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On harmonic oscillators on the two-dimensional sphere S^2 and the hyperbolic plane H^2

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Two Harmonic Oscillators (isotropic and nonisotropic 2:1) are studied on the two-dimensional sphere S^2 and the hyperbolic plane H^2. Both systems are integrable and super-integrable with constants of motion quadratic in the momenta. These properties are shown to derive from a complex factorization for the constants of motion, which holds for arbitrary values of the curvature κ, and the dynamics of the Euclidean harmonic 1:1 and 2:1 oscillators is directly recovered for $\kappa=0$. The harmonic oscillators on either the standard unit sphere (radius $R=1$) or the unit Lobachewski plane ("radius" $R=1$) appear as the particular values of the κ-dependent potentials for the values $\kappa=1$ and $\kappa=-1$. Finally a particular potential is proposed for representing the general spherical (hyperbolic) $n:1$ anisotropic harmonic oscillator on a two-dimensional manifold of constant curvature. © *2002 American Institute of Physics.* [DOI: 10.1063/1.1423402]

I. INTRODUCTION

Dynamics on a nonflat configuration space Q remains as a very partially studied subject, even in the simplest two-dimensional constant curvature cases where Q is either the sphere S^2 or the hyperbolic (Lobachewski) plane H^2 (see Refs. 1, 2, 3 for the case of spherical central potentials). In fact, there exist some noncentral but rather simple problems, e.g., the nonisotropic harmonic oscillators, still awaiting to be studied in manifolds of nonzero curvature.

It is well known that integrable systems, in the classical sense of Arnold–Liouville, must have as many independent constants of motion in involution as degrees of freedom. On the other hand, a system is called super-integrable if it is integrable and, in addition, possesses more independent first integrals than degrees of freedom. It is known that these additional first integrals give rise to a higher degree of regularity in the phase space (e.g., the existence of periodic orbits) since the trajectories are restricted to submanifolds with less than n dimensions. In particular, if a system with n degrees of freedom possesses $2n-1$ independent first integrals, then it is called maximally super-integrable. The Kepler problem, the harmonic oscillator and the Calogero–Moser system are some cases of this very particular class of systems (for other super-integrable systems see Refs. 4–15 and references therein).

In a recent paper[16] the existence of super-integrable systems on the two-dimensional sphere S^2 and on the hyperbolic plane H^2 was analyzed. The study was focused on the quadratic super-integrability, that is, on the existence of systems that, besides the energy, admit two further independent constants of motion quadratic in the velocities (quadratic super-integrability is related with super-separability in the sense of Ref. 14). Some of the spherical (hyperbolic) potentials obtained by this approach were already known, but some others were apparently new. Particularly

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interesting was the existence of a potential that seems to represent the spherical (hyperbolic) version of the Euclidean nonisotropic 2:1 oscillator $4x^2+y^2$ or x^2+4y^2 (for other Euclidean nonisotropic oscillators the additional constant of motion is not quadratic, but higher order in momenta). We recall that the spherical version of the central oscillator was studied by Higgs in Ref. 2 and is known as the Higgs oscillator.

The main objective of this article is to develop a deeper analysis of these two spherical (hyperbolic) oscillators: the central Higgs oscillator and noncentral 2:1 oscillator. We will show, as a basic point, that these two systems are endowed with the same fundamental properties as the standard Euclidean one. This means that those properties to be considered as fundamental (e.g., the existence of constants of motion, super-integrability, complex factorization) are preserved under non-Euclidean deformations. Conversely, the Euclidean Oscillator can be considered as a very particular case (the flat limit) of the general ''curved'' Harmonic Oscillator. We will carry out this approach by considering the curvature as a parameter.

The article is organized as follows: In Sec. II we discuss the fundamental properties of the Euclidean Harmonic Oscillator related with the existence of super-integrability. We are interested in obtaining non-Euclidean deformations of these Euclidean properties. Section III is devoted to the geometry of the Riemannian two-dimensional (2-D) manifolds of constant curvature. We first discuss some properties of the curvilinear systems of coordinates and then we introduce a formalism with the curvature κ as a parameter (this formalism was already used in Ref. 16). This is made so that the Euclidean geometry is directly recovered for $\kappa=0$.

In Secs. IV and V we study the integrability and the super-integrability of both oscillators (isotropic central oscillator and non-isotroipc 2:1 oscillator) in the the sphere S^2 and in the hyperbolic plane H^2 with curvature κ (these two sections have been written in such a way that can be read independently). First, in Sec. IV, the study is presented in ''geodesic polar'' coordinates, and then, in Sec. V, we will obtain the results by using ''geodesic parallel'' coordinates. The dynamics in the Euclidean plane (studied in Sec. II) appears as a very particular case. The harmonic oscillators on either the standard unit sphere (radius $R=1$) or the unit Lobachewski plane (''radius'' $R=1$) correspond to the values $\kappa=1$ and $\kappa=-1$. Finally, in Sec. VI we provide a discussion and an outlook to the results obtained.

II. SUPER-INTEGRABILITY OF THE EUCLIDEAN HARMONIC OSCILLATOR

The two-dimensional harmonic oscillator,

$$L_{\mathrm{HO}}=(\tfrac{1}{2})(v_x^2+v_y^2)-(\tfrac{1}{2})(\omega_1^2x^2+\omega_2^2y^2),$$

is a trivially integrable system, since it is a direct sum of one-degree of freedom systems and, therefore, it has the two one-degree of freedom energies, $I_1=E_x$ and $I_2=E_y$, as involutive integrals. If the oscillator is isotropic then it has the angular momentum as an additional integral of motion. If the oscillator is nonisotroipc then the angular momentum is not preserved but in the very particular case in which the quotient of the two frequencies is rational the system has a third additional nonlinear integral.

In geometric terms the phase space is foliated by tori and every integral curve is a curve with constant slope on a torus. The slope of the curve is determined by the ratio ω_2/ω_1. Thus, if this ratio is irrational the corresponding curve will be dense on the torus.[17,18] If this ratio is rational then the orbit becomes closed and the motion will be periodic.

The super-integrability of the rational case, $\omega_1=n_1\omega_0$, $\omega_2=n_2\omega_0$, with integers n_1,n_2, can be approached by using a complex formalism.[18,19] The following proposition states the existence of the additional constant of motion and give a method for obtaining it explicitly.

Proposition 1: Let J_1, J_2, be the following two functions:

$$J_1=v_x+\mathrm{i}\,n_1\omega_0x,\quad J_2=v_y+\mathrm{i}\,n_2\omega_0y.$$

Then the complex function J_{12} defined as

$$J_{12}=J_1^{n_2}(J_2^*)^{n_1}$$

is a constant of motion.

Proof: The time-evolution of the functions J_1, J_2, is given by

$$\frac{d}{dt}J_1=-\mathrm{i}\,n_1\omega_0 J_1 \quad \frac{d}{dt}J_2=-\mathrm{i}\,n_2\omega_0 J_2 .$$

Hence we have

$$\frac{d}{dt}J_{12}=n_2J_1^{(n_2-1)}(J_2^*)^{n_1}\dot{J}_1+n_1J_1^{n_2}(J_2^*)^{(n_1-1)}\dot{J}_2^*=J_1^{n_2}(J_2^*)^{n_1}(\mathrm{i}\,\omega_0)(n_2n_1-n_1n_2)=0. \tag{1}$$

Therefore J_{12} is a constant of motion. Notice that J_{12}, which can be considered as coupling the two degrees of freedom, depends on the relation between ω_2 and ω_1. As stated above, J_{12} is well defined as a constant of motion only if the quotient ω_2/ω_1 is rational.

Since J_{12} is a complex function it determines two different real first integrals,

$$I_3=\mathrm{Im}(J_{12}), \quad I_4=\mathrm{Re}(J_{12}),$$

which are polynomials in the velocities (momenta) of degree n_1+n_2-1 and n_1+n_2, respectively. Only one of these two functions must be considered as fundamental, because I_1,I_2,I_3,I_4 are functionally dependent (I_3 is independent of I_1 and I_2, but I_4 is a dependent function of I_1, I_2 and I_3).

Next we give the expressions of I_3 and I_4 for the first two rational cases.

(i) Isotropic case $\omega_1=\omega_2=\omega_0$:

$$I_4=\mathrm{Re}(J_{12})=v_xv_y+\omega_0^2xy, \quad I_3=\mathrm{Im}(J_{12})=\omega_0(xv_y-yv_x); \tag{2}$$

$\mathrm{Im}(J_{12})$ is just the angular momentum, and $\mathrm{Re}(J_{12})$ is the nondiagonal component of the Fradkin tensor.[20]

(ii) The nonisotroipc case with $\omega_1=2\omega_0$, $\omega_2=\omega_0$:

$$\begin{aligned} I_4&=\mathrm{Re}(J_{12})=v_xv_y^2+\omega_0^2(4xv_y-yv_x)y,\\ I_3&=\mathrm{Im}(J_{12})=(xv_y-yv_x)v_y-\omega_0^2xy^2. \end{aligned} \tag{3}$$

Summarizing: Proposition 1 states two properties: (i) super-integrability of the rational case, and (ii) complex factorization of the additional constant of motion, that can be considered as the two fundamental features characterizing the two-dimensional harmonic oscillator. In fact, they can even be considered as defining properties. That is, any dynamical system defined in a non-Euclidean space must satisfy the appropriate non-Euclidean versions of points (i) and (ii) to be in fact considered as an (non-Euclidean) Harmonic Oscillator.

