Integrable generalizations of oscillator and Coulomb systems via action-angle variables

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Abstract

Oscillator and Coulomb systems on N-dimensional spaces of constant curvature can be generalized by replacing their angular degrees of freedom with a compact integrable $(N-1)$-dimensional system. We present the action-angle formulation of such models in terms of the radial degree of freedom and the action-angle variables of the angular subsystem. As an example, we construct the spherical and pseudospherical generalization of the two-dimensional superintegrable models introduced by Tremblay, Turbiner and Winternitz and by Post and Winternitz. We demonstrate the superintegrability of these systems and give their hidden constant of motion.

1 Introduction

Prominent in the theory of integrable systems is the Liouville theorem [1], which states that any $2N$-dimensional Hamiltonian system with N mutually commuting constants of motion is integrable. Besides, the theorem states that, if the level surface of these constants of motion are *compact and connected manifolds*, then they are diffeomorphic to N-dimensional tori. This enables one to introduce action-angle variables $(\mathbf{I}, \mathbf{\Phi})$, so that the Hamiltonian depends only on the action variables $\mathbf{I}$, which are constants of motion. The formulation of an integrable system in terms of these variables yields a comprehensive geometric description of its dynamics and is a useful tool for developing perturbation theory [1, 2]. Furthermore, action-angle variables indicate the (non)equivalence of different integrable systems, since they admit only two kinds of freedom:

  – in the functional dependence of the Hamiltonian on the action variables, $H = H(\mathbf{I})$;
  – in the domain of the action variables, $\mathbf{I} \in [\boldsymbol{\beta}^-, \boldsymbol{\beta}^+]$.

Besides the standard textbook problems such as the harmonic oscillator or the Kepler potential, action-angle variables seem to be absent in the literature for the vast variety of known integrable models, such as integrable systems in a curved geometry [3, 4] or coupled to a monopole or instanton background, as well as multi-particle systems of Calogero type [5] (except for rational Calogero models [6]). Therefore, we have recently begun to develop this issue, by analyzing the (one-dimensional) dihedral systems related to the three-particle Calogero model [7] as well as tow-dimensional oscillator-like systems which are relevant to certain models of quantum rings and lenses [8].

The key idea [9, 7] is to pick an integrable system with a $2(N-1)$-dimensional *compact* phase space,

$$\mathcal{H} = \mathcal{H}(I_i), \qquad \{I_i, \Phi_j^0\} = \delta_{ij}, \qquad \Phi_i^0 \in [0, 2\pi), \qquad i, j = 1, \ldots, N-1, \qquad (1.1)$$

in terms of its action-angle variables, and add a *radial* part to it,

$$H = \frac{p_r^2}{2} + \frac{\mathcal{H}(I_i)}{r^2} + V(r), \qquad \{p_r, r\} = 1, \qquad r \in [0, \infty) \quad \text{or} \quad [0, r_0). \qquad (1.2)$$

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Here, we introduced a radial coordinate r and momentum p_r and obtain an extended model with N degrees of freedom. The extended configuration space is a cone over the original compact configuration space. If the latter is just the sphere S^{N-1}, we can obtain, in particular, the three model spaces of constant curvature:

$$S^N \quad : \quad r = r_0 \sin \chi, \qquad p_r = r_0^{-1} p_\chi, \qquad V(r) \to V(r_0 \tan \chi), \tag{1.3}$$

$$\mathbb{R}^N \quad : \quad r = r_0 \chi, \qquad p_r = r_0^{-1} p_\chi, \qquad V(r) \to V(r_0 \chi), \tag{1.4}$$

$$H^N \quad : \quad r = r_0 \sinh \chi, \qquad p_r = r_0^{-1} p_\chi, \qquad V(r) \to V(r_0 \tanh \chi), \tag{1.5}$$

where r_0 is the radial scale and $\{p_\chi, \chi\} = 1$ is a dimensionless canonical pair. Hence, for a particle on the sphere S^N (the sine-cone over S^{N-1}) or on the hyperboloid H^N (the hyperbolic cone over S^{N-1}) obe gets the Hamiltonians

$$H = \frac{p_\chi^2}{2r_0^2} + \frac{\mathcal{H}}{r_0^2 \sin^2 \chi} + V(r_0 \tan \chi) \qquad \text{and} \qquad H = \frac{p_\chi^2}{2r_0^2} + \frac{\mathcal{H}}{r_0^2 \sinh^2 \chi} + V(r_0 \tanh \chi), \tag{1.6}$$

respectively.

As an example, when $\mathcal{H}$ defines the Landau problem, i.e. a particle on S^2 moving in the magnetic field generated by a Dirac monopole located at the center of sphere, we arrive at the particle on $\mathbb{R}^3$ under the influence of this Dirac monopole. The extended system remains integrable for two prominent choices of the radial potential,

$$V(r) = V_{\text{osc}}(r) = \tfrac{1}{2}\omega^2 r^2 \qquad \text{and} \qquad V(r) = V_{\text{cou}}(r) = -\frac{\gamma}{r}, \tag{1.7}$$

with frequency ω and (positive) coupling γ, respectively. For $\mathbb{R}^N$, these are the familiar oscillator and Coulomb potentials, while for S^N they have been named Higgs oscillator [4] and Schrödinger-Coulomb [3], respectively.

