p \right)^{1/p} \tag{2.3.1}$$

In this section we present relative compactness conditions, as well as generic metric entropy upper bounds, in ambient space $L^p(\mathcal{T})$, analogously to the discussion of $C(\mathcal{T})$ in the previous section.

Throughout this section, we will assume $\mathcal{T} \subset \mathbb{R}^d$ and integration is against the Lebesgue measure, but similar results hold for $\mathcal{T} \subset \mathbb{T}$ for a well-behaved metric and measure space $\mathbb{T}$ [GM14].

Proposition 2.10 (Kolmogorov-Riesz theorem [HH10])**.** Let $1 \leq p < \infty$.
Let $\mathcal{T}$ a compact subset of $\mathbb{R}^d$. A subset F of $L^p(\mathcal{T})$ is relatively compact if and only if

1. F is bounded: $\sup_{f \in F} \|f\|_{L^p(\mathcal{T})} < \infty$
2. $\sup_{f \in F} \|f(\boldsymbol{t} + \boldsymbol{\tau}) - f(\boldsymbol{t})\|_{L^p(\mathcal{T})} \xrightarrow{\boldsymbol{\tau} \to 0} 0$

Unbounded domain case: a subset F of $L^p(\mathbb{R}^d)$ is relatively compact if and only if

1. F is bounded: $\sup_{f \in F} \|f\|_{L^p(\mathbb{R})} < \infty$. [4]
2. $\sup_{f \in F} \|f(\boldsymbol{t} + \boldsymbol{\tau}) - f(\boldsymbol{t})\|_{L^p(\mathcal{T})} \xrightarrow{\boldsymbol{\tau} \to 0} 0$
3. $\sup_{f \in F} \int_{\|\boldsymbol{t}\| > T} d\boldsymbol{t} \ |f(\boldsymbol{t})|^p \xrightarrow{T \to \infty} 0$

For simplicity, we will restrict attention to the compact-domain case. Extensions to the case of unbounded domain $L^p(\mathbb{R}^d)$ can be developed analogously to our discussion of $C_0(\mathbb{R})$ in the previous section.

Just as the proof of the Arzela-Ascoli theorem allows to establish a generic upper bound on the metric entropy of subsets of $C(\mathcal{T})$, from the proof of the Kolmogorov-Riesz theorem [HH10] we can extract the following generic upper bound on the metric entropy of subsets of $L^p(\mathcal{T})$.

Proposition 2.11. Let $1 \leq p < \infty$ and $\mathcal{T}$ a compact subset of $\mathbb{R}^d$. Let $F \subset L^p(\mathcal{T})$ relatively compact. Denote

$$\nu(F;\delta) = \sup_{f \in F} \sup_{\|\boldsymbol{\tau}\| \leq \delta} \|f(\boldsymbol{t} + \boldsymbol{\tau}) - f(\boldsymbol{t})\|_{L^p(\mathcal{T})} \tag{2.3.2}$$

By assumption, $\lim_{\delta \to 0} \nu(F;\delta) = 0$.

The approximation number of F is bounded by

$$\forall n \in \mathbb{N}^*, \ a_{n+1}(F; L^p(\mathcal{T})) \leq 2^{d/p} \ \nu(F; 2\varepsilon_n(\mathcal{T}; \mathbb{R}^d)) \tag{2.3.3}$$

(Incidentally, this proves one of the directions of the compact-domain Kolmogorov-Riesz theorem.)

Proof. Let $n \in \mathbb{N}^*$, $\delta > \varepsilon_n(\mathcal{T}; \mathbb{R}^d)$ and $(t_1, ..., t_n)$ a δ-covering of $\mathcal{T}$ in $\mathbb{R}^d$ (for any norm $\|\cdot\|$ on $\mathbb{R}^d$). Abbreviate $B_i = B_{t_i,\delta}$ and $B_0 = B_{0,\delta}$ the balls in $\mathbb{R}^d$.

Since $\mathcal{T} \subset \bigcup_{i=1}^n B_i$, let $(Q_i)_i$ such that $\forall i, \ Q_i \subset B_i$ and $\mathcal{T} = \bigsqcup_{i=1}^n Q_i$ (disjoint union).

For each $f \in F$, denote $\hat{f}$ the function

$$\hat{f}(t) = \sum_{i=1}^n \mathbb{1}_{t \in Q_i} \left[\frac{1}{|B_i|} \int_{B_i} ds \ f(s) \right] \tag{2.3.4}$$

[4] Actually [HHM19] shows that condition 1 is redundant, for the unbounded domain case. The same holds for compact domain, with the subtlety that the functions should still be considered as elements of $L^p(\mathbb{R}^d)$, see the paper.

where $|B_i| = \mathrm{vol}(B_i) = \int_{B_i} dt$. Note that in this sum, only one term is non-zero, for each t. $f \mapsto \hat{f}$ is a linear operator from $L^p(\mathcal{T})$ to itself of rank at most n, since simple functions are L^p-integrable.

To show that this operator approximates identity over F up to controllable error, note that for all $f \in F$:

$$f(t) - \hat{f}(t) = \sum_{i=1}^{n} \mathbb{1}_{t \in Q_i} \frac{1}{|B_i|} \int_{B_i} ds \ \ (f(t) - f(s)) \tag{2.3.5}$$

$$\left| f(t) - \hat{f}(t) \right|^p = \sum_{i=1}^{n} \mathbb{1}_{t \in Q_i} \frac{1}{|B_i|^p} \left| \int_{B_i} ds \ 1 \cdot (f(t) - f(s)) \right|^p \tag{2.3.6}$$

$$\leq \sum_{i=1}^{n} \mathbb{1}_{t \in Q_i} \frac{1}{|B_i|} \int_{B_i} ds \ \ |f(t) - f(s)|^p \tag{2.3.7}$$

by Hoelder's inequality and straightforward algebra. So,

$$\left\| f - \hat{f} \right\|_{L^p}^p = \int_{\mathcal{T}} dt \ \left| f(t) - \hat{f}(t) \right|^p = \sum_{i=1}^{n} \int_{Q_i} dt \ \left| f(t) - \hat{f}(t) \right|^p \tag{2.3.8}$$

$$\leq \sum_{i=1}^{n} \int_{Q_i} dt \ \sum_{j=1}^{n} \mathbb{1}_{t \in Q_j} \frac{1}{|B_j|} \int_{B_j} ds \ \ |f(t) - f(s)|^p \tag{2.3.9}$$

$$= \sum_{i=1}^{n} \int_{Q_i} dt \ \frac{1}{|B_i|} \int_{B_i} ds \ \ |f(t) - f(s)|^p \tag{2.3.10}$$

Now for any t and i such that $t \in Q_i$, then $t \in B_i$ i.e $\|t - t_i\| \leq \delta$, and

$$\forall s, \ s \in B_i \iff \|s - t_i\| \leq \delta \implies \|s - t\| \leq 2\delta \iff s \in t + 2B_0 \tag{2.3.11}$$

Thus, using that $|B_i| = |B_0| = \mathrm{vol}(B_i) = \mathrm{vol}(B_0)$,

$$\left\| f - \hat{f} \right\|_{L^p}^p \leq \sum_{i=1}^{n} \int_{Q_i} dt \ \frac{1}{|B_i|} \int_{2B_0} d\tau \ \ |f(t) - f(t+\tau)|^p \tag{2.3.12}$$

$$= \frac{1}{|B_0|} \int_{\mathcal{T}} dt \ \int_{2B_0} d\tau \ \ |f(t) - f(t+\tau)|^p \tag{2.3.13}$$

$$= \frac{1}{|B_0|} \int_{2B_0} d\tau \ \ \underbrace{\int_{\mathcal{T}} dt \ |f(t) - f(t+\tau)|^p}_{= \|f(t) - f(t+\tau)\|_{L^p}^p \ \leq \ \nu(f; 2\delta)^p} \tag{2.3.14}$$

$$\leq \frac{1}{|B_0|} \ |2B_0| \ \nu(F; 2\delta)^p \tag{2.3.15}$$

$$= \frac{\mathrm{vol}(2B_0)}{\mathrm{vol}(B_0)} \nu(F; 2\delta)^p \tag{2.3.16}$$

$$= 2^d \ \nu(F; 2\delta)^p \tag{2.3.17}$$

by definition of $\nu(F; 2\delta)$.

So, $\left\| f - \hat{f} \right\|_{L^p} \leq 2^{d/p} \ \nu(F; 2\delta)$. The result follows by letting $\delta \to \varepsilon_n(\mathcal{T}; \mathbb{R}^d)$, and using that $\delta \mapsto \nu(F; \delta)$ is right-continuous according to the following lemma. □

Remark 2.6. The factor $2^{d/p}$ that appears in the right-hand-side comes from the simplification of $\left(\frac{\mathrm{vol}(2B_0)}{\mathrm{vol}(B_0)}\right)^{1/p}$. Thus it is not surprising that the generalization to metric and measure space domains $\mathcal{T}$ in [GM14], assumes the measure to be a so-called doubling measure, i.e $0 < \mu(B(t, 2r)) \leq C\mu(B(t, r)) < \infty$ for all $t \in \mathcal{T}$, $r > 0$, and some $C > 0$ called the doubling constant.

Lemma 2.12. Let $1 \leq p < \infty$ and $\mathcal{T}$ a subset of $\mathbb{R}^d$ (not necessarily compact!).

For any $f \in L^p(\mathcal{T})$, denote $\nu(f;\delta) = \sup_{\|\boldsymbol{\tau}\| \leq \delta} \|f(\boldsymbol{t}+\boldsymbol{\tau}) - f(\boldsymbol{t})\|_{L^p(\mathcal{T})}$. For any $F \subset L^p(\mathcal{T})$, denote as in the previous proposition $\nu(F;\delta) = \sup_{f \in F} \nu(f;\delta)$.

Let $f \in L^p(\mathcal{T})$. Then $\delta \mapsto \nu(f;\delta)$ is non-decreasing and continuous from the right at each $\delta \geq 0$.

Let F a relatively compact subset of $L^p(\mathcal{T})$. Then $\delta \mapsto \nu(F;\delta)$ is non-decreasing and continuous from the right at each $\delta \geq 0$.

Proof. The non-decreasing monotonicity is by definition. For right-continuity, we follow similar arguments as in the last paragraph of the proof of [HH10, theorem 5].

Let $f \in L^p(\mathcal{T})$, $\delta_0 \geq 0$ and $\varepsilon > 0$. To show right-continuity of the non-decreasing function $\nu(f;\cdot)$, we want to show that there exists $\sigma > 0$ such that $\nu(f;\delta_0+\sigma) \leq \nu(f;\delta_0) + \varepsilon$.

$C_c^\infty(\mathcal{T})$, the space of infinitely-differentiable functions supported on a compact, is dense in $L^p(\mathcal{T})$. So let $g \in C_c^\infty(\mathcal{T})$, and denote S its support, such that

$$\|f-g\|_{L^p} \leq \varepsilon \tag{2.3.18}$$

First let us show that the mapping $\nu(g;\cdot)$ is right-continuous. Indeed, that mapping is nothing else than the modulus of continuity of a continuous function h at point 0:

$$h(\tau) := \|g(t+\tau) - g(t)\|_{L^p} \tag{2.3.19}$$

$$\omega_0(h;\delta) = \sup_{\|\tau\| \leq \delta} |h(\tau) - h(0)| = \sup_{\|\tau\| \leq \delta} \|g(t+\tau) - g(t)\|_{L^p} = \nu(g;\delta) \tag{2.3.20}$$

To see that h is continuous, use that g is continuous, and the dominated convergence theorem (since $|g(t+\tau) - g(t)|^p \leq [2\sup_S |g|]^p \, \mathbb{1}_{t \in S + B_{0,T}}$ which is integrable, for any $\|\tau\| \leq T$):

$$\lim_{\tau \to \tau_0} h(\tau)^p = \lim_{\tau \to \tau_0} \int_{\mathcal{T}} dt \ |g(t+\tau) - g(t)|^p \tag{2.3.21}$$

$$= \int_{\mathcal{T}} dt \ \lim_{\tau \to \tau_0} |g(t+\tau) - g(t)|^p \tag{2.3.22}$$

$$= \int_{\mathcal{T}} dt \ |g(t+\tau_0) - g(t)|^p \ = h(\tau_0)^p \tag{2.3.23}$$

Thus $\nu(g;\cdot) = \omega_0(h;\cdot)$ for a continuous function h, and so that mapping is right-continuous, by [CS90, proposition 5.4.1]. [5]

Now to show right-continuity of $\nu(f;\cdot)$ itself,

$$f(t+\tau) - f(t) = g(t+\tau) - g(t) \quad + \quad f(t+\tau) - g(t+\tau) \quad - \quad (f(t) - g(t)) \tag{2.3.24}$$

$$\|g(t+\tau) - g(t)\|_{L^p} - 2\varepsilon \leq \|f(t+\tau) - f(t)\|_{L^p} \leq \|g(t+\tau) - g(t)\|_{L^p} + 2\varepsilon \tag{2.3.25}$$

since $\|f(t+\tau) - g(t+\tau)\|_{L^p} = \|f(t) - g(t)\|_{L^p} \leq \varepsilon$ by definition. Taking the sup over $\|\tau\| \leq \delta$ for any δ,

$$\forall \delta, \ \nu(g;\delta) - 2\varepsilon \leq \nu(f;\delta) \qquad \text{and} \qquad \forall \delta, \ \nu(f;\delta) \leq \nu(g;\delta) + 2\varepsilon \tag{2.3.26}$$

Since $\nu(g;\cdot)$ is right-continuous, let $\sigma > 0$ such that $\nu(g;\delta_0+\sigma) \leq \nu(g;\delta_0) + \varepsilon$. Then

$$\nu(f;\delta_0+\sigma) \leq 2\varepsilon + \nu(g;\delta_0+\sigma) \tag{2.3.27}$$

$$\leq 3\varepsilon + \nu(g;\delta_0) \tag{2.3.28}$$

$$\leq 5\varepsilon + \nu(f;\delta_0) \tag{2.3.29}$$

[5] Actually what we use here is a slightly adapted version of [CS90, proposition 5.4.1], since we are considering pointwise instead of uniform modulus of continuity. The proof given in that book relies on the compacity of the domain $\mathcal{T}$ (denoted "X" in the book), but still goes through thanks to our assumption that the domain is contained in a finite-dimensional space, so that bounded subsets of $\mathcal{T}$ are relatively compact.

Thus, as announced, there exists $\sigma > 0$ such that $\nu(f;\delta_0+\sigma) \leq \nu(f;\delta_0)+5\varepsilon$. This concludes the proof that $\nu(f;\cdot)$ is right-continuous.

To show the second part of the lemma, let F relatively compact in $L^p(\mathcal{T})$, $\delta_0 \geq 0$ and $\varepsilon > 0$. We want to show that there exists $\sigma > 0$ such that $\nu(F;\delta_0+\sigma) \leq \nu(F;\delta_0)+\varepsilon$.

Since F is totally bounded, let a finite ε-covering $f_1, ..., f_m$ of F, and simply apply the first part of the lemma to each f_i. The bound on each $\nu(f_i;\delta)$ transfers straightforwardly to $\nu(F;\delta)$. $\square$

2.4 Sampling expansions of band-limited functions

To illustrate the ingredients for the technique of truncating and quantizing a frame decomposition, in this section we discuss the case of band-limited functions. For simplicity we consider the case of square-integrable functions with the $L^2 \to L^2$ Fourier transform, though most of this section would hold without change if we only assume $\hat{f} \in L^1$ and $f \in L^\infty$.

Consider functions $f(t) \in L^2(\mathbb{R})$ that are B-band-limited, i.e whose Fourier transform $\hat{f}(\xi)$ is supported on $[-B, B]$. For $B' \geq B$ and certain functions ψ, it holds for any such f

$$f(t) = \sum_{n\in\mathbb{Z}} f\left(\frac{n}{2B'}\right)\psi\left(t-\frac{n}{2B'}\right) \tag{2.4.1}$$

When $B' = B$ and $\psi(t) = \mathrm{sinc}(2B't) = \frac{\sin(\pi 2B't)}{\pi 2B't}$, this formula is known as the cardinal series expansion, commonly associated with the names of Kotelnikov, Nyquist, Shannon and Whittaker.

For a fixed choice of ψ and any such f, denote

- The N-first-terms approximation $f_N(t) = \sum_{|n|\leq N} f\left(\frac{n}{2B'}\right)\psi\left(t-\frac{n}{2B'}\right)$;
- The truncation error $R_N(t) = f(t) - f_N(t)$;
- The coefficient-sensitivity error $h_{N,\delta}(t) = \sum_{|n|\leq N} \delta_n \psi\left(t-\frac{n}{2B'}\right)$.

The error committed by approximating $f(t)$ by the quantization $g_{N,c}(t) = \sum_{|n|\leq N} c_n\psi\left(t-\frac{n}{2B'}\right)$ is bounded by

$$|f(t) - g_{N,c}(t)| \leq |R_N(t)| + |h_{N,\delta}(t)| \tag{2.4.2}$$

where the $\delta_n = f\left(\frac{n}{2B'}\right) - c_n$ are morally small.

As explained previously, these observations naturally lead to an approximation scheme for B-band-limited functions. In this section, we ask the following questions:

- For what choices of ψ does the above identity hold?
- How can we bound $R_N(t)$ and $h_{N,\delta}(t)$?

We only state results here. For proofs, and a friendlier presentation, see appendix B and appendix C.

2.4.1 Choices of "synthesizer" functions

Definition 2.5. For all $B > 0$, denote $\mathbb{B}_B$ the space of B-band-limited signals:

$$\widehat{\mathbb{B}}_B = \left\{\hat{f} \in L^2(\mathbb{R}); \hat{f}(\xi) = 0 \text{ for } |\xi| > B\right\} \tag{2.4.3}$$

$$\mathbb{B}_B = \left\{f \in L^2(\mathbb{R}); \hat{f} \in \widehat{\mathbb{B}}_B\right\} \tag{2.4.4}$$

For concision (note that this is not standard terminology), let us call a *synthesizer of* $\mathbb{B}_B$ *for* $\frac{1}{2B'}$*-sampling*, or simply (B, B')*-synthesizer*, any function Ψ_0 such that

$$\forall f \in \mathbb{B}_B,\ f(t) = \sum_{n\in\mathbb{Z}} \frac{1}{2B'} f\left(\frac{n}{2B'}\right)\Psi_0\left(t-\frac{n}{2B'}\right) \tag{2.4.5}$$

Proposition 2.13. $\Psi_0(t) \in L^2(\mathbb{R})$ is a synthesizer if and only if it satisfies the *synthesizer condition*: (see Figure 2.2)

$$\widehat{\Psi}_0(\xi) = \begin{cases} 1 & \text{for } |\xi| \leq B \\ 0 & \text{for } \xi \in [\pm B] + 2B'\mathbb{Z}^* := \bigsqcup_{n\in\mathbb{Z}^*} [2nB' - B, 2nB' + B] \\ \text{arbitrary} & \text{everywhere else} \end{cases} \tag{2.4.6}$$

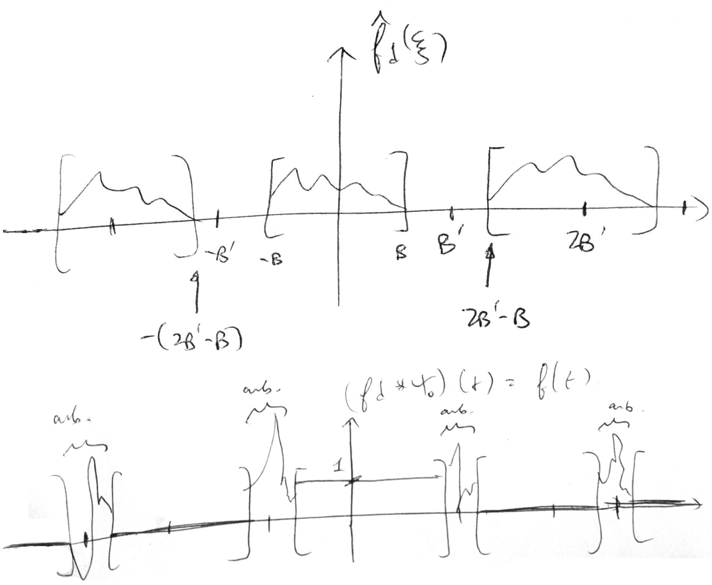


Figure 2.2: Top: spectrum of the discretized signal $f_d(t) = \sum_n \frac{1}{2B'} f\left(\frac{n}{2B'}\right) \delta_{\frac{n}{2B'}}(t)$. Bottom: example spectrum of a Ψ_0 satisfying the synthesizer condition.

We now turn our attention to a class of synthesizers commonly used in practice: (central)-window-based synthesizers.

Definition 2.6. We say that a (B, B')-synthesizer has its *spectrum supported on the central interval*, if (in addition to the synthesizer condition) its Fourier transform is supported on $[-2B', 2B']$. (Figure 2.3)

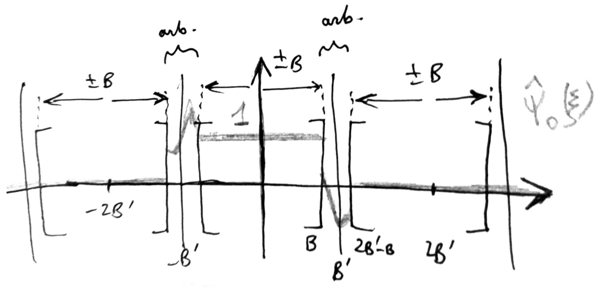


Figure 2.3: Spectrum of a (B, B')-*synthesizer with spectrum supported on the central interval.* It is identically equal to one on the central interval $[-B, B]$, arbitrary in the gaps surrounding the central interval, and zero everywhere else.

Example 2.2. Denote the "canonical" synthesizer

$$\Phi_0^{(B')}(t) = 2B' \operatorname{sinc}(2B't) \tag{2.4.7}$$

$$\widehat{\Phi}_0^{(B)}(\xi) = \mathbb{1}_{|\xi|\le B'} \tag{2.4.8}$$

$\Phi_0^{(B')}$ is a synthesizer and it has spectrum supported on the central interval.

As noted in [PS96, (2.4)], a sufficient condition for Ψ_0 to be a synthesizer is: $\Psi_0(t) = \Phi_0^{(B')}(t)g(t)$ for some well-chosen function g.

Definition 2.7. A *(central) window function* (again, this is not standard terminology) is any function g such that

$$\Psi_0(t) = \Phi_0^{(B')}(t)g(t) \tag{2.4.9}$$

defines a synthesizer with spectrum supported on the central interval.

A *generalized window function* is any function g such that the above formula defines a synthesizer.

For a window function g, $\Psi_0(t) = \Phi_0^{(B')}(t)g(t)$ is called the associated window-based synthesizer.

Proposition 2.14. *$g \in L^2(\mathbb{R})$ is a central window function if and only if*

$$\begin{cases} \operatorname{supp}(\hat{g}) \subset [\pm(B' - B)] \\ g(0) = 1 \end{cases} \tag{2.4.10}$$

Remark 2.7. This condition automatically implies $\|g\|_{L^\infty} \le \|\hat{g}\|_{L^1} \le 2(B' - B)\|\hat{g}\|_{L^2} < \infty$, so $g \in L^\infty(\mathbb{R})$, and so $\Psi_0(t)$ is indeed in $L^2(\mathbb{R})$.

Moreover, note that we restricted our attention to candidate g's in $L^2(\mathbb{R})$; we could in fact consider a more general space ($g \in \mathcal{F}[L^1(\mathbb{R})]$ the range space of the $L^1 \to L^\infty$ Fourier transform).

Proposition 2.15. *$g \in L^2(\mathbb{R})$ is a generalized window function if and only if*

$$\begin{cases} \operatorname{supp}(\hat{g}) \subset [\pm(B' - B)] + 2B'\mathbb{Z} \\ \forall n \neq 0, \int_{(2n-1)B'}^{(2n+1)B'} dy\ \hat{g}(y) = 0 \\ g(0) = \int_{-B'}^{B'} dy\ \hat{g}(y) = 1 \end{cases} \tag{2.4.11}$$

A natural question is then: does the converse hold, i.e can any synthesizer with spectrum supported on the central interval be put in the form $\Phi_0^{(B')}(t)g(t)$ for some window function g? The following proposition answers negatively, because of smoothness considerations.

Proposition 2.16. *Let Ψ_0 a synthesizer with spectrum supported on the central interval.*

Ψ_0 is a window-based synthesizer, if and only if $\widehat{\Psi}_0 \in H^1(\mathbb{R})$ the first-order Sobolev (Hilbert) space, and $\widehat{\Psi}_0'\Big|_{[-2B',0]} = -\ \widehat{\Psi}_0'\Big|_{[0,2B']}$.

2.4.2 Synthesizer functions in higher dimension

Definition 2.8. For any compact subset $K \subset \mathbb{R}^N$, let

$$\widehat{\mathbb{B}}_K = \left\{ \hat{f} \in L^2(\mathbb{R}^N);\ \hat{f}(\boldsymbol{\xi}) = 0 \text{ for } \boldsymbol{\xi} \notin K \right\} \tag{2.4.12}$$

$$\mathbb{B}_K = \left\{ f \in L^2(\mathbb{R}^N);\ \hat{f} \in \widehat{\mathbb{B}}_K \right\} \tag{2.4.13}$$

Note that for any vector $\boldsymbol{B}'$ such that $K \subset [\pm\boldsymbol{B}']$, $\mathbb{B}_K \subset \mathbb{B}_{\boldsymbol{B}'}$.

Let us call a $(K, \boldsymbol{B}')$-*synthesizer* any function $\Psi_{\boldsymbol{0}}(\boldsymbol{t})$ such that

$$\forall f \in \mathbb{B}_K,\ f(\boldsymbol{t}) = \sum_{\boldsymbol{n}\in\mathbb{Z}^N} \frac{1}{(2\boldsymbol{B}')^\times} f\left(\frac{\boldsymbol{n}}{2\boldsymbol{B}'}\right) \Psi_{\boldsymbol{0}}\left(\boldsymbol{t} - \frac{\boldsymbol{n}}{2\boldsymbol{B}'}\right) \tag{2.4.14}$$

Proposition 2.17. $\Psi_{\mathbf{0}}(\boldsymbol{t}) \in L^2(\mathbb{R}^N)$ is a $(K, \boldsymbol{B}')$-synthesizer if and only if it satisfies the *synthesizer condition*:

$$\widehat{\Psi}_{\mathbf{0}}(\boldsymbol{\xi}) = \begin{cases} 1 & \text{for } \boldsymbol{\xi} \in K \\ 0 & \text{for } \boldsymbol{\xi} \in K + 2\boldsymbol{B}' \odot (\mathbb{Z}^N \setminus \{\mathbf{0}\}) := \bigsqcup_{\boldsymbol{n} \in \mathbb{Z}^N \setminus \{\mathbf{0}\}} (2\boldsymbol{n} \odot \boldsymbol{B}' + K) \\ \text{arbitrary} & \text{everywhere else} \end{cases} \tag{2.4.15}$$

The fact that the set $K + 2\boldsymbol{B}' \odot (\mathbb{Z}^N \setminus \{\mathbf{0}\})$ can be written as a disjoint union as above is clear from Figure 2.4.

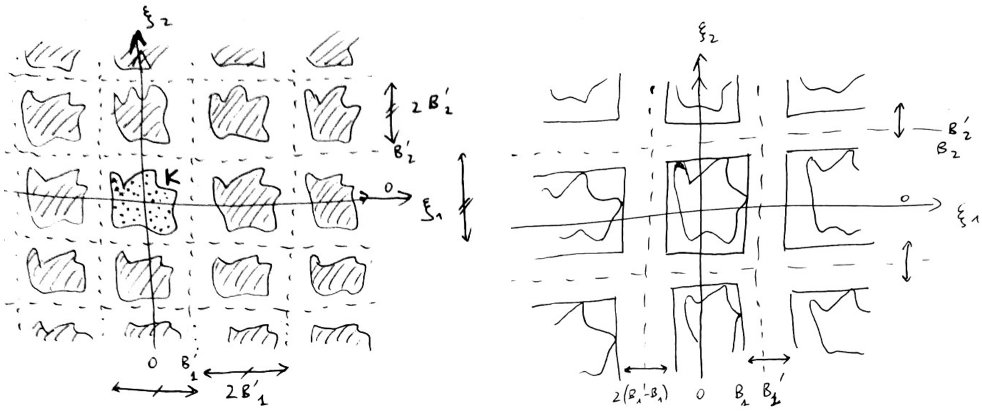


Figure 2.4: Left: $\widehat{\Psi}_{\mathbf{0}}(\boldsymbol{\xi})$ must be $= 1$ on the dotted set, $= 0$ on the hatched set, and can be arbitrary on the set left blank. Right: if we don't like generality, we can always overapproximate K and assume hyperrectangular support $[\pm \boldsymbol{B}] \subset [\pm \boldsymbol{B}']$.

As before, we can define and characterize (central)-window-based synthesizers.

Definition 2.9. We say that a $(K, \boldsymbol{B}')$-synthesizer has its *spectrum supported on the central hyperrectangle*, if (in addition to the synthesizer condition) its Fourier transform is supported on $[\pm 2\boldsymbol{B}']$.

Example 2.3. Denote the "canonical" synthesizer in higher dimension

$$\Phi_{\mathbf{0}}^{(\boldsymbol{B}')}(\boldsymbol{t}) = (2\boldsymbol{B}')^{\times} \operatorname{sinc}^{\times}(2\boldsymbol{B}' \odot \boldsymbol{t}) \tag{2.4.16}$$

$$\widehat{\Phi}_{\mathbf{0}}^{(\boldsymbol{B}')}(\xi) = \mathbb{1}_{\boldsymbol{\xi} \in [\pm 2\boldsymbol{B}']} \tag{2.4.17}$$

$\Phi_{\mathbf{0}}^{(\boldsymbol{B}')}$ is a synthesizer and it has spectrum supported on the central hyperrectangle.

Definition 2.10. A *(central) window function* is any function $g(\boldsymbol{t})$ such that

$$\Psi_{\mathbf{0}}(t) = \Phi_{\mathbf{0}}^{(\boldsymbol{B}')}(\boldsymbol{t}) g(\boldsymbol{t}) \tag{2.4.18}$$

defines a synthesizer with spectrum supported on the central hyperrectangle.

A *generalized window function* is any function $g(\boldsymbol{t})$ such that the above formula defines a synthesizer.

For a window function g, $\Psi_{\mathbf{0}}(t) = \Phi_{\mathbf{0}}^{(\boldsymbol{B}')}(\boldsymbol{t}) g(\boldsymbol{t})$ is called the associated window-based synthesizer.

As in one dimension, we restrict attention to candidates $g(\boldsymbol{t}) \in L^2(\mathbb{R}^N)$, although more general function spaces could be considered.

Proposition 2.18. Let $g(\boldsymbol{t}) \in L^2(\mathbb{R}^N)$. If

$$g(\mathbf{0}) = 1 \tag{2.4.19}$$

$$\operatorname{supp}(\hat{g}) + (-K) \subset [\pm \boldsymbol{B}'] \tag{2.4.20}$$

then $g(\boldsymbol{t})$ is a central window function.

As a partial converse, let $g(\boldsymbol{t}) \in L^2(\mathbb{R}^N)$ such that $\hat{g}(\boldsymbol{\xi})$ has values in $\mathbb{R}_+$. Then the above sufficient condition is also necessary, for $g(\boldsymbol{t})$ to be a central window function. In fact it is even necessary for $g(\boldsymbol{t})$ to be a generalized window function.

The initial assumption $\hat{g}(\boldsymbol{\xi}) \in \mathbb{R}_+$ is very strong (a priori $\hat{g}$ has values in $\mathbb{C}$), so the partial converse statement is not very useful. But its derivation is instructive, in that it gives intuition on what is "missing" for our sufficient condition to also be necessary. (We conjecture that the sufficient condition is also necessarily, but we have not proved it.)

Explicit characterization of allowed $S = \operatorname{supp}(\hat{g})$ by morphological erosion The condition $S + (-K) \subset [\pm \boldsymbol{B}']$ doesn't look very practical; instead, we would like a condition of the form $S \subset Z$.

Now the question of characterizing the tightest possible superset of S subject to this condition, reduces to finding the *erosion* of $[\pm \boldsymbol{B}']$ by $-K$.

Definition 2.11. The erosion of X by Y is the largest set Z such that $Z + Y \subset X$. [6]

In our setting, X is $[\pm \boldsymbol{B}']$ and Y is $-K$. It turns out that since X is a hyperrectangle, the erosion is also just a hyperrectangle.

Proposition 2.19. The erosion of $[\pm \boldsymbol{B}']$ by $-K$ is $[a_1, b_1] \times ... \times [a_N, b_N]$ – in other words,

$$S + (-K) \subset [\pm \boldsymbol{B}'] \iff S \subset [a_1, b_1] \times ... \times [a_N, b_N] \tag{2.4.24}$$

where

$$a_i = \sup_K z_i - B_i' \tag{2.4.25}$$

$$b_i = \inf_K z_i + B_i' \tag{2.4.26}$$

Proof. Simply write

$$\begin{aligned}
S + (-K) \subset [\pm \boldsymbol{B}'] &\iff \forall \boldsymbol{y} \in S, \forall \boldsymbol{z} \in K,\ \boldsymbol{y} - \boldsymbol{z} \in [\pm \boldsymbol{B}'] && (2.4.27)\\
&\iff \forall \boldsymbol{y} \in S, \forall \boldsymbol{z} \in K, \forall i \in [N],\ \ -B_i' \leq y_i - z_i \leq B_i' && (2.4.28)\\
&\iff \forall \boldsymbol{y} \in S, \forall i \in [N],\ \sup_{\boldsymbol{z} \in K} z_i - B_i' \leq y_i \leq \inf_{\boldsymbol{z} \in K} z_i + B_i' && (2.4.29)\\
&\iff S \subset [a_1, b_1] \times ... \times [a_N, b_N] && (2.4.30)
\end{aligned}$$

Note that $a_i \leq b_i$ because $K \subset [\pm \boldsymbol{B}']$. ☐

2.4.3 Bounds on truncation error $R_N(t)$

Notation

We focus on the one-dimensional case, as transposing to higher dimension is not difficult. Note that here N indicates the number of terms kept, rather than dimension.

[6] See `https://en.wikipedia.org/wiki/Erosion_(morphology)`. Actually in the wikipedia article they define the erosion as $Z = \{z;\ (z + Y) \subset X\}$, but it is straightforward to check that $Z' + Y \subset X \iff Z' \subset Z$:

$$\begin{aligned}
z \in Z &\iff \forall y \in Y, z + y \in X && (2.4.21)\\
Z' \subset Z &\iff \forall z \in Z', \forall y \in Y,\ z + y \in X && (2.4.22)\\
&\iff Z' + Y \subset X && (2.4.23)
\end{aligned}$$

Consider the expansion:

$$\forall f \in \mathbb{B}_B,\ f(t) = \sum_{n\in\mathbb{Z}} \frac{1}{2B'} f\left(\frac{n}{2B'}\right)\ \Psi_0\left(t - \frac{n}{2B'}\right) \tag{2.4.31}$$

In this generic expression Ψ_0 denotes some (B, B')-synthesizer to be specified. Denote the truncated expansion and the remainder:

$$f_N(t) := \sum_{|n|\le N} \frac{1}{2B'} f\left(\frac{n}{2B'}\right)\ \Psi_0\left(t - \frac{n}{2B'}\right) \tag{2.4.32}$$

$$R_N(t) := f(t) - f_N(t) = \sum_{|n|>N} \frac{1}{2B'} f\left(\frac{n}{2B'}\right)\ \Psi_0\left(t - \frac{n}{2B'}\right) \tag{2.4.33}$$

For notational convenience, let

$$c_n = \frac{1}{2B'} f\left(\frac{n}{2B'}\right) \qquad f(t) = \sum_n c_n \Psi_0\left(t - \frac{n}{2B'}\right) \tag{2.4.34}$$

$$\tilde{c}_n = c_n \mathbb{1}_{|n|>N} \qquad R_N(t) = \sum_n \tilde{c}_n \Psi_0\left(t - \frac{n}{2B'}\right) \tag{2.4.35}$$

For the L^2 norm

Let any (B, B')-synthesizer Ψ_0.

$$\forall t, |R_N(t)|^2 = \sum_{n,m} \tilde{c}_n \overline{\tilde{c}_m} \Psi_0\left(t - \frac{n}{2B'}\right) \overline{\Psi_0\left(t - \frac{m}{2B'}\right)} \tag{2.4.36}$$

$$\|R_N(t)\|_{L^2}^2 = \sum_{n,m} \tilde{c}_n \overline{\tilde{c}_m} \underbrace{\int_{\mathbb{R}} dt\ \Psi_0\left(t - \frac{n}{2B'}\right) \overline{\Psi_0\left(t - \frac{m}{2B'}\right)}}_{=:A_{nm}} \tag{2.4.37}$$

and

$$A_{nm} = \int_{\mathbb{R}} dt\ \Psi_0\left(t - \frac{n-m}{2B'}\right) \overline{\Psi_0(t)} \tag{2.4.38}$$

$$= \text{Autocorr}\, \Psi_0\left(\frac{n-m}{2B'}\right) \tag{2.4.39}$$

$$= \left\langle \Psi_0\left(t - \frac{n-m}{2B'}\right), \Psi_0 \right\rangle \tag{2.4.40}$$

$$= \left\langle \widehat{\Psi}_0 \cdot e^{-i2\pi \frac{n-m}{2B'}\xi}, \widehat{\Psi}_0 \right\rangle \tag{2.4.41}$$

where Autocorr $\Psi(\tau)$ denotes the autocorrelation function of Ψ.

Since $\tilde{c}_n = 0$ for $|n| \le N$, denoting $\widetilde{A}_{nm} = A_{nm}\mathbb{1}_{|n|>N,|m|>N}$ we have

$$\|R_N(t)\|_{L^2}^2 = \sum_{n,m} \tilde{c}_n \overline{\tilde{c}_m} \widetilde{A}_{nm} \tag{2.4.42}$$

If we don't have any more information on the truncated-sample-sequence $\tilde{c}_n = \frac{1}{2B'} f\left(\frac{n}{2B'}\right)\mathbb{1}_{|n|>N}$, i.e we only know that $(\tilde{c}_n)_n \in \ell^2(\mathbb{Z})$ and $\tilde{c}_n = 0$ for $|n| > N$, then the best we can do is:

$$\|R_N(t)\|_{L^2}^2 \le \left|\left|\left| \widetilde{A} \right|\right|\right| \sum_n |\tilde{c}_n|^2 \tag{2.4.43}$$

where $\left|\left|\left| \widetilde{A} \right|\right|\right| = \sup_{\|d\|_{\ell^2}=1} \left\| \widetilde{A} d \right\|_{\ell^2}$ is the operator norm of $\widetilde{A} : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$.

For the L^∞ norm

Results for the L^∞ norm are morally trickier to obtain, as the Hilbert space point of view is not sufficient. We will need to use specificities of the synthesizer function $\Psi_0(t)$.

Generic truncation bound By Hoelder's inequality, for all $p, q \in [1, \infty]$ such that $\frac{1}{p} + \frac{1}{q} = 1$,

$$|R_N(t)| = \left| \sum_n \tilde{c}_n \mathbb{1}_{|n|>N} \Psi_0 \left(t - \frac{n}{2B'} \right) \right| \tag{2.4.44}$$

$$\leq \|\tilde{\boldsymbol{c}}\|_p \quad \left\| \left(\mathbb{1}_{|n|>N} \Psi_0 \left(t - \frac{n}{2B'} \right) \right)_n \right\|_q \tag{2.4.45}$$

Depending on what information we have on f (or more precisely on its samples $(c_n)_n$), the pair (p, q) should be chosen accordingly. This in turn gives a criterion for choosing Ψ_0.

That is: suppose for example that $\left(\sum_{|n|>N} \left| f\left(\frac{n}{2B'}\right) \right|^p \right)^{1/p} = 2B' \|\tilde{\boldsymbol{c}}\|_p$ goes to zero at a known fast rate when $N \to \infty$. Then, to get the most out of the truncation bound above, one should consider using a synthesizer Ψ_0 such that $\sup_t \left\| \left(\mathbb{1}_{|n|>N} \Psi_0 \left(t - \frac{n}{2B'} \right) \right)_n \right\|_q$ is small.

2.4.4 Example: the Helms-Thomas expansion

The Helms-Thomas expansion [HT62] [Jag66] is: for all $b, B > 0$, $m \in \mathbb{N}^*$ and $B' := B + mb$,

$$\forall f \in \mathbb{B}_B,\ f(t) = \sum_{n \in \mathbb{Z}} f\left(\frac{n}{2B'}\right) \left[\operatorname{sinc}\left(2b \left(t - \frac{n}{2B'} \right) \right) \right]^m \operatorname{sinc}\left(2B' \left(t - \frac{n}{2B'} \right) \right) \tag{2.4.46}$$

We can rewrite this as

$$\forall f \in \mathbb{B}_B,\ f(t) = \sum_{n \in \mathbb{Z}} \frac{1}{2B'} f\left(\frac{n}{2B'}\right)\ \Psi_0 \left(t - \frac{n}{2B'} \right) \tag{2.4.47}$$

with

$$\Psi_0(t) = 2B' \operatorname{sinc}(2Bt') \left[\operatorname{sinc}(2bt) \right]^m \tag{2.4.48}$$

$$= \Phi_0^{(B')}(t) \quad g(t) \tag{2.4.49}$$

(where $\Phi_0^{(B')}(t) = 2B' \operatorname{sinc}(2B't)$, $\widehat{\Phi}_0^{(B')}(\xi) = \mathbb{1}_{|\xi| \leq B'}$ is the "canonical" synthesizer). So we recognize Ψ_0 as the window-based (B, B')-synthesizer with window function

$$g(t) = [\operatorname{sinc}(2bt)]^m \qquad\qquad \hat{g}(\xi) = \left[\underbrace{\frac{\mathbb{1}_{|\xi| \leq b}}{2b} * \ldots * \frac{\mathbb{1}_{|\xi| \leq b}}{2b}}_{m \text{ times}} \right](\xi) \tag{2.4.50}$$

Indeed we can immediately check that $g(0) = 1$, and that $\hat{g}$ is supported on $[\pm b] + \ldots + [\pm b] = m \cdot [\pm b] = [\pm mb] = [\pm(B' - B)]$, since we used $B' = B + mb$ for the sampling rate.

In fact, we remark that $\hat{g}$ is nothing else than (a translated variant of) the box spline of degree $m-1$: see `https://math.stackexchange.com/questions/618272/convolution-of-indicator-function-with-itself`, and `https://commons.wikimedia.org/wiki/File:Convolution_of_box_signal_with_itself2.gif` for an illustration of the case $m = 2$.

Bound on truncation error for the $L^\infty([-W,W])$ semi-norm Denote as in the previous subsection

$$c_n = \frac{1}{2B'} f\left(\frac{n}{2B'}\right) \qquad f(t) = \sum_n c_n \Psi_0\left(t - \frac{n}{2B'}\right) \tag{2.4.51}$$

$$\tilde{c}_n = c_n \mathbb{1}_{|n|>N} \qquad R_N(t) = \sum_n \tilde{c}_n \mathbb{1}_{|n|>N} \Psi_0\left(t - \frac{n}{2B'}\right) \tag{2.4.52}$$

where $\Psi_0(t) = 2B' \operatorname{sinc}(2Bt') \left[\operatorname{sinc}(2bt)\right]^m$, and suppose we want to bound the truncation error uniformly over $[-W,W]$.

Proposition 2.20 ([Zam79, theorem 2])**.** For all $N > 2B'W$,

$$\|R_N\|_{L^\infty([-W,W])} \leq \left(\sup_n |\tilde{c}_n|\right) \sup_{t\in[-W,W]} \sum_{|n|>N} \left|\Psi_0\left(t - \frac{n}{2B'}\right)\right| \tag{2.4.53}$$

$$\leq \left(\sup_n |\tilde{c}_n|\right) \frac{(2B')^{m+1}}{\pi^{m+1}(2b)^m} \frac{2}{m} \left(N - 2B'W\right)^{-m} \tag{2.4.54}$$

Bound on coefficient-sensitivity error for the $L^\infty(\mathbb{R})$ norm

Proposition 2.21. Let, for some coefficients δ_n (that are morally small),

$$h(t) = \sum_{n\in\mathbb{Z}} \delta_n \Psi_0\left(t - \frac{n}{2B'}\right) \tag{2.4.55}$$

Then,

$$\|h\|_{L^\infty(\mathbb{R})} \leq \left(\sup_n |\delta_n|\right) \left(4 + \left(2 + \frac{2}{m}\right)(2b)^{\frac{1}{m+1}} \frac{2B'}{\pi 2b}\right) \tag{2.4.56}$$

2.5 In Reproducing Kernel Hilbert Spaces (RKHS)

In this section, we give pointers to a series of papers on metric entropy estimates in reproducing kernel Hilbert spaces (RKHS). We assume the reader is already familiar with that concept. Let a RKHS $\mathcal{F}_K$ with associated reproducing kernel $K(t,t')$.

By sampling As discussed generically in the first section, upper-estimates on the covering number of a function set $\mathcal{F}_+$ can be found by quantizing the functions' values on a well-chosen sample set $(t_1, ..., t_m)$, and using regularity assumptions.

On the other hand, recall that the usefulness of kernels for non-parametric learning is that, for any sample set $(t_1, ..., t_m)$, the m-dimensional subspace $\operatorname{span}\{K(\cdot, t_i)\}$ completely captures the functions' behaviour on the t_i:

$$\forall f \in \mathcal{F}_K,\ \operatorname*{arg\,min}_{\hat{f}\in H} \sum_i \left|f(t_i) - \hat{f}(t_i)\right|^2 = f_\alpha = \sum_i \alpha_i K(\cdot, t_i) \tag{2.5.1}$$

$$\text{where } \alpha = \left(K(t_i,t_j)\right)_{ij}^{-1} \left(f(t_1)...f(t_m)\right)^T \tag{2.5.2}$$

So by making suitable assumptions, such as regularity of f in the sense of the reproducing kernel space: $\|f\|_{\mathcal{F}_K} \leq R$, and smoothness of the kernel K, and by carefully choosing the set of sample points, morally one can obtain an approximation scheme with controllable error.

Combining these ideas, [Zho02], [Zho03], and [Küh11] derive bounds on the metric entropy of the unit ball of $\mathcal{F}_K$.

"Empirical-norm" covering numbers with m sample points To help avoid confusion, let us mention a related but distinct notion that is also studied in the literature. Given a prespecified number m of sample points and an exponent $1 \leq p < \infty$, define the metric

$$d(f,g)_m = \sup_{t_1,\ldots,t_m \in \mathcal{T}} \|((f-g)(t_i))_i\|_{\ell^p} \tag{2.5.3}$$

Then one may be interested in the metric entropy of subsets of $\mathcal{F}_K$ for this metric [Smo98]. Methods for estimating those quantities are given for example in [WSS01] and [GW02].

For detailed explanations of the significance of entropy numbers for kernel methods and RKHS, we strongly recommend [SC08, chapter 7].

2.6 Compactness of embeddings

In this section we mention an equivalent formulation of metric entropy estimation. We already discussed that equivalence in detail in the general abstract case, in section 1.2; here we make a briefer and more down-to-earth presentation. In a nutshell: any review of metric entropy estimate results should include the keywords "compactness of embeddings".

Let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ normed vector spaces of functions over the same domain. Suppose, with increasing stringentness,

- $E \subset F$.
- The identity embedding $I : E \to F$ is continuous, i.e $\forall x \in E, \|x\|_F \leq |||I||| \, \|x\|_E$ with $|||I||| < \infty$.
  This means that $B^{(E)}$, the unit ball of E, is bounded in $(F, \|\cdot\|_F)$.
- The embedding is compact, i.e $I : E \to F$ is a compact operator.
  This means that $B^{(E)} = I(B^{(E)})$ is relatively compact in $(F, \|\cdot\|_F)$.

Since $I(B^{(E)})$ is relatively compact in the Banach F, a natural question is to estimate its covering number $N_\varepsilon(I(B^{(E)}); F)$, or equivalently its entropy number $\varepsilon_n(I(B^{(E)}); F)$. Now by definition, the latter is just the entropy number of the operator I.

Thus, when the set of interest is some unit ball $B^{(E)}$, estimating its metric entropy in some other norm $\|\cdot\|_F$ is equivalent to estimating the entropy number of the injection $I : E \to F$; in other words, the degree of compactness of the embedding $E \subset F$.

Such cases arise very commonly. For example, the results in [Elb+20, table 1] are all in this case, as well as [Zho02], [Zho03].

The equivalence between metric entropy and degree of compactness of embeddings allows to directly apply a wealth of existing results from functional analysis. In [CS01] for example, they use prior work on embedding relations between Sobolev spaces to estimate the metric entropy of balls in RKHS's for the L^∞ norm, for sufficiently smooth kernels.

Chapter 3

Linear systems

Throughout this chapter we use standard functional analysis notions, recalled in section D.1.

3.1 Specializing the framework

In this section, the systems are assumed to be linear, so they can be viewed as linear operators between vector spaces $S : \mathcal{X} \to \mathcal{Y}$. More precisely, we will use the following variant of the framework (recall from section 1.3):

- $\mathcal{X}$ a vector space of input signals;
- $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$ a normed vector space of output signals;
- $\mathbb{S}$ a vector space of linear mappings from $\mathcal{X}$ to $\mathcal{Y}$, i.e a subspace of $\mathcal{L}(\mathcal{X}, \mathcal{Y})$;
- $X \subset \mathcal{X}$ a set of input signals of interest;
- $\mathbb{S}_+ \subset \mathbb{S}$ a set of systems of interest.

We define a semi-norm over $\mathbb{S}$ by

$$\|S\|_{\infty X} = \sup_{x \in X} \|S[x]\|_{\mathcal{Y}} \tag{3.1.1}$$

(and by making the assumption $\|S\|_{\infty X} < \infty$ for all $S \in \mathbb{S}$). The goal is to estimate $\log N_\varepsilon(\mathbb{S}_+; (\mathbb{S}, \|\cdot\|_{\infty X}))$.

We further make the natural assumptions that

- $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ and $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$ are Banachs;
- $\mathbb{S}$ consists of bounded linear operators from $\mathcal{X}$ to $\mathcal{Y}$, i.e is a subspace of $\mathcal{L}_b(\mathcal{X}, \mathcal{Y})$.

Worst-case error over unit ball and operator norm As a preliminary, let us note right away that since $\mathbb{S}$ consists of linear systems between Banach spaces $\mathcal{X}$ and $\mathcal{Y}$, it seems natural to choose the test input signal set as $X = B^{(\mathcal{X})}$ the unit ball of $\mathcal{X}$. In that case, the worst-case error norm is nothing else than the operator norm:

$$\|S\|_{\infty X} = \sup_{\|x\|_{\mathcal{X}} \leq 1} \|Sx\|_{\mathcal{Y}} = |||S||| \tag{3.1.2}$$

3.2 Bounded sets of linear operators over a relatively compact ball

In this section we show how results on the degree of compactness of embeddings, briefly discussed in section 2.6, directly imply metric entropy estimates for linear systems when the metric is the worst-case error over a relatively compact ball of inputs.

3.2.1 Scalar-valued (linear forms)

Proposition 3.1. Suppose that:

- The output space is simply $\mathcal{Y} = \mathbb{R}$, and $\mathbb{S} = \mathcal{L}_b(\mathcal{X}, \mathbb{R})$. $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ being a Banach space, $\mathbb{S}$ is nothing else than its dual space: $\mathbb{S} = \mathcal{X}'$.
- The set of systems of interest is $\mathbb{S}_+ = B^{(\mathcal{X}')}$ the unit ball of $\mathcal{X}'$.
- Similarly to section 2.6, let $(\mathcal{W}, \|\cdot\|_{\mathcal{W}})$ a compactly embedded subspace of $\mathcal{X}$. That is, the injection $I : \mathcal{W} \to \mathcal{X}$ is a compact operator.

  Moreover, suppose that the worst-case error is measured over unit-$\mathcal{W}$-norm signals: $X = B^{(\mathcal{W})}$. [1]

Then: $\mathcal{W} \subset \mathcal{X} \implies \mathcal{X}' \subset \mathcal{W}'$, and

$$\forall S \in \mathbb{S} = \mathcal{X}', \;\; \|S\|_{\infty X} = \sup_{\|x\|_{\mathcal{W}} \leq 1} |Sx| = \|\!|S|\!\|_{\mathcal{W} \to \mathbb{R}} = \|S\|_{\mathcal{W}'} \tag{3.2.1}$$

and so

$$e_n(\mathbb{S}_+; (\mathbb{S}, \|\cdot\|_{\infty X})) = e_n(B^{(\mathcal{X}')}; (\mathcal{X}', \|\cdot\|_{\mathcal{W}'})) \tag{3.2.2}$$

which can be estimated by

$$e_n(B^{(\mathcal{X}')}; (\mathcal{W}', \|\cdot\|_{\mathcal{W}'})) \leq e_n(B^{(\mathcal{X}')}; (\mathcal{X}', \|\cdot\|_{\mathcal{W}'})) \leq 2e_n(B^{(\mathcal{X}')}; (\mathcal{W}', \|\cdot\|_{\mathcal{W}'})) \tag{3.2.3}$$

Now this last quantity is precisely the dyadic entropy number of the injection operator $I' : \mathcal{X}' \to \mathcal{W}'$. Thus:

$$e_n(\mathbb{S}_+; (\mathbb{S}, \|\cdot\|_{\infty X})) \approx e_n(I' : \mathcal{X}' \to \mathcal{W}') \tag{3.2.4}$$

Moreover if $\mathcal{X}'$ is dense in $\mathcal{W}'$, then the leftmost inequality is an equality, so that: [2]

$$e_n(\mathbb{S}_+; (\mathbb{S}, \|\cdot\|_{\infty X})) = e_n(I' : \mathcal{X}' \to \mathcal{W}') \tag{3.2.5}$$

Note that I' is also the dual operator of $I : \mathcal{W} \to \mathcal{X}$. So [CS90, section 2.5] shows that both are compact (Schauder's theorem), and that $a_n(I) = a_n(I')$. Chapter 3 of that book relates approximation numbers to entropy numbers, which allows to derive estimates of $e_n(I') = e_n(\mathbb{S}_+; (\mathbb{S}, \|\cdot\|_{\infty X}))$ as desired.

Remark 3.1. Using the construction of section 1.2, assuming that X is the unit ball of some compactly embedded Banach subspace $\mathcal{W} \subset \mathcal{X}$ is equivalent to simply assuming X is a relatively compact subset of $\mathcal{X}$. To summarize the construction: given X relatively compact of $\mathcal{X}$, assumed convex and balanced, pose $\mathcal{W} = \overline{\mathbb{R}X}^{\|\cdot\|_X}$ the completion of $\mathbb{R}X = \mathrm{span}(X)$ equipped with $\|x\|_X = \inf\{b > 0;\; x \in bX\}$. Then if X is closed in $\mathcal{X}$, $B^{(\mathcal{W})} = X$ (and if not, we still have $e_n(X; \mathcal{X}) = e_n(B^{(\mathcal{W})}; \mathcal{X})$).

Note that in this construction, $e_n(I : \mathcal{W} \to \mathcal{X}) = e_n(X; \mathcal{X})$. This leads us to conjecture that

$$e_n(B^{(\mathcal{X}')}; (\mathcal{X}', \|\cdot\|_{\infty X})) \approx e_n(X; \mathcal{X}') \tag{3.2.6}$$

This is indeed only a conjecture because we don't know whether $e_n(I' : \mathcal{X}' \to \mathcal{W}') = e_n(I : \mathcal{W} \to \mathcal{X})$.

[1] The same results hold qualitatively if X is bounded and contains a neighborhood of zero in $\mathcal{W}$, and if $\mathbb{S}_+$ is bounded and contains a neighborhood of zero in $\mathcal{X}'$.

[2] If $\mathcal{X}$ is a reflexive Banach space, then $\mathcal{W}$ dense in $\mathcal{X}$ implies $\mathcal{X}'$ dense in $\mathcal{W}'$. The reflexivity assumption is necessary. See `https://math.stackexchange.com/questions/3103618/if-y-subset-x-are-banach-spaces-such-that-y-is-dense-in-x-is-it-true-that`.

Example: modulation spaces In this paragraph we show how to interpret the result of [HPP08] as an estimate on the metric entropy of a certain set of linear systems.

Denote $M^s_{pq}(\mathbb{R}^d)$ the modulation space for exponents $s \in \mathbb{R}$ and $1 \le p, q \le \infty$, and $M^s_p = M^s_{pp}$. The dual space of $M^s_p(\mathbb{R}^d)$ is $M^{-s}_{p'}(\mathbb{R}^d)$, where $1/p + 1/p' = 1$ [GS00] ; in particular it is a reflexive Banach space.

Suppose $\mathbb{S}$ consists of all scalar-valued linear systems over $\mathcal{X} = M^{s_2}_{p_2}(\mathbb{R}^d)$, i.e $\mathbb{S} = \mathcal{X}'$, and that the set of systems under consideration $\mathbb{S}_+$ is the unit ball of $\mathcal{X}'$. On the other hand, suppose that the test input signals are strictly "smoother" than required: $X \subset M^{s_1}_{p_1}(\mathbb{R}^d) = \mathcal{W}$. Moreover suppose X is the unit ball of that space.

Then,

$$e_n\left(\mathbb{S}_+; (\mathbb{S}, \|\cdot\|_{\infty X})\right) \approx e_n\left(I' : (M^{s_2}_{p_2}(\mathbb{R}^d))' \to (M^{s_1}_{p_1}(\mathbb{R}^d))'\right) \tag{3.2.7}$$

$$= e_n\left(\mathrm{id} : M^{-s_2}_{p_2'}(\mathbb{R}^d) \to M^{-s_1}_{p_1'}(\mathbb{R}^d)\right) \tag{3.2.8}$$

According to [HPP08, theorem 4.4], for $1 \le p_1 < p_2 < \infty$ and $s_1 > s_2$, that quantity is asymptotically given by

$$e_n\left(\mathrm{id} : M^{-s_2}_{p_2'}(\mathbb{R}^d) \to M^{-s_1}_{p_1'}(\mathbb{R}^d)\right) \sim n^{1/p_2 - 1/p_1 - (s_1 - s_2)/2d} \tag{3.2.9}$$

3.2.2 Beyond scalar-valued

Suppose again that $(\mathcal{W}, \|\cdot\|_{\mathcal{W}})$ is a compactly embedded subspace of $\mathcal{X}$, and that $X = B^{(\mathcal{W})}$.

This time consider $\mathbb{S} = \mathcal{L}_b(\mathcal{X}, \mathcal{Y})$ for some Banach output space $\mathcal{Y}$, and again $\mathbb{S}_+ = B^{(\mathbb{S})}$. [3]

Proposition 3.2. If $\mathcal{Y}$ is finite-dimensional, and (almost without loss of generality) is isomorphic to $(\mathbb{R}^d, \|\cdot\|_\infty)$,

$$e_{dn+1}(\mathbb{S}_+; (\mathbb{S}, \|\cdot\|_{\infty X})) \le 2e_{n+1}(I' : \mathcal{X}' \to \mathcal{W}') \tag{3.2.10}$$

The factor 2 can be removed if $\mathcal{X}'$ is dense in $\mathcal{W}'$.

Proof. As for the scalar-valued case, $\|S\|_{\infty X} = |||S|||_{\mathcal{W}\to\mathcal{Y}}$ the operator norm, which is the norm on $\mathcal{L}_b(\mathcal{W}, \mathcal{Y})$. So denoting the injection operator $\mathrm{id} : \mathcal{L}_b(\mathcal{X}, \mathcal{Y}) \to \mathcal{L}_b(\mathcal{W}, \mathcal{Y})$,

$$e_n\left(\mathrm{id} : \mathcal{L}_b(\mathcal{X}, \mathcal{Y}) \to \mathcal{L}_b(\mathcal{W}, \mathcal{Y})\right) \;\le\; e_n(\mathbb{S}_+; (\mathbb{S}, \|\cdot\|_{\infty X})) \;\le\; 2e_n\left(\mathrm{id} : \mathcal{L}_b(\mathcal{X}, \mathcal{Y}) \to \mathcal{L}_b(\mathcal{W}, \mathcal{Y})\right) \tag{3.2.11}$$

By decomposing $Sx \in \mathbb{R}^d$ component-wise, one shows that $\mathbb{S} = \mathcal{L}_b(\mathcal{X}, \mathcal{Y}) \simeq (\mathcal{X}')^d$, equipped with the max-over-components norm. So $\mathrm{id} : (\mathcal{X}')^d \to (\mathcal{W}')^d \simeq [I' : \mathcal{X}' \to \mathcal{W}']^{\otimes d}$ (tensor product of d identical copies), and

$$e_n(\mathbb{S}_+; (\mathbb{S}, \|\cdot\|_{\infty X})) \le 2e_n\left([I' : \mathcal{X}' \to \mathcal{W}']^{\otimes d}\right) \tag{3.2.12}$$

Now since $N_\varepsilon(A \times B; \|\cdot\|_{\max}) \le N_\varepsilon(A) \cdot N_\varepsilon(B)$, [4] denoting $\mathbb{B} = I'(B^{(\mathcal{X}')})$ and $\mathbb{B}^{\times d} = \mathbb{B} \times \dots \times \mathbb{B}$,

$$\log N_\varepsilon(\mathbb{B}^{\times d}; (\mathcal{W}')^d) \le d \log N_\varepsilon(\mathbb{B}; \mathcal{W}') \tag{3.2.13}$$

$$e_{dn+1}\left([I' : \mathcal{X}' \to \mathcal{W}']^{\otimes d}\right) \le e_{n+1}(I' : \mathcal{X}' \to \mathcal{W}') \tag{3.2.14}$$

As for the scalar-valued case, the factor 2 can be removed if $\mathcal{X}'$ is dense in $\mathcal{W}'$, since $(\mathcal{X}')^d$ is dense in $(\mathcal{W}')^d$. □

Proposition 3.3. If $\mathcal{Y}$ is infinite-dimensional, then $\mathbb{S}_+$ is not totally bounded for $\|\cdot\|_{\infty X}$.

[3] Again, the same results hold qualitatively if X is bounded and contains a neighborhood of zero in $\mathcal{W}$, and if $\mathbb{S}_+$ is bounded and contains a neighborhood of zero in $\mathcal{L}_b(\mathcal{X}, \mathcal{Y})$.

[4] Moreover this inequality $N_\varepsilon(A \times B; \|\cdot\|_{\max}) \le N_\varepsilon(A) \cdot N_\varepsilon(B)$ is morally quite tight since the same holds for the packing number: $M_\varepsilon(A \times B; \|\cdot\|_{\max}) \ge M_\varepsilon(A) \cdot M_\varepsilon(B)$. These facts are easy to see by a drawing for $A, B \subset \mathbb{R}$ and $A \times B \subset \mathbb{R}^2$.

Pedestrian proof. Let $\mathcal{Y}_d$ any subspace of $\mathcal{Y}$ of dimension d. $\mathcal{Y}_d \simeq \mathbb{R}^d$, and since all norms are equivalent, without loss of generality we assume it is equipped with a multiple of $\|\cdot\|_\infty$. Denote $\mathbb{S}_{+d}$ the subset of $\mathbb{S}_+$ whose elements have values in $\mathcal{Y}_d$, which is also the unit ball of $\mathcal{L}_b(\mathcal{X}, \mathcal{Y}_d)$.

By the same isometry as for the finite-dimensional case, the packing numbers are equal:

$$M_\varepsilon(\mathbb{S}_{+d}; (\mathbb{S}, \|\cdot\|_{\infty X})) = M_\varepsilon(\mathbb{B}^{\times d}; \|\cdot\|_{(\mathcal{W}')^d}) \tag{3.2.15}$$

where $\mathbb{B} = I'(B^{(\mathcal{X}')})$. Now since $M_\varepsilon(A \times B; \|\cdot\|_{\max}) \geq M_\varepsilon(A) \cdot M_\varepsilon(B)$,

$$M_\varepsilon(\mathbb{B}^{\times d}; \|\cdot\|_{(\mathcal{W}')^d}) \geq [M_\varepsilon(\mathbb{B}; \|\cdot\|_{\mathcal{W}'})]^d \tag{3.2.16}$$

which goes to infinity as $d \to \infty$. Besides, for any $\mathbb{S}_{+d} \subset \mathbb{S}_+$,

$$M_\varepsilon(\mathbb{S}_+; (\mathbb{S}, \|\cdot\|_{\infty X})) \geq M_\varepsilon(\mathbb{S}_{+d}; (\mathbb{S}, \|\cdot\|_{\infty X})) \tag{3.2.17}$$

This holds for any d, so we can conclude that for any $\varepsilon > 0$ (small enough so that $M_\varepsilon(\mathbb{B}; \|\cdot\|_{\mathcal{W}'}) > 1$), $M_\varepsilon(\mathbb{S}_+; (\mathbb{S}, \|\cdot\|_{\infty X})) = \infty$. $\square$

So, choosing $\mathbb{S}_+$ as the unit ball of $\mathcal{L}_b(\mathcal{X}, \mathcal{Y})$ is too large to say anything useful. Morally it is because balls in $\mathcal{Y}$ are not compact (Riesz theorem), so at any given point $x_0 \in X$, the set of values that Sx_0 can take is too large (that set is bounded but not compact). In fact this observation is the basis for an alternative, simpler proof.

Alternative simpler proof. As for the scalar-valued case, $\|S\|_{\infty X} = |||S|||_{\mathcal{W} \to \mathcal{Y}}$ the operator norm, which is the norm on $\mathcal{L}_b(\mathcal{W}, \mathcal{Y})$.

Suppose by contradiction $\mathbb{S}_+$ totally bounded for $\|\cdot\|_{\infty X}$, i.e relatively compact in the Banach $\mathcal{L}_b(\mathcal{W}; \mathcal{Y})$.

Let any nonzero $x_0 \in X = B^{(\mathcal{W})}$, and let the evaluation operator

$$E_{x_0} : \begin{bmatrix} \mathcal{L}_b(\mathcal{W}; \mathcal{Y}) \to \mathcal{Y} \\ S \mapsto Sx_0 \end{bmatrix} \tag{3.2.18}$$

The operator E_{x_0} is continuous with operator norm at most $\|x_0\|_{\mathcal{W}} = 1$, by definition (in fact one can show its operator norm is equal to 1, by existence of norming functionals).

So $E_{x_0}(\mathbb{S}_+)$ is relatively compact as the image of the relatively compact set $\mathbb{S}_+$ by the continuous mapping E_{x_0}.

Now, we claim that

$$E_{x_0}(\mathbb{S}_+) = \left\{ Sx_0,\ S \in \mathbb{S}_+ = B^{(\mathcal{L}_b(\mathcal{X};\mathcal{Y}))} \right\} \supset \|x_0\|_{\mathcal{X}} B^{(\mathcal{Y})} \tag{3.2.19}$$

Indeed, by Hahn-Banach theorem there exists $X_0 \in \mathcal{X}'$ a norming functional of x_0 in $\mathcal{X}$, i.e such that $X_0(x_0) = \langle x_0, X_0 \rangle_{\mathcal{X}} = \|x_0\|_{\mathcal{X}} > 0$ and $\|X_0\|_{\mathcal{X}'} = 1$. For any $y \in B^{(\mathcal{Y})}$, letting $S = yX_0$ we have

$$Sx_0 = yX_0(x_0) = \|x_0\|_{\mathcal{X}}\, y \tag{3.2.20}$$

$$\|S\|_{\mathcal{L}_b(\mathcal{X};\mathcal{Y})} = |||yX_0|||_{\mathcal{X} \to \mathcal{Y}} = \|y\|_{\mathcal{Y}} \|X_0\|_{\mathcal{X}'} \leq 1 \quad \text{i.e} \quad S \in \mathbb{S}_+ \tag{3.2.21}$$

Thus $E_{x_0}(\mathbb{S}_+)$ is relatively compact in $\mathcal{Y}$, and contains $\|x_0\|_{\mathcal{X}} B^{(\mathcal{Y})}$. But the latter set is closed and non-compact (by Riesz theorem, $\mathcal{Y}$ being infinite-dimensional). This constitutes a contradiction, so we conclude that $\mathbb{S}_+$ is totally bounded for $\|\cdot\|_{\infty X}$. $\square$

3.3 Kernel representation of linear systems

For the rest of this chapter, we take test input signals from $X = B^{(\mathcal{X})}$ the unit ball of $\mathcal{X}$. As mentioned in the first section, the worst-case error norm is then simply the operator norm:

$$\|S\|_{\infty X} = \sup_{\|x\|_{\mathcal{X}} \leq 1} \|Sx\|_{\mathcal{Y}} = |||S||| \tag{3.3.1}$$

The classical approach for linear system theory relies on the following informal fact:

"Any" linear system mapping, say, $L^p(\mathbb{R})$ to $L^r(\mathbb{R})$, can be put in the form:

$$S : x(t) \mapsto (\Phi_k x)(t) = \int_{\mathbb{R}} d\tau \; k(t, \tau) x(\tau) \tag{3.3.2}$$

for some function $k(t, \tau)$. k is called the impulse response function, because $k(t, \tau) = (S\delta_\tau)(t)$ where δ_τ is the τ-delayed Dirac delta signal (a.k.a unit impulse).

"Any" linear time-invariant system corresponds to a k such that $k(t, \tau) = k(t - \tau)$, so that

$$S : x(t) \mapsto (\Phi_k x)(t) = \int_{\mathbb{R}} d\tau \; k(t - \tau) x(\tau) = (k * x)(t) \tag{3.3.3}$$

i.e S is just the convolution operator with convolution kernel k.