III. GEOMETRY AND DYNAMICS ON THE SPHERE S^2 AND THE HYPERBOLIC PLANE H^2

A two-dimensional manifold M can be described by using different coordinate systems. If we consider it as an imbedded submanifold of $\mathbb{R}^3$, then the points of M can be characterized by the

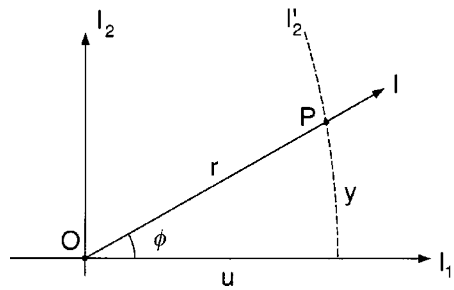

FIG. 1. Polar (r,ϕ) and parallel (u,y) coordinates based on the oriented geodesic l_1 and reference point O. All these coordinates are lengths or angles measured in the intrinsic metric of the space of constant curvature. The figure follows the pattern of a stereographic projection of the sphere from the South Pole, with O at the North Pole, but the geometrical meaning of these coordinates holds for any value of the curvature.

three external coordinates (x,y,z) plus an additional constraint. Nevertheless, in differential geometric terms, a more appropriate approach is to develop the study by using two-dimensional systems of coordinates adapted to M.

On any general two-dimensional Riemannian space (not neccesarily of constant curvature) there are two distinguished types of local coordinate systems: ''geodesic parallel'' and ''geodesic polar'' coordinates. They reduce to the familiar Cartesian and polar coordinates on the Euclidean plane (see Refs. 16 and 21) and both are based on an origin point O and an oriented geodesic l_1 through O (Fig. 1).

For any point P in some suitable neigboorhood of O, there is a unique geodesic l joining O and P. The (geodesic) polar coordinates (r,ϕ) of P, relative to the origin O and the positive geodesic ray of l_1, are the (positive) distance r between O and P measured along l, and the angle ϕ between l and the positive ray l_1, measured around O. These coordinates are defined in a neigborhood of O not extending beyond the cut locus of O; polar coordinates are singular at O, and ϕ is discontinuous on the positive ray of l_1.

Now, consider the geodesic l_2' through P and orthogonal to l_1 and let P_1 be the intersection point of l_2' and l_1 nearest to P. The (geodesic) parallel coordinates (u,y) of P, relative to the origin O and base geodesic l_1, are defined as the distance u between O and P_1, measured along l_1, and the distance y, between P_1 and P, measured along l_2'. Again these coordinates will be regular and without singularities in some suitable strip centered in l_1. If instead of l_1 another line is taken as the base, we obtain another system of geodesic parallel coordinates. Figure 2 also display the particular case with base l_2, orthogonal to l_1 through O; these second sets of parallel coordinates will be denoted (v,x).

These systems are suitable for most general purposes, because the coordinates (r,ϕ), (u,y) and (v,x) have a *direct* geometric significance, as distances and angles measured in the intrinsic metric of the surface. Closed expressions are usually only possible for spaces of *constant curva-*

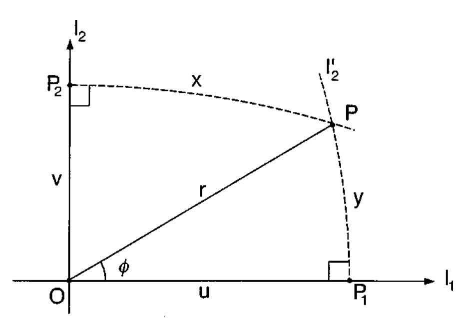

FIG. 2. The three coordinate systems (r,ϕ), (u,y) and (v,x) of a point P. Relationships among these coordinates are discussed in the text for any curvature value κ.

ture. In the constant *positive* curvature case, i.e., the sphere, the geodesics are great circles, and relations among distances and angles are the subject of spherical geometry. Polar coordinates on the sphere are singular at the origin (pole) O and also at its antipodal point (the cut locus of O). Parallel coordinates are singular in the two poles of the base geodesic. While in the Euclidean plane a line orthogonal to both l_1 and l_2 do not exist (nor does it exist in the hyperbolic plane), there is such a line l_3 for the sphere (the polar of the point O); so we have here a third set of parallel coordinates. These three sets are based on three geodesics mutually orthogonal by pairs and the third system with base l_3 is essentially equivalent to the polar coordinates whose center is the pole of l_3.

The notation has been chosen to emphasize the similarities with the Euclidean case. For a point P, r is the distance measured in either S^2 or H^2 (with curvature κ) from P to the origin point O, and ϕ determines the orientation of the line OP through O. On the other side, x, y are the geodesic distances from P to the two "coordinate axes" l_1,l_2; there are other two quantities, u, v which are distances, measured along l_1,l_2, between O and the orthogonal projections of P on l_1,l_2. In the Euclidean case, we have the identities $x=u, y=v$, but once we deal with nonzero curvature these equalities are no longer true; recall that y is the distance from P to the "x" coordinate axis, but u *is not* the distance from P to the "y" coordinate axis. Both polar (r,ϕ) and the two systems of parallel coordinates (u,y) and (v,x) are always *orthogonal*; however, the coordinate system (x,y) made up of the distances to the two coordinate axes is orthogonal in the Euclidean plane, but *not* in S^2 nor in H^2.

For a sphere of radius R the "geographic" coordinates (θ,ϕ) (where θ is the latitude and ϕ the longitude) are closely related to both polar and parallel type coordinate systems: $(R(\pi/2-\theta),\phi)$ are *polar* coordinates with an origin in the North pole, while $(R\phi,R\theta)$ are *parallel* coordinates with the equator as the base line. This equivalence does not exist in the Euclidean and hyperbolic case, where polar and parallel coordinates are very different, so there are reasons to keep their consideration separate, even for the sphere, in the context in which we are working. The fundamental properties of the harmonic oscillator on manifolds of constant curvature will be more clearly seen this way.

The metric of the sphere of curvature $\kappa=1/R^2$ is given in parallel and polar coordinates by

$$ds^2=\cos^2(y/R)\,du^2+dy^2, \quad ds^2=dr^2+R^2\sin^2(r/R)\,d\phi^2,$$

reducing to du^2+dy^2 and $dr^2+r^2\,d\phi^2$ when $R\to\infty$. It is possible to write these expressions in a form which holds simultaneously for the sphere, the Euclidean plane and the hyperbolic plane, by introducing the following "tagged" trigonometric functions:[22]

$$\mathrm{C}_\kappa(x)=\begin{cases}\cos\sqrt{\kappa}\,x, & \text{if } \kappa>0,\\ 1, & \text{if } \kappa=0,\\ \cosh\sqrt{-\kappa}\,x, & \text{if } \kappa<0,\end{cases} \qquad \mathrm{S}_\kappa(x)=\begin{cases}\dfrac{1}{\sqrt{\kappa}}\sin\sqrt{\kappa}\,x, & \text{if } \kappa>0,\\ x, & \text{if } \kappa=0,\\ \dfrac{1}{\sqrt{-\kappa}}\sinh\sqrt{-\kappa}\,x, & \text{if } \kappa<0,\end{cases}$$

and

$$\mathrm{T}_\kappa(x)=\frac{\mathrm{S}_\kappa(x)}{\mathrm{C}_\kappa(x)}.$$

When $\kappa=1$ the three "tagged" functions are the ordinary trigonometrical functions, i.e., $\mathrm{S}_1(x)=\sin x$, $\mathrm{C}_1(x)=\cos x$, $\mathrm{T}_1(x)=\tan x$. For $\kappa=0$ one gets the "parabolic" sine $\mathrm{S}_0(x)=x$, cosine $\mathrm{C}_0(x)=1$, and tangent $\mathrm{T}_0(x)=x$. For $\kappa=-1$, these functions are the hyperbolic cosine, sine, and tangent. Therefore, in the flat case $\kappa=0$ all $\mathrm{C}_\kappa(x)$ are replaced by 1, while all $\mathrm{S}_\kappa(x)$, $\mathrm{T}_\kappa(x)$ are replaced by its variable x; this suggests that in the curved case, $\mathrm{C}_\kappa(x)$ should be looked at as a kind of "curved" deformation of the function 1, while both $\mathrm{S}_\kappa(x)$ and $\mathrm{T}_\kappa(x)$ are two kinds of deformations of the linear function x.

The idea is to obtain relations between the different coordinates (u,y), (v,x), (r,ϕ), in a way that holds regardless of the value of κ. In the cases $\kappa>0$ or $\kappa<0$ they reduce to formulas of spherical or hyperbolic trigonometry, while for $\kappa=0$ they are well-known Euclidean relations.

