

Flat connections and the fundamental group

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In this post we are going to study the correspondence between flat connections and representations of the fundamental group, sometimes called the “Riemann-Hilbert correspondence”. This is a very rich result in that it draws together several fields of Mathematics and leads to new interesting problems like the study of certain moduli spaces or the relationship to Yang-Mills equation. Moreover, this correspondence is the first step to fully understand another correspondence known as the Nonabelian Hodge Theorem.

There are several ways of proving this correspondence. One is through holonomy, and it is maybe the more straightforward since it directly relates flat connections with representations of the fundamental group. In this post however, given that in my last post [CITAR] we already gave a correspondence between representations of the fundamental group and covering spaces, we are going to prove the correspondence by relating flat connections to covering spaces (also called *local systems* in this context).

With this point of view, a general “global” proof of the correspondence can be given by using the Frobenius theorem. We are going to sketch this proof, but I prefer to study in detail the proof for matrix groups, which is more simple and straightforward, and maybe fits better with the “cocycle approach” that I have been maintaining in my previous posts.

Principal bundles and local systems

During all this post we will be using the notions from my post on Torsors and Cocycles [CITAR], and also we will make use of the correspondence between covering spaces and representations that I gave in my previous post [CITAR].

Let me recall that for X a topological space and \mathcal{G} a sheaf of groups over X , a \mathcal{G} torsor is a sheaf of sets on X with nonempty stalks which is endowed with a free and transitive action of \mathcal{G} . What we saw in my post about Torsors and cocycles [CITAR] is that the functor that sends every \mathcal{G} -torsor to its set of transition functions gives an equivalence of categories between the category of \mathcal{G} torsor and the action groupoid $[Z^1(X, \mathcal{G}), C^0(X, \mathcal{G})]$, where these two sets denote the sets of equivalence classes by refinement of pairs (\mathfrak{U}, f) , with \mathfrak{U} an open cover of X and f a Čech 1-cocycle (a Čech 0-cochain, respectively).

For our purposes, we will fix now once and for all X a smooth manifold and G a Lie group, and consider the sheaves \underline{G} , which maps any open set $U \subset X$ to the group $C^\infty(U, G)$ of G -valued smooth functions, and \underline{G} , which maps any open set $U \subset X$ to the group of locally constant functions $U \rightarrow G$.

We will call a G -torsor on X a *principal G -bundle* over X , and we will call a \underline{G} -torsor on X a *G -local system* over X .

Recall that in my previous post [CITAR] we called elements of $Z^1(\mathfrak{U}, \underline{G})$ by the name of \mathfrak{U} -based G -covering spaces. Therefore, a G -local system is an equivalence class by refinement of G -coverings. In that post we also proved the *monodromy theorem* which said that the groupoid $[Z^1(\mathfrak{U}, \underline{G}), C^0(\mathfrak{U}, \underline{G})]$ was equivalent to the groupoid $[\text{Hom}(\pi_1(X, x_0), G), G]$ of G -representations of the fundamental group with the conjugation action, which we called the *Betti groupoid*, \mathfrak{U} being an open cover satisfying certain “good” topological conditions. Since we are on a smooth manifold, every open cover can be refined to one that is “good” in that sense, and thus we get an equivalence between the category of G -local systems and the Betti groupoid.

The purpose of this post is to prove that the category of G -local systems is equivalent to another “differential geometric” category, the category of *flat bundles*. As a consequence, this equivalence will yield an equivalence between this category of flat bundles and the Betti groupoid.

Vector bundles

We will start by proving this equivalence for vector bundles, so we better define what these are.

Consider the *sheaf of smooth complex-valued functions* C_X^∞ on X sending every open set $U \subset X$ to the set $C^\infty(U, \mathbb{C})$. A *sheaf of C_X^∞ modules* is a sheaf of abelian groups E on X such that, on any $U \subset X$, the group $E(U)$ is a $C_X^\infty(U)$ -module and such that the restriction homomorphisms $E(V) \rightarrow E(U)$ are $C_X^\infty(U)$ -linear.

A *complex vector bundle of rank n on X* is a locally free sheaf E of C_X^∞ modules. This means that there exists an open cover \mathfrak{U} of X and isomorphisms $\varphi_U : E|_U \rightarrow C_X^\infty|_U^n$ for every $U \in \mathfrak{U}$. The pair (\mathfrak{U}, φ) , with $\varphi = \{\varphi_U\}_{U \in \mathfrak{U}}$ is called a *trivialization* of E .

To any rank n complex vector bundle E , fixing a trivialization (\mathfrak{U}, φ) , we can associate its set of transition functions $g_{UV} : U \cap V \rightarrow \text{GL}(n, \mathbb{C})$, defined as

$$g_{UV} = \varphi_V|_{U \cap V} \circ \varphi_U^{-1}|_{U \cap V},$$

for $U, V \in \mathfrak{U}$, $U \cap V \neq \emptyset$. Clearly the g_{UV} define a cocycle, so we get a map

$$\{\text{Rank } n \text{ vector bundles}\} \longrightarrow Z^1(\mathfrak{U}, \text{GL}(n, \mathbb{C})).$$

A *gauge transformation* of a vector bundle E is a C_X^∞ -linear sheaf automorphism $\xi : E \rightarrow E$. Fixing a trivialization (\mathfrak{U}, φ) , to any gauge transformation ξ we can associate the element $f \in C^0(\mathfrak{U}, \text{GL}(n, \mathbb{C}))$ defined as

$$f_U = \varphi_U \circ \xi|_U \circ \varphi_U^{-1},$$

for $U \in \mathfrak{U}$.