So since any system S corresponds to a function k, the space $\mathbb{S}$ is morally isomorphic to a function space $\mathcal{K}$, and the subset $\mathbb{S}_+$ to a subset $\mathcal{K}_+$. So provided the norm $\|S\|_{\infty X}$ translates to a norm on k, the question of estimating metric entropy of $\mathbb{S}_+$ in $\mathbb{S}$ morally reduces to estimating metric entropy of $\mathcal{K}_+$ in the function space $\mathcal{K}$.

This motivates us to look more closely into the "fact" stated informally above, keeping in mind that we want to relate $\|S\|_{\infty X} = |||S|||$ to a norm on k.

Notations for this section

- Denote $\mathcal{X} = \mathcal{X}(U)$ a Banach space of input signals. Its elements $x(u)$ are signals over the domain U.
- Likewise, $\mathcal{Y} = \mathcal{Y}(V)$ a Banach space of output signals, with elements $y(v)$ signals over V.
- We consider a linear system mapping input signals to output signals, represented by a bounded operator $S \in \mathcal{L}_b(\mathcal{X}, \mathcal{Y})$.

For concreteness, we will sometimes focus on Lebesgue L^p spaces: the domains are measure spaces and $\mathcal{X}(U) = L^p(U)$, $\mathcal{Y}(V) = L^r(V)$.

Under what conditions can we write S as a kernel-integral operator i.e in the form

$$\text{"}\; (Sx)(v) = \int_U du \; k(v, u) x(u) \;\text{"} \tag{3.3.4}$$

for some function k, and in which space does k live?

Furthermore, what is the relation between $|||S|||$ and k?

3.3.1 Schwartz kernel theorem

A very powerful result from distributions theory:

Proposition 3.4. [5] Let U and V open subsets of $\mathbb{R}^d$. Denote $\mathcal{D}(U) = C_c^\infty(U)$ the space of infinitely-differentiable functions supported on a compact, and $\mathcal{D}'(U)$ the space of distributions over U. For $x \in \mathcal{D}(U), X \in \mathcal{D}'(U)$, denote $\langle x, X \rangle = X(x)$.

For any continuous linear map $S : \mathcal{D}(U) \to \mathcal{D}'(V)$, there exists a distribution $k \in \mathcal{D}'(V \times U)$ such that

$$\forall x \in \mathcal{D}(U), \forall y \in \mathcal{D}(V),\ \ \langle y, Sx \rangle = \langle y \otimes x, k \rangle \tag{3.3.5}$$

where $y \otimes x \in \mathcal{D}(V \times U)$ is defined by $(y \otimes x)(v, u) = y(v)x(u)$.

(Conversely, any such k defines a such S by the above formula.)

Note that with the usual abuse of notation identifying generalized functions (a.k.a distributions) as functions, $\langle y, Sx \rangle = \langle y \otimes x, k \rangle$ becomes:

$$\forall x(u), \forall y(v),\ \ \int_V dv\ (Sx)(v)y(v) = \iint_{V \times U} dv\ du\ k(v, u)y(v)x(u) \tag{3.3.6}$$

and so:

$$\forall x,\ (Sx)(v) = \int_U du\ k(v, u)x(u) \tag{3.3.7}$$

Roughly speaking, for any $\mathcal{X}(U)$ and $\mathcal{Y}(V)$ such that they are densely and continuously embedded in $\mathcal{D}(U)$ and $\mathcal{D}'(V)$ respectively, or the other way around, the above theorem thus provides a kernel-integral representation as desired. This applies in particular for $\mathcal{X} = L^p(U)$ and $\mathcal{Y} = L^r(V)$. [6]

However this does not seem practical for the goal of relating $|||S|||$ to a norm on k. In fact k is a priori a distribution, and the topology of distributions does not come from a norm.

We point out this theorem was one of the starting points of Grothendieck's 1955 work on topological tensor products and nuclear spaces, and that that theory provides a complete answer to this section's question, albeit much too abstract for our purposes. [7] It does not directly give any practical characterization of $|||S|||$ in term of k.

3.3.2 Output signals in $C(V)$ for compact V

Suppose (V, d_V) is a compact metric space and $\mathcal{Y}(V) = C(V)$ the Banach space of real-valued continuous signals over V, equipped with the sup norm.

Proposition 3.5 ([CS90, section 5.13])**.** For any bounded operator $S : \mathcal{X} \to C(V)$, the formula

$$\forall v \in V, K(v) = S'\delta_v \tag{3.3.8}$$

defines a bounded function $K : V \to \mathcal{X}'$ such that

$$\forall x \in \mathcal{X}, \forall v \in V,\ (Sx)(v) = \langle x, K(v) \rangle_{\mathcal{X}} \tag{3.3.9}$$

and $|||S||| = \sup_v \|K(v)\|_{\mathcal{X}'}$.

Moreover, S is a compact operator if and only if K is continuous, and then $|||S||| = \|K\|_{C(V,\mathcal{X}')}$.

Conversely, for any $K \in C(V, \mathcal{X}')$, the above formula defines a compact operator $S : \mathcal{X} \to C(V)$.

[5] https://en.wikipedia.org/wiki/Schwartz_kernel_theorem

[6] See https://mathoverflow.net/questions/2969/what-is-a-rigorous-statement-for-linear-time-invariant-systems-can-be-repre
See also [PP77, theorem 1].

[7] https://en.wikipedia.org/wiki/Integral_linear_operator

When for example $\mathcal{X}(U) = L^p(U)$ and $1 < p < \infty$, then $\mathcal{X}(U)' \simeq L^q(U)$ where $1/p + 1/q = 1$, and $\langle x, X \rangle_{\mathcal{X}} = \int_U du\ x(u)X(u)$; so we can write

$$\forall x \in \mathcal{X}(U), \forall v \in V,\ (Sx)(v) = \int_U du\ k(v,u)x(u) \tag{3.3.10}$$

$$|||S||| = \sup_v \|K(v)\|_{\mathcal{X}'} = \sup_v \left(\int_U du\ \ |k(v,u)|^q \right)^{1/q} \tag{3.3.11}$$

More generally, when $(Sx)(v) = \langle x, K(v) \rangle_{\mathcal{X}}$, we may call K the abstract kernel corresponding to S.

Proof. Let $S : \mathcal{X} \to C(V)$ a bounded operator. Recall that its dual $S' : C(V)' \to \mathcal{X}'$ is defined by

$$\forall (x, Y) \in \mathcal{X} \times C(V)',\ \ \langle x, S'Y \rangle_{\mathcal{X}} = \langle Sx, Y \rangle_{\mathcal{Y}} \tag{3.3.12}$$

For all $v \in V$, the Dirac delta measure $\delta_v : [y \mapsto y(v)]$ belongs to $rca(V) = C(V)'$. [8] So we can define $K(v) = S'\delta_v \in \mathcal{X}(U)'$. Then by definition,

- For all $x \in \mathcal{X}$ and $v \in V$,

$$\langle x, K(v) \rangle_{\mathcal{X}} = \langle x, S'\delta_v \rangle_{\mathcal{X}} = \langle Sx, \delta_v \rangle_{C(V)} = (Sx)(v) \tag{3.3.13}$$

- K is uniformly bounded over V (for the $\mathcal{X}(U)'$-norm) and its sup is the operator norm of S:

$$\begin{aligned} \sup_{v \in V} \|K(v)\|_{\mathcal{X}'} &= \sup_{v \in V} \sup_{\|x\|_{\mathcal{X}} \le 1} |\langle x, K(v) \rangle_{\mathcal{X}}| && (3.3.14) \\ &= \sup_{v \in V} \sup_{\|x\|_{\mathcal{X}} \le 1} |(Sx)(v)| && (3.3.15) \\ &= \sup_{\|x\|_{\mathcal{X}} \le 1} \|Sx\|_{C(V)} && (3.3.16) \\ &= |||S|||_{\mathcal{X} \to C(V)} \quad < \infty && (3.3.17) \end{aligned}$$

Let us characterize the condition under which the bounded operator S is compact, using Arzela-Ascoli theorem.

$$\begin{aligned} & S \text{ is a compact operator} && (3.3.18) \\ \iff\ & S(B^{(\mathcal{X})}) \text{ is a relatively compact subset of } C(V) && (3.3.19) \\ \iff\ & S(B^{(\mathcal{X})}) \text{ is uniformly bounded and equicontinuous} && (3.3.20) \\ \iff\ & (S \text{ is a bounded operator and})\ \lim_{\delta \to 0} \omega \left(S(B^{(\mathcal{X})}); \delta \right) = 0 && (3.3.21) \end{aligned}$$

Now, the modulus of equicontinuity of $S(B^{(\mathcal{X})})$ is equal to the modulus of continuity of K:

$$\begin{aligned} \forall v, w \in V,\ \ \|K(v) - K(w)\|_{\mathcal{X}'} &= \sup_{\|x\|_{\mathcal{X}} \le 1} |\langle x, S'\delta_v - S'\delta_w \rangle| && (3.3.22) \\ &= \sup_{\|x\|_{\mathcal{X}} \le 1} |(Sx)(v) - (Sx)(w)| && (3.3.23) \\ &= \sup_{f \in S(B^{(\mathcal{X})})} |f(v) - f(w)| && (3.3.24) \\ \forall \delta > 0,\quad \sup_{d_V(v,w) \le \delta} \|K(v) - K(w)\|_{\mathcal{X}'} &= \sup_{d_V(v,w) \le \delta} \sup_{f \in S(B^{(\mathcal{X})})} |f(v) - f(w)| && (3.3.25) \\ \omega(K; \delta) &= \omega \left(S(B^{(\mathcal{X})}); \delta \right) && (3.3.26) \end{aligned}$$

[8] `https://regularize.wordpress.com/2011/11/11/dual-spaces-of-continuous-functions/`

So we can conclude that

$$S \text{ is a compact operator} \tag{3.3.27}$$

$$\iff \omega\left(S(B^{(\mathcal{X})});\delta\right) = \omega(K;\delta) \xrightarrow{\delta\to 0} 0 \tag{3.3.28}$$

$$\iff K \text{ is continuous} \tag{3.3.29}$$

Conversely, one can check using the above arguments that, for any $K \in C(V, \mathcal{X}(U)')$, the operator S defined by $\forall x \in \mathcal{X}, v \in V,\ (Sx)(v) = \langle x, K(v)\rangle_{\mathcal{X}}$ has values in $C(V)$ and is compact. □

Remark 3.2. In the introduction of [CS90, chapter 5] they justify giving particular attention to $C([0,1])$-valued operators by mentioning that any separable Banach space $\mathcal{Y}$ can be seen as a subspace of $C([0,1])$ (Banach-Mazur theorem). One could try using this idea to extend this paragraph's scope to any separable $\mathcal{Y}$. Unfortunately, the proof of Banach-Mazur theorem is not constructive, so this would not result in an (abstract)-kernel-integral representation, as discussed in section D.2.

3.3.3 Output signals in $C_0(\mathbb{R})$

Suppose $V = \mathbb{R}$ and $\mathcal{Y}(V) = C_0(\mathbb{R})$ the Banach space of real-valued continuous signals vanishing at infinity, equipped with the sup norm. Our characterization of relatively compact subsets of $C_0(\mathbb{R})$ in section 2.2, extending Arzela-Ascoli theorem, will allow us to make a similar statement as for compact V.

Proposition 3.6. For any bounded operator $S : \mathcal{X} \to C_0(\mathbb{R})$, the formula

$$\forall v \in \mathbb{R}, K(v) = S'\delta_v \tag{3.3.30}$$

defines a bounded function $K : \mathbb{R} \to \mathcal{X}'$ such that

$$\forall x \in \mathcal{X}, \forall v \in \mathbb{R},\ (Sx)(v) = \langle x, K(v)\rangle_{\mathcal{X}} \tag{3.3.31}$$

and $|||S||| = \sup_v \|K(v)\|_{\mathcal{X}'}$.

Moreover, S is a compact operator if and only if K is continuous and vanishes at infinity, and then $|||S||| = \|K\|_{C_0(\mathbb{R},\mathcal{X}')}$.

Conversely, for any $K \in C_0(\mathbb{R}, \mathcal{X}')$, the above formula defines a compact operator $S : \mathcal{X} \to C_0(\mathbb{R})$.

Proof. The Dirac delta measure δ_v is still in the dual space of $\mathcal{Y} = C_0(\mathbb{R})$, [9] so the first part of the proposition is proved by the exact same arguments as for compact V.

Denote $\phi(v) = \sup_{f\in S(B^{(\mathcal{X})})} |f(v)|$, and note that it is related to K simply as:

$$\phi(v) = \sup_{\|x\|_{\mathcal{X}}\le 1} |(Sx)(v)| \tag{3.3.32}$$

$$= \sup_{\|x\|_{\mathcal{X}}\le 1} |\langle x, K(v)\rangle_{\mathcal{X}}| \tag{3.3.33}$$

$$= \|K(v)\|_{\mathcal{X}'} \tag{3.3.34}$$

Recall the definition of pointwise modulus of equicontinuity, and note that it is related to K simply

[9] https://regularize.wordpress.com/2011/11/11/dual-spaces-of-continuous-functions/

as:

$$\begin{aligned}
\omega_v\left(S(B^{(\mathcal{X})});\delta\right) &= \sup_{f\in S(B^{(\mathcal{X})})}\ \sup_{w;d_V(w,v)\le\delta} |f(v)-f(w)| && (3.3.35)\\
&= \sup_{w;d_V(w,v)\le\delta}\ \sup_{\|x\|_{\mathcal{X}}\le 1} |(Sx)(v)-(Sx)(w)| && (3.3.36)\\
&= \sup_{w;d_V(w,v)\le\delta}\ \sup_{\|x\|_{\mathcal{X}}\le 1} |\langle x, K(v)-K(w)\rangle_{\mathcal{X}}| && (3.3.37)\\
&= \sup_{w;d_V(w,v)\le\delta} \|K(v)-K(w)\|_{\mathcal{X}'} && (3.3.38)\\
&= \omega_v(K;\delta) && (3.3.39)
\end{aligned}$$

Let us characterize the condition under which the bounded operator S is compact, using the extension of Arzela-Ascoli theorem from section 2.2.

$$\begin{aligned}
& S \text{ is a compact operator} && (3.3.40)\\
\iff\ & S(B^{(\mathcal{X})}) \text{ is a relatively compact subset of } C_0(\mathbb{R}) && (3.3.41)\\
\iff\ & S(B^{(\mathcal{X})}) \text{ is pointwise equicontinuous and } \phi\in C_0(\mathbb{R}) && (3.3.42)\\
\iff\ & \forall v\in\mathbb{R}, \omega_v\left(S(B^{(\mathcal{X})});\delta\right) \xrightarrow{\delta\to 0} 0 \ \text{ and } \ \phi\in C_0(\mathbb{R}) && (3.3.43)\\
\iff\ & \forall v\in\mathbb{R}, \omega_v(K;\delta) \xrightarrow{\delta\to 0} 0 \ \text{ and } \ [v\mapsto \|K(v)\|_{\mathcal{X}'}]\in C_0(\mathbb{R}) && (3.3.44)\\
\iff\ & K \text{ is continuous and vanishes at infinity} && (3.3.45)
\end{aligned}$$

Conversely, one can check using the above arguments that, for any $K\in C_0(\mathbb{R},\mathcal{X}(U)')$, the operator S defined by $\forall x\in\mathcal{X}, v\in\mathbb{R},\ (Sx)(v)=\langle x,K(v)\rangle_{\mathcal{X}}$ has values in $C_0(\mathbb{R})$ and is compact. □

3.3.4 Output signals in $L^r(V)$

Reminders on Bochner spaces

For background on Bochner spaces, see section D.3. Throughout this paragraph, we assume that the domain Ω is equipped with a σ-finite measure μ. We recall the following facts about their dual spaces from [Kre15]:

Definition 3.1 (Informal)**.** Let G a Banach space and $1\le p\le\infty$. A function $f:\Omega\to G$ belongs to the *Bochner space* $L^p(\Omega;G)$ if the quantity

$$\|f\|_{L^p(\Omega;G)} = \int_\Omega dt\ \ \|f(t)\|_G \tag{3.3.46}$$

is finite. Furthermore $L^p(\Omega;G)$, equipped with that norm, is a Banach space.

Proposition 3.7 (Hoelder's inequality)**.** Let G a Banach space, and $1\le p,q\le\infty$ such that $\frac{1}{p}+\frac{1}{q}=1$.

For all $f\in L^p(\Omega,G)$ and $g\in L^q(\Omega,G')$, the scalar quantity $\int_\Omega dt\ \langle f(t),g(t)\rangle_G$ is well-defined and

$$\left|\int_\Omega dt\ \ \langle f(t),g(t)\rangle_G\right| \le \|f\|_{L^p(\Omega,G)}\ \ \|g\|_{L^q(\Omega,G')} \tag{3.3.47}$$

Proposition 3.8. The above proposition shows that any $g\in L^q(\Omega,G')$ induces a bounded linear form by $\langle f,g\rangle_{L^p(\Omega,G)} := \int_\Omega dt\ \langle f(t),g(t)\rangle_G$, and gives an upper-bound on its dual norm.

When $p<\infty$, that bound is tight, i.e:

$$\|g\|_{(L^p(\Omega,G))'} = \|g\|_{L^q(\Omega,G')} \tag{3.3.48}$$

In other words, for any $1\le p<\infty$, $L^q(\Omega,G')$ embeds isometrically into – i.e is isometrically isomorphic to a subspace of – $(L^p(\Omega,G))'$.

The Radon-Nikodym property is an interesting property satisfied by some (but not all) Banach spaces. We refer to the appendix for its definition.

Proposition 3.9 (Dual space of Bochner space)**.** Let $1 \leq p < \infty$ and G a Banach space with the Radon-Nikodym property.

Then $L^q(\Omega, G')$ is isometrically isomorphic to $(L^p(\Omega, G))'$.

Proposition 3.10 (Sufficient conditions for Radon-Nikodym property)**.** If G is reflexive then it has the Radon-Nikodym property.

If G is the dual space of some Banach space, and G is separable, then G has the Radon-Nikodym property.

A very loose upper-bound

As a motivation for more detailed investigation, let us state a very loose sufficient condition and upper-bound. Namely we show that, if $K \in L^r(V, \mathcal{X}')$, then a similar formula as for the continuous case: $\langle Sx, Y \rangle_{\mathcal{Y}} = \langle xY, K \rangle$, defines a bounded operator S.

Proposition 3.11. Suppose V is equipped with a σ-finite measure μ. Let $1 < r \leq \infty$ and R such that $\frac{1}{r} + \frac{1}{R} = 1$. Consider $\mathcal{Y}(V) = L^r(V)$ the Banach space of r-integrable real-valued signals over V, and $\mathcal{X}$ any Banach space.

Under these conditions, $\widetilde{\mathcal{Y}}(V) = L^R(V)$ is a predual of $\mathcal{Y}$ i.e $(\widetilde{\mathcal{Y}})' = \mathcal{Y}$. Furthermore, $L^r(V, \mathcal{X}')$ embeds isometrically into $(L^R(V, \mathcal{X}))'$.

Let $K \in L^r(V, \mathcal{X}')$. Then the formula

$$\forall x \in \mathcal{X},\ \forall z \in \widetilde{\mathcal{Y}}(V) = L^R(V), \quad \langle z, Sx \rangle_{\widetilde{\mathcal{Y}}} = \langle xz, K \rangle_{L^R(V,\mathcal{X})} \tag{3.3.49}$$

defines a bounded operator $S : \mathcal{X} \to \mathcal{Y}$. Here we denoted $xz : [v \mapsto z(v) \cdot x]$ ($z(v)$ being scalar), which is a "rank-1" element of $L^R(V, \mathcal{X})$.

Moreover the operator norm of S is bounded by $\|\|S\|\|_{\mathcal{X} \to \mathcal{Y}} \leq \|K\|_{L^r(V,\mathcal{X}')}$.

Proof. Let $K \in L^r(V, \mathcal{X}')$. It can also be interpreted as an element of $(L^R(V, \mathcal{X}))'$, as per the isometric embedding.

Clearly, the formula defines a linear operator $S : \mathcal{X} \to \mathcal{Y}$. To show that it is bounded, it suffices to show that $(x, z) \mapsto \langle xz, K \rangle_{L^R(V,\mathcal{X})}$ is a bounded bilinear form:

$$\left| \langle xz, K \rangle_{L^R(V,\mathcal{X})} \right| \leq \|xz\|_{L^R(V,\mathcal{X})} \ \ \|K\|_{(L^R(V,\mathcal{X}))'} \tag{3.3.50}$$

$$= \int_V dv \ \ |z(v)| \, \|x\|_{\mathcal{X}} \ \ \|K\|_{L^r(V,\mathcal{X}')} \tag{3.3.51}$$

$$= \|x\|_{\mathcal{X}} \ \ \|z\|_{L^R(V)} \ \ \|K\|_{L^r(V,\mathcal{X}')} \tag{3.3.52}$$

so that

$$\|\|S\|\|_{\mathcal{X} \to \mathcal{Y}} = \sup_{\|x\|_{\mathcal{X}} \leq 1} \|Sx\|_{\mathcal{Y}} = \sup_{\|x\|_{\mathcal{X}} \leq 1} \|Sx\|_{(\widetilde{\mathcal{Y}})'} = \sup_{\|x\|_{\mathcal{X}} \leq 1} \sup_{\|z\|_{\widetilde{\mathcal{Y}}} \leq 1} |\langle z, Sx \rangle|_{\widetilde{\mathcal{Y}}} \tag{3.3.53}$$

$$= \sup_{\|x\|_{\mathcal{X}} \leq 1} \sup_{\|z\|_{\widetilde{\mathcal{Y}}} \leq 1} |\langle xz, K \rangle|_{L^R(V,\mathcal{X})} \tag{3.3.54}$$

$$\leq \|K\|_{L^r(V,\mathcal{X}')} \tag{3.3.55}$$

This also shows the claimed upper-bound on the operator norm of S. ☐

Remark 3.3.
- For this direction we did not need to assume that $\mathcal{X}$ has the Radon-Nikodym property.

- We needed to consider a predual space of $\mathcal{Y}$ to cover the case $r = \infty$. If we assume $1 < r < \infty$, then $L^r(V)$ is reflexive, so that we can simply use $\mathcal{Y}'$ instead of a $\widetilde{\mathcal{Y}}$.
- The upper-bound of the operator norm comes from

$$\|\|S\|\|_{\mathcal{X}\to\mathcal{Y}} = \sup_{\|x\|_{\mathcal{X}}\leq 1} \sup_{\|z\|_{\widetilde{\mathcal{Y}}}\leq 1} |\langle xz, K\rangle|_{L^R(V,\mathcal{X})} \;\leq\; \|K\|_{(L^R(V,\mathcal{X}))'} = \|K\|_{L^r(V,\mathcal{X}')} \tag{3.3.56}$$

  Which seems extremely loose: on the left-hand side we have a sup over "rank-1" elements of $L^R(V, \mathcal{X})$, while on the right-hand side is a sup over all elements of $L^R(V, \mathcal{X})$.

Tensor product space view

To go further, it is necessary to understand the "geometry" of the Bochner space. For this, it is very helpful to view it as a tensor product of Banach spaces.

Definition 3.2. Denote

$$s(\Omega) = \left\{ \sum_{i=1}^{N} \alpha_i \mathbb{1}_{E_i};\; N \in \mathbb{N}, \alpha_i \in \mathbb{R}, E_i \subset \Omega \text{ measurable} \right\} \tag{3.3.57}$$

$$s(\Omega; G) = \left\{ \sum_{i=1}^{N} x_i \mathbb{1}_{E_i};\; N \in \mathbb{N}, x_i \in G, E_i \subset \Omega \text{ measurable} \right\} \tag{3.3.58}$$

the set of real-valued, resp. G-valued, simple functions over Ω.

For any $1 \leq p < \infty$, $s_p(\Omega)$ (resp. $s_p(\Omega; G)$) denotes $s(\Omega)$ seen as a subspace of $L^p(\Omega)$ (resp. of $L^p(\Omega; G)$), and equipped with the corresponding norm. Note that $s_p(\Omega)$ and $s_p(\Omega; G)$ are normed vector spaces which are not complete.

Proposition 3.12 ([Kre15, proposition 2.14])**.** Let $1 \leq p < \infty$.

Any $f \in L^p(\Omega)$ is the limit (in the $L^p(\Omega)$ norm) of a sequence of simple functions $s_n \in s(\Omega)$, so that

$$L^p(\Omega) = \overline{s_p(\Omega)} \tag{3.3.59}$$

Any $f \in L^p(\Omega, G)$ is the limit (in the $L^p(\Omega, G)$ norm) of a sequence of simple functions $s_n \in s(\Omega; G)$, so that

$$L^p(\Omega, G) = \overline{s_p(\Omega; G)} \tag{3.3.60}$$

Definition 3.3 ([Rya02])**.** $s(\Omega; G)$ is isomorphic to $G \otimes s(\Omega)$, by the identification

$$x \otimes s \in G \otimes s(\Omega) \quad \leftrightarrow \quad xs : [t \mapsto s(t) \cdot x] \in s(\Omega; G) \tag{3.3.61}$$

since $[t \mapsto \sum_i s_i(t) \cdot x_i]$ does not depend on the decomposition $\sum_i x_i \otimes s_i$.

Let $1 \leq p < \infty$, $1/p + 1/q = 1$. Recall that the continuous dual space of $s(\Omega)$ with respect to $\|\cdot\|_{L^p(\Omega)}$ (which is the same as the continuous dual of its completion) is $s_p(\Omega)' = L^p(\Omega)' = L^q(\Omega)$.

The tensor product space $G \otimes s_p(\Omega)$ can be equipped with the *projective norm*:

$$\forall f \in G \otimes s_p(\Omega),\;\; \|f\|_\pi = \inf \left\{ \sum_i \|x_i\|_G \|s_i\|_{L^p(\Omega)};\; x_i \in G, s_i \in s_p(\Omega) \text{ s.t } f = \sum_i x_i \otimes s_i \right\} \tag{3.3.62}$$

or with the *injective norm*:

$$\|f\|_\varepsilon = \sup \left\{ \left| \sum_i \langle x_i, X\rangle \langle s_i, Y\rangle \right|;\; \begin{matrix} \|X\|_{G'} \leq 1, \|Y\|_{L^q(\Omega)} \leq 1, \\ x_i \in G, s_i \in s_p(\Omega) \text{ s.t } f = \sum_i x_i \otimes s_i \end{matrix} \right\} \tag{3.3.63}$$

We denote $G\widehat{\otimes}_\pi s_p(\Omega)$ (resp. $G\widehat{\otimes}_\varepsilon s_p(\Omega)$) the completion of the tensor product space with respect to $\|\cdot\|_\pi$ (resp. $\|\cdot\|_\varepsilon$).

Proposition 3.13 ([DF92, chapter 7])**.** For any $1 \leq p < \infty$, the norm $\|\cdot\|_{L^p(\Omega;G)}$ on $s_p(\Omega; G)$ induces a natural norm on $G \otimes s(\Omega)$, which we denote $\|\cdot\|_{\Delta_p}$ ("natural norm on p-integrable functions"):

$$\left(s(\Omega; G), \|\cdot\|_{L^p(\Omega;G)}\right) \simeq \left(G \otimes s(\Omega), \|\cdot\|_{\Delta_p}\right) \tag{3.3.64}$$

Note that $G\widehat{\otimes}_{\Delta_p} s(\Omega)$, the completion of the tensor product space with respect to $\|\cdot\|_{\Delta_p}$, is isometrically isomorphic to $L^p(\Omega; G)$.

Then for all $f \in G \otimes s_p(\Omega)$, $\|f\|_\varepsilon \leq \|f\|_{\Delta_p} \leq \|f\|_\pi$.

In particular, we have the continuous embeddings:

$$G\widehat{\otimes}_\pi s_p(\Omega) \subset G\widehat{\otimes}_{\Delta_p} s(\Omega) \subset G\widehat{\otimes}_\varepsilon s_p(\Omega) \tag{3.3.65}$$

Remark 3.4. The norm $\|\cdot\|_{\Delta_p}$ on $G \otimes s(\Omega)$ has surprising properties and is well-studied in [DF92, chapter 7]. A further characterization is given in [Cal+16, section 4.2].

A tight equivalence

By applying this tensor product space view to $G = \mathcal{X}'$ and $L^p(\Omega) = L^r(V)$, we are led to a tight, albeit somewhat abstract, equivalence between $L^r(V)$-valued systems and kernel functions.

Proposition 3.14. As previously, denote $\mathcal{Y} = L^r(V)$, R such that $1/r + 1/R = 1$, and $\widetilde{\mathcal{Y}} = L^R(V)$. For simplicity, suppose $1 < r < \infty$, and consider $\mathcal{X}$ a reflexive Banach space.

A linear operator $S : \mathcal{X} \to L^r(V)$ is bounded if and only if there exists $K \in \mathcal{X}'\widehat{\otimes}_\varepsilon L^r(V)$ such that

$$\forall x \in \mathcal{X}, \forall z \in \widetilde{\mathcal{Y}},\ \ \langle z, Sx\rangle_{\widetilde{\mathcal{Y}}} = \langle xz, K\rangle \tag{3.3.66}$$

Moreover it then holds $|||S||| = \|K\|_\varepsilon$.

Informally, the injective norm arises because: for any approximate decomposition $\sum_i X_i \otimes y_i$ of $K \in \mathcal{X}'\widehat{\otimes}_\varepsilon \mathcal{Y}$, [Rya02, (3.3)]

$$\|K\|_\varepsilon \approx \left\|\sum_i X_i \otimes y_i\right\|_\varepsilon = \sup_{\|x\|_{\mathcal{X}} \leq 1} \sup_{\|z\|_{\widetilde{\mathcal{Y}}} \leq 1} \sum_i \langle x, X_i\rangle_{\mathcal{X}} \langle z, y_i\rangle_{\widetilde{\mathcal{Y}}} \tag{3.3.67}$$

which corresponds to the operator norm of $S : \mathcal{X} \to (\widetilde{\mathcal{Y}})'$.

Remark 3.5.

- In particular, the embedding

$$\mathcal{X}'\widehat{\otimes}_{\Delta_r} s(V) \subset \mathcal{X}'\widehat{\otimes}_\varepsilon s_r(V) \tag{3.3.68}$$

  explains that kernels $K \in L^r(V, \mathcal{X}') \simeq \mathcal{X}'\widehat{\otimes}_{\Delta_r} s(V)$ induce operators $S \in \mathcal{L}_b(\mathcal{X}, \mathcal{Y}) \simeq \mathcal{X}'\widehat{\otimes}_\varepsilon s_r(V)$, but that those K's do not describe all bounded operators.

- Also in that connection, the inequality $\|K\|_\varepsilon \leq \|K\|_{\Delta_p}$ recovers the fact that, when $K \in L^r(V, \mathcal{X}')$,

$$|||S||| = \|K\|_\varepsilon \leq \|K\|_{\Delta_p} = \|K\|_{L^r(V,\mathcal{X}')} \tag{3.3.69}$$

As pursuing this discussion would lead us quite far, and linear systems are not our main focus, we leave this topic as an interesting direction for further work. Let us simply mention that, as [Gir02] [GW03] illustrate, characterizing the space of kernel functions corresponding to a given class of linear operators, or even just giving usable sufficient conditions, is a difficult question.

3.3.5 Sufficient conditions for (time-invariant) systems from L^p to L^r: Young's (convolution) inequality

Proposition 3.15 (Young's inequality for integral operators)**.** [10] Let $p, r \in [1, +\infty]$, and take $\mathcal{X}(U) = L^p(U)$, $\mathcal{Y}(V) = L^r(V)$.

Denote $q \in [1, +\infty]$ such that $1/p + 1/q = 1 + 1/r$. (Assuming $1/r \leq 1/p \leq 1 + 1/r$.)

Let $k : U \times V \to \mathbb{R}$ such that

$$\forall u \in U, \;\; \|k(\cdot, u)\|_{L^q(V)} \leq C \tag{3.3.70}$$

$$\forall v \in V, \;\; \|k(v, \cdot)\|_{L^q(U)} \leq C \tag{3.3.71}$$

for some $C < \infty$. Then,

$$(Sx)(v) = \int_U du\ k(v, u)x(u) \tag{3.3.72}$$

defines a bounded linear operator from $L^p(U)$ to $L^r(V)$, and

$$|||S||| \leq C \tag{3.3.73}$$

Definition 3.4. Let $U = V = \mathbb{R}$, so that $\mathcal{X}$ and $\mathcal{Y}$ consist of signals over the same domain.

A linear system S is called *time-invariant* if for all $x \in \mathcal{X}$,

$$\forall \tau \in \mathbb{R}, \;\; S\left[x(\cdot - \tau)\right] = (Sx)(\cdot - \tau) \tag{3.3.74}$$

In other words, if S commutes with the τ-delay operator U_τ for each $\tau \in \mathbb{R}$:

$$U_\tau : x(t) \mapsto x(t - \tau) \tag{3.3.75}$$

$$S \circ U_\tau = U_\tau \circ S \tag{3.3.76}$$

(The operator U_τ being seen both as $\mathcal{X}(\mathbb{R}) \to \mathcal{X}(\mathbb{R})$ and as $\mathcal{Y}(\mathbb{R}) \to \mathcal{Y}(\mathbb{R})$.)

If a system S is linear time-invariant (LTI), and it arises from a kernel function $k(t, s)$ by

$$(Sx)(t) = \int_{\mathbb{R}} ds\ k(t, s)x(s) \tag{3.3.77}$$

then $k(t, s)$ must be a function only of $t - s$. In that case, S is simply a convolution:

$$(Sx)(t) = \int_{\mathbb{R}} ds\ k(t - s)x(s) = (k * x)(t) \tag{3.3.78}$$

Theorem 3.16 (Young's convolution inequality [Gol18, theorem 1.3.10])**.** Let $p, r \in [1, +\infty]$, and take $\mathcal{X} = L^p(\mathbb{R}^d)$, $\mathcal{Y} = L^r(\mathbb{R}^d)$.

Denote $q \in [1, +\infty]$ such that $1/p + 1/q = 1 + 1/r$. (Assuming $1/r \leq 1/p \leq 1 + 1/r$.)

Let $k \in L^q(\mathbb{R}^d)$. Then, for all $x \in L^p(\mathbb{R}^d)$, $(k * x)$ is well-defined and is in $L^r(\mathbb{R}^d)$, and:

$$\|k * x\|_{L^r} \leq \|k\|_{L^q} \|x\|_{L^p} \tag{3.3.79}$$

In other words,

$$(Sx)(t) = (k * x)(t) = \int_{\mathbb{R}^d} d\tau\ k(t - \tau)x(\tau) \tag{3.3.80}$$

defines a bounded linear operator from $L^p(\mathbb{R}^d)$ to $L^r(\mathbb{R}^d)$, and

$$|||S||| \leq \|k\|_{L^q} \tag{3.3.81}$$

[10] `https://en.wikipedia.org/wiki/Young%27s_inequality_for_integral_operators`

This result is very nice, but does not completely answer our questions.

- The theorem allows to define a class of systems based on kernel functions, but doesn't characterize the systems that are in this class.
- The upper-bound on the operator norm is not tight. (In fact the inequality $\|k * x\|_{L^r} \leq \|k\|_{L^q} \|x\|_{L^p}$ itself is not completely tight in general.) [11]

3.3.6 Linear time-invariant (LTI) systems from $C_b(\mathbb{R})$ to itself: fading memory

In the case $\mathcal{X} = \mathcal{Y} = C_b(\mathbb{R})$, [BC85] provides a characterization of the class of LTI systems defined by a kernel function (or rather a kernel measure).

Definition 3.5. We use the same notion of weight function as in section 2.2: a weight function is any $w : \mathbb{R} \to \ ]0, 1]$, such that $\lim_{t\to\pm\infty} w(t) = 0$. As previously, we denote $\|x\|_w = \sup_{t\in\mathbb{R}} |w(t)x(t)|$.

A LTI system $S : C_b(\mathbb{R}) \to C_b(\mathbb{R})$ has w*-fading memory* (a.k.a is w*-myopic*) if its present evaluation is continuous with respect to $\|\cdot\|_w$ on the inputs:

$$\forall\varepsilon > 0, \exists\delta > 0; \quad \sup_{t\in\mathbb{R}} w(t) \left|x(t) - x'(t)\right| \leq \delta \implies \left|(Sx)(0) - (Sx')(0)\right| \leq \varepsilon \tag{3.3.82}$$

In other words, S has w-fading memory if the linear form

$$\delta_0 \circ S : \begin{bmatrix} (C_b(\mathbb{R}), \|\cdot\|_w) \to \mathbb{R} \\ x \mapsto (Sx)(0) \end{bmatrix} \tag{3.3.83}$$

is continuous.

S has *fading memory* if it has w-fading memory for some valid weight function w.

Proposition 3.17. [BC85, theorem 5] Let a LTI system $S : C_b(\mathbb{R}) \to C_b(\mathbb{R})$. S has a convolution representation

$$(Sx)(t) = \int_{\mathbb{R}} h(d\tau)x(t-\tau) \tag{3.3.84}$$

for some measure $h \in ba(\mathbb{R})$, if and only if S has fading memory.

Moreover for any fixed weight function w, for any measure h, the system $S_h : \left[x \mapsto \int_{\mathbb{R}} h(d\tau)x(t-\tau)\right]$ has w-fading memory if and only if $\int_{\mathbb{R}} |h(dt)|\, w(-t)^{-1} < \infty$, and that quantity is the spectral norm of the present evaluation $|||\delta_0 \circ S_h|||_{C_{0,w}(\mathbb{R})\to\mathbb{R}}$.

3.4 Example: a set of LTI systems from L^2 to L^2

In this section, we consider as $\mathbb{S}$ the space of LTI systems $S : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ that can be represented by a L^1 convolution kernel: $F \in L^1(\mathbb{R}^d)$ and

$$\forall x \in L^2(\mathbb{R}^d),\ (Sx)(\boldsymbol{u}) = (F * x)(\boldsymbol{u}) \tag{3.4.1}$$

Young's convolution inequality tells us that S is a bounded linear operator and that $|||S||| \leq \|F\|_{L^1}$.

Suppose we are given a subset $\mathbb{S}_+ \subset \mathbb{S}$, corresponding to convolution kernels in some subset of $L^1(\mathbb{R}^d)$, and we want to estimate its metric entropy. For concreteness, consider the following set of

[11] `https://en.wikipedia.org/wiki/Young%27s_convolution_inequality#Sharp_constant`

systems, inspired by [Zam79]: [12]

$$\mathbb{S}_+(K,M) = \left\{ S_F; \quad \begin{matrix} F \in L^1(\mathbb{R}^d) \\ \operatorname{supp} F \subset K \\ \int_{\mathbb{R}^d} |F|^2 < \infty \\ \left\| \widehat{F} \right\|_{L^\infty} \leq M \end{matrix} \right\} \tag{3.4.2}$$

where the compact $K \subset \mathbb{R}^d$ and the real $M > 0$ are parameters of the problem.

3.4.1 $L^2 - L^2$ operator norm of convolutional systems, W-band-limited input signals

As mentioned previously, Young's convolution inequality only gives an upper-bound on the system's operator norm, in general. In the case of $L^2 \to L^2$ systems, it turns out that the operator norm is easy to characterize.

Proposition 3.18. For all $F \in L^1(\mathbb{R}^d)$ and $S : \begin{bmatrix} L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d) \\ x \mapsto F * x \end{bmatrix}$,

$$|||S||| = \left\| \widehat{F} \right\|_{L^\infty} \tag{3.4.3}$$

where $\widehat{F}$ is the $(L^1 - L^\infty)$ Fourier transform of F.

Proof sketch.

- To show $|||S||| \leq \left\| \widehat{F} \right\|_\infty$, use that the $L^2 - L^2$ Fourier transform is isometric:

$$|||S||| = \sup_{\|x\|_2 = \|\hat{x}\|_2 \leq 1} \|F * x\|_2 = \sup_{\|\hat{x}\|_2 \leq 1} \left\| \widehat{F} \cdot \hat{x} \right\|_2 \leq \left\| \widehat{F} \right\|_\infty \cdot 1 = \left\| \widehat{F} \right\|_\infty \tag{3.4.4}$$

- To show $|||S||| \geq \left\| \widehat{F} \right\|_\infty$, denote $\boldsymbol{u}_0 = \arg\max_{\boldsymbol{u}} \left| \widehat{F}(\boldsymbol{u}) \right|$, and evaluate at $x_n \in L^2(\mathbb{R}^d)$ such that $\hat{x}_n$ is increasingly concentrated around $\boldsymbol{u}_0$ while staying L^2-normalized.

$$|||S||| \geq \left\| \widehat{F} \cdot \hat{x}_n \right\|_2 \xrightarrow{n} \left| \widehat{F}(\boldsymbol{u}_0) \right| = \left\| \widehat{F} \right\|_\infty \tag{3.4.5}$$

□

Furthermore, for W a compact of $\mathbb{R}^d$, we may be interested in the behaviour of the systems only on input signals that are W-band-limited. [13]

Proposition 3.19. Let W a compact subset of $\mathbb{R}^d$. Define the restricted input signal space $\mathcal{X}(W)$, and denote the "operator norm over $\mathcal{X}(W)$" as

$$\mathcal{X}(W) := \left\{ x \in L^2(\mathbb{R}^d);\ \operatorname{supp}(\hat{x}) \subset W \right\} \tag{3.4.6}$$

$$|||S|||_W := \sup_{x \in \mathcal{X}(W), \|x\|_{L^2} \leq 1} \|Sx\|_{L^2} \tag{3.4.7}$$

[12] In [Zam79], they don't consider a "hard constraint" $\operatorname{supp} F \subset K$, instead they assume F vanishing exponentially at infinity.

Furthermore they don't assume $\int_{\mathbb{R}} |F|^2 < \infty$, which we do here for simplicity, same as in section 2.4. That condition could "probably" be removed without modifying the result. The proofs would have to be adapted slightly. Maybe it even works out simply by density arguments ($C(K)$ dense in $L^1(K)$ and in $L^2(K)$).

[13] This is not the path taken in [Zam79]: they take input signals from all of $L^2(\mathbb{R})$, but in return they impose a decrease-rate condition on $\widehat{F}$.

For all $F \in L^1(\mathbb{R}^d)$ and $S : \begin{bmatrix} L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d) \\ x \mapsto F * x \end{bmatrix}$,

$$|||S|||_W = \left\| \widehat{F} \right\|_{L^\infty(W)} := \sup_W \left| \widehat{F} \right| \tag{3.4.8}$$

(which is a semi-norm on $L^1(\mathbb{R}^d)$).

Proof sketch.

- To show $|||S|||_W \leq \left\| \widehat{F} \right\|_{L^\infty(W)}$:

$$|||S|||_W = \sup_{x \in \mathcal{X}(W), \|x\|_2 \leq 1} \|F * x\|_2 = \sup_{\operatorname{supp}(\hat{x}) \subset W, \|\hat{x}\|_2 \leq 1} \left\| \widehat{F} \cdot \hat{x} \right\|_2 \leq \left\| \widehat{F} \right\|_{L^\infty(W)} \tag{3.4.9}$$

- To show $|||S|||_W \geq \left\| \widehat{F} \right\|_{L^\infty(W)}$, denote $\boldsymbol{u}_0 = \arg\max_{\boldsymbol{u} \in W} \left| \widehat{F}(\boldsymbol{u}) \right|$, and evaluate at $x_n \in L^2(\mathbb{R}^d)$ such that $\hat{x}_n$ is increasingly concentrated around $\boldsymbol{u}_0$, while staying L^2-normalized and supported on W.

$$|||S|||_W \geq \left\| \widehat{F} \cdot \hat{x}_n \right\|_2 \xrightarrow{n} \left| \widehat{F}(\boldsymbol{u}_0) \right| = \left\| \widehat{F} \right\|_{L^\infty(W)} \tag{3.4.10}$$

□

3.4.2 Reduction to the metric entropy of a function space

Proposition 3.20. Let the space

$$\mathbb{G}_+(K, M) = \mathbb{B}_K \cap B_{0,M}^{(L^\infty)} = \left\{ f : \mathbb{R}^d \to \mathbb{R}; \quad \begin{array}{c} \hat{f} \in L^1(\mathbb{R}^d) \\ \operatorname{supp} \hat{f} \subset K \\ \int_{\mathbb{R}^d} \left| \hat{f} \right|^2 < \infty \\ \|f\|_{L^\infty} \leq M \end{array} \right\} \tag{3.4.11}$$

The spaces $(\mathbb{S}_+(K, M), |||\cdot|||_W)$ and $\left(\mathbb{G}_+(K, M), \|\cdot\|_{L^\infty(-W)}\right)$ are isometrically isomorphic.
In particular, $\log N_\varepsilon \left(\mathbb{S}_+(K, M), |||\cdot|||_W\right) = \log N_\varepsilon \left(\mathbb{G}_+(K, M), \|\cdot\|_{L^\infty(-W)}\right)$.

Proof sketch.

- Check that, in the definition of $\mathbb{S}$, the correspondence between operators S and convolution kernels F is one to one.
- S with convolution kernel F is mapped to $f := \widetilde{F}$ (inverse Fourier transform of F). Conversely f is mapped to S with convolution kernel $F := \hat{f}$. One can check that this defines a linear bijection between $\mathbb{S}_+(K, M)$ and $\mathbb{G}_+(K, M)$.
- As discussed just above, the operator norm is given by $|||S|||_W = \left\| \widehat{F} \right\|_{L^\infty(W)} = \left\| \widetilde{F} \right\|_{L^\infty(-W)} = \|f\|_{L^\infty(-W)}$, so the mapping is indeed isometric. Here we used that the Fourier transform and inverse Fourier transform only differ by a sign flip of the variable.

□

For simplicity, suppose W symmetric. Now we want to estimate $\log N_\varepsilon \left(\mathbb{G}_+(K, M), \|\cdot\|_{L^\infty(W)}\right)$. Since $\mathbb{G}_+(K, M) \subset \mathbb{B}_K$, we can use the results on band-limited functions developed in section 2.4.

Notation conflict: time vs. frequency variable To avoid confusion, we point out that

- In section 2.4 and appendix A, $f \in \mathbb{B}_K$ could be seen as a signal, and the sampling expansion allowed to recover f from its samples. Here f is a *multiplier* (i.e spectrum of a convolution kernel), the term "signal" being reserved to the input signals $x \in \mathcal{X}(W)$ which f multiplies, and we propose using the sampling expansion on f as a theoretical proof technique.
- In section 2.4 and appendix A, f was seen as a function of the variable t (time), and $\hat{f}$ as a function of the variable ξ (frequency). Here, assuming that the systems are seen as mapping time-signals x to time-signals Sx, it is more natural to take F as a function of time and so $f = \widetilde{F}$ as a function of frequency.

Chapter 4

Volterra series

Volterra series are a well-studied general model for nonlinear systems. They extend the convolution representation of LTI systems to higher orders, and correspond to the signal-to-signal version of polynomials and power series.

The notation and point of view taken here are mostly adapted from [Sch80] [Sch81] (more engineering-oriented), and from [BCD84] [BC85] (more rigorous).

This chapter defines and shows interesting properties of Volterra series. The usefulness of this notion for our purpose (estimating metric entropy of nonlinear systems), will be shown in chapter 6 and chapter 7.

4.1 Definitions

Depending on the assumptions on the signals, different definitions (choices of function space) are possible, though the formulas are the same. Here we chose a setting that balances generality and simplicity, and discuss alternative choices in a dedicated subsection.

Throughout this section, let $1 \leq p, q \leq \infty$ such that $1/p + 1/q = 1$.

4.1.1 Finite-order Volterra series

Definition 4.1. A *Volterra monomial* of order $n \in \mathbb{N}$ is a mapping $\Phi_{k_n} : L^p(\mathbb{R}) \to L^\infty(\mathbb{R})$ defined by

$$\forall x \in L^p,\ \Phi_{k_n}[x](t) = \int_{\mathbb{R}^n} d\tau_1 ... d\tau_n\ k_n(\tau_1, ..., \tau_n) x(t-\tau_1)...x(t-\tau_n) \tag{4.1.1}$$

$$= \int_{\mathbb{R}^n} d\boldsymbol{\tau}\ k_n(\boldsymbol{\tau}) x(t-\tau_1)...x(t-\tau_n) \tag{4.1.2}$$

where $k_n(\boldsymbol{\tau}) \in L^q(\mathbb{R}^n)$ is a function called the Volterra kernel of Φ_{k_n}.

A *Volterra series of finite order* $N \in \mathbb{N}$ is a sum of Volterra monomials of orders $n \leq N$.

In this chapter, we denote by

$$\mathcal{K}_n = L^q(\mathbb{R}^n) \qquad \mathbb{S}_n = \{\Phi_{k_n};\ k_n \in \mathcal{K}_n\} \tag{4.1.3}$$

the vector space of Volterra kernels and Volterra monomials of order n, and by

$$\mathcal{K}_{\leq N} = \mathcal{K}_0 \times ... \times \mathcal{K}_N \qquad \mathbb{S}_{\leq N} = \{\Phi_k;\ k = (k_0, ..., k_N) \in \mathcal{K}_{\leq N}\} \tag{4.1.4}$$

the vector space of Volterra kernels and Volterra series of finite order N, where

$$\forall x \in L^p,\ \Phi_{k_n}[x](t) = \int_{\mathbb{R}^n} d\boldsymbol{\tau}\ k_n(\boldsymbol{\tau})x(t-\tau_1)...x(t-\tau_n) \tag{4.1.5}$$

$$\Phi_k[x](t) = \sum_{n\leq N} \Phi_{k_n}[x](t) \tag{4.1.6}$$

Remark 4.1. Note that $k_n \mapsto \Phi_{k_n}$ is a non-injective linear map, since for example k_n and k'_n : $[\boldsymbol{\tau} \mapsto k_n(\tau_2, \tau_1, \boldsymbol{\tau}_{3:n})]$ define the same operator. The consequences of this non-injectiveness will be discussed in a later section.

Proposition 4.1. The definition of Volterra series is sound, in that Φ_{k_n} maps $L^p(\mathbb{R})$ into $L^\infty(\mathbb{R})$ when $k_n \in L^q(\mathbb{R}^n)$.

In particular, a finite-order Volterra series may be seen as a nonlinear system mapping $\mathcal{X} = L^p(\mathbb{R})$ into $\mathcal{Y} = L^\infty(\mathbb{R})$, and indexed by $\mathcal{K}_{\leq N} = L^q(\mathbb{R}^0) \times ... \times L^q(\mathbb{R}^N)$.

Moreover, for all $k_n \in \mathcal{K}_n$, $k \in \mathcal{K}_{\leq N}$ and $x \in \mathcal{X}$,

$$\|\Phi_{k_n}[x]\|_{\mathcal{Y}} \leq \|k_n\|_{\mathcal{K}_n} \|x\|_{\mathcal{X}}^n \tag{4.1.7}$$

$$\|\Phi_k[x]\|_{\mathcal{Y}} \leq \sum_{n=0}^{N} \|k_n\|_{\mathcal{K}_n} \|x\|_{\mathcal{X}}^n \tag{4.1.8}$$

Proof. For any $x \in L^p(\mathbb{R})$, then $x^{\times n}(\boldsymbol{\tau}) : [\boldsymbol{\tau} \mapsto x(\tau_1)...x(\tau_n)] \in L^p(\mathbb{R}^n)$.

For any $k_n \in L^q(\mathbb{R}^n)$, since $1/p + 1/q = 1$, by Young's convolution inequality

$$\left\|k_n * x^{\times n}\right\|_{L^\infty(\mathbb{R}^n)} \leq \|k_n\|_{L^q} \left\|x^{\times n}\right\|_{L^p} \tag{4.1.9}$$

$$= \|k_n\|_{L^q} \|x\|_{L^p}^n \quad < \infty \tag{4.1.10}$$

and so $(k_n * x^{\times n}) \in L^\infty(\mathbb{R}^n)$.

Now for all $t \in \mathbb{R}$,

$$\Phi_{k_n}[x](t) = \int_{\mathbb{R}^n} d\boldsymbol{\tau}\ k_n(\boldsymbol{\tau})x(t-\tau_1)...x(t-\tau_n) \tag{4.1.11}$$

$$= \int_{\mathbb{R}^n} d\boldsymbol{\tau}\ k_n(\boldsymbol{\tau})x^{\times n}(t\mathbf{1}_n - \boldsymbol{\tau}) \tag{4.1.12}$$

$$= (k_n * x^{\times n})(t\mathbf{1}_n) \tag{4.1.13}$$

where $\mathbf{1}_n = \begin{pmatrix}1 & \dots & 1\end{pmatrix}^T \in \mathbb{R}^n$.

So $\Phi_{k_n}[x]$ is uniformly bounded i.e $\Phi_{k_n}[x] \in L^\infty(\mathbb{R})$, and $\|\Phi_{k_n}[x]\|_{L^\infty} \leq \|k_n\|_{L^q} \|x\|_{L^p}^n$. ☐

Definition 4.2 ([BCD84])**.** The *gain bound function* associated to the Volterra kernels $k = (k_0, ..., k_N) \in \mathcal{K}_{\leq N}$ is the power series with non-negative coefficients

$$f_k(z) = \sum_{n\leq N} \|k_n\|_{L^q} z^n \tag{4.1.14}$$

With this, the bound in the above proposition can be stated as

$$\forall x \in \mathcal{X}, \forall k \in \mathcal{K}_{\leq N},\ \ \|\Phi_k[x]\|_{\mathcal{Y}} \leq f_k(\|x\|_{\mathcal{X}}) \tag{4.1.15}$$

A Volterra series system S can be written as $S = \Phi_k$ for several choices of k, so its gain bound function is defined as

$$f_S(z) = \inf_{k\in\mathcal{K}_{\leq N}; S=\Phi_k} f_k(z) \tag{4.1.16}$$

With this, clearly,

$$\forall x \in \mathcal{X}, \|S[x]\|_{\mathcal{Y}} \leq f_S(\|x\|_{\mathcal{X}}) \tag{4.1.17}$$

Remark 4.2. It is not immediately obvious that $f_S(z)$ is a power series, because of the "inf", but it is indeed the case, as will become clear when we discuss the non-injectiveness of $k \mapsto \Phi_k$. Basically it will turn out that the "inf" is achieved simply at $\overline{k} = \operatorname{Sym} k$ for any k such that $S = \Phi_k$.

4.1.2 Other possible choices of spaces

In this thesis we focus on the setting $\mathcal{X} = L^p(\mathbb{R})$, $\mathcal{K}_n = L^q(\mathbb{R}^n)$ and $\mathcal{Y} = L^\infty(\mathbb{R})$, as introduced in our definitions so far. We dedicate this subsection to briefly discuss other possible choices, before returning to that setting.

Proposition 4.2. The formula defining Volterra monomials:

$$\forall x \in \mathcal{X}, \forall k_n \in \mathcal{K}_n,\ \Phi_{k_n}[x](t) = \int_{\mathbb{R}^n} d\boldsymbol{\tau}\ k(\boldsymbol{\tau}) x(t-\tau_1)...x(t-\tau_n)\ \in \mathcal{Y} \tag{4.1.18}$$

is well-defined for

- $\mathcal{X} = L^p(\mathbb{R})$, $\mathcal{K}_n = L^q(\mathbb{R}^n)$ and $\mathcal{Y} = L^\infty(\mathbb{R})$, where $1/p + 1/q = 1$ (as in the rest of the thesis);
- $\mathcal{X} = L^\infty(\mathbb{R})$, $\mathcal{K}_n = ba(\mathbb{R}^n)$ and $\mathcal{Y} = L^\infty(\mathbb{R})$ (as in [BCD84]);
- $\mathcal{X} = C_b(\mathbb{R})$, $\mathcal{K}_n = L^1(\mathbb{R}^n)$ or $ba(\mathbb{R}^n)$, and $\mathcal{Y} = C_b(\mathbb{R})$ (as in [BC85]);
- $\mathcal{X} = L^{2n}(\mathbb{R})$, $\mathcal{K}_n = L^1(\mathbb{R}^n)$ and $\mathcal{Y} = L^2(\mathbb{R})$;
- $\mathcal{X} = L^n(\mathbb{R})$, $\mathcal{K}_n = L^1(\mathbb{R}^n)$ and $\mathcal{Y} = L^1(\mathbb{R})$;
- $\mathcal{X} = L^p(\mathbb{R})$, $\mathcal{K}_n \subset L^q(\mathbb{R}^n)$, and $\mathcal{Y} = L^r(\mathbb{R})$, where $1/p + 1/q = 1 + 1/(nr)$ and $\mathcal{K}_n$ consists of separable functions: $\forall k_n \in \mathcal{K}_n, \exists \phi_1 ... \phi_n;\ k_n(\boldsymbol{\tau}) = \phi_1(\tau_1)...\phi_n(\tau_n)$ [1]

Moreover there holds an upper-bound: (in the case $\mathcal{K}_n = ba(\mathbb{R}^n)$, the norm is the total variation norm)

$$\forall x \in \mathcal{X}, \forall k_n \in \mathcal{K}_n,\ \ \|\Phi_{k_n}[x]\|_{\mathcal{Y}} \le \|k_n\|_{\mathcal{K}_n} \|x\|_{\mathcal{X}}^n \tag{4.1.19}$$

A different choice of function spaces straightforwardly leads to a corresponding modification in the definition of the gain bound function. Note that it doesn't apply for $\mathcal{X} = L^{2n}$ or L^n, though.

We emphasize the second item, pointing out that the Volterra kernels may not be functions, but rather bounded measures (which may not have a Radon-Nikodym derivative with respect to the usual Lebesgue measure). For example, the nonlinear system $x \mapsto \int_{\mathbb{R}} ds\ x(s)^2$ can be represented as a Volterra monomial with $k_2(\tau_1, \tau_2) = \delta_{\tau_1 - \tau_2 = 0}$, which is not in any L^q space [PP77]. See also [BCD84, examples 1 and 2].

On the right space for the Volterra kernels, see also [PP77, section 3].

Time-varying Volterra series In the same way that the convolution representation of linear systems $Sx(t) = \int_{\mathbb{R}} d\tau\ k(t-\tau)x(\tau)$ can be seen as a special case of the kernel-integral representation $Sx(v) = \int_U du\ k(v,u)x(u)$, one may define the time-varying version of the Volterra series by

$$\Phi_{k_n}[x](v) = \int_{U^n} d\boldsymbol{u}\ k_n(v, \boldsymbol{u}) x(u_1)...x(u_n) \tag{4.1.20}$$

for k_n living in some well-chosen space.

The time-varying Volterra series is not discussed in this thesis, for simplicity. Extending our discussion to the time-varying case would be interesting, but likely quite heavy (in the same way that Young's inequality for integral linear operators was heavier to state than Young's inequality for convolutions, in section 3.3).

[1] Note that this $\mathcal{K}_n$ is not a vector space.

4.1.3 Infinite-order Volterra series

(Henceforth we return to the setting $\mathcal{X} = L^p(\mathbb{R})$, $\mathcal{K}_n = L^q(\mathbb{R}^n)$, $\mathcal{Y} = L^\infty(\mathbb{R})$ with $1 \leq p, q \leq \infty$ and $1/p + 1/q = 1$.)

The definition of the Volterra series is straightforward when there is a finite number N of terms. For $N = \infty$, it is trickier due to convergence issues, but the gain bound function allows a nice characterization of the allowable input domain.

Definition 4.3. For a sequence of functions $k_n \in L^q(\mathbb{R}^n)$ $(n = 0, 1, ...)$, denoted $k = (k_n)_{n\in\mathbb{N}}$, formally define the *infinite-order Volterra series* with kernel functions k as

$$\Phi_k[x](t) = \sum_{n\in\mathbb{N}} \Phi_{k_n}[x](t) \tag{4.1.21}$$

$$= \sum_{n\in\mathbb{N}} \int_{\mathbb{R}^n} d\boldsymbol{\tau}\ k_n(\boldsymbol{\tau}) x(t-\tau_1)...x(t-\tau_n) \tag{4.1.22}$$

and define the associated gain bound function as the power series with non-negative coefficients

$$f_k(z) = \sum_{n\in\mathbb{N}} \|k_n\|_{L^q} z^n \tag{4.1.23}$$

Proposition 4.3. The radius of convergence of $f_k(z)$ is the greatest R such that $\|k_n\|_{L^q} \lesssim \mathrm{cst} \cdot R^{-n}$, i.e $R = \left[\limsup_n \|k_n\|_{L^q}^{1/n}\right]^{-1}$.

The infinite-order Volterra series Φ_k defines a mapping from $B_\rho^{(L^p)}$ to $L^\infty(\mathbb{R})$, for any $0 < \rho < R$. Furthermore,

$$\forall x \in L^p(\mathbb{R}),\ \ \|\Phi_k[x]\|_{L^\infty} \leq f_k(\|x\|_{L^p}) \tag{4.1.24}$$

Proof. The radius of convergence is given by Cauchy-Hadamard theorem.

For any $x \in L^p(\mathbb{R})$ such that $\|x\|_{L^p} < R$, the series

$$\Phi_k[x](t) = \sum_{n\in\mathbb{N}} \int_{\mathbb{R}^n} d\boldsymbol{\tau}\ k_n(\boldsymbol{\tau}) x(t-\tau_1)...x(t-\tau_n) \tag{4.1.25}$$

is absolutely converging in L^∞, by Hoelder inequality and definition of the power series f_k. Furthermore

$$\|\Phi_k[x]\|_{L^\infty} \leq \sum_{n\in\mathbb{N}} \|k_n\|_{L^q} \|x\|_{L^p}^n = f_k(\|x\|_{L^p}) \tag{4.1.26}$$

□

Similarly to the notations $\mathcal{K}_n$, $\mathcal{K}_{\leq N}$ and $\mathbb{S}_n$, $\mathbb{S}_{\leq N}$ for the finite-order case, let us denote by

$$\mathcal{K}_\infty(R) = \left\{ k = (k_0, k_1, ...) \in \mathcal{K}_0 \times \mathcal{K}_1 \times ...;\ \left[\limsup_n \|k_n\|_{\mathcal{K}_n}^{1/n}\right]^{-1} \geq R \right\} \tag{4.1.27}$$

$$\mathbb{S}_\infty(R) = \{\Phi_k;\ k \in \mathcal{K}_\infty(R)\} \tag{4.1.28}$$

the vector space of Volterra kernels and Volterra series of infinite order, whose gain bound function has radius of convergence $\geq R$. (The fact that $\mathcal{K}_\infty(R)$ is a vector space is intuitive and easy to check.)

The above proposition means that $\mathbb{S}_\infty(R)$ can be seen as a space of nonlinear systems from $X = B_\rho^{(L^p)}$ to $L^\infty(\mathbb{R})$, for any $0 < \rho < R$.

4.2 Dealing with symmetry of the Volterra kernels

Remark 4.3. To avoid confusion, note that in this thesis, *we use the term "symmetric function" to refer to permutation-invariant functions* $f(\boldsymbol{\tau}_\sigma) = f(\boldsymbol{\tau})$! (Instead of the common definition $f(-t) = f(t)$.)

Summary

- For Volterra monomials with kernels in $\mathcal{K}_n = L^q(\mathbb{R}^n)$, the quotient space $\mathcal{K}_n/(\operatorname{Ker}\Phi)$ can be identified with $L^q_{\mathrm{Sym}}(\mathbb{R}^n)$ the subset of symmetric functions. Moreover the topologies interact in the intuitive way, i.e $\|k\|_{\mathcal{K}_n/(\operatorname{Ker}\Phi)} = \|\operatorname{Sym} k\|_{L^q_{\mathrm{Sym}}(\mathbb{R}^n)}$ where $\operatorname{Sym} k(\boldsymbol{\tau}) = \frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n} k(\boldsymbol{\tau}_\sigma)$.
- Denote $\Delta_n = \{(\tau_1,...,\tau_n)\in\mathbb{R}^n;\ \tau_1 \le ... \le \tau_n\}$. To construct coverings in the space $L^q_{\mathrm{Sym}}(\mathbb{R}^N)$, it may be helpful to use the isometric isomorphism $\begin{bmatrix} (L^q_{\mathrm{Sym}}(\mathbb{R}^n), \|\cdot\|_{L^q(\mathbb{R}^n)}) \to (L^q(\Delta_n), \|\cdot\|_{L^q(\Delta_n)}) \\ k \mapsto (n!)^{1/q}\, k|_{\Delta_n} \end{bmatrix}$.
- As a toy example, consider sets of lipschitz-continuous kernel functions. If $k \in L^q(\mathbb{R}^n)$ is L-lipschitz (w.r.t any ℓ^p norm on $\mathbb{R}^n$), then $\operatorname{Sym} k$ is too. Furthermore, $k \in L^q_{\mathrm{Sym}}(\mathbb{R}^n)$ is L-lipschitz w.r.t $\|\cdot\|_\infty$ on $\mathbb{R}^n$ if and only if $k|_{\Delta_n}$ is.

4.2.1 Volterra kernels can be assumed symmetric without loss of generality

The map $k_n \mapsto \Phi_{k_n}$ is not injective, because the expression $\int_{\mathbb{R}^n} d\boldsymbol{\tau}\ k_n(\boldsymbol{\tau})x(t-\tau_1)...x(t-\tau_n)$ is symmetric in the variables $\tau_1,...,\tau_n$. This means that, morally, we can assume the Volterra kernels symmetric without loss of generality. This section makes this idea precise.

Definition 4.4. $\mathfrak{S}_n$ denotes the set of permutations σ over $\{1,...,n\}$.

Let the shorthand $\boldsymbol{\tau}_\sigma = \big(\tau_{\sigma(1)},...,\tau_{\sigma(n)}\big)$ for any $\boldsymbol{\tau}\in\mathbb{R}^n$ and $\sigma\in\mathfrak{S}_n$.

A function $f:\mathbb{R}^n\to\mathbb{R}$ is symmetric if $\forall\sigma,\forall\boldsymbol{\tau},\ f(\boldsymbol{\tau}_\sigma) = f(\boldsymbol{\tau})$. Denote $L^q_{\mathrm{Sym}}(\mathbb{R}^n)$ the subset of $L^q(\mathbb{R}^n)$ consisting of symmetric functions.

For any function $f:\mathbb{R}^n\to\mathbb{R}$, denote

$$\operatorname{Sym} f(\boldsymbol{\tau}) = \frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n} f(\boldsymbol{\tau}_\sigma) \tag{4.2.1}$$

Clearly $[f\mapsto \operatorname{Sym} f]$ is linear, and is a projection of $L^q(\mathbb{R}^n)$ onto $L^q_{\mathrm{Sym}}(\mathbb{R}^n)$.

Proposition 4.4. Let $n\in\mathbb{N}$ and recall $\mathcal{K}_n = L^q(\mathbb{R}^n)$ and $\mathbb{S}_n = \{\Phi_{k_n},\ k_n\in\mathcal{K}_n\}$.

The linear map $\Phi : \begin{bmatrix} L^q(\mathbb{R}^n) \to \mathbb{S}_n \\ k_n \mapsto \Phi_{k_n} \end{bmatrix}$ is not injective.

The restriction to the set of symmetric functions, $\Phi|_{L^q_{\mathrm{Sym}}} : \Big[L^q_{\mathrm{Sym}}(\mathbb{R}^n)\to\mathbb{S}_n\Big]$, is injective and defines an isomorphism.

(Equivalently, the quotient space $\mathcal{K}_n/(\operatorname{Ker}\Phi)$ can be identified with $L^q_{\mathrm{Sym}}(\mathbb{R}^n)$.)

Proof. We already saw that Φ is not injective because $k_n(\boldsymbol{\tau})$ and $k_n(\boldsymbol{\tau}_\sigma)$ have the same image by Φ, for any $\sigma\in\mathfrak{S}_n$. In particular note that, by linearity and definition of Sym, $\forall k_n\in L^q(\mathbb{R}^n),\ \Phi_{k_n} = \Phi_{\operatorname{Sym} k_n}$.

To check that the restriction $\Phi|_{L^q_{\mathrm{Sym}}}$ is surjective, simply note that $\Phi = \Phi|_{L^q_{\mathrm{Sym}}}\circ \operatorname{Sym}$, and that Sym is a projection of $L^q(\mathbb{R}^n)$ onto $L^q_{\mathrm{Sym}}(\mathbb{R}^n)$.

It remains to check that $\Phi|_{L^q_{\mathrm{Sym}}}$ is injective. Let $h\in L^q_{\mathrm{Sym}}(\mathbb{R}^n)$ such that $\Phi_h = 0$, we want to show that $h = 0$.

Denote

$$\Phi_h\{x_1,...,x_n\}(t) = \int_{\mathbb{R}^n} d\boldsymbol{\tau}\ h(\tau_1,...,\tau_n)x_1(t-\tau_1)...x_n(t-\tau_n) \tag{4.2.2}$$

the associated n-linear system. $\Phi_h\{x_1,...,x_n\}$ is symmetric in its arguments since h is.

According to [Sch80, section 5.4], the n-linear system $\Phi_h\{...\}$ is completely determined by the Volterra monomial $\Phi_h[...]$, through an algebraic polarization identity: for example if $n=4$, (equation

(5.4-3) of that book) [2]

$$
\begin{aligned}
4!\Phi_h\{x_1, ..., x_4\} &= \Phi_h[x_1 + ... + x_4] && (4.2.3)\\
&- (\Phi_h[x_1 + x_2 + x_3] + ...) && (4.2.4)\\
&+ (\Phi_h[x_1 + x_2] + ...) && (4.2.5)\\
&- (\Phi_h[x_1] + ... + \Phi_h[x_4]) && (4.2.6)
\end{aligned}
$$

The right-hand-side consists of terms which are all 0 by assumption. Alternatively, [BCD84, (2.6) and theorem 2.5.2] states the polarization identity using differentials as:

$$
n!\Phi_h\{x_1, ..., x_n\} = \left.\frac{\partial}{\partial\alpha_1 ... \partial\alpha_n}\right|_{\boldsymbol{\alpha}=0} \Phi_h\left[\sum_{i=1}^{n} \alpha_i x_i\right] \tag{4.2.7}
$$

and the right-hand-side is the differential of an identically zero system. Consequently,

$$
\forall x_1, ..., x_n \in L^p(\mathbb{R}), \Phi_h\{x_1, ..., x_n\} = 0 \tag{4.2.8}
$$

Now evaluate this at $x_1(t) = \mathbb{1}_{t\in A_1}, ..., x_n(t) = \mathbb{1}_{t\in A_n}$ for intervals $A_i \subset \mathbb{R}$. Then we obtain

$$
\Phi_h\{x_1, ..., x_n\}(t = 0) = \int_{\mathbb{R}^n} d\boldsymbol{\tau}\ h_n(-\tau_1, ..., -\tau_n)\mathbb{1}_{\boldsymbol{\tau}\in A_1\times ...\times A_n} = 0 \tag{4.2.9}
$$

Since this holds for all A_i, and hyperrectangles generate the Borel σ-algebra, then $h = 0$, as claimed. □

Thus, we can restrict our attention to k symmetric, without loss of generality in terms of the set of systems Φ_k considered. Let us check that doing so is also without loss of generality in terms of topological information.