In any rectangular triangle, as P_1PO in Fig. 1, the three sides r,u,y, and the angle ϕ at O are related by the following equations:

$$\mathrm{S}_\kappa(y)=\mathrm{S}_\kappa(r)\sin\phi,\quad \mathrm{C}_\kappa(r)=\mathrm{C}_\kappa(u)\mathrm{C}_\kappa(y),$$

$$\mathrm{T}_\kappa(u)=\mathrm{T}_\kappa(r)\cos\phi,\quad \mathrm{T}_\kappa(y)=\mathrm{S}_\kappa(u)\tan\phi,$$

and similar equations for the triangle P_2PO (sides r,v,x, angle $\pi/2-\phi$ at O):

$$\mathrm{S}_\kappa(x)=\mathrm{S}_\kappa(r)\cos\phi,\quad \mathrm{C}_\kappa(r)=\mathrm{C}_\kappa(v)\mathrm{C}_\kappa(x),$$

$$\mathrm{T}_\kappa(v)=\mathrm{T}_\kappa(r)\sin\phi,\quad \mathrm{T}_\kappa(x)\tan\phi=\mathrm{S}_\kappa(v).$$

Starting from these equations we get many relations with a rather symmetrical appearance in the pairs x,y and u,v. The change from *polar* to *parallel* coordinates, in any constant curvature plane, can be read from these equations.

The harmonic oscillator potential will be closely related with the particular function $\mathrm{T}_\kappa^2(r)$ that can be presented, in terms of (y,u) or (x,v), in several alternative ways:

$$\mathrm{T}_\kappa^2(r)=\mathrm{T}_\kappa^2(u)+\frac{\mathrm{T}_\kappa^2(y)}{\mathrm{C}_\kappa^2(u)}=\frac{\mathrm{T}_\kappa^2(u)}{\mathrm{C}_\kappa^2(y)}+\mathrm{T}_\kappa^2(y)=\mathrm{T}_\kappa^2(v)+\frac{\mathrm{T}_\kappa^2(x)}{\mathrm{C}_\kappa^2(v)}=\frac{\mathrm{T}_\kappa^2(v)}{\mathrm{C}_\kappa^2(x)}+\mathrm{T}_\kappa^2(x). \tag{4}$$

These expressions can be considered as different κ-deformed versions of the Pythagorean theorem. In fact for the $\kappa=0$ Euclidean plane we have $u=x$, $v=y$, and they reduce to $r^2=x^2+y^2$.

IV. THE 1:1 AND 2:1 HARMONIC OSCILLATORS ON A 2-D SPACE OF CONSTANT CURVATURE I: POLAR COORDINATES

In the Euclidean plane, central potentials are better discussed in polar coordinates. This is also true for $\kappa\neq 0$. Nevertheless we know that nonisotroipc Euclidean oscillators are better presented in Cartesian coordinates. As we deal with both types when the configuration space is assumed to be a space of constant curvature κ, our idea is to develop both approaches. In this section the study is presented in geodesic polar coordinates, and in the next section we will make use of parallel coordinates (we begin with polar coordinated because it is the usual way of presenting the Higgs oscillator). In any case the properties we are looking for are intrinsic properties of the harmonic oscillator considered as a dynamical system, that is, as a vector field defined on the tangent bundle of a particular manifold (sphere S^2 or hyperbolic plane H^2). So all the issues we are interested in, such as integrability, super-integrability, or complex factorization, will prove to be reached by both approaches. Nevertheless we will find some differences between both approaches; this will be important with a view to facilitate further generalizations to other $n\!:\!1$ oscillators.

The differential element of distance on a manifold Q of constant curvature (S^2 if $\kappa>0$, the Euclidean plane if $\kappa=0$, or H^2 if $\kappa<0$) becomes, when written in "geodesic polar" coordinates:

$$ds^2=dr^2+\mathrm{S}_\kappa^2(r)\,d\phi^2.$$

Thus a general standard Lagrangian has the following form:

$$L(\kappa)=(\tfrac{1}{2})(v_r^2+\mathrm{S}_\kappa^2(r)v_\phi^2)-U(r,\phi,\kappa),$$

in such a way that for $\kappa=0$ we recover the standard Euclidean system,

$$\lim_{\kappa\to 0} L(\kappa)=(\tfrac{1}{2})(v_r^2+r^2v_\phi^2)-V(r,\phi), \quad V(r,\phi)=U(r,\phi,0).$$

The systems we are concerned with are systems endowed with quadratic integrals of motion depending on the curvature κ as a parameter, but we will first recall, as a previous step, the properties of the linear constants, which arise from exact Noether symmetries. An exact Noether symmetry is a complete vector field Y defined on the configuration space Q such that its natural lift Y^t to phase space TQ is an exact symmetry of the Lagrangian, that is, $Y^t(L)=0$. Then, if we denote by θ_L the Cartan one-form,

$$\theta_{\mathrm{L}}=\left(\frac{\partial L}{\partial v_r}\right)dr+\left(\frac{\partial L}{\partial v_\phi}\right)d\phi,$$

the function I defined as $I=i(Y^t)\theta_L$ is a constant of motion. The important point is that if L is a natural Lagrangians of mechanical type (Riemannian metric minus a potential) then the constant I is linear function in the velocities, and the vector field Y must necessarily be symmetry of the kinetic term (isometry of the metric) and symmetry of the potential.

In this particular spherical (hyperbolic) case the kinetic term is endowed with the following three symmetries:

$$Y_{P_1}(\kappa)=(\cos\phi)\frac{\partial}{\partial r}-\left(\frac{\mathrm{C}_\kappa(r)}{\mathrm{S}_\kappa(r)}\sin\phi\right)\frac{\partial}{\partial\phi},$$

$$Y_{P_2}(\kappa)=(\sin\phi)\frac{\partial}{\partial r}+\left(\frac{\mathrm{C}_\kappa(r)}{\mathrm{S}_\kappa(r)}\cos\phi\right)\frac{\partial}{\partial\phi},$$

$$Y_J(\kappa)=\frac{\partial}{\partial\phi}.$$

These three vector fields generate a Lie algebra,

$$[Y_{P_1},Y_{P_2}]=-\kappa Y_J, \quad [Y_{P_1},Y_J]=-Y_{P_2}, \quad [Y_{P_2},Y_J]=Y_{P_1},$$

isomorphic to the Lie algebra of isometries of the spherical (Euclidean, hyperbolic) space; only if $\kappa=0$ (Euclidean plane) Y_{P_1} and Y_{P_2} conmute.

Constants of motion linear in the velocities only appear for some specific potentials. In particular, we have three cases.

(i) If the potential U is of the form $U=U(z(r,\phi))$, $z(r,\phi)=\mathrm{S}_\kappa(r)\sin\phi$, then

$$P_1(\kappa)=i(Y^t_{P_1}(\kappa))\theta_L=(\cos\phi)v_r-(\mathrm{C}_\kappa(r)\mathrm{S}_\kappa(r)\sin\phi)v_\phi$$

is a constant of motion.

(ii) If the potential U is of the form $U=U(z'(r,\phi))$, $z'(r,\phi)=\mathrm{S}_\kappa(r)\cos\phi$, then

$$P_2(\kappa)=i(Y^t_{P_2}(\kappa))\theta_L=(\sin\phi)v_r+(\mathrm{C}_\kappa(r)\mathrm{S}_\kappa(r)\cos\phi)v_\phi$$

is a constant of motion.

(iii) If the potential U depends only on r, i.e., $U=U(r)$; then

$$J(\kappa)=i(Y^t_J(\kappa))\theta_L=\mathrm{S}^2_\kappa(r)v_\phi$$

is a constant of motion.

Notice that, in geometric terms, the two functions z and z' are in fact the "tagged" sines $z = S_k(y)$ and $z' = S_k(x)$. Concerning (iii) it represents the Kepler area law which in this form holds for any κ.

The most general linear constant of motion turns out to be a linear combination of $P_1(\kappa)$, $P_2(\kappa)$, $J(\kappa)$, with constant coefficients:

$$I_{11} = a_1 P_1(\kappa) + a_2 P_2(\kappa) + cJ(\kappa).$$

Suppose now that L has a constant of the motion $I = I(r, \phi, v_r, v_\phi)$ which is quadratic in the velocities

$$I = I_{22} + I_{20}(r, \phi, \kappa), \quad I_{22} = a v_r^2 + 2b v_r v_\phi + c v_\phi^2,$$

where a, b, and c, are functions of r and ϕ (κ-dependent). Then the three functions a, b, and c must take the form

$$a = a_0 \cos^2\phi + c_0 \sin^2\phi + b_0 \sin\phi\cos\phi,$$

$$b = (\tfrac{1}{2}) \mathrm{S}_\kappa(r)\mathrm{C}_\kappa(r)[(c_0 - a_0)\sin 2\phi + b_0 \cos 2\phi] + (\tfrac{1}{2})\mathrm{S}_\kappa^2(r)(-a_1 \cos\phi + c_1 \sin\phi),$$

$$c = \mathrm{S}_\kappa^2(r)\mathrm{C}_\kappa^2(r)(a_0 \sin^2\phi + c_0 \cos^2\phi - b_0 \sin\phi\cos\phi) + \mathrm{S}_\kappa^3(r)\mathrm{C}_\kappa(r)(c_1 \cos\phi + a_1 \sin\phi) + a_2 \mathrm{S}_\kappa^4(r),$$

where $a_0,\ b_0,\ c_0;\ a_1,\ c_1;\ a_2$ are real parameters. The most general form for I_{22} turns out to be a linear combination of binary products of linear constants:

$$I_{22}(\kappa) = a_0 P_1^2(\kappa) + b_0 P_1(\kappa)P_2(\kappa) + c_0 P_2^2(\kappa) + a_1 P_2(\kappa)J(\kappa) + c_1 P_1(\kappa)J(\kappa) + a_2 J^2(\kappa).$$

In the flat $\kappa = 0$ limit, we have $\mathrm{S}_\kappa(r) \to r$ and $\mathrm{C}_\kappa(r) \to 1$, so all these equations coincide with the ones obtained for $Q = E^2$. In the more general κ-dependent approach, the case where the configuration space is Euclidean can be considered, not as a limit case, but simply as the particular case $\kappa = 0$. Generically r appears through $\mathrm{S}_\kappa(r)$, and there are also some factors which in the curved case appear through a tagged cosine of r, $\mathrm{C}_\kappa(r)$, which in the Euclidean case degenerates to $\mathrm{C}_0(r) \equiv 1$ and which therefore becomes invisible; of course these terms turn visible once we deal with the case of nonzero curvature κ. We also notice that the dependence on the curvature κ is only present in the radial part of the functions. The angular functions (e.g., $\cos\phi, \sin\phi$) are κ-independent.

