Entropy numbers of nonlinear systems

Master's thesis presentation

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ETHZ MINS

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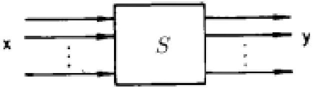
Motivation

"What is a nonlinear system?"



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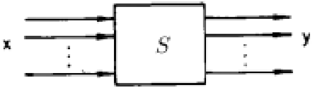
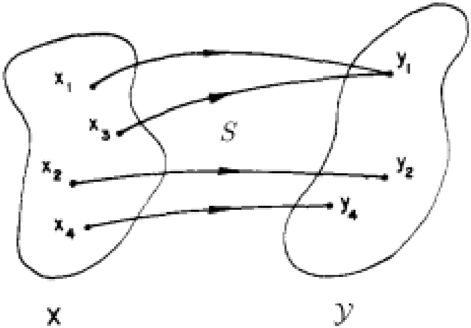
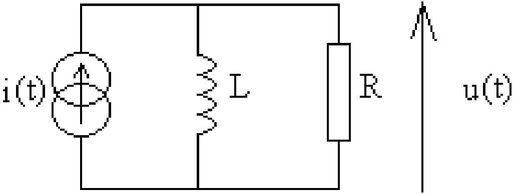




Goal: learn S from observations (x_i, y_i)

Motivation

System identification



Input: $i(t)$, output: $u(t)$. $\quad u(t) = S[\ i(t)\ ]$?

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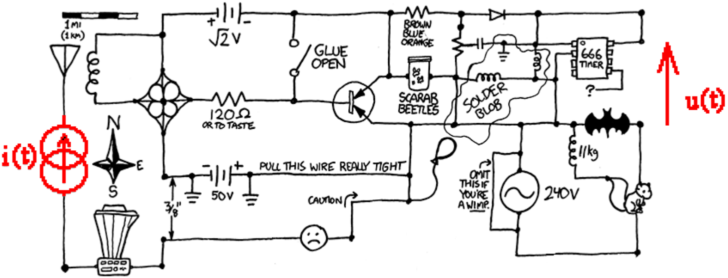

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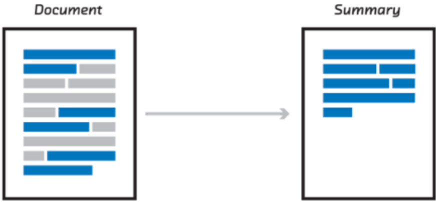
Signal-to-signal tasks in ML

- Text-to-text



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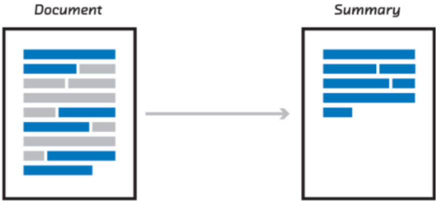
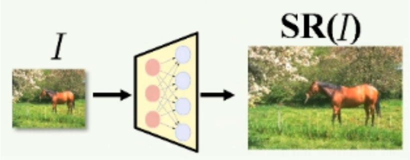


- Super-resolution imaging



Motivation

- Classical (regression/classification):

$$\text{learn} \qquad f : \mathbb{R}^d \to \mathbb{R} \text{ or } \{0, 1\}$$

- Nonlinear system identification:

$$\text{learn (e.g)} \qquad S : L^\infty(\mathbb{R}) \to L^\infty(\mathbb{R})$$

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«How difficult is it to learn a mapping?»

1. Framework for this thesis
2. "Parametrize": LTI systems case
3. "Parametrize": Volterra series
4. Generalize classical techniques

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Metric entropy

« How difficult is it to learn a set of objects $\mathcal{C}$? »

- Learning theory tool: ε-covering number

Metric entropy

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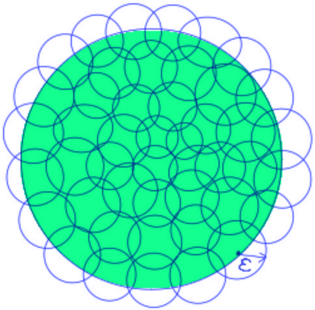
- Learning theory tool: ε-covering number

$$N_\varepsilon(\mathcal{C}; \|\cdot\|) = \min \left\{ n; \quad \exists (p_1, \ldots, p_n) \subset \mathcal{C} \quad \text{s.t} \quad \mathcal{C} \subset \bigcup_i B_{p_i,\varepsilon}^{\|\cdot\|} \right\}$$