Proposition 4.5. Let the norm over $\mathbb{S}_n$ be

$$
\|S\|_{\mathbb{S}_n} = \inf_{k_n\in L^q(\mathbb{R}^n);\ S=\Phi_{k_n}} \|k_n\|_{L^q} \tag{4.2.10}
$$

This is simply the norm induced by the surjective linear map $\Phi : \mathcal{K}_n \to \mathbb{S}_n$.

Then for any $S \in \mathbb{S}_n$, denoting $\overline{k}_n$ its unique antecedent by Φ in $L^q_{\mathrm{Sym}}(\mathbb{R}^n)$, it holds

$$
\|S\|_{\mathbb{S}_n} = \left\|\overline{k}_n\right\|_{L^q} \tag{4.2.11}
$$

In words, the norm induced by Φ coincides with the norm induced by the restricted map $\Phi|_{L^q_{\mathrm{Sym}}(\mathbb{R}^n)}$.

Hence, $\Phi|_{L^q_{\mathrm{Sym}}(\mathbb{R}^n)}$ is an isometric isomorphism between the spaces $(\mathbb{S}_n, \|\cdot\|_{\mathbb{S}_n})$ and $(L^q_{\mathrm{Sym}}(\mathbb{R}^n), \|\cdot\|_{L^q})$.

Proof. Let $S = \Phi_{\overline{k}} \in \mathbb{S}_n$ with $\overline{k} \in L^q_{\mathrm{Sym}}(\mathbb{R}^n)$. By definition we have the inequality $\|S\|_{\mathbb{S}_n} \leq \left\|\overline{k}_n\right\|_{L^q}$.

To show the other direction, first note that

$$
\begin{aligned}
\forall k \in L^q(\mathbb{R}^n),\ S = \Phi_k &\iff \Phi_{\overline{k}} = \Phi_{\mathrm{Sym}\,k} && (4.2.12)\\
&\iff \overline{k} = \mathrm{Sym}\,k && (4.2.13)
\end{aligned}
$$

since $\Phi_k = \Phi_{\mathrm{Sym}\,k}$, and Φ is injective over $L^q_{\mathrm{Sym}}(\mathbb{R}^n)$.

Now, for any $k \in L^q(\mathbb{R}^n)$,

$$
\|\mathrm{Sym}\,k\|_q = \left\|\frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n} k(\boldsymbol{\tau}_\sigma)\right\|_q \leq \frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n} \|k(\boldsymbol{\tau}_\sigma)\|_q \tag{4.2.14}
$$

[2] More generally, see `https://en.wikipedia.org/wiki/Polarization_of_an_algebraic_form`.

and

$$\forall \sigma \in \mathfrak{S}_n,\ \ \|k(\boldsymbol{\tau}_\sigma)\|_q = \left[\int_{\mathbb{R}^n} d\boldsymbol{\tau}\ \ \left|k(\tau_{\sigma(1)},...,\tau_{\sigma(n)})\right|^q\right]^{1/q} = \left[\int_{\mathbb{R}^n} d\boldsymbol{\tau}\ \ |k(\tau_1,...,\tau_n)|^q\right]^{1/q} = \|k\|_q \tag{4.2.15}$$

so that $\|\mathrm{Sym}\, k\|_q \leq \|k\|_q$.

Thus, for any $k \in L^q(\mathbb{R}^n)$ such that $\Phi_{\overline{k}} = \Phi_k$, then $\left\|\overline{k}\right\|_q = \|\mathrm{Sym}\, k\|_q \leq \|k\|_q$. Taking the inf over k yields the required inequality. □

Remark 4.4. We showed that

$$\forall f \in L^q(\mathbb{R}^n),\ \ \|\mathrm{Sym}\, f\|_{L^q} \leq \|f\|_{L^q} \tag{4.2.16}$$

The significance of this inequality can be seen by the following example.

Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is a symmetric function and $g : \mathbb{R}^n \to \mathbb{R}$ is an ε-approximation for the L^q metric, i.e

$$\|f - g\|_{L^q} = \left[\int_{\mathbb{R}^n} d\boldsymbol{\tau}\ \ |f - g|\,(\boldsymbol{\tau})^q\right]^{1/q} \leq \varepsilon \tag{4.2.17}$$

Then $\mathrm{Sym}\, g$ is a better (or at least as good) approximation than g, since

$$\|f - \mathrm{Sym}\, g\|_{L^q} = \|\mathrm{Sym}(f-g)\|_{L^q} \leq \|f - g\|_{L^q} \tag{4.2.18}$$

Yet another way to express this: given an ε-covering $(g_1, ..., g_m)$ of a set of symmetric functions, the family $(\mathrm{Sym}\, g_1, ..., \mathrm{Sym}\, g_m)$ is also an ε-covering – which has the advantage of itself consisting of symmetric functions. So if we wish to construct an ε-covering of a set of symmetric functions, we might as well look for symmetric prototypes.

4.2.2 Restricting symmetric functions to Δ_n

We showed that $\Phi|_{L^q_{\mathrm{Sym}}(\mathbb{R}^n)}$ is an isometric isomorphism between the spaces $(\mathbb{S}_n, \|\cdot\|_{\mathbb{S}_n})$ to $(L^q_{\mathrm{Sym}}(\mathbb{R}^n), \|\cdot\|_{L^q})$. Let us clarify how to work in the latter function space. (Note that we are completely done with spaces of systems, and this subsection discusses symmetry of functions.)

Definition 4.5. Denote the sectors

$$\Delta_n = \{(\tau_1, ..., \tau_n) \in \mathbb{R}^n; \tau_1 \leq ... \leq \tau_n\} \tag{4.2.19}$$

$$\Delta_n^{(\sigma)} = \left\{(\tau_1, ..., \tau_n) \in \mathbb{R}^n; \tau_{\sigma(1)} \leq ... \leq \tau_{\sigma(n)}\right\} \tag{4.2.20}$$

Note that $\Delta_n = \Delta_n^{(\mathrm{id})}$ and $\boldsymbol{\tau} \in \Delta_n^{(\sigma)} \iff \boldsymbol{\tau}_\sigma \in \Delta_n$. Furthermore $\mathbb{R}^n = \bigcup_{\sigma \in \mathfrak{S}_n} \Delta_n^{(\sigma)}$, and the union is disjoint up to hyperplanes.

For the causal case, similarly denote the positive sectors: $\Delta_n^{(\sigma)+} = \Delta_n^{(\sigma)} \cap \mathbb{R}_+^n$.

Proposition 4.6. A symmetric function $k_n(\boldsymbol{\tau})$ is completely specified by its values on the sector Δ_n.

Conversely, any $\tilde{k}_n : \Delta_n \to \mathbb{R}$ induces a unique symmetric $k_n : \mathbb{R}^n \to \mathbb{R}$, by letting $k_n(\boldsymbol{\tau}) = \tilde{k}_n(\boldsymbol{\tau}_{\sigma_{\boldsymbol{\tau}}})$ where $\sigma_{\boldsymbol{\tau}}$ is any permutation such that $\boldsymbol{\tau} \in \Delta_n^{(\sigma_{\boldsymbol{\tau}})}$.

Proof. Let k_n symmetric. $\mathbb{R}^n = \bigcup_{\sigma \in \mathfrak{S}_n} \Delta_n^{(\sigma)}$, and for any σ the value of k_n on $\Delta_n^{(\sigma)}$ is given by

$$\forall \boldsymbol{\tau} \in \Delta_n^{(\sigma)}, k_n(\tau_1, ..., \tau_n) = k_n(\underbrace{\tau_{\sigma(1)}, ..., \tau_{\sigma(n)}}_{\in \Delta_n}) \tag{4.2.21}$$

So k_n is completely specified by its values on Δ_n.

For the converse, let us check that the extension is uniquely defined. Indeed for any $\boldsymbol{\tau} \in \mathbb{R}^n$, if both σ and σ' verify $\boldsymbol{\tau}_\sigma, \boldsymbol{\tau}_{\sigma'} \in \Delta_n$, then

$$\boldsymbol{\tau}_\sigma \in \Delta_n \iff \boldsymbol{\tau} \in \Delta_n^{(\sigma)} \iff \tau_{\sigma(1)} \leq ... \leq \tau_{\sigma(n)} \tag{4.2.22}$$

$$\boldsymbol{\tau}_{\sigma'} \in \Delta_n \iff \boldsymbol{\tau} \in \Delta_n^{(\sigma')} \iff \tau_{\sigma'(1)} \leq ... \leq \tau_{\sigma'(n)} \tag{4.2.23}$$

Then it can be shown by induction that $\forall i \leq n, \tau_{\sigma(i)} = \tau_{\sigma'(i)}$:

- $\tau_{\sigma(1)} = \tau_{\sigma'(1)} = \min_j \tau_j$
- $\tau_{\sigma(i)} = \min_{j \notin \{\sigma(1),...,\sigma(i-1)\}} \tau_j = \min_{j \notin \{\sigma'(1),...,\sigma'(i-1)\}} \tau_j = \tau_{\sigma'(i)}$

More simply put, there is only one way to write the values $\tau_1, ..., \tau_n$ in order. □

Remark 4.5. If k_n corresponds to a causal system, k_n is supported on $\mathbb{R}_+^n$, and the proposition above holds with Δ_n replaced by Δ_n^+. The situation is then easier to visualize. For example for $n = 2$, Δ_n^+ is the cone (in the positive quadrant) delimited by the lines $\mathbb{R}_+(0,1)$ and $\mathbb{R}_+(1,1)$. For $n = 3$, Δ_n^+ is the cone (in the positive "quadrant") delimited by the lines $\mathbb{R}_+(0,0,1)$ and $\mathbb{R}_+(0,1,1)$ and $\mathbb{R}_+(1,1,1)$. See Figure 4.1.

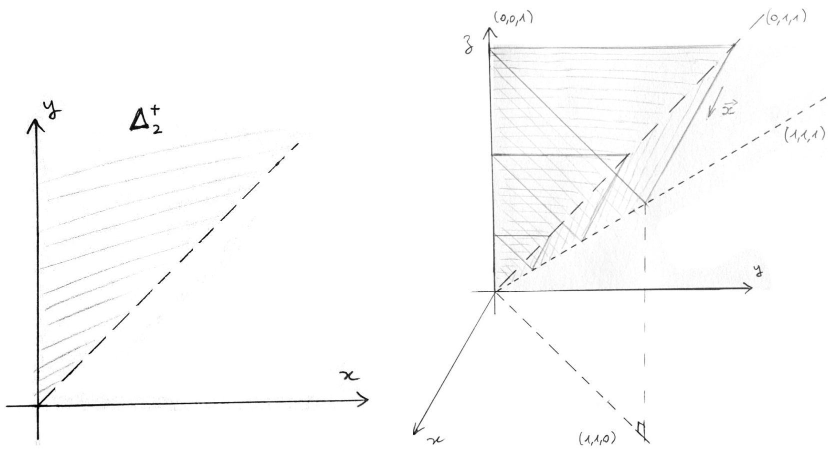


Figure 4.1: Left: the set Δ_2^+ (in $\mathbb{R}^2$). Right: the set Δ_3^+ (in $\mathbb{R}^3$).

Hence, instead of considering $L^q_{\mathrm{Sym}}(\mathbb{R}^n)$ with symmetry constraints, we can equivalently work with the space $L^q(\Delta_n)$. But we need to be careful about the norm used, as simply taking the restriction would incur a constant factor which depends on n.

Proposition 4.7. The following mapping is an isometric isomorphism.

$$\text{Restrict} : \begin{bmatrix} (L^q_{\mathrm{Sym}}(\mathbb{R}^n), \|\cdot\|_{L^q(\mathbb{R}^n)}) \to (L^q(\Delta_n), \|\cdot\|_{L^q(\Delta_n)}) \\ k \mapsto (n!)^{1/q}\, k|_{\Delta_n} \end{bmatrix} \tag{4.2.24}$$

Proof. The fact that Restrict is bijective follows from the previous proposition.

To show that it is isometric, let $k_n \in L^q_{\mathrm{Sym}}(\mathbb{R}^n)$.

$$
\begin{aligned}
n! \ \|k_n\|^q_{L^q(\Delta_n)} &= n! \int_{\Delta_n} d\boldsymbol{\tau} \ |k_n(\boldsymbol{\tau})|^q && (4.2.25)\\
&= \sum_{\sigma \in \mathfrak{S}_n} \int_{\Delta_n^{(\sigma)}} d\boldsymbol{\tau} \ |k_n(\boldsymbol{\tau})|^q && (4.2.26)\\
&= \int_{\mathbb{R}^n} d\boldsymbol{\tau} \ |k_n(\boldsymbol{\tau})|^q && (4.2.27)\\
&= \|k_n\|^q_{L^q(\mathbb{R}^n)} && (4.2.28)\\
(n!)^{1/q} \ \|k_n\|_{L^q(\Delta_n)} &= \|k_n\|_{L^q(\mathbb{R}^n)} && (4.2.29)
\end{aligned}
$$

□

Lipschitz-continuity In this paragraph we essentially show that, to study L-lipschitz Volterra kernels, it suffices to study L-lipschitz functions on Δ_n.

Claim 4.1. Let $\|\cdot\|$ any norm over $\mathbb{R}^n$ such that $\|\boldsymbol{\tau}_\sigma\| = \|\boldsymbol{\tau}\|$. For example $\|\boldsymbol{\tau}\|_\infty = \max_{1 \le i \le n} |\tau_i|$ or $\|\boldsymbol{\tau}\|_\alpha = \left[\sum_{i=1}^n |\tau_i|^\alpha\right]^{1/\alpha}$.

If $k : \mathbb{R}^n \to \mathbb{R}$ is L-lipschitz with respect to $\|\cdot\|$, then $\mathrm{Sym}\, k$ is too.

Proof. Suppose $|k(\boldsymbol{\tau}) - k(\boldsymbol{\tau}')| \le L \|\boldsymbol{\tau} - \boldsymbol{\tau}'\|$. Then

$$
\begin{aligned}
|\mathrm{Sym}\, k(\boldsymbol{\tau}) - \mathrm{Sym}\, k(\boldsymbol{\tau}')| = \frac{1}{n!} \left| \sum_\sigma k(\boldsymbol{\tau}_\sigma) - k(\boldsymbol{\tau}'_\sigma) \right| &\le \frac{1}{n!} \sum_\sigma |k(\boldsymbol{\tau}_\sigma) - k(\boldsymbol{\tau}'_\sigma)| && (4.2.30)\\
&\le \frac{1}{n!} \sum_\sigma L \|\boldsymbol{\tau}_\sigma - \boldsymbol{\tau}'_\sigma\| && (4.2.31)\\
&= \frac{1}{n!} \sum_\sigma L \|\boldsymbol{\tau} - \boldsymbol{\tau}'\| \quad = L \|\boldsymbol{\tau} - \boldsymbol{\tau}'\| && (4.2.32)
\end{aligned}
$$

□

Previously we showed that $\left[k \mapsto (n!)^{1/q} \ k|_{\Delta_n}\right]$ is an isometric isomorphism from $(L^q_{\mathrm{Sym}}(\mathbb{R}^n), \|\cdot\|_{L^q(\mathbb{R}^n)})$ to $(L^q(\Delta_n), \|\cdot\|_{L^q(\Delta_n)})$. Here we show that, up to the normalizing constant, the set of L-lipschitz functions in $L^q_{\mathrm{Sym}}(\mathbb{R}^n)$ is mapped to the set of L-lipschitz functions in $L^q(\Delta_n)$.

Claim 4.2. $k \in L^q_{\mathrm{Sym}}(\mathbb{R}^n)$ is L-lipschitz with respect to $\|\cdot\|_\infty$ if and only if $k|_{\Delta_n}$ is.

Proof. Clearly if $k \in L^q_{\mathrm{Sym}}(\mathbb{R}^n)$ is L-lipschitz then its restriction is too. Now let us show the converse: let $k \in L^q_{\mathrm{Sym}}(\mathbb{R}^n)$ such that $k|_{\Delta_n}$ is L-lipschitz, we want to show that k is L-lipschitz over $\mathbb{R}^n$.

Let $\boldsymbol{\tau}, \boldsymbol{\tau}' \in \mathbb{R}^n$, and σ, σ' such that $\boldsymbol{\tau} \in \Delta_n^{(\sigma)}$ and $\boldsymbol{\tau}' \in \Delta_n^{(\sigma')}$, i.e $\boldsymbol{\tau}_\sigma, \boldsymbol{\tau}'_{\sigma'} \in \Delta_n$. By symmetry of k,

$$
\begin{aligned}
|k(\boldsymbol{\tau}) - k(\boldsymbol{\tau}')| = |k(\boldsymbol{\tau}_\sigma) - k(\boldsymbol{\tau}'_{\sigma'})| &\le L \|\boldsymbol{\tau}_\sigma - \boldsymbol{\tau}'_{\sigma'}\|_\infty && (4.2.33)\\
\|\boldsymbol{\tau} - \boldsymbol{\tau}'\|_\infty = \|\boldsymbol{\tau}_\sigma - \boldsymbol{\tau}'_\sigma\|_\infty &= \left\|\boldsymbol{\tau}_\sigma - (\boldsymbol{\tau}'_{\sigma'})_{\sigma \circ (\sigma')^{-1}}\right\|_\infty && (4.2.34)
\end{aligned}
$$

Hence it suffices to show that

$$
\forall \nu \in \mathfrak{S}_n, \forall \tilde{\boldsymbol{\tau}}, \tilde{\boldsymbol{\tau}}' \in \Delta_n, \|\tilde{\boldsymbol{\tau}} - \tilde{\boldsymbol{\tau}}'\|_\infty \le \|\tilde{\boldsymbol{\tau}} - \tilde{\boldsymbol{\tau}}'_\nu\|_\infty \qquad (4.2.35)
$$

Now this is precisely equivalent to saying that the order statistics are 1-lipschitz for the $\|\cdot\|_\infty$ norm [Wai19, example 2.29], shown for completeness in the lemma just below. □

Lemma 4.8. Given a vector $X = (X_1, ..., X_n) \in \mathbb{R}^n$, denote $\widetilde{X}$ the vector obtained by reordering its entries in a non-decreasing manner: $\{X_1, ..., X_n\} = \left\{\widetilde{X}_1, ..., \widetilde{X}_n\right\}$ (with multiplicity) and $\widetilde{X}_1 \leq ... \leq \widetilde{X}_n$.

For all $X, Y \in \mathbb{R}^n$,

$$\max_{1\leq k\leq n} \left|\widetilde{X}_k - \widetilde{Y}_k\right| \leq \|X - Y\|_\infty \tag{4.2.36}$$

Proof. For concision, abbreviate $\|\cdot\| := \|\cdot\|_\infty$.

Denote σ, σ' such that $\widetilde{X} = X_\sigma$ and $\widetilde{Y} = Y_{\sigma'}$ and $\nu = \sigma \circ (\sigma')^{-1}$, so that $\|X - Y\| = \left\|\widetilde{X} - \widetilde{Y}_\nu\right\|$. We want to show

$$\forall \nu \in \mathfrak{S}_n, \forall k \in [n], \quad \left|\widetilde{X}_k - \widetilde{Y}_k\right| \leq \left\|\widetilde{X} - \widetilde{Y}_\nu\right\| \tag{4.2.37}$$

Suppose by contradiction that $\left|\widetilde{X}_k - \widetilde{Y}_k\right| > \left\|\widetilde{X} - \widetilde{Y}_s\right\|$ for some k. Without loss of generality $\widetilde{X}_k \geq \widetilde{Y}_k$, and so $\widetilde{X}_k > \left\|\widetilde{X} - \widetilde{Y}_\nu\right\| + \widetilde{Y}_k$. Since $\widetilde{X}$ and $\widetilde{Y}$ are ordered this also yields bounds on $\widetilde{X}_i$ and $\widetilde{Y}_j$ for $i \geq k$ and $j \leq k$:

$$\widetilde{X}_n \geq ... \geq \widetilde{X}_k > \widetilde{Y}_k + \left\|\widetilde{X} - \widetilde{Y}_\nu\right\| \geq ... \geq \widetilde{Y}_1 + \left\|\widetilde{X} - \widetilde{Y}_\nu\right\| \tag{4.2.38}$$

Now

- If there exists $i \geq k$ such that $\nu(i) = j \leq k$, then

$$\widetilde{X}_i > \widetilde{Y}_{\nu(i)} + \left\|\widetilde{X} - \widetilde{Y}_\nu\right\| \tag{4.2.39}$$

$$\left(\widetilde{X} - \widetilde{Y}_\nu\right)_i = \widetilde{X}_i - \widetilde{Y}_{\nu(i)} > \left\|\widetilde{X} - \widetilde{Y}_\nu\right\| \tag{4.2.40}$$

  which contradicts the definition of $\|Z\| = \max_k |Z_k|$.

- Otherwise, $\forall i \geq k,\ \nu(i) > k$. Equivalently, $\nu([k, n]) \subset [k + 1, n]$.
  But $\#\nu([k, n]) > \#[k + 1, n]$, so this is also a contradiction.

□

4.3 Continuity properties

Recall that $\mathcal{X} = L^p(\mathbb{R})$, $\mathcal{K}_n = L^q(\mathbb{R}^n)$ with fixed $1 \leq p, q \leq \infty$ such that $1/p + 1/q = 1$, and that

$$\mathcal{K}_\infty(R) = \left\{k = (k_0, k_1, ...) \in \mathcal{K}_0 \times \mathcal{K}_1 \times ...;\ \left[\limsup_n \|k_n\|_{\mathcal{K}_n}^{1/n}\right]^{-1} \geq R\right\} \tag{4.3.1}$$

$$\mathbb{S}_\infty(R) = \{\Phi_k;\ k \in \mathcal{K}_\infty(R)\} \tag{4.3.2}$$

Proposition 4.9. Let $0 < \rho < R$. Let $\Phi_k \in \mathbb{S}_\infty(R)$, with gain bound function f_k. Let $X = B_\rho^{(L^p)}$ the ball of $L^p(\mathbb{R})$ of radius ρ.

Suppose $p = \infty, q = 1$. Φ_k is lipschitz-continuous over X with lipschitz constant $f_k'(\rho)$.

If we assume arbitrary $1 \leq p \leq \infty$, Φ_k is lipschitz-continuous over X with lipschitz constant $f_k'(2\rho)$.

Remark 4.6. This proposition is not surprising at all, by analogy with the scalar-domain case (power series instead of Volterra series). In fact this result could probably be recovered easily, and extended, by using Frechet differentiation arguments.

This proposition generalizes [BCD84, theorem 2.3.2] to arbitrary p. Moreover in the case $p=\infty$, our lipschitz constant is slightly tighter, and has a much simpler expression. Morally it's because they needed to suppose not only that $\|x\|_{L^\infty}$ and $\|\tilde{x}\|_{L^\infty} \leq \rho$, but even $\|x\| + \|\tilde{x} - x\| \leq \rho$.

More precisely, in that paper they announce that S is L-lipschitz over X with $L = \frac{2f(\rho)}{f^{-1}(3f(\rho))-\rho}$. This is looser than our proposition, since

$$f'(r) \leq \frac{2f(r)}{f^{-1}(3f(r)) - r} \tag{4.3.3}$$

Indeed, f is a power series with non-negative coefficients, so f is convex and f^{-1} is concave over $\mathbb{R}_+$. By first-order characterization of concavity,

$$\forall s,t \geq 0,\ f^{-1}(s) - f^{-1}(t) \leq (f^{-1})'(t)\ (s-t) = \frac{s-t}{f'(f^{-1}(t))} \tag{4.3.4}$$

Evaluating at $s = 3f(r)$ and $t = f(r)$,

$$f^{-1}(3f(r)) - r \leq \frac{2f(r)}{f'(r)} \tag{4.3.5}$$

$$\frac{2f(r)}{f^{-1}(3f(r)) - r} \geq f'(r) \tag{4.3.6}$$

since $f'(r) > 0$ and $3f(r) > f(r) \implies f^{-1}(3f(r)) - r > 0$.

Proof. We start by the case $p=\infty$. Let $x, \tilde{x} \in B_\rho^{(\mathcal{X})} = B_\rho^{(L^\infty)}$. We want to bound

$$\|\Phi_k[x] - \Phi_k[\tilde{x}]\|_{\mathcal{Y}} \leq 0 + \sup_t |\Phi_{k_1}[x](t) - \Phi_{k_1}[\tilde{x}](t)| + ... + \sup_t |\Phi_{k_N}[x](t) - \Phi_{k_N}[\tilde{x}](t)| \tag{4.3.7}$$

Now for all $n \in \mathbb{N}$ and $t \in \mathbb{R}$,

$$|\Phi_{k_n}[x] - \Phi_{k_n}[\tilde{x}]|\,(t) = \left| \int_{\mathbb{R}^n} d\boldsymbol{\tau}\ k_n(t\mathbf{1}_n - \boldsymbol{\tau})\,(x(\tau_1)...x(\tau_n) - \tilde{x}(\tau_1)...\tilde{x}(\tau_n)) \right| \tag{4.3.8}$$

$$\leq \|k_n\|_{L^1} \sup_{\tau_1,\dots,\tau_n} |x(\tau_1)...x(\tau_n) - \tilde{x}(\tau_1)...\tilde{x}(\tau_n)| \tag{4.3.9}$$

By the lemma below applied to $x_i = x(\tau_i)$ and $\tilde{x}_i = \tilde{x}(\tau_i)$,

$$|\Phi_{k_n}[x] - \Phi_{k_n}[\tilde{x}]|\,(t) \leq \|k_n\|_{L^1} \sup_{\tau_1,\dots,\tau_n} \sum_{i=1}^{n} |x(\tau_i) - \tilde{x}(\tau_i)|\,\rho^{n-1} \tag{4.3.10}$$

$$= \|k_n\|_{L^1}\,\rho^{n-1} \sum_{i=1}^{n} \sup_{\tau_i} |x(\tau_i) - \tilde{x}(\tau_i)| \tag{4.3.11}$$

$$= \|k_n\|_{L^1}\,\rho^{n-1} n\,\|x - \tilde{x}\|_{L^\infty} \tag{4.3.12}$$

In summary,

$$\|\Phi_k[x] - \Phi_k[\tilde{x}]\|_{\mathcal{Y}} \leq \sum_{n\in\mathbb{N}} \|k_n\|_{L^1}\,\rho^{n-1} n\ \ \|x - \tilde{x}\|_{\mathcal{X}} \tag{4.3.13}$$

$$= f_k'(\rho)\ \ \|x - \tilde{x}\|_{\mathcal{X}} \tag{4.3.14}$$

i.e Φ_k is $f_k'(\rho)$-lipschitz over $B_\rho^{(\mathcal{X})}$, as announced.

Now let arbitrary $1 \leq p \leq \infty$. Let $x, \tilde{x} \in B_\rho^{(\mathcal{X})} = B_\rho^{(L^p)}$. We follow the same reasoning, using Hoelder's inequality: for all $n \in \mathbb{N}$ and $t \in \mathbb{R}$,

$$\left|\Phi_{k_n}[x] - \Phi_{k_n}[\tilde{x}]\right|(t) = \left|\int_{\mathbb{R}^n} d\boldsymbol{\tau}\ k_n(t\mathbf{1}_n - \boldsymbol{\tau})\left(x(\tau_1)...x(\tau_n) - \tilde{x}(\tau_1)...\tilde{x}(\tau_n)\right)\right| \tag{4.3.15}$$

$$\leq \|k_n\|_{L^q}\left[\int_{\mathbb{R}^n} d\boldsymbol{\tau}\ \left|x(\tau_1)...x(\tau_n) - \tilde{x}(\tau_1)...\tilde{x}(\tau_n)\right|^p\right]^{1/p} \tag{4.3.16}$$

By the second part of the lemma below applied to $x_i = x(\tau_i)$ and $\tilde{x}_i = \tilde{x}(\tau_i)$, and using that $\left(\sum_{i=1}^n |\delta_i|\right)^p = \langle \mathbf{1}_n, \boldsymbol{\delta}\rangle^p \leq n^{p-1}\sum_{i=1}^n |\delta_i|^p$ (by Hoelder's inequality),

$$\int_{\mathbb{R}^n} d\boldsymbol{\tau}\ \left|x(\tau_1)...x(\tau_n) - \tilde{x}(\tau_1)...\tilde{x}(\tau_n)\right|^p \tag{4.3.17}$$

$$\leq \int_{\mathbb{R}^n} d\boldsymbol{\tau}\ n^{p-1}\sum_{i=1}^n \left|x(\tau_i) - \tilde{x}(\tau_i)\right|^p \prod_{j\neq i}\max(|x(\tau_j)|, |\tilde{x}(\tau_j)|)^p \tag{4.3.18}$$

$$= n^{p-1}\sum_{i=1}^n \int_{\mathbb{R}} d\tau_i\ \left|x(\tau_i) - \tilde{x}(\tau_i)\right|^p \prod_{j\neq i}\int_{\mathbb{R}} d\tau_j\ \max(|x(\tau_j)|, |\tilde{x}(\tau_j)|)^p \tag{4.3.19}$$

$$= n^{p-1}\ n\ \|x - \tilde{x}\|_{L^p}^p \left(\int_{\mathbb{R}} d\tau\ \max(|x(\tau)|, |\tilde{x}(\tau)|)^p\right)^{n-1} \tag{4.3.20}$$

So in summary, we get the bound:

$$\|\Phi_k[x] - \Phi_k[\tilde{x}]\|_{\mathcal{Y}} \leq \sum_{n\in\mathbb{N}} \|k_n\|_{L^q}\, n\, \|x - \tilde{x}\|_{L^p} \|\max(|x|, |\tilde{x}|)\|_{L^p}^{n-1} \tag{4.3.21}$$

$$= f_k'\left(\|\max(|x|, |\tilde{x}|)\|_{L^p}\right)\ \ \|x - \tilde{x}\|_{L^p} \tag{4.3.22}$$

The annoying quantity can be bounded simply by

$$\|\max(|x|, |\tilde{x}|)\|_{L^p} \leq \||x| + |\tilde{x}|\|_{L^p} \leq \|x\|_{L^p} + \|\tilde{x}\|_{L^p} \leq 2\rho \tag{4.3.23}$$

Since f_k is a power series with non-negative coefficients, f_k' is increasing, and we can conclude that Φ_k is $f_k'(2\rho)$-lipschitz over $B_\rho^{(\mathcal{X})}$, as announced. □

Lemma 4.10 (Modulus of continuity of product of variables)**.** For any $M \in \mathbb{R}_+^n$, and for all $x, \tilde{x} \in \mathbb{R}^n$ such that $|x_j|, |\tilde{x}_j| \leq M_j$ for each j,

$$|x_1...x_n - \tilde{x}_1...\tilde{x}_n| \leq \sum_{i=1}^n |x_i - \tilde{x}_i| \cdot \left(\prod_{j\neq i} M_j\right) \tag{4.3.24}$$

More simply put, for any $x, \tilde{x} \in \mathbb{R}^n$, $|x_1...x_n - \tilde{x}_1...\tilde{x}_n| \leq \sum_{i=1}^n |x_i - \tilde{x}_i| \cdot \left(\prod_{j\neq i}\max(|x_j|, |\tilde{x}_j|)\right)$.

Proof. Let $f : \begin{bmatrix} [\pm\boldsymbol{M}] \to \mathbb{R} \\ x \mapsto x_1...x_n \end{bmatrix}$ where $[\pm\boldsymbol{M}] = [-M_1, M_1] \times ... \times [-M_n, M_n]$. f is infinitely differentiable and

$$\nabla f(x) = \left(\prod_{j\neq 1} x_j\ ,\ ...\ ,\ \prod_{j\neq n} x_j\right)^T \tag{4.3.25}$$

By mean-value theorem (i.e first order Taylor-Lagrange), there exists $\xi \in [x, \tilde{x}] \subset [\pm \boldsymbol{M}]$ such that

$$x_1 ... x_n - \tilde{x}_1 ... \tilde{x}_n = f(x) - f(\tilde{x}) = \nabla f(\xi) \cdot (x - \tilde{x}) \tag{4.3.26}$$

$$|x_1 ... x_n - \tilde{x}_1 ... \tilde{x}_n| \leq \sum_{i=1}^{n} |\nabla f(\xi)_i| \, |x_i - \tilde{x}_i| \tag{4.3.27}$$

$$= \sum_{i=1}^{n} |x_i - \tilde{x}_i| \prod_{j \neq i} |\xi_j| \tag{4.3.28}$$

$$\leq \sum_{i=1}^{n} |x_i - \tilde{x}_i| \left(\prod_{j \neq i} M_j \right) \tag{4.3.29}$$

The simpler reformulation follows immediately from the first part by setting $M_j = \max(|x_j|, |\tilde{x}_j|)$. □

4.4 Time-invariance, L^∞ norm on the output: reduction to scalar-valued (the present)

In this thesis we only consider time-invariant Volterra series, i.e of the form $\int_{\mathbb{R}^n} d\boldsymbol{\tau} \; k_n(t\mathbf{1}_n - \boldsymbol{\tau}) x^{\times n}(\boldsymbol{\tau})$ instead of $\int_{U^n} d\boldsymbol{u} \; k_n(v, \boldsymbol{u}) x^{\times n}(\boldsymbol{u})$, as mentioned in the first section. We already formalized time-invariance for linear systems at the end of section 3.3, and the definition for nonlinear systems is basically identical.

Definition 4.6. Let $\mathcal{X}$ and $\mathcal{Y}$ function spaces over the same domain $\mathcal{T} = \mathbb{R}$ (or $\mathbb{R}^d$).
A system $S : \mathcal{X} \to \mathcal{Y}$ is called *time-invariant* if for all $x \in \mathcal{X}$,

$$\forall \tau \in \mathcal{T}, \; S\left[x(\cdot - \tau)\right] = S[x](\cdot - \tau) \tag{4.4.1}$$

In other words, if S commutes with the τ-delay operator U_τ for each $\tau \in \mathcal{T}$:

$$U_\tau : x(t) \mapsto x(t - \tau) \tag{4.4.2}$$

$$S \circ U_\tau = U_\tau \circ S \tag{4.4.3}$$

(The operator U_τ being seen both as $\mathcal{X} \to \mathcal{X}$ and as $\mathcal{Y} \to \mathcal{Y}$.)

Definition 4.7. A set X of signals $x : \mathcal{T} \to \mathbb{R}$ is *time-invariant* if $U_\tau X = X$ for all $\tau \in \mathcal{T}$, i.e

$$\forall x \in X, \forall \tau \in \mathcal{T}, \; U_\tau x \in X \tag{4.4.4}$$

When the systems and the set of inputs are time-invariant, and the norm on the output is the L^∞ norm, the situation reduces to considering scalar-valued systems [BC85], as we now explain.

Proposition 4.11. Suppose that:

- X is a time-invariant set of inputs.
- $\mathbb{S}$ consists of time-invariant systems from X to $\mathcal{Y}$.
- The output space $\mathcal{Y}$ is equipped with the sup norm: $\|\cdot\|_{\mathcal{Y}} = \|\cdot\|_{L^\infty}$

Then the present-evaluation

$$\begin{bmatrix} (\mathbb{S}, \|\cdot\|_{\infty X}) \to (\mathbb{F}, \|\cdot\|_{\infty X}) \\ S \mapsto F_S = S[\cdot](0) \end{bmatrix} \tag{4.4.5}$$

is an isometric isomorphism, whose inverse function is given by $F \mapsto S_F$ such that

$$\forall x,\ S_F[x](t) = F[U_{-t}x] \tag{4.4.6}$$

Here $\mathbb{F} = \{S[\cdot](0);\ S \in \mathbb{S}\}$ is a set of scalar-valued systems from X to $\mathbb{R}$, equipped with the norm $\|F\|_{\infty X} = \sup_X |F[x]|$.

Proof.
- $S \mapsto F_S$ is surjective by definition.
- To derive the formula for the inverse function, let $S \in \mathbb{S}$ and $F \in \mathbb{F}$ such that for all $x \in X$, $S[x](0) = F[x]$. Now

$$\forall x, t,\ S[U_{-t}x](0) = (U_{-t}S[x])\,(0) = S[x](0+t) \tag{4.4.7}$$

  So, as announced, $\forall x, t,\ S[x](t) = F[U_{-t}x]$.

  This also proves that $S \mapsto F_S$ is injective.
- To show isometry: let $S \in \mathbb{S}$. Since the norm on the output is the L^∞ norm,

$$\begin{aligned}
\|S\|_{\infty X} = \sup_{x\in X} \|S[x]\|_{L^\infty} &= \sup_{x\in X}\sup_t |S[x](t)| && (4.4.8)\\
&= \sup_{x\in X}\sup_t |U_{-t}S[x](0)| && (4.4.9)\\
&= \sup_{x\in X}\sup_t |S[U_{-t}x](0)| && (4.4.10)\\
&= \sup_{x'\in X} |S[x'](0)| && (4.4.11)\\
&= \sup_{x'\in X} |F_S[x']| \ = \|F_S\|_{\infty X} && (4.4.12)
\end{aligned}$$

  by time-invariance of S, time-invariance of X, and definition of F_S.

□

Remark 4.7. "Time-invariant Volterra series" are clearly time-invariant systems, since $\Phi_{k_n}[x(\cdot-\tau)](t) = \int_{\mathbb{R}} d\boldsymbol{s}\ k_n(\boldsymbol{s})x(t-\tau-s_1)...x(t-\tau-s_n) = \Phi_{k_n}[x](t-\tau)$.

4.5 Approximation properties

Volterra series are a useful model of nonlinear systems because they can approximate a large class of systems arbitrarily well. Making this statement precise is not obvious. Almost all past results in this direction ultimately rely on the reduction to scalar-valued systems, explained in the previous section. Indeed, doing so enables using well-known sufficient density conditions for scalar-valued functionals, such as the Stone-Weierstrass theorem [BC85] [SX97], [3] or Frechet differentiation and Schwartz kernel theorem [PP77]. Here we focus on the former path (by Stone-Weierstrass theorem), and leave the latter (Frechet differentiation) for future investigation.

We emphasize that this section's results, based on Stone-Weierstrass theorem, are existential in essence and do not say how to construct an ε-approximation, nor control the order N of the approximating Volterra series. So it seems difficult to use this approach to obtain approximation schemes with guarantees, contrary to the Frechet differentiation approach, for which possible prior knowledge on the smoothness of the systems to approximate can be leveraged.

[3] The paper [SX97] claims to be more general and to capture signal-valued systems, but it simply hides the reduction to scalar-valued within its proofs. It also claims to circumvent the use of Stone-Weierstrass theorem but it simply hides it in its appendix.

4.5.1 Scalar-valued case and time-invariant case

Proposition 4.12. Let $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ the Banach signal space specified below. Let $X \subset \mathcal{X}$ compact.

Let F a continuous functional from $(X, \|\cdot\|_{\mathcal{X}})$ to $\mathbb{R}$.

Then for all $\varepsilon > 0$, there exists a finite-order Volterra series Φ_k such that, denoting the associated functional $F_k = \Phi_k[\cdot](0)$,

$$\|F - F_k\|_{\infty X} \leq \varepsilon \tag{4.5.1}$$

- Recall the definition of $(C_{b,w}(\mathbb{R}), \|\cdot\|_w)$ from section 2.2.

  If $\mathcal{X} = C_{b,w}(\mathbb{R})$, then the Volterra kernels live in $\mathcal{K}_n = w^{\times n}(\boldsymbol{\tau})L^1(\mathbb{R}^n) \subset L^1(\mathbb{R}^n)$.
- If $\mathcal{X} = L^p(\mathbb{R})$ for $1 \leq p < \infty$, then the Volterra kernels live in $\mathcal{K}_n = L^q(\mathbb{R}^n)$ where $1/p + 1/q = 1$.

The discussion of time-invariance in the previous section immediately implies the following corollary.

Corollary 4.13. Let $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ the Banach signal space specified in the proposition. Let $X \subset \mathcal{X}$ compact and time-invariant.

Let S a time-invariant system that is continuous from $(X, \|\cdot\|_{\mathcal{X}})$ to $(\mathcal{Y}, \|\cdot\|_{L^\infty})$. Equivalently, its present-evaluation $F = S[\cdot](0)$ is a continuous functional from $(X, \|\cdot\|_{\mathcal{X}})$ to $\mathbb{R}$.

Then for all $\varepsilon > 0$, there exists a finite-order Volterra series Φ_k such that

$$\|S - \Phi_k\|_{\infty X} \leq \varepsilon \tag{4.5.2}$$

and the Volterra kernels live in the spaces specified in the proposition.

Proof of the proposition. Note that present-evaluation functionals of finite-order Volterra series are F_k of the form

$$F_k[x] = \sum_{n=0}^{N} \int_{\mathbb{R}^n} d\boldsymbol{\tau}\ k_n(-\boldsymbol{\tau}) x(\tau_1) \ldots x(\tau_n) \tag{4.5.3}$$

for some $k \in \mathcal{K}_{\leq N}$.

To prove the proposition, we want to show that there exist $N \in \mathbb{N}$ and $k \in \mathcal{K}_{\leq N}$ such that $\|F - F_k\|_{\infty X} \leq \varepsilon$. That fact is a direct consequence of the following lemmas. □

Lemma 4.14 (Stone-Weierstrass theorem)**.** Suppose $(X, \|\cdot\|_{\mathcal{X}})$ is a compact metric space.

Let $\mathbb{G}$ a set of continuous functionals $G : (X, \|\cdot\|_{\mathcal{X}}) \to \mathbb{R}$ that separates points, i.e

$$\forall x_1 \neq x_2 \in X, \exists G \in \mathbb{G};\ G(x_1) \neq G(x_2) \tag{4.5.4}$$

Then the subalgebra of $C(X; \mathbb{R})$ generated by $\mathbb{G}$ is dense in $C(X; \mathbb{R})$ (for the sup norm $\|\cdot\|_{\infty X}$):

$$\forall F \in C(X; \mathbb{R}), \forall \varepsilon > 0, \exists P \text{ in that subalgebra };\ \ \|F - P\|_{\infty X} \leq \varepsilon \tag{4.5.5}$$

By "the subalgebra generated by $\mathbb{G}$", we mean the set of *polynomial functionals* $P : X \to \mathbb{R}$ of the form [BC85]

$$P(x) = p\left(G_1(x), \ldots, G_M(x)\right) \tag{4.5.6}$$

$$= \sum_{n=0}^{N} \sum_{i_1, \ldots, i_n \leq M} \alpha_{i_1 \ldots i_n} G_{i_1}(x) \ldots G_{i_n}(x) \tag{4.5.7}$$

for some $N, M \in \mathbb{N}$, $G_1, \ldots, G_M \in \mathbb{G}$, and p a M-variate polynomial of degree N with real coefficients.

Lemma 4.15 ([BC85, theorem 1])**.** If $\mathcal{X} = C_{b,w}(\mathbb{R})$, then

$$\mathbb{G} = \left\{ \begin{bmatrix} (\mathcal{X}, \|\cdot\|_w) \to \mathbb{R} \\ x \mapsto \displaystyle\int_{\mathbb{R}} dt\ g(t) x(t) \end{bmatrix} ;\ g \in wL^1(\mathbb{R}) \right\} \tag{4.5.8}$$

consists of continuous functionals, and $\mathbb{G}$ separates points (in $\mathcal{X}$ and a fortiori also in any subset X).

Proof. Let us check that $\mathbb{G}$ consists of continuous functionals. Let $g \in wL^1(\mathbb{R})$, i.e such that $\left\|w^{-1}g\right\|_{L^1} < \infty$, and $x \in C_{b,w}(\mathbb{R})$, i.e such that $wx \in C_b(\mathbb{R})$. Then

$$\left|\int_{\mathbb{R}} dt\ g(t)x(t)\right| \leq \int_{\mathbb{R}} dt\ w^{-1}(t)\left|g(t)\right|\ w(t)\left|x(t)\right| \leq \left\|w^{-1}g\right\|_{L^1}\left\|x\right\|_w \tag{4.5.9}$$

So the linear functional $x \mapsto \int_{\mathbb{R}} gx$ is indeed continuous.

Let us check that $\mathbb{G}$ separates points: let $x_1, x_2 \in C_{b,w}(\mathbb{R})$. Define $g_0(t) = (x_1 - x_2)(t)w(t)^2 e^{-|t|}$.

- $x \mapsto \int_{\mathbb{R}} g_0 x$ belongs to $\mathbb{G}$:

$$\left|w^{-1}(t)g_0(t)\right| = \left|x_1 - x_2\right|(t)w(t)e^{-|t|} \leq \left\|x_1 - x_2\right\|_w e^{-|t|} \tag{4.5.10}$$

  so that $g_0 \in wL^1(\mathbb{R})$, as required.

- $x \mapsto \int_{\mathbb{R}} g_0 x$ separates x_1 and x_2:

$$\left(\int_{\mathbb{R}} g_0 x_1\right) - \left(\int_{\mathbb{R}} g_0 x_2\right) = \int_{\mathbb{R}} (wx_1 - wx_2)(t)^2 e^{-|t|} > 0 \tag{4.5.11}$$

  since wx_1 and wx_2 are continuous and are not equal.

□

Lemma 4.16. If $\mathcal{X} = L^p(\mathbb{R})$ for $1 \leq p < \infty$, and q is such that $1/p + 1/q = 1$, then

$$\mathbb{G} = \left\{ \begin{bmatrix} (\mathcal{X}, \|\cdot\|_{L^p}) \to \mathbb{R} & \\ & x \mapsto \int_{\mathbb{R}} dt\ g(t)x(t) \end{bmatrix} ;\ g \in L^q(\mathbb{R}) \right\} \tag{4.5.12}$$

consists of continuous functionals, and $\mathbb{G}$ separates points (in $\mathcal{X}$ and a fortiori also in any subset X).

Proof. $\mathbb{G}$ consists of continuous functionals because $L^q(\mathbb{R})$ is the dual space of $L^p(\mathbb{R})$, see section D.1.

$\mathbb{G}$ separates points because for all $x_1 \neq x_2 \in L^p(\mathbb{R})$, Hahn-Banach theorem guarantees existence of a norming functional $g_0 \in L^q(\mathbb{R})$ of $x_1 - x_2$. □

4.5.2 Banach-valued time-varying case

Using the result for the case $\mathcal{Y} = \mathbb{R}$ shown above, we can easily deduce an extension to the case of arbitrary $\mathcal{Y}$.

The idea is the same as in the last part of [Ist77]: for any $S \in C(X; \mathcal{Y})$, choose a δ-covering of the compact X where $\omega_X(S; \delta) \leq \varepsilon$, let φ_i (scalar-valued) an adapted continuous partition of unity, approximate S as a linear combination of the φ_i, and approximate the φ_i using polynomial functionals.

Proposition 4.17. Let $\mathcal{X}(U)$ and $\mathcal{Y}(V)$ Banach signal spaces over domains U and V respectively, e.g $U = V = \mathbb{R}$, with $\mathcal{X}(U)$ specified below. Let $X \subset \mathcal{X}$ compact.

Let S a system that is continuous from $(X, \|\cdot\|_{\mathcal{X}})$ to $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$.

Then for all $\varepsilon > 0$, there exist "time-varying Volterra kernels" $k_n : V \times U^n \to \mathbb{R}$ $(0 \leq n \leq N)$, such that

$$\left\|S - \widetilde{\Phi}_k\right\|_{\infty X} \leq \varepsilon \tag{4.5.13}$$

where

$$\widetilde{\Phi}_k[x] = \sum_{n=0}^{N} \widetilde{\Phi}_{k_n}[x] \qquad \widetilde{\Phi}_{k_n}[x](v) = \int_{U^n} d\boldsymbol{u}\ k_n(v, \boldsymbol{u})x(u_1)...x(u_n) \tag{4.5.14}$$

Here, $\forall \boldsymbol{u}, k_n(\cdot, \boldsymbol{u}) \in \mathcal{Y}$ and $\forall v, k_n(v, \cdot) \in \mathcal{K}_n$ with the $\mathcal{K}_n$ as in the previous subsection i.e

- If $\mathcal{X} = C_{b,w}(\mathbb{R})$, then the Volterra kernels live in $\mathcal{K}_n = w^{\times n}(\boldsymbol{u})L^1(\mathbb{R}^n) \subset L^1(\mathbb{R}^n)$.
- If $\mathcal{X} = L^p(\mathbb{R})$ for $1 \leq p < \infty$, then the Volterra kernels live in $\mathcal{K}_n = L^q(\mathbb{R}^n)$ where $1/p + 1/q = 1$.

Proof. Let $S \in C(X; \mathcal{Y})$ and $\varepsilon > 0$. Since X is compact, S is uniformly continuous, so let δ such that $\omega_X(S; \delta) \leq \varepsilon$, and let $(x_1, ..., x_m)$ a δ-covering of X. Let $(\varphi_i)_i$ an adapted continuous partition of unity, i.e such that

- $\varphi_i \in C(X; \mathbb{R})$
- $0 \leq \varphi_i(x) \leq 1$
- $\sum_i \varphi_i(x) = 1$
- $\varphi_i(x) \neq 0 \implies \|x - x_i\|_{\mathcal{X}} \leq \delta$

Let $M = \sup_{x \in X} \|S[x]\|_{\mathcal{Y}}$. For each i, since $\varphi_i \in C(X; \mathbb{R})$, then by the previous subsection there exists $k_i = (k_{in})_n \in \mathcal{K}_{\leq N_i}$ such that

$$\|\varphi_i - F_{k_i}\|_{\infty X} \leq \frac{\varepsilon}{mM} \tag{4.5.15}$$

Denote $y_i = S[x_i] \in \mathcal{Y}(V)$ for each i. Define a system $\widehat{S}$ by:

$$\begin{aligned}
\widehat{S}[x](v) &= \sum_{i=1}^{m} y_i(v) F_{k_i}[x] && (4.5.16)\\
&= \sum_{i=1}^{m} y_i(v) \sum_{n=0}^{N_i} \int_{U^n} d\boldsymbol{u}\ k_{in}(-\boldsymbol{u}) x^{\times n}(\boldsymbol{u}) && (4.5.17)\\
&= \sum_{n=0}^{N} \int_{U^n} d\boldsymbol{u}\ \underbrace{\sum_{i=1}^{m} y_i(v) k_{in}(-\boldsymbol{u})}\, x^{\times n}(\boldsymbol{u}) && (4.5.18)\\
&= \sum_{n=0}^{N} \int_{U^n} d\boldsymbol{u}\ k_n(v, \boldsymbol{u}) x^{\times n}(\boldsymbol{u}) && (4.5.19)
\end{aligned}$$

where $N = \max_i N_i$.

Then, $\widehat{S}$ is of the announced form $\widetilde{\Phi}_k$, with "time-varying Volterra kernels" in the announced spaces.

Finally, one can check that $\left\|S - \widehat{S}\right\|_{\infty X} \leq 2\varepsilon$. Indeed,

- $\|S - \sum_{i=1}^{m} y_i \varphi_i\|_{\infty X} \leq \varepsilon$, by the same reasoning as in section 2.2.
- By definition,

$$\left\|\sum_{i=1}^{m} y_i \varphi_i - \sum_{i=1}^{m} y_i F_{k_i}\right\|_{\infty X} \leq \sum_{i=1}^{m} \|y_i\|_{\mathcal{Y}} \|\varphi_i - F_{k_i}\|_{\infty X} \leq \varepsilon \tag{4.5.20}$$

□

Interestingly, in this section we essentially said that, morally, any system can be approximated by a continuous linear block followed by a no-memory nonlinear block, and that such a model can be put in the form of a Volterra series. In the next section, by contrast, we will consider Volterra series as consisting of a highly nonlinear block with memory (feature map), followed by a linear block in very high dimension (linear layer).

4.6 Relation to polynomial Reproducing Kernel Banach Spaces (RKBS)

As already noticed in [FS06], Volterra and Wiener series are morally nothing else than the elements of a polynomial reproducing kernel Hilbert space (RKHS). That article focused on the simple setting of scalar-valued systems with discrete-time finite-horizon inputs $(x_t)_{1\leq t\leq m}$. [4] In this section we will clarify in what sense Volterra series theory can be viewed in a RKHS-like framework, going beyond that simple setting. We assume the reader is already familiar with RKHS's.

The setting As a natural simplification, we will consider scalar-valued Volterra functionals

$$F_k : \begin{bmatrix} X \to \mathbb{R} \\ x(t) \mapsto \sum_n F_{k_n}[x] = \sum_n \int_{\mathbb{R}^n} d\boldsymbol{\tau}\ k_n(-\boldsymbol{\tau}) x^{\times n}(\boldsymbol{\tau}) \end{bmatrix} \tag{4.6.1}$$

instead of Volterra series systems Φ_k. As argued earlier, this simplification is quite benign when studying time-invariant systems. Here X and the k_n's are defined as in the rest of this chapter; to be clear,

- X is a subset of $\mathcal{X} = L^p(\mathbb{R})$;
- In the finite-order case, $k = (k_0, ..., k_N) \in \mathcal{K}_{\leq N} = \mathcal{K}_0 \times ... \times \mathcal{K}_N$, where $\mathcal{K}_n = L^q(\mathbb{R}^n)$;
- In the infinite-order case, $k = (k_0, k_1, ...) \in \mathcal{K}_\infty(R)$, where $\mathcal{K}_\infty(R)$ is the subset of $\mathcal{K}_0 \times \mathcal{K}_1 \times ...$ such that the gain bound function $f_k(z)$ has radius of convergence $\geq R$;
- $1/p + 1/q = 1$, and for simplicity we assume $1 < p < \infty$ throughout this section.

Also to avoid confusion, we emphasize that here x denotes an input variable belonging to some domain X, and that the mappings that are interpreted in the RKHS framework are the functionals (systems) $F_k : X \to \mathbb{R}$.

The idea First consider the monomial case, with a fixed order n. A Volterra functional F_{k_n} can be viewed as the composition of a highly nonlinear block with memory (feature map ϕ), followed by a linear block in very high dimension (linear layer θ): in very loose notation,

$$F_{k_n}[x] = \int_{\mathbb{R}^n} d\boldsymbol{\tau}\ k_n(-\boldsymbol{\tau}) x(\tau_1)...x(\tau_n) = \sum_{\boldsymbol{\tau}} \theta_{\boldsymbol{\tau}} \phi_{\boldsymbol{\tau}}(x) \tag{4.6.2}$$

where we identify $\theta_{\boldsymbol{\tau}} = k_n(-\boldsymbol{\tau})$ and $\phi_{\boldsymbol{\tau}}(x) = x(\tau_1)...x(\tau_n)$. In this form, one immediately recognizes that the mapping F_{k_n} is morally an element of the RKHS with kernel

$$K(x, \tilde{x}) = \sum_{\boldsymbol{\tau}} \phi_{\boldsymbol{\tau}}(x)\phi_{\boldsymbol{\tau}}(\tilde{x}) = \int_{\mathbb{R}^n} d\boldsymbol{\tau}\ x^{\times n}(\boldsymbol{\tau})\tilde{x}^{\times n}(\boldsymbol{\tau}) = \left(\int_{\mathbb{R}} dt\ x(t)\tilde{x}(t)\right)^n \tag{4.6.3}$$

associated to the feature map $\phi(x) = (\phi_{\boldsymbol{\tau}}(x))_{\boldsymbol{\tau}} = (x^{\times n}(\boldsymbol{\tau}))_{\boldsymbol{\tau}\in\mathbb{R}^n}$ – which has very high dimension to say the least, since its dimension is the cardinality of $\mathbb{R}^n$.

Even more informally, here is another observation that also leads to interpreting Volterra series in the context of RKHS: recall that a Hilbert function space $\mathcal{H}$ over Ω is shown to be a RKHS if and only if the point evaluation functionals $\delta_\omega : f \mapsto f(\omega)$ are bounded for each ω; that is,

$$\forall \omega \in \Omega, \forall f \in \mathcal{H},\ \ |\delta_\omega f| = |f(\omega)| \leq |||\delta_\omega|||_{\mathcal{H}\to\mathbb{R}} \left\|f\right\|_{\mathcal{H}} \quad \text{where} \quad |||\delta_\omega|||_{\mathcal{H}\to\mathbb{R}} < \infty \tag{4.6.4}$$

Now the point evaluation functionals over Volterra series are precisely bounded by the relations $\left\|\Phi_{k_n}[x]\right\|_{\mathcal{Y}} \leq \left\|k_n\right\|_{\mathcal{K}_n} \left\|x\right\|_{\mathcal{X}}^n$ shown in the first section, if we identify $\left\|k_n\right\|_{\mathcal{K}_n}$ to $\left\|f\right\|_{\mathcal{H}}$.

[4] Our earlier discussion of time-invariance and present evaluation functionals shows that their simple setting also covers discrete-time finite-memory time-invariant systems.

4.6.1 Reproducing kernel Banach spaces

Let us formalize the idea of the previous paragraph. Our informal discussion has the crucial problem that the feature vector $\phi(x) = (x^{\times n}(\boldsymbol{\tau}))_{\boldsymbol{\tau}}$ does not live in a Hilbert space; equivalently, the problem is that the expression $\int_{\mathbb{R}} dt\ x(t)\tilde{x}(t)$ is ill-defined when both x and $\tilde{x}$ are in $\mathcal{X} = L^p(\mathbb{R})$. However since the expression is well-defined when $x \in L^p(\mathbb{R})$ and $\tilde{x} \in L^q(\mathbb{R})$, we can fix this issue by simply considering *reproducing kernel Banach spaces* instead.

Reproducing kernel Banach spaces (RKBS) have started to be studied surprisingly recently. In fact several competing definitions and frameworks have been proposed. In this section we will follow the formalism of [LZZ19], which claims to unify previous existing frameworks. For more background on the subject, we refer to the references therein, in particular [XY17], and to recent work by the Xu Yuesheng and Ye Qi clique, as well as by the Zhang Haizhang and Zhang Jun clique; let us also mention [Uns20] which looks into computational aspects.

Definition 4.8 (Reproducing kernel, RKBS [LZZ19, definition 1.2])**.** A *pair of RKBS* is a tuple $(\mathcal{B}_1, \mathcal{B}_2, \langle\cdot,\cdot\rangle_{\mathcal{B}_1\times\mathcal{B}_2})$ where

- $\mathcal{B}_1$ is a Banach space of (real-valued) functions on a set Ω_1, $\mathcal{B}_2$ is a Banach space of (real-valued) functions on a set Ω_2;
- $\langle\cdot,\cdot\rangle_{\mathcal{B}_1\times\mathcal{B}_2}$ is a continuous bilinear form on $\mathcal{B}_1 \times \mathcal{B}_2$, i.e $\langle f, g\rangle_{\mathcal{B}_1\times\mathcal{B}_2} \leq C \|f\|_{\mathcal{B}_1} \|g\|_{\mathcal{B}_2}$ for all $f \in \mathcal{B}_1, g \in \mathcal{B}_2$ and for some constant $C > 0$;
- There exists K a (real-valued) function on $\Omega_1 \times \Omega_2$ such that

$$\forall x \in \Omega_1, K(x,\cdot) \in \mathcal{B}_2 \quad \text{and} \quad \forall x \in \Omega_1, \forall f \in \mathcal{B}_1,\ f(x) = \langle f, K(x,\cdot)\rangle_{\mathcal{B}_1\times\mathcal{B}_2} \tag{4.6.5}$$

$$\forall y \in \Omega_2, K(\cdot,y) \in \mathcal{B}_1 \quad \text{and} \quad \forall y \in \Omega_2, \forall g \in \mathcal{B}_2,\ g(y) = \langle K(\cdot,y), g\rangle_{\mathcal{B}_1\times\mathcal{B}_2} \tag{4.6.6}$$

  One can check that K is then unique given $(\mathcal{B}_1, \mathcal{B}_2, \langle\cdot,\cdot\rangle_{\mathcal{B}_1\times\mathcal{B}_2})$, and is called the *reproducing kernel* of the pair of RKBS.

Note that the reproducing property of K ensures that point evaluation functionals δ_x resp. δ_y are bounded over $\mathcal{B}_1$ resp. $\mathcal{B}_2$ (with operator norm $\leq C \|K(x,\cdot)\|_{\mathcal{B}_2}$ resp. $\leq C \|K(\cdot,y)\|_{\mathcal{B}_1}$).

Note that one can alternatively define RKBS as any Banach space with bounded evaluation functionals, by analogy with the Hilbert case. By contrast here we explicitly defined RKBS as coming in pairs. Given a Banach $\mathcal{B}_1$ with bounded evaluation functionals, specifying $\mathcal{B}_2$ (and $\langle\cdot,\cdot\rangle_{\mathcal{B}_1\times\mathcal{B}_2}$) seems unavoidable to get a well-founded framework, since $\mathcal{B}_1 = C([0,1])$ for example is shown to admit several distinct reproducing kernels [LZZ19, example 3.7].

Proposition 4.18 (RKBS from feature maps [LZZ19, theorem 2.1])**.** Let $\mathcal{W}_1$, $\mathcal{W}_2$ two Banach spaces, $\langle\cdot,\cdot\rangle_{\mathcal{W}_1\times\mathcal{W}_2}$ a continuous bilinear form on $\mathcal{W}_1 \times \mathcal{W}_2$, and two mappings $\phi_1 : \Omega_1 \to \mathcal{W}_1$, $\phi_2 : \Omega_2 \to \mathcal{W}_2$. Suppose that the linear spans of $\phi_1(\Omega_1)$ and of $\phi_2(\Omega_2)$ are dense in the sense that

$$\left\{v \in \mathcal{W}_2;\ \forall a \in \phi_1(\Omega_1), \langle a, v\rangle_{\mathcal{W}_1\times\mathcal{W}_2} = 0\right\} = \{0\} \tag{4.6.7}$$

$$\left\{u \in \mathcal{W}_1;\ \forall b \in \phi_2(\Omega_2), \langle u, b\rangle_{\mathcal{W}_1\times\mathcal{W}_2} = 0\right\} = \{0\} \tag{4.6.8}$$

Then

$$\mathcal{B}_1 := \left\{f_v = \langle\phi_1(\cdot), v\rangle_{\mathcal{W}_1\times\mathcal{W}_2}\ ;\ v \in \mathcal{W}_2\right\} \quad \text{equipped with} \quad \|f_v\|_{\mathcal{B}_1} := \|v\|_{\mathcal{W}_2} \tag{4.6.9}$$

$$\mathcal{B}_2 := \left\{g_u = \langle u, \phi_2(\cdot)\rangle_{\mathcal{W}_1\times\mathcal{W}_2}\ ;\ u \in \mathcal{W}_1\right\} \quad \text{equipped with} \quad \|g_u\|_{\mathcal{B}_2} := \|u\|_{\mathcal{W}_1} \tag{4.6.10}$$

$$\langle f_v, g_u\rangle_{\mathcal{B}_1\times\mathcal{B}_2} := \langle u, v\rangle_{\mathcal{W}_1\times\mathcal{W}_2} \tag{4.6.11}$$

defines a pair of RKBS, and the associated kernel is given by $K(x,y) = \langle\phi_1(x), \phi_2(y)\rangle_{\mathcal{W}_1\times\mathcal{W}_2}$.

Note that a converse holds trivially: given any pair of RKBS $(\mathcal{B}_1, \mathcal{B}_2, \langle\cdot,\cdot\rangle_{\mathcal{B}1\times\mathcal{B}_2})$, it can be interpreted as being constructed from a pair of feature maps by simply setting $\mathcal{W}_1 = \mathcal{B}_2$, $\mathcal{W}_2 = \mathcal{B}_1$, $\phi_1(x) = K(x,\cdot)$, $\phi_2(y) = K(\cdot,y)$ and $\langle u, v\rangle_{\mathcal{W}_1\times\mathcal{W}_2} = \langle v, u\rangle_{\mathcal{B}_1\times\mathcal{B}_2}$.

4.6.2 Volterra monomials as RKBS

Let us now check that the space of Volterra functionals over $\mathcal{X} = L^p(\mathbb{R})$ falls into the RKBS framework proposed in [LZZ19] with an explicit pair of feature maps. We start by the case of Volterra monomials of fixed order n.

Proposition 4.19. Let

- $\Omega_1 = X = B_{\rho_1}^{(\mathcal{X})}$ a centered ball of $\mathcal{X} = L^p(\mathbb{R})$, [5] $\mathcal{W}_1 = L^p_{\mathrm{Sym}}(\mathbb{R}^n)$ and $\phi_1 : \begin{bmatrix} \Omega_1 \to \mathcal{W}_1 \\ x(t) \mapsto x^{\times n}(\boldsymbol{t}) \end{bmatrix}$;
- $\Omega_2 = \widetilde{X} = B_{\rho_2}^{(\widetilde{\mathcal{X}})}$ a centered ball of $\widetilde{\mathcal{X}} = L^q(\mathbb{R})$, $\mathcal{W}_2 = L^q_{\mathrm{Sym}}(\mathbb{R}^n)$ and $\phi_2 : \begin{bmatrix} \Omega_2 \to \mathcal{W}_2 \\ \tilde{x}(t) \mapsto \tilde{x}^{\times n}(\boldsymbol{t}) \end{bmatrix}$;
- $\langle \cdot, \cdot \rangle_{\mathcal{W}_1 \times \mathcal{W}_2}$ defined by: (note that, up to isometric isomorphism, it coincides with the duality bracket of $L^p(\mathbb{R}^n)$)

$$\forall x_n \in \mathcal{W}_1 = L^p_{\mathrm{Sym}}(\mathbb{R}^n), \forall \tilde{x}_n \in \mathcal{W}_2 = L^q_{\mathrm{Sym}}(\mathbb{R}^n),\ \ \langle x_n, \tilde{x}_n \rangle_{\mathcal{W}_1 \times \mathcal{W}_2} = \int_{\mathbb{R}^n} d\boldsymbol{\tau}\ \tilde{x}_n(-\boldsymbol{\tau}) x_n(\boldsymbol{\tau}) \tag{4.6.12}$$

Then the bilinear form $\langle \cdot, \cdot \rangle_{\mathcal{W}_1 \times \mathcal{W}_2}$ is continuous, and the linear spans of $\phi_1(\Omega_1)$ and of $\phi_2(\Omega_2)$ are dense in the sense of the previous proposition. So $(\mathcal{B}_1, \mathcal{B}_2, \langle \cdot, \cdot \rangle_{\mathcal{B}_1 \times \mathcal{B}_2})$ defined as in the previous proposition is a pair of RKBS.

Moreover, note that

$$\mathcal{B}_1 = \left\{ \langle \phi_1(\cdot), v \rangle_{\mathcal{W}_1 \times \mathcal{W}_2} ;\ v \in \mathcal{W}_2 \right\} = \left\{ F_{k_n} : \left[x \mapsto \int_{\mathbb{R}^n} d\boldsymbol{\tau}\ k_n(-\boldsymbol{\tau}) x^{\times n}(\boldsymbol{\tau}) \right] ;\ k_n \in L^q_{\mathrm{Sym}}(\mathbb{R}^n) \right\} \tag{4.6.13}$$

So $\mathcal{B}_1$ is just the space of Volterra monomials of order n over signals $x \in L^p(\mathbb{R})$, and symmetrically $\mathcal{B}_2$ is the space of Volterra monomials of order n over signals $\tilde{x} \in L^q(\mathbb{R})$.

Also note that the associated kernel is simply

$$K(x, \tilde{x}) = \langle \phi_1(x), \phi_2(\tilde{x}) \rangle_{\mathcal{W}_1 \times \mathcal{W}_2} = \left(\int_{\mathbb{R}} dt\ \tilde{x}(-t) x(t) \right)^n \tag{4.6.14}$$

Proof. The bilinear form $\langle \cdot, \cdot \rangle_{\mathcal{W}_1 \times \mathcal{W}_2}$ is is clearly continuous.

We want to check that

$$\begin{aligned} & \left\{ v \in \mathcal{W}_2;\ \forall a \in \phi_1(\Omega_1), \langle a, v \rangle_{\mathcal{W}_1 \times \mathcal{W}_2} = 0 \right\} && (4.6.15) \\ &= \left\{ k_n \in L^q_{\mathrm{Sym}}(\mathbb{R}^n);\ \forall x \in X, \langle \phi_1(x), k_n(-\boldsymbol{\tau}) \rangle_{L^p(\mathbb{R}^n)} = 0 \right\} \ = \{0\} && (4.6.16) \\ \text{and}\ \ & \left\{ u \in \mathcal{W}_1;\ \forall b \in \phi_2(\Omega_2), \langle u, b \rangle_{\mathcal{W}_1 \times \mathcal{W}_2} = 0 \right\} && (4.6.17) \\ &= \left\{ \tilde{k}_n \in L^p_{\mathrm{Sym}}(\mathbb{R}^n);\ \forall \tilde{x} \in \widetilde{X}, \left\langle \tilde{k}_n, \phi_2(\tilde{x})(-\boldsymbol{\tau}) \right\rangle_{L^p(\mathbb{R}^n)} = 0 \right\} \ = \{0\} && (4.6.18) \end{aligned}$$

Now since $\langle \phi_1(x), k_n(-\boldsymbol{\tau}) \rangle_{L^p(\mathbb{R}^n)} = F_{k_n}[x]$, the first equality simply expresses that the mapping $k_n \mapsto F_{k_n}$ restricted to the set of symmetric Volterra kernels is injective, which we proved earlier in this chapter. The second equality holds for the same reason, exchanging the roles of p and q. □

The fact that all Volterra monomials can be obtained using this construction, is not so surprising considering the following observation (which is ultimately equivalent to the same arguments as in the proposition just above).

