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Dynamical symmetries in a spherical geometry I

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Abstract. The two potentials for which a particle moving non-relativistically in a spherical space under the action of a conservative central force executes closed orbits are found. When the curvature is zero they reduce to the familiar Coulomb and isotropic oscillator potentials of Euclidean geometry. The corresponding vector (for the former) and symmetric tensor (for the latter) constants of motion are constructed. For each system in N dimensions the Poisson bracket algebra in classical mechanics, and the commutator algebra in quantum mechanics, of these constants of motion and the angular momentum components are constructed. It is proved that by an appropriate choice of independent constants of motion these Poisson bracket algebras may be transformed into Lie algebraic structures, those of the symmetry groups $SO(N+1)$ and $SU(N)$ respectively: the Hamiltonian of each system is expressed as a function of the Casimir operators of its symmetry group. The corresponding transformations of the quantum mechanical commutator algebras are performed only for $N = 2$: the corresponding expressions for the Hamiltonians as functions of the Casimir operators yield the energy levels of the two systems.

1. Introduction

It is well known (Bacry, Ruegg and Souriau 1966, Stehle and Han 1967) that the Hamiltonian of a particle moving non-relativistically in N-dimensional Euclidean space under the action of a conservative central force is symmetric under a Lie group larger than $SO(N)$, the group of rotations about the force centre, in just two special cases: these are the Kepler problem, for which the higher symmetry group is $SO(N+1)$ for bound states and $SO(N, 1)$ for scattering states, and the isotropic oscillator, for which it is $SU(N)$. In classical mechanics the special feature that distinguishes these two systems is that all the bounded orbits are closed (Bertand 1873), a feature which permits the construction of constants of the motion which specify the orientation of an orbit within its plane. In the Kepler problem these constants are cartesian components of the Runge–Lenz vector (Laplace 1827, Runge 1919, Lenz 1925), which is parallel to the major axis of the orbit: in the oscillator problem they are cartesian components of a symmetric second order tensor, which is coplanar with the orbit and has the same principal axes (Fradkin 1965). In each case the higher symmetry is revealed when one constructs the Poisson bracket algebra of the constants of the motion (including the angular momentum components, which specify the plane of the orbit): if the Runge–Lenz vector (Fradkin tensor) is suitably normalised, this algebra has the structure of the Lie algebra of $SO(N+1)(SU(N))$. These higher symmetries are not geometric (that is, they cannot be expressed as mappings of configuration space alone); they are symmetries in phase space, for which the term 'dynamical symmetries' is often used.

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In quantum mechanics, where the Poisson brackets are replaced by commutators, standard techniques of matrix representation theory enable one to calculate, for these two systems, both the spectrum of the Hamiltonian and the degeneracy of each energy level with respect to angular momentum eigenvalues, which must be that of an irreducible representation of the higher symmetry group (Pauli 1926, Fock 1935, Jauch and Hill 1940, Baker 1956).

In this paper the corresponding dynamical systems which result from the replacement of N-dimensional Euclidean geometry by that of the N-sphere are studied. In § 2 it is shown that in a certain co-ordinate system, which is associated with a certain projection of the N-sphere (embedded in Euclidean space of $N+1$ dimensions) onto a tangent N-plane, closed orbits occur for the same potentials as in Euclidean geometry. The corresponding vector or tensor constants of the motion and their Poisson bracket algebras are constructed in § 3. In § 4 the corresponding quantum mechanical operators and commutator algebras are found.

The rest of the paper concentrates on two-dimensional systems, since these exhibit the essential features of the dynamical symmetries (SO(3) and SU(2)) without the complications which arise for $N > 2$ from the non-trivial structure of the rotation group SO(N). In § 5 the Hamiltonians of the quantum mechanical Kepler and isotropic oscillator systems on a 2-sphere are expressed as functions of the Casimir operators of SO(3) and SU(2) respectively. In § 6 it is shown how the generators of the dynamical symmetries of these systems may be used, in a Schrödinger representation, to construct wavefunctions. Further implications of the algebra are discussed in § 7. A subsequent paper (Leemon 1979) will deal with the generalisation of these results to N dimensions and, in particular, will discuss the physically most relevant dimension, $N = 3$.

The quantum mechanical Kepler problem on a 3-sphere has previously been discussed by Schrödinger (1940)†, who found the energy levels and their degeneracy in angular momentum by factorising the second order differential operator in the radial Schrödinger equation. The isotropic oscillator on a 3-sphere has been studied, in the guise of 'a soluble non-linear chiral model' by Lakshmanan and Eswaran (1975), who solved the radial Schrödinger equation. It was the recognition that the angular momentum degeneracy of the energy levels found by these authors was just that of the Euclidean isotropic oscillator levels which originally motivated this investigation of dynamical symmetries in a spherical geometry.

2. Classical dynamics on a sphere

There are several co-ordinate systems on a sphere which are useful generalisations of the cartesian systems of Euclidean geometry; they correspond to various projections of the N-sphere, embedded in a Euclidean space of $N+1$ dimensions, onto the tangent N-plane at the chosen origin. One such system, employed by Lakshmanan and Eswaran (1975), is obtained simply by imposing the constraint

$$q_0^2 + q_i q_i = \lambda^{-1}, \tag{1}$$

where λ is the curvature of the sphere, upon the $N+1$ cartesian co-ordinates q_0, q_i ($i = 1 \ldots N$). The N independent variables q_i are then cartesian co-ordinates of an

† I am grateful to Professor Asim Barut for bringing this paper to my attention.

orthogonal projection of the sphere: their domain is the disc $q_i q_i \leqslant \lambda^{-1}$ and each set $\{q_i\}$ corresponds to two points of the sphere.