An ε-covering

Metric entropy

- ε-covering number $N_\varepsilon(\mathcal{C}; \|\cdot\|)$
- Metric entropy $\log_2 N_\varepsilon(\mathcal{C}; \|\cdot\|)$

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Why metric entropy? Intuition:

Proposition

For any "bitstring length" $\ell \in \mathbb{N}$, consider encoder/decoder scheme

$$E : \mathcal{C} \to \{0,1\}^\ell \qquad D : \{0,1\}^\ell \to \mathcal{C}$$

$\log_2 N_\varepsilon(\mathcal{C}; \|\cdot\|)$ is the minimum ℓ s.t

$$\inf_{E,D} \sup_{c \in \mathcal{C}} \|c - D(E(c))\| \;\leq\; \varepsilon$$

("best-obtainable worst-case error")

Metric entropy

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("best-obtainable worst-case error")

$\to$ quantifies "massiveness"

$\to$ fundamental bound on compressibility / learnability

Entropy numbers

$N_\varepsilon(\mathcal{C}; \|\cdot\|)$ ε-covering number $\mathbb{R}_+^* \to \mathbb{N}$

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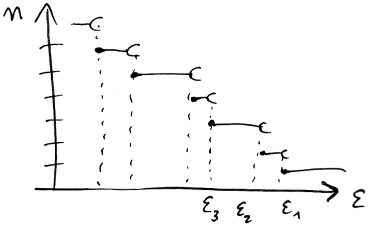

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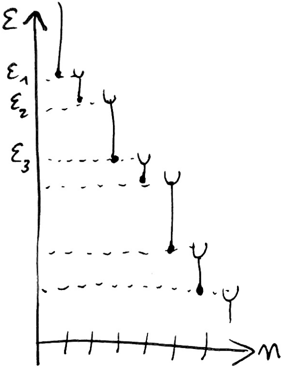

"Metric entropy" $\equiv$ "Entropy number"

Framework

- $\mathcal{X}$ space of input signals $x(t)$ (e.g $L^2(\mathbb{R}), C(\mathbb{R}), ...$)
- $\mathcal{Y}$ space of output signals $y(t)$
- $\mathbb{S}$ space of systems S

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1. Worst-case error

$$\mathbb{S} = \{S : \mathcal{X} \to \mathcal{Y}\} \qquad \left\| S - \widehat{S} \right\|_\infty = \sup_x \left\| S[x] - \widehat{S}[x] \right\|_{\mathcal{Y}}$$

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$$\mathbb{S} = \{S : (\mathcal{X}, \Sigma, \mathbb{P}) \to \mathcal{Y}\} \quad \left\| S - \widehat{S} \right\|_{L^1_{\mathbb{P}}} = \mathbb{E}_x \left\| S[x] - \widehat{S}[x] \right\|_{\mathcal{Y}}$$

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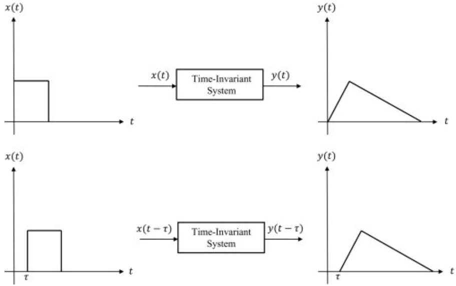
Goal:

$$\text{Given } \mathbb{S}_+ \subset \mathbb{S}, \quad \text{estimate } N_\varepsilon(\mathbb{S}_+; \|\cdot\|_{\infty U})$$

LTI systems

LTI = Linear Time-Invariant

$$S(\lambda x_1 + x_2) = \lambda S x_1 + S x_2$$



Convolution representation

Theorem (Schwartz kernel theorem)

For all $S : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ LTI, there exists $k \in \mathcal{D}'(\mathbb{R})$ s.t