[5] Note that the construction would actually work for any X absorbing in $\mathcal{X}$.

Lemma 4.20 (Density criterion using Hahn-Banach theorem [BR19, corollary 2 in section 1.1])**.** Let E a Banach space and $\underline{E}$ a subspace.

$$\underline{E} \text{ is dense in } E \quad \Longleftrightarrow \quad \{X \in E';\ \forall x \in \underline{E}, \langle x, X \rangle_E = 0\} = \{0_{E'}\} \tag{4.6.19}$$

Proposition 4.21. The set $\{\tilde{x}^{\times n}(\boldsymbol{t});\ \tilde{x} \in L^q(\mathbb{R})\}$ has its linear span dense in $L^q_{\mathrm{Sym}}(\mathbb{R}^n)$.

Proof. Apply the lemma to $\underline{E} = \operatorname{span}\{\tilde{x}^{\times n}(\boldsymbol{t});\ \tilde{x} \in L^q(\mathbb{R})\}$ and $E = L^q_{\mathrm{Sym}}(\mathbb{R}^n) \simeq L^q(\Delta_n)$. (Recall from earlier in the chapter the notation $\Delta_n = \{(\tau_1, ..., \tau_n) \in \mathbb{R}^n;\ \tau_1 \leq ... \leq \tau_n\}$.) Note that E' is isometrically isomorphic to $L^p(\Delta_n)$ where $1/p + 1/q = 1$.

Let $\tilde{k}_n \in E' \simeq L^p(\Delta_n)$ such that

$$\forall \tilde{x} \in L^q(\mathbb{R}),\ \left\langle \tilde{x}^{\times n}(\boldsymbol{\tau}), \tilde{k}_n \right\rangle_{L^q(\Delta^n)} = \mathrm{cst} \cdot \left\langle \tilde{x}^{\times n}(\boldsymbol{\tau}), \tilde{h}_n(-\boldsymbol{\tau}) \right\rangle_{L^q(\mathbb{R}^n)} = 0 \tag{4.6.20}$$

where $\tilde{h}_n(-\boldsymbol{\tau}) = \operatorname{Sym} \tilde{k}_n$ denotes the only symmetric extension of $\tilde{k}_n$ to all of $\mathbb{R}^n$.

Now $\left[\tilde{x} \mapsto \left\langle \tilde{x}^{\times n}(\boldsymbol{\tau}), \tilde{h}_n(-\boldsymbol{\tau}) \right\rangle_{L^q(\mathbb{R}^n)} = \int_{\mathbb{R}^n} d\boldsymbol{\tau}\ \tilde{x}^{\times n}(\boldsymbol{\tau}) \tilde{h}_n(-\boldsymbol{\tau})\right]$ is nothing else than $F_{\tilde{h}_n}$ the Volterra monomial with Volterra kernel $\tilde{h}_n$. Furthermore, as proved earlier in this chapter, the mapping $\tilde{h}_n \mapsto F_{\tilde{h}_n}$ restricted to the set of symmetric Volterra kernels is injective. This shows that $\tilde{h}_n$, and so $\tilde{k}_n$, must be identically zero. $\square$

Extrapolating to the space $C_{b\,\mathrm{Sym}}(\mathbb{R}^n)$ of symmetric continuous bounded multivariate functions instead of $L^q_{\mathrm{Sym}}(\mathbb{R}^n)$, and by using the formalism used in [BCD84] where the Volterra kernels are bounded measures and the signals live in $C_b(\mathbb{R})$, this shows that *any* $f(\boldsymbol{t}) \in C_{b\,\mathrm{Sym}}(\mathbb{R}^n)$ *is arbitrarily well uniformly approximated by a finite sum of functions of the form* $\sum_i f_i(t_1)...f_i(t_n)$ with $f_i \in C_b(\mathbb{R})$, which we find nice and surprising. That result seems hard to prove via Stone-Weierstrass and symmetrization, which would arguably be the obvious path. [6] It applies for example to $f(\boldsymbol{t}) = \mathbb{1}_{\boldsymbol{t} \in [0,1]^n}(t_1 + ... + t_n)$, which we find especially surprising. (We advise the reader to double-check this result though, precisely because we find it so surprising.)

4.6.3 Volterra series as RKBS

From the interpretation of Volterra monomials as elements of a RKBS, we can immediately state a corresponding interpretation for Volterra series of fixed finite order N, by simply concatenating the feature maps for $0 \leq n \leq N$. The case of infinite-order Volterra series is less obvious but morally not much more difficult, as we now check, by a construction analogous to dot-product kernels.

Pose the notation for the space of *symmetric* Volterra kernels:

$$\overline{\mathcal{K}}_n = L^q_{\mathrm{Sym}}(\mathbb{R}^n) \tag{4.6.21}$$

$$\overline{\mathcal{K}}_\infty(R) = \left\{ k = (k_0, k_1, ...) \in \overline{\mathcal{K}}_0 \times \overline{\mathcal{K}}_1 \times ...;\ \left[\limsup_n \|k_n\|^{1/n}_{\overline{\mathcal{K}}_n}\right]^{-1} \geq R \right\} \tag{4.6.22}$$

where we recall that $\left[\limsup_n \|k_n\|^{1/n}_{\overline{\mathcal{K}}_n}\right]^{-1}$ is just the radius of convergence of the gain bound function $f_k(z) = \sum_n \|k_n\|_{\overline{\mathcal{K}}_n} z^n$, so that

$$F_k[x] = \sum_{n=0}^{\infty} F_{k_n}[x] = \sum_{n=0}^{\infty} \int_{\mathbb{R}^n} d\boldsymbol{\tau}\ k_n(-\boldsymbol{\tau}) x^{\times n}(\boldsymbol{\tau}) \tag{4.6.23}$$

[6] Using Stone-Weierstrass theorem shows that any (symmetric) continuous function $f(\boldsymbol{t})$ is arbitrarily well uniformly approximated *over any compact* by a finite sum of functions of the form $\sum_i f_{i1}(t_1)...f_{in}(t_n)$, *with* $f_{i1} \neq f_{i2} \neq ...f_{in}$ *for each* i a priori; and symmetrizing such a sum does not lead to an expresssion of the form $\sum_i f_i(t_1)...f_i(t_n)$. (Think of $t_1 t_2^3 + t_1^3 t_2$ vs. $t_1^2 t_2^2$.)

is well-defined for all $k \in \overline{\mathcal{K}}_\infty(R)$ and $\|x\|_{\mathcal{X}} = \|x\|_{L^p(\mathbb{R})} < R$. Moreover for any fixed $r < R$, denote $\overline{\mathcal{K}}_\infty(R,r)$ the space $\overline{\mathcal{K}}_\infty(R)$ equipped with the norm

$$\|k\|_{\overline{\mathcal{K}}_\infty(R,r)} = \sum_{n=0}^{\infty} \|k_n\|_{\overline{\mathcal{K}}_n} r^n \tag{4.6.24}$$

Finally, denote $\tilde{\overline{\mathcal{K}}}_n = L^p_{\mathrm{Sym}}(\mathbb{R}^n)$ and $\tilde{\overline{\mathcal{K}}}_\infty(R',r')$ defined likewise but with q replaced by p.

In the spirit of transparency, note that **the sequel presumes that $\overline{\mathcal{K}}_\infty(R,r)$ is a Banach space, which I haven't checked**.

Proposition 4.22. Let $0 < \rho < r < R$, and

- $\Omega_1 = X = B_\rho^{(\mathcal{X})}$ the centered ball of $\mathcal{X} = L^p(\mathbb{R})$ of radius ρ, $\mathcal{W}_1 = \tilde{\overline{\mathcal{K}}}_\infty(\rho^{-1}, r^{-1})$, and $\phi_1 : \begin{bmatrix} \Omega_1 \to \mathcal{W}_1 \\ x(t) \mapsto (\phi_{1n}(x))_n \end{bmatrix}$, where $\phi_{1n}(x) = x^{\times n}(\boldsymbol{t})$;
- $\Omega_2 = \widetilde{X} = B_{R^{-1}}^{(\widetilde{\mathcal{X}})}$ the centered ball of $\widetilde{\mathcal{X}} = L^q(\mathbb{R})$ of radius R^{-1}, $\mathcal{W}_2 = \overline{\mathcal{K}}_\infty(R,r)$, and $\phi_2 : \begin{bmatrix} \Omega_2 \to \mathcal{W}_2 \\ \tilde{x}(t) \mapsto (\phi_{2n}(\tilde{x}))_n \end{bmatrix}$, where $\phi_{2n}(\tilde{x}) = \tilde{x}^{\times n}(\boldsymbol{t})$;
- $\langle \cdot, \cdot \rangle_{\mathcal{W}_1 \times \mathcal{W}_2}$ defined by:

$$\forall \tilde{k} \in \mathcal{W}_1 = \tilde{\overline{\mathcal{K}}}_\infty(\rho^{-1}, r^{-1}), \forall k \in \mathcal{W}_2 = \overline{\mathcal{K}}_\infty(R,r),\ \langle \tilde{k}, k \rangle_{\mathcal{W}_1 \times \mathcal{W}_2} = \sum_{n=0}^{\infty} \int_{\mathbb{R}^n} d\boldsymbol{\tau}\ k_n(-\boldsymbol{\tau}) \tilde{k}_n(\boldsymbol{\tau}) \tag{4.6.25}$$

Then the bilinear form $\langle \cdot, \cdot \rangle_{\mathcal{W}_1 \times \mathcal{W}_2}$ is continuous, and the linear spans of $\phi_1(\Omega_1)$ and of $\phi_2(\Omega_2)$ are dense in the sense of the previous proposition. So $(\mathcal{B}_1, \mathcal{B}_2, \langle \cdot, \cdot \rangle_{\mathcal{B}_1 \times \mathcal{B}_2})$ defined as in the previous proposition is a pair of RKBS.

Moreover, note that

$$\mathcal{B}_1 = \left\{ \langle \phi_1(\cdot), v \rangle_{\mathcal{W}_1 \times \mathcal{W}_2} ;\ v \in \mathcal{W}_2 \right\} = \left\{ F_k : \left[x \mapsto \sum_{n=0}^{\infty} \int_{\mathbb{R}^n} d\boldsymbol{\tau}\ k_n(-\boldsymbol{\tau}) x^{\times n}(\boldsymbol{\tau}) \right] ;\ k \in \overline{\mathcal{K}}_\infty(R) \right\} \tag{4.6.26}$$

So $\mathcal{B}_1$ is just the space of Volterra series over signals $x \in X = B_\rho^{(L^p(\mathbb{R}))}$ whose gain bound function has radius of convergence $\geq R$, and symmetrically $\mathcal{B}_2$ is the space of Volterra series over signals $\tilde{x} \in \widetilde{X} = B_{R^{-1}}^{(L^q(\mathbb{R}))}$ whose gain bound function has radius of convergence $\geq \rho^{-1}$.

Also note that the associated kernel is simply

$$K(x, \tilde{x}) = \langle \phi_1(x), \phi_2(\tilde{x}) \rangle_{\mathcal{W}_1 \times \mathcal{W}_2} = \sum_{n=0}^{\infty} \left(\int_{\mathbb{R}} dt\ \tilde{x}(-t) x(t) \right)^n = \frac{1}{1 - \left[\int_{\mathbb{R}} dt\ \tilde{x}(-t) x(t) \right]} \tag{4.6.27}$$

Proof. Let us show that the bilinear form $\langle \cdot, \cdot \rangle_{\mathcal{W}_1 \times \mathcal{W}_2}$ is continuous. Abbreviate $\|k_n\| := \|k_n\|_{\overline{\mathcal{K}}_n}$, $\|\tilde{k}\| := \|\tilde{k}_n\|_{\tilde{\overline{\mathcal{K}}}_n}$ and $\langle \tilde{k}_n, k_n \rangle := \int_{\mathbb{R}^n} d\boldsymbol{\tau}\ k_n(-\boldsymbol{\tau}) \tilde{k}_n(\boldsymbol{\tau})$. Recall by definition

$$\|\tilde{k}\|_{\mathcal{W}_1} = \sum_{n=0}^{\infty} \|\tilde{k}_n\| r^{-n} \qquad \|k\|_{\mathcal{W}_2} = \sum_{n=0}^{\infty} \|k_n\| r^n \qquad \langle \tilde{k}, k \rangle_{\mathcal{W}_1 \times \mathcal{W}_2} = \sum_{n=0}^{\infty} \langle \tilde{k}_n, k_n \rangle \tag{4.6.28}$$

and we want to show that $\left\langle \tilde{k}, k \right\rangle_{\mathcal{W}_1 \times \mathcal{W}_2} \leq \text{cst} \cdot \left\| \tilde{k} \right\|_{\mathcal{W}_1} \|k\|_{\mathcal{W}_2}$. Indeed:

$$\left\langle \tilde{k}, k \right\rangle_{\mathcal{W}_1 \times \mathcal{W}_2} = \sum_{n=0}^{\infty} \left\langle \tilde{k}_n, k_n \right\rangle \leq \sum_{n=0}^{\infty} \left\| \tilde{k}_n \right\| \|k_n\| \, r^{-n} r^n \tag{4.6.29}$$

$$\leq \left[\sum_{n=0}^{\infty} \left\| \tilde{k}_n \right\| r^{-n} \right] \left[\sum_{m=0}^{\infty} \|k_m\| \, r^m \right] = \left\| \tilde{k} \right\|_{\mathcal{W}_1} \|k\|_{\mathcal{W}_2} \tag{4.6.30}$$

Next, we want to check that

$$\left\{ v \in \mathcal{W}_2;\ \forall a \in \phi_1(\Omega_1), \langle a, v \rangle_{\mathcal{W}_1 \times \mathcal{W}_2} = 0 \right\} \tag{4.6.31}$$

$$= \left\{ k \in \overline{\mathcal{K}}_\infty(R);\ \forall x \in X, \sum_{n=0}^{\infty} \langle \phi_{1n}(x), k_n(-\boldsymbol{\tau}) \rangle_{L^p(\mathbb{R}^n)} = 0 \right\} = \{0\} \tag{4.6.32}$$

$$\text{and} \quad \left\{ u \in \mathcal{W}_1;\ \forall b \in \phi_2(\Omega_2), \langle u, b \rangle_{\mathcal{W}_1 \times \mathcal{W}_2} = 0 \right\} \tag{4.6.33}$$

$$= \left\{ \tilde{k}_n \in \widetilde{\overline{\mathcal{K}}}_\infty(\rho^{-1});\ \forall \tilde{x} \in \widetilde{X}, \sum_{n=0}^{\infty} \left\langle \tilde{k}_n, \phi_{2n}(\tilde{x})(-\boldsymbol{\tau}) \right\rangle_{L^p(\mathbb{R}^n)} = 0 \right\} = \{0\} \tag{4.6.34}$$

Now since $\langle \phi_{1n}(x), k_n(-\boldsymbol{\tau}) \rangle_{L^p(\mathbb{R}^n)} = F_{k_n}[x]$, the first equality simply expresses that the mapping $k \mapsto F_k$ restricted to the set of symmetric Volterra kernel sequences is injective, which is a straightforward consequence of the corresponding statement for Volterra monomials. The second equality holds for the same reason, exchanging the roles of p and q. □

Chapter 5

Wiener series

Besides Volterra series, another widely used general model for nonlinear systems are Wiener series. They are especially adapted to settings with stochastic inputs. In this chapter,

- We show in what sense the topology of the set of Wiener series is equivalent to the one of $L^2_{\text{Sym}}(\mathbb{R}^n)$, when inputs are white Gaussian noise functions.
- We clarify how symmetry interacts with product-form ONB's of $L^2(\mathbb{R}^n)$. Namely, an ONB of $L^2(\mathbb{R})$ induces an isometry between $L^2_{\text{Sym}}(\mathbb{R}^n)$ and an appropriately weighted version of $\ell^2(\mathbb{N})$, with an awkward but explicit correspondence.

The presentation in this chapter is largely based on the engineering-oriented texts [Sch80] [Sch81], and as such we will sweep most mathematical details under the carpet. While this allows for a simpler presentation, it has the great downside that the objects manipulated are not well defined. Making this chapter's discussion mathematically well-grounded would be an interesting direction for future work.

The paper [PP77] provides a nice rigorous presentation of Wiener series. Introduction to necessary background on stochastic processes (Brownian/Gaussian/Wiener motion/process) can be found here `https://realnotcomplex.com/probability-and-statistics/stochastic-processes` or in the course "Brownian motion and stochastic calculus" at ETH.

5.1 Definitions

5.1.1 White Gaussian noise (WGN)

Definition 5.1. A *Gaussian noise function* is a sample path $z(t)$ of some centered stationary Gaussian process, i.e whose covariance function $K(t,t')$ depends only on $t - t'$ and $\mathbb{E}z(t) = 0$.

A white Gaussian noise (WGN) function of amplitude $A > 0$ is a sample path $x(t)$ of a (generalized) Gaussian process with covariance function $K_{WGN}(t,t') = A\delta_0(t - t')$.

Remark 5.1.

- Here we take a very naive approach following [Sch80]. See [KS47] for a systematic characterization of stationary Gaussian processes.
- In a sense that is difficult to formalize, a WGN process corresponds to the derivatives of a Wiener a.k.a Brownian process (with respect to time), which is a Gaussian process with covariance function $K_W(t,t') = \min(t,t')$.

Definition 5.2. The *time-average* of a function $g : [-T,T] \to \mathbb{R}$ is defined as the scalar

$$\overline{g(t)}^t = \frac{1}{2T}\int_{-T}^{T} dt\ g(t) \tag{5.1.1}$$

The time-average of a function $g : \mathbb{R} \to \mathbb{R}$ is the limit of the above expression when $T \to \infty$ (when it exists); that is, $\overline{g(t)}^t = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} dt\ g(t)$.

The *time amplitude density* of a signal $x : [-T, T] \to \mathbb{R}$ is

$$P_x(x_0) = \frac{1}{2T} \frac{d}{dx} \mu \left\{t;\ x(t) \in [x_0, x_0 + dx]\right\} \tag{5.1.2}$$

Note that $\int_{\mathbb{R}} dx_0\ P_x(x_0) = 1$. The time amplitude density of a signal $x : \mathbb{R} \to \mathbb{R}$ is the limit of the above expression when $T \to \infty$ (when it exists).

Definition 5.3 ([Sch80, section 19.5]). A stochastic process $\{z(t)\}$ is called *quasi-ergodic* (to all orders) if

$$\overline{f\left(\hat{z}(t), \hat{z}(t+\tau_1), ..., \hat{z}(t+\tau_{n-1})\right)}^t = \mathbb{E}_z f\left(z(t_0), z(t_0+\tau_1), ..., z(t_0+\tau_{n-1})\right) \tag{5.1.3}$$

for any $f : \mathbb{R}^n \to \mathbb{R}$, any $t_0, \tau_1, ..., \tau_{n-1} \in \mathbb{R}$, and almost any sample path $\hat{z}(t)$.

In words, $\{z(t)\}$ is quasi-ergodic if the joint time amplitude density of $(\hat{z}(t), \hat{z}(t+\tau_1), ..., \hat{z}(t+\tau_{n-1}))$ is equal to the probability density of the joint random variable $(z(t_0), z(t_0+\tau_1), ..., z(t_0+\tau_{n-1}))$, for any $t_0, \tau_1, ..., \tau_{n-1} \in \mathbb{R}$ and almost any sample path $\hat{z}(t)$.

(Note that this implies that the latter density does not depend on t_0, so any quasi-ergodic $\{z(t)\}$ by our definition is also stationary in this sense.)

Proposition 5.1. Gaussian noise functions are quasi-ergodic. Let $z(t)$ a Gaussian noise function.

- The time amplitude density P_z is centered Gaussian (with variance $K(0)$ where z is drawn from the Gaussian process with covariance function $K(t-t')$).
- The average of products of time-delayed $z(t)$ with itself is: [Sch80, section 10.2, appendix A]

$$\overline{z(t-\tau_1)...z(t-\tau_{2M+1})}^t = 0 \tag{5.1.4}$$

$$\overline{z(t-\tau_1)...z(t-\tau_{2M})}^t = \sum \prod \overline{z(t-\tau_i)z(t-\tau_j)}^t \tag{5.1.5}$$

  where the sum is over all distinct pairings of the factors (i.e all distinct partitionings of the $2M$ factors into pairs).

- In particular, the autocorrelation function of z is: $\phi_{zz}(\tau) := \overline{z(t)z(t+\tau)}^t$. ($= K(\tau)$ when z is drawn from the Gaussian process with covariance function $K(t-t')$.) So combined with the previous item, we see that z is essentially characterized by its autocorrelation function.
- For any $h : \mathbb{R} \to \mathbb{R}$, $(h * z)(t)$ is also a Gaussian noise function [Sch80, section 11.4].

WGN functions are quasi-ergodic. Let $x(t)$ a WGN function of amplitude $A > 0$.

- $x(t)$ has autocorrelation function $\phi_{xx}(\tau) := \overline{x(t)x(t+\tau)}^t = A\delta_0(\tau)$ and power density spectrum $\hat{\phi}_{xx}(\xi) = A$.
- The time amplitude density of P_x is centered Gaussian with variance A:

$$P_x(x) = \mathcal{N}(x; 0, A) = \frac{1}{\sqrt{2\pi A}} e^{-\frac{x^2}{2A}} \tag{5.1.6}$$

- For any $h : \mathbb{R} \to \mathbb{R}$, $z(t) = (h * x)(t)$ is the Gaussian noise function with autocorrelation $\phi_{zz}(\tau) = A\phi_{hh}(\tau)$ [Sch80, (11.2-6)].

  In particular, for two convolution kernels h_i, h_j, [Sch80, section 18.5]

$$z_i(t) := (h_i * x)(t) \qquad z_j(t) := (h_j * x)(t) \tag{5.1.7}$$

$$\overline{z_i(t)z_j(t)}^t = A \int_{\mathbb{R}} h_i(\tau)h_j(\tau)d\tau \tag{5.1.8}$$

Remark 5.2 (Non-white Gaussian inputs). The Wiener series model can be adapted to other types of stochastic inputs.

In particular when the inputs are non-white Gaussian noise functions $z(t)$, under certain assumptions on the covariance function of the associated Gaussian process, we can reformulate the setting to assume WGN input. The idea is the following: for any $h : \mathbb{R} \to \mathbb{R}$ living in a "nice" function space and x WGN function, we know that $z(t) = (h*x)(t)$ is a Gaussian noise function with autocorrelation function $\phi_{zz}(\tau) \propto \phi_{hh}(\tau)$. Now, recall that the autocorrelation function of a Gaussian noise function $z(t)$ with covariance function $K(t-t')$ is given by $\phi_{zz}(\tau) = K(\tau)$. So for any covariance function K such that there exists h satisfying $K(\tau) \propto \phi_{hh}(\tau)$, drawing a sample path z from the associated Gaussian process is equivalent to drawing a WGN function x and setting $z = h * x$. Note that the condition on K for there to exist such a h, is that the spectrum $\widehat{K}(\xi)$ can be factorized as $\widehat{K}(\xi) \propto \hat{h}(\xi)\overline{\hat{h}(\xi)}$ (where $\overline{\cdot}$ denotes complex conjugate), for a $\hat{h}$ in a "nice" function space. For a more detailed discussion of that case, see [Sch80, section 15.3].

For adaptations of the Wiener series model to other types of stochastic inputs, see [Sch80, chapter 21] and [PP78].

5.1.2 Wiener G-functionals, Wiener series

Definition 5.4 ([Sch80, chapter 12])**.** For $i \in \mathbb{N}$ and $k_i : \mathbb{R}^i \to \mathbb{R}$, let $H_i[k_i; \cdot]$ denote the ith-order Volterra monomial with kernel k_i.

For any $k_n \in L^2(\mathbb{R}^n)$, the nth-order *G-functional* with *leading Wiener kernel* k_n is the system:

$$x(t) \mapsto G_n[k_n; x(t)] = \sum_{i=0}^{n} H_i[k_{i(n)}; x(t)] \tag{5.1.9}$$

where the *derived kernels* $k_{i(n)}$ are determined by the defining property of G-functionals:

$$\forall m < n, \overline{H_m[x(t)]G_n[k_n; x(t)]}^t = 0 \tag{5.1.10}$$

for any mth-order Volterra monomial H_m and any WGN $x(t)$.

A *Wiener series* of order N is a sum of G-functionals of orders $n \leq N$.

Proposition 5.2. The derived kernels $k_{i(n)}$ are uniquely determined by the leading kernel k_n [Sch80, (13.9-12)].

The formulas giving $k_{i(n)}$ from k_n are linear, so G-functionals are linear with respect to the leading kernel. That is, for all k_n, h_n,

$$G_n[k_n + \lambda h_n; \cdot] = G_n[k_n; \cdot] + \lambda G_n[h_n; \cdot] \tag{5.1.11}$$

Moreover, the derived kernels are symmetric if the leading kernel is symmetric.

Proposition 5.3. For all $m \neq n$ and WGN function $x(t)$,

$$\overline{G_n[k_n; x(t)]G_m[h_m; x(t)]}^t = 0 \tag{5.1.12}$$

For all symmetric kernels k_n, h_n, and WGN function $x(t)$ of amplitude A, [Sch80, chapter 14]

$$\overline{G_n[k_n; x(t)]G_n[h_n; x(t)]}^t = n!A^n \langle k_n, h_n \rangle_{L^2(\mathbb{R}^n)} \tag{5.1.13}$$

In particular, $\overline{G_n[k_n; x(t)]^2}^t = n!A^n \|k_n\|_{L^2}^2$.

More generally, for k_n and $h_n \in L^2(\mathbb{R}^n)$ not necessarily symmetric,

$$\overline{G_n[k_n; x(t)]G_n[h_n; x(t)]}^t = n!A^n \langle \mathrm{Sym}[k_n], \mathrm{Sym}[h_n] \rangle_{L^2} \tag{5.1.14}$$

Symmetric kernels have minimum norm In the same way as for Volterra series, we can restrict our attention to symmetric Wiener kernels, without loss of generality in the set of systems considered nor in terms of topological information.

Proposition 5.4. Denote

- The space of G-functionals of order n, $\mathbb{S}_n = \{G_n[k_n;\cdot];\ k_n \in L^2(\mathbb{R}^n)\}$;
- The surjective operator $G_n : \begin{bmatrix} L^2(\mathbb{R}^n) \to \mathbb{S}_n \\ k_n \mapsto G_n[k_n;\cdot] \end{bmatrix}$, and its restriction $G_n|_{L^2_{\mathrm{Sym}}(\mathbb{R}^n)}$;
- The (renormalized) norm induced by G_n: $\|S\|_{\mathbb{S}_n} = \inf_{k \in L^2(\mathbb{R}^n); S = G_n[k_n;\cdot]} \sqrt{n!A^n}\,\|k\|_{L^2(\mathbb{R}^n)}$.

Then $G_n|_{L^2_{\mathrm{Sym}}(\mathbb{R}^n)}$ is injective, its image space is $\mathbb{S}_n$, and the norm on $\mathbb{S}_n$ induced by G_n is equal to the one induced by $G_n|_{L^2_{\mathrm{Sym}}(\mathbb{R}^n)}$.

In other words $G_n|_{L^2_{\mathrm{Sym}}(\mathbb{R}^n)}$ is an isometric isomorphism between the spaces $L^2_{\mathrm{Sym}}(\mathbb{R}^n)$ and $(\mathbb{S}_n, \|\cdot\|_{\mathbb{S}_n})$.

Proof.

Surjective For a $S = G_n[h_n;\cdot]$, just take $k_n = \mathrm{Sym}[h_n]$; then $S = G_n[k_n;\cdot]$ and $k_n \in L^2_{\mathrm{Sym}}(\mathbb{R}^n)$.

Injective Let $k_n \in L^2_{\mathrm{Sym}}(\mathbb{R}^n)$ such that $G_n[k_n;\cdot] = 0$. Then by evaluating at a WGN $x(t)$ of amplitude A, since $\overline{G_n[k_n;x(t)]^2}^t = n!A^n \|k_n\|^2_{L^2}$ by the previous proposition, then $k_n = 0$.

Isometric For all $k_n \in L^2(\mathbb{R}^n)$, as already seen in section 4.2, triangle inequality implies

$$\|\mathrm{Sym}[k_n]\|_{L^2} \leq \|k_n\|_{L^2} \tag{5.1.15}$$

So for any $S = G_n[k_n;\cdot] \in \mathbb{S}_n$ with $k_n \in L^2(\mathbb{R}^n)$,

$$\inf_{h_n \in L^2; S = G_n[h_n;\cdot]} \|h_n\|_{L^2} = \|\mathrm{Sym}[k_n]\|_{L^2} \tag{5.1.16}$$

which proves the isometry.

□

5.1.3 Worst-case or average error norm

Worst-case error norm over a set of WGN functions Wiener series models are particularly well-suited for system identification with Gaussian inputs, since $\overline{G_n[k_n;x(t)]G_n[h_n;x(t)]}^t = n!A^n \langle k_n, h_n\rangle_{L^2(\mathbb{R}^n)}$ for any symmetric k_n, h_n and any WGN $x(t)$ of amplitude A.

In terms of our framework from section 1.3, this means that, denoting:

- $\mathcal{Y} = L^2_{\mathrm{avg}}(\mathbb{R})$ the set of signals with finite squared-time-average, with the inner product

$$\langle y(t), z(t)\rangle_{\mathcal{Y}} = \overline{y(t)z(t)}^t \tag{5.1.17}$$

- $\mathbb{S}_n$ the set of Wiener G-functionals of order n as defined above, and $\|\cdot\|_{\mathbb{S}_n}$ the norm induced by the operator $G_n : L^2_{\mathrm{Sym}}(\mathbb{R}^n) \to \mathbb{S}_n$;
- X a set of WGN functions of amplitude A. Note that the $x \in X$ are not necessarily drawn from the same Gaussian process; we don't consider any interaction between the signals x, we only use each of them separately as test input signals.

Then for each $x \in X$, since x is a WGN of amplitude A,

$$\|G_n[k_n;x(t)]\|_{\mathcal{Y}} = \sqrt{\overline{G_n[k_n;x(t)]^2}^t} = \sqrt{n!A^n}\,\|k_n\|_{L^2} \tag{5.1.18}$$

for any $k_n \in L^2_{\mathrm{Sym}}(\mathbb{R}^n)$. So, the error norm over X is simply equal to

$$\|S\|_{\infty X} = \|S\|_{\mathbb{S}_n} \tag{5.1.19}$$

$L^2_{\mathbb{P}_x}$-average error norm Since WGN functions are quasi-ergodic, another possible point of view is the following. Denote:

- $\mathcal{Y} = L^2_{\text{avg}}(\mathbb{R})$ as before;
- $\mathbb{S}_n$ the set of Wiener G-functionals of order n as before;
- $\mathbb{P}_x$ the distribution of the stochastic process associated to WGN functions of amplitude A.

Recall that the $L^2_{\mathbb{P}}$-average error norm is defined by $\|S\|_{2\mathbb{P}} = \sqrt{\mathbb{E}_{x\sim\mathbb{P}} \|S[x]\|^2_{\mathcal{Y}}}$. Then by quasi-ergodicity, it also holds that

$$\|S\|_{2\mathbb{P}_x} = \|S\|_{\mathbb{S}_n} \tag{5.1.20}$$

Remark 5.3. This subsection focused on systems modeled by a single G-functional, but the discussion transfers straightforwardly to Wiener series of finite order.

5.2 Orthonormal expansion of the Wiener kernels

Since the leading Wiener kernels k_n are in $L^2(\mathbb{R}^n)$, they may be decomposed on an orthonormal basis (ONB) of that space. However Wiener kernels should be taken as symmetric for the isometry results presented above to hold.

5.2.1 Motivation, two levels of orthogonality

Let $N \in \mathbb{N}$ and consider a Wiener series of order N, i.e a system of the form $S = \sum_{n=0}^{N} G_n[h_n; \cdot]$ where $h_n \in L^2_{\text{Sym}}(\mathbb{R}^n)$ for each $n \leq N$.

For each n, let $(\psi_{nk})_k$ an ONB of $L^2_{\text{Sym}}(\mathbb{R}^n)$. (And not of $L^2(\mathbb{R}^n)$!)

Decompose each h_n as: $\forall n,\ h_n = \sum_{k=0}^{\infty} a_{nk}\psi_{nk}$, where $a_{nk} = \langle h_n, \psi_{nk}\rangle_{L^2}$. Then,

$$S = \sum_{n=0}^{N} G_n[h_n; \cdot] = \sum_{n=0}^{N}\sum_{k=0}^{\infty} a_{nk} G_n[\psi_{nk}; \cdot] \tag{5.2.1}$$

Note that the operators $(G_n[\psi_{nk}; \cdot])_{n,k}$ are orthogonal on two distinct levels: for any WGN $x(t)$,

- For all $n \neq n'$, and any $k, k' \in \mathbb{N}$, $\langle G_n[\psi_{nk}; x(t)], G_{n'}[\psi_{n'k'}; x(t)]\rangle_{L^2_{\text{avg}}} = 0$ by the defining property of Wiener G-functionals.
- When $n = n'$, for any $k, k' \in \mathbb{N}$, $\langle G_n[\psi_{nk}; x(t)], G_n[\psi_{nk'}; x(t)]\rangle_{L^2_{\text{avg}}} = n!A^n \langle \psi_{nk}, \psi_{nk'}\rangle_{L^2(\mathbb{R}^n)} = n!A^n \mathbb{1}_{k=k'}$ by symmetry and orthonormality of $(\psi_{nk})_k$.

In summary: $\langle G_n[\psi_{nk}; x(t)], G_n[\psi_{n'k'}; x(t)]\rangle_{L^2_{\text{avg}}} = n!A^n \mathbb{1}_{(n,k)=(n',k')}$ for any WGN $x(t)$. As a consequence:

$$\|S[x]\|^2_{L^2_{\text{avg}}(\mathbb{R})} = \sum_{n=0}^{N}\sum_{k=0}^{\infty} n!A^n |a_{nk}|^2 \tag{5.2.2}$$

Thus the (worst-case error over any set of WGN functions) norm of the Wiener series S is a weighted ℓ^2 norm of the coefficients a_{nk} of the Wiener kernels h_n in the bases ψ_{nk}. So, studying the metric entropy of sets of systems modeled by Wiener series reduces, by isometry, to studying the metric entropy in a weighted ℓ^2 space. However in our construction we needed to use an ONB of $L^2_{\text{Sym}}(\mathbb{R}^n)$, which is difficult to manipulate as we now discuss.

Orthonormal expansions of symmetric functions Thus we would like to consider an ONB of the space $L^2_{\mathrm{Sym}}(\mathbb{R}^n)$. We know that such an ONB exists, because $L^2_{\mathrm{Sym}}(\mathbb{R}^n)$ is a separable Hilbert space: for any total family $(\phi_k)_{k\in\mathbb{N}}$ of $L^2(\mathbb{R}^n)$, then $(\mathrm{Sym}[\phi_k])_k$ is a total family of $L^2_{\mathrm{Sym}}(\mathbb{R}^n)$, and so an ONB $(\psi_k)_k$ can be obtained by Gram-Schmidt orthonormalization. However these ψ_k are a priori very complicated, even when the ϕ_k have a simple form, so decomposing the kernel functions onto that ONB is difficult.

In fact, searching "orthogonal basis of symmetric functions" on any web browser quickly indicates that even finding such a basis is a difficult topic. So we fall back to considering orthonormal expansions of the (symmetric) Wiener kernels in $L^2(\mathbb{R}^n)$, and see what can be said.

5.2.2 Orthonormal basis (ONB) of $L^2(\mathbb{R}^n)$ with functions in product-form

It is often convenient to work with ONB's of functions in product-form, in the following sense.

Proposition 5.5. For $(l_a)_{a\in\mathbb{N}}$ an ONB of $L^2(\mathbb{R})$, the family $(l_{a_1,\dots,a_n})_{a_1,\dots,a_n\in\mathbb{N}} = (l_{\boldsymbol{a}})_{\boldsymbol{a}\in\mathbb{N}^n}$ defined by $l_{a_1,\dots,a_n}(t_1,\dots,t_n) = l_{a_1}(t_1)\dots l_{a_n}(t_n)$ is an ONB of $L^2(\mathbb{R}^n)$.

Consider the decomposition of some $k_n \in L^2(\mathbb{R}^n)$ along that basis:

$$k_n = \sum_{a_1,\dots,a_n} c_{a_1,\dots,a_n} l_{a_1}(t_1)\dots l_{a_n}(t_n) \tag{5.2.3}$$

Then k_n is symmetric if and only if its coefficients are, i.e if $\forall\sigma\in\mathfrak{S}_n,\ c_{a_1,\dots,a_n} = c_{a_{\sigma(1)},\dots,a_{\sigma(n)}}$

Proof. The family $(l_{\boldsymbol{a}})_{\boldsymbol{a}\in\mathbb{N}^n}$ is clearly orthonormal. To show that it is a total family i.e its span is dense in $L^2(\mathbb{R}^n)$, it suffices to check that its orthogonal is $\{0\}$ [Gol+19, corollary 10.3]:

$$\left[\forall a_1,\dots,a_n, \int_{\mathbb{R}^n} dt_1\dots dt_n\ g(t_1,\dots,t_n) l_{a_1}(t_1)\dots l_{a_n}(t_n) = 0\right] \implies g = 0 \text{ almost everywhere} \tag{5.2.4}$$

Now this fact is clear by induction, so $(l_{\boldsymbol{a}})_{\boldsymbol{a}\in\mathbb{N}^n}$ is indeed an ONB of $L^2(\mathbb{R}^n)$.

The characterization of the symmetry of k_n is clear from the expression of the coefficients as inner products: $c_{a_1,\dots,a_n} = \langle k_n, l_{a_1,\dots,a_n}\rangle$. □

Remark 5.4 (Product-form ONB of $L^2(\mathbb{R}^n)$ does not induce an ONB of $L^2_{\mathrm{Sym}}(\mathbb{R}^n)$). Consider an ONB of $L^2(\mathbb{R}^n)$ in product-form $(l_{\boldsymbol{a}})_{\boldsymbol{a}\in\mathbb{N}^n}$, as above. Denote for all $(a_1,\dots,a_n)\in\mathbb{N}^n$

$$\bar{l}_{a_1,\dots,a_n} = \mathrm{Sym}[l_{a_1,\dots,a_n}] = \frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n} l_{a_{\sigma(1)},\dots,a_{\sigma(n)}} \in L^2_{\mathrm{Sym}}(\mathbb{R}^n) \tag{5.2.5}$$

Then for $(a_1,\dots,a_n),\ (b_1,\dots,b_n)\in\mathbb{N}^n$,

$$\left\langle \bar{l}_{a_1,\dots,a_n}, \bar{l}_{b_1,\dots,b_n}\right\rangle_{L^2} = \frac{1}{(n!)^2}\sum_{\sigma,\nu\in\mathfrak{S}_n}\left\langle l_{a_{\sigma(1)},\dots,a_{\sigma(n)}}, l_{b_{\nu(1)},\dots,b_{\nu(n)}}\right\rangle \tag{5.2.6}$$

$$= \frac{1}{(n!)^2}\sum_{\sigma,\nu\in\mathfrak{S}_n}\left\langle l_{a_1,\dots,a_n}, l_{b_{(\sigma^{-1}\circ\nu)(1)},\dots,b_{(\sigma^{-1}\circ\nu)(n)}}\right\rangle \tag{5.2.7}$$

$$= \frac{1}{n!}\sum_{s\in\mathfrak{S}_n}\left\langle l_{a_1,\dots,a_n}, l_{b_{s(1)},\dots,b_{s(n)}}\right\rangle \tag{5.2.8}$$

$$= \frac{1}{n!}\sum_{s\in\mathfrak{S}_n}\mathbb{1}_{a_1=b_{s(1)}}\dots\mathbb{1}_{a_n=b_{s(n)}} \tag{5.2.9}$$

So, $\left\langle \bar{l}_{a_1,\dots,a_n}, \bar{l}_{b_1,\dots,b_n}\right\rangle = 0$ when $\boldsymbol{a}$ and $\boldsymbol{b}$ have "disjoint support" i.e $\{a_1,\dots,a_n\}\cap\{b_1,\dots,b_n\} = \varnothing$. But it is not the case for vectors $\boldsymbol{a}$, $\boldsymbol{b}$ that have only distinct support i.e $\{a_1,\dots,a_n\}\neq\{b_1,\dots,b_n\}$. So it is not straightforward to find a family of functions in $L^2_{\mathrm{Sym}}(\mathbb{R}^n)$ that is both orthogonal and total, by this approach. (One can always be found by Gram-Schmidt-orthonormalizing the $\bar{l}_{\boldsymbol{a}}$, but the resulting family will be complicated.)

5.2.3 Convenient choices for the product-form ONB

Convenient bases $(l_a)_{a\in\mathbb{N}}$ for $L^2(\mathbb{R})$ are provided by weighted polynomial families. For example [Sch80, chapter 16] and [BC85, section 5.2] use Laguerre polynomials. This subsection is a brief digression to clarify, more generally, under what conditions on the weighting function the procedure described there can be applied.

ONB of polynomials of $L^2_w(\mathbb{R})$

Proposition 5.6 ([Gol+19, proposition 10.7])**.** Let a function $w : \mathbb{R} \to \mathbb{R}_+^*$ such that for all $k \in \mathbb{N}$,

$$\int_{\mathbb{R}} dt\ t^k w(t) < \infty \tag{5.2.10}$$

and such that $\int_{\mathbb{R}} dt\ e^{-st} w(t)$ (its two-sided Laplace transform) is defined for all $s \in \mathbb{R}$.

Consider the space $L^2_w(\mathbb{R})$ of functions such that $\|f\|^2_{L^2_w} := \int_{\mathbb{R}} dt\ \ |f(t)|^2\, w(t) < \infty$, equipped with the inner product $\langle f, g\rangle_w := \int_{\mathbb{R}} dt\ f(t)g(t)w(t)$.

Then the set of all polynomial functions is dense in $L^2_w(\mathbb{R})$.

Proof. It suffices to show that the orthogonal of the set of polynomial functions is $\{0\}$. Let f orthogonal to all t^k, $k \in \mathbb{N}$, i.e $\forall k, \int_{\mathbb{R}} dt\ f(t)t^k w(t) = 0$.

Since $\int_{\mathbb{R}} |f|\, w \leq \|\sqrt{w}\|_{L^2}\, \|f\sqrt{w}\|_{L^2} < \infty$ by Cauchy-Schwarz inequality, then $f(t)w(t)$ is in $L^1(\mathbb{R})$. Let $g(\xi)$ its $(L^1 - L^\infty)$ Fourier transform, defined pointwisely for each ξ.

Then for each ξ we can write

$$g(\xi) := \int_{\mathbb{R}} dt\ e^{-i\xi t} f(t)w(t) \tag{5.2.11}$$

$$= \int_{\mathbb{R}} dt\ \sum_{k=0}^{\infty} \frac{(-i\xi)^k}{k!} f(t)t^k w(t) \tag{5.2.12}$$

$$= \sum_{k=0}^{\infty} \frac{(-i\xi)^k}{k!} \int_{\mathbb{R}} dt\ f(t)t^k w(t) \tag{5.2.13}$$

The summation-integration interversion is justified by Fubini theorem thanks to the assumption on w.

So $f(t)w(t) = 0$ since its Fourier transform, $g(\xi)$, is zero; so $f(t) = 0$ since $w(t) > 0$. □

Since $(t^k)_{k\in\mathbb{N}}$ is a total family of $L^2_w(\mathbb{R})$, by Gram-Schmidt orthonormalization we obtain an ONB $(P_0, ..., P_k, ...)$ consisting of polynomials, with deg $P_k = k$.

Example 5.1 ([Gol+19, examples 10.18, 10.19]).

- If $w(t) = \mathbb{1}_{t\in[-1,1]}$, P_k are the Legendre polynomials.
- If $w(t) = \mathbb{1}_{t\in[-1,1]}\sqrt{1-t^2}$, P_k are the Chebychev polynomials.
- If $w(t) = e^{-t^2/2}$, P_k are the (normalized probabilists') Hermite polynomials.
- If $w(t) = \mathbb{1}_{t\geq 0}e^{-at}$ $(a > 0)$, P_k are the Laguerre polynomials with scale factor a.

Remark 5.5. The condition that $\int_{\mathbb{R}} dt\ e^{-st} w(t)$ is defined for all $s \in \mathbb{R}$ is excessively strong; in fact it is not verified for $w(t) = \mathbb{1}_{t\geq 0}e^{-at}$. It would be interesting to look for the weakest assumption that makes the summation-integration interversion in the proof go through. Although this is bound to be a solved question, I was unable to find the answer in reference resources known to me.

ONB of weighted polynomials $P(t)\sqrt{w(t)}$ of $L^2(\mathbb{R})$

Proposition 5.7. For any ONB $(P_0, ..., P_k, ...)$ of $(L^2_w(\mathbb{R}), \|\cdot\|_{L^2_w})$, then $\left(P_0(t)\sqrt{w(t)}, ..., P_k(t)\sqrt{w(t)}, ...\right)$ is an ONB of $(L^2(\mathbb{R}), \|\cdot\|_{L^2})$.

Proof. Orthonormality is trivial. For any $\phi \in L^2(\mathbb{R})$, then $\frac{\phi(t)}{\sqrt{w(t)}} \in L^2_w(\mathbb{R})$, so we can expand it as $\frac{\phi(t)}{\sqrt{w(t)}} = \sum_{k=0}^{\infty} a_k P_k(t)$, and so $\phi(t) = \sum_{k=0}^{\infty} a_k P_k(t)\sqrt{w(t)}$. □

Example 5.2. The ONB of $L^2(\mathbb{R})$ corresponding to a particular family of polynomials is named accordingly, e.g "Laguerre functions" designates the functions $l_k(t) = P_k(t)\sqrt{w(t)}$ where $w(t) = \mathbb{1}_{t\geq 0} e^{-at}$ and P_k is kth Laguerre polynomial. One can check that the construction of Laguerre functions in [Sch80, chapter 16] is exactly the one described above, with scale factor a (in our notation) equal to $2p$ (in theirs).

5.2.4 Indexing equivalence classes of the product-form ONB

Proposition 5.8. Let $(l_a)_{a\in\mathbb{N}}$ ONB of $L^2(\mathbb{R})$ and $(l_{\boldsymbol{a}})_{\boldsymbol{a}\in\mathbb{N}^n}$ the corresponding product-form ONB of $L^2(\mathbb{R}^n)$.

Let $k_n \in L^2_{\text{Sym}}(\mathbb{R}^n)$. Consider its decomposition on that ONB: $k_n(\boldsymbol{\tau}) = \sum_{\boldsymbol{a}\in\mathbb{N}^n} c_{\boldsymbol{a}} l_{\boldsymbol{a}}(\boldsymbol{\tau})$. Since k_n is symmetric, then $c_{\boldsymbol{a}} = c_{\boldsymbol{a}_\sigma}$ for any permutation $\sigma \in \mathfrak{S}_n$.

The decomposition can be rewritten to only involve free parameters, as follows:

$$k_n(\boldsymbol{\tau}) = \sum_{\overline{\boldsymbol{a}}} c_{\overline{\boldsymbol{a}}} \sum_{\boldsymbol{a}\in\mathbb{N}^n:\boldsymbol{a}\sim\overline{\boldsymbol{a}}} l_{\boldsymbol{a}}(\boldsymbol{\tau}) \tag{5.2.14}$$

where "$\sim$" is the equivalence relation over $\mathbb{N}^n$ defined by $\boldsymbol{a} \sim \boldsymbol{b} \iff \exists \nu \in \mathfrak{S}_n; \boldsymbol{a} = \boldsymbol{b}_\nu$, and $\sum_{\overline{\boldsymbol{a}}}$ denotes summation over all distinct ordered vectors of $\mathbb{N}^n$, i.e over the equivalence classes under "$\sim$". Equivalently, by applying on both sides the linear operator Sym : $f \mapsto \frac{1}{n!}\sum_\sigma f(\boldsymbol{\tau}_\sigma)$,

$$k_n(\boldsymbol{\tau}) = \sum_{\overline{\boldsymbol{a}}} c_{\overline{\boldsymbol{a}}} \frac{d_{\overline{\boldsymbol{a}}}}{n!} \sum_{\sigma\in\mathfrak{S}_n} l_{\overline{\boldsymbol{a}}}(\boldsymbol{\tau}_\sigma) \tag{5.2.15}$$

where $d_{\overline{\boldsymbol{a}}}$ is the number of distinct orderings of the vector $\overline{\boldsymbol{a}}$, i.e the cardinality of its equivalence class.

Moreover, with these notations,

$$\|k_n\|^2_{L^2} = \sum_{\boldsymbol{a}\in\mathbb{N}^n} |c_{\boldsymbol{a}}|^2 = \sum_{\overline{\boldsymbol{a}}} d_{\overline{\boldsymbol{a}}}\ |c_{\overline{\boldsymbol{a}}}|^2 \tag{5.2.16}$$

Finally, note that in terms of the G-functionals,

$$G_n[k_n; x(t)] = \sum_{\overline{\boldsymbol{a}}} c_{\overline{\boldsymbol{a}}} \sum_{\boldsymbol{a}\in\mathbb{N}^n:\boldsymbol{a}\sim\overline{\boldsymbol{a}}} G_n[l_{\boldsymbol{a}}; x(t)] = \sum_{\overline{\boldsymbol{a}}} c_{\overline{\boldsymbol{a}}}\ d_{\overline{\boldsymbol{a}}} G_n[l_{\overline{\boldsymbol{a}}}; x(t)] \tag{5.2.17}$$

Thus we expanded k_n as a combination of the family $\left(\frac{d_{\overline{\boldsymbol{a}}}}{n!}\sum_{\sigma\in\mathfrak{S}_n} l_{\overline{\boldsymbol{a}}}(\boldsymbol{\tau}_\sigma)\right)_{\overline{\boldsymbol{a}}}$ indexed by distinct ordered vectors $\overline{\boldsymbol{a}} \in \mathbb{N}^n$. Note that this family is not orthogonal, as remarked in an earlier subsection.

It remains to understand what the equivalence classes under "$\sim$" look like.

Definition 5.5 (Indexing by distinct indices and multiplicity)**.** The equivalence class of $\boldsymbol{a} = (a_1, ..., a_n) \in \mathbb{N}^n$ is characterized by the distinct values taken by the a_i and by the number of times each value is taken. In other words $\overline{\boldsymbol{a}}$ can be represented by $(m_1, k_1), ..., (m_N, k_N)$ with $m_j \neq m_{j'}$ and $k_j > 0$. N

denotes the number of distinct indices i.e $N = |\{a_1, ..., a_n\}|$. The set of equivalence classes is then described by the set

$$\left\{ \{(m_1,k_1),...,(m_N,k_N)\}; \begin{array}{r} N \in \mathbb{N} \\ \forall j,j',\, m_j \neq m_{j'} \\ \forall j,\, k_j > 0 \\ k_1 + ... + k_N = n \end{array} \right\} \tag{5.2.18}$$

Denote, as in [Sch80, (18.1-3)]: for all $(m_1,k_1),...,(m_N,k_N)$ with $m_j \neq m_{j'}$ and $k_j > 0$,

$$l_{m_1}^{k_1}...l_{m_N}^{k_N} = \mathrm{Sym}\ l\underbrace{m_1,...,m_1,}_{k_1 \text{ times}}\ ...,\ \underbrace{m_N,...,m_N}_{k_N \text{ times}} \in L^2_{\mathrm{Sym}}(\mathbb{R}^n) \tag{5.2.19}$$

Definition 5.6 (Indexing by a sequence that sums to n)**.** It is also convenient to represent $\overline{\boldsymbol{a}}$ as a sequence $\alpha \in \mathbb{N}^{\mathbb{N}}$ such that $|\alpha| := \sum_{m=0}^{\infty} \alpha_m = n$. Namely, α is obtained from $\overline{\boldsymbol{a}}$ via its representation $(m_1,k_1),...,(m_N,k_N)$, by setting

- $\forall 1 \leq j \leq N,\ \alpha_{m_j} = k_j$
- $\alpha_{m'} = 0$ at all other entries.

The set of equivalence classes is then described by the set $\{\alpha \in \mathbb{N}^{\mathbb{N}};\ |\alpha| = n\}$.

Accordingly we denote: $l_\alpha = l_{m_1}^{k_1}...l_{m_N}^{k_N} \in L^2_{\mathrm{Sym}}(\mathbb{R}^n)$.

In the sequel we will often use the following loose but obvious notation: $l_\alpha = \prod_{m=0}^{\infty} l_m^{\alpha_m}$. Note that there are always at most n factors different from 1, since all others have a 0 in the exponent.

Remark 5.6. The quantity $d_{\overline{\boldsymbol{a}}} = |\{\boldsymbol{a} \in \mathbb{N}^n : \boldsymbol{a} \sim \overline{\boldsymbol{a}}\}|$ can be easily expressed in these notations as the multinomial coefficients: (recall that the numbers appearing in the denominator sum to n)

$$d_{\overline{\boldsymbol{a}}} = \frac{n!}{k_1!...k_N!} = \binom{n}{k_1,...,k_N} =: \binom{n}{\alpha} \tag{5.2.20}$$

Indeed, choosing a vector $\boldsymbol{a}$ such that $\boldsymbol{a} \sim \overline{\boldsymbol{a}}$ is just choosing an anagram of the "word" $m_1...m_1\ \ldots\ m_N...m_N$, with "letter" m_j appearing k_j times.

In summary, with these two indexing methods, we have

$$\|k_n\|^2_{L^2} = \sum_{\overline{\boldsymbol{a}}} d_{\overline{\boldsymbol{a}}}\ c^2_{\overline{\boldsymbol{a}}} = \sum_{|\alpha|=n} \binom{n}{\alpha} c^2_\alpha \tag{5.2.21}$$

$$G_n[k_n; x(t)] = \sum_{\overline{\boldsymbol{a}}} c_{\overline{\boldsymbol{a}}}\ d_{\overline{\boldsymbol{a}}}\ G_n[l_{\overline{\boldsymbol{a}}}; x(t)] = \sum_{|\alpha|=n} \binom{n}{\alpha} c_\alpha\ G_n[l_\alpha; x(t)] \tag{5.2.22}$$

For the purpose of mapping entropy-numbers in $\mathbb{S}$ into entropy-numbers in sequence spaces (and conversely), the ideas presented so far are actually sufficient; so we could stop our discussion here. Since the above decomposition of the G-functional leads to interesting insights on the Wiener model, we keep discussing it in the next section.

5.3 Orthogonal decomposition of the G-functionals

In this section, we present and exploit properties of G-functionals presented in [Sch80, chapters 18, 19, 20]. It seems that those properties are remarkable results, rather than simply consequences of the defining property of G-functionals, so for their proof we refer to the aforementioned chapters.

Preliminary: Hermite polynomials In this chapter we will use the same convention for the Hermite polynomials as in [Sch80]. [1] Equivalently:

[1] There is no clearly established naming convention, apparently, but extrapolating from `https://en.wikipedia.org/wiki/Hermite_polynomials` we can call our H_n the "(unnormalized) probabilists' Hermite polynomials with variance A".

- For any $z \in \mathbb{R}$, $H_n''(z) - \frac{z}{A}H_n'(z) + \frac{n}{A}H_n(z) = 0$.
- For all $n < m$, $\mathbb{E}_{\zeta\sim\mathcal{N}(0,A)}[\zeta^m H_n(\zeta)] = 0$, and $\mathbb{E}_{\zeta\sim\mathcal{N}(0,A)}[\zeta^n H_n(\zeta)] = n!A^n$, and $\deg H_n = n$.
- For all $n < m$, $\mathbb{E}_{\zeta\sim\mathcal{N}(0,A)}[H_m(\zeta)H_n(\zeta)] = 0$, and $\mathbb{E}_{\zeta\sim\mathcal{N}(0,A)}[H_n(\zeta)^2] = n!A^n$, and $\deg H_n = n$.

5.3.1 Remarkable properties of G-functionals

Proposition 5.9. For any ONB $(l_a)_{a\in\mathbb{N}}$ of $L^2(\mathbb{R})$,

- [Sch80, (18.1-18)]

$$G_{k_1+\ldots+k_N}\left[l_{m_1}^{k1}\ldots l_{m_N}^{k_N}; x(t)\right] = G_{k_1}\left[l_{m_1}^{k_1}; x(t)\right]\ldots G_{k_N}\left[l_{m_N}^{k_N}; x(t)\right] \tag{5.3.1}$$

$$G_n[l_\alpha; x(t)] = G_n\left[\prod_{i=0}^{\infty} l_i^{\alpha_i}; x(t)\right] = \prod_{i=0}^{\infty} G_{\alpha_i}[l_i^{\alpha_i}; x(t)] \tag{5.3.2}$$

  We stress that here, in accordance with our indexing convention described above, the m_i are pairwise distinct.

- [Sch80, (18.3-4)]

$$G_n[l_i^n; x(t)] = H_n(z_i(t)) \tag{5.3.3}$$

  where $z_i(t) = (l_i * x)(t)$ (which is a Gaussian noise function).

- For all $\alpha \neq \beta \in \mathbb{N}^{\mathbb{N}}$ such that $|\alpha| = |\beta| = n$,

$$\overline{G_n[l_\alpha; x(t)]G_n[l_\beta; x(t)]}^t = 0 \tag{5.3.4}$$

Proof. For the first two items we refer to the book [Sch80]. For the third item, the proof given in the book is convoluted, and introduces an unnecessary notion. [2] Let us reformulate the proof concisely.

- If $\mathrm{supp}(\alpha) \neq \mathrm{supp}(\beta)$, let $i_0 \in \mathrm{supp}(\beta) \setminus \mathrm{supp}(\alpha)$. Denote e_{i_0} the sequence equal to 1 at i_0 and to 0 at all other entries.

$$G_n[l_\alpha; x(t)]\ G_n[l_\beta; x(t)] = G_n[l_\alpha; x(t)]\ G_{n-\beta_{i_0}}\left[l_\beta/l_{i_0}^{\beta_{i_0}}; x(t)\right]\ G_{\beta_{i_0}}\left[l_{i_0}^{\beta_{i_0}}; x(t)\right] \tag{5.3.5}$$

$$= G_{n+\beta_{i_0}}\left[l_\alpha l_{i_0}^{\beta_{i_0}}; x(t)\right]\ G_{n-\beta_{i_0}}\left[l_\beta/l_{i_0}^{\beta_{i_0}}; x(t)\right] \tag{5.3.6}$$

  (The notation is justified since $|\alpha + \beta_{i_0}e_{i_0}| = n + \beta_{i_0}$ and $|\beta - \beta_{i_0}e_{i_0}| = n - \beta_{i_0}$.) Since $n + \beta_{i_0} \neq n - \beta_{i_0}$, the time average of this product of G-functionals is zero.

- If $\mathrm{supp}(\alpha) = \mathrm{supp}(\beta)$ and $\alpha \neq \beta$, let i_0 such that $\alpha_{i_0} < \beta_{i_0}$.

$$G_n[l_\alpha; x(t)]\ G_n[l_\beta; x(t)] \tag{5.3.7}$$

$$= G_{n-\alpha_{i_0}}\left[l_\alpha/l_{i_0}^{\alpha_{i_0}}; x(t)\right]\ G_{\alpha_{i_0}}\left[l_{i_0}^{\alpha_{i_0}}; x(t)\right]\ \cdot\ G_{n-\beta_{i_0}}\left[l_\beta/l_{i_0}^{\beta_{i_0}}; x(t)\right]\ G_{\beta_{i_0}}\left[l_{i_0}^{\beta_{i_0}}; x(t)\right] \tag{5.3.8}$$

$$= G_{n-\alpha_{i_0}+\beta_{i_0}}\left[l_\alpha l_{i_0}^{\beta_{i_0}-\alpha_{i_0}}; x(t)\right]\ \cdot\ G_{n-\beta_{i_0}+\alpha_{i_0}}\left[l_\beta l_{i_0}^{\alpha_{i_0}-\beta_{i_0}}; x(t)\right] \tag{5.3.9}$$

  Since $n - \alpha_{i_0} + \beta_{i_0} \neq n - \beta_{i_0} + \alpha_{i_0}$, the time average of this product of G-functionals is zero.

$\square$

[2] The proof of the third item is done in [Sch80, section 20.1] by introducing objects called "Q-polynomials". This notion does not seem to add anything to the discussion. They are not used anywhere outside of that book (the term "q-polynomial" exists but means something completely different).

In the previous section, we noted that any $k_n \in L^2_{\text{Sym}}(\mathbb{R}^n)$ could be expanded as a combination of the family $\left(\frac{d_{\overline{\boldsymbol{a}}}}{n!}\sum_{\sigma\in\mathfrak{S}_n} l_{\overline{\boldsymbol{a}}}(\boldsymbol{\tau}_\sigma)\right)_{\overline{\boldsymbol{a}}} \subset L^2_{\text{Sym}}(\mathbb{R}^n)$, but that the expansion was not orthogonal. In the indexing convention using sequences $\alpha \in \mathbb{N}^{\mathbb{N}}$ with $|\alpha| = n$, that family is written simply as: $\left(\binom{n}{\alpha} l_\alpha\right)_\alpha$.

The third item of the proposition just above essentially says that, even though the family $(l_\alpha)_\alpha$ is not orthogonal in $L^2_{\text{Sym}}(\mathbb{R}^n)$, the corresponding family of G-functionals $(G_n[l_\alpha; x])_\alpha$ is orthogonal in $\mathcal{Y} = L^2_{\text{avg}}(\mathbb{R})$ for any WGN x.

5.3.2 Sanity-check: norm of a G-functional using the decomposition

We can do a sanity-check that the norm of $G_n[k_n; x(t)]$ as calculated by this decomposition, is the same as the one announced at the beginning of the chapter: $\overline{G_n[k_n; x(t)]^2}^t = n!A^n \|k_n\|^2_{L^2}$.

By the decomposition shown in the previous section,

$$G_n[k_n; x(t)] = \sum_{|\alpha|=n} \binom{n}{\alpha} c_\alpha \; G_n[l_\alpha; x(t)] \tag{5.3.10}$$

$$\overline{G_n[k_n; x(t)]^2} = \sum_{|\alpha|=n}\sum_{|\beta|=n} c_\alpha \binom{n}{\alpha} c_\beta \binom{n}{\beta} \; \overline{G_n[l_\alpha; x(t)] G_n[l_\beta; x(t)]} \tag{5.3.11}$$

$$= \sum_{|\alpha|=n} c_\alpha^2 \binom{n}{\alpha}^2 \; \overline{G_n[l_\alpha; x(t)]^2} \tag{5.3.12}$$

$$= \sum_{|\alpha|=n} c_\alpha^2 \binom{n}{\alpha}^2 \; \overline{\prod_{i=0}^{\infty} G_{\alpha_i}\left[l_i^{\alpha_i}; x(t)\right]^2} \tag{5.3.13}$$

By the remarkable properties of G-functionals, those factors can be expressed using Hermite polynomials and $z_i(t) = (l_i * x)(t)$:

$$\overline{G_n[k_n; x(t)]^2} = \sum_{|\alpha|=n} c_\alpha^2 \binom{n}{\alpha}^2 \; \overline{\prod_{i=0}^{\infty} H_{\alpha_i}[z_i(t)]^2} \tag{5.3.14}$$

Now, by the properties of Gaussian noise functions, and orthonormality of the l_i,

$$\overline{\prod_j z_{m_j}^{k_j}(t)} = \prod_j \overline{z_{m_j}^{k_j}(t)} \tag{5.3.15}$$

In other words, not only is $z_i(t)$ interpretable as a quasi-ergodic ensemble for each i, but the family $(z_i(t))_{i\in\mathbb{N}}$ consists of statistically independent ensembles. From this statistical interpretation, morally it follows that

$$\overline{\prod_{i=0}^{\infty} H_{\alpha_i}[z_i(t)]^2} = \prod_{i=0}^{\infty} \overline{H_{\alpha_i}[z_i(t)]^2} \tag{5.3.16}$$

More prosaically, the above equality follows simply by expanding the product on both sides, using that the product of averages is the average of products of monomials $z_{m_j}^{k_j}$, and identifying each term (recall that expanding the product poses no difficulty since there are only a finite number of factors different from 1, since $|\alpha| = n$).

Moreover, for each i, $z_i(t) = (l_i * x)(t)$ is a Gaussian noise function with autocorrelation $\phi_{z_i z_i}(\tau) = A\phi_{l_i l_i}(\tau)$ and variance $\phi_{z_i z_i}(0) = A \int_{\mathbb{R}} dt \; l_i(t) l_i(t) = A$. So by quasi-ergodicity,

$$\overline{H_{\alpha_i}[z_i(t)]^2} = \mathbb{E}_{\zeta\sim\mathcal{N}(0,A)}[H_{\alpha_i}(\zeta)^2] = \alpha_i! A^{\alpha_i} \tag{5.3.17}$$

So, putting everything together,

$$\overline{G_n[k_n;x(t)]^2}^t = \sum_{|\alpha|=n} c_\alpha^2 \binom{n}{\alpha}^2 \prod_{i=0}^{\infty} \overline{H_{\alpha_i}[z_i(t)]^2}^t \tag{5.3.18}$$

$$= \sum_{|\alpha|=n} c_\alpha^2 \binom{n}{\alpha}^2 \prod_{i=0}^{\infty} \alpha_i! A^{\alpha_i} \tag{5.3.19}$$

$$= \sum_{|\alpha|=n} c_\alpha^2 \binom{n}{\alpha}^2 A^n k_1!...k_N! \tag{5.3.20}$$

$$= n!A^n \sum_{|\alpha|=n} c_\alpha^2 \binom{n}{\alpha} \tag{5.3.21}$$

since $\binom{n}{\alpha} = \frac{n!}{k_1!...k_N!}$ by definition. So, in conjunction with $\|k_n\|_{L^2}^2 = \sum_{|\alpha|=n} \binom{n}{\alpha} c_\alpha^2$ from the previous section, we indeed find

$$\overline{G_n[k_n;x(t)]^2}^t = n!A^n \|k_n\|_{L^2}^2 \tag{5.3.22}$$

as expected.

5.4 Approximation properties

One can show that Wiener series are universal approximators for the $L^2_{\mathbb{P}_x}$ norm, i.e for the squared-average error norm over inputs drawn from a WGN process. Showing that result, or even stating it rigorously, require a deeper understanding of stochastic process theory than this thesis has.

For an informal discussion of the class of systems that are arbitrarily-well represented as a Wiener series (the "Wiener class of systems"), see [Sch80, section 15.6]. For a rigorous discussion, we refer to [PP77] and [PP78]; as a sample of those papers' results, we reproduce one of their main theorems below.

Theorem 5.10 ([PP77, theorem 2], informally restated)**.** Let $\mathbb{P}_x$ the distribution of the stochastic process associated to WGN functions over $[0,1]$.

Denote $L^2_{\mathbb{P}_x}$ the set of scalar-valued functionals F such that $\|F\|_{2\mathbb{P}_x} := \sqrt{\mathbb{E}_x |F[x]|^2} < \infty$ equipped with the inner product $\langle F, G\rangle = \mathbb{E}_x F[x]G[x]$.

Then the set of scalar Wiener G-functionals $\{G_n[k_n;\cdot](0);\ n \in \mathbb{N}, k_n \in L^2(\mathbb{R}^n)\}$ is a complete set of $L^2_{\mathbb{P}_x}$, i.e its linear span is dense in $L^2_{\mathbb{P}_x}$. That is, any $F \in L^2_{\mathbb{P}_x}$ is ε-close in $L^2_{\mathbb{P}_x}$ norm to some finite-order ("scalar") Wiener series.

In fact for any ONB $(l_a)_{a\in\mathbb{N}}$ of $L^2(\mathbb{R})$, $\{G_n[l_\alpha;\cdot](0);\ n \in \mathbb{N}, \alpha \in \mathbb{N}^{\mathbb{N}} \text{ s.t } |\alpha| = n\}$ is a complete orthonormal set of $L^2_{\mathbb{P}_x}$.

Chapter 6

Systems parametrized by kernel functions

In the three explicit models we have seen (kernel-integral representation of linear systems, Volterra series, Wiener series), the systems are characterized by (sequences of) so-called kernel functions. More generally, we can consider systems that explicitly derive from kernel functions: $\mathbb{S} = \{\Phi_k, k \in \mathcal{K}\}$, for some function space $\mathcal{K}$ and some linear map $k \mapsto \Phi_k$. Studying the metric entropy of these systems morally reduces to the metric entropy of the kernel functions, under certain transferability conditions, as we discuss abstractly in this chapter.

6.1 Specializing the framework

For concreteness we focus on the first variant of the framework (systems defined on X with worst-case error $\|\cdot\|_{\infty X}$), but the reasoning extends straightforwardly to the other variants. That is, we suppose given

- X a set of input signals;
- $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$ a normed vector space of output signals;
- $\mathbb{S}$ a vector space of systems $S : X \to \mathcal{Y}$;
- $\mathbb{S}_+ \subset \mathbb{S}$ a set of systems of interest.

And we are interested in estimating the metric entropy $\log N_\varepsilon(\mathbb{S}, \|\cdot\|_{\infty X})$ for the worst-case error norm $\|S\|_{\infty X} = \sup_{x \in X} \|S[x]\|_{\mathcal{Y}}$.

6.1.1 The induced norm

Suppose that $\mathbb{S}$ is of the form

$$\mathbb{S} = \{\Phi_k;\ k \in \mathcal{K}\} \tag{6.1.1}$$

for some normed space $(\mathcal{K}, \|\cdot\|_{\mathcal{K}})$ and some linear map $\Phi : k \mapsto \Phi_k$.