If the system is spherically symmetric, i.e. S^{N-1} invariant, the compact Hamiltonian $\mathcal{H}$ is just given by the $SO(N)$ Casimir function J^2, which defines the kinetic energy of a free particle on S^{N-1}. Deviations from spherical symmetry are encoded in $\mathcal{H}$. In other words, replacing J^2 by the Hamiltonian of some compact $N{-}1$-dimensional integrable system defines a deformation of the N-dimensional oscillator and Coulomb systems.

Particular examples with $N{=}2$ are the so-called Tremblay-Turbiner-Winternitz (TTW) [10] and Post-Winternitz (PW) [11] models, defined on $\mathbb{R}^2$, which have attracted some interest recently (see, e.g. [12] and references therein). There the compact subsystem on the circle S^1 is just the famous Pöschl-Teller system [14],

$$\mathcal{H} = \mathcal{H}_{\text{PT}} = \frac{p_\varphi^2}{2} + \frac{k^2 \alpha_1^2}{2 \sin^2 k\varphi} + \frac{2k^2 \alpha_1^2}{\cos^2 k\varphi} \qquad \text{with} \qquad k \in \mathbb{N}. \tag{1.8}$$

The paper is arranged as follows. In Section 2 we give the necessary information on action-angle variables and present the general formulae for the systems with Hamiltonians (1.2) and (1.6). Explicit expressions for action-angle variables for these systems with the potentials (1.7) are computed in Section 3. In Section 4, we construct the spherical and pseudospherical generalizations of the TTW and PW systems. We demonstrate the superintegrability of these systems write down their hidden constants of motion. In an Appendix we provide the action-angle variables for a free particle on the $(N{-}1)$-dimensional sphere, which yields the complete set of action-angle variables for the N-dimensional oscillator and Coulomb systems as well as their spherical and pseudospherical analogs.

2 Action-angle variables

The well-known Liouville theorem gives an exact criterion for the integrability of an N-dimensional mechanical system: the existence of N mutually commuting constants of motion

$$\mathbf{F} = (F_1 {\equiv} H, F_2, \ldots, F_N) \qquad \text{with} \qquad \{F_\mu, F_\nu\} = 0 \qquad \text{for} \quad \mu, \nu = 1, \ldots, N. \tag{2.1}$$

The theorem also states that the compact and connected level surfaces $M_f = \big((p_\mu, q_\nu) : F_\mu = \text{const}\big)$ are diffeomorphic to N-dimensional tori T^N. The particle performs a free motion in the natural angular coordinates

$\boldsymbol{\Phi} = (\Phi_1, \ldots, \Phi_N)$ parameterizing such a torus. Their conjugate momenta, the so-called action variables $\mathbf{I} = (I) \equiv (I_1, \ldots, I_N)$, are conserved and thus are functions of the constants of motion, $\mathbf{I} = \mathbf{I}(\mathbf{F})$. Hence, there exists a canonical transformation $(\mathbf{p}, \mathbf{q}) \mapsto (\mathbf{I}, \boldsymbol{\Phi})$, after which the Hamiltonian depends on the (constant) action variables only. Consequently, the equations of motion read

$$\frac{d\mathbf{I}}{dt} = 0 \qquad \text{and} \qquad \frac{d\boldsymbol{\Phi}}{dt} = \frac{\partial H(I)}{\partial \mathbf{I}} \qquad \text{for} \quad \boldsymbol{\Phi} \in T^N. \tag{2.2}$$

The general construction of action-angle variables proceeds as follows [1]. First, fix a level surface $\mathbf{F} = \mathbf{c}$. Second, introduce the generating function S for the canonical transformation $(\mathbf{p}, \mathbf{q}) \mapsto (\mathbf{I}, \boldsymbol{\Phi})$ as the indefinite integral

$$S(\mathbf{c}, \mathbf{q}) = \int_{\mathbf{F} = \mathbf{c}} \mathbf{p} \cdot d\mathbf{q}, \tag{2.3}$$

where $\mathbf{p} = \mathbf{p}(\mathbf{c}, \mathbf{q})$ by use of the equations of motion. The action variables $\mathbf{I}$ can be obtained via

$$I_\mu(\mathbf{c}) = \frac{1}{2\pi} \oint_{\gamma_\mu} \mathbf{p} \cdot d\mathbf{q}, \tag{2.4}$$

where γ_μ is some homology cycle of the level surface $\mathbf{F} = \mathbf{c}$. Third, invert these relations to get $\mathbf{c} = \mathbf{c}(\mathbf{I})$. Fourth, the angle variables $\boldsymbol{\Phi}$ can be found from the expression

$$\boldsymbol{\Phi}(\mathbf{c}, \mathbf{q}) = \left. \frac{\partial S(\mathbf{c}(\mathbf{I}), \mathbf{q})}{\partial \mathbf{I}} \right|_{\mathbf{I} \to \mathbf{I}(\mathbf{c})}. \tag{2.5}$$

Fifth, one may restore $\mathbf{c} = \mathbf{c}(\mathbf{p}, \mathbf{q})$ to arrive at explicit formulae for the canonical transformation.

