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The Tremblay–Turbiner–Winternitz system on spherical and hyperbolic spaces: superintegrability, curvature-dependent formalism and complex factorization

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Abstract

The higher order superintegrability of the Tremblay–Turbiner–Winternitz system (related to the harmonic oscillator) is studied on the two-dimensional spherical and hyperbolic spaces, S_κ^2 ($\kappa > 0$) and H_κ^2 ($\kappa < 0$). The curvature κ is considered as a parameter and all the results are formulated in explicit dependence on κ. The idea is that the additional constant of motion can be factorized as the product of powers of two particular rather simple complex functions (here denoted by M_r and N_ϕ). This technique leads to a proof of the superintegrability of the Tremblay–Turbiner–Winternitz system on S_κ^2 ($\kappa > 0$) and H_κ^2 ($\kappa < 0$), and to the explicit expression of the constants of motion.

Keywords: integrability on spaces of constant curvature, nonlinear oscilators, superintegrability, higher order constants of motion, complex factorization
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1. Introduction

It is well known that systems that admit Hamilton–Jacobi (or Schrödinger in the quantum case) separability in more than one coordinate system are superintegrable with quadratic in the momenta constants of motion (in some particular cases the constant is determined by an exact Noether symmetry and then it is linear). For example, the following potential, known as the Smorodinsky–Winternitz (S-W) potential [1–3], and representing a two dimensional isotonic oscillator [4, 5],

$$V_{\rm sw} = \frac{1}{2}\,\omega_0^2(x^2+y^2) + \frac{k_2}{x^2} + \frac{k_3}{y^2}, \tag{1}$$

is separable in Cartesian and polar coordinates and it is, therefore, superintegrable with three quadratic constants of motion (see [6] for a recent review on superintegrability).

The potential $V_{\rm sw}$ admits two generalizations. The first one

$$V(n_x, n_y) = \frac{1}{2}\,{\omega_0}^2\big(n_x^2x^2 + n_y^2y^2\big) + \frac{k_1}{2x^2} + \frac{k_2}{2y^2}, \tag{2}$$

that preserves the separability in Cartesian coordinates, is also superintegrable [7–9] but with a polynomial of higher order than two as a third integral of motion. The second generalization of $V_{\rm sw}$, that takes the form

$$V_{\rm ttw}(r,\phi) = \frac{1}{2}\,{\omega_0}^2r^2 + \frac{1}{2\,r^2}\left(\frac{\alpha}{\cos^2(m\phi)} + \frac{\beta}{\sin^2(m\phi)}\right), \tag{3}$$

was first studied by Tremblay, Turbiner, and Winternitz [10, 11], and then by other authors [12–19]. When $m = 1$ it reduces to $V_{\rm sw}$, but in the general $m \neq 1$ case (m must be an integer or rational number) it is only separable in polar coordinates; therefore, the third integral is not quadratic in the momenta but a polynomial of higher order than two (the degree of the polynomial depends of the value of m).

The idea that the harmonic oscillator (and also the Kepler problem) can be correctly defined on spaces of constant curvature appears in a book of Riemannian geometry of 1905 by Liebmann [20]; but it was Higgs [21] who studied this system in detail (the study by Higgs was limited to a spherical geometry but his approach can be extended, introducing the appropriate changes, to the hyperbolic space). The Tremblay–Turbiner–Winternitz (TTW) system is directly related to the harmonic oscillator; so it seems natural to also study the TTW system on constant curvature spaces. Actually, this question has been recently considered in [22] (the TTW system but without the harmonic potential part) and in [18, 23] (action-angle variables and perturbation theory).

The aim of this paper is to study the TTW system on the spaces of constant curvature S_κ^2 ($\kappa > 0$) and H_κ^2 ($\kappa < 0$), and to prove the superintegrability for all the values of κ. Two important points are:

(a) All the mathematical expressions will depend on the curvature κ as a parameter, in such a way that considering values $\kappa > 0$, $\kappa = 0$, or $\kappa < 0$, we will obtain the corresponding property particularized for the system on the sphere S_κ^2, on the Euclidean space $\mathbb{E}^2$, or on the hyperbolic space H_κ^2, respectively. This curvature-dependent formalism was already used in [24, 25] (and in [26, 27] for the quantum oscillator); other papers making use of this κ-dependent formalism are [28–36].