A. Isotropic oscillator

Let us consider the following spherical (Euclidean, hyperbolic) Lagrangian with curvature κ,

$$L = (\tfrac{1}{2})(v_r^2 + \mathrm{S}_\kappa^2(r) v_\phi^2) - (\tfrac{1}{2})\omega_0^2\, U_{11}(r, \phi, \kappa), \quad U_{11} = \mathrm{T}_\kappa^2(r),$$

so that the standard oscillator potential on the unit sphere (Higgs oscillator), on the Euclidean plane, or on the unit Lobachewski plane, arise as the following three particular cases:

$$U_{11}(1) = \tan^2 r, \quad V_{11} = U_{11}(0) = r^2, \quad U_{11}(-1) = \tanh^2 r.$$

The Euclidean oscillator V_{11} (parabolic potential without singularities) appears in this formalism as making a separation between two different situations (see Fig. 3). The spherical potential is

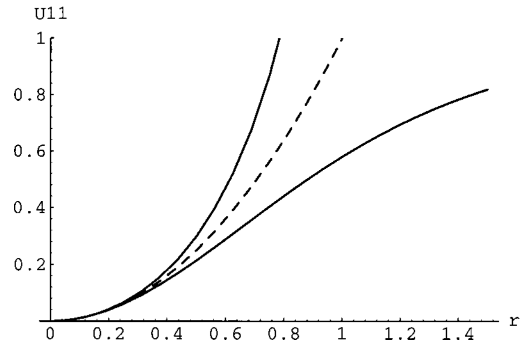

FIG. 3. Plot of U_{11} as a function of r, for $\kappa=-1$ (lower curve), $\kappa=0$ (dashed line) and $\kappa=1$ (upper curve).

represented by a well with singularities on the border (impenetrable walls at the equatorial circle $r=\pi/2\sqrt{\kappa}$ if the potential center is at the poles), and the hyperbolic potential by a well with finite depth since $U_{11}\to 1/|\kappa|$ when $r\to\infty$.

The dynamics is given by the following κ-dependent vector field:

$$X_{11}=v_r\frac{\partial}{\partial r}+v_\phi\frac{\partial}{\partial\phi}+f_r\frac{\partial}{\partial v_r}+f_\phi\frac{\partial}{\partial v_\phi},$$

$$f_r=(\mathrm{S}_\kappa(r)\mathrm{C}_\kappa(r))v_\phi^2-\omega_0^2\left(\frac{\mathrm{S}_\kappa(r)}{\mathrm{C}_\kappa^3(r)}\right),$$

$$f_\phi=-2\left(\frac{\mathrm{C}_\kappa(r)}{\mathrm{S}_\kappa(r)}\right)v_r v_\phi .$$

This is an integrable system endowed with two fundamental quadratic integrals of motion:

$$I_1(\kappa)=P_1^2(\kappa)+\omega_0^2(\mathrm{T}_\kappa(r)\cos\phi)^2,\quad I_2(\kappa)=P_2^2(\kappa)+\omega_0^2(\mathrm{T}_\kappa(r)\sin\phi)^2.$$

These constants are sum of two squares (as in the Euclidean plane) so they can be interpreted as the modulus of appropriate complex functions. Notice also that they can be considered as the κ-deformed versions of the two Euclidean one-degree of freedom energies; nevertheless, for $\kappa\neq 0$, the sum $I_1(\kappa)+I_2(\kappa)$ does not represent the total energy.

We begin our analysis by considering the time-derivative of the the two κ-dependent functions $P_1(\kappa)$ and $P_2(\kappa)$. They are given by

$$\frac{d}{dt}P_1(\kappa)=-\omega_0^2\left(\frac{\mathrm{S}_\kappa(r)}{\mathrm{C}_\kappa^3(r)}\right)\cos\phi,\quad \frac{d}{dt}P_2(\kappa)=-\omega_0^2\left(\frac{\mathrm{S}_\kappa(r)}{\mathrm{C}_\kappa^3(r)}\right)\sin\phi\ .$$

In a similar way, the time-derivative of the two velocity-independent functions $\mathrm{T}_\kappa(r)\cos\phi$, $\mathrm{T}_\kappa(r)\sin\phi$, is given by

$$\frac{d}{dt}(\mathrm{T}_\kappa(r)\cos\phi)=\frac{P_1(\kappa)}{\mathrm{C}_\kappa^2(r)},\quad \frac{d}{dt}(\mathrm{T}_\kappa(r)\sin\phi)=\frac{P_2(\kappa)}{\mathrm{C}_\kappa^2(r)}.$$

The following proposition states the super-integrability of the system and proves the existence of a complex factorization.

Proposition 2: Let K_1, K_2, be the following two functions:

$$K_1 = P_1(\kappa) + \mathrm{i}\omega_0(\mathrm{T}_\kappa(r)\cos\phi), \quad K_2 = P_2(\kappa) + \mathrm{i}\omega_0(\mathrm{T}_\kappa(r)\sin\phi).$$

Then (i) The modulus of K_1 and K_2 are constants of motion and coincide with $I_1(\kappa), I_2(\kappa)$:

$$|K_1|^2 = I_1(\kappa), \quad |K_2|^2 = I_2(\kappa).$$

(ii) The complex function K_{12} defined as

$$K_{12} = K_1 K_2^*$$

is also a constant of motion.

Proof: (i) follows directly from the definition of K_1 and K_2. For proving (ii) we analyze the time-evolution of K_1 and K_2. We have

$$\frac{d}{dt}K_1 = \frac{d}{dt}P_1(\kappa) + \mathrm{i}\,\omega_0\frac{d}{dt}(\mathrm{T}_\kappa(r)\cos\phi) = -\omega_0^2\left(\frac{\mathrm{S}_\kappa(r)}{\mathrm{C}_\kappa^3(r)}\right)\cos\phi + \mathrm{i}\,\omega_0\frac{P_1(\kappa)}{\mathrm{C}_\kappa^2(r)} = \left(\frac{\mathrm{i}\omega_0}{\mathrm{C}_\kappa^2(r)}\right)K_1,$$

and a similar calculus leads to

$$\frac{d}{dt}K_2 = \left(\frac{\mathrm{i}\omega_0}{\mathrm{C}_\kappa^2(r)}\right)K_2.$$

Thus we obtain

$$\frac{d}{dt}(K_1 K_2^*) = 0,$$

which states the function K_{12} as an additional constant of motion. It can be decomposed into real and imaginary parts as follows:

$$K_{12} = I_4(\kappa) + \mathrm{i}I_3(\kappa),$$

where $I_4(\kappa)$, $I_3(\kappa)$ are real constants of motion, respectively, quadratic and linear in the velocities, and given by

$$I_4(\kappa) = P_1(\kappa)P_2(\kappa) + \omega_0^2(\mathrm{T}_\kappa^2(r)\cos\phi\,\sin\phi), \quad I_3(\kappa) = \omega_0 J(\kappa).$$

These are the general κ-dependent versions of the flat space constants denoted I_4, I_3. A direct calculation shows that

$$I_1(\kappa) + I_2(\kappa) + \kappa(J(\kappa))^2 = (\tfrac{1}{2})[v_r^2 + \mathrm{S}_\kappa^2(r)v_\phi^2 + \omega_0^2 U_{11}(r,\phi,\kappa)],$$

which means that in any space of nonzero constant curvature κ the total energy can be written as a sum of *three* summands, one of which carries the spherical (Euclidean, hyperbolic) angular momentum with the curvature κ as a coefficient and vanishes into the limit $\kappa \to 0$.

Summarizing: the isotropic harmonic oscillator is super-integrable for any value of the curvature κ (positive, zero or negative) and an additional constant of motion K_{12} can be obtained by complex factorization. The fundamental integral of motion $I_3(\kappa)$ represents the spherical or hyperbolic version of the angular momentum (because the potential U_{11} is central independently of the value of the curvature). Concerning $I_4(\kappa)$, it represents the κ-dependent spherical or hyperbolic versions of the so–called Fradkin constant.[20]. Next, we give the expressions of these two constants for three particular cases.

(i) Higgs oscillator in the unit sphere (radius $R=1$). In this case we have

$$P_1=(\cos\phi)v_r-(\cos r\ \sin r\sin\phi)\,v_\phi,$$
$$P_2=(\sin\phi)v_r+(\cos r\ \sin r\cos\phi)v_\phi,$$
$$J=\sin^2 r\,v_\phi,$$

and then I_4, I_3, are given by

$$I_4=P_1P_2+\omega_0^2\,(\tan^2 r\cos\phi\sin\phi),\quad I_3=\omega_0 J.$$

(ii) Isotropic oscillator in the unit Lobachewski plane (“radius” $R=1$). In this case we have

$$P_1=(\cos\phi)v_r-(\cosh r\sinh r\sin\phi)v_\phi,$$
$$P_1=(\sin\phi)v_r+(\cosh r\sinh r\cos\phi)v_\phi,$$
$$J=\sinh^2 r v_\phi,$$

and then I_4, I_3, are given by

$$I_4=P_1P_2+\omega_0^2(\tanh^2 r\cos\phi\sin\phi),\quad I_3=\omega_0 J.$$

(iii) The expressions for $\kappa=0$ become the usual formulas for the standard isotropic oscillator.