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Proof:

$$\begin{aligned} Sx(t) &= S \int_{\mathbb{R}} d\tau \; x(\tau) \; \delta_\tau(t) \\ &= \int_{\mathbb{R}} d\tau \; x(\tau) \; \underbrace{S\delta_\tau(t)} = \int_{\mathbb{R}} d\tau \; x(\tau) \; k(t,\tau) \end{aligned}$$

Time-invariance $\implies$ $k(t,\tau) = k(t-\tau)$

Convolution representation and norm

$$\begin{aligned}\mathbb{S} &= \left\{S : L^2(\mathbb{R}) \to L^2(\mathbb{R}) \ \ \text{LTI}\right\} \\ &= \{L_k : x \mapsto (k * x), \quad k \in \mathcal{K}\} \simeq \mathcal{K}\end{aligned}$$

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Theorem

Suppose $U = B^{(L^2(\mathbb{R}))}$.

$$\forall k \in L^1(\mathbb{R}), \ \ \|L_k\|_{\infty U} = |||L_k||| = \left\|\hat{k}\right\|_{L^\infty(\mathbb{R})}$$

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Conclusion: reduced to metric entropy in function space ☺

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Volterra series

- LTI system: $k(t)$

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$k = (k_0, k_1, \ldots)$

$k_n(t_1, \ldots, t_n)$

$$V_k[x] = \sum_{n=0}^{\infty} \int_{\mathbb{R}^n} d\boldsymbol{\tau} \; k_n(\boldsymbol{\tau}) \; x(t-\tau_1) \ldots x(t-\tau_n)$$

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Where do x, k, $V_k[x]$ live? i.e: $\mathcal{X}, \mathcal{K}, \mathcal{Y}$?

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$$\|V_k[x]\|_{\mathcal{Y}} \leq \sum_{n=0}^{\infty} \|k_n\|_{\mathcal{K}_n} \|x\|_{\mathcal{X}}^n$$

- $\mathcal{X} = L^p(\mathbb{R}),\ \ \mathcal{K}_n = L^q(\mathbb{R}^n),\ \ \mathcal{Y} = L^\infty(\mathbb{R})$ $\quad (1/p + 1/q = 1)$

or

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(Convergence of $\sum_{n=0}^{\infty}$?)

Volterra series

For simplicity assume finite order N

$$V_k[x] = \sum_{n=0}^{N} \int_{\mathbb{R}^n} d\boldsymbol{\tau}\ k_n(\boldsymbol{\tau})\ x(t-\tau_1)...x(t-\tau_n)$$

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"Parametrize" path

$$\mathbb{S} = \{\Phi_k, \quad k \in \mathcal{K}\}$$

Summary: if $\quad \mathbb{S}_+ = \{\Phi_k, \quad k \in \mathcal{K}_+\} \quad$ then ☺

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Problems:

- Upper estimate may be too loose
- Given $\mathbb{S}_+$ specified differently, how to find $\mathcal{K}_+$?

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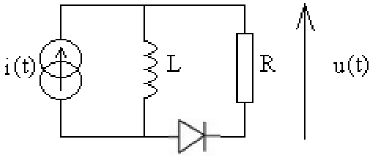
$$\mathbb{S} = \{\Phi_k, \quad k \in \mathcal{K}\}$$

Summary: if $\quad \mathbb{S}_+ = \{\Phi_k, \quad k \in \mathcal{K}_+\} \quad$ then ☺

$$\|\Phi_k\|_{\infty U} \leq c\,\|k\|_{\mathcal{K}}$$

Problems:

- Upper estimate may be too loose
- Given $\mathbb{S}_+$ specified differently, how to find $\mathcal{K}_+$?



$$\mathbb{S}_+ = \{[i(t) \mapsto u(t)]\,;\; R \in [R_{min}, R_{max}], L \in [L_{min}, L_{max}]\} \qquad \rightsquigarrow \qquad \mathcal{K}_+??$$

Generalize classical techniques

- Different approach: adapt techniques from the case $\mathcal{F}_+ \subset \mathcal{F} = \left\{ f : \mathbb{R}^d \to \mathbb{R} \right\}$

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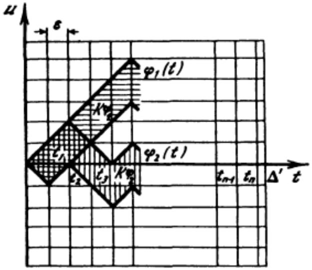

