See Exercise 2.8 for the proof.

CI assumptions allow us to build large probabilistic models from small pieces. We will see many examples of this throughout the book. In particular, in Section 3.5, we discuss naive Bayes classifiers, in Section 17.2, we discuss Markov models, and in Chapter 10 we discuss graphical models; all of these models heavily exploit CI properties.

2.2.5 Continuous random variables

So far, we have only considered reasoning about uncertain discrete quantities. We will now show (following (Jaynes 2003, p107)) how to extend probability to reason about uncertain continuous quantities.

Suppose X is some uncertain continuous quantity. The probability that X lies in any interval $a \leq X \leq b$ can be computed as follows. Define the events $A = (X \leq a)$, $B = (X \leq b)$ and $W = (a < X \leq b)$. We have that $B = A \vee W$, and since A and W are mutually exclusive, the sum rules gives

$$p(B) = p(A) + p(W) \tag{2.17}$$

and hence

$$p(W) = p(B) - p(A) \tag{2.18}$$

Define the function $F(q) \triangleq p(X \leq q)$. This is called the **cumulative distribution function** or **cdf** of X. This is obviously a monotonically increasing function. See Figure 2.3(a) for an example. Using this notation we have

$$p(a < X \leq b) = F(b) - F(a) \tag{2.19}$$

Now define $f(x) = \frac{d}{dx}F(x)$ (we assume this derivative exists); this is called the **probability density function** or **pdf**. See Figure 2.3(b) for an example. Given a pdf, we can compute the probability of a continuous variable being in a finite interval as follows:

$$P(a < X \leq b) = \int_a^b f(x)dx \tag{2.20}$$

As the size of the interval gets smaller, we can write

$$P(x \leq X \leq x + dx) \approx p(x)dx \tag{2.21}$$

We require $p(x) \geq 0$, but it is possible for $p(x) > 1$ for any given x, so long as the density integrates to 1. As an example, consider the **uniform distribution** $\text{Unif}(a, b)$:

$$\text{Unif}(x|a, b) = \frac{1}{b - a}\mathbb{I}(a \leq x \leq b) \tag{2.22}$$

If we set $a = 0$ and $b = \frac{1}{2}$, we have $p(x) = 2$ for any $x \in [0, \frac{1}{2}]$.