Assume now that the compact subsystem is already formulated in terms of action-angle variables (I_i, Φ_i^0), with $i = 1, \ldots, N-1$, while the radial part is given by (p_r, r). We characterize the level sets by $(H \equiv E, I_i)$. The generating function for the extended system (1.2) then reads

$$S(E, I_i, r, \Phi_i^0) = \sqrt{2} \int dr \sqrt{E - \frac{\mathcal{H}(I)}{r^2} - V(r)} + \sum_{i=1}^{N-1} I_i \Phi_i^0. \tag{2.6}$$

¿From this function we immediately get the action variables $I_i = I_i$ and

$$I_r(E, I_i) = \frac{\sqrt{2}}{2\pi} \oint dr \sqrt{E - \frac{\mathcal{H}(I)}{r^2} - V(r)}. \tag{2.7}$$

The corresponding angle variables are given by

$$\Phi_r = \frac{1}{\sqrt{2}} \frac{\partial E}{\partial I_r} \int \frac{dr}{\sqrt{E - \frac{\mathcal{H}(I)}{r^2} - V(r)}} \quad \text{and} \quad \Phi_i = \Phi_i^0 + \frac{\partial E/\partial I_i}{\partial E/\partial I_r} \Phi_r - \frac{1}{\sqrt{2}} \frac{\partial \mathcal{H}(I)}{\partial I_i} \int \frac{dr}{r^2 \sqrt{E - \frac{\mathcal{H}(I)}{r^2} - V(r)}}. \tag{2.8}$$

Making in (2.6) and (2.8) the replacements described in (1.3) or (1.5), we shall get the system on the N-sphere or -pseudosphere. Of course, for the full construction of the action-angle variables, we need to provide the action-angle variables of the subsystem $\mathcal{H}$.

3 Deformed oscillator and Coulomb problems

In this section we present the action-angle variables (I_r, Φ_r, Φ_i) for the deformed oscillator and Coulomb systems given by the expressions (1.2)–(1.7). The action variables I_i of the "angular Hamiltonian" $\mathcal{H}$ remain unchanged, while the angle variables Φ_i^0 receive corrections, as seen in (2.8). For notational simplicity we abbreviate $H(\mathbf{p}, \mathbf{q}) = E$, put $r_0 = 1$ and drop the argument I_i of $\mathcal{H}$. In the following, we list the results for each of the six combinations in the table below:

radial potential		oscillator	Coulomb
metric cone:	$\mathbb{R}^N$	Euclidean oscillator	Euclidean Coulomb
sine-cone:	S^N	spherical Higgs oscillator	spherical Schrödinger-Coulomb
hyperbolic cone:	H^N	pseudospherical Higgs oscillator	pseudospherical Schrödinger-Coulomb

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Euclidean oscillator

$$H_{\text{osc}} = \frac{p_r^2}{2} + \frac{\mathcal{H}}{r^2} + \frac{\omega^2 r^2}{2} = \omega\big(2I_r + \sqrt{2\mathcal{H}}\big), \tag{3.1}$$

$$I_r = \frac{E}{2\omega} - \sqrt{\frac{\mathcal{H}}{2}} \qquad \text{and} \qquad \Phi_r = -\arcsin\Big(\frac{E - r^2\omega^2}{\sqrt{E^2 - 2\mathcal{H}\omega^2}}\Big), \tag{3.2}$$

$$\Phi_i = \Phi_i^0 + \frac{1}{2\sqrt{2\mathcal{H}}}\,\frac{\partial \mathcal{H}}{\partial I_i}\Big[\Phi_r - \arcsin\Big(\frac{E\,r^2 - 2\mathcal{H}}{r^2\sqrt{E^2 - 2\omega^2\mathcal{H}}}\Big)\Big]. \tag{3.3}$$

Euclidean Coulomb

$$H_{\text{cou}} = \frac{p_r^2}{2} + \frac{\mathcal{H}}{r^2} - \frac{\gamma}{r} = -\frac{\gamma^2}{2\big(I_r + \sqrt{2\mathcal{H}}\big)^2}, \tag{3.4}$$

$$I_r = \frac{\gamma}{\sqrt{-2E}} - \sqrt{2\mathcal{H}} \qquad \text{and} \qquad \Phi_r = -\frac{2}{\gamma}\sqrt{E\mathcal{H} - E\,r(E\,r + \gamma)} - \arcsin\Big(\frac{2E\,r + \gamma}{\sqrt{4E\mathcal{H} + \gamma^2}}\Big), \tag{3.5}$$

$$\Phi_i = \Phi_i^0 + \sqrt{\frac{2}{\mathcal{H}}}\,\frac{\partial \mathcal{H}}{\partial I_i}\Big[\Phi_r - \frac{1}{2}\arcsin\Big(\frac{\gamma\,r - 2\mathcal{H}}{r\sqrt{4E\mathcal{H} + \gamma^2}}\Big)\Big]. \tag{3.6}$$