B. Nonisotropic 2:1 oscillator

As mentioned in the Introduction, systems on the two-dimensional sphere (Euclidean, hyperbolic plane) with the first integrals quadratic in the velocities were studied in Ref. 16. Several different potentials were obtained as solutions of a system of two coupled differential equations depending on the parameter κ. One of the potentials obtained in this approach was

$$U_{21}(r,\phi,\kappa)=\frac{1}{1-\kappa(\mathrm{S}_\kappa(r)\sin\phi)^2}\left[4\left(\frac{\mathrm{T}_\kappa(r)\cos\phi}{1-\kappa(\mathrm{T}_\kappa(r)\cos\phi)^2}\right)^2+(\mathrm{S}_\kappa(r)\sin\phi)^2\right],$$

which satisfies

$$\lim_{\kappa\to 0} U_{21}=4r^2\cos^2\phi+r^2\sin^2\phi$$

and was interpreted as representing the potential of the spherical (hyperbolic) version of the 2:1 harmonic oscillator.

So, let us now study the following spherical (hyperbolic) Lagrangian with curvature κ:

$$L=(\tfrac{1}{2})(v_r^2+\mathrm{S}_\kappa^2(r)v_\phi^2)-(\tfrac{1}{2})\,\omega_0^2U_{21}(r,\phi,\kappa).$$

The dynamics is given by the following vector field:

$$X_{21}=v_r\frac{\partial}{\partial r}+v_\phi\frac{\partial}{\partial\phi}+f_r\frac{\partial}{\partial v_r}+f_\phi\frac{\partial}{\partial v_\phi},$$

where the κ-dependent forces f_r and f_ϕ are given by

$$f_r = \mathrm{S}_\kappa(r)\mathrm{C}_\kappa(r)v_\phi^2 - (\omega_0^2\mathrm{S}_\kappa(r)\mathrm{C}_\kappa(r))\left[\frac{\mathrm{C}_\kappa^2(r)(4\cos^2\phi+\sin^2\phi)+\kappa\mathrm{S}_\kappa(r)(3+\cos^2\phi)\cos^2\phi}{(\mathrm{C}_\kappa^2(r)-\kappa\mathrm{S}_\kappa^2(r)\cos^2\phi)^3}\right],$$

$$f_\phi = -2\left(\frac{\mathrm{C}_\kappa(r)}{\mathrm{S}_\kappa(r)}\right)v_r v_\phi + (\omega_0^2\sin\phi\cos\phi)\left[\frac{3\mathrm{C}_\kappa^2(r)+\kappa\mathrm{S}_\kappa^2(r)\cos^2\phi}{(\mathrm{C}_\kappa^2(r)-\kappa\mathrm{S}_\kappa^2(r)\cos^2\phi)^3}\right].$$

It was proved in Ref. 16 that this system possesses the following two quadratic integrals of motion:

$$I_1(\kappa) = P_1^2(\kappa) + 4\omega_0^2\left(\frac{\mathrm{T}_\kappa(r)\cos\phi}{1-\kappa(\mathrm{T}_\kappa(r)\cos\phi)^2}\right)^2,$$

$$I_2(\kappa) = P_2^2(\kappa) + \kappa J^2(\kappa) + \omega_0^2[1+\kappa(\mathrm{T}_\kappa(r)\cos\phi)^2]\left(\frac{\mathrm{T}_\kappa(r)\sin\phi}{1-\kappa(\mathrm{T}_\kappa(r)\cos\phi)^2}\right)^2. \tag{5}$$

For further convenience, it will be useful to write the constant I_2 as

$$I_2(\kappa) = P_2^2(\kappa) + \kappa J^2(\kappa) + \omega_0^2 F(r,\phi,\kappa)(\mathrm{T}_\kappa(r)\sin\phi)^2,$$

where we have denoted by $F(r,\phi,\kappa)$ the following function:

$$F(r,\phi,\kappa) = \frac{1+\kappa(\mathrm{T}_\kappa(r)\cos\phi)^2}{(1-\kappa\mathrm{T}_\kappa^2(r)\cos^2\phi)^2}.$$

Notice that if we denote by $I_0(\kappa)$ the trivial constant of motion (energy),

$$I_0(\kappa) = (\tfrac{1}{2})(v_r^2 + \mathrm{S}_\kappa^2(r)v_\phi^2) + (\tfrac{1}{2})\omega_0^2 U_{21}(r,\phi,\kappa);$$

then

$$2I_0(\kappa) = I_1(\kappa) + I_2(\kappa).$$

Let us now denote by K_1 the following complex function:

$$K_1 = P_1(\kappa) + (2\mathrm{i}\omega_0)\left(\frac{\mathrm{T}_\kappa(r)\cos\phi}{1-\kappa(\mathrm{T}_\kappa(r)\cos\phi)^2}\right).$$

By using

$$\frac{d}{dt}P_1(\kappa) = -\left(\frac{4\omega_0^2}{\mathrm{C}_\kappa^2(r)}\right)F(r,\phi,\kappa)\left[\frac{\mathrm{T}_\kappa(r)\cos\phi}{1-\kappa\,\mathrm{T}_\kappa^2(r)\sin^2\phi}\right],$$

$$\frac{d}{dt}\left(\frac{\mathrm{T}_\kappa(r)\cos\phi}{1-\kappa\mathrm{T}_\kappa^2(r)\sin^2\phi}\right) = \left(\frac{1}{\mathrm{C}_\kappa^2(r)}\right)F(r,\phi,\kappa)\,P_1(\kappa),$$

we get the time-evolution of the function K_1 given by

$$\frac{d}{dt}K_1 = \left(\frac{2\mathrm{i}\omega_0}{\mathrm{C}_\kappa^2(r)}\right)F(r,\phi,\kappa)\,K_1.$$

It seems that the following step must be the analysis of the time evolution of $P_2(\kappa)$ (as we have done for the case of the central Higgs oscillator). Nevertheless instead of considering $P_2(\kappa)$ by itself, we will consider two other related functions obtained by addition (subtraction) of a new term related with the spherical (hyperbolic) version of the angular momentum:

$$\frac{d}{dt}[P_2(\kappa)+\sqrt{\kappa}\,J(\kappa)]=-\left(\frac{\omega_0}{\mathrm{C}_\kappa(r)}\right)^2\left(\frac{1}{1+\sqrt{\kappa}\,\mathrm{T}_\kappa(r)\cos\phi}\right)^2\left[\frac{\mathrm{T}_\kappa(r)\sin\phi}{1+\sqrt{\kappa}\mathrm{T}_\kappa(r)\cos\phi}\right],$$

$$\frac{d}{dt}[P_2(\kappa)-\sqrt{\kappa}J(\kappa)]=-\left(\frac{\omega_0}{\mathrm{C}_\kappa(r)}\right)^2\left(\frac{1}{1-\sqrt{\kappa}\mathrm{T}_\kappa(r)\cos\phi}\right)^2\left[\frac{\mathrm{T}_\kappa(r)\sin\phi}{1-\sqrt{\kappa}\mathrm{T}_\kappa(r)\cos\phi}\right].$$

On the other hand, the time-derivative of the two velocity-independent functions,

$$\left[\frac{\mathrm{T}_\kappa(r)\sin\phi}{1+\sqrt{\kappa}\mathrm{T}_\kappa(r)\cos\phi}\right],\quad\left[\frac{\mathrm{T}_\kappa(r)\sin\phi}{1-\sqrt{\kappa}\mathrm{T}_\kappa(r)\cos\phi}\right],$$

is given by

$$\frac{d}{dt}\left[\frac{\mathrm{T}_\kappa(r)\sin\phi}{1+\sqrt{\kappa}\mathrm{T}_\kappa(r)\cos\phi}\right]=\left(\frac{1}{\mathrm{C}_\kappa^2(r)}\right)\left(\frac{1}{1+\sqrt{\kappa}\,\mathrm{T}_\kappa(r)\cos\phi}\right)^2[P_2(\kappa)+\sqrt{\kappa}\,J(\kappa)],$$

$$\frac{d}{dt}\left[\frac{\mathrm{T}_\kappa(r)\sin\phi}{1-\sqrt{\kappa}\,\mathrm{T}_\kappa(r)\cos\phi}\right]=\left(\frac{1}{\mathrm{C}_\kappa^2(r)}\right)\left(\frac{1}{1-\sqrt{\kappa}\,\mathrm{T}_\kappa(r)\cos\phi}\right)^2[P_2(\kappa)-\sqrt{\kappa}\,J(\kappa)].$$

These relations to us suggests to define not one, but two functions, similar to K_2:

$$K_2^+=P_2(\kappa)+\sqrt{\kappa}J(\kappa)+\mathrm{i}\omega_0\left(\frac{\mathrm{T}_\kappa(r)\sin\phi}{1+\sqrt{\kappa}\mathrm{T}_\kappa(r)\cos\phi}\right),$$

$$K_2^-=P_2(\kappa)-\sqrt{\kappa}J(\kappa)+\mathrm{i}\,\omega_0\left(\frac{\mathrm{T}_\kappa(r)\sin\phi}{1-\sqrt{\kappa}\mathrm{T}_\kappa(r)\cos\phi}\right),$$

whose time derivatives are given by

$$\frac{d}{dt}K_2^+=\left(\frac{\mathrm{i}\omega_0}{\mathrm{C}_\kappa^2(r)}\right)\left[\frac{1}{(1+\sqrt{\kappa}\mathrm{T}_\kappa(r)\cos\phi)^2}\right]K_2^+,$$

$$\frac{d}{dt}K_2^-=\left(\frac{\mathrm{i}\omega_0}{\mathrm{C}_\kappa^2(r)}\right)\left[\frac{1}{(1-\sqrt{\kappa}\mathrm{T}_\kappa(r)\cos\phi)^2}\right]K_2^-.$$

Then we have the following proposition.