(b) It is well known that the two dimensional harmonic oscillator with rational quotient of frequencies admits a third integral. The important point is that this additional integral can be obtained as the product of two simple complex functions [37] (see also [9]). The superintegrability of the standard Euclidean TTW system was proved in [16] by using this technique. Now, in this paper, we present a generalization of this method to the $\kappa \neq 0$ case.

The paper is organized as follows. In section 2 we first introduce the κ-dependent formalism and then we study the superintegrability of the harmonic oscillator and the S-W potential on spaces of constant curvature. In section 3 we prove the superintegrabilty of the TTW system on spherical and hyperbolic spaces. Finally in section 4 we make some comments and we present some open questions.

2. The harmonic oscillator on spaces of constant curvature

2.1. κ-dependent formalism

On a two-dimensional Riemannian space (M, g) (not necessarily of constant curvature) there are two distinguished types of coordinate systems, 'geodesic parallel' and 'geodesic polar' coordinates, that reduce to the familiar Cartesian (x, y) and polar coordinates (r, ϕ) on the Euclidean plane [38]. Here we only consider the geodesic polar coordinates that are based on a point O and an oriented geodesic l_0 through O. For any point P in some suitable neighborhood a point O (that represents the origin) there is a unique geodesic l joining O and P. The geodesic polar coordinates (r, ϕ) of P are the distance r between O and P measured along l, and the angle ϕ between l and the positive ray l_0 measured at O. These coordinates are singular at O and ϕ is discontinuous on the positive ray of l_0.

In what follows we will make use of the following κ-dependent trigonometric-hyperbolic functions

$$\mathrm{C}_\kappa(x) = \begin{cases} \cos\sqrt{\kappa}\,x & \text{if } \kappa > 0, \\ 1 & \text{if } \kappa = 0, \\ \cosh\sqrt{-\kappa}\,x & \text{if } \kappa < 0, \end{cases} \qquad \mathrm{S}_\kappa(x) = \begin{cases} \frac{1}{\sqrt{\kappa}}\sin\sqrt{\kappa}\,x & \text{if } \kappa > 0, \\ x & \text{if } \kappa = 0, \\ \frac{1}{\sqrt{-\kappa}}\sinh\sqrt{-\kappa}\,x & \text{if } \kappa < 0, \end{cases} \tag{4}$$

and $\mathrm{T}_\kappa(x) = \mathrm{S}_\kappa(x)/\,\mathrm{C}_\kappa(x)$ [24–36]. Then the following κ-dependent expression

$$\mathrm{d}s_\kappa^2 = \mathrm{d}r^2 + \mathrm{S}_\kappa^2(r)\,\mathrm{d}\phi^2, \tag{5}$$

represents the expression, in geodesic polar coordinates (r, ϕ), of the differential line element on the spaces $(S_\kappa^2, \mathbb{E}^2, H_\kappa^2)$ with constant curvature κ. This metric reduces to

$$\mathrm{d}s_1^2 = \mathrm{d}r^2 + (\sin^2 r)\,\mathrm{d}\phi^2, \quad \mathrm{d}s_0^2 = \mathrm{d}r^2 + r^2\,\mathrm{d}\phi^2, \quad \mathrm{d}s_{-1}^2 = \mathrm{d}r^2 + (\sinh^2 r)\,\mathrm{d}\phi^2,$$

in the three particular cases of the unit sphere $\kappa = 1$, Euclidean plane $\kappa = 0$, and 'unit' Lobachewski plane $\kappa = -1$.