Spherical Higgs oscillator

$$H_{\text{s-higgs}} = \frac{p_\chi^2}{2} + \frac{\mathcal{H}}{\sin^2\chi} + \frac{\omega^2\tan^2\chi}{2} = \frac{1}{2}\Big(2I_\chi + \sqrt{2\mathcal{H}} + \omega\Big)^2 - \frac{\omega^2}{2}, \tag{3.7}$$

$$I_\chi = \frac{1}{2}\Big(\sqrt{2E + \omega^2} - \sqrt{2\mathcal{H}} - \omega\Big) \qquad \text{and}$$

$$\Phi_\chi = -2\arcsin\Big(\frac{(2E + \omega^2)\cos 2\chi + 2\mathcal{H} - \omega^2}{\sqrt{(2E + \omega^2)^2 - 2(2\mathcal{H} + \omega^2)(2E + \omega^2) + (2\mathcal{H} - \omega^2)^2}}\Big), \tag{3.8}$$

$$\Phi_i = \Phi_i^0 + \frac{1}{2\sqrt{2\mathcal{H}}}\,\frac{\partial \mathcal{H}}{\partial I_i}\Big[\Phi_\chi + \arctan\Big(\frac{(E + \mathcal{H})\cos 2\chi - E + 3\mathcal{H}}{\sqrt{2\mathcal{H}}\,\sqrt{2E - 4\mathcal{H} - \omega^2 - (4\mathcal{H} - 2\omega^2)\cos 2\chi - (2E + \omega^2)\cos^2 2\chi}}\Big)\Big]. \tag{3.9}$$

Spherical Schrödinger-Coulomb

$$H_{\text{s-sch-cou}} = \frac{p_\chi^2}{2} + \frac{\mathcal{H}}{\sin^2\chi} - \gamma\cot\chi = \frac{1}{2}\Big(I_\chi + \sqrt{2\mathcal{H}}\Big)^2 - \frac{\gamma^2}{2(I_\chi + \sqrt{2\mathcal{H}})^2}, \tag{3.10}$$

$$I_\chi = \sqrt{E + \sqrt{E^2 + \gamma^2}} - \sqrt{2\mathcal{H}} \qquad \text{and} \qquad \Phi_\chi = \text{Im}\Big[\frac{2\sqrt{\mathcal{H}(E + \mathrm{i}\gamma)}}{\sqrt{E + \sqrt{E^2 + \gamma^2}}}\log\zeta\Big] \tag{3.11}$$

$$\Phi_i = \Phi_i^0 + \frac{1}{\sqrt{2\mathcal{H}}}\,\frac{\partial \mathcal{H}}{\partial I_i}\Big[\Phi_\chi + \arcsin\Big(\frac{2\mathcal{H}\cot\chi - \gamma}{\sqrt{4(E - \mathcal{H})\mathcal{H} + \gamma^2}}\Big)\Big], \tag{3.12}$$

$$\text{where} \qquad \zeta = \frac{4}{\sqrt{\mathcal{H}}}\mathrm{e}^{\mathrm{i}(\chi + \frac{\pi}{2})}\sin\chi\Big(1 + \sqrt{E - \frac{\mathcal{H}}{\sin^2\chi} + \gamma\cot\chi}\Big) + \frac{(4\mathcal{H} - 2\mathrm{i}\gamma)\sqrt{E + \mathrm{i}\gamma}}{\sqrt{\mathcal{H}(E^2 + \gamma^2)}}. \tag{3.13}$$

Pseudospherical Higgs oscillator

$$H_{\text{ps-higgs}} = \frac{p_\chi^2}{2} + \frac{\mathcal{H}}{\sinh^2\chi} + \frac{\omega^2 \tanh^2\chi}{2} = \frac{\omega^2}{2} - \frac{1}{2}\left(2I_\chi + \sqrt{2\mathcal{H}} - \omega\right)^2, \tag{3.14}$$

$$I_\chi = \frac{1}{2}\left(\omega - \sqrt{2\mathcal{H}} - \sqrt{\omega^2 - 2E}\right), \qquad \text{and}$$

$$\Phi_\chi = -2\arctan\left(\frac{\left(1 - \sqrt{1-\eta^2}\right)\left(E + \mathcal{H} - \omega^2\right)}{\eta\,\omega\sqrt{\omega^2 - 2E}} + \frac{\sqrt{(E+\mathcal{H})^2 - 2\mathcal{H}\omega^2}}{\omega\sqrt{\omega^2 - 2E}}\right), \tag{3.15}$$

$$\Phi_i = \Phi_i^0 + \frac{1}{2\sqrt{2\mathcal{H}}}\,\frac{\partial\mathcal{H}}{\partial I_i}\left[\Phi_\chi - 2\arctan\left(\frac{\left(1 - \sqrt{1-\eta^2}\right)\left(E + \mathcal{H}\right)}{\eta\,\omega\sqrt{2\mathcal{H}}} + \frac{\sqrt{(E+\mathcal{H})^2 - 2\mathcal{H}\omega^2}}{\omega\sqrt{2\mathcal{H}}}\right)\right], \tag{3.16}$$

$$\text{where} \qquad \eta = \frac{\omega^2\tanh^2\chi - (E+\mathcal{H})}{\sqrt{(E+\mathcal{H})^2 - 2\mathcal{H}\omega^2}}. \tag{3.17}$$