Thus, if we denote by \mathbf{Vect}_n the category whose objects are vector bundles on X of rank n and where morphisms are given by gauge transformations, by making a choice of trivialization on any vector bundle, we have defined a functor

$$\mathbf{Vect}_n \longrightarrow [Z^1(X, \text{GL}(n, \mathbb{C})), C^0(X, \text{GL}(n, \mathbb{C}))].$$

This functor is fully faithful. Fix E is a vector bundle and we fix (\mathfrak{U}, φ) a trivialization of E . Clearly, the map $\xi \mapsto f$, with f defined as above gives a bijection between gauge transformations and elements of $C^0(\mathfrak{U}, \text{GL}(n, \mathbb{C}))$.

This functor is essentially surjective. Let (\mathfrak{U}, g) , with $g \in Z^1(\mathfrak{U}, \text{GL}(n, \mathbb{C}))$ be a pair representing an element of $Z^1(X, \text{GL}(n, \mathbb{C}))$. The way to recover now the vector bundle is similar to how we recovered a torsor from a cocycle. Define the presheaf

$$E(U) = \coprod_{V \in \mathfrak{U}} C_X^\infty(U \cap V)^n / \sim,$$

where $f \in C_X^\infty(U \cap V)^n$ and $f' \in C_X^\infty(U \cap V')^n$, with $V \cap V'$ are related by \sim if

$$f|_{U \cap V \cap V'} = g_{VV'} f'|_{U \cap V \cap V'}.$$

This presheaf verifies the sheaf condition by construction and it is clearly locally free of rank n . If (\mathfrak{V}, φ) is any trivialization of E , then it is easy to check that, after passing to a common refinement, its transition functions are on the same orbit as the refinement of the cocycle g by a 0-cochain.

Therefore, the category \mathbf{Vect}_n is equivalent to that of $\text{GL}(n, \mathbb{C})$ -torsors (that is, principal $\text{GL}(n, \mathbb{C})$ -bundles).

Connections in vector bundles

Let E be a vector bundle and consider the bundles Ω_X^k consisting on complex-valued smooth differential k -forms on X . For example, Ω_X^1 is the *cotangent bundle* of X (actually, it is the sheaf of sections of the cotangent bundle, but we are already regarding bundles as locally free sheaves).

Definition. A *connection* D on E is a \mathbb{C} -linear operator

$$D : E \rightarrow E \otimes \Omega_X^1$$

such that

$$D(fs) = sdf + fDs,$$

for $f \in C_X^\infty(U)$ and $s \in E(U)$, for every open subset $U \subset X$.

Let D be a connection on a vector bundle E and take an open set $U \in \mathfrak{U}$ in some trivialization (\mathfrak{U}, φ) of E . We define a *frame of E in U* to be a basis $\{e_1, \dots, e_n\}$ of $E(U)$, given that it is a free $C_X^\infty(U)$ -module. For any e_i of the frame, the connection acts as

$$De_i = \sum_j e_j A_i^j,$$

for $A_i^j \in \Omega_X^1(U)$. Using matrix notation, regarding $e = (e_i)$ as a row vector and $A = (A_i^j)$ as a square matrix, we get

$$De = eA.$$

Now, given any other section $s \in E(U)$, we can write $s = \sum_i s^i e_i$, for $s^i \in C_X^\infty(U)$ and we have

$$Ds = \sum_i (ds^i e_i + s^i De_i) = \sum_i ds^i e_i + s^i e_j A_j^i = (d + A)s.$$

The matrix A is called the *connection 1-form* of D on U .

Definition. Let D be a connection on a vector bundle E . We define the *curvature* of D as the operator

$$D^2 : E \rightarrow E \otimes \Omega_X^2.$$

The curvature is a C_X^∞ -linear map since

$$D^2(fs) = D(sdf + fDs) = Ds \wedge df + df \wedge Ds + fD^2s = fD^2s,$$

for $s \in E(U)$ and $f \in C_X^\infty(U)$.

Locally, in some trivializing open set, we have

$$D^2(e) = D(eA) = De \wedge A + edA = e(A \wedge A + dA) = eF_A,$$

for $F_A = dA + A \wedge A$ a matrix of 2-forms which we call the *curvature 2-form*.