For the purposes of this paper another projection is more useful, for reasons which will shortly be made clear: this is (in cartographers' jargon) the gnomonic projection, which is the projection onto the tangent plane from the centre of the sphere in the embedding space. Cartesian coordinates of this projection will be denoted by x_i. The relation between the two projections is

$$q_i = x_i(1+\lambda r^2)^{-1/2} \tag{2}$$

where

$$r^2 = x_i x_i.$$

From equations (1) and (2) one readily obtains the metric

$$\mathrm{d}s^2 = (1+\lambda r^2)^{-2}(\boldsymbol{x}\,.\,\mathrm{d}\boldsymbol{x}/r)^2 + (1+\lambda r^2)^{-1}[\mathrm{d}\boldsymbol{x}\,.\,\mathrm{d}\boldsymbol{x} - (\boldsymbol{x}\,.\,\mathrm{d}\boldsymbol{x}/r)^2]. \tag{3}$$

The advantage of this projection over all others for the analysis of the motion of a particle on a sphere stems from the fact that free particle motion (uniform motion on a great circle) projects into rectilinear, but non-uniform motion on the tangent plane. That is, the projected free particle orbits are the same as in Euclidean geometry: the curvature affects only the speed of the projected motion. It will now be shown that this feature persists in the presence of a central force derived from a potential $V(r)$.

The Lagrangian for the non-relativistic motion of a particle of unit mass under such a force is $\frac{1}{2}\dot{s}^2 - V$, where $\dot{s}^2$ is defined in (3). The momentum conjugate to $\boldsymbol{x}$ is thus

$$\boldsymbol{p} = (1+\lambda r^2)^{-2}\frac{\boldsymbol{x}(\boldsymbol{x}\,.\,\dot{\boldsymbol{x}})}{r^2} + (1+\lambda r^2)^{-1}(\dot{\boldsymbol{x}} - \boldsymbol{x}(\boldsymbol{x}\,.\,\dot{\boldsymbol{x}})/r^2) \tag{4}$$

so the angular momentum tensor is

$$L_{ij} = x_i p_j - x_j p_i = (1+\lambda r^2)^{-1}(x_i \dot{x}_j - x_j \dot{x}_i) \tag{5}$$

and the Hamiltonian is

$$H = (1+\lambda r^2)\{p^2 + \lambda(\boldsymbol{x}\,.\,\boldsymbol{p})^2\} + V(r) \tag{6}$$

Rotational symmetry of H implies the constancy of L_{ij}, so every projected orbit lies in a plane $L_{ij}x_j = 0$. In terms of polar co-ordinates (r, θ) in that plane angular momentum and energy conservation take the forms

$$(1+\lambda r^2)^{-1} r^2 \dot{\theta} = L, \tag{5a}$$

where

$$\boldsymbol{L}^2 = \tfrac{1}{2} L_{ij} L_{ij},$$

and

$$\tfrac{1}{2}[(1+\lambda r^2)^{-2}\dot{r}^2 + (1+\lambda r^2)^{-1} r^2 \dot{\theta}^2] + V(r) = E. \tag{6a}$$

From the last two equations one finds the differential equation of the orbit

$$\tfrac{1}{2}\boldsymbol{L}^2[r^{-4}(\mathrm{d}r/\mathrm{d}\theta)^2 + r^{-2}] + V(r) = E - \tfrac{1}{2}\lambda \boldsymbol{L}^2. \tag{7}$$

Clearly, since the curvature appears here only in the combination $E - \frac{1}{2}\lambda\boldsymbol{L}^2$, the projected orbits are the same, for a given $V(r)$, as in Euclidean geometry.

In particular, it follows that the orbits are closed only if

$$V(r) = -\mu r^{-1} \qquad \text{(Kepler problem)} \tag{8}$$

or

$$V(r) = \tfrac{1}{2}\omega^2 r^2 \qquad \text{(isotropic oscillator)} \tag{9}$$

where μ and ω are constants.

In terms of the angular coordinate χ on the sphere, defined by

$$\lambda^{1/2} r = \tan \chi,$$

these potentials have the forms

$$V = -\mu\lambda^{1/2} \cot \chi \tag{8a}$$

$$V = \tfrac{1}{2}\omega^2\lambda^{-1} \tan^2 \chi. \tag{9a}$$

In the form $(8a)$ the Kepler (or Coulomb) potential is clearly antisymmetric between the two hemispheres: if μ is taken positive, it has an attractive singularity at the origin $\chi = 0$ (north pole) and an equal repulsive singularity at the antipodal point $\chi = \pi$ (south pole). Moreover, unlike the Euclidean counterpart, here *all* the orbits are closed, since the sphere is compact – those projections which do not close (hyperbolae) correspond to closed orbits which cross the equator ($\chi = \frac{1}{2}\pi$). In the limit $\lambda \to 0$ one recovers the Euclidean Kepler (attractive Coulomb) orbits, both bounded and unbounded, from the northern hemisphere and the unbounded orbits of the repulsive Coulomb potential from the southern hemisphere.