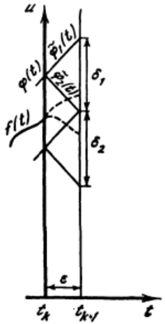

Proof of metric entropy estimate for the set of lipschitz-continuous functions

(Lipschitz)-continuous functions

Lipschitz-continuous functions over a compact interval

$$\mathcal{F}_+ \subset \left\{ f : [0,1] \to \mathbb{R}; \qquad \forall t, t', \quad \left| f(t) - f(t') \right| \leq L \left| t - t' \right| \right\}$$

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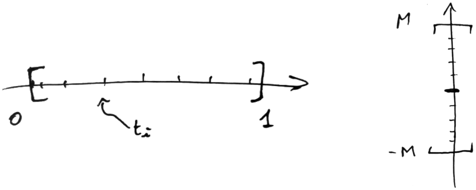

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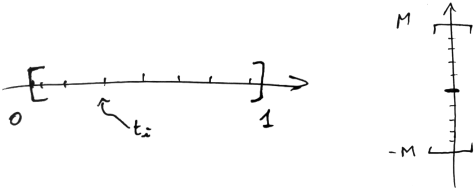

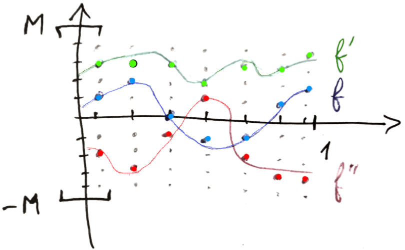

(Lipschitz)-continuous systems

Lipschitz-continuous system over a compact metric space

$$\mathbb{S}_+ \subset \left\{ S : (U, d) \to (\mathcal{Y}, \|\cdot\|_{\mathcal{Y}});\ \forall x, x', \left\| S[x] - S[x'] \right\|_{\mathcal{Y}} \leq L d(x, x') \right\}$$

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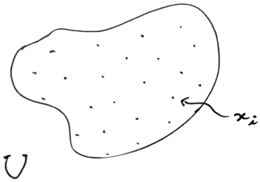

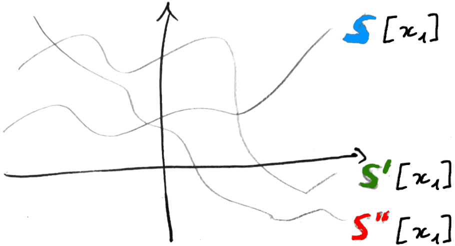

Quantize the output set $\{S[x];\ S \in \mathbb{S}_+, x \in U\}$?

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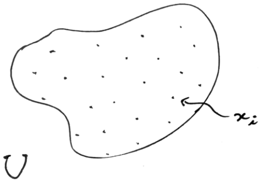

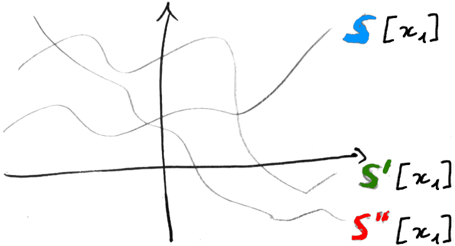

Theorem (Banach-valued Arzela-Ascoli)

$\mathbb{S}_+$ *relatively compact in* $C(U; \mathcal{Y})$ *iff*

- $\mathbb{S}_+$ *equicontinuous* ($\Leftarrow$ *L-lipschitz*)
- $\mathbb{S}_+$ *"equicompact":* $\{S[x];\ S \in \mathbb{S}_+, x \in U\}$ *relatively compact*

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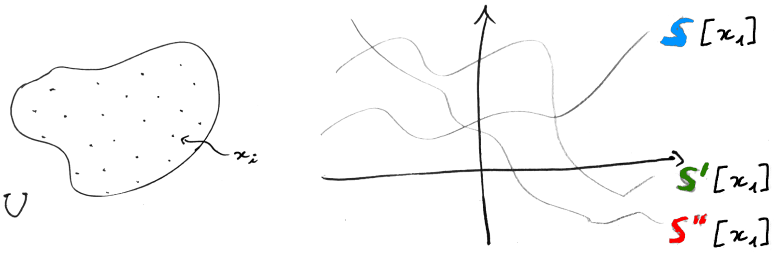