Proposition 3: Let the complex functions K_1, K_2^+, K_2^- *be defined as above. Then the complex function* K_{122} *defined as*

$$K_{122}=K_1\,K_2^{+*}\,K_2^{-*}$$

is a constant of motion.

Proof: We have already obtained the time derivatives of everyone of the three functions K_1, K_2^+, K_2^-. Because of this, the time evolution of K_{122} is given by

$$\frac{d}{dt}K_{122}=\dot{K}_x K_2^{+*} K_2^{-*}+K_1 \dot{K}_y^{+*} K_2^{-*}+K_1 K_2^{+*} \dot{K}_2^{-*}$$

$$=\left(\frac{\mathrm{i}\,\omega_0}{\mathrm{C}_\kappa^2(r)}\right)\left[2F(r,\phi,\kappa)-\frac{1}{(1+\sqrt{\kappa}\,\mathrm{T}_\kappa(r)\cos\phi)^2}-\frac{1}{(1-\sqrt{\kappa}\,\mathrm{T}_\kappa(r)\cos\phi)^2}\right]K_{122}\ .$$

But the function F is such that the term in square brackets vanishes identically, and we arrive at

$$\frac{d}{dt}K_{122}=0.$$

Thus we have shown that complex function $K_{122}=I_4(\kappa)+\mathrm{i}\,I_3(\kappa)$ as well as the two associated real functions, $I_4(\kappa)$ and $I_3(\kappa)$, are all of them constants of motion. After some computations (we omit the details) we have obtained

$$I_4(\kappa)=[P_2^2(\kappa)-\kappa J^2(\kappa)]\,P_1(\kappa)+\omega_0^2\,I_{41}(\kappa),$$

$$I_3(\kappa)=P_2(\kappa)\,J(\kappa)-\omega_0^2\,I_{30}(\kappa),$$

with

$$I_{41}(\kappa)=\left(\frac{\mathrm{T}_\kappa(r)\sin\phi)}{(1-\kappa\,\mathrm{T}_\kappa^2(r)\cos^2\phi)^2}\right)[I_{41}^a(\kappa)+\kappa\,I_{41}^b(\kappa)],$$

$$I_{41}^a(\kappa)=4P_2(\kappa)\mathrm{T}_\kappa(r)\cos\phi-P_1(\kappa)\mathrm{T}_\kappa(r)\sin\phi,$$

$$I_{41}^b(\kappa)=(\mathrm{T}_\kappa^2(r)\cos^2\phi)(4J(\kappa)+P_1(\kappa)\mathrm{T}_\kappa(r)\sin\phi),$$

$$I_{30}(\kappa)=\frac{\mathrm{T}_\kappa^3(r)\cos\phi\sin^2\phi}{(1-\kappa\,\mathrm{T}_\kappa^2(r)\cos^2\phi)^2}.$$

We close this section observing that, although these κ-dependent polar coordinates (r,ϕ) prove to be convenient for the central U_{11} case, the results of this noncentral subsection [actually, even the function $U_{21}(r,\phi)$ itself] suggest the convenience of a new study in a new and more appropriate system of coordinates.

V. THE 1:1 AND 2:1 HARMONIC OSCILLATORS ON A 2-D SPACE OF CONSTANT CURVATURE II: PARALLEL COORDINATES

The dynamics of a system defined in a two-dimensional space with constant curvaure κ can also be studied by using the ‘‘geodesic parallel’’ system of coordinates introduced in Sec. II. In this case the differential element of distance is

$$ds^2=\mathrm{C}_\kappa^2(y)\,du^2+dy^2,$$

so a standard Lagrangian (kinetic term minus potential function) has the following form:

$$L(\kappa)=(\tfrac{1}{2})(\mathrm{C}_\kappa^2(y)v_u^2+v_y^2)-U(u,y,\kappa),$$

in such a way that the Euclidean system is just given by the particular value of $L(\kappa)$ in $\kappa=0$,

$$\lim_{\kappa\to 0} L(\kappa)=(\tfrac{1}{2})\,(v_x^2+v_y^2)-V(x,y),\quad V(x,y)=U(x,y,0)\ .$$

The kinetic term remains invariant under the action of the three κ-dependent vector fields Y_{P_1}, Y_{P_2}, $Y_J(\kappa)$, whose expressions in parallel coordinates are

$$Y_{P_1}(\kappa)=\frac{\partial}{\partial u},\quad Y_{P_2}(\kappa)=\kappa\,\mathrm{S}_\kappa(u)\mathrm{T}_\kappa(y)\,\frac{\partial}{\partial u}+\mathrm{C}_\kappa(u)\,\frac{\partial}{\partial y},$$

$$Y_J(\kappa)=\mathrm{C}_\kappa(u)\mathrm{T}_\kappa(y)\,\frac{\partial}{\partial u}-\mathrm{S}_\kappa(u)\,\frac{\partial}{\partial y}.$$

Moreover, the potentials U_1, U_2, U_J, now characterized by the following dependence:

$$U_1=U(y),\quad U_2=U(\mathrm{S}_\kappa(u)\mathrm{C}_k(y)),\quad \text{and}\quad U_J=U\left(\frac{\mathrm{T}^2_\kappa(u)}{\mathrm{C}^2_\kappa(y)}+\mathrm{T}^2_\kappa(y)\right)$$

(remark that these expressions embody the same dependence as derived in the previous section) are endowed with $Y_{P_1}(\kappa)$, $Y_{P_2}(\kappa)$, $Y_J(\kappa)$, as exact Noether symmetries. The associated linear constants of motion are given by

$$P_1(\kappa)=\mathrm{C}^2_\kappa(y)\,v_u,\quad P_2(\kappa)=\kappa\,\mathrm{S}_\kappa(u)\mathrm{C}_\kappa(y)\mathrm{S}_\kappa(y)\,v_u+\mathrm{C}_\kappa(u)\,v_y,$$
$$J(\kappa)=\mathrm{C}_\kappa(u)\mathrm{C}_\kappa(y)\mathrm{S}_\kappa(y)\,v_u-\mathrm{S}_\kappa(u)\,v_y, \tag{6}$$

and hence the more general form for a linear constant of motion I_{11}, is as a linear combination of these three functions, $I_{11}=a_1P_1(\kappa)+a_2P_2(\kappa)+cJ(\kappa)$.

The constants of motion which are quadratic in the velocities arise from generalized Noether symmetries (hidden symmetries) of $L(\kappa)$, and they have the following expression:

$$I=I_{22}+I_{20}(u,y,\kappa),$$

with the term I_{22} again given by a linear combination of quadratic pairings,

$$I_{22}(\kappa)=a_0P^2_1(\kappa)+b_0P_1(\kappa)P_2(\kappa)+c_0P^2_2(\kappa)+a_1P_2(\kappa)J(\kappa)+c_1P_1(\kappa)J(\kappa)+a_2J^2(\kappa).$$

In contrast to the formalism in polar coordinates (r,ϕ), the κ-dependence is now present in both coordinates, u and y, because both are lengths.

A. Isotropic oscillator

The Lagrangian of isotropic spherical (Euclidean, hyperbolic) oscillator with curvature κ is

$$L=\left(\frac{1}{2}\right)(\mathrm{C}^2_\kappa(y)v^2_u+v^2_y)-\left(\frac{1}{2}\right)\omega^2_0\,U_{11}(u,y,\kappa),\quad U_{11}=\frac{\mathrm{T}^2_\kappa(u)}{\mathrm{C}^2_\kappa(y)}+\mathrm{T}^2_\kappa(y).$$

This potential is indeed the same discussed with polar coordinates as $\mathrm{T}^2_\kappa(r)$ [see (4)]. In Sec. III we pointed out that the fuction $\mathrm{T}^2_\kappa(r)$ admits several alternative expressions in terms of the coordinates (u,y) which can be considered as curvature versions of the Pythagorean theorem. Because of this, the potential $U_{11}=U_{11}(u,y,\kappa)$ can also be written as

$$U_{11}=\mathrm{T}_\kappa^2(u)+\frac{\mathrm{T}_\kappa^2(y)}{\mathrm{C}_\kappa^2(u)}.$$

The dynamics is given by the following κ-dependent vector field:

$$X_{11}=v_u\frac{\partial}{\partial u}+v_y\frac{\partial}{\partial y}+f_u\frac{\partial}{\partial v_u}+f_y\frac{\partial}{\partial v_y},$$

$$f_u=2\,\kappa\,\mathrm{T}_\kappa(y)\,v_u v_y-\omega_0^2\left(\frac{\mathrm{S}_\kappa(u)}{\mathrm{C}_\kappa^3(u)\,\mathrm{C}_\kappa^4(y)}\right),$$

$$f_y=-\,\kappa\,\mathrm{C}_\kappa(y)\,\mathrm{S}_\kappa(y)\,v_u^2-\omega_0^2\left(\frac{\mathrm{S}_\kappa(y)}{\mathrm{C}_\kappa^2(u)\,\mathrm{C}_\kappa^3(y)}\right).$$

The following two κ-dependent functions remain constant along the trajectories of X_{11}:

$$I_1(\kappa)=P_1^2(\kappa)+\omega_0^2\,\mathrm{T}_\kappa^2(u),\quad I_2(\kappa)=P_2^2(\kappa)+\omega_0^2\left(\frac{\mathrm{T}_\kappa(y)}{\mathrm{C}_\kappa(u)}\right)^2. \tag{7}$$

They are integrals of motion quadratic in the velocities (momenta in the Hamiltonian formalism) which correspond to the κ-deformed versions of the two Euclidean one-degree of freedom energies.