On the other hand, in the form $(9a)$ the oscillator potential is clearly symmetric between the two hemispheres and singular on the equator. Possible orbits are thus confined to one hemisphere or the other and take the same form on each. As in Euclidean space, solutions of the orbit equation exist only if ω^2 is non-negative. It should be noted that, on account of the singularity at the equator, in the limit $\omega \to 0$ one does not recover free particle motion but rather the motion of a particle which is free apart from a reflecting barrier at the equator.

3. Constants of the motion

3.1. *The Kepler problem*

In Euclidean space the Runge–Lenz vector, which at every point on a Kepler orbit lies parallel to the major axis, has cartesian components

$$R_i = -L_{ij}p_j + \mu x_i / r. \tag{10}$$

Their Poisson brackets with the Hamiltonian and with each other are

$$\{R_i, H\} = 0 \tag{11}$$

$$\{R_i, R_j\} = -2HL_{ij}. \tag{12}$$

The length of the vector is given by

$$\boldsymbol{R}^2 = \mu^2 + 2H\boldsymbol{L}^2. \tag{13}$$

It is not difficult to find the generalisation of the expression (10) which is appropriate to the sphere. The first term in R_i, which is conserved in the motion of a free particle, is constructed from the generators L_{ij} and p_j of the Euclidean group E(N). In free particle motion on the sphere, conservation of linear momentum is replaced by conservation of the vector

$$\boldsymbol{\pi} = \boldsymbol{p} + \lambda \boldsymbol{x}(\boldsymbol{x} \cdot \boldsymbol{p}) \tag{14}$$

whose components are proportional to the corresponding generators of the geometrical symmetry group SO($N+1$), which are angular momentum components in the embedding space:

$$\pi_i = \lambda^{\frac{1}{2}} L_{0i}. \tag{15}$$

On making the same replacement in the expression (10) one obtains the required generalisation

$$R_i = -L_{ij}\pi_j + \mu x_i/r, \tag{16}$$

from which follow the generalisations of (12) and (13):

$$\{R_i, R_j\} = (-2H + 2\lambda \boldsymbol{L}^2)L_{ij}, \tag{17}$$

$$\boldsymbol{R}^2 = \mu^2 + 2H\boldsymbol{L}^2 - \lambda(\boldsymbol{L}^2)^2. \tag{18}$$

The Hamiltonian may be written in the form

$$H = \tfrac{1}{2}(\boldsymbol{\pi}^2 + \lambda \boldsymbol{L}^2) - \mu/r \tag{19}$$

since the free particle Hamiltonian is proportional to the quadratic Casimir operator $(L_{0i}L_{0i} + \boldsymbol{L}^2)$ of the geometric SO($N+1$) group.

In Euclidean space it is a trivial task to construct a normalised Runge–Lenz vector $\boldsymbol{M}$ such that the Poisson bracket algebra of L_{ij} and M_i has the structure of the Lie algebra of SO($N+1$). Inspection of (11) and (12) indicates that the vector

$$\boldsymbol{M} = (-2H)^{-1/2}\boldsymbol{R}, \tag{20}$$

which is a real dynamic variable for the bounded orbits ($E<0$)†, has the required Poisson brackets

$$\{M_i, H\} = 0 \tag{21}$$

$$\{M_i, M_j\} = L_{ij}. \tag{22}$$

Equation (13) then allows one to write H in terms of the Casimir operator‡ of the dynamical symmetry group:

$$H = -\tfrac{1}{2}\mu^2/C \tag{23}$$

where

$$C = \boldsymbol{L}^2 + \boldsymbol{M}^2.$$

† For the unbounded orbits ($E>0$) one defines instead

$$\boldsymbol{M}' = (+2H)^{-1/2}\boldsymbol{R}$$

and obtains the algebra of SO(N, 1).

‡ The realisation of the generators of SO($N+1$) by the dynamic variables L_{ij} and M_i has only one independent Casimir operator on account of the identities, such as

$$L_{ij}M_k + L_{jk}M_i + L_{ki}M_j \equiv 0,$$

which they satisfy.

On the sphere, the occurrence of $\boldsymbol{L}^2$ in the Poisson brackets (17) makes the construction of $\boldsymbol{M}$ a non-trivial task. The requirement that $\boldsymbol{M}$ be a vector under SO(N) which is a constant of the motion dictates the form

$$\boldsymbol{M} = \boldsymbol{R} f(\boldsymbol{L}^2, H). \tag{24}$$

The Poisson brackets of components of such a vector are

$$\{M_i, M_j\} = -L_{ij} \frac{\partial}{\partial(\boldsymbol{L}^2)} \boldsymbol{M}^2(\boldsymbol{L}^2, H),$$

so the Poisson brackets (22) are obtained if

$$\boldsymbol{L}^2 + \boldsymbol{M}^2 = C(H), \tag{25}$$

where $C(H)$ is, so far, an arbitrary function. Now if $\boldsymbol{M}$, like $\boldsymbol{R}$, is to be well defined for all orbits, it must, like $\boldsymbol{R}$, be zero when the orbit is circular. Thus $\boldsymbol{L}^2 = C(E)$ must be the expression for the angular momentum of a circular orbit of energy E, which is found by setting $\boldsymbol{R}^2 = 0$ in (18)†. Therefore the solution of (25) for the Hamiltonian in terms of the Casimir operator C for *all* orbits is determined by (18) to be

$$\mu^2 + 2HC - \lambda C^2 = 0.$$

Explicitly, the analogue of (23) for the sphere is

$$H = \tfrac{1}{2}\lambda C - \tfrac{1}{2}\mu^2 C^{-1} \tag{26}$$

and there is *no* analogue of the corresponding SO(N, 1) relation which is valid for $E > 0$ in Euclidean space.