A general standard Lagrangian (κ-dependent kinetic term minus a potential) has the following form

$$L(r, \phi, v_r, v_\phi; \kappa) = \tfrac{1}{2}\big(v_r^2 + \mathrm{S}_\kappa^2(r)v_\phi^2\big) - U(r, \phi; \kappa),$$

in such a way that for $\kappa = 0$ we recover the expression of a standard Lagrangian in the Euclidean space. The two linear momenta, reducing to p_x and p_y, in the Euclidean case, are given by

$$P_1(\kappa) = (\cos\phi)v_r - (\mathrm{C}_\kappa(r)\,\mathrm{S}_\kappa(r)\sin\phi)v_\phi$$
$$P_2(\kappa) = (\sin\phi)v_r + (\mathrm{C}_\kappa(r)\,\mathrm{S}_\kappa(r)\cos\phi)v_\phi$$

and the κ-dependent expression for the angular momentum is

$$J(\kappa) = \mathrm{S}_\kappa^2(r)v_\phi.$$

2.2. The harmonic oscillator on spaces of constant curvature

The following (spherical, Euclidean, hyperbolic) Lagrangian with curvature κ,

$$L(\kappa) = \tfrac{1}{2}\left(v_r^2 + \mathrm{S}_\kappa^2(r)v_\phi^2\right) - U(r; \kappa), \quad U(r; \kappa) = \tfrac{1}{2}\,\omega_0^2\,\mathrm{T}_\kappa^2(r), \tag{6}$$

represents the κ-dependent version of the harmonic oscillator [24, 25]; the potential $U(r; \kappa)$ reduces to

$$U_1 = \tfrac{1}{2}\omega_0^2\tan^2 r, \quad U_0 = V = \tfrac{1}{2}\omega_0^2 r^2, \quad U_{-1} = \tfrac{1}{2}\omega_0^2\tanh^2 r,$$

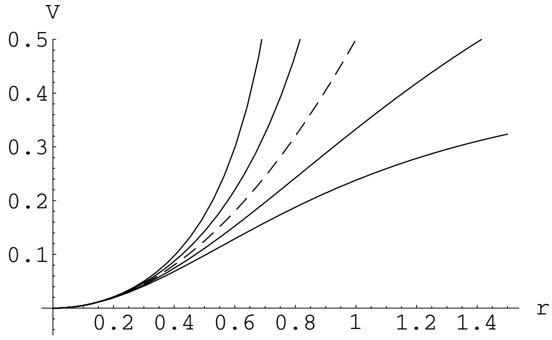

Figure 1. Plot of the potential $U(r,k) = (1/2)\,\omega_0^2\,\mathrm{T}_\kappa^2(r)$, $\omega_0 = 1$, as a function of r, for $\kappa < 0$ (lower curves), $\kappa = 0$ (dash line), and $\kappa > 0$ (upper curves).

in the three particular cases of the unit sphere ($\kappa = 1$), Euclidean plane ($\kappa = 0$), and 'unit' Lobachewski plane ($\kappa = -1$); the Euclidean function $V(r)$ appears in this formalism as making separation between two different behaviors (see figure 1); of course, the domain of r depends of the value of κ; we have $r \in [0,\infty)$ for $\kappa \leqslant 0$ and $r \in [0, \pi/2\sqrt{\kappa}]$ for $\kappa > 0$. It is known [24, 25] that this system is superintegrable for all the values of the curvature κ since, in addition to the angular momentum $J(\kappa)$, it is endowed with the following two quadratic constants of motion

$$I_1(\kappa) = P_1^2(\kappa) + \omega_0^2(\mathrm{T}_\kappa(r)\cos\phi)^2,$$
$$I_2(\kappa) = P_2^2(\kappa) + \omega_0^2(\mathrm{T}_\kappa(r)\sin\phi)^2,$$

in such a way that the energy can be written as follows

$$E(\kappa) = \tfrac{1}{2}(I_1(\kappa) + I_2(\kappa) + \kappa J^2(\kappa)).$$

An additional interesting property is the existence of the following fourth integral of motion

$$I_4(\kappa) = P_1(\kappa)P_2(\kappa) + \omega_0^2\big(\mathrm{T}_\kappa^2(r)\cos\phi\sin\phi\big).$$

The reason is that, although it is not functionally independent since it satisfies the following relation

$$I_4^2(\kappa) = I_1(\kappa)I_2(\kappa) - \omega_0^2 J^2(\kappa),$$

the set of the three κ-dependent functions $\{I_1(\kappa), I_2(\kappa), I_4(\kappa)\}$ can be considered as the three components of the κ-dependent version of the Fradkin tensor [39].

