Flat connections and the fundamental group

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In this post we are going to study the correspondence between flat connections and representations of the fundamental group, sometimes called the “Riemann-Hilbert correspondence”. This is a very rich result in that it draws together several fields of Mathematics and leads to new interesting problems like the study of certain moduli spaces or the relations with Yang-Mills equations. Moreover, this correspondence is the first step to fully understand another correspondence known as the Nonabelian Hodge Theorem.

There are several ways of proving this correspondence. One is through holonomy, and it is maybe the more straightforward since it directly relates flat connections with representations of the fundamental group. In this post however, given that in my last post we already gave a correspondence between representations of the fundamental group and covering spaces, we are going to prove the correspondence by relating flat connections to covering spaces (also called *local systems* in this context).

With this point of view, a general “global” proof of the correspondence can be given by using the Frobenius theorem. We are going to sketch this proof, but I prefer to study in detail the proof for matrix groups, which is more simple and straightforward, and maybe fits better with the “cocycle approach” that I have been mantaining in my previous posts.

Principal bundles and local systems

During all this post we will be using the notions from my post on torsors and cocycles, and also we will make use of the correspondence between covering spaces and representations that I gave in my last post.

Let me recall that for X a topological space and $\mathcal{G}$ a sheaf of groups over X, a $\mathcal{G}$ torsor is a sheaf of sets on X with nonempty stalks which is endowed with a free and transitive action of $\mathcal{G}$. What we saw in my post on torsors and cocycles is that the functor that sends every $\mathcal{G}$-torsor to its set of transition functions gives an equivalence of categories between the category of $\mathcal{G}$ torsor and the action groupoid $[Z^1(X,\mathcal{G}), C^0(X,\mathcal{G})]$, where these two sets denote the sets of equivalence classes by refinement of pairs $(\mathfrak{U}, f)$, with $\mathfrak{U}$ an open cover of X and f a Čech 1-cocycle (a Čech 0-cochain, respectively).

For our purposes, we will fix now once and for all X a smooth manifold and G a Lie group, and consider the sheaves G, which maps any open set $U \subset X$ to the group $C^\infty(U,G)$ of G-valued smooth functions, and $\underline{G}$, which maps any open set $U \subset X$ to the group of locally constant functions $U \to G$.

We will call a G-torsor on X a *principal G-bundle* over X, and we will call a $\underline{G}$-torsor on X a *G-local system* over X.

Recall that in my last post we called elements of $Z^1(\mathfrak{U},\underline{G})$ by the name of $\mathfrak{U}$-based G-covering spaces. Therefore, a G-local system is an equivalence class by refinement of G-coverings. In that post we also proved the *monodromy theorem* which said that the groupoid $[Z^1(\mathfrak{U},\underline{G}), C^0(\mathfrak{U},\underline{G})]$ was equivalent to the groupoid $[\mathrm{Hom}(\pi_1(X,x_0),G),G]$ of G-representations of the fundamental group with the conjugation action, which we called the *Betti groupoid*, $\mathfrak{U}$ being an open cover satisfying certain “good” topological conditions. Since we are on a smooth manifold, every open cover can be refined to one that is “good” in that sense, and thus we get an equivalence between the category of G-local systems and the Betti groupoid. In particular, the *Betti moduli set* $\mathcal{M}_B(G) = \mathrm{Hom}(\pi_1(X,x_0),G)/G$ parametrizes isomorphism classes of G-local systems.

The purpose of this post is to prove that the category of G-local systems is equivalent to another “differential geometric” category, the category of *flat bundles.* As a consequence, this equivalence will yield an equivalence between this category of flat bundles and the Betti groupoid.

Vector bundles

We will start by proving this equivalence for vector bundles, so we better define what these are.

Consider the *sheaf of smooth complex-valued functions* C^∞_X on X sending every open set $U \subset X$ to the set $C^\infty(X, \mathbb{C})$. A *sheaf of* C^∞_X *modules* is a sheaf of abelian groups E on X such that, on any $U \subset X$, the group $E(U)$ is a $C^\infty_X(U)$-module and such that the restriction homomorphisms $E(V) \to E(U)$ are $C^\infty_X(U)$-linear.

A *complex vector bundle of rank n on X* is a locally free sheaf E of C^∞_X modules. This means that there exist an open cover $\mathfrak{U}$ of X and isomorphisms $\varphi_U : E|_U \to C^\infty_X|_U^n$ for every $U \in X$. The pair $(\mathfrak{U}, \varphi)$, with $\varphi = \{\varphi_U\}_{U\in\mathfrak{U}}$ is called a *trivialization* of E.