3.2. The isotropic oscillator

In Euclidean space a symmetric tensor‡ whose principal axes are those of the orbit has cartesian components

$$S_{ij} = p_i p_j + \omega^2 x_i x_j. \tag{27}$$

Their Poisson brackets with the oscillator Hamiltonian and with each other are

$$\{S_{ij}, H\} = 0 \tag{28}$$

$$\{S_{ij}, S_{kl}\} = \omega^2 (L_{ik}\delta_{jl} + L_{il}\delta_{jk} + L_{jk}\delta_{il} + L_{jl}\delta_{ik}). \tag{29}$$

The components (27) clearly form a matrix of rank two, so only two independent scalars can be constructed from them. They are

$$I_1 = S_{ii} = 2H \tag{30}$$

$$I_2 = S_{ij}S_{ji} - S_{ii}S_{jj} = -2\omega^2 \boldsymbol{L}^2. \tag{31}$$

On the sphere, once again all that is necessary to construct the corresponding constants of the motion is to replace $\boldsymbol{p}$ in S_{ij} by $\boldsymbol{\pi}$:

$$S_{ij} = \pi_i \pi_j + \omega^2 x_i x_j. \tag{32}$$

† The explicit expression so obtained is

$$C(E) = \lambda^{-1}[E + (E^2 + \lambda\mu^2)^{1/2}].$$

‡ This tensor is twice that defined by Fradkin (1965).

The Poisson brackets (29) are replaced by

$$\{S_{ij}, S_{kl}\} = \omega^2(L_{ik}\delta_{jl} + L_{il}\delta_{jk} + L_{jk}\delta_{il} + L_{jl}\delta_{ik}) + \lambda(L_{ik}S_{jl} + L_{il}S_{jk} + L_{jk}S_{il} + L_{jl}S_{ik}). \tag{33}$$

Equation (30) is replaced by

$$S_{ii} = 2H - \lambda \boldsymbol{L}^2 \tag{34}$$

but (31) is unchanged.

In Euclidean space the algebra (29) leads trivially to that of the dynamical symmetry group SU(N), whose generators are L_{ij} and a traceless symmetric tensor N_{ij}, such that

$$\{N_{ij}, H\} = 0 \tag{35}$$

$$\{N_{ij}, N_{kl}\} = L_{ik}\delta_{jl} + L_{il}\delta_{jk} + L_{jk}\delta_{il} + L_{jl}\delta_{ik}. \tag{36}$$

Clearly one may choose

$$N_{ij} = \omega^{-1}(S_{ij} - N^{-1}S_{kk}\delta_{ij}). \tag{37}$$

Equations (30) and (31) now lead to the expression

$$N_{ij}N_{ji} = 4(N-1)H^2/N\omega^2 - 2\boldsymbol{L}^2$$

for the only independent scalar which can be constructed from N_{ij}. Hence H is obtained as a function of the Casimir operator† of the dynamical group SU(N):

$$H = \omega C^{1/2} \tag{38}$$

where

$$4(1-N^{-1})C = N_{ij}N_{ij} + L_{ij}L_{ij}.$$

On the sphere, the construction of N_{ij} from S_{ij} is non-trivial, due to the non-linearity of the expressions for the Poisson brackets (33), but may be accomplished by a procedure similar to that used in the Kepler problem. Since the matrix S_{ij} has rank two, the most general symmetric tensor with the same principal axes, which is also of rank two as a matrix, is

$$T_{ij} = f(\boldsymbol{L}^2, H)S_{ij} + g(\boldsymbol{L}^2, H)(S_{im}S_{mj} - S_{mm}S_{ij}) = fS_{ij} + \omega^2 g L_{im}L_{mj}. \tag{39}$$

The corresponding two scalars are

$$J_1 = T_{ii} = fI_1 + gI_2 \tag{40}$$

$$J_2 = T_{ij}T_{ji} - T_{ii}T_{jj} = (f^2 - fgI_1 - \tfrac{1}{2}g^2 I_2)I_2. \tag{41}$$

The Poisson brackets of components of such a tensor are

$$\{T_{ij}, T_{kl}\} = L_{ik}U_{jl} + L_{il}U_{jk} + L_{jk}U_{il} + L_{jl}U_{ik}$$

where

$$U_{ij} = \delta_{ij}(\omega^2 J_2/I_2) - T_{ij}\,\partial J_1/\partial(\boldsymbol{L}^2) - L_{im}L_{mj}\,\partial(\omega^2 J_2/I_2)/\partial(\boldsymbol{L}^2). \tag{42}$$

† Once again, the realisation of SU(N) by the dynamic variables L_{ij} and N_{ij} has only one independent Casimir operator, on account of the algebraic relations among them.