Proposition 6.1. If Φ is injective then it defines an isomorphism between $\mathcal{K}$ and $\mathbb{S}$, and an isometry when $\mathbb{S}$ is equipped with the induced norm $\|S\|_{\mathbb{S}} = \left\|\Phi^{-1}(S)\right\|_{\mathcal{K}}$.

So if Φ is a bijection, the situation is considerable simpler; but if not, we can still say the following.

Proposition 6.2. $\mathbb{S}$ is isomorphic to $\mathcal{K}/(\operatorname{Ker}\Phi)$ and can be equipped with the induced semi-norm

$$\|S\|_{\mathbb{S}} = \inf_{k \in \mathcal{K}; \Phi_k = S} \|k\|_{\mathcal{K}} \tag{6.1.2}$$

Proposition 6.3. Suppose $\Phi : \mathcal{K} \to \mathbb{S}$ is continuous when $\mathbb{S}$ is equipped with the topology of pointwise convergence; that is,

$$\left[k_{(n)} \xrightarrow{n} k_{(\infty)} \text{ in } \mathcal{K}\right] \implies \left[\forall x \in X,\ \Phi_{k_{(n)}}[x] \xrightarrow{n} \Phi_{k_{(\infty)}}[x] \text{ in } \mathcal{Y}\right] \tag{6.1.3}$$

Then $\|\cdot\|_{\mathbb{S}}$ is positive definite, i.e is a norm.

If in addition $\mathcal{K}$ is a Banach space, then $(\mathbb{S}, \|\cdot\|_{\mathbb{S}})$ is also a Banach space.

Proof. Suppose $\|S\|_{\mathbb{S}} = \inf_{k; S=\Phi_k} \|k\|_{\mathcal{K}} = 0$. For each n, let $k_{(n)}$ such that $S = \Phi_{k_{(n)}}$ and $\left\|k_{(n)}\right\|_{\mathcal{K}} \leq 1/n$. $k_{(n)} \to 0$ so for all $x \in X$, $S[x] = \Phi_{k_{(n)}}[x] \to 0$, so S is identically zero.

Since $(\mathbb{S}, \|\cdot\|_{\mathbb{S}}) \simeq \mathcal{K}/(\operatorname{Ker}\Phi)$, assuming $\mathcal{K}$ is a Banach it suffices to show that $\operatorname{Ker}\Phi$ is closed.

Let $k_{(n)}$ a sequence of elements of $\operatorname{Ker}\Phi$ that converges in $\mathcal{K}$ to some $k_{(\infty)}$. To show that $k_{(\infty)} \in \operatorname{Ker}\Phi$, let $x \in X$. Since Φ is continuous with respect to the topology of pointwise convergence then by definition

$$\Phi_{k_{(\infty)}}[x] = \lim_{n\to\infty} \Phi_{k_{(n)}}[x] = 0 \tag{6.1.4}$$

Which concludes the proof. □

6.2 Transferring coverings and packings

Under certain *transferability conditions*, we can translate ε-coverings in the kernel function space $\mathcal{K}$ to ε-coverings in $\mathbb{S}$.

Denote $\|\cdot\|_{\mathcal{K}/(\operatorname{Ker}\Phi)}$ the semi-norm over $\mathcal{K}$ defined by

$$\|k\|_{\mathcal{K}/(\operatorname{Ker}\Phi)} = \inf_{k'; \Phi_k = \Phi_{k'}} \|k'\|_{\mathcal{K}} \tag{6.2.1}$$

Proposition 6.4 (Upper-bounds (transferring coverings))**.** If $\Phi : (\mathcal{K}, \|\cdot\|_{\mathcal{K}}) \to (\mathbb{S}, \|\cdot\|_{\infty X})$ is continuous with operator norm M, i.e

$$\forall k \in \mathcal{K}, \forall x \in X,\ \ \|\Phi_k[x]\|_{\mathcal{Y}} \leq M\, \|k\|_{\mathcal{K}} \tag{6.2.2}$$

then for $\mathbb{S}_+ = \Phi(\mathcal{K}_+)$,

$$N_{M\delta}(\mathbb{S}_+, \|\cdot\|_{\infty X}) \leq N_\delta(\mathcal{K}_+, \|\cdot\|_{K/(\operatorname{Ker}\Phi)}) \tag{6.2.3}$$

Proof. Direct consequence of the useful facts on metric entropy in section 1.1.

To be more explicit: if $(k_1, ..., k_p)$ is a δ-covering of $\mathcal{K}_+$ for the semi-norm $\|\cdot\|_{\mathcal{K}/(\operatorname{Ker}\Phi)}$, i.e satisfies

$$\forall k \in \mathcal{K}_+, \exists i \leq p;\ \ \|k - k_i\|_{K/(\operatorname{Ker}\Phi)} \leq \delta \tag{6.2.4}$$

then letting $S_i = \Phi_{k_i}$,

$$\forall S = \Phi_k \in \mathbb{S}_+, \exists i \leq p;\ \ \|S - S_i\|_{\infty X} = \|\Phi_{k-k_i}\|_{\infty X} \leq M \cdot \inf_{h'; \Phi_{h'} = \Phi_{k-k_i}} \|h'\|_{\mathcal{K}} \tag{6.2.5}$$

$$= M \cdot \|k - k_i\|_{\mathcal{K}/(\operatorname{Ker}\Phi)} \ \leq M \cdot \delta \tag{6.2.6}$$

i.e $(S_1, ..., S_p)$ is a $M\delta$-covering of $\mathbb{S}_+$. □

Proposition 6.5 (Lower-bounds (transferring packings))**.** If "injectivized" Φ is bounded away from 0 by a positive constant m, i.e

$$\forall k \in \mathcal{K}, \exists x \in X;\ \ \|\Phi_k[x]\|_{\mathcal{Y}} \geq m\, \|k\|_{\mathcal{K}/(\operatorname{Ker}\Phi)} \tag{6.2.7}$$

then for $\mathbb{S}_+ = \Phi(\mathcal{K}_+)$,

$$M_{m\varepsilon}(\mathbb{S}_+, \|\cdot\|_{\infty X}) \geq M_\varepsilon(\mathcal{K}_+, \|\cdot\|_{\mathcal{K}/(\operatorname{Ker}\Phi)}) \tag{6.2.8}$$

Proof. Direct consequence of the useful facts on metric entropy in section 1.1.

To be more explicit: if $(k_1, ..., k_p)$ is a δ-packing of $\mathcal{K}_+$ for the semi-norm $\|\cdot\|_{\mathcal{K}/(\mathrm{Ker}\,\Phi)}$, i.e satisfies

$$\forall i \neq j, \|k_i - k_j\|_{\mathcal{K}/(\mathrm{Ker}\,\Phi)} > \delta \tag{6.2.9}$$

then letting $S_i = \Phi_{k_i}$,

$$\forall i \neq j, \ \sup_x \|S_i[x] - S_j[x]\|_{\mathcal{Y}} = \sup_x \left\|\Phi_{k_i - k_j}[x]\right\|_{\mathcal{Y}} \tag{6.2.10}$$

$$\geq m \cdot \|k_i - k_j\|_{\mathcal{K}/(\mathrm{Ker}\,\Phi)} \ > m \cdot \delta \tag{6.2.11}$$

i.e $(S_1, ..., S_p)$ is a $m\delta$-packing of $(\mathbb{S}_+, \|\cdot\|_{\infty X})$. □

Hence, our problem reduces simply to estimating metric entropy of sets in the function space $\mathcal{K}$ or $\mathcal{K}/(\mathrm{Ker}\,\Phi)$.

6.3 Example: Volterra series

6.3.1 Upper bounds

Denote $\Phi_{k_n}[x](t) = \int_{\mathbb{R}^n} d\boldsymbol{\tau}\ k_n(\boldsymbol{\tau}) x(t-\tau_1)...x(t-\tau_n)$ the Volterra monomial with Volterra kernel k_n.

As discussed in section 4.1, for each order $n \in \mathbb{N}$ we have the upper-bound

$$\|\Phi_{k_n}[x]\|_{\mathcal{Y}} \leq \|k_n\|_{\mathcal{K}_n} \|x\|_{\mathcal{X}}^n \tag{6.3.1}$$

and for the infinite-order case,

$$\|\Phi_k[x]\|_{\mathcal{Y}} \leq \sum_{n=0}^{\infty} \|k_n\|_{\mathcal{K}_n} \|x\|_{\mathcal{X}}^n = f_k(\|x\|_{\mathcal{X}}) \tag{6.3.2}$$

Assume $X \subset B_\rho^{(\mathcal{X})}$ for some $0 < \rho < R$ where R is the least radius of convergence of the f_k's, so that the specialized framework of the first section applies.

- $\Phi : (\mathcal{K}, \|\cdot\|_{\mathcal{K}}) \to (\mathbb{S}, \|\cdot\|_{\infty X})$ is continuous with operator norm bounded by $f_k(\rho)$.
- In particular $\Phi : \mathcal{K} \to \mathbb{S}$ is continuous when $\mathbb{S}$ is equipped with the topology of pointwise convergence.
- $\mathcal{K}/(\mathrm{Ker}\,\Phi)$ is isomorphic to the subset of $\mathcal{K}$ consisting of symmetric functions.

6.3.2 Lower bounds

Here is an attempt to satisfy the lower-bound transferability condition; unfortunately it is not satisfactory, and I haven't found any workaround. Recall that the condition is:

$$\forall k \in \mathcal{K}, \exists x \in X; \ \|\Phi_k[x]\|_{\mathcal{Y}} \geq m \|k\|_{\mathcal{K}/(\mathrm{Ker}\,\Phi)} \tag{6.3.3}$$

Check that it can be weakened to

$$\forall k \in \Delta\mathcal{K}_+, \exists x \in X; \ \|\Phi_k[x]\|_{\mathcal{Y}} \geq m \|k\|_{\mathcal{K}/(\mathrm{Ker}\,\Phi)} \tag{6.3.4}$$

where $\Delta\mathcal{K}_+ := \{k_a - k_b;\ k_a, k_b \in \mathcal{K}_+\}$.

Consider quadratic Volterra monomials (with the L^∞ norm for both input and output):

$$\mathbb{S} = \left\{ \Phi_k : \begin{bmatrix} L^\infty(\mathbb{R}) \to L^\infty(\mathbb{R}) \\ x(t) \mapsto \iint_{\mathbb{R}^2} d\tau_1 d\tau_2 k(\tau_1, \tau_2) x(t-\tau_1) x(t-\tau_2) \end{bmatrix},\ k(\tau_1, \tau_2) \in L^1_{\mathrm{Sym}}(\mathbb{R}^2) \right\} \tag{6.3.5}$$

Let $k \in \Delta\mathcal{K}_+$, we want a lower-bound on $\|\Phi_k[x]\|_{\mathcal{Y}}$. Since $\mathcal{Y} = L^\infty(\mathbb{R})$, we can simplify the search by considering

$$\|\Phi_k[x]\|_{\mathcal{Y}} \geq |\Phi_k[x](t=0)| = \left| \iint_{\mathbb{R}^2} d\boldsymbol{\tau} k(\boldsymbol{\tau}) x(-\tau_1) x(-\tau_2) \right| \tag{6.3.6}$$

(Assuming X is time-invariant, this simplification is without loss of generality, as explained in section 4.4.)

Suppose the function $\mathrm{sign}(k(\tau_1, \tau_2))$ is separable, i.e there exist ϕ_1, ϕ_2 such that $\mathrm{sign}(k(\tau_1, \tau_2)) = \phi_1(\tau_1)\phi_2(\tau_2)$. Then by symmetry of k, necessarily $\phi_1 = \phi_2 =: \phi$, and ϕ has values in $\{-1, +1\}$.

Evaluating the system at the input $x(t) = \phi(-t)$, we get

$$\iint_{\mathbb{R}^2} d\boldsymbol{\tau} k(\boldsymbol{\tau}) x(-\tau_1) x(-\tau_2) = \iint_{\mathbb{R}^2} d\boldsymbol{\tau} \, |k(\boldsymbol{\tau})| \; \mathrm{sign}(k(\boldsymbol{\tau}))\phi(\tau_1)\phi(\tau_2) \tag{6.3.7}$$

$$= \iint_{\mathbb{R}^2} d\boldsymbol{\tau} \, |k(\boldsymbol{\tau})| \; \mathrm{sign}(k(\boldsymbol{\tau}))^2 \; = \|k\|_{L^1} \tag{6.3.8}$$

so that

$$\|\Phi_k[x]\|_{\mathcal{Y}} \geq |\Phi_k[x](t=0)| = \|k\|_{L^1} \tag{6.3.9}$$

In other words, we verified the condition with $m = 1$ and $x(t) = \phi(-t)$. The two assumptions used in our derivation are that

1. The function $\mathrm{sign}(k(\tau_1, \tau_2))$ is separable as $\phi(\tau_1)\phi(\tau_2)$;
2. The function ϕ is in X.

The second assumption is quite reasonable, for example if $X \supset B_r^{(L^\infty(\mathbb{R}))}$ for some $r > 0$ then it clearly holds (if $r < 1$, evaluate at $x(t) = r\ \phi(-t)$ instead).

The first assumption is much more questionable however. Effectively we are requiring that the sign of $k(\tau_1, \tau_2)$ is "by block", as in Figure 6.1.

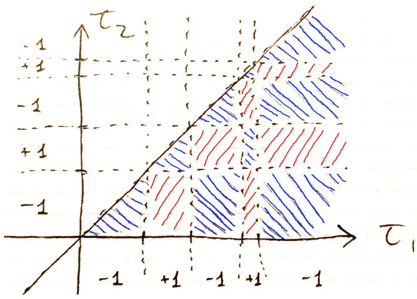


Figure 6.1: A separable sign function of a quadratic Volterra kernel. $\mathrm{sign}(k(\tau_1, \tau_2)) = \phi(\tau_1)\phi(\tau_2)$ is positive on the blue areas and negative on the red areas. Only the bottom-right half-space is represented since the function is symmetric.

A reasonable sufficient condition for this to happen is to assume $k(\tau_1, \tau_2)$ itself to be separable, i.e $\exists \psi;\ k(\tau_1, \tau_2) = \psi(\tau_1)\psi(\tau_2)$. However, this is not a reasonable assumption at all: if we suppose that all $k_a - k_b \in \Delta\mathcal{K}_+$ are of the form $(k_a - k_b)(\tau_1, \tau_2) = \phi(\tau_1)\phi(\tau_2)$ for some ϕ, then for any $k_c \in \mathcal{K}_+$ we also must have

$$(k_a - k_b)(\tau_1, \tau_2) = \phi(\tau_1)\phi(\tau_2) \tag{6.3.10}$$

$$(k_b - k_c)(\tau_1, \tau_2) = \psi(\tau_1)\psi(\tau_2) \tag{6.3.11}$$

$$(k_a - k_c)(\tau_1, \tau_2) = \phi(\tau_1)\phi(\tau_2) + \psi(\tau_1)\psi(\tau_2) \tag{6.3.12}$$

It is simple to check that the only way for this last function to also be separable is if $\psi \propto \phi$. In other words, necessarily $\mathcal{K}_+$ consists of multiples of a single (separable) Volterra kernel k_0. So this case is trivial and uninteresting.

Chapter 7

Continuous systems over compact input sets

7.1 Totally bounded sets of continuous systems: Banach-valued Arzela-Ascoli theorem

For (X,d) a compact metric space and $\mathcal{Y}$ a Banach space, denote $C(X;\mathcal{Y})$ the space of continuous mappings from X to $\mathcal{Y}$ equipped with the sup norm $\|S\|_{\infty X} = \sup_{x\in X}\|S[x]\|_{\mathcal{Y}}$. In this section we recall the conditions for a subset $\mathbb{S}_+$ of $C(X;\mathcal{Y})$ to be totally bounded.

Totally bounded sets of continuous systems are characterized by the Arzela-Ascoli theorem. We already presented that result for real-valued continuous functions over a real compact domain, in section 2.2. Let us nonetheless write out again the definitions for the slightly more general setting of arbitrary compact metric domain and Banach output space, to fix ideas and notation.

Definition 7.1. Let (X,d) a metric space, $\mathcal{Y}$ a Banach space, and $\mathbb{S}_+$ a set of systems $S : X \to \mathcal{Y}$.

$\mathbb{S}_+$ is *equicontinuous at point* $x \in X$ if $\forall\varepsilon > 0, \exists\delta_x > 0;\ \forall S \in \mathbb{S}_+, \forall x' \in X,\ d(x,x') \le \delta_x \implies \|S[x] - S[x']\|_{\mathcal{Y}} \le \varepsilon$. It is *pointwise equicontinuous* if equicontinuous every point.

$\mathbb{S}_+$ is *uniformly equicontinuous* if $\forall\varepsilon > 0, \exists\delta > 0;\ \forall S \in \mathbb{S}_+, \forall x,x' \in X,\ d(x,x') \le \delta \implies \|S[x] - S[x']\|_{\mathcal{Y}} \le \varepsilon$.

Definition 7.2. Let $S : X \to \mathcal{Y}$.

- The *modulus of continuity of* S *at point* $x \in X$ is $\omega_x(S;\delta) = \sup_{x'\in X:d(x,x')\le\delta}\|S[x] - S[x']\|_{\mathcal{Y}}$
- The *modulus of (uniform) continuity* of S is $\omega(S;\delta) = \sup_{x,x'\in X;d(x,x')\le\delta}\|S[x] - S[x']\|_{\mathcal{Y}}$

Note that S is continuous at $x \in X$ if and only if $\lim_{\delta\to0}\omega_x(S;\delta) = 0$, and that S is uniformly continuous if and only if $\lim_{\delta\to0}\omega(S;\delta) = 0$.

Let $\mathbb{S}_+ \subset C(X;\mathcal{Y})$.

- The *modulus of equicontinuity of* $\mathbb{S}_+$ *at point* $x \in X$ is $\omega_x(\mathbb{S}_+;\delta) = \sup_{S\in\mathbb{S}_+}\omega_t(S;\delta)$
- The *modulus of (uniform) equicontinuity* of $\mathbb{S}_+$ is $\omega(\mathbb{S}_+;\delta) = \sup_{S\in\mathbb{S}_+}\omega(S;\delta)$

Note that $\mathbb{S}_+$ is equicontinuous at $x \in X$ if and only if $\lim_{\delta\to0}\omega_x(\mathbb{S}_+;\delta) = 0$, and that $\mathbb{S}_+$ is uniformly equicontinuous if and only if $\lim_{\delta\to0}\omega(\mathbb{S}_+;\delta) = 0$.

Proposition 7.1. Let $L > 0$. $S \in C(X;\mathcal{Y})$ is called *L-lipschitz* if $\forall x,x' \in X, \|S[x] - S[x']\|_{\mathcal{Y}} \le L\ d(x,x')$. S is L-lipschitz if and only if

$$\forall\delta > 0,\ \omega(S;\delta) \le L\delta \tag{7.1.1}$$

Theorem 7.2 (Arzela-Ascoli for Banach-valued functions)**.** Suppose (X, d) is a compact metric space. Then $C(X; \mathcal{Y})$ equipped with the sup norm $\|\cdot\|_{\infty X}$ is a Banach, and

1. A subset $\mathbb{S}_+$ of $C(X; \mathcal{Y})$ is pointwise equicontinuous if and only if uniformly equicontinuous.
2. $\mathbb{S}_+$ is relatively compact in $C(X; \mathcal{Y})$ if and only if it is (pointwise or uniformly) equicontinuous and the set $\mathbb{S}_+(x) = \{S[x];\ S \in \mathbb{S}_+\}$ is relatively compact in $\mathcal{Y}$ for each $x \in X$.
3. $\mathbb{S}_+$ is relatively compact in $C(X; \mathcal{Y})$ if and only if it is (pointwise or uniformly) equicontinuous and the set $\mathbb{S}_+(X) = \{S[x]; S \in \mathbb{S}_+, x \in X\}$ is relatively compact in $\mathcal{Y}$.

Remark 7.1. In comparison to the real-valued case, pointwise equiboundedness of $\mathbb{S}_+$ is replaced by the "pointwise equicompactness" assumption that each $\mathbb{S}_+(x)$ is relatively compact. That is a strictly stronger condition, since relatively compact sets are bounded but the converse is false for infinite-dimensional $\mathcal{Y}$ (Riesz theorem).

The third item essentially says that, just as the equiboundedness requirement in the real-valued Arzela-Ascoli theorem could be understood indifferently as pointwise or uniform, so can the "equicompactness" requirement in the Banach-valued version.

Proof. The first item follows by the same proof as for the real-valued analog in section 2.2.

The second item is proved in "Real and Functional Analysis" (Ch. III.3) by Serge Lang.

The third item follows from the second one and the lemma just below. □

Lemma 7.3. [1] Suppose (X, d) is a compact metric space. Let $\mathbb{S}_+$ *be* a (pointwise or uniformly) equicontinuous subset of $C(X; \mathcal{Y})$. The two conditions are equivalent:

(*i*) The set $\mathbb{S}_+(x) = \{S[x];\ S \in \mathbb{S}_+\}$ is relatively compact in $\mathcal{Y}$ for each $x \in X$.

(*ii*) The set $\mathbb{S}_+(X) = \{S[x]; S \in \mathbb{S}_+, x \in X\}$ is relatively compact in $\mathcal{Y}$.

Proof. The implication $(ii) \implies (i)$ is obvious, since $\mathbb{S}_+(x) \subset \mathbb{S}_+(X)$.

Now assume (i) holds. In particular this means that $\mathbb{S}_+$ is relatively compact in $C(X; \mathcal{Y})$, by the second item of the theorem above (recall that only its third item remains to be proved).

The mapping $\begin{bmatrix} X \times C(X; \mathcal{Y}) \to \mathcal{Y} \\ \hfill (x, S) \mapsto S[x] \end{bmatrix}$ is continuous, $X \times C(X; \mathcal{Y})$ being equipped with the max metric: $d((x, S), (x', S')) = \max\left(d_X(x, x'), \|S - S'\|_{\infty X}\right)$. Indeed, let any $(x_0, S_0) \in X \times C(X; \mathcal{Y})$ and $\varepsilon > 0$. Since S_0 is uniformly continuous there exists δ such that $\omega(S_0; \delta) \leq \varepsilon$. For any (x, S) such that $\max\left(d_X(x, x_0), \|S - S_0\|_{\infty X}\right) \leq \min(\varepsilon, \delta)$,

$$\|S - S_0\|_{\infty X} \leq \varepsilon \tag{7.1.2}$$
$$d_X(x, x_0) \leq \delta \quad \text{so that} \quad \|S_0[x] - S_0[x_0]\|_{\mathcal{Y}} \leq \omega(S_0; \delta) \tag{7.1.3}$$

and so

$$\begin{aligned} \|S[x] - S_0[x_0]\|_{\mathcal{Y}} &\leq \|S[x] - S_0[x]\|_{\mathcal{Y}} + \|S_0[x] - S_0[x_0]\|_{\mathcal{Y}} && (7.1.4)\\ &\leq \|S - S_0\|_{\infty X} + \omega(S_0; \delta) && (7.1.5)\\ &\leq 2\varepsilon && (7.1.6) \end{aligned}$$

Thus the mapping $(x, S) \mapsto S[x]$ is continuous. Moreover it maps $X \times \mathbb{S}_+$ to $\mathbb{S}_+(X)$, by definition. So to conclude that the latter is relatively compact, it suffices to check that the former is. And indeed, $\overline{X \times \mathbb{S}_+} = \overline{X} \times \overline{\mathbb{S}_+} = X \times \overline{\mathbb{S}_+}$ is compact as a cartesian product of compacts. □

[1] The author is indebted to Thea Kosche for this lemma and its proof.

7.2 Specializing the framework

We use a slightly more detailed version of our first framework variant from section 1.3: let

- (X, d) a metric space of input signals;
- $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$ a Banach space of outputs;
- $\mathbb{S}$ a vector space of systems $S : X \to \mathcal{Y}$;
- $\mathbb{S}_+ \subset \mathbb{S}$ a set of systems of interest.

And we make the assumption that $\|S\|_{\infty X} < \infty$ for all $S \in \mathbb{S}$, without loss of generality for our purpose, which is to estimate the metric entropy $\log N_\varepsilon(\mathbb{S}_+; \|\cdot\|_{\infty X})$.

The assumptions In addition to the above, we make the following assumptions:

A1. The metric space (X, d) is compact; [2]

A2. Each $S \in \mathbb{S}_+$ is (uniformly) continuous over the compact X; in other words, $\mathbb{S}_+ \subset C(X; \mathcal{Y})$;

A3. $\mathbb{S}_+$ is (uniformly) equicontinuous over X, i.e the modulus of continuity goes to 0 uniformly over $\mathbb{S}_+$: $\lim_{\delta \to 0} \omega(\mathbb{S}_+; \delta) = \lim_{\delta \to 0} \sup_{S \in \mathbb{S}_+} \omega(S; \delta) = 0$;

A4. For each $x \in X$, $\mathbb{S}_+(x)$ is relatively compact in $\mathcal{Y}$;

A5. Each $S \in \mathbb{S}$ is continuous over X, and any continuous system can be approximated by a model in $\mathbb{S}$, arbitrarily well uniformly over X. In other words, $\mathbb{S}$ is a dense subspace of $C(X; \mathcal{Y})$.

Remark 7.2. Given A1-A2, A3-A4 simply express that $\mathbb{S}_+$ is totally bounded for the worst-case error norm $\|\cdot\|_{\infty X}$, as characterized by Arzela-Ascoli theorem. So these assumptions are very reasonable, as otherwise the metric entropy of $\mathbb{S}_+$ would be infinite for too small ε.

The goal Since $\mathbb{S}$ is dense in $C(X; \mathcal{Y})$, then as shown in section 1.1,

$$\varepsilon_n(\mathbb{S}_+; \mathbb{S}) = \varepsilon_n(\mathbb{S}_+; C(X; \mathcal{Y})) \tag{7.2.1}$$

Note that if we forgo assumption A5, since the metric of interest is always the same, we can still simply use that

$$N_{2\varepsilon}^{C(X;\mathcal{Y})}(\mathbb{S}_+) \leq M_{2\varepsilon}(\mathbb{S}_+) \leq N_\varepsilon^{\mathbb{S}}(\mathbb{S}_+) \leq M_\varepsilon(\mathbb{S}_+) \leq N_{\varepsilon/2}^{C(X;\mathcal{Y})}(\mathbb{S}_+) \tag{7.2.2}$$

$$(1/2)\varepsilon_n(\mathbb{S}_+; C(X; \mathcal{Y})) \leq \varepsilon_n(\mathbb{S}_+; \mathbb{S}) \leq 2\varepsilon_n(\mathbb{S}_+; C(X; \mathcal{Y})) \tag{7.2.3}$$

And if we assume $\mathbb{S} \subset C(X; \mathcal{Y})$, $N_\varepsilon^{C(X;\mathcal{Y})}(\mathbb{S}_+) \leq N_\varepsilon^{\mathbb{S}}(\mathbb{S}_+)$ so that

$$\varepsilon_n(\mathbb{S}_+; C(X; \mathcal{Y})) \leq \varepsilon_n(\mathbb{S}_+; \mathbb{S}) \leq 2\varepsilon_n(\mathbb{S}_+; C(X; \mathcal{Y})) \tag{7.2.4}$$

With this reduction in mind, in the rest of this chapter, we turn our attention to the entropy number of $\mathbb{S}_+$ in ambient space $C(X; \mathcal{Y})$.

[2]Assumption A1 could be generalized with minimal effort to (X, d) totally bounded, since uniformly continuous mappings then have a unique continuous extension to its completion $\overline{X}$ by Cauchy-continuity. `https://en.wikipedia.org/wiki/Cauchy-continuous_function`

7.3 Evaluation operator formalism

In this section we show how to bridge our setting, described in the previous section, to the setting of operator approximation theory. Formulating that bridge precisely will allow to directly apply known results from the study of $C(X)$-valued operators in [CS90, chapter 5].

The bridge is via the Minkowski functional construction already seen in section 1.2. Basically: letting the embedding operator $I : \mathbb{S} \to C(X;\mathcal{Y})$, then $\varepsilon_n(\mathbb{S}_+; (\mathbb{S}, \|\cdot\|_{\infty X}))$ is (almost) equal to $\varepsilon_n(I(\mathbb{S}_+); C(X;\mathcal{Y}))$. Under further simplifying assumptions, the question thus reduces to estimating the entropy number of the linear operator I.

7.3.1 An alternate set of assumptions

Consider the setting where assumptions A1-A5 are replaced by B0-B5:

B0. $\mathbb{S}_+$ is convex, balanced, absorbing and bounded [3] in $(\mathbb{S}, \|\cdot\|_{\infty X})$. For simplicity, $\mathbb{S}_+$ is also closed;

B1. (same as A1) The metric space (X, d) is compact;

B2. (same as A2) $\mathbb{S}_+ \subset C(X;\mathcal{Y})$;

B3. (same as A3) $\mathbb{S}_+$ is (uniformly) equicontinuous over X;

B4. (same as A4) For each $x \in X$, $\mathbb{S}_+(x)$ is relatively compact in $\mathcal{Y}$;

B5. (same as A5) $\mathbb{S}$ is a dense subspace of $C(X;\mathcal{Y})$.

Recall that assumption B0 implies that $\|S\|_{\mathbb{S}_+} = \inf\{b > 0; S/b \in \mathbb{S}_+\}$ defines a norm on $\mathbb{R}\mathbb{S}_+ = \mathbb{S}$, and that $(\mathbb{S}, \|\cdot\|_{\mathbb{S}_+})$ is a Banach whose unit ball is $\mathbb{S}_+$.

Now consider the following linear operators.

Definition 7.3. For each $x \in X$, let the evaluation operator $E_x : \begin{bmatrix} \mathbb{S} \to \mathcal{Y} \\ S \mapsto S[x] \end{bmatrix}$. Note that by definition, $\mathbb{S}_+(x) = E_x(\mathbb{S}_+)$.

Also let the embedding operator $I : \begin{bmatrix} (\mathbb{S}, \|\cdot\|_{\mathbb{S}_+}) \to C(X;\mathcal{Y}) \\ S \mapsto S \end{bmatrix}$. Note that $\mathbb{S}_+ = I(\mathbb{S}_+)$, the image of the unit ball of $(\mathbb{S}, \|\cdot\|_{\mathbb{S}_+})$.

Then assumptions B2-B4 can be equivalently replaced by:

B2'. I is well-defined as a $C(X;\mathcal{Y})$-valued operator;

B3'. $\mathbb{S}_+$ is relatively compact. Equivalently since $\mathbb{S}_+ = I(B^{(\mathbb{S},\|\cdot\|_{\mathbb{S}_+})})$, the embedding operator $I : (\mathbb{S}, \|\cdot\|_{\mathbb{S}_+}) \to C(X;\mathcal{Y})$ is compact;

B4'. For each x, E_x is a compact operator. (Though this is already captured by B3'.)

Furthermore since $\mathbb{S}_+ = I(B^{(\mathbb{S},\|\cdot\|_{\mathbb{S}_+})})$, the entropy number of the set $\mathbb{S}_+$ in $C(X;\mathcal{Y})$ is just the entropy number of the embedding operator I

$$\varepsilon_n(\mathbb{S}_+; C(X;\mathcal{Y})) = \varepsilon_n(I : (\mathbb{S}, \|\cdot\|_{\mathbb{S}_+}) \to C(X;\mathcal{Y})) \tag{7.3.1}$$

Remark 7.3. By considering the entropy and approximation numbers of the operator I, rather than of the set $\mathbb{S}_+$, we add a layer of indirection, as well as the assumption B0 on $\mathbb{S}_+$. So aside from the purpose of directly applying results from operator approximation theory, the evaluation operator formalism presented in this subsection is pointlessly restrictive, compared to our setting with A1-A5.

[3] Note that B1-B4 imply $\mathbb{S}_+$ relatively compact in $C(X;\mathcal{Y})$ and so $\mathbb{S}_+(X)$ relatively compact in $\mathcal{Y}$, so the boundedness assumption in B0 is redundant.

In section 1.2 we mentioned that convex, balanced, absorbing and closed subsets of infinite-dimensional Banach spaces contain a neighborhood of zero and so are not totally bounded. But $(\mathbb{S}, \|\cdot\|_{\infty X})$ is precisely not assumed Banach since it is dense in $C(X;\mathcal{Y})$ (assumption B5), so $\mathbb{S}_+$ *absorbing in* $\mathbb{S}$ may still be totally bounded for $\|\cdot\|_{\infty X}$.

7.3.2 Tools from the book [CS90]

The results of [CS90, chapter 5] can be applied on the compact $C(X;\mathcal{Y})$-valued operator I, when $\mathcal{Y}=\mathbb{R}$. For ease of reference we copied below a few relevant theorems from that book.

Definition 7.4. The modulus of continuity of an operator $T : E \to C(X;\mathbb{R})$ is the "modulus of equicontinuity" of $T(B^{(E)})$, i.e

$$\omega(T;\delta) = \sup_{\|a\|_E \leq 1} \omega(Ta;\delta) \tag{7.3.2}$$

T is a compact operator if and only if $\lim_{\delta\to 0}\omega(T;\delta) = 0$.

Theorem 7.4 (Theorem 5.6.1 and (3.1.8) of [CS90])**.** (Their most general theorem.)
Let (X,d) compact metric space, E Banach space, $T : E \to C(X;\mathbb{R})$ compact operator. Then

$$e_n(T) \leq c_p \left[|\!|\!| T |\!|\!| n^{-1/p} + 2^{1/p} \sup_{1\leq k\leq n} \left(\frac{k}{n}\right)^{1/p} \omega(T;\varepsilon_k(X)) \right] \tag{7.3.3}$$

for arbitrary $p > 0$, where c_p is a known constant depending only on p.

Theorem 7.5 (Theorem 5.7.1 of [CS90])**.** (A specialization of the above theorem under a simplifying assumption, which they argue is quite common.)
Let (X,d) compact metric space, E Banach space, $T : E \to C(X;\mathbb{R})$ compact operator such that there exists $\sigma > 0, \rho \geq 1$ such that

$$\forall n \in \mathbb{N}^*, \forall k \leq n, \left(\frac{k}{n}\right)^{\sigma} \omega(T;\varepsilon_k(X)) \leq \rho\omega(T;\varepsilon_n(X)) \tag{7.3.4}$$

Then

$$e_n(T) \leq C(p,\rho,\sigma)\cdot \left[|\!|\!| T |\!|\!| n^{-1/p} + \omega(T;\varepsilon_n(X)) \right] \tag{7.3.5}$$

for arbitrary $p > 0$, where $C(p,\rho,\sigma)$ is a known constant depending only on p,ρ,σ.

Proposition 7.6 (Proposition 5.8.4 of [CS90])**.** (A further specialization of the above, which is of most interest to us.) [4]
Let X compact and convex subset of a normed space $\mathcal{X}$ with more than one point, E Banach space, $T : E \to C(X;\mathbb{R})$ compact operator with $\mathrm{rank}(T) > 1$. Then

$$e_n(T) \leq C_0 \frac{|\!|\!| T |\!|\!|}{\omega(T;\varepsilon_1(X))} \omega(T;\varepsilon_n(X)) \tag{7.3.6}$$

where C_0 is a known universal constant. (Note that $\varepsilon_1(X)$ is just the half-diameter of the compact set $X \subset \mathcal{X}$.)

For ease of reference, the constants used in those theorems are:

- $c_p = 2^7(16(2+1/p))^{1/p}$ for Banachs over the field $\mathbb{R}$ and $c_p = 2^7(32(2+1/p))^{1/p}$ for Banachs over the field $\mathbb{C}$
- $C(p,\rho,\sigma) = \begin{cases} \rho c_p 2^{1/p} \text{ for } p \leq 1/\sigma \\ \rho c_{1/\sigma} 2^{\sigma} \text{ for } p > 1/\sigma \end{cases}$
- $C_0 = 15C(1,13,1)$

[4][CS90, (5.8.11)] provides better estimates for the case where X is contained in a finite-dimensional subspace of $\mathcal{X}$ of known dimension m.

Beyond scalar-valued We would like to apply this last proposition on $T = I$, but replacing $C(X;\mathbb{R})$ by $C(X;\mathcal{Y})$ for a Banach output signal space $\mathcal{Y}$.

To extend [CS90, chapter 5] to the case of $C(X;\mathcal{Y})$ (with $\mathcal{Y}$ not necessarily $= \mathbb{R}$), adaptations are needed for:

- Proposition 5.2.1 and preceding (partition of unity),
- Theorem 5.6.1 (relies heavily on partition of unity).

Note that such an extension is essentially given by the derivations of the next section.

7.4 Simple generic bounds

We now return to the setting described at the beginning of the chapter, with the assumptions A1-A5. In this section we illustrate how to give simple bounds on the entropy number of $\mathbb{S}_+$ in $C(X;\mathcal{Y})$.

7.4.1 Scalar-valued case

In the case where $\mathcal{Y} = \mathbb{R}$, the same reasoning as in the real-domain real-valued case holds, with trivial modifications: replace $|\cdot - \cdot|$ by $d_X(\cdot,\cdot)$. To fix ideas, we state the result again with the notation of this chapter.

Proposition 7.7 ([CS90, theorem 5.6.1])**.** Let $\mathbb{S}_+$ a relatively compact subset of $C(X;\mathbb{R})$, where X is a compact metric space. The approximation number of $\mathbb{S}_+$ is bounded by

$$\forall n \in \mathbb{N}^*,\ a_{n+1}(\mathbb{S}_+; C(X;\mathcal{Y})) \leq \omega(\mathbb{S}_+; \varepsilon_n^{\mathrm{self}}(X)) \tag{7.4.1}$$

As explained in section 4.4, if we assume the systems and the set X are time-invariant and $\mathcal{Y} = L^\infty(\mathbb{R})$, then we may without loss of generality consider $\mathcal{Y} = \mathbb{R}$. So just considering the scalar-valued case already gives a usable result for such systems.

7.4.2 Approximating the outputs by linear subspaces

Proposition 7.8. Let X and $\mathbb{S}_+$ satisfying A1-A4, i.e X compact metric space and $\mathbb{S}_+ \subset C(X;\mathcal{Y})$ relatively compact. In particular, the set $\mathbb{S}_+(X) = \{S[x]; S \in \mathbb{S}_+, x \in X\}$ is relatively compact in $\mathcal{Y}$.

The approximation and Kolmogorov numbers of $\mathbb{S}_+$ are bounded by

$$\forall n, m \in \mathbb{N}^*,\ a_{nm+1}(\mathbb{S}_+; C(X;\mathcal{Y})) \leq \omega(\mathbb{S}_+; \varepsilon_n^{\mathrm{self}}(X)) + a_{m+1}(\mathbb{S}_+(X);\mathcal{Y}) \tag{7.4.2}$$

$$\forall n, m \in \mathbb{N}^*,\ d_{nm+1}(\mathbb{S}_+; C(X;\mathcal{Y})) \leq \omega(\mathbb{S}_+; \varepsilon_n^{\mathrm{self}}(X)) + d_{m+1}(\mathbb{S}_+(X);\mathcal{Y}) \tag{7.4.3}$$

Proof. Proceeding as in the proof of the scalar-valued case, let $(x_1, ..., x_n)$ a δ-covering of X for some $\delta \geq \varepsilon_n^{\mathrm{self}}(X)$, and $(\varphi_i)_{i \leq n}$ an associated partition of unity.

Now also let T a linear operator of rank at most m such that $\forall y \in \mathbb{S}_+(X),\ \ \|y - Ty\| \leq \sigma$, for some $\sigma \geq a_{m+1}(\mathbb{S}_+(X);\mathcal{Y})$.

Consider the map $S \mapsto \widehat{S} = \sum_{i=1}^n TS[x_i]\varphi_i$. This defines a linear operator of rank at most nm, and it is straightforward to show that $\forall x \in X, \left\|S[x] - \widehat{S}[x]\right\|_{\mathcal{Y}} \leq \omega(\mathbb{S}_+;\delta) + \sigma$.

Thus, $a_{nm+1}(\mathbb{S}_+; C(X;\mathcal{Y})) \leq \omega(\mathbb{S}_+;\delta) + \sigma$. Since $\omega(\mathbb{S}_+;\cdot)$ is right-continuous [CS90, proposition 5.4.1], we conclude by letting $\delta \to \varepsilon_n^{\mathrm{self}}(X)$ and $\sigma \to a_{m+1}(\mathbb{S}_+(X);\mathcal{Y})$ from the right.

The statement on the Kolmogorov number is proved analogously. $\square$

By our brief review of the relation between compacity and approximation quantities in section 1.1, note that, since $\mathbb{S}_+(X)$ is totally bounded, $\lim_{m\to\infty} d_m(\mathbb{S}_+(X);\mathcal{Y}) = 0$. However $\lim_{m\to\infty} a_m(\mathbb{S}_+(X);\mathcal{Y}) \neq 0$ a priori.

7.4.3 Taking a covering of the outputs

Proposition 7.9. Let X and $\mathbb{S}_+$ satisfying A1-A4, i.e X compact metric space and $\mathbb{S}_+ \subset C(X;\mathcal{Y})$ relatively compact. In particular, the set $\mathbb{S}_+(X) = \{S[x]; S \in \mathbb{S}_+, x \in X\}$ is relatively compact in $\mathcal{Y}$.

The entropy number of $\mathbb{S}_+$ is bounded by

$$\forall n, m \in \mathbb{N}^*,\ \varepsilon_{m^n}(\mathbb{S}_+; C(X;\mathcal{Y})) \leq \omega(\mathbb{S}_+; \varepsilon_n^{\text{self}}(X)) + \varepsilon_m(\mathbb{S}_+(X);\mathcal{Y}) \tag{7.4.4}$$

Proof. Proceeding as in the proof of the scalar-valued case, let $(x_1, ..., x_n)$ a δ-covering of X for some $\delta \geq \varepsilon_n^{\text{self}}(X)$, and $(\varphi_i)_{i \leq n}$ an associated partition of unity.

Now also let $(y_1, ..., y_m) \subset \mathcal{Y}$ a σ-covering of $\mathbb{S}_+(X)$ for some $\sigma \geq \varepsilon_m(\mathbb{S}_+(X);\mathcal{Y})$.

Consider the collection $\left\{\sum_{i=1}^n y_{f(i)}\varphi_i;\ f : [1,n] \to [1,m]\right\}$. This is a collection of cardinality m^n, which constitutes a $(\omega(\mathbb{S}_+;\delta) + \sigma)$-covering of $\mathbb{S}_+$.

Thus, $\varepsilon_{m^n}(\mathbb{S}_+; C(X;\mathcal{Y})) \leq \omega(\mathbb{S}_+;\delta) + \sigma$. Since $\omega(\mathbb{S}_+;\cdot)$ is right-continuous [CS90, proposition 5.4.1], we conclude by letting $\delta \to \varepsilon_n^{\text{self}}(X)$ and $\sigma \to \varepsilon_m(\mathbb{S}_+(X);\mathcal{Y})$ from the right. $\square$

Towards lower-bounds and tighter upper-bounds In terms of the proof techniques discussed in section 2.1, this chapter's derivations are all instances of "sampling on a δ-covering of the inputs under smoothness assumptions".

As we discussed in that chapter, much better bounds can be obtained by quantizing the joint sample sequences $(S[x_1], ..., S[x_n])$, instead of taking a uniform covering of the output set $\mathbb{S}_+(X)$. Information on those sample-sequences are also necessary to obtain lower bounds on the entropy numbers.

For an example application of that refined proof technique for the case of lipschitz-continuous systems, see [KT59, section 9.2].

Chapter 8

Directions for future work

Summary In this thesis, we discussed how to estimate metric entropy of nonlinear systems. We followed two distinct paths as explained in chapter 0: either by leveraging explicit general "parametric" models (the kernel-integral representation of linear systems, the Volterra series, the Wiener series), or by "abstractly" extending techniques for estimating metric entropy in (scalar-domain, scalar-valued) function spaces to spaces of systems.

Although those two paths are conceptually distinct, they intersect in the sense that they both required a discussion of classical results for metric entropy estimates in function spaces (for approximating the kernel functions in the "parametric" path, and for extending the proof techniques in the "abstract" path). Moreover, explicit models helped for intuition in abstract discussions.

Technical loose ends Here we list interesting technical questions that we encountered during the thesis. Most of them are not particularly crucial for our overarching motivation however, contrary to the next section.

section 1.2 Does it ever happen that $\mathbb{R}A \neq \mathbb{R}\overline{A}$, when A is a convex, balanced and bounded set and the ambient space is a Banach?

section 2.5 What are the techniques for estimating metric entropy in reproducing kernel Hilbert spaces (e.g [Küh11])? Does that setting completely capture the "truncate and quantize a frame decomposition" technique, as [NW91] and [KK08] could indicate?

section 3.2 For $\mathcal{X}$ a Banach space and X a relatively compact subset, assumed convex and balanced, under what conditions can we write $e_n(B^{(\mathcal{X}')}; (\mathcal{X}', \|\cdot\|_{\infty X})) \approx e_n(X; \mathcal{X}')$?

section 3.3 Does the abstract theory of integral linear operators lead to usable insights for metric entropy estimates of linear systems?

Can we extend the fading-memory characterization of LTI systems admitting a convolution representation, from the $C_b(\mathbb{R}) \to C_b(\mathbb{R})$ setting [BC85] to other settings e.g LTI systems from $L^p(\mathbb{R})$ to $L^r(\mathbb{R})$?

chapter 4 Relation of Volterra series theory to Carleman linearization, and to the dynamical systems framework $\frac{dx(t)}{dt} = f(x(t), u(t))$ [Ban88]?

section 4.6 Is $\overline{\mathcal{K}}_\infty(R, r)$ a Banach space?

Is it really true that any symmetric function $f(\boldsymbol{t}) \in C_{b\,\mathrm{Sym}}(\mathbb{R}^n)$ is arbitrarily well uniformly approximated by a finite sum of functions of the form $\sum_i f_i(t_1)...f_i(t_n)$, with $f_i \in C_b(\mathbb{R})$?

section 5.2 What condition on the weight $w(t)$ ensures that the set of polynomials in complete in $L^2_w(\mathbb{R})$?

section 5.4 Formal statement for the approximation properties of Wiener series [PP77]? (Requires a deeper understanding of Gaussian stochastic processes.)

section 7.4 Tighter bounds by jointly quantizing the sample sequences, as in [KT59, section 9.2], instead of uniformly covering the output set?

section B.5 Justification for the last step of the proof for the sufficient condition?

8.1 Kernel methods for nonlinear system identification

A noticeable gap in our discussion of metric entropy estimation techniques is the case of reproducing kernel Hilbert spaces (RKHS). They were only briefly mentioned in section 2.5, even though we argued that they are the natural framework to further the idea of "sampling under smoothness assumptions". So the natural next step, starting from the present thesis, would be to investigate the references mentioned in that section to see what additional insight can be drawn.

That investigation, in turn, may well lead to a third "non-parametric" path towards metric entropy estimates for nonlinear systems. That is, in the same way that kernel methods in the classical $\mathbb{R}^d \to \mathbb{R}$ setting provide a non-parametric learning framework, one may be interested in non-parametric system identification – as opposed to the "parametric" path based on identifying a functional parameter k (the convolution/Volterra/Wiener kernel function).

Kernel-based *linear* system identification, drawing from a machine learning perspective, have been the object of much attention in the past decade [Pil+14] [LCM20]. Conceptually though, non-parametric learning of a linear system is equivalent to non-parametric learning of its impulse response function, so arguably this brings nothing fundamentally new on the theory side. The nonlinear case has been much less explored; in fact googling "kernel methods for system identification" returns mostly articles implicitly concerned with linear systems. The only two relevant references we found on the topic are quite recent: [Pil18] [CP19].

So, we believe that an interesting direction for further work would be to take a kernel-based approach to the question of estimating entropy numbers for nonlinear systems. We recall that, in section 4.6, we showed how the Volterra series model could be interpreted in the context of polynomial reproducing kernel Banach spaces (RKBS). This suggests that, if a $L^p(\mathbb{R}) \to L^r(\mathbb{R})$ framework is considered, the RKBS formalism is most adapted. This also raises the following questions (ordered by decreasing expected interest):

- Non-parametric learning in RKBS – the straightforward generalization of kernel ridge regression with RKHS – is studied in [LZZ19], [WX20], [Uns20]. In the case of polynomial RKBS (of finite order, say), how does it relate to classical Volterra-series-based identification schemes [Sch81]?
- In our derivation of Volterra series as polynomial RKBS, we used that the input signals were in $\mathcal{X} = L^p(\mathbb{R})$ for $1 < p < \infty$. How can the cases $p = 1$ and $p = \infty$, and $\mathcal{X} = C_b(\mathbb{R})$, be treated?
- In our derivation of Volterra series as polynomial RKBS, we used that the systems were time-invariant and assumed L^∞ norm on the outputs (i.e the signals were viewed pointwise), so that we could reduce the discussion to scalar outputs. Could the discussion be extended to vector outputs? (This may require background theory on vector-valued reproducing kernel spaces.)
- We saw that the set of Volterra series could be viewed as a RKBS over $X \subset L^p(\mathbb{R})$ with reproducing kernel $K(x, \tilde{x}) = \sum_{n=0}^{\infty} \langle x, \tilde{x} \rangle_{L^p}$. What happens if we want instead $K(x, \tilde{x}) = \sum_{n=0}^{\infty} a_n \langle x, \tilde{x} \rangle_{L^p}$ for some positive coefficients a_n, as in [FS06, (19)]?

Appendix A

An elementary view of the sampling expansion for band-limited functions: basic standard results

An elementary view of the Nyquist-Shannon sampling expansion for band-limited square-integrable signals, part 1/3.

A.1 Introduction

In this document we discuss the (Kotelnikov-) (Whittaker-) (Nyquist-) (Shannon-) sampling theorem, its variants, and some consequences, using only elementary notions on Hilbert spaces. Namely we only require basic knowledge of the Fourier transform, and the Fourier series expansion for square-integrable signals over an interval.

As a first part, we present the well-known Nyquist-Shannon sampling theorem in its simplest form. In section A.2 we derive the formulas for the one-dimensional version. Then we explain how it relates to connected topics: the discrete-time Fourier transform (DTFT) in section A.3, and the Poisson summation formula in section A.4. In section A.5 we informally present an alternative derivation based on studying the spectrum of the discretized signal. In section A.6 we develop the analogous formulas for the multidimensional setting.

As a second part, we develop certain variants of the classical sampling expansion, known as *oversampling with design freedom.* In section B.1 we treat the one-dimensional case completely, by defining and characterizing so-called *synthesizer* functions. Still in one dimension, in section B.2 we study a particular class of practically convenient synthesizers, corresponding to the use of *window functions.* The Helms-Thomas expansion is introduced in section B.3 as a well-known example of this scheme. In section B.4 we define and characterize synthesizer functions for the multidimensional setting. Window-based synthesizers for multidimensional signals are studied in section B.5.

As a third part, we discuss techniques to bound the truncation error of (oversampled) sampling expansions. In section C.1 we briefly present the complex-analysis point of view on the sampling expansion. Bounds on truncation errors are derived in section C.2, for errors measured in L^2-norm or in L^∞-norm. Example truncation error and coefficient-sensitivity error bounds, in L^∞-norm for the Helms-Thomas expansion, are derived in section C.3.

Relation to previous work The starting points for this document were reading notes for the following papers:

- [Zam79] which uses the Helms-Thomas expansion to compute the ε-entropy of certain linear time-invariant systems.

  Some of the references therein, which provide the required results on the Helms-Thomas expansion: [Jag66] and [HT62].

- [PS96] which contains a principled discussion of other choices of window-based synthesizers (alternatives to the Helms-Thomas expansion), with the perspective of minimizing truncation and coefficient-sensitivity error.

  Some of the references therein, that make very interesting comments on the sampling expansion in general (but without the goal of controlling approximation error in mind): [Cam68], [Hig85].

There is less literature on the multidimensional sampling expansion available than one could expect. The most relevant references we could find are [Bos82], [Jer77, section IV.A], [Hig85, story Five], and [PM62], all of which are quite old.

As far as we are aware, the discussion of the possible choices for window functions in section B.2 and section B.5 is not contained in any publicly available article.

A.1.1 Summary of construction of the Fourier transform

(This subsection can be skipped without loss.)

For context, recall that the Fourier transform is typically defined as follows [Gol+19].

- It is defined as a linear operator $L^1(\mathbb{R}) \to L^\infty(\mathbb{R})$ by: $\forall f(t) \in L^1(\mathbb{R}), \hat{f}(\xi) = \int_{\mathbb{R}} dt\ f(t)e^{-i2\pi\xi t}$. Since the integrand is absolutely summable, by properties of (Lebesgue) integration,
  - This formula defines a function $\hat{f} \in L^\infty(\mathbb{R})$, and we have $\left\|\hat{f}(\xi)\right\|_{L^\infty} \leq \|f(t)\|_{L^1}$.
  - $\hat{f}$ is actually uniformly continuous (by dominated convergence). [1]
  - $\hat{f}$ vanishes at infinity: $\lim_{|\xi|\to\infty} \hat{f}(\xi) = 0$ (Riemann-Lebesgue theorem).
- For any $f \in L^1(\mathbb{R})$ such that $\hat{f} \in L^1(\mathbb{R})$, we have the inversion formula: $f(t) = \int_{\mathbb{R}} d\xi\ \hat{f}(\xi)e^{i2\pi\xi t}$. (The condition $\hat{f} \in L^1(\mathbb{R})$ is necessary for the right-hand-side even to make sense.)
- The Fourier transform operator is remarked to have a nice behaviour for $f(t) \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, in that $\hat{f}$ is then also square-integrable and that $\int_{\mathbb{R}} dt\ |f(t)|^2 = \int_{\mathbb{R}} d\xi\ \left|\hat{f}(\xi)\right|^2$. This is Plancherel's identity. [2] It is typically proved via density arguments. [Gol18, section 1.3]
  - $C_c(\mathbb{R})$ (the set of continuous functions supported on a compact) is dense in $L^2(\mathbb{R})$ for the L^2 norm.
  - $C_c^\infty(\mathbb{R})$ (the set of infinitely-differentiable functions supported on a compact) is dense in $C_c(\mathbb{R})$ for the sup norm; this is easily shown by convolution by mollifiers.
  - $C_c^\infty(\mathbb{R})$ is dense in $L^2(\mathbb{R})$ for the L^2 norm.
  - $C_c(\mathbb{R})$ and $C_c^\infty(\mathbb{R})$ are both contained in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$.

  Plancherel's identity is relatively straightforward to derive for $f(t) \in C_c^\infty(\mathbb{R})$, and transfers to functions in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ by L^2-norm density. (Simpler derivations, not involving distributions, are possible by rearranging the arguments [Gol+19, section 11.2], but messier to present.)

[1] `https://math.stackexchange.com/questions/68642/fourier-transform-is-uniformly-continuous`
[2] `https://en.wikipedia.org/wiki/Plancherel_theorem`

- The space $C_c(\mathbb{R})$ is contained in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and is dense in $L^2(\mathbb{R})$ for the L^2 norm. So the Fourier transform seen as a bounded linear operator $C_c(\mathbb{R}) \to L^2(\mathbb{R})$ (both spaces being equipped with the L^2 norm) (bounded since it's in fact isometric), can be extended by continuity to an operator $L^2(\mathbb{R}) \to L^2(\mathbb{R})$. By construction it is clear that this extended operator is still isometric.
- For $f \in L^2(\mathbb{R})$, the inversion formula "$f(t) = \int_{\mathbb{R}} d\xi\ \hat{f}(\xi)e^{i2\pi\xi t}$" makes sense i.e both sides are well-defined, but it still needs to be proved. This can be done by density arguments.
  - Approximate $f \in L^2(\mathbb{R})$ by $f_\varepsilon \in C_c^\infty(\mathbb{R})$ (for the L^2 norm).
  - Show that $\hat{f}_\varepsilon \in L^1(\mathbb{R})$ (see below). This shows the inversion formula for f_ε.
  - Let $\varepsilon \to 0$ and check that both sides converge to the desired quantities in the L^2 sense (basically trivial by isometry of $L^2 - L^2$ Fourier transform).

  For all k, if $f \in C_c^k(\mathbb{R})$ then $f, f^{(k)} \in L^1(\mathbb{R})$ so $\hat{f}$ and $\widehat{f^{(k)}} = (i2\pi\xi)^k \hat{f}$ vanish at infinity (by Riemann-Lebesgue), and so $(1 + |\xi|^k)\hat{f}(\xi) = o_{|\xi|\to\infty}(1)$ which implies $\hat{f}(\xi) \in L^1$.

The same kind of arguments allows one to define and study the Fourier transform on a variety of spaces, besides L^1 and L^2: the Lebesgue space L^p for general $1 \leq p < \infty$, the Schwartz space $\mathcal{S}$, the generalized function spaces $\mathcal{E}'$ and $\mathcal{S}'$. For a summary accessible at the level of this document, see e.g Chapter 2 of [Yan13].

For the rest of the document, we consider the Fourier transform as an operator $L^2(\mathbb{R}) \to L^2(\mathbb{R})$, which we know is an isometric isomorphism with inversion formula $f(t) = \int_{\mathbb{R}} d\xi\ \hat{f}(\xi)e^{i2\pi\xi t}$.

A.1.2 Real-valued case

Throughout this document, all functions considered are complex-valued. It is straightforward, though somewhat tedious, to adapt this document to the real-valued case.

- Since complex-valued functions include real-valued as a special case, all the formulas presented still hold without modification. The only statements invalidated by adding the real-valued constraint, are those on the set of sample-sequences $\mathbb{A}_{B/B'}$, and on the conditions for $\psi_0(t)$ to be a synthesizer.
- If we constrain f real-valued, then $\hat{f}$ is conjugate symmetric: $\hat{f}(-\xi) = \overline{\hat{f}(\xi)}$. So we may define $\widehat{\mathbb{B}}_B^{(r)}$ the subspace of $[-B, B]$-supported L^2 signals that are conjugate symmetric. The functions $\hat{\varphi}_n^{(B)}(\xi) = \frac{e^{-i2\pi\frac{n}{2B}\xi}}{\sqrt{2B}}$ do not form an ONB of $\widehat{\mathbb{B}}_B^{(r)}$; indeed we must take into account the constraint $\left\langle \hat{f}, \hat{\varphi}_n^{(B)} \right\rangle \in \mathbb{R}$.
- If we constrain $\hat{f}$ real-valued, then f is conjugate symmetric. (This is the case in [Zam79].) So we may define $\widehat{\mathbb{B}}_B^{(\hat{r})}$ the subspace of $[-B, B]$-supported L^2 signals that are real-valued. The functions $\hat{\varphi}_n^{(B)}(\xi)$ do not form an ONB of $\widehat{\mathbb{B}}_B^{(\hat{r})}$; indeed we must take into account the constraint $\left\langle \hat{f}, \hat{\varphi}_n^{(B)} \right\rangle = \overline{\left\langle \hat{f}, \hat{\varphi}_{-n}^{(B)} \right\rangle}$.

A.1.3 Notable notation

Convention for Fourier transform and sinc **function** For the Fourier transform we use the convention

$$\hat{f}(\xi) = \int_{\mathbb{R}} dt\ f(t)e^{-i2\pi\xi t} \tag{A.1.1}$$

$$f(t) = \int_{\mathbb{R}} d\xi\ \hat{f}(\xi)e^{i2\pi\xi t} \tag{A.1.2}$$

with t time variable and ξ frequency variable. It is straightforward to translate this document to other conventions (involving the angular frequency ω instead of ξ) by using the correspondences outlined here: `https://en.wikipedia.org/wiki/Fourier_transform#Other_conventions`.

The sinc function (sinus cardinal) is defined by $\operatorname{sinc}(t) := \frac{\sin \pi t}{\pi t}$.

Shorthands for multidimensional variables Throughout our discussions of the multidimensional setting we will use a number of shorthand notations. All of them will be introduced in section A.6 as they appear for the first time. They are also summarized here for convenience.

- For $B > 0$, $[\pm B] := [-B, B]$.
- For $\boldsymbol{B} \in (\mathbb{R}_+)^N$, $[\pm \boldsymbol{B}] := [-B_1, B_1] \times ... \times [-B_N, B_N]$ is the centered hyperrectangle of sides $2\boldsymbol{B}$.
- For $\boldsymbol{t}, \boldsymbol{\xi} \in \mathbb{R}^N$, $\boldsymbol{\xi} \cdot \boldsymbol{t} := \xi_1 t_1 + ... + \xi_N t_N$ is the dot product.
- For $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^N$, $\boldsymbol{a} \odot \boldsymbol{b} := (a_1 b_1, ..., a_N b_N) \in \mathbb{R}^N$ denotes coordinate-wise multiplication.
- For $\boldsymbol{B} \in (\mathbb{R}_+)^N$, $\boldsymbol{n} \in \mathbb{Z}^N$, $\frac{\boldsymbol{n}}{2\boldsymbol{B}} := \left(\frac{n_1}{2B_1}, ..., \frac{n_N}{2B_N}\right)^T \in \mathbb{R}^N$ is the coordinate-wise division.
- For $\boldsymbol{t} \in \mathbb{R}^N$, $\operatorname{sinc}^\times(\boldsymbol{t}) := \operatorname{sinc}(t_1)... \operatorname{sinc}(t_N) \in \mathbb{R}$ is the product of the coordinate-wise sinc's.
- For $\boldsymbol{B} \in (\mathbb{R}_+)^N$, $(2\boldsymbol{B})^\times := 2B_1...2B_N$. It will often arise as a normalizing factor because $(2\boldsymbol{B})^\times = \mu^N([\pm \boldsymbol{B}])$ the volume of the hyperrectangle.
- For $\boldsymbol{B} \in (\mathbb{R}_+)^N$, $\sqrt{2\boldsymbol{B}}^\times := \sqrt{2B_1}...\sqrt{2B_N} \in \mathbb{R}_+$ is the product of the coordinate-wise $\sqrt{\cdot}$'s. Note that $\sqrt{2\boldsymbol{B}}^\times = \sqrt{(2\boldsymbol{B})^\times} = \sqrt{\mu^N([\pm \boldsymbol{B}])}$ the square root of the volume of the hyperrectangle.

Time-domain sampling rate vs. band-limitation upper-bound In this document,

- B corresponds to the *actual* band-limitation of the signal considered.
- B' corresponds to "what the experimenter believes" is (an upper-bound on) the band-limitation of the signal.
- The formulas correspond to sampling the signal at time-intervals of $\frac{1}{2B'}$.

Many authors choose to present the formulas in terms of T (the time-domain sampling rate) instead of B' (the band-limitation upper-bound), with $T = \frac{1}{2B'}$.

A.2 One-dimensional sampling expansion

In this section we derive the one-dimensional sampling expansion, a.k.a cardinal series representation, a.k.a Nyquist-Shannon sampling theorem, a.k.a Whittaker-Shannon formula... [3]

Definition A.1. For all $B > 0$, let $\mathbb{B}_B$ the space of B-band-limited signals i.e

$$\widehat{\mathbb{B}}_B = \left\{\hat{f} \in L^2(\mathbb{R}); \hat{f}(\xi) = 0 \text{ for } |\xi| > B\right\} \tag{A.2.1}$$

$$\mathbb{B}_B = \left\{f \in L^2(\mathbb{R}); \hat{f} \in \widehat{\mathbb{B}}_B\right\} \tag{A.2.2}$$

The Fourier transform is an isometric isomorphism $L^2(\mathbb{R}) \to L^2(\mathbb{R})$, so elements $f \in \mathbb{B}_B$ correspond one-to-one to elements $\hat{f} \in \widehat{\mathbb{B}}_B$. It is not hard to already see (and the results of this section anyway imply) that $\widehat{\mathbb{B}}_B$ is a closed subspace of the separable Hilbert space $L^2(\mathbb{R})$, so it is also a separable Hilbert space; $\mathbb{B}_B$ too, by isometry.

[3] See `https://en.wikipedia.org/wiki/Nyquist%E2%80%93Shannon_sampling_theorem` for a history of the terminology.

A.2.1 Exact sampling

$\widehat{\mathbb{B}}_B$ is the set of L^2 functions supported on $[-B, B]$, so $\widehat{\mathbb{B}}_B \simeq L^2([-B, B])$. Now $\hat{\varphi}_n^{(B)}(\xi) := \frac{1}{\sqrt{2B}} e^{-i2\pi \frac{n}{2B}\xi}$ is an ONB of that space. Indeed by a simple calculation

$$\left\langle \hat{\varphi}_n^{(B)}, \hat{\varphi}_m^{(B)} \right\rangle_{L^2([-B,B])} = \frac{1}{2B} \int_{-B}^{B} d\xi \; e^{-i2\pi \frac{n-m}{2B}\xi} = \mathbb{1}_{m=n} \tag{A.2.3}$$

and by Fourier series we know that

$$\forall \hat{g} \in L^2([-B, B]),\ \exists (c_n)_{n\in\mathbb{Z}};\ \hat{g}(\xi) = \sum_{n\in\mathbb{Z}} c_n e^{-i2\pi \frac{n}{2B}\xi} \tag{A.2.4}$$

So $\hat{\phi}_n^{(B)}(\xi) := \frac{1}{\sqrt{2B}} \mathbb{1}_{|\xi|\leq B} e^{-i2\pi \frac{n}{2B}\xi}$ is an ONB of $\widehat{\mathbb{B}}_B$, and so

$$\forall \hat{f} \in \widehat{\mathbb{B}}_B, \hat{f}(\xi) = \sum_{n\in\mathbb{Z}} \left\langle \hat{f}, \hat{\phi}_n^{(B)} \right\rangle \hat{\phi}_n^{(B)}(\xi) \tag{A.2.5}$$

$$\forall f \in \mathbb{B}_B, f(t) = \sum_{n\in\mathbb{Z}} \left\langle f, \phi_n^{(B)} \right\rangle \phi_n^{(B)}(t) \tag{A.2.6}$$

where $\phi_n^{(B)}(t)$ is the inverse Fourier transform of $\hat{\phi}_n^{(B)}$. (Since the $L^2 - L^2$ Fourier transform is isometric i.e $\langle g, h\rangle = \left\langle \hat{g}, \hat{h} \right\rangle$.)

- Compute $\phi_n^{(B)}$:

$$\hat{\phi}_n^{(B)}(\xi) = \frac{1}{\sqrt{2B}} \mathbb{1}_{|\xi|\leq B} e^{-i2\pi \frac{n}{2B}\xi} \tag{A.2.7}$$

$$= \hat{\phi}_0^{(B)}(\xi) e^{-i2\pi \frac{n}{2B}\xi} \tag{A.2.8}$$

$$\phi_n^{(B)}(t) = \phi_0^{(B)}\left(t - \frac{n}{2B}\right) \tag{A.2.9}$$

  Indeed $\phi_n^{(B)}(t) = \int_{\mathbb{R}} d\xi\ \hat{\phi}_0^{(B)}(\xi) e^{-i2\pi \frac{n}{2B}\xi}\ e^{i2\pi\xi t} = \int_{\mathbb{R}} d\xi\ \hat{\phi}_0^{(B)}(\xi) e^{i2\pi\xi(t-\frac{n}{2B})} = \phi_0^{(B)}(t - \frac{n}{2B})$.
  Compute $\phi_0^{(B)}$:

$$\hat{\phi}_0^{(B)}(\xi) = \frac{1}{\sqrt{2B}} \mathbb{1}_{|\xi|\leq B} \cdot 1 \tag{A.2.10}$$

$$\phi_0^{(B)}(t) = \frac{1}{\sqrt{2B}} \int_{-B}^{B} d\xi\ 1 \cdot e^{i2\pi t\xi} = \frac{1}{\sqrt{2B}} \frac{2i\sin(2\pi t B)}{i2\pi t} \tag{A.2.11}$$

$$= \sqrt{2B}\,\mathrm{sinc}(2tB) \tag{A.2.12}$$

  where $\mathrm{sinc}(x) := \frac{\sin(\pi x)}{\pi x}$.

- Compute $\left\langle f, \phi_n^{(B)} \right\rangle$:

$$\left\langle f, \phi_0^{(B)} \right\rangle = \left\langle \hat{f}, \hat{\phi}_0^{(B)} \right\rangle = \frac{1}{\sqrt{2B}} \int_{-B}^{B} \hat{f}(\xi) e^{i2\pi 0\cdot\xi} d\xi \tag{A.2.13}$$

$$= \frac{1}{\sqrt{2B}} \int_{\mathbb{R}} \hat{f}(\xi) e^{i2\pi 0\cdot\xi} d\xi \tag{A.2.14}$$

$$= \frac{1}{\sqrt{2B}} f(0) \tag{A.2.15}$$

and so

$$\left\langle f, \phi_n^{(B)} \right\rangle = \left\langle f(t), \phi_0^{(B)}(t - \frac{n}{2B}) \right\rangle \tag{A.2.16}$$
$$= \left\langle f(t + \frac{n}{2B}), \phi_0^{(B)}(t) \right\rangle \tag{A.2.17}$$
$$= \frac{1}{\sqrt{2B}} f\left(\frac{n}{2B}\right) \tag{A.2.18}$$

Summary Note that the functions $\phi_n^{(B)}$, $\psi_n^{(B,B')}$ defined here will be used many times throughout the document.