Theorem (Banach-valued Arzela-Ascoli)

$\mathbb{S}_+$ *can be quantized iff*

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Conclusion

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  - "Parametric"

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$$f : \mathbb{R}^d \to \mathbb{R} \qquad \rightsquigarrow \qquad S : \mathcal{X} \to \mathcal{Y}$$

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- Direction for future work: "non-parametric" path?
  Kernel methods for nonlinear system identification

Illustrations adapted from

- "Nonlinear system modeling based on the Wiener theory" (Schetzen 1981)
- `https://commons.wikimedia.org/wiki/File:Circuit_L_R_parall%C3%A8le_-_courant_en_entr%C3%A9e_et_tension_en_sortie.png`
- `https://xkcd.com/730/`
- `https://towardsdatascience.com/8a3fbfdc5e9b`
- "'Zero-Shot' Super-Resolution using Deep Internal Learning" (Shocher et al. 2017)
- *Neural Network Theory lecture notes* HS2019 ETHZ
- *Introduction to Digital Communications*, Chapter 3 (Grami 2016)
- "ε-Entropy and ε-Capacity of Sets In Functional Spaces" (Kolmogorov and Tikhomirov 1959)

Appendix

The idea

- Time-invariant system $\rightarrow$ scalar-valued functional (cf report)

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- Volterra monomial with fixed n

$$V_{k_n}[x](t) = \int_{\mathbb{R}^n} d\boldsymbol{\tau}\ k_n(\boldsymbol{\tau})\ x(t-\tau_1)...x(t-\tau_n)$$

$$F_{\theta_n}[x] = \int_{\mathbb{R}^n} d\boldsymbol{\tau}\ \theta_n(\boldsymbol{\tau})x(\tau_1)...x(\tau_n) = \int_{\mathbb{R}^n} d\boldsymbol{\tau}\ \theta_n(\boldsymbol{\tau})x^{\times n}(\boldsymbol{\tau})$$

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- **Idea:** view as linear combination of feature map

$$F_{\theta_n}[x] = \int_{\mathbb{R}^n} d\boldsymbol{\tau}\ \theta_n(\boldsymbol{\tau}) \phi(x)(\boldsymbol{\tau}) = \langle \phi(x), \theta_n \rangle$$

$$x \in L^p(\mathbb{R}) \quad \mapsto \quad \phi(x) = x^{\times n} \in L^p_{\mathrm{Sym}}(\mathbb{R}^n)$$

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- So $F_{\theta_n} \in$ RKHS with

$$K(x, \tilde{x}) = \langle \phi(x), \phi(\tilde{x}) \rangle = \left(\int_{\mathbb{R}} x\ \tilde{x} \right)^n$$

Problem: ill-defined when $x, \tilde{x} \in L^p(\mathbb{R})$...

RKBS

RKBS = Reproducing Kernel *Banach* Space

Definition (Lin et al. 2019)

Pair of RKBS: a tuple $(\mathcal{B}_1, \mathcal{B}_2, \langle \cdot, \cdot \rangle_{\mathcal{B}_1 \times \mathcal{B}_2})$ s.t

- $\mathcal{B}_i$ Banach space of (real-valued) functions on Ω_i
- $\langle \cdot, \cdot \rangle_{\mathcal{B}_1 \times \mathcal{B}_2}$ continuous bilinear form on $\mathcal{B}_1 \times \mathcal{B}_2$

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- $\exists K : \Omega_1 \times \Omega_2 \to \mathbb{R}$, called *reproducing kernel*, s.t

$$\begin{aligned} K(x, \cdot) &\in \mathcal{B}_2 \qquad \forall f \in \mathcal{B}_1,\ f(x) = \langle f, K(x, \cdot) \rangle_{\mathcal{B}_1 \times \mathcal{B}_2} \\ K(\cdot, y) &\in \mathcal{B}_1 \qquad \forall g \in \mathcal{B}_2,\ g(y) = \langle K(\cdot, y), g \rangle_{\mathcal{B}_1 \times \mathcal{B}_2} \end{aligned}$$

(K is unique)

RKBS from feature maps

Proposition (Lin et al. 2019)