2.3. The S-W potential on spaces of constant curvature

The following (spherical, Euclidean, hyperbolic) κ-dependent potential

$$U(r,\phi;\kappa) = \frac{1}{2}\,\omega_0^2\,\mathrm{T}_\kappa^2(r) + \frac{k_2}{(\mathrm{S}_\kappa(r)\cos\phi)^2} + \frac{k_3}{(\mathrm{S}_\kappa(r)\sin\phi)^2}, \tag{7}$$

that is well defined for all the values of κ, represents the spherical ($k > 0$) and hyperbolic ($\kappa < 0$) version of the Euclidean potential $V_{\rm sw}$ ($\kappa = 0$); it reduces to

$$U_1 = \frac{1}{2}\,\omega_0^2\,\tan^2 r + \frac{1}{\sin^2 r}\left(\frac{k_2}{\cos^2\phi} + \frac{k_3}{\sin^2\phi}\right),$$
$$U_{-1} = \frac{1}{2}\,\omega_0^2\,\tanh^2 r + \frac{1}{\sinh^2 r}\left(\frac{k_2}{\cos^2\phi} + \frac{k_3}{\sin^2\phi}\right),$$

in the particular cases of the unit sphere ($\kappa = 1$) and 'unit' Lobachewski plane ($\kappa = -1$). It is endowed with the following three quadratic constants of motion

$$\begin{aligned} I_1(\kappa) &= P_1^2(\kappa) + \omega_0^2(\mathrm{T}_\kappa(r)\cos\phi)^2 + \frac{2k_2}{(\mathrm{T}_\kappa(r)\cos\phi)^2}, \\ I_2(\kappa) &= P_2^2(\kappa) + \omega_0^2(\mathrm{T}_\kappa(r)\sin\phi)^2 + \frac{2k_3}{(\mathrm{T}_\kappa(r)\sin\phi)^2}, \\ I_3(\kappa) &= J^2(\kappa) + \frac{2k_2}{\cos^2\phi} + \frac{2k_3}{\sin^2\phi} \end{aligned}$$

and, therefore, it is a superintegrable system for all the values of κ.

3. The TTW system on spaces of constant curvature

In the following, we will make use of the Hamiltonian formalism; therefore, the time derivative $\mathrm{d}/\mathrm{d}t$ of a function means the Poisson bracket of the function with the Hamiltonian.

We have seen, in the previous section, that in both the harmonic oscillator and the S-W potential the curvature κ modifies many things but preserves the fundamental property of superintegrability. Now in this section we will prove that this is also true for the TTW system.

It is well known that if $F(\phi)$ is an arbitrary function then the following Hamiltonian (harmonic oscillator plus an angular deformation introduced by F)

$$H = \frac{1}{2}\left(p_r^2 + \frac{p_\phi^2}{r^2}\right) + \frac{1}{2}\,{\omega_0}^2 r^2 + \frac{1}{2}\,\frac{F(\phi)}{r^2} \tag{8}$$

is separable in polar coordinates and it is therefore endowed with the following two constants of motion

$$\begin{aligned} J_1 &= p_r^2 + \frac{p_\phi^2}{r^2} + {\omega_0}^2 r^2 + \frac{F(\phi)}{r^2} \\ J_2 &= p_\phi^2 + F(\phi). \end{aligned}$$

The following proposition states this property for spherical ($\kappa > 0$) and hyperbolic ($\kappa < 0$) spaces.

Proposition 1. *The Hamiltonian*

$$H(\kappa) = \frac{1}{2}\left(p_r^2 + \frac{p_\phi^2}{\mathrm{S}_\kappa^2(r)}\right) + \frac{1}{2}\,\omega_0^2\,\mathrm{T}_\kappa^2(r) + \frac{1}{2}\,\frac{F(\phi)}{(\mathrm{S}_\kappa(r))^2} \tag{9}$$

is separable in geodesic polar coordinates (r, ϕ) *and it is endowed with the following two constants of motion*

$$\begin{aligned} J_1 &= p_r^2 + \frac{p_\phi^2}{\mathrm{S}_\kappa^2(r)} + \omega_0^2\,\mathrm{T}_\kappa^2(r) + \frac{F(\phi)}{(\mathrm{S}_\kappa(r))^2} \\ J_2 &= p_\phi^2 + F(\phi). \end{aligned}$$

This property is true for all the values of the curvature κ.