Pseudospherical Schrödinger-Coulomb

$$H_{\text{ps-sch-cou}} = \frac{p_\chi^2}{2} + \frac{\mathcal{H}}{\sinh^2\chi} - \gamma\coth\chi = -\frac{1}{2}\left(I_\chi + \sqrt{2\mathcal{H}}\right)^2 - \frac{\gamma^2}{2(I_\chi + \sqrt{2\mathcal{H}})^2}, \tag{3.18}$$

$$I_\chi = \frac{1}{\sqrt{2}}\left(\sqrt{-E+\gamma} - \sqrt{-E-\gamma} - 2\sqrt{\mathcal{H}}\right) \qquad \text{and}$$

$$\Phi_\chi = \frac{\sqrt{-E+\gamma}}{\sqrt{2}(\sqrt{-E-\gamma} - \sqrt{-E+\gamma})}\arctan\left(\frac{\sqrt{4\mathcal{H}(E+\mathcal{H}) + \gamma^2} + (\gamma - 2\mathcal{H})\eta}{2\sqrt{\mathcal{H}(-E-\gamma)}\sqrt{1-\eta^2}}\right)$$

$$\qquad - \frac{\sqrt{-E-\gamma}}{\sqrt{2}(\sqrt{-E-\gamma} - \sqrt{-E+\gamma})}\arctan\left(\frac{\sqrt{4\mathcal{H}(E+\mathcal{H}) + \gamma^2} + (\gamma + 2\mathcal{H})\eta}{2\sqrt{\mathcal{H}(-E+\gamma)}\sqrt{1-\eta^2}}\right), \tag{3.19}$$

$$\Phi_i = \Phi_i^0 + \frac{1}{\sqrt{2\mathcal{H}}}\,\frac{\partial\mathcal{H}}{\partial I_i}\left[\Phi_\chi + \frac{1}{\sqrt{2}}\arcsin\left(\frac{2\mathcal{H}\cot\chi - \gamma}{\sqrt{4(E+\mathcal{H})\mathcal{H} + \gamma^2}}\right)\right]. \tag{3.20}$$

4 Generalizations of the Tremblay-Turbiner-Winternitz system

As mentioned in the Introduction, action-angle variables elegantly explain the superintegrability of the recently suggested deformation of the two-dimensional oscillator system introduced by Tremblay-Turbiner-Winternitz (TTW) [10] and also of the Coulomb versions treated by Post-Winternitz (PW) [11]. They also allow us to construct analogous deformations of other superintegrable systems.

Our generalizations of the TTW and PW systems are defined by (1.2)–(1.7) with N=2, where the one-dimensional "angular" Hamiltonian $\mathcal{H}$ is given by the generalized Pöschl-Teller system on the circle (1.8) [14]. The action-angle variables of this subsystem are given by [7]

$$I_{\text{PT}} = \tfrac{1}{k}\sqrt{2\mathcal{H}_{\text{PT}}} - (\alpha_1 + \alpha_2) \qquad \text{and} \qquad \Phi_{\text{PT}} = \tfrac{1}{2}\arcsin\left\{\tfrac{1}{a}\left[\cos 2k\varphi + b\right]\right\}, \tag{4.1}$$

where

$$a = \sqrt{1 - \frac{k^2(\alpha_1^2 + \alpha_2^2)}{\mathcal{H}_{\text{PT}}} + \left(\frac{k^2(\alpha_1^2 - \alpha_2^2)}{2\mathcal{H}_{\text{PT}}}\right)^2} \qquad \text{and} \qquad b = \frac{k^2(\alpha_2^2 - \alpha_1^2)}{2\mathcal{H}_{\text{PT}}}, \tag{4.2}$$

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so that the Hamiltonian reads

$$\mathcal{H}_{\mathrm{PT}} = \tfrac{1}{2}(k\tilde{I}_{\mathrm{PT}})^2 \qquad \text{with} \qquad \tilde{I}_{\mathrm{PT}} \equiv I_{PT} + \alpha_1 + \alpha_2 \ \in [\alpha_1+\alpha_2, \infty). \tag{4.3}$$

Clearly, in action-angle variables, the Pöschl-Teller Hamiltonian coincides with the Hamiltonian of a free particle on a circle of radius k, but with a different domain for the action variable. Hence, choosing the potential in (1.2) to be of oscillator or Coulomb type, the extended system will be superintegrable. More precisely, in the variables $\big(p_r, r, I_{\mathrm{PT}}(p_\varphi, \varphi), \Phi_{\mathrm{PT}}(p_\varphi, \varphi)\big)$, this system takes the form of a conventional two-dimensional oscillator or Coulomb system on the cone. Hence, for rational values of k these systems possess hidden symmetries. For the oscillator case, the hidden constants of motion have been constructed in [13]. Here, we extend their results to the Coulomb case [11] as well as to the TTW- and PW-like systems on spheres and pseudospheres.