Definition A.2. We denote $\phi_n^{(B)}(t)$ the function of $L^2(\mathbb{R})$ defined by

$$\hat{\phi}_0^{(B)}(\xi) = \frac{1}{\sqrt{2B}} \mathbb{1}_{|\xi| \le B} \tag{A.2.19}$$
$$\text{i.e } \phi_0^{(B)}(t) = \sqrt{2B}\, \text{sinc}(2tB) \tag{A.2.20}$$

where $\text{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$, and

$$\hat{\phi}_n^{(B)}(\xi) = \hat{\phi}_0^{(B)}(\xi) e^{-i2\pi \frac{n}{2B} \xi} \tag{A.2.21}$$
$$\text{i.e } \phi_n^{(B)}(t) = \phi_0^{(B)}\left(t - \frac{n}{2B}\right) \tag{A.2.22}$$

Proposition A.1. The following reconstruction formula holds:

$$\forall f \in \mathbb{B}_B, f(t) = \sum_{n \in \mathbb{Z}} \left\langle f, \phi_n^{(B)} \right\rangle \phi_n^{(B)}(t) \tag{A.2.23}$$
$$= \sum_{n \in \mathbb{Z}} \frac{1}{\sqrt{2B}} f\left(\frac{n}{2B}\right) \sqrt{2B}\, \text{sinc}\left(2B\left(t - \frac{n}{2B}\right)\right) \tag{A.2.24}$$
$$= \sum_{n \in \mathbb{Z}} f\left(\frac{n}{2B}\right) \text{sinc}\left(2B\left(t - \frac{n}{2B}\right)\right) \tag{A.2.25}$$

A.2.2 Oversampling (without design freedom)

Still consider $\mathbb{B}_B$, $\widehat{\mathbb{B}}_B$, and let $B' \ge B$. Then clearly $\mathbb{B}_B \subset \mathbb{B}_{B'}$ and $\widehat{\mathbb{B}}_B \subset \widehat{\mathbb{B}}_{B'}$ (i.e if f is B-band-limited then a fortiori it is B'-band-limited). So

$$\forall \hat{f} \in \widehat{\mathbb{B}}_B, \hat{f}(\xi) = \sum_{n \in \mathbb{Z}} \left\langle \hat{f}, \hat{\phi}_n^{(B')} \right\rangle \hat{\phi}_n^{(B')}(\xi) \tag{A.2.26}$$
$$= \sum_{n \in \mathbb{Z}} \left\langle \hat{f}, \hat{\phi}_n^{(B')} \right\rangle \underbrace{\hat{\phi}_n^{(B')}(\xi)\ \mathbb{1}_{|\xi| \le B}}_{=:\hat{\psi}_n^{(B,B')}(\xi)} \tag{A.2.27}$$

since $\hat{f}(\xi) = 0$ for $|\xi| > B$, where

$$\hat{\psi}_n^{(B,B')}(\xi) = \frac{1}{\sqrt{2B'}} e^{-i2\pi \frac{n}{2B'} \xi} \mathbb{1}_{|\xi| \le B'}\ \mathbb{1}_{|\xi| \le B} \tag{A.2.28}$$
$$= \frac{1}{\sqrt{2B'}} e^{-i2\pi \frac{n}{2B'} \xi} \mathbb{1}_{|\xi| \le B} \tag{A.2.29}$$

So,

$$\forall \hat{f} \in \widehat{\mathbb{B}}_B, \hat{f}(\xi) = \sum_{n\in\mathbb{Z}} \left\langle \hat{f}, \hat{\phi}_n^{(B')} \right\rangle \hat{\psi}_n^{(B,B')}(\xi) \tag{A.2.30}$$

$$\forall f \in \mathbb{B}_B, f(t) = \sum_{n\in\mathbb{Z}} \left\langle f, \phi_n^{(B')} \right\rangle \psi_n^{(B,B')}(t) \tag{A.2.31}$$

$$= \sum_{n\in\mathbb{Z}} \frac{1}{\sqrt{2B'}} f\left(\frac{n}{2B'}\right) \psi_n^{(B,B')}(t) \tag{A.2.32}$$

where $\psi_n^{(B,B')}(t)$ is the inverse Fourier transform of $\hat{\psi}_n^{(B,B')}(\xi)$:

$$\hat{\psi}_n^{(B,B')}(\xi) = \hat{\psi}_0^{(B,B')}(\xi)\ e^{-i2\pi\frac{n}{2B'}\xi} \tag{A.2.33}$$

$$\psi_n^{(B,B')}(t) = \psi_0^{(B,B')}\left(t - \frac{n}{2B'}\right) \tag{A.2.34}$$

$$\psi_0^{(B,B')}(t) = \int_{\mathbb{R}} d\xi\ \frac{1}{\sqrt{2B'}} \mathbb{1}_{|\xi|\le B} e^{i2\pi t\xi} \tag{A.2.35}$$

$$= \frac{1}{\sqrt{2B'}} \int_{-B}^{B} d\xi\ e^{i2\pi t\xi} \tag{A.2.36}$$

$$= \frac{1}{\sqrt{2B'}} 2B\,\mathrm{sinc}(2Bt) \tag{A.2.37}$$

Summary

Definition A.3. We denote $\psi_n^{(B,B')}(t)$ the function of $L^2(\mathbb{R})$ defined by

$$\hat{\psi}_0^{(B,B')}(\xi) = \frac{1}{\sqrt{2B'}} \mathbb{1}_{|\xi|\le B} \tag{A.2.38}$$

$$\psi_0^{(B,B')}(t) = \frac{2B}{\sqrt{2B'}}\,\mathrm{sinc}(2Bt) \tag{A.2.39}$$

and

$$\hat{\psi}_n^{(B,B')}(\xi) = \hat{\psi}_0^{(B,B')}(\xi) e^{-i2\pi\frac{n}{2B'}\xi} \tag{A.2.40}$$

$$\psi_n^{(B,B')}(t) = \psi_0^{(B,B')}\left(t - \frac{n}{2B'}\right) \tag{A.2.41}$$

Proposition A.2. Let $B' \ge B$. The following reconstruction formula holds:

$$\forall f \in \mathbb{B}_B, f(t) = \sum_{n\in\mathbb{Z}} \left\langle f, \phi_n^{(B')} \right\rangle \psi_n^{(B,B')}(t) \tag{A.2.42}$$

$$= \sum_{n\in\mathbb{Z}} \frac{1}{\sqrt{2B'}} f\left(\frac{n}{2B'}\right)\ \frac{2B}{\sqrt{2B'}}\,\mathrm{sinc}\left(2B\left(t - \frac{n}{2B}\right)\right) \tag{A.2.43}$$

$$= \sum_{n\in\mathbb{Z}} \frac{2B}{2B'} f\left(\frac{n}{2B'}\right) \mathrm{sinc}\left(2B\left(t - \frac{n}{2B'}\right)\right) \tag{A.2.44}$$

Remark A.1. It may appear somewhat unintuitive (but it is true) that

$$\forall f \in \mathbb{B}_B,\ f(t) = \sum_{n\in\mathbb{Z}} \left\langle f, \phi_n^{(B')} \right\rangle \phi_n^{(B')}(t) = \sum_{n\in\mathbb{Z}} \left\langle f, \phi_n^{(B')} \right\rangle \psi_n^{(B,B')}(t) \tag{A.2.45}$$

The left-hand side is the exact-sampling expansion of f seen as an element of $\mathbb{B}_{B'}$; the right-hand side is the oversampled expansion of f seen as an element of $\mathbb{B}_B \subset \mathbb{B}_{B'}$.

In terms of frame theory, $(\psi_n^{(B,B')})_n$ and $(\phi_n^{(B')})_n$ are both dual frames of $(\phi_n^{(B')})_n$ for the Hilbert space $\mathbb{B}_B$, for any $B' \geq B$ [Böl20, section 1.4.1]. Furthermore, intuitively the functions of $\mathbb{B}_B$ are oversampled since the samples $f\left(\frac{n}{2B}\right)$ would allow one to reconstruct functions not only in $\mathbb{B}_B$ but in all of $\mathbb{B}_{B'}$; in other words, the frame $(\phi_n^{(B')})_{n\in\mathbb{Z}}$ is overcomplete for the space $\mathbb{B}_B$.

A.2.3 A particular case of oversampling with design freedom

Our elementary view already allows us to discuss a particular case of design freedom [Böl20, section 1.4.2].

Let $\hat{f} \in \widehat{\mathbb{B}}_B$. Then for any function $\hat{h}_{arb}(\xi)$, (both sides are zero when $|\xi| > B$)

$$\forall \xi \in \mathbb{R}, \hat{f}(\xi) = \hat{f}(\xi)\left[\mathbb{1}_{|\xi|\leq B} + \mathbb{1}_{|\xi|>B}\hat{h}_{arb}(\xi)\right] \tag{A.2.46}$$

So, for any $B' \geq B$,

$$\begin{aligned}
\hat{f}(\xi) &= \sum_{n\in\mathbb{Z}} \left\langle \hat{f}, \hat{\phi}_n^{(B')} \right\rangle \hat{\phi}_n^{(B')}(\xi) && \text{(A.2.47)}\\
&= \sum_{n\in\mathbb{Z}} \frac{1}{\sqrt{2B'}} f\left(\frac{n}{2B'}\right) \frac{1}{\sqrt{2B'}} e^{-i2\pi\frac{n}{2B'}\xi} \mathbb{1}_{|\xi|\leq B'} && \text{(A.2.48)}\\
&= \sum_{n\in\mathbb{Z}} \frac{1}{\sqrt{2B'}} f\left(\frac{n}{2B'}\right) \frac{1}{\sqrt{2B'}} e^{-i2\pi\frac{n}{2B'}\xi} \mathbb{1}_{|\xi|\leq B'} \left[\mathbb{1}_{|\xi|\leq B} + \mathbb{1}_{|\xi|>B}\hat{h}_{arb}(\xi)\right] && \text{(A.2.49)}
\end{aligned}$$

Now $\mathbb{1}_{|\xi|\leq B'}\left[\mathbb{1}_{|\xi|\leq B} + \mathbb{1}_{|\xi|>B}\hat{h}_{arb}(\xi)\right] = \mathbb{1}_{|\xi|\leq B} + \mathbb{1}_{B<|\xi|\leq B'}\hat{h}_{arb}(\xi)$, so

$$\hat{f}(\xi) = \sum_{n\in\mathbb{Z}} \frac{1}{\sqrt{2B'}} f\left(\frac{n}{2B'}\right) \hat{\psi}_n^{(h,B,B')}(\xi) \tag{A.2.50}$$

where

$$\begin{aligned}
\hat{\psi}_n^{(h,B,B')}(\xi) &= \frac{1}{\sqrt{2B'}} e^{-i2\pi\frac{n}{2B'}\xi}\left(\mathbb{1}_{|\xi|\leq B} + \mathbb{1}_{B<|\xi|\leq B'}\hat{h}(\xi)\right) && \text{(A.2.51)}\\
&= \hat{\psi}_0^{(h,B,B')}(\xi) e^{-i2\pi\frac{n}{2B'}\xi} && \text{(A.2.52)}\\
\psi_n^{(h,B,B')}(t) &= \psi_0^{(h,B,B')}\left(t - \frac{n}{2B'}\right) && \text{(A.2.53)}
\end{aligned}$$

Summary

Proposition A.3. Let $B' \geq B$. The following reconstruction formula holds:

$$\forall f \in \mathbb{B}_B, f(t) = \sum_{n\in\mathbb{Z}} \frac{1}{\sqrt{2B'}} f\left(\frac{n}{2B'}\right) \psi_0^{(h,B,B')}\left(t - \frac{n}{2B'}\right) \tag{A.2.54}$$

where $\psi_0^{(h,B,B')}(t)$ is any function whose spectrum is of the form $\frac{1}{\sqrt{2B'}}\mathbb{1}_{|\xi|\leq B} + \mathbb{1}_{B<|\xi|\leq B'}\hat{h}(\xi)$; equivalently, any function whose spectrum satisfies

$$\hat{\psi}_0^{(h,B,B')}(\xi) = \begin{cases} \frac{1}{\sqrt{2B'}} & \text{for } |\xi| \leq B \\ \text{arbitrary} & \text{for } B < |\xi| \leq B' \\ 0 & \text{for } B' < |\xi| \end{cases} \tag{A.2.55}$$

See [Böl20, Figure 1.7] for an illustration.

A.3 Relation to the Discrete-Time Fourier Transform (DTFT)

A.3.1 Reminders on the DTFT

Definition A.4. The (1-unit) DTFT of a sequence $(a_n)_n \in \ell^2(\mathbb{Z})$ is [4]

$$\mathcal{F}_d[a](x) = \sum_{n\in\mathbb{Z}} a_n e^{-i2\pi nx} \qquad \in L^2([-1/2, 1/2]) \tag{A.3.1}$$

In other words the DTFT is just the Fourier series with coefficients a_n. The inverse operation (inverse DTFT) is: for a function $g \in L^2([-1/2, 1/2])$,

$$\left(\mathcal{F}_d^{-1}[g]\right)_n = \int_{-1/2}^{1/2} dx\ g(x) e^{i2\pi nx} \qquad \in \ell^2(\mathbb{Z}) \tag{A.3.2}$$

i.e it simply consists in computing its Fourier series coefficients.

Recall that $\hat{\varphi}_n^{(B)}(\xi) := \frac{1}{\sqrt{2B}} e^{-i2\pi \frac{n}{2B}\xi}$ is an ONB of $L^2([-B, B])$; so for $\hat{f}$ in that space we have

$$\forall \xi \in [-B, B],\ \hat{f}(\xi) = \sum_{n\in\mathbb{Z}} \left\langle \hat{f}, \hat{\varphi}_n^{(B)} \right\rangle \hat{\varphi}_n^{(B)}(\xi) = \sum_{n\in\mathbb{Z}} c_n \frac{e^{-i2\pi \frac{n}{2B}\xi}}{\sqrt{2B}} \tag{A.3.3}$$

$$\text{where} \qquad c_n := \left\langle \hat{f}, \hat{\varphi}_n^{(B)} \right\rangle \tag{A.3.4}$$

To see the correspondence with the usual form of the DTFT, it can be convenient to define the renormalized quantities (for $f \in \mathbb{B}_B$)

$$\underline{t} = 2Bt\ (= t/T) \tag{A.3.5}$$

$$\underline{\xi} = \frac{\xi}{2B} \tag{A.3.6}$$

$$\underline{f}(\underline{t}) = \frac{1}{\sqrt{2B}} f(t) = \frac{1}{\sqrt{2B}} f\left(\frac{\underline{t}}{2B}\right) \tag{A.3.7}$$

$$\underline{\hat{f}}(\underline{\xi}) = \sqrt{2B}\hat{f}(\xi)\ = \sqrt{2B}\hat{f}(2B\xi) \tag{A.3.8}$$

This notation is consistent with the notation for the Fourier transform since

$$\int_{\mathbb{R}} d\underline{t}\ \underline{f}(\underline{t}) e^{-i2\pi\underline{\xi}\underline{t}} = \frac{1}{\sqrt{2B}} \int_{\mathbb{R}} d\underline{t}\ f\left(\frac{\underline{t}}{2B}\right) e^{-i2\pi\underline{\xi}\cdot\underline{t}} \tag{A.3.9}$$

$$= \frac{1}{\sqrt{2B}} \int_{\mathbb{R}} dt\ 2B\ f(t)\ e^{-i2\pi\ 2B\underline{\xi}\cdot t} \tag{A.3.10}$$

$$= \sqrt{2B}\hat{f}(2B\underline{\xi}) = \underline{\hat{f}}(\underline{\xi}) \tag{A.3.11}$$

With this, the above can be rewritten as a (1-unit) Fourier series

$$\forall \xi \in [-B, B],\ \hat{f}(\xi) = \sum_{n\in\mathbb{Z}} c_n \frac{e^{-i2\pi \frac{n}{2B}\xi}}{\sqrt{2B}} \tag{A.3.12}$$

$$\iff \quad \forall \underline{\xi} \in [-1/2, 1/2],\ \underline{\hat{f}}(\underline{\xi}) = \sum_{n\in\mathbb{Z}} c_n e^{-i2\pi n\underline{\xi}} \tag{A.3.13}$$

[4] Here we chose to define the DTFT as a square-integrable signal over an interval I. It is often defined alternatively as an $|I|$-periodic signal (over $\mathbb{R}$), square-integrable on any compact. The two representations are of course equivalent.

and (recall $\langle f, g\rangle = \int_I f\overline{g}$ for complex-valued functions)

$$c_n = \left\langle \hat{f}, \hat{\varphi}_n^{(B)} \right\rangle = \int_{-B}^{B} d\xi\ \hat{f}(\xi) \frac{e^{+i2\pi\frac{n}{2B}\xi}}{\sqrt{2B}} \tag{A.3.14}$$

$$= \int_{-1/2}^{1/2} d\underline{\xi}\ \underline{\hat{f}}(\underline{\xi}) e^{i2\pi n\underline{\xi}} \tag{A.3.15}$$

so we recover the fact that c_n are the coefficients of the Fourier series expansion of $\underline{\hat{f}} \in L^2([-1/2, 1/2])$.

Definition A.5. The $2B$-unit DTFT is: (this is not standard terminology)

$$\mathcal{F}_{d,2B}[a](\xi) := \sum_{n\in\mathbb{Z}} a_n \frac{e^{-i2\pi\frac{n}{2B}\xi}}{\sqrt{2B}} \qquad \in L^2([-B, B]) \tag{A.3.16}$$

and the $2B$-unit inverse DTFT

$$\left(\mathcal{F}_{d,2B}^{-1}\left[\hat{f}\right]\right)_n := \int_{-B}^{B} d\xi\ \hat{f}(\xi) \frac{e^{i2\pi\frac{n}{2B}\xi}}{\sqrt{2B}} \qquad \in \ell^2(\mathbb{Z}) \tag{A.3.17}$$

In summary, the DTFT operators $\mathcal{F}_d : \ell^2(\mathbb{Z}) \to L^2([-1/2, 1/2])$ and $\mathcal{F}_{d,2B} : \ell^2(\mathbb{Z}) \to L^2([-B, B])$ are isometric isomorphisms.

A.3.2 Equivalent reformulations of the sampling theorem

The exact-sampling expansion was obtained by considering the ($2B$-unit) Fourier series expansion of $\hat{f}$ seen as an element of $L^2([-B, B])$:

$$\forall \xi \in [-B, B],\ \hat{f}(\xi) = \sum_{n\in\mathbb{Z}} \left\langle \hat{f}, \hat{\phi}_n^{(B)} \right\rangle \hat{\phi}_n^{(B)}(\xi) = \sum_{n\in\mathbb{Z}} c_n \frac{e^{-i2\pi\frac{n}{2B}\xi}}{\sqrt{2B}} \tag{A.3.18}$$

and the coefficients $c_n := \left\langle \hat{f}, \hat{\phi}_n^{(B)} \right\rangle$ turn out to be equal to $\frac{1}{\sqrt{2B}} f\left(\frac{n}{2B}\right)$.

Thus, the sampling theorem may be reformulated equivalently as: for $f \in \mathbb{B}_B$,

- The ($2B$-unit) Fourier series coefficients of $\hat{f}$ seen as an element of $L^2([-B, B])$, are given by $c_n := \frac{1}{\sqrt{2B}} f\left(\frac{n}{2B}\right)$.
- The ($2B$-unit) DTFT of $c_n := \frac{1}{\sqrt{2B}} f\left(\frac{n}{2B}\right)$ is nothing else than $\hat{f}\Big|_{[-B,B]}$.
- The (continuous-time) Fourier transform of f is equal to the discrete-time Fourier transform of its (renormalized) sample-sequence $\frac{1}{\sqrt{2B}} f\left(\frac{n}{2B}\right)$, over $[-B, B]$.

A.3.3 The set of sample-sequences for oversampled signals

Proposition A.4. The set

$$\left\{ (c_n)_n = \left(f\left(\frac{n}{2B'}\right) \right)_n,\ f \in \mathbb{B}_B \right\} \tag{A.3.19}$$

is equal to

$$\left\{ (c_n)_n \in \ell^2(\mathbb{Z}),\ \mathcal{F}_{d,2B'}[c](\xi) \text{ is supported on } [-B, B] \right\} \tag{A.3.20}$$

which is by definition also equal to

$$\left\{ (c_n)_n \in \ell^2(\mathbb{Z})\ ;\ \mathcal{F}_d[c](\underline{\xi}) \text{ is supported on } \left[-\frac{B}{2B'}, \frac{B}{2B'}\right] \right\} \tag{A.3.21}$$

In words, the set of possible sample-sequences is equal to the set of sequences in $\ell^2(\mathbb{Z})$ whose (1-unit) DTFT is supported on $\left[-\frac{B}{2B'}, \frac{B}{2B'}\right]$.

In the sequel we denote this set by $\mathbb{A}_{B/B'}$. It is clearly a linear subspace of $\ell^2(\mathbb{Z})$.

Indeed, for all $f \in \mathbb{B}_{B'}$, the above discussion applied to B' shows that

$$\mathcal{F}_{d,2B'}\left[\frac{1}{\sqrt{2B'}} f\left(\frac{n}{2B'}\right)\right](\xi) = \hat{f}(\xi) \tag{A.3.22}$$

So,

- For all $f \in \mathbb{B}_B$ $(\subset \mathbb{B}_{B'})$, $\mathcal{F}_{d,2B'}\left[f\left(\frac{n}{2B'}\right)\right](\xi) = \sqrt{2B'}\hat{f}(\xi)$ which is supported on $[-B, B]$. This shows the first inclusion.
- Conversely, if the sequence $(c_n)_n$ is such that $\mathcal{F}_{d,2B'}[c](\xi)$ is supported on $[-B, B]$, then simply letting $\hat{f}(\xi) := \mathcal{F}_{d,2B'}[c](\xi)$ yields an f such that $f \in \mathbb{B}_B$ (by assumption) and $c_n = \frac{1}{\sqrt{2B'}} f\left(\frac{n}{2B'}\right)$ (by the sampling theorem). This is exactly the converse inclusion (modulo normalization, but both sets are linear spaces anyway).

A.4 Deducing (a special case of) the Poisson summation formula

Proposition A.5. For all $B, b > 0$, it holds

$$\forall f \in \mathbb{B}_B, \ \sum_{n\in\mathbb{Z}} \hat{f}(\xi + 2nb) = \sum_{n\in\mathbb{Z}} \frac{1}{2b} f\left(\frac{n}{2b}\right) e^{-i2\pi\frac{n}{2b}\xi} \tag{A.4.1}$$

This is a special case of the Poisson summation formula, which actually holds for much more general hypotheses on f. [5] To prove this identity, we proceed in three steps.

For $b \geq B$ Both sides of the formula are $2b$-periodic with respect to ξ so it suffices to show it for $\xi \in [-b, b]$.

Suppose $b \geq B$. Then $\mathbb{B}_B \subset \mathbb{B}_b$, and the left-hand-side reduces to (at most) a single term:

$$\sum_{n\in\mathbb{Z}} \hat{f}(\xi + 2nb) = \hat{f}(\xi) \tag{A.4.2}$$

for $\xi \in [-b, b]$. (Otherwise it is equal to $\hat{f}(\xi \mod 2b)$ where $\xi \mod 2b$ is the unique element of $[-b, b] \cap (\xi + 2b\mathbb{Z})$.) So it suffices to show that

$$\hat{f}(\xi) = \sum_{n\in\mathbb{Z}} \frac{1}{2b} f\left(\frac{n}{2b}\right) e^{-i2\pi\frac{n}{2b}\xi} \tag{A.4.3}$$

holds for all $\xi \in [-b, b]$.

It is indeed the case since $f \in \mathbb{B}_B \subset \mathbb{B}_b$ so that, as we already showed,

$$\hat{f}(\xi) = \sum_{n\in\mathbb{Z}} \left\langle \hat{f}, \hat{\phi}_n^{(b)} \right\rangle \hat{\phi}_n^{(b)}(\xi) \tag{A.4.4}$$

$$= \sum_{n\in\mathbb{Z}} \frac{1}{\sqrt{2b}} f\left(\frac{n}{2b}\right) \frac{1}{\sqrt{2b}} e^{-i2\pi\frac{n}{2b}\xi} \mathbb{1}_{|\xi|\leq b} \tag{A.4.5}$$

In words, $f \in \mathbb{B}_B \subset \mathbb{B}_b$ so that by the sampling theorem, *the $2b$-unit DTFT of $c_n := \frac{1}{\sqrt{2b}} f\left(\frac{n}{2b}\right)$ is nothing else than $\hat{f}\Big|_{[-b,b]}$*.

From now on, suppose that $b < B$.

[5] `https://en.wikipedia.org/wiki/Poisson_summation_formula` and `https://mathoverflow.net/questions/14568/truth-of-the-poisson-summation-formula` and [Gol+19, section 9.4]

General case: split into subintervals We show how to conclude by using that the formula is linear in f.

Since we supposed $B > b$, the idea is to split the interval $[-B, B]$ into subintervals of length $2b$. More precisely, denote

$$\forall m \in \mathbb{N}, I_m = [2bm, 2b(m+1)] \tag{A.4.6}$$

$$\mathbb{R} = \bigcup_{m \in \mathbb{N}} I_m \tag{A.4.7}$$

$$\hat{f}_m(\xi) = \hat{f}(\xi) \mathbb{1}_{\xi \in I_m} \tag{A.4.8}$$

Let M such that $[-B, B] \subset \bigcup_{-M \leq m \leq M} I_m$. Then

$$f = \sum_{-M \leq m \leq M} f_m \tag{A.4.9}$$

Since the formula to be proved is linear in f, it suffices to show it for each f_m, and the general case will follow by summing over $-M \leq m \leq M$.

Remark A.2. The value of B actually doesn't matter, as long as it is finite so that there are only finitely many subintervals I_m to consider. This observation can be used to weaken the hypotheses need on f, namely it easily follows that the formula holds for any $f \in \mathbb{B} := \bigcup_{B>0} \mathbb{B}_B$. By looking more carefully at the function spaces considered, it can probably be generalized to the closure of that space with respect to some norm to be determined.

For $f \in \mathbb{B}_I$ We show that the result still holds when the spectrum is translated by a constant, which concludes the proof.

Let us show that for any interval I of length $2b$, if $\hat{f}$ is supported on I (which we can denote $f \in \mathbb{B}_I$) then the formula holds.

Denote $I = [a_0 - b, a_0 + b]$. If $\hat{f} \in \widehat{\mathbb{B}}_I$ then $\hat{f}(\cdot - a_0) \in \widehat{\mathbb{B}}_b$, and its inverse Fourier transform is $f(t)e^{i2\pi a_0 t}$. We showed in the first point that the formula applies for that function:

$$\sum_{n \in \mathbb{Z}} \hat{f}(\xi + 2nb - a_0) = \sum_{n \in \mathbb{Z}} \frac{1}{2b} f\left(\frac{n}{2b}\right) e^{i2\pi a_0 \frac{n}{2b}} e^{-i2\pi \frac{n}{2b}\xi} \tag{A.4.10}$$

$$= \sum_{n \in \mathbb{Z}} \frac{1}{2b} f\left(\frac{n}{2b}\right) e^{-i2\pi \frac{n}{2b}(\xi - a_0)} \tag{A.4.11}$$

This holds for all $\xi \in \mathbb{R}$, so by evaluating at $\xi' = \xi + a_0$ we get the desired formula.

A.5 Classic but less elementary view: spectrum of the discretized signal

In this section we show that the spectrum of the discretized signal consists of non-overlapping translated repetitions of the original spectrum. This point of view also allows to derive the sampling expansion; actually it seems to be the usual way of introducing it, e.g [Böl20, section 1.4], `https://en.wikipedia.org/wiki/Nyquist%E2%80%93Shannon_sampling_theorem#Derivation_as_a_special_case_of_Poisson_summation`.

However, contrary to the L^2 point of view which we took so far, working with the discretized signal rigorously requires notions on distributions: Dirac delta function, Fourier transform of (tempered) distributions, Poisson summation formula i.e Dirac comb. Hence, *we will not be very careful to specify under what conditions the formulas and calculations hold, in this subsection.* Our goal will be mainly to explain what is happening informally. This being said, the reader familiar with the relevant topics will have no difficulty filling in the blanks.

Remark A.3 (Scope of this section). In fact, the arguments presented in this section would allow to derive the sampling expansion for a more general class of functions: whereas in the rest of this document we are concerned with the L^2 case

$$f \in \mathbb{B}_B = \left\{ f \in L^2(\mathbb{R});\ \operatorname{supp} \hat{f} \subset [-B, B] \right\} \subset L^2(\mathbb{R}) \tag{A.5.1}$$

(with $\hat{f}$ the $L^2 - L^2$ Fourier transform), this section applies for any

$$f \in \left\{ \widetilde{F};\ F \in L^1(\mathbb{R}) \text{ s.t } \operatorname{supp} F \subset [-B, B] \right\} \subset \mathcal{F}[L^1(\mathbb{R})] \subset L^\infty(\mathbb{R}) \tag{A.5.2}$$

(with $\widetilde{F}$ the $L^1 - L^\infty$ inverse Fourier transform, by which we mean $\widetilde{F}(t) := \widehat{F}(-t)$ since the formulas for Fourier transform and inverse Fourier transform only differ by a sign flip of the variable). The former space is contained in the latter since $f \in \mathbb{B}_B \implies \left\| \hat{f} \right\|_{L^1} \leq \sqrt{2B} \left\| \hat{f} \right\|_{L^2} < \infty$. So the latter class of functions is indeed more general.

Thus this section's less-elementary approach leads to more general results. There might be ways to prove those more general results starting from the L^2 case, though, by using density arguments. Anyway, this document does not aim for generality in the first place.

A.5.1 Spectrum of the discretized signal

Let $f \in \mathbb{B}_B$. Define the *discretized signal*

$$f_d(t) = \sum_{n \in \mathbb{Z}} T f(nT) \delta_{nT}(t) \tag{A.5.3}$$

$$= \sum_{n \in \mathbb{Z}} \frac{1}{2B'} f\left(\frac{n}{2B'}\right) \delta_{\frac{n}{2B'}}(t) \tag{A.5.4}$$

where $\delta_\tau(t) := \delta(t - \tau)$ and $\delta(t)$ is the Dirac delta distribution defined by $\forall g, \int_{\mathbb{R}} dt\ g(t)\delta(t) = g(0)$. (We purposefully don't specify what space the test functions g live in, because that's where the unrigorousness arises...) The choice of normalization is such that $f_d(t)$ and f have approximately the same integral over each compact, see Figure A.1.

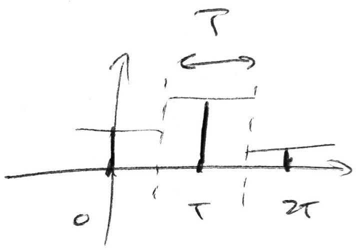


Figure A.1: Justification for the normalization. $f_d(t)$ has the same "mass" as the rectangle approximation $\sum_{n \in \mathbb{Z}} f(nT) \mathbb{1}_{|t - nT| \leq T/2}$.

We have for all $\xi \in \mathbb{R}$

$$\hat{\delta}(\xi) = \int_{\mathbb{R}} dt\ e^{-i2\pi\xi t} \delta(t) = 1 \tag{A.5.5}$$

$$\hat{\delta}_\tau(\xi) = \int_{\mathbb{R}} dt\ e^{-i2\pi\xi t} \delta(t - \tau) = e^{-i2\pi\tau\xi} \tag{A.5.6}$$

$$\hat{f}_d(\xi) = \sum_{n \in \mathbb{Z}} \frac{1}{2B'} f\left(\frac{n}{2B'}\right) e^{-i2\pi \frac{n}{2B'} \xi} \tag{A.5.7}$$

We recognize that the right-hand-side is the same as in the Poisson summation formula, so that for all $\xi \in \mathbb{R}$

$$\hat{f}_d(\xi) = \sum_{n\in\mathbb{Z}} \hat{f}(\xi + 2nB') \tag{A.5.8}$$

By assumption $\hat{f}$ is supported on $[-B, B]$, so each term $\hat{f}(\xi + 2nB')$ is supported on $[2nB' - B, 2nB' + B]$. Thus the terms have disjoint support, and we showed that *the spectrum of the discretized signal is just non-overlapping translated repetitions of the original spectrum* (Figure A.2).

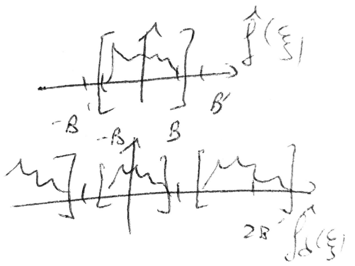


Figure A.2: The spectrum of f_d consists of non-overlapping $2B'$-translated repetitions of the spectrum of f.

A.5.2 Alternative derivation of the sampling expansion

The sampling expansion can be rederived starting from the above considerations, as follows. On the one hand,

$$\forall g,\ (g * \delta_\tau)(t) = \int_{\mathbb{R}} ds\ g(t-s)\delta_\tau(s) = g(t-\tau) \tag{A.5.9}$$

so that for all $\Psi(t)$,

$$(f_d * \Psi)(t) = \sum_{n\in\mathbb{Z}} \frac{1}{2B'} f\left(\frac{n}{2B'}\right) \Psi\left(t - \frac{n}{2B'}\right) \tag{A.5.10}$$

On the other hand,

- If $\forall \xi \in \text{supp}(\hat{f}_d),\ \widehat{\Psi}_1(\xi) = 1$, then

$$\hat{f}_d \widehat{\Psi}_1(\xi) = \hat{f}_d(\xi) \tag{A.5.11}$$
$$(f_d * \Psi_1)(t) = f_d(t) \tag{A.5.12}$$

  Note that since $\text{supp}(\hat{f}) \subset [-B, B]$, then $\text{supp}(\hat{f}_d) \subset \bigsqcup_{n\in\mathbb{Z}} [2nB' - B, 2nB' + B]$.

  Thus, for all $\Psi_1(t)$ such that $\widehat{\Psi}_1(\xi) = 1$ on that set,

$$f_d(t) = \sum_{n\in\mathbb{Z}} \frac{1}{2B'} f\left(\frac{n}{2B'}\right) \Psi_1\left(t - \frac{n}{2B'}\right) \tag{A.5.13}$$

- If $\widehat{\Psi}_0(\xi)$ is a multiplier that preserves $\hat{f}_d(\xi)$ on $[-B, B]$ (where it is equal to $\hat{f}(\xi)$), and "zeroes out" $\hat{f}_d(\xi)$ everywhere else, then

$$\hat{f}_d \widehat{\Psi}_0(\xi) = \hat{f}(\xi) \tag{A.5.14}$$
$$(f_d * \Psi_0)(t) = f(t) \tag{A.5.15}$$

Note that this condition on $\widehat{\Psi}_0$ can be written as (see Figure A.3)

$$\widehat{\Psi}_0(\xi) = \begin{cases} 1 & \text{for } \xi \in [-B, B] \\ 0 & \text{for } \xi \in \operatorname{supp}(\hat{f}_d) \setminus \{-B, B\} \\ & \qquad = \bigsqcup_{n \in \mathbb{Z}^*} [2nB' - B, 2nB' + B] \\ \text{arbitrary} & \text{everywhere else} \end{cases} \tag{A.5.16}$$

Thus, for all $\Psi_0(t)$ satisfying the above condition, we have the reconstruction formula

$$f(t) = \sum_{n \in \mathbb{Z}} \frac{1}{2B'} f\left(\frac{n}{2B'}\right) \Psi_0\left(t - \frac{n}{2B'}\right) \tag{A.5.17}$$

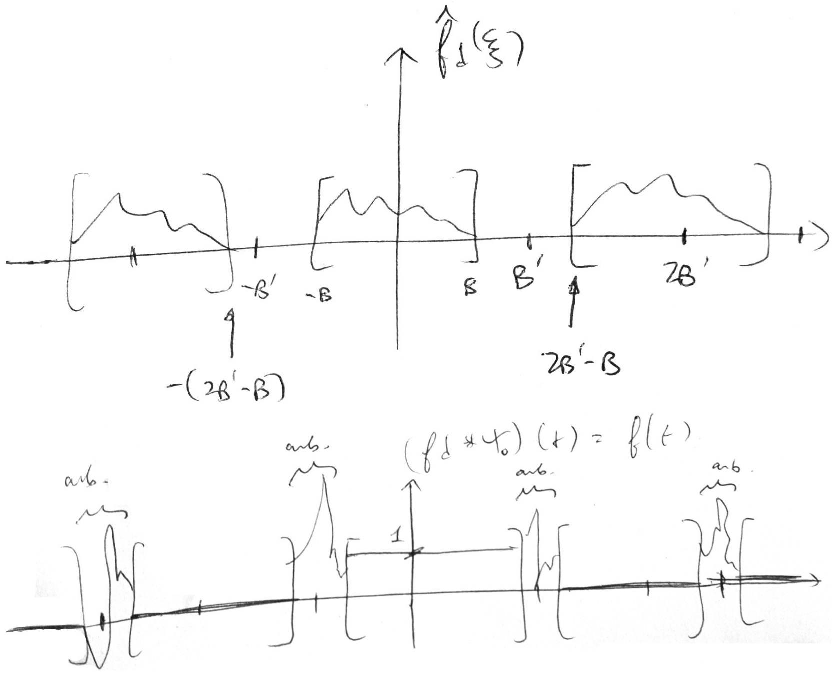


Figure A.3: Condition on $\widehat{\Psi}_0$ for reconstruction formula

One can check that the functions $\phi_0^{(B)}$ and $\psi_0^{(B,B')}$, as well as the $\psi_0^{(h,B,B')}$ implementing the particular case of design freedom discussed in section A.2, verify this condition (modulo normalization). In section B.1 we will prove that this condition is necessary and sufficient for the reconstruction formula to hold; the discussion of this section already gives all the intuition as to where it comes from, though.

We emphasize again that the derivations of this section were not rigorously justified; we showed these calculations mainly to give intuition, and because they are quite common in texts on the sampling expansion.

A.6 N-dimensional sampling expansion

Throughout our discussions of the multidimensional setting we will use a number of shorthand notations. All of them will be introduced in this section as they appear. They are also summarized for convenience in the introductory section of this document.

The Fourier transform in N dimensions is given by: $(\boldsymbol{t}, \boldsymbol{\xi} \in \mathbb{R}^N)$

$$\hat{f}(\boldsymbol{\xi}) = \int_{\mathbb{R}^N} d\boldsymbol{t}\; f(\boldsymbol{t}) e^{-i2\pi \boldsymbol{\xi}\cdot\boldsymbol{t}} \tag{A.6.1}$$

$$f(\boldsymbol{t}) = \int_{\mathbb{R}^N} d\boldsymbol{\xi}\; \hat{f}(\boldsymbol{\xi}) e^{i2\pi \boldsymbol{\xi}\cdot\boldsymbol{t}} \tag{A.6.2}$$

where $\boldsymbol{\xi} \cdot \boldsymbol{t} := \xi_1 t_1 + ... + \xi_N t_N$ is the dot product.

Definition A.6. For all $\boldsymbol{B} \in \mathbb{R}_+^N$, let the space of $\boldsymbol{B}$-band-limited multivariate signals

$$\widehat{\mathbb{B}}_{\boldsymbol{B}} = \left\{ \hat{f} \in L^2(\mathbb{R}^N); \hat{f}(\boldsymbol{\xi}) = 0 \text{ for } \boldsymbol{\xi} \notin [\pm \boldsymbol{B}] \right\} \tag{A.6.3}$$

$$\mathbb{B}_{\boldsymbol{B}} = \left\{ f \in L^2(\mathbb{R}^N); \hat{f} \in \widehat{\mathbb{B}}_{\boldsymbol{B}} \right\} \tag{A.6.4}$$

where we used the shorthand $[\pm\boldsymbol{B}] := [-B_1, B_1] \times ... \times [-B_N, B_N]$ for the centered hyperrectangle of sides $2\boldsymbol{B}$.

A.6.1 Exact sampling

Clearly

$$\hat{\phi}_{\boldsymbol{n}}^{(\boldsymbol{B})}(\boldsymbol{\xi}) = e^{-i2\pi \frac{\boldsymbol{n}}{2\boldsymbol{B}}\cdot\boldsymbol{\xi}} \frac{\mathbb{1}_{\boldsymbol{\xi}\in[\pm\boldsymbol{B}]}}{\sqrt{2\boldsymbol{B}}^{\times}} \tag{A.6.5}$$

$$= e^{-i2\pi\left(\frac{n_1}{2B_1}\xi_1 + ... + \frac{n_N}{2B_N}\xi_N\right)} \frac{\mathbb{1}_{|\xi_1|\le B_1,...,|\xi_N|\le B_N}}{\sqrt{2B_1...2B_N}} \tag{A.6.6}$$

$$= \hat{\phi}_{n_1}^{(B_1)}(\xi_1)...\hat{\phi}_{n_N}^{(B_N)}(\xi_N) \tag{A.6.7}$$

is an ONB of $\widehat{\mathbb{B}}_{\boldsymbol{B}}$. We used the shorthands $\frac{\boldsymbol{n}}{2\boldsymbol{B}} := \left(\frac{n_1}{2B_1}, ..., \frac{n_N}{2B_N}\right)^T$ and $\sqrt{2\boldsymbol{B}}^{\times} := \sqrt{2B_1}...\sqrt{2B_N}$; note that $\sqrt{2\boldsymbol{B}}^{\times} = \sqrt{\mu^N([\pm\boldsymbol{B}])}$ the square root of the volume of the hyperrectangle. Therefore,

$$\forall \hat{f} \in \widehat{\mathbb{B}}_{\boldsymbol{B}}, \hat{f}(\boldsymbol{\xi}) = \sum_{\boldsymbol{n}\in\mathbb{Z}^N} \left\langle \hat{f}, \hat{\phi}_{\boldsymbol{n}}^{(\boldsymbol{B})} \right\rangle \hat{\phi}_{\boldsymbol{n}}^{(\boldsymbol{B})}(\boldsymbol{\xi}) \tag{A.6.8}$$

$$\forall f \in \mathbb{B}_{\boldsymbol{B}}, f(\boldsymbol{t}) = \sum_{\boldsymbol{n}\in\mathbb{Z}^N} \left\langle f, \phi_{\boldsymbol{n}}^{(\boldsymbol{B})} \right\rangle \phi_{\boldsymbol{n}}^{(\boldsymbol{B})}(\boldsymbol{t}) \tag{A.6.9}$$

- Compute $\phi_{\boldsymbol{n}}^{(\boldsymbol{B})}(\boldsymbol{t})$:

$$\hat{\phi}_{\boldsymbol{n}}^{(\boldsymbol{B})}(\boldsymbol{\xi}) = \hat{\phi}_{n_1}^{B_1}(\xi_1)...\hat{\phi}_{n_N}^{B_N}(\xi_N) \tag{A.6.10}$$

$$\phi_{\boldsymbol{n}}^{(\boldsymbol{B})}(\boldsymbol{t}) = \phi_{n_1}^{B_1}(t_1)...\phi_{n_N}^{B_N}(t_N) \tag{A.6.11}$$

$$= \sqrt{2B_1}...\sqrt{2B_N}\, \mathrm{sinc}\left(2B_1\left(t_1 - \frac{n_1}{2B_1}\right)\right) ... \mathrm{sinc}\left(2B_N\left(t_N - \frac{n_N}{2B_N}\right)\right) \tag{A.6.12}$$

$$= \phi_{\boldsymbol{0}}^{(\boldsymbol{B})}\left(\boldsymbol{t} - \frac{\boldsymbol{n}}{2\boldsymbol{B}}\right) \tag{A.6.13}$$

$$\phi_{\boldsymbol{0}}^{(\boldsymbol{B})}(\boldsymbol{t}) = \sqrt{2\boldsymbol{B}}^{\times}\, \mathrm{sinc}^{\times}(2\boldsymbol{B} \odot \boldsymbol{t}) \tag{A.6.14}$$

using the shorthand $\mathrm{sinc}^{\times}(\boldsymbol{t}) := \mathrm{sinc}(t_1)...\mathrm{sinc}(t_N)$, and $\odot$ denotes coordinate-wise multiplication.

- Compute $\left\langle f, \phi_{\boldsymbol{n}}^{(\boldsymbol{B})} \right\rangle$:

$$\left\langle f, \phi_{\boldsymbol{0}}^{(\boldsymbol{B})} \right\rangle = \left\langle \hat{f}, \hat{\phi}_{\boldsymbol{0}}^{(\boldsymbol{B})} \right\rangle = \frac{1}{\sqrt{2\boldsymbol{B}^{\times}}} \left\langle \hat{f}, \mathbb{1}_{\boldsymbol{\xi} \in [\pm \boldsymbol{B}]} e^{i2\pi \cdot 0} \right\rangle \tag{A.6.15}$$

$$= \frac{1}{\sqrt{2\boldsymbol{B}^{\times}}} \int_{[\pm \boldsymbol{B}]} \hat{f}(\boldsymbol{\xi}) e^{i2\pi \cdot 0} d\boldsymbol{\xi} \tag{A.6.16}$$

$$= \frac{1}{\sqrt{2\boldsymbol{B}^{\times}}} \int_{\mathbb{R}^N} \hat{f}(\boldsymbol{\xi}) e^{i2\pi \cdot 0} d\boldsymbol{\xi} \tag{A.6.17}$$

$$= \frac{1}{\sqrt{2\boldsymbol{B}^{\times}}} f(\boldsymbol{0}) \tag{A.6.18}$$

since $\hat{f}$ is supported on $[\pm \boldsymbol{B}]$, and so

$$\left\langle f, \phi_{\boldsymbol{n}}^{(\boldsymbol{B})} \right\rangle = \left\langle f(\boldsymbol{t}), \phi_{\boldsymbol{0}}^{(\boldsymbol{B})} \left(\boldsymbol{t} - \frac{\boldsymbol{n}}{2\boldsymbol{B}} \right) \right\rangle \tag{A.6.19}$$

$$= \left\langle f \left(\boldsymbol{t} + \frac{\boldsymbol{n}}{2\boldsymbol{B}} \right), \phi_{\boldsymbol{0}}^{(\boldsymbol{B})}(\boldsymbol{t}) \right\rangle \tag{A.6.20}$$

$$= \frac{1}{\sqrt{2\boldsymbol{B}^{\times}}} f \left(\frac{\boldsymbol{n}}{2\boldsymbol{B}} \right) \tag{A.6.21}$$

Summary

Definition A.7. Denote

$$\phi_{\boldsymbol{n}}^{(\boldsymbol{B})}(\boldsymbol{t}) = \phi_{n_1}^{(B_1)}(t_1) ... \phi_{n_N}^{(B_N)}(t_N) \tag{A.6.22}$$

Proposition A.6.

$$\forall f \in \mathbb{B}_{\boldsymbol{B}}, f(\boldsymbol{t}) = \sum_{\boldsymbol{n} \in \mathbb{Z}^N} \left\langle f, \phi_{\boldsymbol{n}}^{(\boldsymbol{B})} \right\rangle \phi_{\boldsymbol{n}}^{(\boldsymbol{B})}(\boldsymbol{t}) \tag{A.6.23}$$

$$= \sum_{\boldsymbol{n} \in \mathbb{Z}^N} \frac{1}{\sqrt{2\boldsymbol{B}^{\times}}} f \left(\frac{\boldsymbol{n}}{2\boldsymbol{B}} \right) \sqrt{2\boldsymbol{B}}^{\times} \operatorname{sinc}^{\times} \left(2\boldsymbol{B} \odot \left(\boldsymbol{t} - \frac{\boldsymbol{n}}{2\boldsymbol{B}} \right) \right) \tag{A.6.24}$$

$$= \sum_{\boldsymbol{n} \in \mathbb{Z}^N} f \left(\frac{\boldsymbol{n}}{2\boldsymbol{B}} \right) \operatorname{sinc}^{\times} \left(2\boldsymbol{B} \odot \left(\boldsymbol{t} - \frac{\boldsymbol{n}}{2\boldsymbol{B}} \right) \right) \tag{A.6.25}$$

More explicitly, expanding the shorthands,

$$\forall f \in \mathbb{B}_{\boldsymbol{B}},\ f(\boldsymbol{t}) = \sum_{\boldsymbol{n} \in \mathbb{Z}^N} f \left(\frac{\boldsymbol{n}}{2\boldsymbol{B}} \right) \operatorname{sinc} \left(2B_1 \left(t_1 - \frac{n_1}{2B_1} \right) \right) ... \operatorname{sinc} \left(2B_N \left(t_N - \frac{n_N}{2B_N} \right) \right) \tag{A.6.26}$$

A.6.2 Oversampling (without design freedom)

For hyperrectangle-supported spectrum

Let $\boldsymbol{B}' \geq \boldsymbol{B}$ i.e $\forall i, B_i' \geq B_i$. Then $\mathbb{B}_{\boldsymbol{B}} \subset \mathbb{B}_{\boldsymbol{B}'}$ so

$$\forall \hat{f} \in \widehat{\mathbb{B}}_{\boldsymbol{B}}, \hat{f}(\boldsymbol{\xi}) = \sum_{\boldsymbol{n} \in \mathbb{Z}^N} \left\langle \hat{f}, \hat{\phi}_{\boldsymbol{n}}^{(\boldsymbol{B}')} \right\rangle \hat{\phi}_{\boldsymbol{n}}^{(\boldsymbol{B}')}(\boldsymbol{\xi}) \tag{A.6.27}$$

$$= \sum_{\boldsymbol{n} \in \mathbb{Z}^N} \left\langle \hat{f}, \hat{\phi}_{\boldsymbol{n}}^{(\boldsymbol{B}')} \right\rangle \underbrace{\hat{\phi}_{\boldsymbol{n}}^{(\boldsymbol{B}')}(\boldsymbol{\xi}) \mathbb{1}_{\boldsymbol{\xi} \in [\pm \boldsymbol{B}]}}_{=: \hat{\psi}_{\boldsymbol{n}}^{(\boldsymbol{B}, \boldsymbol{B}')}(\boldsymbol{\xi})} \tag{A.6.28}$$

since $\hat{f}(\boldsymbol{\xi}) = 0$ for $\boldsymbol{\xi} \notin [\pm \boldsymbol{B}]$, where we defined $\hat{\psi}_{\boldsymbol{n}}^{(\boldsymbol{B}, \boldsymbol{B}')}(\boldsymbol{\xi}) = \hat{\phi}_{\boldsymbol{n}}^{(\boldsymbol{B}')}(\boldsymbol{\xi}) \mathbb{1}_{\boldsymbol{\xi} \in [\pm \boldsymbol{B}]}$.

Clearly, its inverse Fourier transform is given by

$$\hat{\psi}_{\boldsymbol{n}}^{(\boldsymbol{B},\boldsymbol{B}')}(\boldsymbol{\xi}) = \hat{\phi}_{n_1}^{(B_1)}(\xi_1)...\hat{\phi}_{n_N}^{(B_N)}(\xi_N)\mathbb{1}_{|\xi_1|\le B_1}...\mathbb{1}_{|\xi_N|\le B_N} \tag{A.6.29}$$

$$= \hat{\psi}_{n_1}^{(B_1,B_1')}(\xi_1)...\hat{\psi}_{n_N}^{(B_N,B_N')}(\xi_N) \tag{A.6.30}$$

$$\psi_{\boldsymbol{n}}^{(\boldsymbol{B},\boldsymbol{B}')}(\boldsymbol{t}) = \psi_{n_1}^{(B_1,B_1')}(t_1)...\psi_{n_N}^{(B_N,B_N')}(t_N) \tag{A.6.31}$$

$$= \frac{2B_1...2B_N}{\sqrt{2B_1'...2B_N'}} \operatorname{sinc}\left(2B_1\left(t_1 - \frac{n_1}{2B_1'}\right)\right)...\operatorname{sinc}\left(2B_N\left(t_N - \frac{n_N}{2B_N'}\right)\right) \tag{A.6.32}$$

$$= \frac{(2\boldsymbol{B})^\times}{\sqrt{2\boldsymbol{B}'}^\times} \operatorname{sinc}^\times\left(2\boldsymbol{B}\odot\left(\boldsymbol{t} - \frac{\boldsymbol{n}}{2\boldsymbol{B}'}\right)\right) \tag{A.6.33}$$

using the shorthand $(2\boldsymbol{B})^\times := 2B_1...2B_N$.

Summary

Definition A.8. Denote

$$\psi_{\boldsymbol{n}}^{(\boldsymbol{B},\boldsymbol{B}')}(\boldsymbol{t}) = \psi_{n_1}^{(B_1,B_1')}(t_1)...\psi_{n_N}^{(B_N,B_N')}(t_N) \tag{A.6.34}$$

Proposition A.7.

$$\forall f \in \mathbb{B}_{\boldsymbol{B}}, f(\boldsymbol{t}) = \sum_{\boldsymbol{n}\in\mathbb{Z}^N} \left\langle f, \phi_{\boldsymbol{n}}^{(\boldsymbol{B})}\right\rangle \psi_{\boldsymbol{n}}^{(\boldsymbol{B},\boldsymbol{B}')}(\boldsymbol{t}) \tag{A.6.35}$$

$$= \sum_{\boldsymbol{n}\in\mathbb{Z}^N} \frac{1}{\sqrt{2\boldsymbol{B}}^\times} f\left(\frac{\boldsymbol{n}}{2\boldsymbol{B}}\right) \frac{(2\boldsymbol{B})^\times}{\sqrt{2\boldsymbol{B}'}^\times} \operatorname{sinc}^\times\left(2\boldsymbol{B}\odot\left(\boldsymbol{t} - \frac{\boldsymbol{n}}{2\boldsymbol{B}'}\right)\right) \tag{A.6.36}$$

$$= \sum_{\boldsymbol{n}\in\mathbb{Z}^N} \frac{(2\boldsymbol{B})^\times}{(2\boldsymbol{B}')^\times} f\left(\frac{\boldsymbol{n}}{2\boldsymbol{B}}\right) \operatorname{sinc}^\times\left(2\boldsymbol{B}\odot\left(\boldsymbol{t} - \frac{\boldsymbol{n}}{2\boldsymbol{B}'}\right)\right) \tag{A.6.37}$$

More explicitly, expanding the shorthands,

$$\forall f \in \mathbb{B}_{\boldsymbol{B}},\ f(\boldsymbol{t}) = \sum_{\boldsymbol{n}\in\mathbb{Z}^N} \frac{2B_1...2B_N}{2B_1'...2B_N'} f\left(\frac{\boldsymbol{n}}{2\boldsymbol{B}}\right) \operatorname{sinc}\left(2B_1\left(t_1 - \frac{n_1}{2B_1'}\right)\right)...\operatorname{sinc}\left(2B_N\left(t_N - \frac{n_N}{2B_N'}\right)\right) \tag{A.6.38}$$

For compact-supported spectrum

For any compact subset $K \subset \mathbb{R}^N$, let

$$\widehat{\mathbb{B}}_K = \left\{\hat{f} \in L^2(\mathbb{R}^N);\ \hat{f}(\boldsymbol{\xi}) = 0 \text{ for } \boldsymbol{\xi} \notin K\right\} \tag{A.6.39}$$

$$\mathbb{B}_K = \left\{f \in L^2(\mathbb{R}^N);\ \hat{f} \in \widehat{\mathbb{B}}_K\right\} \tag{A.6.40}$$

Let $\boldsymbol{B}'$ any vector such that $K \subset [\pm\boldsymbol{B}']$. Then $\mathbb{B}_K \subset \mathbb{B}_{\boldsymbol{B}'}$ so for any $f \in \mathbb{B}_K$,

$$\hat{f}(\boldsymbol{\xi}) = \sum_{\boldsymbol{n}\in\mathbb{Z}^N} \left\langle \hat{f}, \hat{\phi}_{\boldsymbol{n}}^{(\boldsymbol{B}')}\right\rangle \hat{\phi}_{\boldsymbol{n}}^{(\boldsymbol{B}')}(\boldsymbol{\xi}) \tag{A.6.41}$$

$$= \sum_{\boldsymbol{n}\in\mathbb{Z}^N} \left\langle \hat{f}, \hat{\phi}_{\boldsymbol{n}}^{(\boldsymbol{B}')}\right\rangle \underbrace{\hat{\phi}_{\boldsymbol{n}}^{(\boldsymbol{B}')}(\boldsymbol{\xi})\mathbb{1}_{\boldsymbol{\xi}\in K}}_{=:\hat{\psi}_{\boldsymbol{n}}^{(K,\boldsymbol{B}')}(\boldsymbol{\xi})} \tag{A.6.42}$$

(same as for the hyperrectangle-support case).

The function $\hat{\psi}_{\boldsymbol{n}}^{(K,\boldsymbol{B'})}$ and its inverse Fourier transform are given by

$$\hat{\psi}_{\boldsymbol{n}}^{(K,\boldsymbol{B'})}(\boldsymbol{\xi}) = \hat{\phi}_{\boldsymbol{n}}^{(\boldsymbol{B'})}(\boldsymbol{\xi})\mathbb{1}_{\boldsymbol{\xi}\in K} = \frac{1}{\sqrt{2\boldsymbol{B'}}^{\times}} e^{-i2\pi \frac{\boldsymbol{n}}{2\boldsymbol{B'}}\cdot\boldsymbol{\xi}}\ \mathbb{1}_{\boldsymbol{\xi}\in K} \tag{A.6.43}$$

$$= e^{-i2\pi \frac{\boldsymbol{n}}{2\boldsymbol{B'}}\cdot\boldsymbol{\xi}}\ \hat{\psi}_{\boldsymbol{0}}^{(K,\boldsymbol{B'})}(\boldsymbol{\xi}) \tag{A.6.44}$$

$$\psi_{\boldsymbol{n}}^{(K,\boldsymbol{B'})}(\boldsymbol{t}) = \psi_{\boldsymbol{0}}^{(K,\boldsymbol{B'})}\left(\boldsymbol{t} - \frac{\boldsymbol{n}}{2\boldsymbol{B'}}\right) \tag{A.6.45}$$

Summary

Definition A.9. Denote $\psi_{\boldsymbol{0}}^{(K,\boldsymbol{B'})}(\boldsymbol{t})$ the inverse Fourier transform of $\hat{\psi}_{\boldsymbol{0}}^{(K,\boldsymbol{B'})}(\boldsymbol{\xi}) = \frac{1}{\sqrt{2\boldsymbol{B'}}^{\times}}\mathbb{1}_{\boldsymbol{\xi}\in K}$.

Proposition A.8. Let $K \subset \mathbb{R}^N$ a compact and $\boldsymbol{B'} \in \mathbb{R}_+^N$ such that $K \subset [\pm\boldsymbol{B'}]$. The following reconstruction formula holds:

$$\hat{f}(\boldsymbol{\xi}) = \sum_{\boldsymbol{n}\in\mathbb{Z}^N} \left\langle \hat{f}, \hat{\phi}_{\boldsymbol{n}}^{(\boldsymbol{B'})} \right\rangle \hat{\psi}_{\boldsymbol{n}}^{(K,\boldsymbol{B'})}(\boldsymbol{\xi}) \tag{A.6.46}$$

$$f(\boldsymbol{t}) = \sum_{\boldsymbol{n}\in\mathbb{Z}^N} \left\langle f, \phi_{\boldsymbol{n}}^{(\boldsymbol{B'})} \right\rangle \psi_{\boldsymbol{n}}^{(K,\boldsymbol{B'})}(\boldsymbol{t}) \tag{A.6.47}$$

$$= \sum_{\boldsymbol{n}\in\mathbb{Z}^N} \frac{1}{\sqrt{2\boldsymbol{B'}}^{\times}} f\left(\frac{\boldsymbol{n}}{2\boldsymbol{B'}}\right)\ \psi_{\boldsymbol{0}}^{(K,\boldsymbol{B'})}\left(\boldsymbol{t} - \frac{\boldsymbol{n}}{2\boldsymbol{B'}}\right) \tag{A.6.48}$$

A.6.3 A particular case of oversampling with design freedom

Let $f \in \mathbb{B}_K$ and $\boldsymbol{B'}$ such that $K \subset [\pm\boldsymbol{B'}]$. By the same arguments as for the one-dimensional case, for any function $\hat{h}_{arb}(\boldsymbol{\xi})$,

$$\hat{f}(\boldsymbol{\xi}) = \hat{f}(\boldsymbol{\xi})\left[\mathbb{1}_{\boldsymbol{\xi}\in K} + \mathbb{1}_{\boldsymbol{\xi}\notin K}\hat{h}_{arb}(\boldsymbol{\xi})\right] \tag{A.6.49}$$

$$\hat{f}(\boldsymbol{\xi}) = \sum_{\boldsymbol{n}\in\mathbb{Z}^N} \left\langle f, \phi_{\boldsymbol{n}}^{(\boldsymbol{B'})} \right\rangle \hat{\phi}_{\boldsymbol{n}}^{(\boldsymbol{B'})}(\boldsymbol{\xi}) \tag{A.6.50}$$

$$= \sum_{\boldsymbol{n}\in\mathbb{Z}^N} \left\langle f, \phi_{\boldsymbol{n}}^{(\boldsymbol{B'})} \right\rangle \hat{\phi}_{\boldsymbol{n}}^{(\boldsymbol{B'})}(\boldsymbol{\xi})\left[\mathbb{1}_{\boldsymbol{\xi}\in K} + \mathbb{1}_{\boldsymbol{\xi}\notin K}\hat{h}_{arb}(\boldsymbol{\xi})\right] \tag{A.6.51}$$

$$= \sum_{\boldsymbol{n}\in\mathbb{Z}^N} \left\langle f, \phi_{\boldsymbol{n}}^{(\boldsymbol{B'})} \right\rangle \hat{\psi}_{\boldsymbol{n}}^{(h,K,\boldsymbol{B'})}(\boldsymbol{\xi}) \tag{A.6.52}$$

where

$$\hat{\psi}_{\boldsymbol{n}}^{(h,K,\boldsymbol{B'})}(\boldsymbol{\xi}) = \frac{1}{\sqrt{2\boldsymbol{B'}}^{\times}} e^{-i2\pi \frac{\boldsymbol{n}}{2\boldsymbol{B'}}\cdot\boldsymbol{\xi}}\left(\mathbb{1}_{|\boldsymbol{\xi}|\in K} + \mathbb{1}_{\boldsymbol{\xi}\in[\pm\boldsymbol{B'}]\setminus K}\hat{h}(\boldsymbol{\xi})\right) \tag{A.6.53}$$

$$= \hat{\psi}_{\boldsymbol{0}}^{(h,K,\boldsymbol{B'})}(\boldsymbol{\xi})e^{-i2\pi \frac{\boldsymbol{n}}{2\boldsymbol{B'}}\cdot\boldsymbol{\xi}} \tag{A.6.54}$$

$$\psi_{\boldsymbol{n}}^{(h,K,\boldsymbol{B'})}(\boldsymbol{t}) = \psi_{\boldsymbol{0}}^{(h,K,\boldsymbol{B'})}\left(\boldsymbol{t} - \frac{\boldsymbol{n}}{2\boldsymbol{B'}}\right) \tag{A.6.55}$$

Summary

Proposition A.9. Let $K \subset \mathbb{R}^N$ a compact and $\boldsymbol{B'} \in \mathbb{R}_+^N$ such that $K \subset [\pm\boldsymbol{B'}]$. The following reconstruction formula holds:

$$\forall f \in \mathbb{B}_K, f(\boldsymbol{t}) = \sum_{\boldsymbol{n}\in\mathbb{Z}^N} \frac{1}{\sqrt{2\boldsymbol{B'}}^{\times}} f\left(\frac{\boldsymbol{n}}{2\boldsymbol{B'}}\right)\psi_{\boldsymbol{0}}^{(h,K,\boldsymbol{B'})}\left(\boldsymbol{t} - \frac{\boldsymbol{n}}{2\boldsymbol{B'}}\right) \tag{A.6.56}$$

where $\psi_{\mathbf{0}}^{(h,K,\boldsymbol{B}')}(\boldsymbol{t})$ is any function such that its spectrum satisfies

$$\hat{\psi}_{\mathbf{0}}^{(h,K,\boldsymbol{B}')}(\boldsymbol{\xi}) = \begin{cases} \frac{1}{\sqrt{2\boldsymbol{B}'^{\times}}} & \text{for } \boldsymbol{\xi} \in K \\ \text{arbitrary} & \text{for } \boldsymbol{\xi} \in [\pm \boldsymbol{B}'] \setminus K \\ 0 & \text{for } \boldsymbol{\xi} \notin [\pm \boldsymbol{B}'] \end{cases} \tag{A.6.57}$$

A.6.4 The set of sample-sequences for oversampled signals

Recall that, for the one-dimensional case, the set

$$\mathbb{A}_{B/B'} := \left\{ (c_n)_n = \left(f\left(\frac{n}{2B'} \right) \right)_n, \ f \in \mathbb{B}_B \right\} \tag{A.6.58}$$

was shown to be equal to

$$\mathbb{A}_{B/B'} = \left\{ (c_n)_n \in \ell^2(\mathbb{Z}), \ \mathcal{F}_{d,2B'}[c](\xi) \text{ is supported on } [-B, B] \right\} \tag{A.6.59}$$

$$= \left\{ (c_n)_n \in \ell^2(\mathbb{Z}) \ ; \ \mathcal{F}_d[c](\underline{\xi}) \text{ is supported on } \left[-\frac{B}{2B'}, \frac{B}{2B'} \right] \right\} \tag{A.6.60}$$

(The second equality holds by definition of the DTFT $\mathcal{F}_d$.)

Definition A.10. Define the N-dimensional DTFT: for $(a_{\boldsymbol{n}})_{\boldsymbol{n} \in \mathbb{Z}^N}$,

$$\mathcal{F}_d[a](\boldsymbol{x}) = \sum_{\boldsymbol{n} \in \mathbb{Z}^N} a_{\boldsymbol{n}} e^{-i2\pi \boldsymbol{n} \cdot \boldsymbol{x}} \qquad \in L^2\left([-1/2, 1/2]^N \right) \tag{A.6.61}$$

and the $2\boldsymbol{B}$-unit DTFT

$$\mathcal{F}_{d,2\boldsymbol{B}}[a](\xi) := \sum_{\boldsymbol{n} \in \mathbb{Z}^N} a_{\boldsymbol{n}} \frac{e^{-i2\pi \frac{\boldsymbol{n}}{2\boldsymbol{B}} \cdot \boldsymbol{\xi}}}{\sqrt{2B_1 ... 2B_N}} \qquad \in L^2\left([-B_1, B_1] \times ... \times [-B_N, B_N] \right) \tag{A.6.62}$$

$$= \sum_{\boldsymbol{n} \in \mathbb{Z}^N} a_{\boldsymbol{n}} \frac{e^{-i2\pi \frac{\boldsymbol{n}}{2\boldsymbol{B}} \cdot \boldsymbol{\xi}}}{\sqrt{2\boldsymbol{B}}^{\times}} \qquad \in L^2([\pm \boldsymbol{B}]) \tag{A.6.63}$$

Then likewise, we have

Proposition A.10. For any compact $K \subset \mathbb{R}^N$ and $\boldsymbol{B}'$ such that $K \subset [\pm \boldsymbol{B}']$, the set

$$\mathbb{A}_{K/\boldsymbol{B}'} := \left\{ (c_{\boldsymbol{n}})_{\boldsymbol{n}} = \left(f\left(\frac{\boldsymbol{n}}{2\boldsymbol{B}'} \right) \right)_{\boldsymbol{n}}, \ f \in \mathbb{B}_K \right\} \tag{A.6.64}$$

is equal to

$$\mathbb{A}_{K/\boldsymbol{B}'} = \left\{ (c_{\boldsymbol{n}})_{\boldsymbol{n}} \in \ell^2(\mathbb{Z}^N) \ ; \ \ \mathcal{F}_{d,2\boldsymbol{B}'}[c](\boldsymbol{\xi}) \text{ is supported on } K \right\} \tag{A.6.65}$$

$$= \left\{ (c_{\boldsymbol{n}})_{\boldsymbol{n}} \in \ell^2(\mathbb{Z}^N) \ ; \ \mathcal{F}_d[c](\underline{\boldsymbol{\xi}}) \text{ is supported on } K/(2\boldsymbol{B}') \right\} \tag{A.6.66}$$

where $K/(2\boldsymbol{B}') := \left\{ \left(\frac{\xi_1}{2B'_1}, ..., \frac{\xi_N}{2B'_N} \right)^T, \ \boldsymbol{\xi} \in K \right\} \subset [-1/2, 1/2]^N$.

A.6.5 Poisson summation formula

Recall the result for the one-dimensional case:

$$\forall B, b > 0, \ \forall f \in \mathbb{B}_B, \ \sum_{n \in \mathbb{Z}} \hat{f}(\xi + 2nb) = \sum_{n \in \mathbb{Z}} \frac{1}{2b} f\left(\frac{n}{2b} \right) e^{-i2\pi \frac{n}{2b} \xi} \tag{A.6.67}$$

If $f \in \mathbb{B}_K$ for some compact $K \subset \mathbb{R}^N$, by applying the one-dimensional formula for each dimension sequentially, we directly obtain that

$$\forall \boldsymbol{b} \in \mathbb{R}_+^N, \ \sum_{\boldsymbol{n} \in \mathbb{Z}^N} \hat{f}(\boldsymbol{\xi} + 2\boldsymbol{n} \odot \boldsymbol{b}) = \sum_{\boldsymbol{n} \in \mathbb{Z}^N} \frac{1}{2b_1 ... 2b_N} f\left(\frac{\boldsymbol{n}}{2\boldsymbol{b}} \right) e^{-i2\pi \frac{\boldsymbol{n}}{2\boldsymbol{b}} \cdot \boldsymbol{\xi}} \tag{A.6.68}$$

where $\odot$ denotes pointwise multiplication.

Appendix B

An elementary view of the sampling expansion for band-limited functions: oversampling with design freedom

An elementary view of the Nyquist-Shannon sampling expansion for band-limited square-integrable signals, part 2/3.

B.1 Characterizing design freedom in one dimension, synthesizer functions

B.1.1 Definition, notation

Definition B.1. A function Ψ_0 is called a *synthesizer of* $\mathbb{B}_B$ *for* $\frac{1}{2B'}$*-sampling*, or simply (B, B')-*synthesizer*, if it holds: (note that this is not standard terminology)

$$\forall f \in \mathbb{B}_B, f(t) = \sum_{n \in \mathbb{Z}} \frac{1}{2B'} f\left(\frac{n}{2B'}\right) \Psi_0\left(t - \frac{n}{2B'}\right) \tag{B.1.1}$$

Remark B.1. Compared to our discussion so far, we use a different convention for the normalization. As a general rule, functions denoted by lowercase letters use the previous normalization scheme, while uppercase letters indicate the use of the following unnormalized variant: $\Psi_0(t) \leftrightsquigarrow \sqrt{2B'}\psi_0(t)$.