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- "span $\phi_i(\Omega_i)$ are dense":

$$\left\{v \in \mathcal{W}_2;\ \forall x \in \Omega_1, \langle \phi_1(x), v \rangle_{\mathcal{W}_1 \times \mathcal{W}_2} = 0\right\} = \{0\}$$
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This induces a pair of RKBS

$$\mathcal{B}_1 := \left\{F_v = \langle \phi_1(\cdot), v \rangle_{\mathcal{W}_1 \times \mathcal{W}_2};\ v \in \mathcal{W}_2\right\} \qquad \|F_v\|_{\mathcal{B}_1} := \|v\|_{\mathcal{W}_2}$$

$$\mathcal{B}_2 := \left\{G_u = \langle u, \phi_2(\cdot) \rangle_{\mathcal{W}_1 \times \mathcal{W}_2};\ u \in \mathcal{W}_1\right\} \qquad \|G_u\|_{\mathcal{B}_2} := \|u\|_{\mathcal{W}_1}$$

$$\langle F_v, G_u \rangle_{\mathcal{B}_1 \times \mathcal{B}_2} := \langle u, v \rangle_{\mathcal{W}_1 \times \mathcal{W}_2}$$

and $K(x, \tilde{x}) = \langle \phi_1(x), \phi_2(\tilde{x}) \rangle_{\mathcal{W}_1 \times \mathcal{W}_2}$.

Volterra series as polynomial RKBS

Proposition

- $\phi_1 : L^p(\mathbb{R}) \to L^p_{\mathrm{Sym}}(\mathbb{R}^n),\ x(t) \mapsto x^{\times n}(\boldsymbol{t})$
  $\phi_2 : L^q(\mathbb{R}) \to L^q_{\mathrm{Sym}}(\mathbb{R}^n),\ \tilde{x}(t) \mapsto \tilde{x}^{\times n}(\boldsymbol{t})$
- $\langle u_n(\boldsymbol{t}), \tilde{u}_n(\boldsymbol{t})\rangle_{L^p} = \int_{\mathbb{R}^n} d\boldsymbol{t}\ u_n(\boldsymbol{t})\tilde{u}_n(\boldsymbol{t})$ (Rk: $\equiv$ duality bracket)

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$$\mathcal{B}_1 = \left\{F_{\theta_n} : \left[x \mapsto \int_{\mathbb{R}^n} d\boldsymbol{\tau}\ \theta_n(\boldsymbol{\tau})x^{\times n}(\boldsymbol{\tau})\right];\ \theta_n \in L^q_{\text{Sym}}(\mathbb{R}^n)\right\}$$

$\|F_{\theta_n}\|_{\mathcal{B}_1} = \|\theta_n\|_{L^q_{\text{Sym}}(\mathbb{R}^n)}$ and $K(x,\tilde{x}) = \left(\int_{\mathbb{R}} x\ \tilde{x}\right)^n$.

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- Learning in RKBS vs. classical Volterra-series-based system identification?
- What if we want $K(x, \tilde{x}) = \sum_{n=0}^{\infty} a_n \left(\int_{\mathbb{R}} x \, \tilde{x} \right)^n$ for some $a_n \geq 0$?

Entropy numbers

$$N_\varepsilon(\mathcal{C}; \|\cdot\|) = \min\left\{n; \quad \exists(p_1, ..., p_n) \subset \mathcal{C} \quad \text{s.t} \quad \mathcal{C} \subset \bigcup_i B_{p_i,\varepsilon}^{\|\cdot\|}\right\}$$

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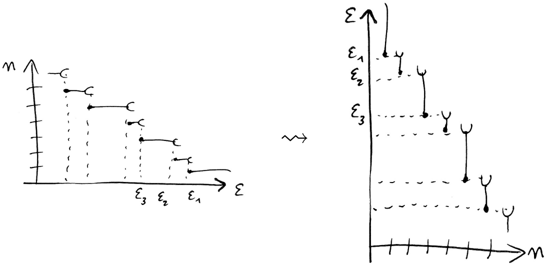

$$N_\varepsilon(\mathcal{C}; \|\cdot\|) \leq n_0 \iff \varepsilon_n(\mathcal{C}; \|\cdot\|) \leq \varepsilon_0$$

"Metric entropy" $\equiv$ "Entropy number"