For the three spaces of constant curvature and for the oscillator potential, the action-angle Hamiltonians are

$$H_\omega = \begin{cases} \omega\,(2I_\chi + k\tilde{I}_{\mathrm{PT}}) & \text{for} \quad \mathbb{R}^2 \\ \tfrac{1}{2}(2I_\chi + k\tilde{I}_{\mathrm{PT}} + \omega)^2 - \frac{\omega^2}{2} & \text{for} \quad S^2 \\ -\tfrac{1}{2}(2I_\chi + k\tilde{I}_{\mathrm{PT}} - \omega)^2 + \frac{\omega^2}{2} & \text{for} \quad H^2 \end{cases} \tag{4.4}$$

and depend only on the combination $2I_\chi + k\tilde{I}_{\mathrm{PT}}$. Thus, the evolution of the angle variables is given by

$$\Phi_\chi(t) = 2\,\Omega\,t \qquad \text{and} \qquad \Phi_\varphi(t) = k\,\Omega\,t \qquad \text{with} \qquad \Omega = \frac{dH_\omega}{d(2I_\chi + k\tilde{I}_{\mathrm{PT}})}. \tag{4.5}$$

For rational values of k the trajectories are closed. It then follows that the hidden constant of motion is

$$I_{\mathrm{hidden}} = \cos\big(m\Phi_\chi - 2n\Phi_\varphi\big) \qquad \text{for} \quad k = m/n. \tag{4.6}$$

Explicitly, this hidden constant of motion reads:

Euclidean TTW system

$$I_{add} = CM_m\left(\frac{Er^2 - 2\mathcal{H}_{PT}}{r^2\sqrt{E^2 - 2\omega^2\mathcal{H}_{PT}}}\right)CM_n\left(\tfrac{1}{a}\left[\cos 2k\varphi + b\right]\right)$$

$$+ SM_m\left(\frac{Er^2 - 2\mathcal{H}_{PT}}{r^2\sqrt{E^2 - 2\omega^2\mathcal{H}_{PT}}}\right)SM_n\left(\tfrac{1}{a}\left[\cos 2k\varphi + b\right]\right) \tag{4.7}$$

where we denoted

$$CM_n(x) = \cos(n\arcsin x) = \sum_{i=0}^{[\frac{n}{2}]}(-1)^i C_n^{2i} x^{2i}\sqrt{1-x^2}^{\,n-2i},$$

$$SM_n(x) = \sin(n\arcsin x) = \sum_{i=0}^{[\frac{n-1}{2}]}(-1)^i C_n^{2i+1} x^{2i+1}\sqrt{1-x^2}^{\,n-2i-1}. \tag{4.8}$$

Spherical TTW system

$$I_{add} = CM_m\left(\frac{\xi}{\sqrt{\xi^2+1}}\right)CM_n\left(\tfrac{1}{a}\left[\cos 2k\varphi + b\right]\right) - SM_m\left(\frac{\xi}{\sqrt{\xi^2+1}}\right)SM_n\left(\tfrac{1}{a}\left[\cos 2k\varphi + b\right]\right) \tag{4.9}$$

where

$$\xi = \frac{(E + \mathcal{H}_{PT})\cos 2\chi - E + 3\mathcal{H}_{PT}}{\sqrt{2\mathcal{H}_{PT}}\sqrt{2E - 4\mathcal{H}_{PT} - \omega^2 - (4\mathcal{H}_{PT} - 2\omega^2)\cos 2\chi - (2E + \omega^2)\cos^2 2\chi}}. \tag{4.10}$$

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Pseudospherical TTW system

$$I_{add} = CM_{2m}\left(\frac{\xi}{\sqrt{\xi^2+1}}\right)CM_n\left(\tfrac{1}{a}\left[\cos 2k\varphi + b\right]\right) + SM_{2m}\left(\frac{\xi}{\sqrt{\xi^2+1}}\right)SM_n\left(\tfrac{1}{a}\left[\cos 2k\varphi + b\right]\right) \qquad (4.11)$$

where

$$\xi = \frac{\sqrt{(E+\mathcal{H}_{PT})^2 - 2\mathcal{H}_{PT}\omega^2} - \sqrt{2(E+\mathcal{H}_{PT})\omega^2\tanh^2\chi - \omega^4\tanh^4\chi - 2\mathcal{H}_{PT}\omega^2}}{\omega^2\tanh^2\chi - (E+\mathcal{H}_{PT})}\frac{E+\mathcal{H}_{PT}}{\omega\sqrt{2\mathcal{H}_{PT}}}$$

$$+ \frac{\sqrt{(E+\mathcal{H}_{PT})^2 - 2\mathcal{H}_{PT}\omega^2}}{\omega\sqrt{2\mathcal{H}_{PT}}}. \qquad (4.12)$$

Thus, choosing the Higgs oscillator on the (pseudo)sphere, we get a superintegrable (pseudo)spherical analog of the TTW oscillator.