As we comment in the introduction, the TTW system is separable in the Euclidean plane in polar coordinates. Now we see that it admits a generalization to the spaces S_κ^2 ($\kappa > 0$) and H_κ^2 ($\kappa < 0$) that appears as a particular case of the Hamiltonian (9); therefore, it is also separable (and therefore integrable) in spherical and hyperbolic spaces.

The following proposition proves the superintegrability of the TTW system on spaces of constant curvature and presents a method for obtaining the explicit expression of the third integral of motion.

Proposition 2. *Consider the nonlinear harmonic oscillator-related potential*

$$U_m(r,\phi) = \frac{1}{2}\,\omega_0^2\,\mathrm{T}_\kappa^2(r) + \frac{1}{2}\,\frac{F_m(\phi)}{(\mathrm{S}_\kappa(r))^2},\quad F_m(\phi) = \frac{k_a}{\sin^2(m\phi)} + k_b\left(\frac{\cos(m\phi)}{\sin^2(m\phi)}\right), \tag{10}$$

where k_a and k_b are arbitrary constants. Let J_1 and J_2 represent the two quadratic constants of motion associated to the Liouville integrability

$$J_1 = p_r^2 + \frac{p_\phi^2}{\mathrm{S}_\kappa^2(r)} + \omega_0^2\,\mathrm{T}_\kappa^2(r) + \frac{F_m}{\mathrm{S}_\kappa^2(r)}$$

$$J_2 = p_\phi^2 + F_m$$

and let M_r and N_ϕ be the complex functions $M_r = M_{r1} + \mathrm{i}\,M_{r2}$ and $N_\phi = N_{\phi 1} + \mathrm{i}\,N_{\phi 2}$ with real and imaginary parts, M_{ra} and $N_{\phi a}$, $a = 1, 2$, be defined as

$$M_{r1} = \frac{2}{\mathrm{T}_\kappa(r)}p_r\sqrt{J_2},\quad M_{r2} = p_r^2 + \omega_0^2\,\mathrm{T}_\kappa^2(r) - \frac{J_2}{\mathrm{T}_\kappa^2(r)} = J_1 - \frac{1+\mathrm{C}_\kappa^2(r)}{\mathrm{S}_\kappa^2(r)}J_2,$$

$$N_{\phi 1} = \frac{k_b}{2} + J_2\cos(m\phi),\quad N_{\phi 2} = \sqrt{J_2}p_\phi\,\sin(m\phi).$$

Then, the complex function K_m defined as

$$K_m = M_r^m (N_\phi^*)^2$$

is a (complex) constant of motion.

Proof. First, let us comment that the functions M_{r1} and M_{r2} are κ-dependent but they satisfy the appropriate Euclidean limit [16]

$$\lim_{\kappa\to 0} M_{r1} = \frac{2}{r}p_r\sqrt{J_2},\quad \lim_{\kappa\to 0} M_{r2} = p_r^2 + \omega_0^2 r^2 - \frac{J_2}{r^2} = J_1 - \frac{2}{r^2}J_2.$$

The expressions of the functions $N_{\phi 1}$ and $N_{\phi 2}$ are the same as in the Euclidean plane.

The time derivative (Poisson bracket with $H(\kappa)$) of the function M_{r1} is proportional to M_{r2} and the time derivative of the M_{r2} is proportional to M_{r1} but with the opposite sign

$$\frac{\mathrm{d}}{\mathrm{d}t}M_{r1} = -2\lambda_\kappa M_{r2},\quad \frac{\mathrm{d}}{\mathrm{d}t}M_{r2} = 2\lambda_\kappa M_{r1},$$

and this property is also true for the angular functions

$$\frac{\mathrm{d}}{\mathrm{d}t}N_{\phi 1} = -m\lambda_\kappa N_{\phi 2},\quad \frac{\mathrm{d}}{\mathrm{d}t}N_{\phi 2} = m\lambda_\kappa N_{\phi 1},$$

where the common factor λ_κ takes the value

$$\lambda_\kappa = \frac{1}{\mathrm{S}_\kappa^2(r)}\sqrt{J_2},\quad \lambda_0 = \frac{1}{r^2}\sqrt{J_2}.$$