Thus the Poisson brackets (36) are obtained if

$$N_{ij} = T_{ij} - N^{-1} T_{kk} \delta_{ij}$$

provided that in (39) the functions f and g are chosen so that

$$\begin{aligned} J_1 &= A(H) \\ J_2 &= -2\boldsymbol{L}^2 \end{aligned} \tag{43}$$

where $A(H)$ is, so far, an arbitrary function. Then the quadratic Casimir operator of $\mathrm{SU}(N)$ is given by

$$N_{ij}N_{ij} + L_{ij}L_{ij} = 4(1 - N^{-1})C(H)$$

where

$$C(H) = \tfrac{1}{4}[A(H)]^2. \tag{44}$$

Once again, the function $A(H)$ is determined by considering the circular orbits. For the oscillator these are characterised by degeneracy of the two non-zero eigenvalues of S_{ij}, which occurs when

$$I_1^2 + 2I_2 = 0.$$

This condition has the explicit form

$$E = \tfrac{1}{2}\lambda \boldsymbol{L}^2 + \omega(\boldsymbol{L}^2)^{1/2}. \tag{45}$$

The same degeneracy condition for T_{ij},

$$J_1^2 + 2J_2 = 0,$$

has the explicit form

$$\tfrac{1}{4}[A(E)]^2 = \boldsymbol{L}^2. \tag{46}$$

The requirement that relations (45) and (46) agree determines $A(E)$ to be such that (44) is equivalent to

$$H = \tfrac{1}{2}\lambda C + \omega C^{1/2}. \tag{47}$$

This is the required generalisation of the Euclidean relation (38).

4. Quantum dynamics on a sphere

The quantum mechanical Hamiltonian H_0 for a free particle on a sphere is obtained from its classical counterpart by substituting for the classical Casimir operator of the geometrical $\mathrm{SO}(N+1)$ group its quantum mechanical counterpart. The requirement of symmetry removes ordering ambiguities other than those which would give rise to an unobservable additive constant in H_0. Thus

$$H_0 = \tfrac{1}{2}(\boldsymbol{\pi}^2 + \lambda \boldsymbol{L}^2)$$

in which the classical definition (14) of $\boldsymbol{\pi}$ must be replaced by its hermitean counterpart

$$\boldsymbol{\pi} = \boldsymbol{p} + \tfrac{1}{2}\lambda\{\boldsymbol{x}(\boldsymbol{x}\cdot\boldsymbol{p}) + (\boldsymbol{p}\cdot\boldsymbol{x})\boldsymbol{x}\}. \tag{14a}$$

4.1. *The Kepler problem*

It is easily verified that the quantum Hamiltonian (19) commutes with the hermitean version

$$R_i = -\tfrac{1}{2}(L_{ij}\pi_j - \pi_j L_{ji}) + \mu x_i/r \tag{16a}$$

of the Runge–Lenz vector (16). The Poisson brackets (17) are replaced by the commutators†

$$[R_i, R_j] = \mathrm{i}L_{ij}[-2H + \lambda\{2\boldsymbol{L}^2 + \tfrac{1}{4}(N-3)^2\}] \tag{17a}$$

and the length of $\boldsymbol{R}$ is now given by

$$\boldsymbol{R}^2 = \mu^2 + (2H - \lambda\boldsymbol{L}^2)\{\boldsymbol{L}^2 + \tfrac{1}{4}(N-1)^2\} - \lambda\boldsymbol{L}^2. \tag{18a}$$

Just as (17) and (18) imply the relation (26) between the classical Hamiltonian and the quadratic Casimir operator of the dynamical SO($N+1$) group, so also (17a) and (18a) imply a relation between the quantum Hamiltonian and the corresponding quantum Casimir operator. In a subsequent paper (Leemon 1979) it will be shown that this relation is

$$H = \tfrac{1}{2}\lambda C - \tfrac{1}{2}\mu^2[C + \tfrac{1}{4}(N+1)^2]^{-1}. \tag{26a}$$

In this paper, in § 5, this result will be proved for $N = 2$, where the algebra has a particularly simple structure.

4.2. *The isotropic oscillator*

Similarly, the quantum Hamiltonian

$$H = H_0 + \tfrac{1}{2}\omega^2 r^2$$

commutes with the hermitean symmetric tensor

$$S_{ij} = \tfrac{1}{2}(\pi_i\pi_j + \pi_j\pi_i) + \omega^2 x_i x_j. \tag{32a}$$

The components have the commutators

$$[S_{ij}, S_{kl}] = \mathrm{i}[(\omega^2 - \tfrac{1}{4}\lambda^2)(L_{ik}\delta_{jl} + L_{il}\delta_{jk} + L_{jk}\delta_{il} + L_{jl}\delta_{ik}) + \tfrac{1}{2}\lambda(L_{ik}S_{jl} + L_{il}S_{jk} + L_{jk}S_{il} + L_{jl}S_{ik})$$
$$+ \tfrac{1}{2}\lambda(S_{ik}L_{jl} + S_{il}L_{jk} + S_{jk}L_{il} + S_{jl}L_{ik})] \tag{33a}$$

and the scalars formed from them are

$$I_1 = S_{ii} = 2H - \lambda\boldsymbol{L}^2 \tag{34}$$

as before and

$$I_2 = S_{ij}S_{ji} - S_{ii}S_{jj} = -\omega^2\{2\boldsymbol{L}^2 + N(N-1)\} - \lambda\{2(N-1)H - (N+\tfrac{1}{2})\lambda\boldsymbol{L}^2\}. \tag{31a}$$

Again, just as (31), (33) and (34) lead to the relation (47) between the classical oscillator Hamiltonian and the quadratic Casimir operator of the dynamical SU(N) group, so also (31a), (33a) and (34) lead to the relation

$$H = \tfrac{1}{2}\lambda(C + \tfrac{1}{2}N) + [(\omega^2 + \tfrac{1}{4}\lambda^2)(C + \tfrac{1}{4}N^2)]^{1/2} \tag{47a}$$

† Natural units, with $\hbar = 1$, are used from now on.

between the quantum Hamiltonian and the corresponding quantum Casimir operator. This will be proved for $N = 2$ in the next section and for general N in the subsequent paper (Leemon 1979).