The construction of superintegrable deformations of the Coulomb system, i.e. the PW model and its generalization to the (pseudo)spherical environment, proceeds completely similarly. The Hamiltonians

$$H_\gamma = \begin{cases} -\frac{\gamma^2}{2}(I_\chi + k\tilde{I}_{\mathrm{PT}})^{-2} & \text{for} \quad \mathbb{R}^2 \\ -\frac{\gamma^2}{2}(I_\chi + k\tilde{I}_{\mathrm{PT}})^{-2} + \frac{1}{2}(I_\chi + k\tilde{I}_{\mathrm{PT}})^2 & \text{for} \quad S^2 \\ -\frac{\gamma^2}{2}(I_\chi + k\tilde{I}_{\mathrm{PT}})^{-2} - \frac{1}{2}(I_\chi + k\tilde{I}_{\mathrm{PT}})^2 & \text{for} \quad H^2 \end{cases} \qquad (4.13)$$

depend only on the combination $I_\chi + k\tilde{I}_{\mathrm{PT}}$, and for rational $k = m/n$ the trajectories are closed, supporting

$$I_{\mathrm{hidden}} = \cos\left(m\Phi_\chi - n\Phi_\varphi\right). \qquad (4.14)$$

Explicitly this constant of motion reads:

Euclidean PW system

$$I_{add} = CM_{2m}\left(\frac{\gamma r - 2\mathcal{H}_{PT}}{r\sqrt{4E\mathcal{H}_{PT}+\gamma^2}}\right)CM_n\left(\tfrac{1}{a}\left[\cos 2k\varphi + b\right]\right)$$

$$+ SM_{2m}\left(\frac{\gamma r - 2\mathcal{H}_{PT}}{r\sqrt{4E\mathcal{H}_{PT}+\gamma^2}}\right)SM_n\left(\tfrac{1}{a}\left[\cos 2k\varphi + b\right]\right). \qquad (4.15)$$

Spherical PW system

$$I_{add} = CM_{2m}\left(\frac{2\mathcal{H}_{PT}\cot\chi - \gamma}{\sqrt{4(E-\mathcal{H}_{PT})\mathcal{H}_{PT}+\gamma^2}}\right)CM_n\left(\tfrac{1}{a}\left[\cos 2k\varphi + b\right]\right)$$

$$- SM_{2m}\left(\frac{2\mathcal{H}_{PT}\cot\chi - \gamma}{\sqrt{4(E-\mathcal{H}_{PT})\mathcal{H}_{PT}+\gamma^2}}\right)SM_n\left(\tfrac{1}{a}\left[\cos 2k\varphi + b\right]\right). \qquad (4.16)$$

Pseudospherical PW system

$$I_{add} = CM_{2m}\left(\frac{2\mathcal{H}_{PT}\coth\chi - \gamma}{\sqrt{4(E+\mathcal{H}_{PT})\mathcal{H}_{PT}+\gamma^2}}\right)CM_n\left(\tfrac{1}{a}\left[\cos 2k\varphi + b\right]\right)$$

$$- SM_{2m}\left(\frac{2\mathcal{H}_{PT}\coth\chi - \gamma}{\sqrt{4(E+\mathcal{H}_{PT})\mathcal{H}_{PT}+\gamma^2}}\right)SM_n\left(\tfrac{1}{a}\left[\cos 2k\varphi + b\right]\right). \qquad (4.17)$$

Thus, choosing the Schrödinger-Coulomb system on the (pseudo)sphere, we get a superintegrable (pseudo)spherical analog of the PW model.

5 Summary and outlook

We have presented the action-angle variables for a particular class of integrable deformations of N-dimensional oscillator and Coulomb systems, on flat space as well as on the sphere and the pseudosphere. These integrable systems were obtained by replacing the angular part of the ordinary (spherically symmetric) oscillator or Coulomb systems with a suitable compact integrable system formulated in terms of its action-angle variables.

As the application with $N=2$, we constructed the spherical and pseudospherical generalization of the Tremblay-Turbiner-Winternitz (TTW) and Post-Winternitz (PW) models, demonstrated their superintegrability and computed their hidden constant of motion. For completeness, we also provided the action-angle variables of the undeformed angular subsystem, i.e. for a free particle moving on the $(N-1)$-sphere.

An obvious task is to extend the above example to higher dimensions, by employing generalizations of the Pöschl-Teller systems to construct higher-dimensional analogs of the TTW and PW models.

Appendix: Free particle on S^{N-1}

In this Appendix we recollect the action-angle variables for the "angular Hamiltonian" $\mathcal{H}_{PT}$ appearing in every spherically symmetric N-dimensional system and defining the free motion of a particle on S^{N-1} with radius $r_0 = 1$. It is given by the Casimir function L_N^2 of SO(N),

$$\mathcal{H} = \tfrac{1}{2} L_N^2. \tag{A.1}$$

The embedding of the unit $(N-1)$-sphere into $\mathbb{R}^N$ is given by a set of polar coordinates,

$$
\left.
\begin{aligned}
x_1 &= s_{N-1}\, s_{N-2} \cdots s_3\, s_2\, s_1 \\
x_2 &= s_{N-1}\, s_{N-2} \cdots s_3\, s_2\, c_1 \\
x_3 &= s_{N-1}\, s_{N-2} \cdots s_3\, c_2 \\
&\vdots \\
x_{N-1} &= s_{N-1}\, c_{N-2} \\
x_N &= c_{N-1}
\end{aligned}
\right\}
\qquad
\begin{aligned}
&\text{with} && s_k := \sin\theta_k \quad \text{and} \quad c_k := \cos\theta_k \\
&\text{for} && \theta_1 \in [0, 2\pi)\,, \quad \theta_{k>1} \in [0, \pi) \\
&\text{and} && k = 1, 2, \ldots, N-1.
\end{aligned}
\tag{A.2}
$$

In these coordinates, we have the recursion

$$L_N^2 = p_{N-1}^2 + \frac{L_{N-1}^2}{s_{N-1}^2} \tag{A.3}$$

where p_{N-1} is the momentum conjugate to θ_{N-1}. It is easy to see that the L_k^2 for $k = 1, \ldots, N$ are in involution with each other and, therefore, can be used for constructing action-angle variables. Each variable θ_k defines an independent homology cycle S_k^1 of the torus T^N. The level surfaces $L_k^2 = \text{constant} =: j_k$ are diffeomorphic to T^N.