Example B.1. From our earlier discussion of basic standard results, the following functions are (B, B')-synthesizers:

- $\Phi_0^{(B')}(t) = 2B' \operatorname{sinc}(2B't)$ $\widehat{\Phi}_0^{(B')}(\xi) = \mathbb{1}_{|\xi| \leq B'}$
- $\Psi_0^{(B,B')}(t) = 2B \operatorname{sinc}(2Bt)$ $\widehat{\Psi}_0^{(B,B')}(\xi) = \mathbb{1}_{|\xi| \leq B}$
- Any $\Psi_0^{(h,B,B')}(t)$ whose spectrum satisfies

$$\widehat{\Psi}_0^{(h,B,B')}(\xi) = \begin{cases} 1 & \text{for } |\xi| \leq B \\ \text{arbitrary} & \text{for } B < |\xi| \leq B' \\ 0 & \text{for } B' < |\xi| \end{cases} \tag{B.1.2}$$

As already stated in our unrigorous discussion of section A.5 based on the spectrum of the discretized signal, we have the following characterization.

Proposition B.1. $\Psi_0(t)$ is a synthesizer if and only if: (see Figure B.1)

$$\widehat{\Psi}_0(\xi) = \begin{cases} 1 & \text{for } |\xi| \leq B \\ 0 & \text{for } \xi \in [\pm B] + 2B'\mathbb{Z}^* := \bigsqcup_{n\in\mathbb{Z}^*}[2nB' - B, 2nB' + B] \\ \text{arbitrary} & \text{everywhere else} \end{cases} \tag{B.1.3}$$

We will refer to this condition as *the synthesizer condition.*

Remark B.2. The equality $[\pm B] + 2B'\mathbb{Z}^* := \bigsqcup_{n\in\mathbb{Z}^*}[2nB' - B, 2nB' + B]$ (where $\sqcup$ denotes disjoint union) simply reflects the fact that, since $B' \geq B$, the repetitions of the spectrum have disjoint support. In terms of Figure B.1, that set is represented by the union of the bracketed intervals (except the one around zero), which are clearly disjoint.

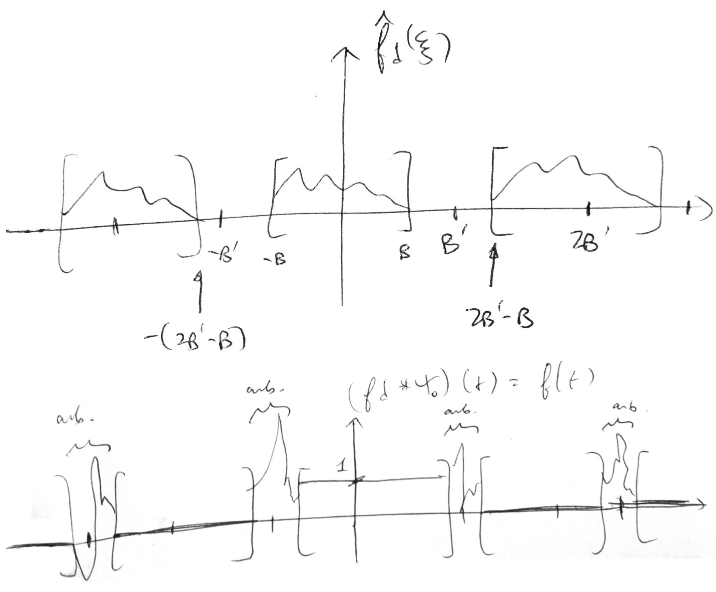


Figure B.1: Condition on $\widehat{\Psi}_0$ for reconstruction formula

Remark B.3. In the literature, people don't seem to care that much about this synthesizer condition, perhaps because there is a particular class of practically useful synthesizers that uses a different representation, discussed momentarily. The only paper that we are aware of that contains a discussion of the synthesizer condition, is [Cam68].

This section answers the following question:

> (Given fixed $B' \geq B$,) what are the functions $\Psi_0(t)$ such that it holds $\forall f \in \mathbb{B}_B, f(t) = \sum_{n\in\mathbb{Z}} \frac{1}{2B'} f\left(\frac{n}{2B'}\right) \Psi_0\left(t - \frac{n}{2B'}\right)$?

A related and more general question would be:

> What are the families $(\Psi_n(t))_n$ such that it holds $\forall f \in \mathbb{B}_B, f(t) = \sum_{n\in\mathbb{Z}} \frac{1}{2B'} f\left(\frac{n}{2B'}\right) \Psi_n(t)$?

Unfortunately, the derivations to follow cannot be extended to approach this question. It will also be touched on briefly in section B.3 but without satisfactory conclusion.

B.1.2 Proof for the L^2 point of view

Here we prove the characterization of synthesizer functions for the L^2 setting. The calculations will look exactly the same as in our unrigorous discussion of section A.5: by using the Poisson summation formula, we are led to "multiply out" the repetitions of the spectrum, except for the original one. The only difference is that, instead of seeing the Poisson summation formula as a statement on the Fourier transform of Dirac combs, we see it as a statement on the (periodicized) DTFT.

Necessity Fix $B' \geq B$. Let $\Psi_0(t)$ such that

$$\forall f \in \mathbb{B}_B,\ f(t) = \sum_{n\in\mathbb{Z}} \frac{1}{2B'} f\left(\frac{n}{2B'}\right)\ \Psi_0\left(t - \frac{n}{2B'}\right) \tag{B.1.4}$$

$$\text{i.e}\ \ \hat{f}(\xi) = \sum_{n\in\mathbb{Z}} \frac{1}{2B'} f\left(\frac{n}{2B'}\right)\ \widehat{\Psi}_0(\xi) e^{-i2\pi\frac{n}{2B'}\xi} \tag{B.1.5}$$

Denote the "canonical" synthesizer $\Phi_0^{(B')}(t) = 2B' \operatorname{sinc}(2B't)$ ($\widehat{\Phi}_0^{(B)}(\xi) = \mathbb{1}_{|\xi|\leq B'}$), and $\Delta\Psi_0(t) = \Psi_0(t) - \Phi_0^{(B')}(t)$ (and $\Delta\widehat{\Psi}_0(\xi) = \widehat{\Delta\Psi_0}(\xi) = \widehat{\Psi}_0(\xi) - \widehat{\Phi}_0^{(B')}(\xi)$). We showed previously that $\Phi_0^{(B')}(t)$ satisfies the equality above, so that equivalently we want to characterize the $\Delta\Psi_0(t)$ such that

$$\forall f \in \mathbb{B}_B,\ \sum_{n\in\mathbb{Z}} f\left(\frac{n}{2B'}\right)\ \Delta\widehat{\Psi}_0(\xi) e^{-i2\pi\frac{n}{2B'}\xi} = 0 \tag{B.1.6}$$

Notice that this expression involves the $2B'$-unit DTFT of the sample-sequence, i.e the right-hand-side of the Poisson summation formula (up to constant normalizing factors).

$$\forall f \in \mathbb{B}_B,\ \forall \xi \in \mathbb{R},\ \underbrace{\sum_{n\in\mathbb{Z}} f\left(\frac{n}{2B'}\right)\ e^{-i2\pi\frac{n}{2B'}\xi}}\ \Delta\widehat{\Psi}_0(\xi) = 0 \tag{B.1.7}$$

$$\mathcal{F}_{d,2B'}\left[\left(f\left(\frac{n}{2B'}\right)\right)_n\right](\xi)\ \Delta\widehat{\Psi}_0(\xi) = 0 \tag{B.1.8}$$

$$\sum_{n\in\mathbb{Z}} \hat{f}(\xi + 2nB')\ \Delta\widehat{\Psi}_0(\xi) = 0 \tag{B.1.9}$$

Since this is an equality that holds for (almost) all $\xi \in \mathbb{R}$ and for all $\hat{f} \in \widehat{\mathbb{B}}_B$, it implies that $\Delta\widehat{\Psi}_0(\xi) = 0$ for all $\xi \in \bigsqcup_{n\in\mathbb{Z}}[2nB' - B, 2nB' + B]$.

Sufficiency Conversely, clearly $\Delta\widehat{\Psi}_0(\xi) = 0$ on that set implies the above equality.

This concludes the proof.

B.2 Window-based synthesizers in one dimension

We now turn our attention to a class of synthesizers commonly used in practice: (central)-window-based synthesizers.

Synthesizers with spectrum supported on the central interval

Definition B.2. We say that a synthesizer has its spectrum supported on the central interval, if (in addition to the synthesizer condition) its Fourier transform is supported on $[-2B', 2B']$.

Example B.2. The "canonical" synthesizer $\Phi_0^{(B')}(t) = 2B' \operatorname{sinc}(2B't)$, $\widehat{\Phi}_0^{(B')}(\xi) = \mathbb{1}_{|\xi|\leq B'}$ has spectrum supported on the central interval.

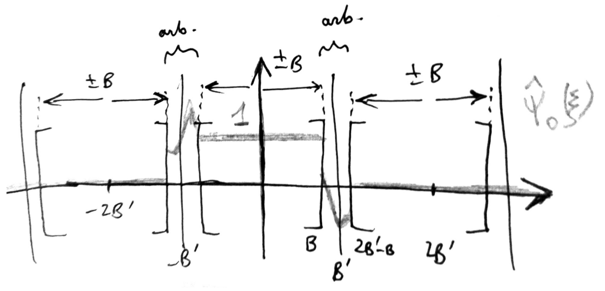


Figure B.2: Spectrum of a (B, B')-*synthesizer with spectrum supported on the central interval.* It is $= 1$ on the central interval $[-B, B]$, arbitrary in the gaps surrounding the central interval, and zero everywhere else.

Remark B.4. One can check that a function $\Psi_0 \in L^2(\mathbb{R})$ is a *synthesizer with spectrum supported on the central interval* if and only if: (see Figure B.2)

$$\widehat{\Psi}_0(\xi) = \begin{cases} 1 & \text{for } |\xi| \le B \\ \text{arbitrary} & \text{for } B < |\xi| \le 2B' - B \\ 0 & \text{for } 2B' - B < |\xi| \end{cases} \tag{B.2.1}$$

(Central)-Window-based synthesizers As noted in [PS96, (2.4)], a sufficient condition for Ψ_0 to be a synthesizer is: $\Psi_0(t) = \Phi_0^{(B')}(t)g(t)$ for some well-chosen function g.

Definition B.3. A *(central) window function* [1] is any function g such that

$$\Psi_0(t) = \Phi_0^{(B')}(t)g(t) \tag{B.2.2}$$

defines a synthesizer with spectrum supported on the central interval.

A *generalized window function* is any function g such that the above formula defines a synthesizer.

For window function g, $\Psi_0(t) = \Phi_0^{(B')}(t)g(t)$ is called the associated window-based synthesizer.

B.2.1 Characterization of central window functions

Proposition B.2. $g \in L^2(\mathbb{R})$ is a central window function if and only if

$$\begin{cases} \operatorname{supp}(\hat{g}) \subset [\pm(B' - B)] \\ g(0) = 1 \end{cases} \tag{B.2.3}$$

Remark B.5. This condition automatically implies $\|g\|_{L^\infty} \le \|\hat{g}\|_{L^1} \le 2(B' - B)\|\hat{g}\|_{L^2} < \infty$, so $g \in L^\infty(\mathbb{R})$, and so $\Psi_0(t)$ is indeed in $L^2(\mathbb{R})$.

Moreover, note that we restricted our attention to candidate g's in $L^2(\mathbb{R})$; we could in fact consider a more general space ($g \in \mathcal{F}[L^1(\mathbb{R})]$ the range space of the $L^1 \to L^\infty$ Fourier transform).

Necessity Let $g \in L^2(\mathbb{R})$ such that $\Psi_0(t) = \Phi_0^{(B')}(t)g(t)$ is a synthesizer with spectrum supported on the central interval. We want to show that g satisfies the announced condition.

This can be proved using the same arguments as the ones detailed in the next subsection, so we leave it as a (rather trivial) exercise to make the necessary adaptations. The only difference is that the weak differential is zero, and so $\hat{g}$ is periodic, on a larger set of ξ.

[1] This is consistent with the terminology used in [PS96], but not with [Böl20, section 1.5], which discusses quite a different context, and where "window function" morally corresponds to what we called synthesizer.

Sufficiency Let $g \in L^2(\mathbb{R})$ satisfying the announced condition, and let $\Psi_0(t) = \Phi_0^{(B')}(t)g(t)$. Then, $\widehat{\Psi}_0(\xi) = (\widehat{\Phi}_0^{(B')} * \hat{g})(\xi)$.

Recall the following property of convolution: if f is supported on A and g is supported on B, then $f * g$ is supported on $A + B := \{a + b;\ a \in A, b \in B\}$ (Minkowski sum). Indeed,

$$(f * g)(x) = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} dy_1\ dy_2\ f(y_1)g(y_2)\mathbb{1}_{y_1+y_2=x} \tag{B.2.4}$$

$$= \int_A \int_B dy_1\ dy_2\ f(y_1)g(y_2) \qquad \underbrace{\mathbb{1}_{y_1+y_2=x}}_{=0 \text{ for all } y_1,y_2 \text{ if } x\notin A+B} \tag{B.2.5}$$

Thus $\widehat{\Psi}_0$ is supported on $\mathrm{supp}(\hat{\phi}_0^{(B')}) + \mathrm{supp}(\hat{g}) \subset [\pm B'] + [\pm(B' - B)] = [\pm(2B' - B)]$.

Furthermore, for any $|\xi| \leq B$,

$$\widehat{\Psi}_0(\xi) = \left(\widehat{\Phi}_0^{(B')} * \hat{g}\right)(\xi) \tag{B.2.6}$$

$$= \int_{-\infty}^{\infty} dy\ \hat{g}(y)\mathbb{1}_{|\xi-y|\leq B'} \tag{B.2.7}$$

$$= \int_{\xi-B'}^{\xi+B'} dy\ \hat{g}(y) \tag{B.2.8}$$

Now since $B' \geq B$ and $|\xi| \leq B$,

$$[\xi - B', \xi + B'] \supset [B - B', -B + B'] \tag{B.2.9}$$

$$= [\pm(B' - B)] \tag{B.2.10}$$

$$\supset \mathrm{supp}(\hat{g}) \tag{B.2.11}$$

so

$$\widehat{\Psi}_0(\xi) = \int_{-\infty}^{\infty} dy\ \hat{g}(y)e^{i2\pi\cdot 0} = 1 \tag{B.2.12}$$

Thus Ψ_0 satisfies the conditions characterizing synthesizers with spectrum supported on the central interval.

This concludes the proof.

B.2.2 Characterization of generalized window functions

Proposition B.3. *$g \in L^2(\mathbb{R})$ is a generalized window function if and only if*

$$\begin{cases} \mathrm{supp}(\hat{g}) \subset [\pm(B' - B)] + 2B'\mathbb{Z} \\ \forall n \neq 0, \int_{(2n-1)B'}^{(2n+1)B'} dy\ \hat{g}(y) = 0 \\ g(0) = \int_{-B'}^{B'} dy\ \hat{g}(y) = 1 \end{cases} \tag{B.2.13}$$

Necessity Let $g \in L^2(\mathbb{R})$ such that $\Psi_0(t) = \Phi_0^{(B')}(t)g(t)$ satisfies the synthesizer condition. Then

$$\forall \xi \in \mathbb{R},\ \widehat{\Psi}_0(\xi) = (\widehat{\Phi}_0^{(B')} * \hat{g})(\xi) = (\mathbb{1}_{|\xi|\leq B'} * \hat{g})(\xi) \tag{B.2.14}$$

$$= \int_{\xi-B'}^{\xi+B'} dy\ \hat{g}(y) \tag{B.2.15}$$

Evaluating at $\xi_n = 2nB'$, we get as announced

$$\forall n \neq 0, \int_{2nB'-B'}^{2nB'+B'} dy\ \hat{g}(y) = \widehat{\Psi}_0(2nB') = 0 \tag{B.2.16}$$

$$\int_{-B'}^{B'} dy\ \hat{g}(y) = \widehat{\Psi}_0(0) = 1 \tag{B.2.17}$$

Furthermore, since $\mathbb{R} = \bigsqcup_{n\in\mathbb{Z}}[(2n-1)B', (2n+1)B']$,

$$g(0) = \int_{\mathbb{R}} dt\ \hat{g}(y)e^{i2\pi\cdot 0} = \sum_{n\in\mathbb{Z}} \int_{2nB'-B'}^{2nB'+B'} dy\ \hat{g}(y) = 1 \tag{B.2.18}$$

Besides, $\widehat{\Psi}_0(\xi)$ is weakly differentiable with $\widehat{\Psi}_0'(\xi) = \hat{g}(\xi + B') - \hat{g}(\xi - B')$, and since $\widehat{\Psi}_0$ is constant on each interval $[\pm B] + 2nB'$ $(n \in \mathbb{Z})$,

$$\forall n \in \mathbb{Z}, \forall\, |\xi - 2nB'| \leq B,\ \hat{g}(\xi + B') - \hat{g}(\xi - B') = 0 \tag{B.2.19}$$

By staring long enough at this condition, and with the help of Figure B.3, we let the reader convince themselves that it is equivalent to:

$$\text{the restriction } \ \hat{g}|_{(2n+1)B'+[\pm B]} \ \text{ is independent of } n \in \mathbb{Z} \tag{B.2.20}$$

Now, in particular, since those sets are disjoint (whenever $B' \geq B$) (as easily seen from the figure),

$$\forall n \in \mathbb{Z}, \int_{(2n+1)B'+[\pm B]} dy\ \ |\hat{g}(y)|^2 = a_0 \text{ independent of } n \tag{B.2.21}$$

$$\infty > \int_{\mathbb{R}} dy\ \ |\hat{g}(y)|^2 \geq \sum_{n\in\mathbb{Z}} \int_{(2n+1)B'+[\pm B]} dy\ \ |\hat{g}(y)|^2 = \sum_{n\in\mathbb{Z}} a_0 \tag{B.2.22}$$

So for all n, $\int_{(2n+1)B'+[\pm B]} dy\ \ |\hat{g}(y)|^2 = 0$ and $\hat{g}$ is zero on those sets. In other words, $\hat{g}$ is supported on their complement, which is easily seen from the figure to be equal to the announced set:

$$\text{supp}(\hat{g}) \subset \mathbb{R} \setminus \left(\bigcup_{n\in\mathbb{Z}} (2n+1)B' + [\pm B] \right) = \bigcup_{n\in\mathbb{Z}} 2nB' + [\pm(B' - B)] = [\pm(B' - B)] + 2B'\mathbb{Z} \tag{B.2.23}$$

Sufficiency Let $g \in L^2(\mathbb{R})$ satisfying the announced condition, and let $\Psi_0(t) = \Phi_0^{(B')}(t)g(t)$. Starting again from $\widehat{\Psi}_0(\xi) = (\widehat{\Phi}_0^{(B')} * \hat{g})(\xi) = \int_{\xi-B'}^{\xi+B'} dy\ \hat{g}(y)$, the idea is to compute this integral (for the relevant ξ) by distinguishing whether y can be in $\text{supp}(\hat{g})$. Again, it is very helpful to refer to Figure B.3.

For all $n \in \mathbb{Z}$ and $\xi \in [\pm B] + 2nB'$,

$$\begin{aligned}
\widehat{\Psi}_0(\xi) &= \int_{[\xi-B',\xi+B']} dy\ \hat{g}(y) && \text{(B.2.24)}\\
&= \int_{[\xi-B',2nB'-(B'-B)]} dy\ \hat{g}(y) + \int_{[2nB'-(B'-B),2nB'+(B'-B)]} dy\ \hat{g}(y) + \int_{[2nB'+(B'-B),\xi+B']} dy\ \hat{g}(y) && \text{(B.2.25)}\\
&= 0 + \int_{[2nB'-(B'-B),2nB'+(B'-B)]} dy\ \hat{g}(y) + 0 && \text{(B.2.26)}\\
&= \int_{[2nB'-B',2nB'+B']} dy\ \hat{g}(y) && \text{(B.2.27)}\\
&= \mathbb{1}_{n=0} && \text{(B.2.28)}
\end{aligned}$$

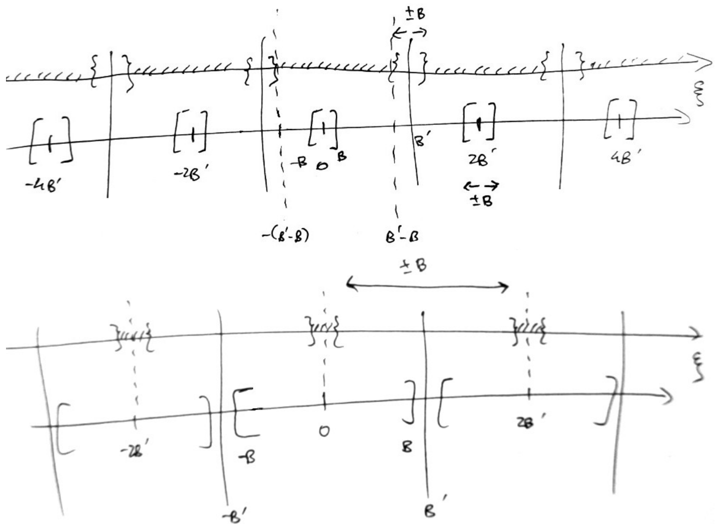


Figure B.3: Top: when $B < B'/2$. Bottom: when $B'/2 < B < B'$.
Bottom axis for each setting: the synthesizer condition is that $\widehat{\Psi}_0$ must be zero on the bracketed intervals, except the one around zero. Top axis for each setting: the sets of interest $(2n+1)B' + [\pm B]$ are the curly-bracketed intervals.

And this is exactly the synthesizer condition for Ψ_0. Let us now explain the above calculation step by step.

The first equality is just a reformulation of the premise $\Psi_0(t) = \Phi_0^{(B')}(t)g(t)$.

In the second equality, one can check that the three integrals are over proper intervals (i.e of the form $[a,b]$ with $a \leq b$), using that $\xi \in [\pm B] + 2nB'$ i.e $2nB' - B \leq \xi \leq 2nB' + B$, and that $B' \geq B$.

In the third equality, we notice that $[\xi - B', 2nB' - (B' - B)] \cap \operatorname{supp}(\hat{g}) = [2nB' + (B' - B), \xi + B'] \cap \operatorname{supp}(\hat{g}) = \varnothing$ so only the second term remains. (These disjointness statements are not difficult to check with the help of the figure.)

In the fourth equality, also using the assumption on the support of $\hat{g}$ (and also with the help of the figure), we see that $\hat{g}$ is zero on $[2nB' - B', 2nB' + B'] \setminus [2nB' - (B' - B), 2nB' + (B' - B)]$, so that the second term can be written as above.

In the fifth equality, we simply make use of the conditions on the value of g.

This concludes the proof.

B.2.3 Characterization of (central)-window-based synthesizers among synthesizers with spectrum supported on the central interval

A natural question is then: does the converse hold, i.e can any synthesizer with spectrum supported on the central interval be put in the form $\Phi_0^{(B')}(t)g(t)$ for some window function g? The following proposition answers negatively, because of smoothness considerations.

Proposition B.4. Let Ψ_0 a synthesizer with spectrum supported on the central interval.

Ψ_0 is a window-based synthesizer, if and only if $\widehat{\Psi}_0 \in H^1(\mathbb{R})$ the first-order Sobolev (Hilbert) space, [2] and $\widehat{\Psi}_0'\Big|_{[-2B',0]} = -\widehat{\Psi}_0'\Big|_{[0,2B']}$.

[2]For a reminder on Sobolev (Hilbert) spaces and their relation to the Fourier transform, see e.g `https://en.wikipedia.org/wiki/Sobolev_space#One-dimensional_case`.

Necessity Indeed if Ψ_0 is such that

$$\widehat{\Psi}_0(\xi) = (\widehat{\Phi}_0^{(B')} * \hat{g})(\xi) \quad \text{where} \quad \begin{cases} g \in L^2(\mathbb{R}) \\ \mathrm{supp}(\hat{g}) \subset [\pm(B'-B)] \\ g(0) = 1 \end{cases} \tag{B.2.29}$$

(where $\widehat{\Phi}_0^{(B')}(\xi) = \mathbb{1}_{|\xi|\le B'}$) then

$$\forall \xi \in \mathbb{R},\ \widehat{\Psi}_0(\xi) = \int_{\xi-B'}^{\xi+B'} dy\ \hat{g}(y) \tag{B.2.30}$$

This implies that $\widehat{\Psi}_0$ is weakly differentiable and that its differential is square-integrable, i.e $\widehat{\Psi}_0 \in H^1(\mathbb{R})$, with

$$\forall \xi \in \mathbb{R},\ \widehat{\Psi}_0'(\xi) = \hat{g}(\xi + B') - \hat{g}(\xi - B') \tag{B.2.31}$$

(This could also be seen by a property of convolution, as $(\mathbb{1}_{|\xi|\le B'} * \hat{g})'(\xi) = \left((\mathbb{1}_{|\xi|\le B'})' * \hat{g}\right)(\xi) = \left((\delta_{-B'} - \delta_{B'}) * \hat{g}\right)(\xi) = \hat{g}(\xi + B') - \hat{g}(\xi - B')$, where δ_a is the a-delayed Dirac delta measure.)

Now, since $\mathrm{supp}(\hat{g}) \subset [\pm(B'-B)] \subset [\pm B']$, then $\hat{g}(\xi + B') \subset [-2B', 0]$ and $\hat{g}(\xi - B') \subset [0, 2B']$. So $\widehat{\Psi}_0'(\xi)$ can be seen as the superposition of two disjoint translated repetitions of $\hat{g}$ (up to sign), as illustrated in Figure B.4.

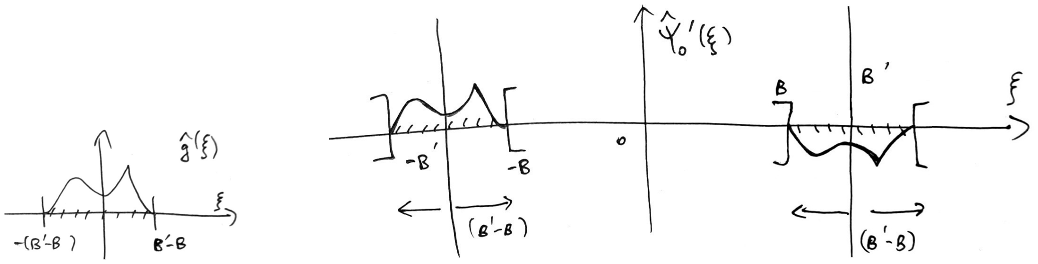


Figure B.4: $\widehat{\Psi}_0'(\xi)$ can be seen as the superposition of two disjoint translated repetitions of $\hat{g}$, by $-B'$ and $+B'$ respectively.

Sufficiency Conversely, it is straightforward to check that, given a synthesizer with spectrum supported on the central interval Ψ_0 such that additionally $\widehat{\Psi}_0 \in H^1(\mathbb{R})$ and $\left.\widehat{\Psi}_0'\right|_{[-2B',0]} = -\left.\widehat{\Psi}_0'\right|_{[0,2B']}$, the formulas

$$\forall\, |\xi| > B' - B,\ \hat{g}(\xi) = 0 \tag{B.2.32}$$

$$\forall \xi \in \mathbb{R},\ \hat{g}(\xi + B') - \hat{g}(\xi - B') = \widehat{\Psi}_0'(\xi) \tag{B.2.33}$$

define a unique $\hat{g} \in L^2(\mathbb{R})$ such that $\Psi_0(t) = \phi_0^{(B')}(t) g(t)$, and that $g(0) = 1$.

B.3 Helms-Thomas expansion

B.3.1 Result and original derivation

Proposition B.5. For all $b, B > 0$, $m \in \mathbb{N}^*$ and $B' := B + mb$, the following reconstruction formula holds:

$$\forall f \in \mathbb{B}_B,\ f(t) = \sum_{n\in\mathbb{Z}} f\left(\frac{n}{2B'}\right) \left[\mathrm{sinc}\left(2b\left(t - \frac{n}{2B'}\right)\right)\right]^m \mathrm{sinc}\left(2B'\left(t - \frac{n}{2B'}\right)\right) \tag{B.3.1}$$

This expansion was first introduced in [HT62]. It is studied in [Jag66], [Zam79], and [PS96] (among others). We follow those three papers in calling it the Helms-Thomas expansion.

The original paper [HT62] derives this formula using the classical sampling expansion and a trick. [Jag66] and [Zam79] reprove it essentially the same way. Here we reproduce their derivation, adapted to our notation.

Proof. Let $b, B > 0$, $m \in \mathbb{N}^*$, and $B' \geq B + mb$. Let $f \in \mathbb{B}_B$.

Recall that if F is supported on A and G is supported on B, then $F * G$ is supported on $A + B := \{a + b;\ a \in A, b \in B\}$.

Denote

$$\hat{g}_{s,b}(\xi) = \mathbb{1}_{|\xi| \leq b} e^{-i2\pi\xi s} \tag{B.3.2}$$
$$g_{s,b}(t) = 2b \operatorname{sinc}(2b(t - s)) \tag{B.3.3}$$

Then since $f \in \mathbb{B}_B$, $\widehat{f g_{s,b}}(\xi) = \left(\hat{f} * \hat{g}_{s,b}\right)(\xi)$ is supported on $[-B - b, B + b]$. Further, let

$$g_{s,b,m}(t) = g_{s,b}(t)^m \tag{B.3.4}$$
$$\hat{g}_{s,b,m}(\xi) = \underbrace{(\hat{g}_{s,b} * \ldots * \hat{g}_{s,b})}_{m \text{ times}}(\xi) \tag{B.3.5}$$

Then $\widehat{f g_{s,b,m}}(\xi)$ is supported on $[-B - mb, B + mb]$.

Thus the product $f(t)\left[\operatorname{sinc}(2b(t - s))\right]^m$ is $(B + mb)$-band-limited. The one-dimensional sampling expansion (oversampling without design freedom, using $B' \geq (B + mb)$) yields

$$f(t)\left[\operatorname{sinc}(2b(t - s))\right]^m = \sum_{n \in \mathbb{Z}} \frac{2(B + mb)}{2B'} f\left(\frac{n}{2B'}\right)\left[\operatorname{sinc}\left(2b\left(\frac{n}{2B'} - s\right)\right)\right]^m \operatorname{sinc}\left(2(B + mb)\left(t - \frac{n}{2B'}\right)\right) \tag{B.3.6}$$

This holds for all s and t, so in particular for $s = t$ and by symmetry of sinc,

$$f(t) = \sum_{n \in \mathbb{Z}} \frac{2(B + mb)}{2B'} f\left(\frac{n}{2B'}\right)\left[\operatorname{sinc}\left(2b\left(t - \frac{n}{2B'}\right)\right)\right]^m \operatorname{sinc}\left(2(B + mb)\left(t - \frac{n}{2B'}\right)\right) \tag{B.3.7}$$

The Helms-Thomas expansion is this equality for the exact sampling case i.e $B' := B + mb$. □

As a byproduct of the proof, we have

Proposition B.6. For all $b, B > 0$, $m \in \mathbb{N}^*$ and $B' \geq B + mb$, the following reconstruction formula holds:

$$\forall f \in \mathbb{B}_B,\ f(t) = \sum_{n \in \mathbb{Z}} \frac{2(B + mb)}{2B'} f\left(\frac{n}{2B'}\right)\left[\operatorname{sinc}\left(2b\left(t - \frac{n}{2B'}\right)\right)\right]^m \operatorname{sinc}\left(2(B + mb)\left(t - \frac{n}{2B'}\right)\right) \tag{B.3.8}$$

We will refer to this formula as the *oversampled variant of the Helms-Thomas expansion.*

Remark B.6. By the same kind of reasoning as in the above derivation: let $f \in \mathbb{B}_B$ and $g \in \mathbb{B}_b$. Then the product $f(t)g(t)$ is $(B + b)$-band-limited. The sampling theorem yields, for all $B' \geq (B + b)$,

$$f(t)g(t) = \sum_{n \in \mathbb{Z}} \frac{2(B + b)}{2B'} f\left(\frac{n}{2B'}\right) g\left(\frac{n}{2B'}\right) \operatorname{sinc}\left(2(B + b)\left(t - \frac{n}{2B'}\right)\right) \tag{B.3.9}$$

$$f(t) = \sum_{n \in \mathbb{Z}} \frac{2(B + b)}{2B'} f\left(\frac{n}{2B'}\right) \frac{g\left(\frac{n}{2B'}\right)}{g(t)} \operatorname{sinc}\left(2(B + b)\left(t - \frac{n}{2B'}\right)\right) \tag{B.3.10}$$

Using a complex-analysis viewpoint, one may show that any $g \in \mathbb{B}_b$ for any $b > 0$ has a "small" number of zeros, in that its zeros are isolated. Indeed, g is analytic on $\mathbb{R}$.

So the right-hand-side makes sense for almost all t, as the quotient of an L^2-convergent series $\sum_{\mathbb{Z}} f\left(\frac{n}{2B}\right) g\left(\frac{n}{2B}\right) \operatorname{sinc}\left(2B\left(t - \frac{n}{2B'}\right)\right)$ and an analytic function $g(t)$. However nothing guarantees that the series $\sum_{\mathbb{Z}} f\left(\frac{n}{2B}\right) \frac{g\left(\frac{n}{2B}\right)}{g(t)} \operatorname{sinc}\left(2B\left(t - \frac{n}{2B'}\right)\right)$ converges in L^2.

B.3.2 Non-shift-invariant oversampling expansions?

In section B.1, we completely characterized *"shift-invariant" oversampling expansions.* That is, we showed that for any $\Psi_0 \in L^2(\mathbb{R})$, the identity $\forall f \in \mathbb{B}_B,\ f(t) = \sum_{n\in\mathbb{Z}} \frac{1}{2B'} f\left(\frac{n}{2B'}\right) \Psi_0\left(t - \frac{n}{2B'}\right)$ holds if and only if Ψ_0 satisfies the synthesizer condition: $\widehat{\Psi}_0(\xi) = \begin{cases} 1 & \text{for } |\xi| \le B \\ 0 & \text{for } \xi \in [\pm B] + 2B'\mathbb{Z}^* \\ \text{arbitrary} & \text{everywhere else} \end{cases}$.

It is tempting to assert that this also completely captures *"all oversampling expansions"*; that is, that any family $(\Psi_n(t))_n$ such that

$$\forall f \in \mathbb{B}_B,\ f(t) = \sum_{n\in\mathbb{Z}} \frac{1}{2B'} f\left(\frac{n}{2B}\right) \Psi_n(t) \tag{B.3.11}$$

must be of the form $\Psi_n(t) = \Psi_0(t - nB')$ with Ψ_0 verifying the properties listed above. Indeed one may be tempted to write:

For all $f \in \mathbb{B}_B$,

$$2B' f(t) = \sum_{n\in\mathbb{Z}} f\left(\frac{n}{2B'}\right) \Psi_n(t) \tag{B.3.12}$$

Besides, for all $m \in \mathbb{Z}$, $f\left(t + \frac{m}{2B'}\right)$ is also in $\mathbb{B}_B$ so

$$2B' f\left(t + \frac{m}{2B'}\right) = \sum_{n\in\mathbb{Z}} f\left(\frac{n+m}{2B'}\right) \Psi_n(t) \tag{B.3.13}$$

$$= \sum_{n\in\mathbb{Z}} f\left(\frac{n}{2B'}\right) \psi_{n-m}(t) \tag{B.3.14}$$

$$2B' f(t) = \sum_{n\in\mathbb{Z}} f\left(\frac{n}{2B'}\right) \Psi_{n-m}\left(t - \frac{m}{2B'}\right) \tag{B.3.15}$$

Thus

$$\sum_{n\in\mathbb{Z}} f\left(\frac{n}{2B'}\right) \Psi_n(t) = \sum_{n\in\mathbb{Z}} f\left(\frac{n}{2B'}\right) \Psi_{n-m}\left(t - \frac{m}{2B'}\right) \tag{B.3.16}$$

so for all $n, m \in \mathbb{Z}$**,**

$$\Psi_n(t) = \Psi_{n-m}\left(t - \frac{m}{2B'}\right) = \Psi_0\left(t - \frac{n}{2B'}\right) \tag{B.3.17}$$

However this last step is unsound. All that we can deduce, is that for all $m \in \mathbb{Z}$, the family $\left(\Psi_n(t) - \Psi_{n-m}(t - \frac{m}{2B'})\right)_n$ is in the null space of some operator.

Interestingly, in the previous subsection we showed that for all $f \in \mathbb{B}_B$ and $g \in \mathbb{B}_b$ and $B' \ge B + b$,

$$f(t) = \sum_{n\in\mathbb{Z}} \frac{2(B+b)}{2B'} f\left(\frac{n}{2B'}\right) \frac{g\left(\frac{n}{2B'}\right)}{g(t)} \operatorname{sinc}\left(2(B+b)\left(t - \frac{n}{2B'}\right)\right) \tag{B.3.18}$$

So in particular, for all $B' \ge B$, letting $b := B' - B$ and $g(t) = \operatorname{sinc}(2bt)$ yields: for all $f \in \mathbb{B}_B$,

$$f(t) = \sum_{n\in\mathbb{Z}} f\left(\frac{n}{2B'}\right) \frac{\operatorname{sinc}\left(2(B'-B)\frac{n}{2B'}\right)}{\operatorname{sinc}(2(B'-B)t)} \operatorname{sinc}\left(2B'\left(t - \frac{n}{2B'}\right)\right) \tag{B.3.19}$$

This does not quite constitute a counter-example, though, since the functions $\Psi_n(t)$ on the right-hand-side are not in L^2. (Or at least they do not look like they are, to check).

B.3.3 Helms-Thomas expansion as window-based oversampling

We can rewrite the Helms-Thomas expansion as: for all $B' := B + mb$,

$$\forall f \in \mathbb{B}_B,\ f(t) = \sum_{n\in\mathbb{Z}} f\left(\frac{n}{2B'}\right)\left[\operatorname{sinc}\left(2b\left(t - \frac{n}{2B'}\right)\right)\right]^m \operatorname{sinc}\left(2B'\left(t - \frac{n}{2B'}\right)\right) \tag{B.3.20}$$

$$= \sum_{n\in\mathbb{Z}} \frac{1}{2B'} f\left(\frac{n}{2B'}\right)\ \Psi_0\left(t - \frac{n}{2B'}\right) \tag{B.3.21}$$

where

$$\Psi_0(t) = \underbrace{2B' \operatorname{sinc}(2Bt')}\ \underbrace{[\operatorname{sinc}(2bt)]^m} = \Phi_0^{(B')}(t)\ g(t) \tag{B.3.22}$$

with $\Phi_0^{(B')}(t) = 2B' \operatorname{sinc}(2B't)$, $\widehat{\Phi}_0^{(B')}(\xi) = \mathbb{1}_{|\xi|\leq B'}$ the "canonical" synthesizer. So we recognize Ψ_0 as the window-based (B, B')-synthesizer with window function

$$g(t) = [\operatorname{sinc}(2bt)]^m \qquad\qquad \hat{g}(\xi) = \left[\underbrace{\frac{\mathbb{1}_{|\xi|\leq b}}{2b} * \dots * \frac{\mathbb{1}_{|\xi|\leq b}}{2b}}_{m\text{ times}}\right](\xi) \tag{B.3.23}$$

Indeed we can immediately check that $g(0) = 1$, and that $\hat{g}$ is supported on $m \cdot [\pm b] = [\pm mb] = [\pm(B' - B)]$, since we used $B' = B + mb$ for the sampling rate.

In fact, we remark that $\hat{g}$ is nothing else than (a translated variant of) the box spline of degree $m-1$: see `https://math.stackexchange.com/questions/618272/convolution-of-indicator-function-with-itself`, and `https://commons.wikimedia.org/wiki/File:Convolution_of_box_signal_with_itself2.gif` for an illustration of the case $m = 2$. See also Figure B.5 for an illustration of the resulting synthesizer's spectrum.

B.3.4 Oversampled variant of the Helms-Thomas expansion as window-based "over-oversampling"

We can rewrite the oversampled variant of the Helms-Thomas expansion as: for all $B' \geq B + mb$,

$$\forall f \in \mathbb{B}_B,\ f(t) = \sum_{n\in\mathbb{Z}} \frac{2(B+mb)}{2B'} f\left(\frac{n}{2B'}\right)\left[\operatorname{sinc}\left(2b\left(t - \frac{n}{2B'}\right)\right)\right]^m \operatorname{sinc}\left(2(B+mb)\left(t - \frac{n}{2B'}\right)\right) \tag{B.3.24}$$

$$= \sum_{n\in\mathbb{Z}} \frac{1}{2B'} f\left(\frac{n}{2B'}\right)\ \Psi_0\left(t - \frac{n}{2B'}\right) \tag{B.3.25}$$

with

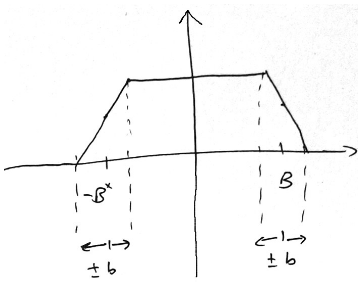
$$\Psi_0(t) = \underbrace{2(B+mb)\operatorname{sinc}(2(B+mb)t)}\ \underbrace{[\operatorname{sinc}(2bt)]^m} \tag{B.3.26}$$

$$= \Phi_0^{(B+mb)}(t) \cdot [\operatorname{sinc}(2bt)]^m \tag{B.3.27}$$

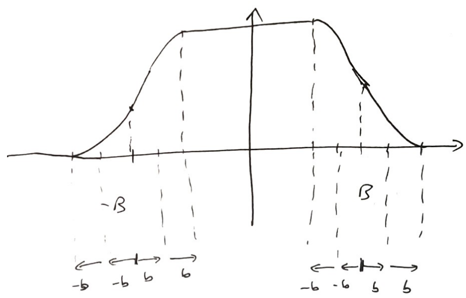
It is tempting to say that $\Psi_0(t)$ is just a window-based synthesizer with window function

$$g(t) := [\operatorname{sinc}(2bt)]^m \qquad\qquad \hat{g}(\xi) = \left[\underbrace{\frac{\mathbb{1}_{|\xi|\leq b}}{2b} * \dots * \frac{\mathbb{1}_{|\xi|\leq b}}{2b}}_{m\text{ times}}\right](\xi) \tag{B.3.28}$$

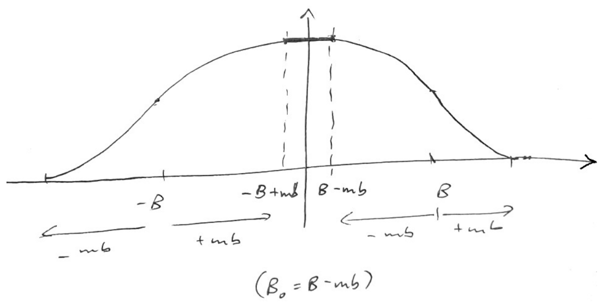
However, following our terminology of section B.2, $\Psi_0(t)$ is neither a $(B + mb, B')$-synthesizer, nor a $(B, B + mb)$-synthesizer, nor a (B, B')-window-based-synthesizer:



(a) $\widehat{\Psi}_0(\xi)$ for $m = 1$



(b) $\widehat{\Psi}_0(\xi)$ for $m = 2$



(c) $\widehat{\Psi}_0(\xi)$ for any m such that $B - mb = B_0 > 0$

Figure B.5: The resulting synthesizer's spectrum for various values of m. It is obtained (up to normalizing factor) as the iterated convolution of indicator functions: $\widehat{\Psi}_0(\xi) \propto \mathbb{1}_{|\xi|<B} * \left[\mathbb{1}_{|\xi|\leq b} * ... * \mathbb{1}_{|\xi|\leq b}\right]$, with the indicator $\mathbb{1}_{|\xi|\leq b}$ appearing m times. So denoting $h_m(\xi) = \widehat{\Psi}_0(\xi)$ for a specific value of m, the drawings were obtained by picturing the iterated convolutions $h_{m+1} = h_m * \mathbb{1}_{|\xi|\leq b}$.

- A $(B + mb, B')$-synthesizer by definition gives a reconstruction formula that holds for any $f \in \mathbb{B}_{B+mb}$; while the above expansion only holds for $f \in \mathbb{B}_B \subsetneq \mathbb{B}_{B+mb}$.
- A $(B, B+mb)$-synthesizer by definition reconstructs f from $\left(f\left(\frac{n}{2(B+mb)}\right)\right)_n$, its samples spaced by $\frac{1}{2(B+mb)}$; while the above expansion uses a sampling rate of $\frac{1}{2B'}$.
- The (B, B')-window-based-synthesizer with window function $g(t)$ would be $2B' \operatorname{sinc}(2B't)g(t)$; while here $\Psi_0(t) = 2(B + mb) \operatorname{sinc}(2(B + mb)t)g(t)$.

Thus the oversampled variant of the Helms-Thomas expansion does not fall naturally in the class of window-based synthesizers as we defined them in section B.2. Essentially, the difference with the (B, B')-window-based-synthesizers studied in that section, is that the present expansion results from windowing the "sub-canonical" synthesizer $\Phi_0^{(\overline{B})}$, for an intermediary frequency $\overline{b} := (B + mb)\overline{B} \in \ ]B, B'[$, instead of windowing the canonical synthesizer $\Phi_0^{(B')}$. The conditions on the window $g(t)$ must of course be adapted: in the latter case we showed that it's essentially $\operatorname{supp}(\hat{g}) \subset [\pm(B' - B]$, while in the former case we conjecture that it's $\operatorname{supp}(\hat{g}) \subset [\pm(\overline{B} - B)]$. See Figure B.6 for an illustration of this window-based "over-oversampling" situation.

Can Ψ_0 nonetheless be interpreted as a (B, B')-window-based-synthesizer? One can check that $\widehat{\Psi}_0(\xi)$ is supported on the central interval $[-2B', 2B']$ and is in $H^1(\mathbb{R})$, since it's a variant of the box spline as illustrated in Figure B.5. So we may use the characterization developed in section B.2. Now, by comparing Figure B.6 to Figure B.4, clearly the weak differential $\left.\widehat{\Psi}_0'\right|_{[-2B',0]}$ is not equal

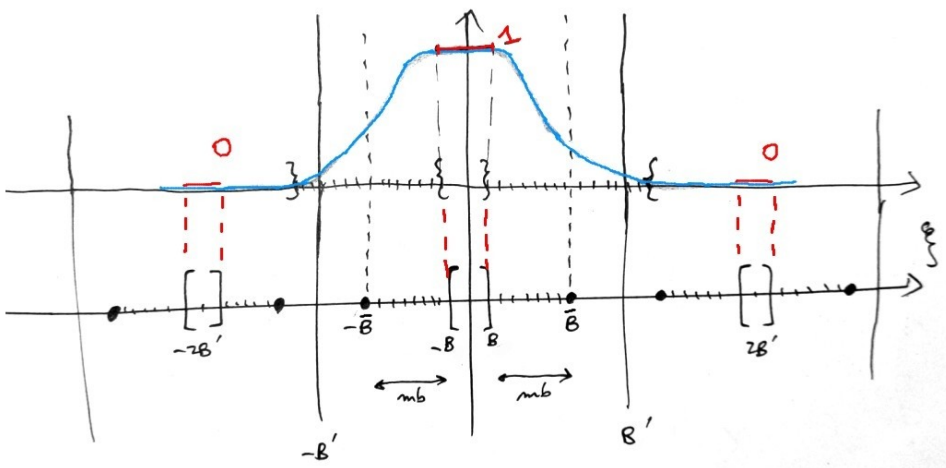


Figure B.6: Visual support for the window-based "over-oversampling".
Bottom axis: the repeated spectrum of $f \in \mathbb{B}_B$ is supported on the bracketed intervals; the bullets mark the choice of an intermediary $\overline{B} \in \ ]B, B'[$. We consider synthesizers obtained by windowing $\Phi_0^{(\overline{B})}$, i.e $\mathbb{1}_{|\xi|\leq\overline{B}}$ in frequency space, by a window function g.
Top axis: the curly-bracketed intervals are the same as the bracketed intervals from the bottom axis, except they are repeated by symmetry around $-\overline{B}$ and $\overline{B}$. Up to a translation by $\pm\overline{B}$, each dotted interval $[\pm(\overline{B} - B)]$ represents the allowed support of $\hat{g}$. Convolving $\widehat{\Phi}_0^{(\overline{B})}$ by such a $\hat{g}$ results in a $\widehat{\Psi}_0$ (curve in blue) that satisfies the (B, B') synthesizer condition (in red).

to a translation of $-\left.\widehat{\Psi}_0'\right|_{[0,2B']}$. Thus, we can answer negatively: the oversampled variant of the Helms-Thomas expansion is not captured by our notion of window-based sampling as defined in this document. It would be an interesting exercise to extend it to cover this case.

B.4 Characterizing design freedom in the multidimensional setting

Definition B.4. Let $K \subset \mathbb{R}^N$ a compact and $\boldsymbol{B'} \in \mathbb{R}_+^N$ such that $K \subset [\pm\boldsymbol{B'}]$. Recall that we denote $\widehat{\mathbb{B}}_K = \left\{\hat{f} \in L^2(\mathbb{R}^N);\ \hat{f}(\boldsymbol{\xi}) = 0 \text{ for } \boldsymbol{\xi} \notin K\right\}$ and $\mathbb{B}_K = \left\{f \in L^2(\mathbb{R}^N);\ \hat{f} \in \widehat{\mathbb{B}}_K\right\}$.

The function $\Psi_{\boldsymbol{0}}(\boldsymbol{t})$ is called a $(K, \boldsymbol{B'})$-synthesizer if

$$\forall f \in \mathbb{B}_K, f(\boldsymbol{t}) = \sum_{\boldsymbol{n}\in\mathbb{Z}^N} \frac{1}{(2\boldsymbol{B'})^\times} f\left(\frac{\boldsymbol{n}}{2\boldsymbol{B'}}\right) \Psi_{\boldsymbol{0}}\left(\boldsymbol{t} - \frac{\boldsymbol{n}}{2\boldsymbol{B'}}\right) \tag{B.4.1}$$

Proposition B.7. $\Psi_{\boldsymbol{0}}(\boldsymbol{t})$ is a $(K, \boldsymbol{B'})$-synthesizer if and only if

$$\widehat{\Psi}_{\boldsymbol{0}}(\boldsymbol{\xi}) = \begin{cases} 1 & \text{for } \boldsymbol{\xi} \in K \\ 0 & \text{for } \boldsymbol{\xi} \in K + 2\boldsymbol{B'} \odot (\mathbb{Z}^N \setminus \{\boldsymbol{0}\}) := \bigsqcup_{\boldsymbol{n}\in\mathbb{Z}^N\setminus\{\boldsymbol{0}\}} (2\boldsymbol{n} \odot \boldsymbol{B'} + K) \\ \text{arbitrary} & \text{everywhere else} \end{cases} \tag{B.4.2}$$

Proof. Same reasoning as for the one-dimensional case. □

The fact that the set $K + 2\boldsymbol{B'} \odot (\mathbb{Z}^N \setminus \{\boldsymbol{0}\})$ can be written as a disjoint union as above is clear from Figure B.7.

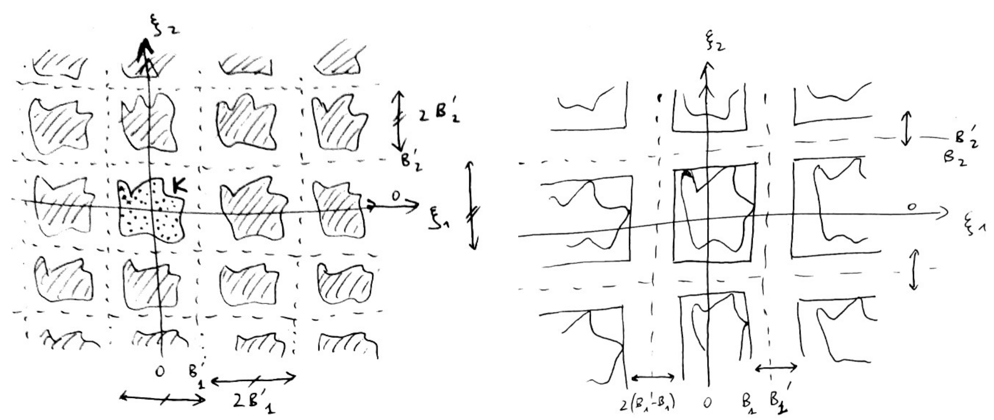


Figure B.7: Left: $\widehat{\Psi}_{\mathbf{0}}(\boldsymbol{\xi})$ must be $= 1$ on the dotted set, $= 0$ on the hatched set, and can be arbitrary on the set left blank. Right: if we don't like generality, we can always overapproximate K and assume hyperrectangular support $[\pm \boldsymbol{B}] \subset [\pm \boldsymbol{B}']$.

B.4.1 Cartesian product of synthesizers

Let $\boldsymbol{B} \leq \boldsymbol{B}'$. For all $\Psi_{(1)}(t)$ (B_1, B'_1)-synthesizer, ..., $\Psi_{(N)}(t)$ (B_N, B'_N)-synthesizer,

$$\forall f \in \mathbb{B}_{\boldsymbol{B}},\ f(\boldsymbol{t}) = \sum_{n_1 \in \mathbb{Z}} \frac{1}{2B'_1} \Psi_{(1)} \left(t_1 - \frac{n_1}{2B'_1} \right) f \left(\frac{n_1}{2B'_1}, t_2, ..., t_N \right) = ... \tag{B.4.3}$$

$$= \sum_{\boldsymbol{n} \in \mathbb{Z}^N} \frac{1}{(2\boldsymbol{B}')^{\times}} f \left(\frac{\boldsymbol{n}}{2\boldsymbol{B}'} \right) \prod_{i=1}^{N} \Psi_{(i)} \left(t_i - \frac{n_i}{2B'_i} \right) \tag{B.4.4}$$

So that $\Psi(\boldsymbol{t}) := \Psi_{(1)}(t_1)...\Psi_{(N)}(t_N)$ is a $(\boldsymbol{B}, \boldsymbol{B}')$-synthesizer.

More generally, let $[N] = I_1 \sqcup ... \sqcup I_s$ a partition of the indexes, and for all $\boldsymbol{t} \in \mathbb{R}^N$, denote $\boldsymbol{t}_I := (t_i)_{i \in I}$ so that $\boldsymbol{t} \simeq \boldsymbol{t}_{I_1} \times ... \times \boldsymbol{t}_{I_s}$. For all $\Psi_{(I_1)}(\boldsymbol{t}_{I_1})$ $(K_1, \boldsymbol{B}'_{I_1})$-synthesizer, ..., $\Psi_{(I_s)}(\boldsymbol{t}_{I_s})$ $(K_s, \boldsymbol{B}'_{I_s})$-synthesizer,

$$\forall f \in \mathbb{B}_K,\ f(\boldsymbol{t}) = \sum_{\boldsymbol{n}_{I_1} \in \mathbb{Z}^{I_1}} \frac{1}{(2\boldsymbol{B}'_{I_1})^{\times}} \Psi_{(I_1)} \left(\boldsymbol{t}_{I_1} - \frac{\boldsymbol{n}_{I_1}}{2\boldsymbol{B}'_{I_1}} \right) f \left(\frac{\boldsymbol{n}_{I_1}}{2\boldsymbol{B}'_{I_1}}, (t_i)_{i \notin I_1} \right) = ... \tag{B.4.5}$$

$$= \sum_{\boldsymbol{n} \in \mathbb{Z}^N} \frac{1}{(2\boldsymbol{B}')^{\times}} f \left(\frac{\boldsymbol{n}}{2\boldsymbol{B}'} \right) \prod_{k=1}^{s} \Psi_{(I_k)} \left(\boldsymbol{t}_{I_k} - \frac{\boldsymbol{n}_{I_k}}{2\boldsymbol{B}'_{I_k}} \right) \tag{B.4.6}$$

So that $\Psi(\boldsymbol{t}) := \Psi_{(I_1)}(\boldsymbol{t}_{I_1})...\Psi_{(I_N)}(\boldsymbol{t}_{I_N})$ is a $(K, \boldsymbol{B}')$-synthesizer, where

$$K := \left\{ \boldsymbol{t} \in \mathbb{R}^N;\ \boldsymbol{t}_{I_1} \in K_1, ..., \boldsymbol{t}_{I_N} \in K_s \right\}\ (\simeq K_1 \times ... \times K_s) \tag{B.4.7}$$

B.5 Window-based synthesizers in the multidimensional setting

As for the one-dimensional case, we can define and characterize (central)-window-based synthesizers.

Synthesizers with spectrum supported on the central hyperrectangle

Definition B.5. We call a $(K, \boldsymbol{B}')$-*synthesizer with spectrum supported on the central hyperrectangle* any $\Psi_{\mathbf{0}}(\boldsymbol{t})$ that satisfies the synthesizer condition and such that $\mathrm{supp}(\widehat{\Psi}_{\mathbf{0}}(\boldsymbol{\xi})) \subset [\pm 2\boldsymbol{B}']$.

Example B.3. The "canonical" synthesizer in dimension N: $\Phi_{\mathbf{0}}^{(\boldsymbol{B}')}(\boldsymbol{t}) = (2\boldsymbol{B}')^{\times}\operatorname{sinc}^{\times}(2\boldsymbol{B}' \odot \boldsymbol{t})$, $\widehat{\Phi}_{\mathbf{0}}^{(\boldsymbol{B}')}(\xi) = \mathbb{1}_{\boldsymbol{\xi}\in[\pm 2\boldsymbol{B}']}$, has spectrum supported on the central hyperrectangle.

(Central)-Window-based synthesizers

Definition B.6. A *(central) window function* is any function $g(\boldsymbol{t})$ such that

$$\Psi_{\mathbf{0}}(t) = \Phi_{\mathbf{0}}^{(\boldsymbol{B}')}(\boldsymbol{t})g(\boldsymbol{t}) \tag{B.5.1}$$

defines a synthesizer with spectrum supported on the central hyperrectangle.

A *generalized window function* is any function $g(\boldsymbol{t})$ such that the above formula defines a synthesizer.

For window function $g(\boldsymbol{t})$, $\Psi_{\mathbf{0}}(t) = \Phi_{\mathbf{0}}^{(\boldsymbol{B}')}(\boldsymbol{t})g(\boldsymbol{t})$ is called the associated window-based synthesizer.

B.5.1 Partial characterization of central window functions

As in one dimension, we restrict attention to candidates $g(\boldsymbol{t}) \in L^2(\mathbb{R}^N)$, although more general function spaces could be considered.

Proposition B.8. Let $g(\boldsymbol{t}) \in L^2(\mathbb{R}^N)$ such that $\hat{g}(\boldsymbol{\xi})$ has values in $\mathbb{R}_+$. Then for $g(\boldsymbol{t})$ to be a central window function, it is necessary that

$$g(\mathbf{0}) = 1 \tag{B.5.2}$$

$$\operatorname{supp}(\hat{g}) + (-K) \subset [\pm\boldsymbol{B}'] \tag{B.5.3}$$

In fact this condition is even necessary for $g(\boldsymbol{t})$ to be a generalized window function.

Conversely, let $g(\boldsymbol{t}) \in L^2(\mathbb{R}^N)$. The above condition is sufficient for $g(\boldsymbol{t})$ to be a central window function.

The initial assumption $\hat{g}(\boldsymbol{\xi}) \in \mathbb{R}_+$ is very strong (a priori $\hat{g}$ has values in $\mathbb{C}$), so the necessary condition is not very useful. But its derivation is instructive, in that it gives intuition on what is "missing" for our sufficient condition to also be necessary. (We conjecture that the sufficient condition is also necessarily, but we have not proved it.)

The rest of this subsection is dedicated to proving the above proposition.

(Partial) necessity Let $g(\boldsymbol{t}) \in L^2(\mathbb{R}^N)$ such that $\hat{g}(\boldsymbol{\xi})$ has values in $\mathbb{R}_+$. Suppoe $g(\boldsymbol{t})$ is a central window function i.e $\Psi_{\mathbf{0}}(\boldsymbol{t}) := \Phi_{\mathbf{0}}^{(\boldsymbol{B}')}(\boldsymbol{t})g(\boldsymbol{t})$ is a synthesizer.

Denote $S := \operatorname{supp}(\hat{g})$. Our starting point is that

$$\widehat{\Phi}_{\mathbf{0}}(\boldsymbol{\xi}) = (\mathbb{1}_{\cdot\in[\pm\boldsymbol{B}']} * \hat{g})(\boldsymbol{\xi}) = \int_{\mathbb{R}^N} d\boldsymbol{y}\ \hat{g}(\boldsymbol{y})\mathbb{1}_{\boldsymbol{\xi}-\boldsymbol{y}\in[\pm\boldsymbol{B}']} \tag{B.5.4}$$

$$= \int_{\mathbb{R}^N} d\boldsymbol{y}\ \hat{g}(\boldsymbol{y})\mathbb{1}_{\boldsymbol{y}\in\boldsymbol{\xi}+[\pm\boldsymbol{B}']} \tag{B.5.5}$$

$$= \int_{\boldsymbol{\xi}+[\pm\boldsymbol{B}']} d\boldsymbol{y}\ \hat{g}(\boldsymbol{y}) \tag{B.5.6}$$

and $\widehat{\Phi}_{\mathbf{0}}(\boldsymbol{\xi})$ must satisfy the synthesizer condition

$$\widehat{\Psi}_{\mathbf{0}}(\boldsymbol{\xi}) = \begin{cases} 1 & \text{for } \boldsymbol{\xi} \in K \\ 0 & \text{for } \boldsymbol{\xi} \in K + 2\boldsymbol{B}' \odot (\mathbb{Z}^N \setminus \{\mathbf{0}\}) \\ \text{arbitrary} & \text{everywhere else} \end{cases} \tag{B.5.7}$$

Claim B.1. $g(\mathbf{0}) = 1$

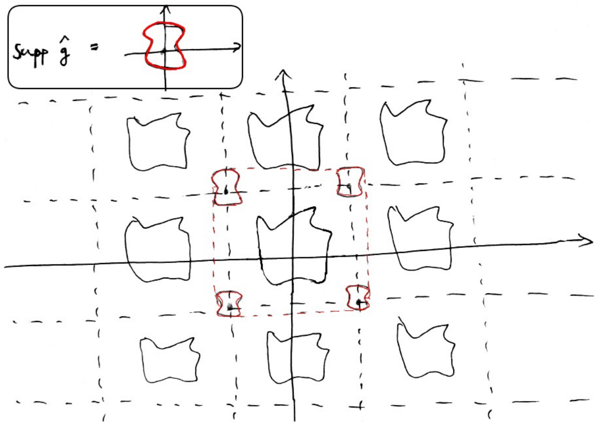


Figure B.8: The dashed black lines represent the partition $\mathbb{R}^N = \bigsqcup_{\boldsymbol{n}\in\mathbb{Z}^N}[\pm\boldsymbol{B}'] + 2\boldsymbol{n}\odot\boldsymbol{B}'$; each hyperrectangle in that partition contains a translated repetition of K. The shape of (a superset of) the support $\operatorname{supp}(\widehat{\Psi}_{\mathbf{0}})$ is represented by the dashed red lines. (Its corners are not accurately represented.) One of the challenges will be to determine under what condition that set is disjoint from the non-original repetitions $K + 2\boldsymbol{n}\odot\boldsymbol{B}'$, $\boldsymbol{n}\in\mathbb{Z}^N\setminus\{\mathbf{0}\}$.

Proof. Exact same idea as for the one-dimensional case:

$$g(\mathbf{0}) = \int_{\mathbb{R}^N} d\boldsymbol{\xi}\ \hat{g}(\boldsymbol{\xi})e^{i2\pi\cdot 0} \tag{B.5.8}$$

$$= \sum_{\boldsymbol{n}\in\mathbb{Z}^N}\int_{2\boldsymbol{n}\odot\boldsymbol{B}'+[\pm\boldsymbol{B}']} d\boldsymbol{\xi}\ \hat{g}(\boldsymbol{\xi}) \tag{B.5.9}$$

$$= \sum_{\boldsymbol{n}\in\mathbb{Z}^N}\widehat{\Phi}_{\mathbf{0}}(2\boldsymbol{n}\odot\boldsymbol{B}') \tag{B.5.10}$$

$$= 1 + \sum_{\boldsymbol{n}\in\mathbb{Z}^N\setminus\{\mathbf{0}\}} 0 \tag{B.5.11}$$

□

Claim B.2. $\forall\boldsymbol{\xi}\in K, S\subset\boldsymbol{\xi}+[\pm\boldsymbol{B}']$. (In particular, S is compact.) Equivalently this can be written as $S+(-K)\subset[\pm\boldsymbol{B}']$.

Proof. S is closed by definition, the first part of the claim implies that S is bounded, and the ambient space $\mathbb{R}^N$ has finite dimension; hence compacity of S.

The announced reformulation is indeed equivalent by definition of Minkowski sum.

It remains to show the first part of the claim. Let $\boldsymbol{\xi}\in K$; by the synthesizer condition,

$$\widehat{\Phi}_{\mathbf{0}}(\boldsymbol{\xi}) = 1 = g(\mathbf{0}) \tag{B.5.12}$$

$$\int_{\boldsymbol{\xi}+[\pm\boldsymbol{B}']} d\boldsymbol{y}\ \hat{g}(\boldsymbol{y}) = \int_{\mathbb{R}^N} d\boldsymbol{y}\ \hat{g}(\boldsymbol{y}) \tag{B.5.13}$$

By the lemma stated and proved just below, we conclude that $\operatorname{supp}\hat{g} = S\subset\boldsymbol{\xi}+[\pm\boldsymbol{B}']$. □

Lemma B.9. Let $f\in L^1_{loc}(\mathbb{R}^N)$ a locally integrable function that has values in $\mathbb{R}_+$. Suppose additionally that $\int_{\mathbb{R}^N} d\boldsymbol{y}\ f(\boldsymbol{y}) < \infty$ (so that f is actually integrable, since $f(\boldsymbol{y}) = |f(\boldsymbol{y})|$).

If

$$\int_{[0,1]^N} d\boldsymbol{y}\ f(\boldsymbol{y}) = \int_{\mathbb{R}^N} d\boldsymbol{y}\ f(\boldsymbol{y}) \tag{B.5.14}$$

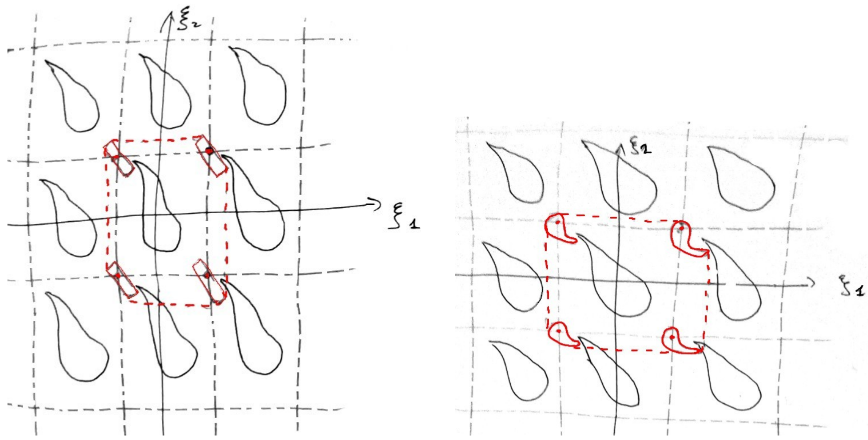


Figure B.9: Further examples. (Actually, was meant as visual aid for a previous version of the proof, which turned out to be unnecessary.)

then $\mathrm{supp}(f) \subset [0,1]^N$.

Proof. By definition of the support of an L^1_{loc} function, $\mathrm{supp}(f)$ is the smallest closed subset in the complement of which $f = 0$ almost everywhere. So it suffices to show that $[0,1]^N$ is such a set.

It is obviously closed. It is also clear that $f = 0$ a.e on its complement, since $f \geq 0$ and $\int_{\mathbb{R}^N \setminus [0,1]^N} d\boldsymbol{y}\; f(\boldsymbol{y}) = 0$ by assumption. □

Remark B.7. Functions $f \in L^p(\mathbb{R}^N)$ are defined only almost everywhere, so the usual definition of support $\mathrm{supp}(f) = \{\boldsymbol{x};\ f(\boldsymbol{x}) \neq 0\}$ doesn't apply. Somehow until now we managed to avoid this technicality, and contended ourselves with the intuitive idea of support inherited from the case of functions defined everywhere.

In the proof above, for the first time in this document we had to use the rigorous definition of the support of an L^p function. This is also the last and only time that we have to do so, so we chose to restrict our discussion of that concept to the present remark.

Clearly any $f \in L^p(\mathbb{R}^N)$ is locally integrable. The support set of a locally integrable function $f \in L^1_{loc}(\mathbb{R}^N)$ is defined as [Gol18, definition 1.3.3]

$$\mathrm{supp}(f) := \bigcap_{\Omega \in \mathcal{O}(f)} (\mathbb{R}^N \setminus \Omega) \qquad \text{where} \quad \mathcal{O}(f) := \left\{\Omega \text{ open subset of } \mathbb{R}^N;\ f = 0 \text{ a.e on } \Omega\right\} \tag{B.5.15}$$

In words, it is "the smallest closed subset in the complement of which f is a.e equal to 0". [3]

Sufficiency Let $g(\boldsymbol{t}) \in L^2(\mathbb{R}^N)$ (with values in $\mathbb{C}$) that satisfies the condition

$$g(\mathbf{0}) = 1 \tag{B.5.16}$$

$$\mathrm{supp}(\hat{g}) + (-K) \subset [\pm \boldsymbol{B}'] \tag{B.5.17}$$

We want to show that g is a central window function, i.e that $\Psi_{\mathbf{0}}(\boldsymbol{t}) = \Phi_{\mathbf{0}}^{(\boldsymbol{B}')}(\boldsymbol{t})g(\boldsymbol{t})$ is a synthesizer with spectrum supported on the central hyperrectangle. By our characterization of synthesizer functions, equivalently we want to show that

$$\widehat{\Psi}_{\mathbf{0}}(\boldsymbol{\xi}) = \int_{\boldsymbol{\xi}+[\pm \boldsymbol{B}']} d\boldsymbol{y}\; \hat{g}(\boldsymbol{y}) = \begin{cases} 1 & \text{for } \boldsymbol{\xi} \in K \\ 0 & \text{for } \boldsymbol{\xi} \in K + 2\boldsymbol{B}' \odot (\mathbb{Z}^N \setminus \{\mathbf{0}\}) \\ \text{arbitrary} & \text{everywhere else} \end{cases} \tag{B.5.18}$$

[3] `https://en.wikipedia.org/wiki/Distribution_(mathematics)#Support_of_a_distribution`

and that it has its spectrum supported on the central hyperrectangle i.e $\mathrm{supp}(\widehat{\Psi}_{\mathbf{0}}) \subset [\pm 2\boldsymbol{B}']$.