5. Energy levels of two dimensional systems

In two dimensions the angular momentum has only one component L_{12}, and both vectors and traceless symmetric tensors have only two components each, which may be chosen so as to be raising and lowering operators for the eigenvalues of L_{12}. These simplifying features make the solution of the algebraic problems associated with the quantum Kepler and oscillator systems relatively straightforward.

5.1. The Kepler problem

The commutators of the angular momentum with a vector operator

$$[L_{ij}, R_k] = \mathrm{i}(\delta_{ik}R_j - \delta_{jk}R_i)$$

reduce in two dimensions to

$$[L, R_\pm] = \pm R_\pm$$

where

$$L = L_{12}$$

$$R_\pm = R_1 \pm \mathrm{i}R_2.$$

These commutators generalise to

$$f(L)R_\pm = R_\pm f(L \pm 1). \tag{48}$$

Now (17*a*) and (18*a*) may be written

$$\tfrac{1}{2}(R_+R_- - R_-R_+) = L\{-2H + \lambda(2L^2 + \tfrac{1}{4})\}$$

$$\tfrac{1}{2}(R_+R_- + R_-R_+) = \mu^2 + (2H - \lambda L^2)(L^2 + \tfrac{1}{4}) - \lambda L^2,$$

so

$$R_+R_- = F(L - \tfrac{1}{2}), \qquad R_-R_+ = F(L + \tfrac{1}{2})$$

where

$$F(x) = \mu^2 + 2Hx^2 - \lambda x^2(x^2 - \tfrac{1}{4}). \tag{49}$$

Let the normalised Runge–Lenz vector be

$$\boldsymbol{M} = \tfrac{1}{2}(\boldsymbol{R}f(L, H) + f(L, H)\boldsymbol{R}).$$

Condition (22) requires that its components have the commutator

$$[M_+, M_-] = 2L. \tag{50}$$

But using (48) and (49) one finds

$$[M_+, M_-] = [\phi(L - \tfrac{1}{2})]^2 F(L - \tfrac{1}{2}) - [\phi(L + \tfrac{1}{2})]^2 F(L + \tfrac{1}{2}) \tag{51}$$

where

$$\phi(L + \tfrac{1}{2}) = \tfrac{1}{2}(f(L, H) + f(L + 1, H)).$$

Comparing (50) and (51), one gets a difference equation whose solution is

$$(\phi(x))^2 = (A(H) - x^2)/F(x),$$

where the arbitrary function $A(H)$ may be expressed in terms of the SO(3) Casimir operator:

$$C = L^2 + \boldsymbol{M}^2 = A(H) - \tfrac{1}{4}.$$

Thus the normalisation factor is given by

$$(\phi(x))^2 = \frac{C(H) + \frac{1}{4} - x^2}{\mu^2 + 2Hx^2 - \lambda x^2(x^2 - \frac{1}{4})}.$$

The function $C(H)$ is determined by the requirement that ϕ^2 be positive† to be such that the denominator contains the numerator as a factor:

$$\mu^2 + (2H - \lambda C)(C + \tfrac{1}{4}) = 0.$$

Thus the quantum counterpart of the classical relation (26) is

$$H = \tfrac{1}{2}\lambda C - \tfrac{1}{2}\mu^2(C + \tfrac{1}{4})^{-1}. \tag{52}$$

The well known irreducible representations of SO(3) now lead to the energy levels

$$E_n = \tfrac{1}{2}\lambda n(n+1) - \tfrac{1}{2}\mu^2(n + \tfrac{1}{2})^{-2} \tag{53}$$

where n is a non-negative integer.

Each level contains angular momentum eigenvalues

$$l = -n, -n+1, \ldots n. \tag{54}$$

5.2. *The isotropic oscillator*

The algebra of the two dimensional oscillator may be analysed in the same way. The commutators of the angular momentum with a symmetric tensor operator may be written

$$[L, S_\pm] = \pm 2S_\pm$$

where

$$S_\pm = \tfrac{1}{2}(S_{11} - S_{22}) \pm \mathrm{i}S_{12}.$$

More generally, for any $f(L)$,

$$f(L)S_\pm = S_\pm f(L \pm 2). \tag{55}$$

From (31*a*), (33*a*) and (34) one gets

$$\tfrac{1}{2}(S_+S_- - S_-S_+) = L[2(\omega^2 - \tfrac{1}{4}\lambda^2) + \lambda(2H - \lambda L^2)]$$

$$\tfrac{1}{2}(S_+S_- + S_-S_+) = \tfrac{1}{4}(2H - \lambda L^2)^2 + [\omega^2(L^2+1) + \lambda(H - \tfrac{5}{4}\lambda L^2)]$$

† That ϕ^2 must be non-negative follows from the requirement that $\boldsymbol{M}$ be hermitean. The stronger condition $\phi^2 > 0$, follows from the requirement that the relation between $\boldsymbol{R}$ and $\boldsymbol{M}$ be non-singular. The singularity which is to be avoided would otherwise occur when a matrix element of the relation involved eigenstates of L corresponding to its highest eigenvalue (given C), for which the numerator function $C + \frac{1}{4} - x^2$ vanishes. These eigenstates are the quantum mechanical counterparts of the circular orbits which were used in § 3 to obtain the classical relation between H and C.