The following proposition states the super-integrability of the system and proves the existence of a complex factorization.

Proposition 4: Let K_1, K_2, be the following two functions:

$$K_1=P_1(\kappa)+\mathrm{i}\,\omega_0\,\mathrm{T}_\kappa(u),\quad K_2=P_2(\kappa)+\mathrm{i}\,\omega_0\left(\frac{\mathrm{T}_\kappa(y)}{\mathrm{C}_\kappa(u)}\right).$$

Then the complex function K_{12} defined as

$$K_{12}=K_1K_2^*$$

is a constant of motion.

Proof: The time-evolution of the functions K_1, K_2 is given by

$$\frac{d}{dt}K_1\equiv X_{11}(K_1)=X_{11}[\mathrm{C}_\kappa^2(y)\,v_u]+\mathrm{i}\,\omega_0\,X_{11}[\mathrm{T}_\kappa(u)]=\left(\frac{\mathrm{i}\,\omega_0}{\mathrm{C}_\kappa^2(u)\mathrm{C}_\kappa^2(y)}\right)K_1,$$

$$\frac{d}{dt}K_2\equiv X_{11}(K_2)=X_{11}[P_2(\kappa)]+\mathrm{i}\,\omega_0\,X_{11}\left[\frac{\mathrm{T}_\kappa(y)}{\mathrm{C}_\kappa(u)}\right]=\left(\frac{\mathrm{i}\,\omega_0}{\mathrm{C}_\kappa^2(u)\mathrm{C}_\kappa^2(y)}\right)K_2.$$

Hence we have

$$\frac{d}{dt}K_{12}=X_{11}(K_1)K_2^*+K_1\,X_{11}(K_2^*)=0,$$

which states the function K_{12} as an additional constant of motion. The associated real integrals of motion $I_4, \ I_3$ are

$$I_4 = \mathrm{Re}(K_1 K_2^*) = P_1(\kappa) P_2(\kappa) + \omega_0^2 \, \mathrm{T}_\kappa(u) \left(\frac{\mathrm{T}_\kappa(y)}{\mathrm{C}_\kappa(u)} \right),$$

$$I_3 = \mathrm{Im}(K_1 K_2^*) = \omega_0 \, P_2(\kappa) \mathrm{T}_\kappa(u) - \omega_0 \, P_1(\kappa) \left(\frac{\mathrm{T}_\kappa(y)}{\mathrm{C}_\kappa(u)} \right) = \omega_0 \, J(\kappa).$$

We close with two properties. First, the relation between the modulus of the functions $K_1, \ K_2$, and the two fundamental constants of motion is the same that of the Euclidean plane,

$$|K_1|^2 = I_1(\kappa), \quad |K_2|^2 = I_2(\kappa).$$

Second, the three constants of motion $I_1(\kappa), I_2(\kappa), I_3(\kappa)$ are functionally independent, and the total energy is related to them as

$$(\tfrac{1}{2})(\mathrm{C}_\kappa^2(y) v_u^2 + v_y^2) + (\tfrac{1}{2}) \omega_0^2 \, U_{11}(u,y,\kappa) = (\tfrac{1}{2})(I_1(\kappa) + I_2(\kappa) + \kappa \, J^2(\kappa)),$$

because of

$$P_1^2(\kappa) + P_2^2(\kappa) + \kappa \, J^2(\kappa) = \mathrm{C}_\kappa^2(y) v_u^2 + v_y^2.$$

B. Nonisotropic 2:1 oscillator

In parallel coordinates the Lagrangian of the spherical (hyperbolic) 2:1 harmonic oscillator with curvature κ is

$$L = (\tfrac{1}{2})(\mathrm{C}_\kappa^2(y) v_u^2 + v_y^2) - (\tfrac{1}{2}) \omega_0^2 \, U_{21}(u,y,\kappa),$$

$$U_{21} = \frac{\mathrm{T}_\kappa^2(2\,u)}{\mathrm{C}_\kappa^2(y)} + \mathrm{T}_\kappa^2(y) = \mathrm{T}_\kappa^2(2\,u) + \frac{\mathrm{T}_\kappa^2(y)}{\mathrm{C}_\kappa^2(2\,u)}.$$

Therefore, the dynamics is represented by the following κ-dependent vector field:

$$X_{21} = v_u \frac{\partial}{\partial u} + v_y \frac{\partial}{\partial y} + f_u \frac{\partial}{\partial v_u} + f_y \frac{\partial}{\partial v_y},$$

$$f_u = 2\,\kappa \, \mathrm{T}_\kappa(y) v_u v_y - 2\,\omega_0^2 \left(\frac{\mathrm{S}_\kappa(2\,u)}{\mathrm{C}_\kappa^3(2\,u) \mathrm{C}_\kappa^4(y)} \right),$$

$$f_y = -\kappa \, \mathrm{C}_\kappa(y) \mathrm{S}_\kappa(y) \, v_u^2 - \omega_0^2 \left(\frac{\mathrm{S}_\kappa(y)}{\mathrm{C}_\kappa^2(2\,u) \mathrm{C}_\kappa^3(y)} \right).$$

This system possesses two integrals of motion quadratic in the velocities:

$$I_1(\kappa) = P_1(\kappa)^2 + 4\,\omega_0^2 \left(\frac{\mathrm{T}_\kappa(u)}{1 - \kappa \, \mathrm{T}_\kappa^2(u)} \right)^2,$$

$$I_2(\kappa) = P_2^2(\kappa) + \kappa \, J^2(\kappa) + \omega_0^2 \left(\frac{\mathrm{T}_\kappa(y)}{\mathrm{C}_\kappa(2\,u)} \right)^2 = [v_y^2 + \kappa \, \mathrm{C}_\kappa^2(y) \mathrm{S}_\kappa^2(y) v_u^2] + \omega_0^2 \left(\frac{\mathrm{T}_\kappa(y)}{\mathrm{C}_\kappa(2\,u)} \right)^2.$$

In the two-dimensional sphere S^2 (hyperbolic plane) with curvature κ, the expression for the tangent of the double-angle is $\mathrm{T}_\kappa(2\,\alpha)=2\,\mathrm{T}_\kappa(\alpha)/(1-\kappa\,\mathrm{T}_\kappa^2(\alpha))$; thus the function $I_1(\kappa)$ can also be rewritten as follows:

$$I_1(\kappa)=P_1(\kappa)^2+\omega_0^2\,\mathrm{T}_\kappa^2(2\,u).$$

The dynamics is κ-dependent and so are the two integrals of motion. Hence the following two equations:

$$X_{21}(I_1(\kappa))=0,\quad X_{21}(I_2(\kappa))=0,$$

remain true for any value (positive, negative or null) of the curvature κ. Moreover, if we denote by $I_0(\kappa)$ the trivial constant of motion:

$$I_0(\kappa)=(\tfrac{1}{2})(\mathrm{C}_\kappa^2(y)v_u^2+v_y^2)+(\tfrac{1}{2})\omega_0^2\,U_{21}(u,y,\kappa),$$

then we have

$$2I_0(\kappa)=I_1(\kappa)+I_2(\kappa).$$

So the total energy $I_0(\kappa)$ splits as a sum of two terms, $I_1(\kappa)$ and $I_2(\kappa)$, as was the case in the Euclidean plane. Nevertheless, for $\kappa\neq 0$ the second integral contains an additional term proportional to the angular momentum. For the zero curvature limit, this additional term vanishes, and we obtain the correct Euclidean expressions,

$$\lim_{\kappa\to 0} I_1(\kappa)=v_x^2+4\,\omega_0^2\,x^2,\qquad \lim_{\kappa\to 0} I_2(\kappa)=v_y^2+\omega_0^2\,y^2.$$

Let us denote by K_1, K_2^+, K_2^-, the following three complex functions:

$$K_1=P_1(\kappa)+\mathrm{i}\,\omega_0\,\mathrm{T}_\kappa(2\,u),$$

$$K_2^+=[P_2(\kappa)+\sqrt{\kappa}\,J(\kappa)]+\mathrm{i}\,\omega_0(\mathrm{C}_\kappa(u)+\sqrt{\kappa}\,\mathrm{S}_\kappa(u))\left(\frac{\mathrm{T}_\kappa(y)}{\mathrm{C}_\kappa(2\,u)}\right),$$

$$K_2^-=[P_2(\kappa)-\sqrt{\kappa}\,J(\kappa)]+\mathrm{i}\,\omega_0(\mathrm{C}_\kappa(u)-\sqrt{\kappa}\,\mathrm{S}_\kappa(u))\left(\frac{\mathrm{T}_\kappa(y)}{\mathrm{C}_\kappa(2\,u)}\right).$$

The first function can be considered as a ‘‘curved’’ version of the Euclidean function J_1 (see the notation of Sec. II), and K_2^+, K_2^-, two different κ-dependent deformations of the Euclidean function J_2.