Basically we simply go through the same arguments as for necessity. Again denote for convenience $S := \mathrm{supp}(\hat{g})$.

Claim B.3. For all $\boldsymbol{\xi} \in K$, $\widehat{\Psi}_{\mathbf{0}}(\boldsymbol{\xi}) = 1$

Proof. $S + (-K) \subset [\pm \boldsymbol{B}']$ means that, for all $\boldsymbol{\xi} \in K$, $S \subset \boldsymbol{\xi} + [\pm \boldsymbol{B}']$ and so

$$\widehat{\Psi}_{\mathbf{0}}(\boldsymbol{\xi}) = \int_{\boldsymbol{\xi}+[\pm \boldsymbol{B}']} d\boldsymbol{y}\ \hat{g}(\boldsymbol{y}) \tag{B.5.19}$$

$$= \int_{S} d\boldsymbol{y}\ \hat{g}(\boldsymbol{y}) + 0 \tag{B.5.20}$$

$$= \int_{\mathbb{R}^N} d\boldsymbol{y}\ \hat{g}(\boldsymbol{y}) \tag{B.5.21}$$

$$= g(\mathbf{0}) = 1 \tag{B.5.22}$$

□

Claim B.4. $\mathrm{supp}(\widehat{\Psi}_{\mathbf{0}}) \subset [\pm 2\boldsymbol{B}']$

Proof. Let $\boldsymbol{\xi} \notin [\pm 2\boldsymbol{B}']$. Then clearly $(\boldsymbol{\xi} + [\pm \boldsymbol{B}']) \cap [\pm \boldsymbol{B}'] = \varnothing$. Since $S \subset [\pm \boldsymbol{B}']$, in particular we also have $(\boldsymbol{\xi} + [\pm \boldsymbol{B}']) \cap S = \varnothing$. Hence $\widehat{\Psi}_{\mathbf{0}}(\boldsymbol{\xi}) = \int_{\boldsymbol{\xi}+[\pm \boldsymbol{B}']} d\boldsymbol{y}\ \hat{g}(\boldsymbol{y}) = 0$. □

Claim B.5. For all $\boldsymbol{n} \in \mathbb{Z}^N \setminus \{\mathbf{0}\}$, $(S + \ ]\pm \boldsymbol{B}'[\ ) \cap (K + \boldsymbol{n} \odot 2\boldsymbol{B}') = \varnothing$
Since $\mathrm{supp}(\widehat{\Psi}_{\mathbf{0}}) \subset S + [\pm \boldsymbol{B}']$, in particular $\widehat{\Psi}_{\mathbf{0}} = 0$ on $K + \boldsymbol{n} \odot 2\boldsymbol{B}'$ for all $\boldsymbol{n} \in \mathbb{Z}^N \setminus \{\mathbf{0}\}$.

Proof. Let $\boldsymbol{n} \in \mathbb{Z}^N \setminus \{\mathbf{0}\}, \boldsymbol{y} \in S, \boldsymbol{z} \in K$. To prove the first part of the claim we want to show that for all $\boldsymbol{\xi} \in \ ]\pm \boldsymbol{B}'[\ $, $\boldsymbol{y} + \boldsymbol{\xi} \neq \boldsymbol{z} + \boldsymbol{n} \odot 2\boldsymbol{B}'$; in other words, that $-\boldsymbol{y} + \boldsymbol{z} + \boldsymbol{n} \odot 2\boldsymbol{B}' \notin \ ]\pm \boldsymbol{B}'[$.

Since $S + (-K) \in [\pm \boldsymbol{B}']$, which is symmetric,

$$\boldsymbol{y} - \boldsymbol{z} \in [\pm \boldsymbol{B}'] \tag{B.5.23}$$

$$-\boldsymbol{y} + \boldsymbol{z} \in [\pm \boldsymbol{B}'] \tag{B.5.24}$$

$$-\boldsymbol{y} + \boldsymbol{z} + \boldsymbol{n} \odot 2\boldsymbol{B}' \in [\pm \boldsymbol{B}'] + \boldsymbol{n} \odot 2\boldsymbol{B}' \tag{B.5.25}$$

Now since $\boldsymbol{n} \neq \mathbf{0}$, it is geometrically obvious that $]\pm \boldsymbol{B}'[\ \cap ([\pm \boldsymbol{B}'] + \boldsymbol{n} \odot 2\boldsymbol{B}') = \varnothing$, and so $-\boldsymbol{y} + \boldsymbol{z} + \boldsymbol{n} \odot 2\boldsymbol{B}' \notin \ ]\pm \boldsymbol{B}'[$ as required.

The fact that $\mathrm{supp}(\widehat{\Psi}_{\mathbf{0}}) \subset S + [\pm \boldsymbol{B}']$ comes from the equality $\widehat{\Psi}_{\mathbf{0}} = \widehat{\Phi}_{\mathbf{0}}^{(\boldsymbol{B}')} * \hat{g}$ and the general property of convolution $\mathrm{supp}(f * g) \subset \mathrm{supp}(f) + \mathrm{supp}(g)$. (Recall that $\mathrm{supp}(\widehat{\Phi}_{\mathbf{0}}^{(\boldsymbol{B}')}) = [\pm \boldsymbol{B}']$ and $\mathrm{supp}(\hat{g}) = S$ by definition.)

For the second part of the claim, in a sense we just need to deal with the boundary i.e the difference between $S + \ ]\pm \boldsymbol{B}'[$ and $S + [\pm \boldsymbol{B}']$. Formally, for any $\boldsymbol{n} \in \mathbb{Z}^N \setminus \{\mathbf{0}\}$,

- On one hand,

$$(S + \ ]\pm \boldsymbol{B}'[\ ) \cap (K + \boldsymbol{n} \odot 2\boldsymbol{B}') = \varnothing \implies K + \boldsymbol{n} \odot 2\boldsymbol{B}' \subset \mathbb{R}^N \setminus (S + \ ]\pm \boldsymbol{B}'[\ ) \tag{B.5.26}$$

- On the other hand,

$$\mathrm{supp}(\widehat{\Psi}_{\mathbf{0}}) \subset S + [\pm \boldsymbol{B}'] \implies \widehat{\Psi}_{\mathbf{0}} = 0 \text{ a.e on } \mathbb{R}^N \setminus (S + [\pm \boldsymbol{B}']) \tag{B.5.27}$$

Since the "almost everywhere" is with respect to the Lebesgue measure,

$$\widehat{\Psi}_{\mathbf{0}} = 0 \text{ a.e on } \overline{\mathbb{R}^N \setminus (S + [\pm \boldsymbol{B}'])} \tag{B.5.28}$$

Now by basic topology, this set contains in particular

$$\overline{\mathbb{R}^N \setminus (S + [\pm \boldsymbol{B}'])} \supset \mathbb{R}^N \setminus (S + \ ]\pm \boldsymbol{B}'[\ ) \tag{B.5.29}$$
$$\supset K + \boldsymbol{n} \odot 2\boldsymbol{B}' \tag{B.5.30}$$

and so $\widehat{\Psi}_{\mathbf{0}} = 0$ a.e on $K + \boldsymbol{n} \odot 2\boldsymbol{B}'$, as claimed. □

Remark B.8. The careful reader will have remarked that the last two steps of the proof just above require more justification, which unfortunately I am unable to provide.

- Denoting temporarily $A = \mathbb{R}^N \setminus (S + [\pm \boldsymbol{B}'])$, we argued that $\widehat{\Psi}_{\mathbf{0}} = 0$ a.e on A and so also a.e on $\overline{A}$. However this argument requires that the boundary $\partial A = \overline{A} \setminus \mathring{A}$ has Lebesgue measure zero; equivalently since the boundary of a set equals the boundary of its complement, that the boundary of $S + [\pm \boldsymbol{B}']$ has Lebesgue measure zero. Which is not obvious... I expect it to be true, though, because $S + [\pm \boldsymbol{B}']$ is morally smooth (even when S is ill-behaved).

- The inclusion $\overline{\mathbb{R}^N \setminus (S + [\pm \boldsymbol{B}'])} \supset \mathbb{R}^N \setminus (S + \ ]\pm \boldsymbol{B}'[\ )$ is not entirely clear. It boils down to $\text{interior}(S + [\pm \boldsymbol{B}']) \subset S + \ ]\pm \boldsymbol{B}'[$, which is shown true by `https://math.stackexchange.com/questions/3823633/interior-of-minkowski-sum-of-closed-convex-sets` when S is convex, but not for arbitrary (closed) S.

B.5.2 Explicit condition for allowed $S = \mathrm{supp}(\hat{g})$ by morphological erosion

The condition $S + (-K) \subset [\pm \boldsymbol{B}']$ doesn't look very practical; instead, we would like a condition of the form $S \subset Z$. Now the question of characterizing the tightest possible superset of S subject to this condition, reduces to finding the *erosion* of $[\pm \boldsymbol{B}']$ by $-K$.

Definition B.7. The erosion of X by Y is the set $Z = \{z;\ (z + Y) \subset X\}$. [4]

The erosion of X by Y is characterized as the largest set Z such that $Z + Y \subset X$, in the sense of inclusion. That is, for any set Z',

$$Z' + Y \subset X \iff Z' \subset Z \tag{B.5.31}$$

Indeed:

$$z \in Z \iff \forall y \in Y, z + y \in X \tag{B.5.32}$$
$$Z' \subset Z \iff \forall z \in Z', \forall y \in Y,\ z + y \in X \tag{B.5.33}$$
$$\iff Z' + Y \subset X \tag{B.5.34}$$

In our setting, X is $[\pm \boldsymbol{B}']$ and Y is $-K$. It turns out that since X is a hyperrectangle, the erosion is also just a hyperrectangle.

Proposition B.10. The erosion of $[\pm \boldsymbol{B}']$ by $-K$ is $[a_1, b_1] \times ... \times [a_N, b_N]$ – in other words,

$$S + (-K) \subset [\pm \boldsymbol{B}'] \iff S \subset [a_1, b_1] \times ... \times [a_N, b_N] \tag{B.5.35}$$

where

$$a_i = \sup_K z_i - B'_i \tag{B.5.36}$$

$$b_i = \inf_K z_i + B'_i \tag{B.5.37}$$

[4] `https://en.wikipedia.org/wiki/Erosion_(morphology)`

Proof. Simply write

$$
\begin{aligned}
S + (-K) \subset [\pm \boldsymbol{B}'] &\iff \forall \boldsymbol{y} \in S, \forall \boldsymbol{z} \in K,\ \boldsymbol{y} - \boldsymbol{z} \in [\pm \boldsymbol{B}'] && \text{(B.5.38)}\\
&\iff \forall \boldsymbol{y} \in S, \forall \boldsymbol{z} \in K, \forall i \in [N],\ \ - B_i' \leq y_i - z_i \leq B_i' && \text{(B.5.39)}\\
&\iff \forall \boldsymbol{y} \in S, \forall i \in [N],\ \sup_{\boldsymbol{z} \in K} z_i - B_i' \leq y_i \leq \inf_{\boldsymbol{z} \in K} z_i + B_i' && \text{(B.5.40)}\\
&\iff S \subset [a_1, b_1] \times ... \times [a_N, b_N] && \text{(B.5.41)}
\end{aligned}
$$

Note that $a_i \leq b_i$ because $K \subset [\pm \boldsymbol{B}']$. □

Appendix C

An elementary view of the sampling expansion for band-limited functions: truncation error bounds

An elementary view of the Nyquist-Shannon sampling expansion for band-limited square-integrable signals, part 3/3.

C.1 Complex analysis viewpoint

Our space $\mathbb{B}_B$ is just the restriction to the real line of the Paley-Wiener space PW_{2B}. This point of view requires complex analysis at a level which is beyond me, but here is what I was able to gather. For a rigorous self-contained presentation, accessible at about the level of this document, see e.g Chapters 2 and 3 of [Yan13].

This complex analysis viewpoint is interesting in its own right, but is also the basis for certain techniques to bound the truncation error [HT62].

C.1.1 Analytic continuation

Let $F \in L^2(\mathbb{R})$ supported on $[-B, B]$. Let $f \in L^2(\mathbb{R})$ its inverse Fourier transform, i.e $\hat{f} = F$:

$$f(t) = \int_{-B}^{B} d\xi \; F(\xi) e^{i2\pi\xi t} \tag{C.1.1}$$

This equality holds in the L^2 sense i.e for almost every $t \in \mathbb{R}$. Further,

$$f(t) = \int_{-B}^{B} d\xi \; F(\xi) \sum_{k=0}^{\infty} \frac{(i2\pi\xi t)^k}{k!} \tag{C.1.2}$$

$$= \sum_{k=0}^{\infty} t^k \int_{-B}^{B} d\xi \; F(\xi) \frac{(i2\pi\xi)^k}{k!} \tag{C.1.3}$$

This $\sum / \int$ inversion is justified by Fubini-Tonelli theorem, for each t, since $\left(F(\xi)\frac{(i2\pi\xi t)^k}{k!}\right)_{k\in\mathbb{N},\xi\in[-B,B]}$ is absolutely summable:

$$\sum_{k=0}^{\infty}\int_{-B}^{B}d\xi\ \left|F(\xi)\frac{(i2\pi\xi t)^k}{k!}\right| \leq \sum_{k=0}^{\infty}\left(\sup_{-B\leq\xi\leq B}\left|\frac{(i2\pi\xi t)^k}{k!}\right|\right)\left(\int_{-B}^{B}d\xi\ |F(\xi)|\right) \tag{C.1.4}$$

$$= \sum_{k=0}^{\infty}\frac{(2\pi B\,|t|)^k}{k!}\,\|F\|_{L^1} \qquad < \infty \tag{C.1.5}$$

since $\|F\|_{L^1} \leq \sqrt{2B}\,\|F\|_{L^2} < \infty$.

Consider the power series

$$g(z) = \sum_{k=0}^{\infty} z^k \int_{-B}^{B} d\xi\ F(\xi)\frac{(i2\pi\xi)^k}{k!} \tag{C.1.6}$$

Denote a_k its k-th coefficient. We have that $|a_k| \leq \frac{(2\pi B)^k}{k!}\,\|F\|_{L^1}$, so the power series converges absolutely for all $z\in\mathbb{C}$ i.e $g(z)$ has convergence radius ∞. Thus it is holomorphic on all of $\mathbb{C}$, a.k.a analytic on all of $\mathbb{C}$, a.k.a an entire function, which we denote $g(z)\in\mathrm{Hol}(\mathbb{C})$.

Now $f(t)$ coincides with $g(z)$ on the non-discrete set $\mathbb{R}$, so f may be prolonged analytically to $\mathbb{C}$ as $f(z) := g(z)$. In summary we have, in the sense of analytic functions i.e pointwise

$$\forall z\in\mathbb{C},\ f(z) = \int_{-B}^{B} d\xi\ F(\xi)e^{i2\pi\xi z} \tag{C.1.7}$$

Relation to Laplace transform Recall that the bilateral Laplace transform of F is:

$$\mathcal{L}F(s) = \int_{-\infty}^{\infty} d\xi\ F(\xi)e^{-s\xi} = \int_{-B}^{B} d\xi\ F(\xi)e^{-s\xi} \tag{C.1.8}$$

defined for all $s\in\mathbb{C}$ such that the integral converges. Note that

$$f(z) = \int_{-B}^{B} d\xi\ F(\xi)e^{i2\pi\xi z} = \mathcal{L}F(-i2\pi z) \tag{C.1.9}$$

In particular, we recover that for F supported on a compact, the (bilateral) Laplace transform is defined for all $s\in\mathbb{C}$.

Exponential type Denoting $z = \Re(z) + i\Im(z)$, [Rud87, section 19.1]

$$|f(z)| \leq \int_{-B}^{B} d\xi\ \left|F(\xi)e^{i2\pi\xi z}\right| \tag{C.1.10}$$

$$\leq \|F\|_{L^1}\sup_{\xi\in[-B,B]}\left|e^{i2\pi\xi z}\right| \tag{C.1.11}$$

$$= \|F\|_{L^1}\sup_{\xi\in[-B,B]} e^{-2\pi\xi\Im(z)} \tag{C.1.12}$$

$$= \|F\|_{L^1}\ e^{2\pi B|\Im(z)|} \tag{C.1.13}$$

In particular, $\forall z\in\mathbb{C}, |f(z)| \leq \mathrm{cst}\cdot e^{2\pi B|z|}$. We say that f has *exponential type* $2\pi B$. [1]

[1] `https://en.wikipedia.org/wiki/Exponential_type`

C.1.2 A Paley-Wiener space

Denote as PW_{2B} the space of entire functions of exponential type $2\pi B$ that are square-integrable over the real line: (note that this is not standard terminology, most often that space is not given a name)

$$PW_{2B}(C) = \left\{ f \in \mathrm{Hol}(\mathbb{C});\ \begin{matrix} \int_{-\infty}^{\infty} dt\ |f(t)|^2 < \infty \\ \forall z \in \mathbb{C}, |f(z)| \leq Ce^{2\pi B|z|} \end{matrix} \right\} \tag{C.1.14}$$

$$PW_{2B} = \bigcup_{C>0} PW_{2B}(C) \tag{C.1.15}$$

In the previous subsection, we saw that any $f(t) \in \mathbb{B}_B$ can be seen as the restriction to $\mathbb{R}$ of a function $f(z) \in PW_{2B}$. It turns out that the converse is true:

$$\mathbb{B}_B = \{\, f|_{\mathbb{R}}\ ;\ f(z) \in PW_{2B}\} \tag{C.1.16}$$

This is the Paley-Wiener theorem [HP05, theorem A.3.46] [Rud87, theorem 19.3].

Relation to Hardy spaces See [HP05, appendix A.3], `https://en.wikipedia.org/wiki/Paley%E2%80%93Wiener_theorem`, and `https://en.wikipedia.org/wiki/H_square` for more information on Hardy spaces.

Definition C.1 (verbatim from [HP05, appendix A.3.4])**.** For $1 \leq p \leq \infty$ and $\alpha \in \mathbb{R}$ denote by $H^p(\mathbb{C}_{\Re>\alpha})$ the space of all analytic functions u on $\mathbb{C}_{\Re>\alpha} = \{z \in \mathbb{C}; \Re(z) > \alpha\}$ satisfying $\|u\|_{H^p(\mathbb{C}_{\Re>\alpha})} < \infty$ where

$$\|u\|_{H^p(\mathbb{C}_{\Re>\alpha})} = \begin{cases} \sup_{\beta>\alpha} \left(\int_{-\infty}^{\infty} d\omega\, |u(\beta + i\omega)|^p \right)^{1/p} & \text{if } p < \infty \\ \sup_{\Re(s)>\alpha} |u(s)| & \text{if } p = \infty \end{cases} \tag{C.1.17}$$

It is known that this defines norms on the vector spaces $H^p(\mathbb{C}_{\Re>\alpha})$, and provided with these norms they are Banach spaces.

By [HP05, proposition A.3.45], since any $F \in L^2(\mathbb{R})$ supported on $[-B, B]$ is also in $L^p(\mathbb{R})$ for any $1 \leq p \leq 2$, then

$$PW_{2B} \subset \bigcap_{1\leq p\leq 2} H^p(\mathbb{C}_{\Re>-\infty}) \tag{C.1.18}$$

C.2 Bounds on the truncation error

Notation We focus on the one-dimensional case, as transposing to higher dimension is not difficult. Note that here N indicates the number of terms kept, rather than dimension.

For any fixed (B, B')-synthesizer $\sqrt{2B'}\psi_0(t)$, and $f \in \mathbb{B}_B$,

$$f(t) = \sum_{n\in\mathbb{Z}} \left\langle f, \phi_n^{(B)} \right\rangle \psi_n(t) \tag{C.2.1}$$

$$= \sum_{n\in\mathbb{Z}} \frac{1}{\sqrt{2B'}} f\left(\frac{n}{2B'}\right)\ \psi_0\left(t - \frac{n}{2B'}\right) \tag{C.2.2}$$

Denote the truncated expansion and the remainder: [2]

$$f_N(t) := \sum_{|n|\leq N} \frac{1}{\sqrt{2B'}} f\left(\frac{n}{2B'}\right)\ \psi_0\left(t - \frac{n}{2B'}\right) \tag{C.2.3}$$

$$R_N(t) := f(t) - f_N(t) = \sum_{|n|>N} \frac{1}{\sqrt{2B'}} f\left(\frac{n}{2B'}\right)\ \psi_0\left(t - \frac{n}{2B'}\right) \tag{C.2.4}$$

[2] More generally, we could consider $R_I(t) = \sum_{n\notin I}[\ldots]$. For simplicity we stick to the case $I = [-N, N]$.

For notational convenience, let

$$c_n = \frac{1}{\sqrt{2B'}} f\left(\frac{n}{2B'}\right) \qquad f(t) = \sum_n c_n \psi_0\left(t - \frac{n}{2B'}\right) \tag{C.2.5}$$

$$\tilde{c}_n = c_n \mathbb{1}_{|n|>N} \qquad R_N(t) = \sum_n \tilde{c}_n \psi_0\left(t - \frac{n}{2B'}\right) \tag{C.2.6}$$

Recall from section A.3 that $(c_n)_n \in \mathbb{A}_{B/B'}$ for all $f \in \mathbb{B}_B$. Crucially, note that $(\tilde{c}_n)_n \notin \mathbb{A}_{B/B'}$, in general.

The goal Establish upper-bounds on the *truncation error* $\|R_N(t)\|$ (as measured by some norm or semi-norm $\|\cdot\|$ to be specified), as a function of assumptions on f and of the choice of ψ_0.

Sources, further reading See [HT62] and [Jag66] for L^∞-norm truncation error bounds for the classical sampling expansion, and for the Helms-Thomas expansion (the classical, not the oversampled variant).

See [KK08], on "sampling in shift-invariant spaces", for more generalizations of the sampling expansion. That paper also takes the point of view of Hilbert spaces. It is much more general, but also much more abstract. It's very possible that most of the present document's content can be seen as a special case of it.

See [PS96] for a principled exploration of possible choices of window-based synthesizers i.e of the form $\Psi_0(t) = \Phi_0^{(B')}(t)g(t)$ (in one dimension); along with truncation bounds for the L^∞ norm. In fact our discussion of section B.2 was initially based on that paper. In particular, in Theorem 2 and Lemma 5 they describe a generic technique for deriving truncation bounds for window-based synthesizers, which essentially generalizes the approach of [Jag66].

C.2.1 For the L^2 norm

In this subsection, we discuss how to control the L^2-norm truncation error $\|R_N\|_{L^2}$. Considering the L^2 norm is morally the easiest, since all of our preliminary derivations in appendix A and appendix B have been from the Hilbert space point of view.

Let any (B, B')-synthesizer $\sqrt{2B'}\psi_0$, and $f \in \mathbb{B}_B$.

$$\forall t,\ |R_N(t)|^2 = \sum_{n,m} \tilde{c}_n \overline{\tilde{c}_m} \psi_0\left(t - \frac{n}{2B'}\right) \overline{\psi_0\left(t - \frac{m}{2B'}\right)} \tag{C.2.7}$$

$$\|R_N(t)\|_{L^2}^2 = \sum_{n,m} \tilde{c}_n \overline{\tilde{c}_m} \underbrace{\int_{\mathbb{R}} dt\ \psi_0\left(t - \frac{n}{2B'}\right) \overline{\psi_0\left(t - \frac{m}{2B'}\right)}}_{=:A_{nm}} \tag{C.2.8}$$

and

$$\begin{aligned} A_{nm} &= \int_{\mathbb{R}} dt\ \psi_0\left(t - \frac{n-m}{2B'}\right) \overline{\psi_0(t)} && \text{(C.2.9)} \\ &= \operatorname{Autocorr} \psi_0 \left(\frac{n-m}{2B'}\right) && \text{(C.2.10)} \\ &= \left\langle \psi_0\left(t - \frac{n-m}{2B'}\right), \psi_0 \right\rangle && \text{(C.2.11)} \\ &= \left\langle \hat{\psi}_0 \cdot e^{-i2\pi\frac{n-m}{2B'}\xi}, \hat{\psi}_0 \right\rangle && \text{(C.2.12)} \end{aligned}$$

where $\operatorname{Autocorr} \psi(\tau)$ denotes the autocorrelation function of ψ.

For example, if we use $\psi_0(t) = \psi_0^{(B,B')}(t)$ where $\hat{\psi}_0^{(B,B')}(\xi) = \frac{1}{\sqrt{2B'}}\mathbb{1}_{|\xi|\leq B}$, then

$$A_{nm} = \frac{1}{2B'}\int_{-B}^{B} d\xi \; e^{-i2\pi\frac{n-m}{2B'}\xi} \tag{C.2.13}$$

$$= \frac{2B}{2B'}\operatorname{sinc}\left((n-m)\frac{B}{B'}\right) \tag{C.2.14}$$

Since $\tilde{c}_n = 0$ for $|n| \leq N$, denoting $\widetilde{A}_{nm} = A_{nm}\mathbb{1}_{|n|>N,|m|>N}$ we have

$$\|R_N(t)\|_{L^2}^2 = \sum_{n,m}\tilde{c}_n\overline{\tilde{c}_m}\widetilde{A}_{nm} \tag{C.2.15}$$

If we don't have any more information on the truncated-sample-sequence $\tilde{c}_n = \frac{1}{\sqrt{2B'}}f\left(\frac{n}{2B'}\right)\mathbb{1}_{|n|>N}$, i.e we only know that $(\tilde{c}_n)_n \in \ell^2(\mathbb{Z})$ and $\tilde{c}_n = 0$ for $|n| > N$, then the best we can do is:

$$\|R_N(t)\|_{L^2}^2 \leq \left|\left|\left|\widetilde{A}\right|\right|\right| \sum_n |\tilde{c}_n|^2 \tag{C.2.16}$$

where $\left|\left|\left|\widetilde{A}\right|\right|\right| = \sup_{\|d\|_{\ell^2}=1}\left\|\widetilde{A}d\right\|_{\ell^2}$ is the operator norm of $\widetilde{A} : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$.

Computing the coefficients A_{nm}? Recall from our characterization of synthesizers in section B.1 that ψ_0 can always be put in the form $\hat{\psi}_0(\xi) = \hat{\phi}_0^{(B')}(\xi) + \hat{h}(\xi)$ for some h such that $\hat{h}(\xi) = 0$ on $[-B,B] + 2B'\mathbb{Z}$. So

$$A_{nm} = \left\langle \hat{\psi}_0 \cdot e^{-i2\pi\frac{n-m}{2B'}\xi}, \hat{\psi}_0 \right\rangle \tag{C.2.17}$$

$$= \left\langle \hat{\phi}_0^{(B')} \cdot e^{-i2\pi\frac{n-m}{2B'}\xi}, \hat{\phi}_0^{(B')} \right\rangle \tag{C.2.18}$$

$$+ \left\langle \hat{\phi}_0^{(B')} \cdot e^{-i2\pi\frac{n-m}{2B'}\xi}, \hat{h} \right\rangle + \left\langle \hat{h} \cdot e^{-i2\pi\frac{n-m}{2B'}\xi}, \hat{\phi}_0^{(B')} \right\rangle \tag{C.2.19}$$

$$+ \left\langle \hat{h} \cdot e^{-i2\pi\frac{n-m}{2B'}\xi}, \hat{h} \right\rangle \tag{C.2.20}$$

- We already know that the first term is equal to $\mathbb{1}_{n=m}$.
- For the second line, we can use that $\hat{\phi}_0^{(B')}(\xi)$ is real-valued and properties of the hermitian inner product, and obtain

$$\left\langle \hat{\phi}_0^{(B')} \cdot e^{-i2\pi\frac{n-m}{2B'}\xi}, \hat{h} \right\rangle + \left\langle \hat{h} \cdot e^{-i2\pi\frac{n-m}{2B'}\xi}, \hat{\phi}_0^{(B')} \right\rangle = \left\langle 2\Re\hat{h}(\xi)e^{-i2\pi\frac{n-m}{2B'}\xi}, \hat{\phi}_0^{(B')} \right\rangle \tag{C.2.21}$$

$$= \frac{1}{\sqrt{2B'}}\int_{-B'}^{B'} d\xi \; 2\Re\hat{h}(\xi)\; e^{-i2\pi\frac{n-m}{2B'}\xi} \tag{C.2.22}$$

  where $\Re\hat{h}$ denotes the real part of $\hat{h}$. Note that since $\hat{h}(\xi) = 0$ on $[-B,B] + 2B'\mathbb{Z}$, the integrand is nonzero only for $\xi \in [-B',-B] \cup [B,B']$. Also note that this quantity can be interpreted using the inverse DTFT of $\hat{h}\Big|_{[-B',B']}$:

$$\frac{1}{\sqrt{2B'}}\int_{-B'}^{B'} d\xi \; 2\Re\hat{h}(\xi)\; e^{+i2\pi\frac{m-n}{2B'}\xi} = \left(\mathcal{F}_{d,2B'}^{-1}\left[2\Re\hat{h}\Big|_{[-B',B']}\right]\right)_{m-n} \tag{C.2.23}$$

- The last term can be written as

$$\left\langle \hat{h} \cdot e^{-i2\pi \frac{n-m}{2B'}\xi}, \hat{h} \right\rangle = \int_{\mathbb{R}} d\xi \, \left| \hat{h}(\xi) \right|^2 e^{-i2\pi \frac{n-m}{2B'}\xi} \tag{C.2.24}$$

$$= \sum_{k\in\mathbb{Z}} \int_{-B'}^{B'} d\xi \, \left| \hat{h}(\xi + 2kB') \right|^2 e^{-i2\pi \frac{n-m}{2B'}\xi} \tag{C.2.25}$$

Again, for each k, the integrand is nonzero only for $\xi \in [-B', -B] \cup [B, B']$. The integral on the first line is absolutely convergent since $\hat{h} \in L^2(\mathbb{R})$, so we can also write

$$\left\langle \hat{h} \cdot e^{-i2\pi \frac{n-m}{2B'}\xi}, \hat{h} \right\rangle = \int_{-B'}^{B'} \left(\sum_{k\in\mathbb{Z}} \left| \hat{h}(\xi + 2kB') \right|^2 \right) e^{-i2\pi \frac{n-m}{2B'}\xi} \tag{C.2.26}$$

(And the integrand is still nonzero only for $\xi \in [-B', -B] \cup [B, B']$.)

C.2.2 For the L^∞ norm

Results for the L^∞ norm are morally trickier to obtain, as the Hilbert space point of view is not sufficient. We will need to use specificities of the chosen synthesizer function $\psi_0(t)$.

Generic truncation bound By Hoelder's inequality, for all $p, q \in [1, \infty]$ such that $\frac{1}{p} + \frac{1}{q} = 1$,

$$|R_N(t)| = \left| \sum_n \tilde{c}_n \psi_0 \left(t - \frac{n}{2B'} \right) \right| \tag{C.2.27}$$

$$\leq \|\tilde{\boldsymbol{c}}\|_p \, \left\| \left(\mathbb{1}_{|n|>N} \psi_0 \left(t - \frac{n}{2B'} \right) \right)_n \right\|_q \tag{C.2.28}$$

Depending on what information we have on f (or more precisely on its samples $(c_n)_n$), the pair (p, q) should be chosen accordingly. This in turn gives a criterion for choosing ψ_0.

That is: suppose for example that $\left(\sum_{|n|>N} \left| f\left(\frac{n}{2B'}\right) \right|^p \right)^{1/p} = \sqrt{2B'} \, \|\tilde{\boldsymbol{c}}\|_p$ goes to zero at a known fast rate when $N \to \infty$. Then, to get the most out of the truncation bound above, one should consider using a synthesizer ψ_0 such that $\sup_t \left\| \left(\mathbb{1}_{|n|>N} \psi_0 \left(t - \frac{n}{2B'} \right) \right)_n \right\|_q$ is small.

Remark C.1. Almost all existing results that we reviewed are continuations of this generic truncation bound (with the exception of [HT62] which uses the complex analysis viewpoint).

- [Jag66, theorems 1, 2] give a bound on $|R_N(t)|$ for the exact-sampling case (i.e $\psi_0(t) = \phi_0^{(B')}(t) = \sqrt{2B'}\,\mathrm{sinc}(2B't)$), using $p = q = 2$. In other words they give an upper-bound in terms of $\sqrt{2B'}\,\|\tilde{c}\|_2 = \left[\sum_{|n|>N} \left| f\left(\frac{n}{2B'}\right) \right|^2 \right]^{1/2}$.
- [Jag66, theorem 5] presents a bound on $|R_N(t)|$ for the Helms-Thomas expansion using $p = \infty$ and $q = 1$. In other words they give an upper-bound in terms of $\sqrt{2B'}\,\|\tilde{c}\|_\infty = \sup_{|n|>N} \left| f\left(\frac{n}{2B'}\right) \right|$.
- [PS96, theorem 2, lemma 5] describe a generic technique for deriving truncation bounds for window-based synthesizers, which essentially generalizes the approach of [Jag66].

Remark C.2 (Decay of the signal (resp. synthesizer) and smoothness of its spectrum). For all ψ, $\psi(t) = o_{|t|\to\infty}(t^{-k})$, if and only if $\hat{\psi}(\xi)$ has an integrable (weak) derivative of order k.

This corresponds to the well-known fact that, informally, *the Fourier transform swaps smoothness for decay at infinity.* [3] Actually that statement can be misleading, since more precisely here "smoothness" means *integrability* of the derivatives, not continuity; so for example $e^{-|t|}$ is infinitely "smooth" in that sense.

[3] `https://math.stackexchange.com/questions/206362/smoothness-and-decay-property-of-fourier-transformation`

Consequently, we have sufficient conditions for finiteness of each factor in the generic truncation bound:

- For fixed t, $\left|\psi_0\left(t - \frac{n}{2B'}\right)\right|^q = o_{|n|\to\infty}(n^{-qk})$. So by Riemann's criterion, the infinite sum $\sum_n \left|\psi_0\left(t - \frac{n}{2B'}\right)\right|^q$ is finite if $qk > 1$, i.e $\hat{\psi}_0$ has integrable (weak) derivative of order $1 + \lfloor 1/q \rfloor$.
- Likewise, the infinite sum $\sum_n \left|f\left(\frac{n}{2B'}\right)\right|^p$ is finite if $pk > 1$, i.e $\hat{f}$ has integrable (weak) derivative of order $1 + \lfloor 1/p \rfloor$.

Difficulty in higher-dimension It's not difficult to show that, in d dimensions, the above generic truncation bound generalizes to

$$|R_{\boldsymbol{N}}(\boldsymbol{t})| \leq \frac{1}{\sqrt{2B'_1 ... 2B'_d}} \left[\sum_{\exists i; |n_i| > N_i} \left|f\left(\frac{\boldsymbol{n}}{2\boldsymbol{B'}}\right)\right|^p\right]^{1/p} \left[\sum_{\exists i; |n_i| > N_i} \left|\psi_0\left(\boldsymbol{t} - \frac{\boldsymbol{n}}{2\boldsymbol{B'}}\right)\right|^q\right]^{1/q} \tag{C.2.29}$$

(In the case $1 < p, q < \infty$, with the appropriate modifications if p or $q = \infty$.)

It's quite a bit trickier to derive precise bounds, because the sums involved are over $\left\{\boldsymbol{n} \in \mathbb{Z}^d; \exists i, |n_i| > N_i\right\}$, which is tricky to work with. Indeed *it is wrong to write*

$$\left|\sum_{\exists i; |n_i| > N_i} [...]\right| \leq \sum_{|n_1| > N_1} ... \sum_{|n_d| > N_d} |[...]| \tag{C.2.30}$$

A natural first approach is to use the union-bound, though it seems quite brutal:

$$\left|\sum_{\exists i; |n_i| > N_i} [...]\right| \leq \sum_i \sum_{\boldsymbol{n} \in \mathbb{Z}^d; |n_i| > N_i} |[...]| \tag{C.2.31}$$

C.3 Example: the Helms-Thomas expansion

Recall from appendix B that the Helms-Thomas expansion is: for all $b, B > 0$, $m \in \mathbb{N}^*$ and $B' := B + mb$,

$$\forall f \in \mathbb{B}_B,\ f(t) = \sum_{n \in \mathbb{Z}} f\left(\frac{n}{2B'}\right) \left[\operatorname{sinc}\left(2b\left(t - \frac{n}{2B'}\right)\right)\right]^m \operatorname{sinc}\left(2B'\left(t - \frac{n}{2B'}\right)\right) \tag{C.3.1}$$

Use a slightly different normalization convention compared to the previous section: write $\Psi_0(t)$ instead of $\sqrt{2B'}\psi_0(t)$, and denote

$$c_n = \frac{1}{2B'} f\left(\frac{n}{2B'}\right) \qquad f(t) = \sum_n c_n \Psi_0\left(t - \frac{n}{2B'}\right) \tag{C.3.2}$$

$$\tilde{c}_n = c_n \mathbb{1}_{|n| > N} \qquad R_N(t) = \sum_n \tilde{c}_n \mathbb{1}_{|n| > N} \Psi_0\left(t - \frac{n}{2B'}\right) \tag{C.3.3}$$

where

$$\Psi_0(t) = 2B' \operatorname{sinc}(2Bt') \left[\operatorname{sinc}(2bt)\right]^m \tag{C.3.4}$$

C.3.1 Bound on truncation error for the $L^\infty([-W, W])$ semi-norm

Suppose we want to bound the truncation error uniformly over $[-W, W]$.

Proposition C.1 ([Zam79, theorem 2])**.** For all $N > 2B'W$,

$$\|R_N\|_{L^\infty([-W,W])} \leq \left(\sup_n |\tilde{c}_n|\right) \sup_{t\in[-W,W]} \sum_{|n|>N} \left|\Psi_0\left(t - \frac{n}{2B'}\right)\right| \tag{C.3.5}$$

$$\leq \left(\sup_n |\tilde{c}_n|\right) \frac{(2B')^{m+1}}{\pi^{m+1}(2b)^m} \frac{2}{m} \left(N - 2B'W\right)^{-m} \tag{C.3.6}$$

Proof. The first inequality is by Hoelder's inequality with $p = \infty$, $q = 1$.

Let $N > 2B'W$ and $t \in [-W, W]$.

$$\sum_{|n|>N} \left|\Psi_0\left(t - \frac{n}{2B'}\right)\right| = 2B' \sum_{|n|>N} \left|\operatorname{sinc}\left(2B'\left(t - \frac{n}{2B'}\right)\right)\right| \left|\operatorname{sinc}\left(2b\left(t - \frac{n}{2B'}\right)\right)\right|^m \tag{C.3.7}$$

$$\leq 2B' \frac{1}{2B'\pi} \frac{1}{(2b\pi)^m} \sum_{|n|>N} \frac{1}{\left|t - \frac{n}{2B'}\right|^{m+1}} \tag{C.3.8}$$

$$= \frac{1}{\pi^{m+1}(2b)^m} (2B')^{m+1} \sum_{|n|>N} \frac{1}{|2B't - n|^{m+1}} \tag{C.3.9}$$

since $\operatorname{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$ and $|\sin(x)| \leq 1$.

Since $|2B't| \leq 2B'W < N$, then for all $|n| > N$,

$$|2B't - n| \geq |n| - 2B'W > 0 \tag{C.3.10}$$

So, the summand being monotone decreasing in n,

$$\sum_{|n|>N} \frac{1}{|2B't - n|^{m+1}} \leq \sum_{|n|>N} \frac{1}{(|n| - 2B'W)^{m+1}} \tag{C.3.11}$$

$$= 2 \sum_{n>N} \frac{1}{(n - 2B'W)^{m+1}} \tag{C.3.12}$$

$$\leq 2 \int_N^\infty dx \, \frac{1}{(x - 2B'W)^{m+1}} \tag{C.3.13}$$

$$= 2 \frac{1}{m} \left(N - 2B'W\right)^{-m} \tag{C.3.14}$$

In summary,

$$\sum_{|n|>N} \left|\Psi_0\left(t - \frac{n}{2B'}\right)\right| \leq \frac{1}{\pi^{m+1}(2b)^m} (2B')^{m+1} 2 \frac{1}{m} \left(N - 2B'W\right)^{-m} \tag{C.3.15}$$

$$= \frac{2}{m} \frac{(2B')^{m+1}}{\pi^{m+1}(2b)^m} \left(N - 2B'W\right)^{-m} \tag{C.3.16}$$

□

C.3.2 Bound on coefficient-sensitivity error for the $L^\infty(\mathbb{R})$ norm

Lemma C.2 ([Zam79, lemma 1])**.** Let, for some coefficients δ_n (that are morally small),

$$h(t) = \sum_{n\in\mathbb{Z}} \delta_n \Psi_0\left(t - \frac{n}{2B'}\right) \tag{C.3.17}$$

For any $p \in \mathbb{N}^*$,

$$\|h\|_{L^\infty(\mathbb{R})} \leq \left(\sup_n |\delta_n|\right) \sup_t \sum_{n\in\mathbb{Z}} \left|\Psi_0\left(t - \frac{n}{2B'}\right)\right| \tag{C.3.18}$$

$$\leq \left(\sup_n |\delta_n|\right) \left((2p+1) + \frac{(2B')^{m+1}}{\pi^{m+1}(2b)^m} \frac{2}{m}\left(p - \frac{1}{2}\right)^{-m}\right) \tag{C.3.19}$$

Proof. The first inequality is by Hoelder's inequality with $p = \infty$, $q = 1$.
For each $t \in \mathbb{R}$, denote $k_t \in \mathbb{Z}$ such that $|2B't - k_t| \leq \frac{1}{2}$, and $\tilde{t} = t - \frac{k_t}{2B'}$, so that $\left|2B'\tilde{t}\right| \leq 1/2$.
Fix a $p \in \mathbb{N}^*$. For each $t \in \mathbb{R}$, translate the sum as:

$$\sum_{n\in\mathbb{Z}} \left|\Psi_0\left(t - \frac{n}{2B'}\right)\right| = \sum_{n\in\mathbb{Z}} \left|\Psi_0\left(t - \frac{n+k_t}{2B'}\right)\right| = \sum_{n\in\mathbb{Z}} \left|\Psi_0\left(\tilde{t} - \frac{n}{2B'}\right)\right| \tag{C.3.20}$$

$$= 2B' \sum_{n\in\mathbb{Z}} \left|\operatorname{sinc}\left(2B'\left(\tilde{t} - \frac{n}{2B'}\right)\right)\right| \left|\operatorname{sinc}\left(2b\left(\tilde{t} - \frac{n}{2B'}\right)\right)\right|^m \tag{C.3.21}$$

Now split the sum according to $|n| \gtrless p$, and bound the first part by $|\operatorname{sinc}(x)| \leq 1$:

$$\frac{1}{2B'} \sum_{n\in\mathbb{Z}} \left|\Psi_0\left(t - \frac{n}{2B'}\right)\right| \leq (2p+1) + \sum_{|n|>p} \left|\operatorname{sinc}\left(2B'\left(\tilde{t} - \frac{n}{2B'}\right)\right)\right| \left|\operatorname{sinc}\left(2b\left(\tilde{t} - \frac{n}{2B'}\right)\right)\right|^m \tag{C.3.22}$$

$$\leq (2p+1) + \frac{1}{2B'\pi} \frac{1}{(2b\pi)^m} \sum_{|n|>p} \frac{1}{\left|\tilde{t} - \frac{n}{2B'}\right|^{m+1}} \tag{C.3.23}$$

$$= (2p+1) + \frac{1}{\pi^{m+1} 2B'(2b)^m} (2B')^{m+1} \sum_{|n|>p} \frac{1}{\left|2B'\tilde{t} - n\right|^{m+1}} \tag{C.3.24}$$

Note that $\left|2B'\tilde{t}\right| \leq 1/2 < p$, so we can use the same argument as for the truncation error. We obtain

$$\sum_{|n|>p} \frac{1}{\left|2B'\tilde{t} - n\right|^{m+1}} \leq \frac{2}{m} (p - 1/2)^{-m} \tag{C.3.25}$$

In summary,

$$\sum_{n\in\mathbb{Z}} \left|\Psi_0\left(t - \frac{n}{2B'}\right)\right| \leq 2B'(2p+1) + 2B' \frac{1}{\pi^{m+1} 2B'(2b)^m} (2B')^{m+1} \frac{2}{m} (p - 1/2)^{-m} \tag{C.3.26}$$

$$\leq 2B'(2p+1) + \frac{(2B')^{m+1}}{\pi^{m+1}(2b)^m} \frac{2}{m} (p - 1/2)^{-m} \tag{C.3.27}$$

□

Proposition C.3. Let, for some coefficients δ_n (that are morally small),

$$h(t) = \sum_{n\in\mathbb{Z}} \delta_n \Psi_0\left(t - \frac{n}{2B'}\right) \tag{C.3.28}$$

Then,

$$\|h\|_{L^\infty(\mathbb{R})} \leq \left(\sup_n |\delta_n|\right) \inf_{p>1/2} \left((2p+3) + \frac{(2B')^{m+1}}{\pi^{m+1}(2b)^m} \frac{2}{m}\left(p - \frac{1}{2}\right)^{-m}\right) \tag{C.3.29}$$

$$= \left(\sup_n |\delta_n|\right) \left(4 + \left(2 + \frac{2}{m}\right)(2b)^{\frac{1}{m+1}} \frac{2B'}{\pi 2b}\right) \tag{C.3.30}$$

Proof. Fix a $p > 1/2$. Apply the lemma using $\lceil p \rceil$. Then, since $p \leq \lceil p \rceil \leq p+1$ and $[p \mapsto 2p+1]$ is increasing and $[p \mapsto (p-1/2)^{-m}]$ is decreasing, the claimed upper-bound follows.

It remains to compute the inf, which is straightforward by setting the derivative of $p \mapsto (2p+3) + \frac{(2B')^{m+1}}{\pi^{m+1}(2b)^m} \frac{2}{m} \left(p - \frac{1}{2}\right)^{-m}$ to zero. The claimed bound follows. $\square$

Appendix D

Bounded linear operators between Banach signal spaces

D.1 Duality in Banach spaces

This section can be used as a short (and rather dry) primer on some aspects of Banach spaces, for readers used to working only in Hilbert spaces. As will become apparent, Banach spaces can be seen as a generalization of Hilbert spaces, the difference being essentially that the (topological) dual space is not necessarily isomorphic to the primal.

Definition D.1 (Basic notions)**.** A metric space (E, d) is complete if all Cauchy sequences $(u_n)_n \in E^{\mathbb{N}}$ converge in E.

A Banach space $(E, \|\cdot\|_E)$ is a vector space equipped with a norm for which it is a complete space.

A Hilbert space $(H, \langle\cdot,\cdot\rangle_H)$ is a vector space equipped with an inner product that is complete for the induced norm $\|x\|_H^2 = \langle x, x\rangle_H$.

The unit ball of a normed space $(E, \|\cdot\|_E)$ is denoted $B_{0,1}^{(E)} = B^{(E)} := \{x \in E;\ \ \|x\|_E \leq 1\}$.

A continuous linear mapping $T : E \to F$ between Banach spaces is called a bounded operator, and its operator norm is the finite quantity $|||T||| = |||T|||_{E\to F} = \sup_{\|x\|_E \leq 1} \|Tx\|_F$. The set of bounded operators from E to F equipped with the operator norm $(\mathcal{L}_b(E, F), |||\cdot|||)$ is itself a Banach space.

The bounded operator $T : E \to F$ is called compact if it sends the unit ball into a relatively compact set, i.e $T(B^{(E)})$ is a relatively compact set of F, i.e $\overline{T(B^{(E)})}$ is compact where $\overline{\cdot}$ denotes closure w.r.t the topology of F.

Definition D.2 (Duality)**.** The (topological) dual space of a Banach space E is the space of bounded linear forms $E' = \mathcal{L}_b(E, \mathbb{R})$. It is equipped with the norm $\|X\|_{E'} := \sup_{\|x\|_E \leq 1} |X(x)|$. E' is itself a Banach space.

The duality bracket is the bilinear operator $\langle\cdot,\cdot\rangle_E : E \times E' \to \mathbb{R}$ defined by $\langle x, X\rangle_E = X(x)$.

The bidual space is $E'' = (E')'$. E can be embedded into E'' by $x \mapsto x''$, where x'' is defined by: $\forall X \in E',\ \langle X, x''\rangle_{E'} = \langle x, X\rangle_E = X(x)$. It is not hard to show (using existence of norming functionals, see below) that this embedding is isometric i.e $\|x\|_{(E')'} = \|x\|_E$.

E is called a reflexive Banach space if the converse holds, i.e any element of the bidual can also be seen as an element of the primal, i.e $E'' \simeq E$.

In this document, elements of the primal space will typically be denoted by lowercase letters e.g $x \in E, y \in F$, and elements of the dual by uppercase letters e.g $X \in E', Y \in F'$.

Remark D.1. The duality bracket is very similar to the physicists' bra-ket notation; except that here the primal is on the left and the dual is on the right, instead of the opposite.

When E is reflexive, then all the shorthands from the bra-ket notation can be used. That is, a dual element X can be denoted without ambiguity as $\langle \cdot, X \rangle_E$, and a primal element $x = x''$ as $\langle x, \cdot \rangle_E$. Moreover, for a bounded operator $T : E \to F$, we can write without ambiguity $\langle Tx, Y \rangle_F = \langle x | T | Y \rangle$. However since there are many interesting Banach spaces that are not reflexive, we will not use such shorthands.

Example D.1 (Hilbert spaces). For a Hilbert space $E = H$,

- The dual space is isomorphic to H itself by the Riesz representation theorem, i.e any bounded linear form F can be put in the form: $\forall g \in H,\ F(g) = \langle g, f \rangle_H$. (If H is a complex Hilbert space, we use the mathematicians' convention that the Hermitian inner product $\langle \cdot, \cdot \rangle_H$ is left-linear and right-antilinear.)
- The duality bracket is just the inner product.
- The bidual space is also just $H'' \simeq H' \simeq H$.

Example D.2 (Lebesgue L^p spaces). Let $1 \leq p \leq \infty$. The Lebesgue space $L^p(\mathbb{R}) = L^p_\mu(\mathbb{R})$ is a Banach. (The measure μ used for integration will often be implicit. It will typically be the Lebesgue measure.)

- For $p = 2$, $L^2(\mathbb{R})$ is a (separable) Hilbert space.
- For $1 < p < \infty$, the dual space of $L^p(\mathbb{R})$ is $(L^p(\mathbb{R}))' \simeq L^q(\mathbb{R})$ where $\frac{1}{p} + \frac{1}{q} = 1$ ("conjugate exponent"). For $x \in L^p(\mathbb{R}), X \in L^q(\mathbb{R})$, $\langle x, X \rangle_{L^p} = \int_{\mathbb{R}} du\ x(u) X(u)$. [1]
- For $p = 1$, if the measure μ used for integration is σ-finite, then the dual space is $(L^1(\mathbb{R}))' \simeq L^\infty(\mathbb{R})$, and for $x \in L^1(\mathbb{R}), X \in L^\infty(\mathbb{R})$, $\langle x, X \rangle_{L^1} = \int_{\mathbb{R}} du\ x(u) X(U)$. [2]
- For $p = \infty$, the dual space is $(L^\infty(\mathbb{R}))' \simeq ba(\mathbb{R})$ the space of bounded finitely-additive measures that are absolutely continuous w.r.t μ. (Note that finite-additivity is weaker than the usual countable-additivity condition on what is typically called "measure".)

In particular, $L^p(\mathbb{R})$ is a reflexive Banach space for $1 < p < \infty$, and one can check that $L^1(\mathbb{R})$ and $L^\infty(\mathbb{R})$ are not reflexive.

Example D.3. The space of real-valued continuous functions over a compact metric space $(\mathcal{T}, d)$ is denoted $C(\mathcal{T})$, and is equipped with the sup norm $\|y\|_{C(\mathcal{T})} = \sup_{t \in \mathcal{T}} |y(t)|$. $C(\mathcal{T})$ is a Banach and its dual is the space of regular Borel measures $rca(\mathcal{T})$. [3]

For further examples, see `https://en.wikipedia.org/wiki/List_of_Banach_spaces#Classical_Banach_spaces`.

Theorem D.1 (Hahn-Banach theorem). See `https://en.wikipedia.org/wiki/Hahn%E2%80%93Banach_theorem`.

As one of the many important consequences of that theorem, we have the existence of norming functionals.

Definition D.3 (Norming functional). For all $x \in E \setminus \{0_E\}$, there exists $X \in E'$ such that $X(x) = \langle x, X \rangle_E = \|x\|_E$ and $\|X\|_{E'} = 1$. X is then called a *norming functional* of x.

Importantly, the norming functional X is not unique in general, and there is no generic way to construct it (the proof is not constructive). (Contrast this with the Hilbert space case, where the norming functional is unique and given by the Riesz representation theorem.)

Definition D.4 (Dual operator). For a bounded operator $T : E \to F$ between Banach spaces, the dual operator is $T' : F' \to E'$ defined by $\forall Y \in F',\ T'Y : x \mapsto \langle Tx, Y \rangle_F$. In other words,

$$\forall (x, Y) \in E \times F',\ \ \langle Tx, Y \rangle_F = \langle x, T'Y \rangle_E \tag{D.1.1}$$

[1] `https://www.math.ucdavis.edu/~hunter/m206/measure_notes.pdf` Section 7.5
[2] `https://www.math.ksu.edu/~nagy/real-an/4-06-dual-lp.pdf` Theorem 6.4
[3] `https://regularize.wordpress.com/2011/11/11/dual-spaces-of-continuous-functions/` Section 3

Proposition D.2 (Specifying an operator by its action on F')**.** Denote $B_T : \begin{bmatrix} (E \times F') \to \mathbb{R} \\ \quad (x, Y) \mapsto \langle Tx, Y \rangle_F = \langle x, T'Y \rangle_E \end{bmatrix}$.
Also denote $\mathcal{B}_b(E \times F', \mathbb{R})$ the space of continuous bilinear forms, equipped with the norm $|||B|||_{\mathcal{B}} = \sup_{\|x\|_E \le 1} \sup_{\|Y\|_{F'} \le 1} |B(x, Y)|$.

Then $\begin{bmatrix} (\mathcal{L}_b(E, F), |||\cdot|||_{E \to F}) \to (\mathcal{B}_b(E, F'), |||\cdot|||_{\mathcal{B}}) \\ T \mapsto B_T \end{bmatrix}$ is an injective isometric linear map between Banach spaces. It is bijective if $F'' = F$.

Proof. Everything is trivial, except perhaps for the completeness of $(\mathcal{B}_b(E, F'), |||\cdot|||_{\mathcal{B}})$, for which we refer to `https://math.stackexchange.com/questions/185103/completeness-of-the-space-of-bounded-bilinear-maps`.

For injectiveness and isometry, use the existence of norming functionals (Hahn-Banach theorem).

For surjectiveness when F is reflexive, notice that any $B \in \mathcal{B}_B(E, F')$ defines an operator $T : E \to F''$ by $\forall Y \in F', \ \langle Y, Tx \rangle_{F'} = B(x, Y)$. ☐

Proposition D.3 (Specifying an operator by its action on a predual)**.** Suppose that there exists a Banach $\widetilde{F}$ such that $F = (\widetilde{F})'$, i.e $\widetilde{F}$ is *a predual space* of F.

Then any operator $T : E \to F = (\widetilde{F})'$ is fully characterized by the action of Tx on $\widetilde{F}$ for each $x \in E$, i.e by the bilinear form $\widetilde{B}_T : \begin{bmatrix} E \times \widetilde{F} \to \mathbb{R} \\ (x, z) \mapsto \langle z, Tx \rangle_{\widetilde{F}} \end{bmatrix}$. Furthermore, $\widetilde{B}_T$ is bounded with $\left|\left|\left|\widetilde{B}_T\right|\right|\right|_{\mathcal{B}} := \left|\left|\left|\widetilde{B}_T\right|\right|\right|_{\mathcal{B}(E \times \widetilde{F}, \mathbb{R})} = |||T|||_{E \to F}$.

In other words, $\begin{bmatrix} (\mathcal{L}_b(E, F), |||\cdot|||_{E \to F}) \to (\mathcal{B}_b(E, \widetilde{F}), |||\cdot|||_{\mathcal{B}}) \\ T \mapsto \widetilde{B}_T \end{bmatrix}$ is an injective isometric linear map between Banach spaces. In fact it is even bijective (without need for additional assumption).

Proof. Same as the previous proposition. ☐

Definition D.5 (Space of continuous functions over a compact)**.** The space of continuous functions over a compact metric space $(\mathcal{T}, d)$ with values in a Banach space G is denoted $C(\mathcal{T}, G)$ or $C_G(\mathcal{T})$, and is equipped with the sup norm $\|y\|_{C(\mathcal{T}, G)} = \sup_{t \in \mathcal{T}} \|y(t)\|_G$. It is a Banach space.

Its dual space is the space of G'-valued vector measures, see `https://mathoverflow.net/questions/354944/dual-space-of-continuous-banach-space-valued-functions` (I don't really understand this, I just mention it for completeness).

When $G = \mathbb{R}$, we will abbreviate $C_{\mathbb{R}}(\mathcal{T}) = C(\mathcal{T})$, and its dual space is $rca(\mathcal{T})$ as noted above.

Other notions that are less basic will be introduced in each section as needed (Banach-Mazur theorem, Bochner spaces, tensor product spaces).

D.2 An attempt at the general case via Banach-Mazur theorem

The Banach-Mazur theorem essentially reduces all separable Banach spaces to $C([0, 1])$. In this section we illustrate that that reduction is too abstract for the purpose of understanding the structure of linear operators with values in an arbitrary separable Banach.

Problem statement This section should be understood with the context of chapter 3 in mind. Let us briefly recall it. First pose (or recall) the notations:

- Denote $\mathcal{X} = \mathcal{X}(U)$ a Banach space of input signals; its elements will typically be denoted as $x(u)$, and $u \in U$ corresponds to the variable of the input signals.

- Likewise, $\mathcal{Y} = \mathcal{Y}(V)$ is a Banach space of output signals, with elements typically denoted as $y(v)$.
- We consider a linear system mapping input signals to output signals, represented by a bounded operator $T : \mathcal{X}(U) \to \mathcal{Y}(V)$.

The question we ask is (informally):

Under what conditions can we write T as an integral operator i.e in the form

$$\text{" } (Tx)(v) = \int_U du\ K(v,u)x(u) \text{ "} \tag{D.2.1}$$

for some kernel function K in some space to be specified?

One way to formulate this problem rigorously can be found in the first three paragraphs of `https://encyclopediaofmath.org/wiki/Integral_representations_of_linear_operators`. But here we take a more naive approach and interpret this question as an open-ended question.

In the introduction of [CS90, chapter 5] they justify giving particular attention to $C([0,1])$-valued operators by mentioning that any separable Banach space $\mathcal{Y}$ can be seen as a subspace of $C([0,1])$.

Definition D.6. A Banach space E is called separable if it contains a dense generating family, i.e if there exists $(e_n)_{n\in\mathbb{N}} \in E^{\mathbb{N}}$ such that $\mathrm{span}((e_n)_n)$ is dense in E.

Theorem D.4 (Banach-Mazur theorem)**.** Every separable Banach space is isometrically isomorphic to a subspace of $C([0,1])$.

This theorem seems to indicate that, to study separable Banach spaces, it is in some sense sufficient to study $C([0,1])$. In this section, we try to leverage this idea along with our previous study of $C(V)$-valued operators with V compact (chapter 3), to treat the case of $\mathcal{Y}$-valued operators with $\mathcal{Y}$ separable.

We will argue that, unfortunately, *the reduction suggested by the Banach-Mazur theorem is too abstract for our purpose.* For concreteness, think of $\mathcal{Y}(V) = L^r(\mathbb{R})$ for some $1 \leq r < \infty$. This is clearly a separable Banach space, with generating family $\left\{ \mathbb{1}_{\left[\frac{k}{2^n}, \frac{k+1}{2^n}\right]},\ k \in \mathbb{Z}, n \in \mathbb{N} \right\}$ since simple functions are dense in $L^r(\mathbb{R})$ [Rud87, theorem 3.13].

Idea of the would-be reduction Let a compact operator $T : \mathcal{X}(U) \to \mathcal{Y}(V)$, with $\mathcal{Y}(V)$ a separable Banach, and $J : \mathcal{Y}(V) \to \mathcal{C}([0,1])$ an isometric embedding. Then $J \circ T : \mathcal{X}(U) \to \mathcal{C}([0,1])$ is compact (since J is continuous). By our discussion of chapter 3, it can be represented using an abstract kernel $K \in C([0,1], \mathcal{X}')$ as:

$$(JTx)(t) = \langle x, K(t) \rangle_{\mathcal{X}(U)} =: (S_K x)(t) \tag{D.2.2}$$

$$J \circ T = S_K \text{ i.e } T = J^{-1} \circ S_K \tag{D.2.3}$$

(This is well-defined since J is injective). Now, for the goal of determining whether T itself can be written in the form of an (abstract) integral operator, this idea is only helpful if J^{-1} also has a nice form.

Proof of Banach-Mazur's theorem is not constructive For a proof of the theorem, see `https://fr.wikipedia.org/wiki/Th%C3%A9or%C3%A8me_de_Banach-Mazur` (or `https://de.wikipedia.org/wiki/Satz_von_Banach-Mazur` which is almost exactly the same) and references therein. For intuition, see also `https://mathoverflow.net/questions/82720/banach-mazur-applied-to-a-hilbert-space`. All of the proofs I've seen follow the same outline, which we now describe.

An isometric embedding of a separable Banach $\mathcal{Y}$ into $C([0,1])$ is constructed in three steps. Note that the difficulties of the second step can be swept under the carpet at my level of rigor; the third step is obvious (using the existence of norming functionals); the real difficulty lies in the first step.

1. Show that there exists a surjection $\varphi : \Delta \to B^{(\mathcal{Y}')}$ that is continuous when $B^{(\mathcal{Y}')}$ is equipped with the weak-* topology (topology of pointwise convergence of the linear forms).

   Here Δ denotes the Cantor set (the set of $t \in [0,1]$ that can be written as $t = \sum_{n=1}^{\infty} t_n 3^{-n}$ with $t_n \in \{0,2\}$).

2. Argue that φ can be extended to $[0,1]$ (and still verifying the same properties), by interpolating linearly from $\varphi|_\Delta$.

3. Check that this defines an isometric injective linear map: $J : \begin{bmatrix} \mathcal{Y} \to C([0,1]) \\ y \mapsto (Jy)(t) = \langle y, \varphi(t) \rangle_{\mathcal{Y}} \end{bmatrix}$.

To be more explicit, $\varphi : [0,1] \to B^{(\mathcal{Y}')}$ continuous w.r.t weak-* topology means that: for any convergent sequence $t_n \to t_\infty$ in $[0,1]$, then $\forall y \in \mathcal{Y},\ \langle y, \varphi(t_n) \rangle \to \langle y, \varphi(t_\infty) \rangle$.

Finding a "nice" surjective φ is hopeless Recall that we would want J^{-1} to have a nice form. Since it's defined by $(Jy)(t) = \langle y, \varphi(t) \rangle_{\mathcal{Y}}$, we already see that there is not much chance for that, unless φ itself has a particularly nice form. But as we argued above, there is also not much chance of finding such a construction.

Thus the Banach-Mazur theorem does not seem to be helpful for transferring the result of the previous subsection to more general cases.

D.3 Background on Bochner spaces

For a rigorous presentation of Bochner integration and discussion of the duals of Bochner spaces, accessible at the level of this document, we refer to Chapter 2 of [Kre15]. Here we reproduce only the necessary notions and results, while leaving some aspects non-rigorous. In particular, note that some of the statements below may be wrong strictly speaking, in that some assumptions or technical considerations may be missing.

Throughout this paragraph, let (Ω, Σ, μ) a "nice" measure space, and G a Banach space.

Definition D.7 (Simple function). A simple function is a function $s : \Omega \to G$ of the form $s(t) = \sum_{i=1}^{N} x_i \mathbb{1}_{E_i}(t)$ for some $N \in \mathbb{N}$, $x_i \in G$, and pairwise disjoint measurable $E_i \subset \Omega$.

Definition D.8 (Bochner integral). The Bochner integral of a simple function over Ω is defined in the usual way. (It is an element of G.)

A function $f : \Omega \to G$ is measurable if it is the pointwise a.e limit of a sequence of simple functions.

A measurable function f is Bochner-integrable if it is the pointwise a.e limit of a sequence of simple functions $(s_n)_n$ such that $\int_\Omega dt\ \left\|(f - s_n)(t)\right\|_G \xrightarrow{n\to\infty} 0$. Then the quantity $\int_\Omega dt\ f(t)$ is defined in the usual way, after checking that it does not depend on the specific sequence $(s_n)_n$ that converges to f.

Proposition D.5 (Bochner's theorem). A measurable function $f : \Omega \to G$ is Bochner-integrable if and only if $\|f(t)\| \in L^1(\Omega, \mathbb{R})$, and we have the triangle inequality $\left\|\int_\Omega dt\ f(t)\right\|_G \leq \int_\Omega dt\ \|f(t)\|_G$.

Definition D.9 (Bochner space a.k.a vector-valued L^p space). Let $1 \leq p < \infty$. The Bochner space $L^p(\Omega, G)$ consists of the functions $f : \Omega \to G$ such that $\|f\|_{L^p(\Omega,G)} := \int_\Omega dt\ \|f(t)\|_G^p < \infty$. Equipped with this norm, it is a Banach space.

Proposition D.6. If $1 \leq p < \infty$, for any $f \in L^p(\Omega, G)$, then there exists a sequence of simple functions converging to f in $L^p(\Omega, G)$ norm. Moreover it can be chosen such that $\|s_n(t)\|_G \leq 2\,\|f(t)\|_G$ pointwise a.e.

Example D.4 ("Rank-1" elements). For any $x \in G$ and $f \in L^p(\Omega, \mathbb{R})$, xf is the element of $L^p(\Omega, G)$ defined by $(xf)(t) = f(t) \cdot x$.

Proposition D.7 (Hoelder's inequality)**.** Let $1 \leq p, q \leq \infty$ s.t $\frac{1}{p} + \frac{1}{q} = 1$. For all $f \in L^p(\Omega, G)$ and $g \in L^q(\Omega, G')$, the scalar quantity $\int_\Omega dt \ \langle f(t), g(t) \rangle_G$ is well-defined and

$$\left| \int_\Omega dt \ \langle f(t), g(t) \rangle_G \right| \leq \int_\Omega dt \ |\langle f(t), g(t) \rangle_G| \tag{D.3.1}$$

$$\leq \int_\Omega dt \ \|f(t)\|_G \, \|g(t)\|_{G'} \tag{D.3.2}$$

$$\leq \left(\int_\Omega dt \ \|f(t)\|_G^p \right)^{1/p} \left(\int_\Omega dt \ \|g(t)\|_{G'}^q \right)^{1/q} \tag{D.3.3}$$

$$= \|f\|_{L^p(\Omega,G)} \ \|g\|_{L^q(\Omega,G')} \tag{D.3.4}$$

Thus clearly any $g \in L^q(\Omega, G')$ induces a bounded linear form by $\langle f, g \rangle_{L^p(\Omega,G)} := \int_\Omega dt \ \langle f(t), g(t) \rangle_G$. By the above calculation its dual norm is $\|g\|_{(L^p(\Omega,G))'} \leq \|g\|_{L^q(\Omega,G')}$; as it turns out, equality holds when $p < \infty$ [Kre15, proposition 2.18].

D.3.1 Dual space of Bochner spaces

Proposition D.8. As discussed just above, for any $1 \leq p < \infty$, $L^q(\Omega, G')$ embeds isometrically into (i.e is isometrically isomorphic to a subspace of) $(L^p(\Omega, G))'$.

Definition D.10 (Radon-Nikodym property)**.** A Banach space G satisfies the Radon-Nikodym property if for any measure space (Ω, Σ, μ), "the vector-valued variant of the Radon-Nikodym theorem holds", i.e: for any vector measure ν with values in G that has bounded variation and is absolutely continuous w.r.t μ, there exists a function $f \in L^1_\mu(\Omega, G)$ such that $\nu(A) = \int_A f d\mu$ for all A.

(For the definition of a vector measure and variation and absolute continuity of vector measures, we refer the reader to [Kre15] or wikipedia.)

Proposition D.9 (Sufficient conditions for Radon-Nikodym property)**.** If G is reflexive then it has the Radon-Nikodym property.

If G is the dual space of some Banach space, and G is separable, then G has the Radon-Nikodym property.

Proposition D.10 (Dual space of Bochner space)**.** Let $1 \leq p < \infty$ and G a Banach space with the Radon-Nikodym property. If $p = 1$ we further require that μ is σ-finite, same as in the scalar-valued case.

We already saw that any $g \in L^q(\Omega, G')$ can be seen as a bounded linear form i.e an element of $(L^p(\Omega, G))'$, with the same norm. Under the conditions above, we have the converse: $(L^p(\Omega, G))'$ is isometrically isomorphic to $L^q(\Omega, G')$.

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