so

$$S_+S_- = G(L-1), \qquad S_-S_+ = G(L+1)$$

where

$$G(x) = H^2 - (\omega^2 + \tfrac{1}{4}\lambda^2 + \lambda H)x^2 + \tfrac{1}{4}\lambda^2 x^4. \tag{56}$$

Let the normalised traceless tensor which is to be constructed have components

$$N_\pm = \tfrac{1}{2}\{S_\pm f(L, H) + f(L, H)S_\pm\}.$$

Condition (36) determines their commutator to be

$$[N_+, N_-] = 4L. \tag{57}$$

But from (55) and (56) one gets

$$[N_+, N_-] = (\phi(L-1))^2 G(L-1) - (\phi(L+1))^2 G(L+1) \tag{58}$$

where

$$\phi(L+1) = \tfrac{1}{2}\{f(L, H) + f(L+2, H)\}.$$

Comparing (57) and (58), one obtains a difference equation whose solution is

$$(\phi(x))^2 = (A(H) - x^2)/G(x),$$

where now the arbitrary function $A(H)$ is related to the Casimir operator of SU(2):

$$\begin{aligned} C &= \tfrac{1}{2}(N_+N_- + N_-N_+) + L^2 \\ &= A(H) - 1. \end{aligned}$$

Thus the normalisation factor is given by

$$(\phi(x))^2 = \frac{C + 1 - x^2}{H^2 - (\omega^2 + \frac{1}{4}\lambda^2 + \lambda H)x^2 + \frac{1}{4}\lambda^2 x^4}.$$

Positivity again requires the denominator to contain the numerator as a factor, whence

$$H = \tfrac{1}{2}\lambda(C+1) + [(\omega^2 + \tfrac{1}{4}\lambda^2)(C+1)]^{1/2}. \tag{59}$$

Since the eigenvalues of C for SU(2) are $n(n+2)$, where the integer n is non-negative, the energy levels are

$$E_n = \tfrac{1}{2}\lambda(n+1)^2 + (n+1)\omega' \tag{60}$$

where

$$\omega' = (\omega^2 + \tfrac{1}{4}\lambda^2)^{1/2}. \tag{61}$$

Each level contains angular momentum eigenvalues

$$l = -n, -n+2, \ldots n.$$

6. Wavefunctions

The simultaneous eigenfunctions of H and $\boldsymbol{L}^2$ for these systems may be found, as in Euclidean space, by using the vector $\boldsymbol{R}_i$ or the tensor S_{ij} in a Schrödinger representation to generate recurrence relations among the radial functions. The angular eigenfunctions are, of course, hyperspherical harmonics.

For example, since $\boldsymbol{R}$ is a vector operator under SO(N) which is orthogonal to the angular momentum and commutes with H_{Kepler}, its non-vanishing matrix elements are $\langle n, l\pm 1|\boldsymbol{R}|n, l\rangle$. So it may be used to relate the radial functions for fixed n and adjacent values of l.

In particular, for $N=2$, the algebra of § 5.1 provides an explicit relation between $\boldsymbol{R}$ and the normalised vector $\boldsymbol{M}$:

$$R_+ = M_+(\phi(L+\tfrac{1}{2}))^{-1} = M_+[\lambda(L+\tfrac{1}{2})^2+\mu^2/(C+\tfrac{1}{4})]^{1/2}$$
$$R_- = M_-(\phi(L-\tfrac{1}{2}))^{-1} = M_-[\lambda(L-\tfrac{1}{2})+\mu^2/(C+\tfrac{1}{4})]^{1/2}.$$

Since $M_\pm$ are the raising and lowering operators for L in the algebra of SO(3), one obtains (with the standard phase convention)

$$R_\pm|n, l\rangle = \{(n\mp l)(n\pm l+1)[\lambda(l\pm\tfrac{1}{2})^2+\mu^2/(n+\tfrac{1}{2})^2]\}^{1/2}|n, l\pm 1\rangle. \tag{62}$$

Now the definitions (14a) and (16a) lead to the following expressions for the operators $R_\pm$ in a Schrödinger representation where the angular variables χ (colatitude) and ϕ (longitude) are chosen as co-ordinates:

$$R_\pm = \lambda^{1/2}\exp(\pm\mathrm{i}\phi)\left[\left(\frac{\partial}{\partial\chi}\pm\mathrm{i}\cot\chi\frac{\partial}{\partial\phi}\right)\left(\mp\mathrm{i}\frac{\partial}{\partial\phi}+\frac{1}{2}\right)+\alpha\right] \tag{63}$$

where

$$\alpha = \mu/\lambda^{1/2}. \tag{64}$$

By combining (62) and (63) one gets the recurrence relations for the wavefunctions $X_{nl}(\chi)\exp(\mathrm{i}l\phi)$:

$$[(\pm l+\tfrac{1}{2})(\mathrm{d}/\mathrm{d}x\mp l\cot\chi)+\alpha]X_{nl} = [(n\mp l)(n\pm l+1)\{(\chi l\pm\tfrac{1}{2})^2+\alpha^2/(n+\tfrac{1}{2})^2\}]^{1/2}X_{n,l\pm 1}. \tag{65}$$

In particular, the condition $R_+|n, n\rangle = 0$ provides the differential equation

$$[(n+\tfrac{1}{2})(\mathrm{d}/\mathrm{d}\chi - n\cot\chi)+\alpha]X_{nn} = 0 \tag{66}$$

satisfied by the eigenfunction corresponding to maximum l. Its solution is

$$X_{nn} = a_n(\sin\chi)^n\exp[-\alpha\chi/(n+\tfrac{1}{2})] \tag{67}$$

where a_n is determined by the normalisation

$$2\pi\int_0^\pi |X_{nn}|^2\sin\chi\,\mathrm{d}\chi = 1. \tag{68}$$

The recurrence relations (65) now determine the other eigenfunctions. An expression for X_{nl} in closed form has been found by Leemon (1979) for general dimension N.