Therefore, the time evolution of the complex functions M_r and N_ϕ is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}M_r = \mathrm{i}2\lambda_\kappa M_r,\quad \frac{\mathrm{d}}{\mathrm{d}t}N_\phi = \mathrm{i}m\lambda_\kappa N_\phi.$$

Thus we have

$$\frac{\mathrm{d}}{\mathrm{d}t}K_m = \frac{\mathrm{d}}{\mathrm{d}t}\big(M_r^m (N_\phi^*)^2\big) = M_r^{(m-1)}N_\phi^*(m\dot{M}_r N_\phi^* + 2M_r \dot{N}_\phi{}^*)$$
$$= M_r^{(m-1)}N_\phi^*(m\mathrm{i}2\lambda_\kappa M_r N_\phi^* + 2M_r(-\mathrm{i}m\lambda_\kappa N_\phi^*)) = 0.$$

Finally, let us comment that the moduli of these two complex functions (that are constant of motion of fourth order in the momenta) are given by

$$|M_r|^2 = 4\big(H^2 - \omega_0^2 J_2\big) + \kappa\,(\kappa J_2 - 4H)J_2$$
$$|N_\phi|^2 = J_2^2 - k_a J_2 + \frac{k_b^2}{4}.$$

□

Summarizing, the TTW is superintegrable for any value of the curvature (positive, zero or negative) and the additional constant of motion K_m can be obtained by complex factorization. Since the function K_m is complex it can be written as $K_m = J_3 + \mathrm{i}\,J_4$ with J_3 and J_4 real constants of motion, that is, $\mathrm{d}J_3/\mathrm{d}t = 0$, $\mathrm{d}J_4/\mathrm{d}t = 0$. One of them, for example J_3, can be chosen as the third fundamental integral of motion.

The function $F_m(\phi)$ in (10) can be considered, at first sight, as not the same as the angular function in the original TTW potential (3). Nevertheless the following trigonometric equality can be proved

$$\frac{2(\alpha+\beta)}{\sin^2(2m\phi)} + 2(\beta-\alpha)\left(\frac{\cos(2m\phi)}{\sin^2(2m\phi)}\right) = \frac{\alpha}{\cos^2(m\phi)} + \frac{\beta}{\sin^2(m\phi)}.$$

Thus, the above proposition 2 is also true for the potential U_m rewritten with the angular function as in (3). More specifically, let us now consider the following (spherical, Euclidean, hyperbolic) potential

$$U'_m(r,\phi) = \frac{1}{2}\,\omega_0^2\,\mathrm{T}_\kappa^2(r) + \frac{1}{2}\,\frac{G_m(\phi)}{\mathrm{S}_\kappa^2(r)}, \quad G_m(\phi) = \frac{\alpha}{\cos^2(m\phi)} + \frac{\beta}{\sin^2(m\phi)} \tag{11}$$

and let us denote by J'_1 and J'_2 the two constants of motion J_1 and J_2 but now rewritten as functions of G_m

$$J'_1 = p_r^2 + \frac{p_\phi^2}{\mathrm{S}_\kappa^2(r)} + \omega_0^2\,\mathrm{T}_\kappa^2(r) + \frac{G_m}{\mathrm{S}_\kappa^2(r)}$$
$$J'_2 = p_\phi^2 + G_m.$$

Then if we also write with primes the new functions M_r and N_ϕ

$$M'_{r1} = \frac{2}{\mathrm{T}_\kappa(r)}\,p_r\sqrt{J'_2}, \qquad M'_{r2} = p_r^2 + \omega_0^2\,\mathrm{T}_\kappa^2(r) - \frac{J'_2}{\mathrm{T}_\kappa^2(r)},$$

$$N'_{\phi 1} = \beta - \alpha + J'_2\cos(2m\phi), \qquad N'_{\phi 2} = \sqrt{J'_2}\,p_\phi\sin(2m\phi),$$

the complex constant of motion for the potential (11) is now given by

$$K'_m = (M'_r)^{2m}(N'^{*}_\phi)^2.$$

4. Final comments

The following two points summarize the main results proved in this paper.