Proposition 5: Let the complex functions K_1, K_2^+, K_2^-, *be defined as above. Then*

(i) *The modulus of* K_1 *is a constant of motion and coincides with* $I_1(\kappa)$; *the sums of the modulus of* K_2^+ *and* K_2^- *is a constant of motion and coincides with* $I_2(\kappa)$.

(ii) *The complex function* K_{122}, *defined as*

$$K_{122}=K_1\,K_2^{+*}\,K_2^{-*},$$

is a constant of motion.

Proof: (i) The modulus of the first function K_1 is the constant value of the first integral of motion,

$$|K_1|^2 = P_1^2(\kappa) + \omega_0^2[\mathrm{T}_\kappa(2\,u)]^2 = I_1(\kappa),$$

and the sum of the modulus of K_2^+ and K_2^- coincides with $2I_2(\kappa)$:

$$|K_2^+|^2 + |K_2^-|^2 = 2\left(P_2^2(\kappa) + \kappa\, J^2(\kappa) + \omega_0^2\left[\frac{\mathrm{T}_\kappa(y)}{\mathrm{C}_\kappa(2\,u)}\right]^2\right) = 2\,I_2(\kappa).$$

(ii) The time evolution of the first function K_1 is given by

$$\frac{d}{dt}K_1 \equiv X_{21}(K_1) = \frac{d}{dt}(\mathrm{C}_\kappa^2(y)v_u) + (\mathrm{i}\,\omega_0)\frac{d}{dt}\mathrm{T}_\kappa(2\,u) = \left[\frac{2\,\mathrm{i}\,\omega_0}{\mathrm{C}_\kappa^2(2\,u)\mathrm{C}_\kappa^2(y)}\right]K_1,$$

and a similar calculus leads to

$$\frac{d}{dt}K_2^+ \equiv X_{21}(K_2^+) = \mathrm{i}\,\omega_0\left[\frac{\mathrm{C}_\kappa(u) + \sqrt{\kappa}\,\mathrm{S}_\kappa(u)}{\mathrm{C}_\kappa(2\,u)\mathrm{C}_\kappa(y)}\right]^2 K_2^+,$$

$$\frac{d}{dt}K_2^- \equiv X_{21}(K_2^-) = \mathrm{i}\,\omega_0\left[\frac{\mathrm{C}_\kappa(u) - \sqrt{\kappa}\,\mathrm{S}_\kappa(u)}{\mathrm{C}_\kappa(2\,u)\mathrm{C}_\kappa(y)}\right]^2 K_2^-.$$

The important point is that we get the following expression for the time evolution of the function product $K_{22} = K_2^+ K_2^-$:

$$\begin{aligned}\frac{d}{dt}(K_2^+K_2^-) &= X_{21}(K_2^+)K_2^- + K_2^+X_{21}(K_2^-)\\ &= \left[\frac{\mathrm{i}\,\omega_0}{\mathrm{C}_\kappa^2(2\,u)\mathrm{C}_\kappa^2(y)}\right][(\mathrm{C}_\kappa(u) + \sqrt{\kappa}\,\mathrm{S}_\kappa(u))^2 + (\mathrm{C}_\kappa(u) - \sqrt{\kappa}\,\mathrm{S}_\kappa(u))^2](K_2^+K_2^-)\\ &= \left[\frac{2\,\mathrm{i}\,\omega_0}{\mathrm{C}_\kappa^2(2\,u)\mathrm{C}_\kappa^2(y)}\right](K_2^+K_2^-).\end{aligned}$$

Thus we arrive at the property

$$\frac{d}{dt}K_{122} = X_{21}(K_1)(K_2^{+*}K_2^{-*}) + K_1\,X_{21}(K_2^{+*}K_2^{-*}) = 0$$

and the function K_{122} is an integral of motion. As it is complex, we obtain two real κ-dependent constants, $I_3(\kappa)$ and $I_4(\kappa)$, defined in the usual form

$$K_{122} = I_4(\kappa) + \mathrm{i}I_3(\kappa).$$

After some simplification the expressions for these two constants become

$$I_3(\kappa) = J(\kappa)P_2(\kappa) + \omega_0^2\left(\frac{\mathrm{S}_\kappa(u)\mathrm{C}_\kappa(u)}{\mathrm{C}_\kappa^2(2\,u)}\right)\mathrm{T}_\kappa^2(y),$$

$$I_4(\kappa) = [P_2^2(\kappa) - \kappa\,J^2(\kappa)]P_1(\kappa) + \omega_0^2[2\,\mathrm{T}_\kappa(2\,u)\,v_y - \mathrm{C}_\kappa(y)\mathrm{S}_\kappa(y)\,v_u]\left(\frac{\mathrm{T}_\kappa(y)}{\mathrm{C}_\kappa(2\,u)}\right).$$

We close this section with the following observations.

(i) Although K_2^+ and K_2^- are two different κ-dependent functions, $K_2^+ \neq K_2^-$, they have the same function J_2 as the Euclidean limit. We have

$$\lim_{\kappa\to 0} K_1 = v_x + \mathrm{i}\,\omega_0\,(2x), \qquad \lim_{\kappa\to 0} K_2^+ = \lim_{\kappa\to 0} K_2^- = v_y + \mathrm{i}\,\omega_0\, y.$$

The coefficient $n_1=2$ in the Euclidean function J_1 (see the notation of Sec. II) is now present, not as a global multiplicative factor on the imaginary part of K_1, but as a coefficient inside the argument of the tangent function.

(ii) The Euclidean function J_2 is κ-deformed in two different ways, K_2^+ and K_2^-, in such a way that the Euclidean square factor J_2^2 becomes the product $K_2^+ K_2^-$.

(iii) The transition of U_{11} to U_{21} is very simple: just the change $\mathrm{T}_\kappa(u)$ by $\mathrm{T}_\kappa(2\,u)$ in the potential expressed in parallel coordinates.

VI. FINAL COMMENTS AND OUTLOOK

We have started with a discussion of the curvilinear systems of coordinates on 2-D spaces of constant curvature and then we have studied the two oscillators with quadratic super-integrability. We have proved that they can be considered as κ-deformations of the Euclidean oscillator or, alternatively, that the classic and well-known Euclidean oscillator appears just as a very particular case of a much more general ''curved'' system. Concerning our approach, two important points are, first, that the results have been obtained for a general value of the curvature κ, in such a way that they cover simultaneously the case of a Euclidean plane ($\kappa=0$), the 2-sphere ($\kappa>0$) and the hyperbolic plane ($\kappa<0$); and, second, that all the computations have been carried out in both ''geodesic polar'' and ''geodesic parallel'' coordinates. The ''geodesic parallel'' coordinates may seem rather unusual, but they have proved to be the more appropriate ones for the study of the nonisotropic case.

The results obtained in Sec. V suggest that the appropriate potential $U_{n1}(u,y,\kappa)$ for representing the general nonisotropic $n{:}1$ oscillator, with an n integer, on a 2-D manifold of constant curvature κ, is given by

$$U_{n1} = \frac{\mathrm{T}_\kappa^2(n\,u)}{\mathrm{C}_\kappa^2(y)} + \mathrm{T}_\kappa^2(y) = \mathrm{T}_\kappa^2(n\,u) + \frac{\mathrm{T}_\kappa^2(y)}{\mathrm{C}_\kappa^2(n\,u)}.$$

This potential satisfies the appropriate Euclidean limit,

$$\lim_{\kappa\to 0} U_{n1}(\kappa) = (nx)^2 + y^2,$$

and is integrable for arbitrary values of the integer n. The two fundamental integrals of motion are both quadratic in the velocities

$$I_1(\kappa) = P_1^2(\kappa) + \omega_0^2\,[\mathrm{T}_\kappa(n\,u)]^2,$$

$$I_2(\kappa) = [P_2^2(\kappa) + \kappa\, J^2(\kappa)] + \omega_0^2 \left(\frac{\mathrm{T}_\kappa(y)}{\mathrm{C}_\kappa(n\,u)}\right)^2,$$

and can be considered as κ-deformations of the two quadratic Euclidean energies:

$$\lim_{\kappa\to 0} I_1(\kappa) = v_x^2 + \omega_0^2\,(nx)^2, \qquad \lim_{\kappa\to 0} I_2(\kappa) = v_y^2 + \omega_0^2\, y^2.$$

It seems natural to conjecture that U_{n1} (for any integer value of n) is super-integrable as well, and that the corresponding additional (nonquadratic) integral can also be obtained through a complex constant similar to the one obtained for the $2{:}1$ case in Sec. V. A more difficult problem seems to be the obtaining of the spherical (hyperbolic) version of the rational $n_1{:}n_2$ oscillator, since a direct generalization will lead to problems with the domain and range of the potential (in this general case, the spherical system looks more complicated than the hyperbolic one). We think that these are open questions that must be investigated.

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