A similar construction of the Schrödinger wavefunctions for the two-dimensional oscillator may be carried out by using the relations

$$S_\pm = \tfrac{1}{2}\lambda N_\pm[\{2\gamma+(C+1)^{1/2}\}^2-(L\pm 1)^2]^{1/2} \tag{69}$$

where

$$\gamma = \lambda^{-1}(\omega^2+\tfrac{1}{4}\lambda^2)^{1/2}, \tag{70}$$

between components of the Fradkin tensor S_{ij} and the normalised tensor N_{ij}, which follow from the algebra of § 5.2. On expressing $S_\pm$ as Schrödinger operators and recalling that $\frac{1}{2}N_\pm$ are raising and lowering operators for $\frac{1}{2}L$ in the algebra of SU(2), one obtains recurrence relations between the wavefunctions corresponding to states $|nl\rangle$ and $|n, l\pm 2\rangle$. Again, Leemon (1979) has constructed explicit expressions for these wavefunctions, for any value of the dimension N.

7. Discussion

The two dynamical systems, the algebra of whose constants of motion has been analysed in the previous sections, share many of the features of their better known Euclidean counterparts. For example, the constants of motion R_i and S_{ij} are quadratic functions of the generators π_i and L_{ij} of the geometric SO($N+1$) symmetry, with co-ordinate dependence only in the terms of degree zero. This is indeed what one would expect on the basis of the general analysis of dynamical symmetries in Euclidean geometry which was performed by Makarov *et al* (1967). It is also to be expected that, like their Euclidean counterparts, the Hamilton–Jacobi and Schrödinger equations of the two systems will be separable in certain families of elliptic co-ordinate systems on a sphere†.

In one respect the systems are simpler than their Euclidean counterparts: each of them contains two parameters, a curvature and a force constant, and as ,the force constants tends to zero the structure of the symmetry group remains the same. Thus it is, in principle, straightforward to relate the wavefunctions of the Coulomb system to those of the free particle on a sphere by a unitary transformation, whereas the corresponding Euclidean problem is complicated by the continuous spectra of the Hamiltonians. It should be noted however that, as was remarked in § 2, the limit $\omega \to 0$ of the oscillator is not the free particle on a sphere but a particle which moves freely apart from a barrier at the equator. The relation between the symmetry group SU(N) of this system and the SO($N+1$) symmetry of the completely free particle has been previously discussed by Ravenhall *et al* (1967) for $N=3$.

Finally, it is worth remarking that the quantum oscillator system has the symmetry SU(N) even for negative values of ω^2, provided that

$$\omega^2 + \tfrac{1}{4}\hbar^2\lambda^2 = \omega'^2 \geqslant 0.$$

When $\omega' = 0$ a simplification of the algebra occurs. In particular, for $N=2$ the relation (69) which specifies the normalisation of the Fradkin tensor has the simpler form

$$S_\pm = \tfrac{1}{2}\lambda N_\pm[(C+1)^2 - (L\pm 1)^2]^{1/2}, \tag{71}$$

from which one obtains the action of $S_\pm$ as raising and lowering operators:

$$S_\pm|n, l\rangle = \tfrac{1}{2}\lambda(n\mp l)(n\pm l+2)|n, l\pm 2\rangle. \tag{72}$$

If the co-ordinates (q_1, q_2) of the orthogonal projection onto the tangent plane at the origin are used and the scalar product of two Schrödinger wavefunctions is taken to be $\int \psi_1^*(q)\psi_2(q)\, \mathrm{d}^2q$, where the domain of integration is the circular disc into which the sphere is projected, the Schrödinger equation of the system has the form

$$\left[\frac{1}{4}\left(q_i\frac{\partial}{\partial q_i} + \frac{\partial}{\partial q_i}q_i\right)^2 - \lambda^{-1}\frac{\partial^2}{\partial q_i^2}\right]\psi = (n+1)^2\psi. \tag{73}$$

† A full analysis of this property of the three-dimensional Euclidean Kepler problem is to be found in Kalnins *et al* 1976. For a discussion of the three-dimensional Euclidean oscillator see Boyer *et al* (1975).

For $\lambda = 1$ this is the partial differential equation satisfied by the Zernike polynomials of degree n (Zernike 1934)†. Equation (72) then provides relations between Zernike polynomials with angular dependence $\exp(\mathrm{i}l\phi)$ and $\exp \mathrm{i}(l \pm 2)\phi$. For $N > 2$, the eigenvalue in the Schrödinger equation (73) becomes $(n + \frac{1}{2}N)^2$: its solutions are polynomials which are generalisations to the interior of the N-sphere, $\lambda q_i q_i \leqslant 1$, of the Zernike circle polynomials.

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† These polynomials have extensive application in optics. For a review of their properties see Tango (1977).