Following the standard procedure we should compute the N integrals

$$I_k = \frac{1}{2\pi} \oint_{S_k^1} \mathbf{p} \cdot d\mathbf{q} = \frac{1}{2\pi} \oint_{S_k^1} p_k \, d\theta_k = \int_{\theta_k^-}^{\theta_k^+} \sqrt{j_k - \frac{j_{k-1}}{\sin^2\theta_k}} \, d\theta_k, \tag{A.4}$$

where in the second equality we used that the θ_k are mutually orthogonal and the cycles S_k^1 are independent. The integration ranges $[\theta_k^-, \theta_k^+]$ are determined from the condition that the radicants should be non-negative. Substituting

$$u_k = \sqrt{\frac{j_k}{j_k - j_{k-1}}} \, \cos\theta_k, \tag{A.5}$$

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we arrive at

$$I_k = 2\frac{j_k - j_{k-1}}{2\pi\sqrt{j_k}} \int\limits_{-1}^{1} \frac{\sqrt{1-u_k^2}\,du_k}{1 - \frac{j_k - j_{k-1}}{j_k}u_k^2} = \sqrt{j_k} - \sqrt{j_{k-1}}, \qquad \text{so that} \qquad \sqrt{j_k} = \sum_{m=1}^{k} I_m. \tag{A.6}$$

For the generating function we obtain

$$S = \sum_{l=1}^{N-1} S_l \qquad \text{where} \qquad S_l = \int d\theta_l \sqrt{\left(\sum_{m=1}^{l} I_m\right)^2 - \sin^{-2}\theta_l\left(\sum_{m=1}^{l-1} I_m\right)^2}, \tag{A.7}$$

from which we get the angle variables

$$\Phi_k^0 = \frac{\partial S}{\partial I_k} = \frac{\partial S_k}{\partial I_k} + \sum_{l=k+1}^{N-1} \frac{\partial S_l}{\partial I_k} = \int \frac{\sqrt{j_k}\,d\theta_k}{\sqrt{j_k - \frac{j_{k-1}}{\sin^2\theta_k}}} + \sum_{l=k+1}^{N-1} \int \frac{d\theta_l}{\sqrt{j_l - \frac{j_{l-1}}{\sin^2\theta_l}}}\left(\sqrt{j_l} - \frac{\sqrt{j_{l-1}}}{\sin^2\theta_l}\right). \tag{A.8}$$

The first integral can be included in the first part of the sum (as $l=k$), which yields

$$\sum_{l=k}^{N-1} \int \frac{\sqrt{j_l}\,d\theta_l}{\sqrt{j_l - \frac{j_{l-1}}{\sin^2\theta_l}}} = \sum_{l=k}^{N-1} \arcsin u_l. \tag{A.9}$$

After the substitution and abbreviation

$$u_l = \frac{2t_l}{(1+t_l)^2} \qquad \text{and} \qquad a = \sqrt{\frac{j_l - j_{l-1}}{j_l}} < 1, \tag{A.10}$$

respectively, the second part of the sum in (A.8) becomes

$$\sum_{l=k+1}^{N-1} \sqrt{\frac{j_{l-1}}{j_l}} \int dt_l \left[\frac{1}{(t_l-a)^2 + 1 - a^2} + \frac{1}{(t_l+a)^2 + 1 - a^2}\right] = \sum_{l=k+1}^{N-1} \sqrt{\frac{j_{l-1}}{j_l(1-a^2)}}\left[\arctan\frac{t_l-a}{\sqrt{1-a^2}} + \arctan\frac{t_l+a}{\sqrt{1-a^2}}\right]. \tag{A.11}$$

Pulling all together, we finally find

$$\Phi_k^0 = \sum_{l=k}^{N-1} \arcsin u_l + \sum_{l=k+1}^{N-1} \arctan\left(\sqrt{\frac{j_{l-1}}{j_l}}\frac{u_l}{\sqrt{1-u_l^2}}\right). \tag{A.12}$$

To summarize, the action-angle variables for a free particle on S^{N-1} are given by (A.6) and (A.12), with $j_l = L_l^2(p_1,\ldots,p_l,\theta_1,\ldots,\theta_l)$. The angular Hamiltonian (A.1) can be expressed as

$$\mathcal{H} = \tfrac{1}{2}L_N^2 = \frac{1}{2}\left(\sum_{m=1}^{N-1} I_m\right)^2. \tag{A.13}$$

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