Fix E a vector bundle and consider the group \mathcal{G}_E of gauge transformations of E and the set \mathcal{A}_E of all connections on E . We have a natural action of \mathcal{G}_E on \mathcal{A}_E by conjugation: if $s \in E(U)$,

$$(\xi \cdot D)(s) = \xi \circ D \circ \xi^{-1}.$$

Now, take $U \in \mathfrak{U}$ for (\mathfrak{U}, φ) some trivialization and $f_U = \varphi_U \circ \xi|_U \circ \varphi_U^{-1}$. If we choose the local frame e as the inverse image through φ_U of the canonical basis and we consider A the connection 1-form of D in this frame we have

$$e(\xi \cdot A) = (\xi \cdot D)(e) = \xi D(\xi^{-1}(e)) = \xi D(e f_U^{-1}) = \xi (D e f_U^{-1} + e d f_U^{-1}) = e(f_U A f_U^{-1} + f_U d f_U^{-1}),$$

so

$$\xi \cdot A = f_U A f_U^{-1} + f_U d f_U^{-1}.$$

The curvature of the gauge-transformed connection now is

$$(\xi \cdot D)^2 = \xi \circ D \circ \xi^{-1} \circ \xi \circ D \circ \xi^{-1} = \xi \circ D^2 \circ \xi^{-1}.$$

Thus, in the frame defined above,

$$\begin{aligned} e F_{\xi \cdot A} &= \xi(D^2(\xi^{-1}(e))) = D^2(e f_U^{-1}) f_U = D(D(e) f_U^{-1} + e d f_U^{-1}) f_U \\ &= (D^2(e) f_U^{-1} - D(e) \wedge d f_U^{-1} + D(e) \wedge d f_U^{-1}) f_U = D^2(e) = e F_A. \end{aligned}$$

Flat connections

Definition. We say that a connection D on a vector bundle E is flat if its curvature vanishes, that is, $D^2 = 0$. A *flat vector bundle* is a pair (E, D) , where E is a vector bundle and D is a flat connection on E .

If we denote by \mathcal{F}_E the set of flat connections on a vector bundle E , the formulas above show that the action of the group of gauge transformations \mathcal{G}_E on \mathcal{A}_E descends to an action on \mathcal{F}_E . And in consequence the functor relating vector bundles and principal $\mathrm{GL}(n, \mathbb{C})$ -bundles descends to a functor

$$\mathbf{FlatVect}_n \longrightarrow [Z^1(X, \mathrm{GL}(n, \mathbb{C})), C^0(X, \mathrm{GL}(n, \mathbb{C}))],$$

where $\mathbf{FlatVect}_n$ denotes the category whose objects are flat vector bundles of rank n and its morphisms are gauge transformations. The category $\mathbf{FlatVect}_n$ is clearly a groupoid and it is also called the *de Rham groupoid*. The set of equivalence classes of this category is called the *de Rham moduli set* $\mathcal{M}_{\mathrm{de Rham}}$.

We can now state and prove the main theorem of this post:

Theorem. The functor we just gave factors through the inclusion of local systems in principal bundles

$$[Z^1(X, \underline{\mathrm{GL}(n, \mathbb{C})}), C^0(X, \underline{\mathrm{GL}(n, \mathbb{C})})] \hookrightarrow [Z^1(X, \mathrm{GL}(n, \mathbb{C})), C^0(X, \mathrm{GL}(n, \mathbb{C}))],$$

and the resulting functor

$$\mathbf{FlatVect}_n \longrightarrow [Z^1(X, \underline{\mathrm{GL}(n, \mathbb{C})}), C^0(X, \underline{\mathrm{GL}(n, \mathbb{C})})],$$

is an equivalence of categories. In particular, we get a bijection $\mathcal{M}_{\mathrm{de Rham}} \cong \mathcal{M}_{\mathrm{Betti}}$ between the de Rham and the Betti moduli sets.

Since this is all a bit messy, let us state precisely what we are going to prove:

1. Provided any vector bundle E , with a trivialization (\mathfrak{U}, φ) and a flat connection D on E , we have to find locally constant functions $h_{UV} : U \cap V \rightarrow \text{GL}(n, \mathbb{C})$, for $U, V \in \mathfrak{U}$ and $U \cap V \neq \emptyset$ and functions $f_U : U \rightarrow \text{GL}(n, \mathbb{C})$ for $U \in \mathfrak{U}$ such that

$$h_{UV} = f_U g_{UV} f_V^{-1},$$

where g_{UV} are the transition functions of E in the trivialization (\mathfrak{U}, φ) . Moreover, we have to prove that g_{UV} and g'_{UV} are related by some 0-cochain $f \in C^0(\mathfrak{U}, \text{GL}(n, \mathbb{C}))$ if and only if the corresponding h_{UV} and h'_{UV} are related by some 0-cochain $\tilde{f} \in C^0(\mathfrak{U}, \underline{\text{GL}(n, \mathbb{C})})$. This accounts for the factorization and for being fully faithful.

2. Given any pair (\mathfrak{U}, h) , with $h \in Z^1(\mathfrak{U}, \underline{\text{GL}(n, \mathbb{C})})$, we have to construct a vector bundle E over X and a flat connection D on E such that the transition functions of E are given precisely by h . This would show that the resulting functor is essentially surjective.

Let us start by proving 1 by finding the mentioned locally constant functions h_{UV} . Suppose that we could find, for every $U \in \mathfrak{U}$, a frame

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