- The TTW system is not a specific characteristic of the Euclidean space but it is well defined in all the three spaces of constant curvature. Moreover, we have represented the TTW system by a unique Hamiltonian (with potential (10) or with potential (11)) that is a smooth function of the curvature κ and, in this way, we can say that there are not three different TTW systems but only one that is defined, at the same time, in the three different manifolds.

- The TTW system is superintegrable in the three spaces of constant curvature. The additional third integral of motion can be explicitly obtained as the product of powers of two particular rather simple complex functions (here denoted by M_r and N_ϕ). This factorization, that is valid for all the values of κ, generalizes the Euclidean property previously proved in [16].

We conclude with the following two comments: First, the superintegrabilty of another Euclidean system, known as the PW system, similar to the TTW but related with the Kepler problem [40, 41] has been recently proved. We think that the PW system can also be studied on spaces of constant curvature by making use of the curvature-dependent formalism. Second, the TTW system is also important at the quantum level. The properties of the functions M_r and N_ϕ are probably interesting (changing functions for operators) for the study of the quantum Schrödinger equation using the method of factorization and ladder operators.

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References

[1] Fris T I, Mandrosov V, Smorodinsky Y A, Uhlir M and Winternitz P 1965 On higher symmetries in quantum mechanics *Phys. Lett.* **16** 354–6
[2] Evans N W 1990 Superintegrability of the Smorodinsky–Winternitz system *Phys. Lett.* **147** 483–6
[3] Grosche C, Pogosyan G S and Sissakian A N 1995 Path integral discussion for Smorodinsky Winternitz potentials: I. Two- and three-dimensional Euclidean space *Fortschr. Phys.* **43** 453–521
[4] Weissman Y and Jortner J 1979 The isotonic oscillator *Phys. Lett.* A **70** 177–79
[5] Zhu D 1987 A new potential with the spectrum of an isotonic oscillator *J. Phys. A: Math. Gen.* **20** 4331–36
[6] Miller W, Post S and Winternitz P 2013 Classical and quantum superintegrability with applications *J. Phys. A: Math. Theor.* **46** 423001
[7] Evans N W and Verrier P E 2008 Superintegrability of the caged anisotropic oscillator *J. Math. Phys.* **49** 092902
[8] Rodríguez M A, Tempesta P and Winternitz P 2008 Reduction of superintegrable systems: the anisotropic harmonic oscillator *Phys. Rev.* E **78** 046608
[9] Rañada M F, Rodríguez M A and Santander M 2010 A new proof of the higher-order superintegrability of a noncentral oscillator with inversely quadratic nonlinearities *J. Math. Phys.* **51** 042901
[10] Tremblay F, Turbiner A V and Winternitz P 2009 An infinite family of solvable and integrable quantum systems on a plane *J. Phys. A: Math. Theor.* **42** 242001
[11] Tremblay F, Turbiner A V and Winternitz P 2010 Periodic orbits for an infinite family of classical superintegrable systems *J. Phys. A: Math. Theor.* **43** 015202
[12] Quesne C 2010 Superintegrability of the Tremblay–Turbiner–Winternitz quantum Hamiltonians on a plane for odd k *J. Phys. A: Math. Theor.* **43** 082001
[13] Quesne C 2010 N=2 supersymmetric extension of the Tremblay–Turbiner–Winternitz Hamiltonians on a plane *J. Phys A: Math. Theor.* **43** 305202
[14] Kalnins E G, Kress J M and Miller W 2010 Superintegrability and higher order constants for quantum systems *J. Phys. A: Math. Theor.* **43** 265205
[15] Calzada J A, Celeghini E, del Olmo M A and Velasco M A 2012 Algebraic aspects of TTW Hamiltonian system *J. Phys.: Conf. Ser.* **343** 012029
[16] Ranada M F 2012 A new approach to the higher-order superintegrability of the Tremblay–Turbiner–Winternitz system *J. Phys. A: Math. Theor.* **45** 465203
[17] Gonera C 2012 On superintegrability of TTW model *Phys. Lett.* A **376** 2341–3

[18] Hakobyan T, Lechtenfeld O, Nersessian A, Saghatelian A and Yeghikyan V 2012 Integrable generalizations of oscillator and Coulomb systems via action-angle variables *Phys. Lett.* A **376** 679–86
[19] Celeghini E, Kuru S, Negro J and del Olmo M A 2013 A unified approach to quantum and classical TTW systems based on factorizations *Ann. Phys.* **332** 27–37
[20] Liebmann H 1905 *Nichteuklidische Geometrie* 1st edn (Leipzig: Göschen'sch Verlag)
Liebmann H 1923 *Nichteuklidische Geometrie* 3rd edn (Berlin: De Gruyter)
[21] Higgs P W 1979 Dynamical symmetries in a spherical geometry: I *J. Phys. A: Math. Gen.* **12** 309–23
[22] Maciejewski A J, Przybylska M and Yoshida H 2010 Necessary conditions for super-integrability of a certain family of potentials in constant curvature spaces *J. Phys. A: Math. Theor.* **43** 382001
[23] Gonera C and Kaszubska M 2013 Superintegrable systems on spaces of constant curvature arXiv:1311.0729v2
[24] Rañada M F and Santander M 2002 On the harmonic oscillator on the two-dimensional sphere S^2 and the hyperbolic plane H^2 *J. Math. Phys.* **43** 431–51
[25] Rañada M F and Santander M 2003 On the harmonic oscillator on the two-dimensional sphere S^2 and the hyperbolic plane H^2: II *J. Math. Phys.* **44** 2149–67
[26] Cariñena J F, Rañada M F and Santander M 2011 The harmonic oscillator on three-dimensional spherical and hyperbolic spaces: curvature dependent formalism and quantization *Int. J. Theor. Phys.* **50** 2170–8
[27] Cariñena J F, Rañada M F and Santander M 2012 Curvature-dependent formalism, Schrödinger equation and energy levels for the harmonic oscillator on three-dimensional spherical and hyperbolic spaces *J. Phys. A: Math. Theor.* **45** 265303
[28] Dombrowski P and Zitterbarth J 1991 On the planetary motion in the 3-dim standard spaces of constant curvature *Demonstratio Math.* **24** 375–458
[29] Ballesteros A, Herranz F J, del Olmo M A and Santander M 1993 Quantum structure of the motion groups of the two-dimensional Cayley–Klein geometries *J. Phys. A: Math. Gen.* **26** 5801–23
[30] Rañada M F and Santander M 1999 Superintegrable systems on the two-dimensional sphere S2 and the hyperbolic plane H2 *J. Math. Phys.* **40** 5026–57
[31] Herranz F J, Ortega R and Santander M 2000 Trigonometry of spacetimes: a new self-dual approach to a curvature/signature (in)dependent trigonometry *J. Phys. A: Math. Gen.* **33** 4525–51
[32] Herranz F J and Ballesteros A 2006 Superintegrability on three-dimensional Riemannian and relativistic spaces of constant curvature *SIGMA* **2** 010
[33] Chanu C, Degiovanni L and Rastelli G 2011 First integrals of extended Hamiltonians in n+1 dimensions generated by powers of an operator *SIGMA* **7** 038
[34] Chanu C, Degiovanni L and Rastelli G 2012 Generalizations of a method for constructing first integrals of a class of natural Hamiltonians and some remarks about quantization *J. Phys.: Conf. Ser.* **343** 012101
[35] Diacu F, Perez-Chavela E and Santoprete M 2012 The n-body problem in spaces of constant curvature. part I: relative equilibria *J. Nonlinear Sci.* **22** 247–66
[36] Ballesteros A, Herranz F J and Musso F 2013 The anisotropic oscillator on the 2D sphere and the hyperbolic plane *Nonlinearity* **26** 971–90
[37] Jauch J M and Hill E L 1940 On the problem of degeneracy in quantum mechanics *Phys. Rev.* **57** 641–645
[38] Klinberger W 1978 A course in differential geometry *(Graduate texts in Mathematics)* (New York: Springer)
[39] Fradkin D M 1965 Three-dimensional isotropic harmonic oscillator and SU3 *Am. J. Phys.* **33** 207–11
[40] Post S and Winternitz P 2010 An infinite family of superintegrable deformations of the Coulomb potential *J. Phys. A: Math. Theor.* **43** 222001
[41] Rañada M F 2013 Higher order superintegrability of separable potentials with a new approach to the Post-winternitz system *J. Phys. A: Math. Theor.* **46** 125206