To any rank n complex vector bundle E, fixing a trivialization $(\mathfrak{U}, \varphi)$, we can associate its set of transition functions $g_{UV} : U \cap V \to \mathrm{GL}(n, \mathbb{C})$, defined as

$$g_{UV} = \varphi_V|_{U\cap V} \circ \varphi_U^{-1}|_{U\cap V},$$

for $U, V \in \mathfrak{U}$, $U \cap V \neq \varnothing$. Clearly the g_{UV} define a cocycle, so we get a map

$$\{\text{Rank } n \text{ vector bundles}\} \longrightarrow Z^1(\mathfrak{U}, \mathrm{GL}(n, \mathbb{C})).$$

A *gauge transformation* between two vector bundles E_1 and E_2 is a C^∞_X-linear sheaf isomorphism $\xi : E_1 \to E_2$. Fixing trivializations $(\mathfrak{U}, \varphi_1)$ and $(\mathfrak{U}, \varphi_2)$ of E_1 and E_2 with the same trivializing open cover, to any gauge transformation ξ we can associate the element $f \in C^0(\mathfrak{U}, \mathrm{GL}(n, \mathbb{C}))$ defined as

$$f_U = \varphi_{2,U} \circ \xi|_U \circ \varphi_{1,U}^{-1},$$

for $U \in \mathfrak{U}$.

Thus, if we denote by $\mathbf{Vect}_n$ the category whose objects are vector bundles on X of rank n and where morphisms are given by gauge transformations, by making a choice of trivialization on any vector bundle, we have defined a functor

$$\mathbf{Vect}_n \longrightarrow [Z^1(X, \mathrm{GL}(n, \mathbb{C})), C^0(X, \mathrm{GL}(n, \mathbb{C}))].$$

This functor is fully faithful. Fix E_1 and E_2 vector bundle with trivializations $(\mathfrak{U}, \varphi_1)$ and $(\mathfrak{U}, \varphi_2)$. Clearly, the map $\xi \mapsto f$, with f defined as above gives a bijection between gauge transformations and elements of $C^0(\mathfrak{U}, \mathrm{GL}(n, \mathbb{C}))$.

This functor is essentially surjective. Let $(\mathfrak{U}, g)$, with $g \in Z^1(\mathfrak{U}, \mathrm{GL}(n, \mathbb{C}))$, be a pair representing an element of $Z^1(X, \mathrm{GL}(n, \mathbb{C}))$. The way to recover now the vector bundle is similar to how we recovered a torsor from a cocycle. Define the presheaf

$$E_g(U) = \coprod_{V\in\mathfrak{U}} C^\infty_X(U \cap V)^n / \sim,$$

where $f \in C^\infty_X(U \cap V)^n$ and $f' \in C^\infty_X(U \cap V')^n$, with $V \cap V'$ are related by $\sim$ if

$$f|_{U\cap V\cap V'} = g_{VV'} f'|_{U\cap V\cap V'}.$$

This presheaf verifies the sheaf condition by construction and it is clearly locally free of rank n. If $(\mathfrak{V}, \varphi)$ is any trivialization of E_g, then it is easy to check that, after passing to a common refinement, its transition functions are on the same orbit as the refinement of the cocycle g by a 0-cochain.

Therefore, the category $\mathbf{Vect}_n$ is equivalent to that of $\mathrm{GL}(n, \mathbb{C})$-torsors (that is, principal $\mathrm{GL}(n, \mathbb{C})$-bundles).

Connections in vector bundles

Let E be a vector bundle and consider the bundles Ω^k_X consisting on complex-valued smooth differential k-forms on X. For example, Ω^1_X is the *cotangent bundle* of X (actually, it is the sheaf of sections of the cotangent bundle, but we are already regarding bundles as locally free sheaves).

Definition. A *connection* D on E is a $\mathbb{C}$-linear operator

$$D : E \to E \otimes \Omega^1_X$$

such that

$$D(fs) = sdf + fDs,$$

for $f \in C^\infty_X(U)$ and $s \in E(U)$, for every open subset $U \subset X$.

Let D be a connection on a vector bundle E and take an open set $U \in \mathfrak{U}$ in some trivialization $(\mathfrak{U}, \varphi)$ of E. We define a *frame of E in U* to be a basis $\{e_1, ..., e_n\}$ of $E(U)$, given that it is a free $C^\infty_X(U)$-module. For any e_i of the frame, the connection acts as

$$De_i = \sum_j e_j A^j_i,$$

for $A^j_i \in \Omega^1_X(U)$. Using matrix notation, regarding $e = (e_i)$ as a row vector and $A = (A^j_i)$ as a square matrix, we get

$$De = eA.$$

Now, given any other section $s \in E(U)$, we can write $s = \sum_i s^i e_i$, for $s^i \in C^\infty_X(U)$ and we have

$$Ds = \sum_i (ds^i e_i + s^i De_i) = \sum_i ds^i e_i + s^i e_j A^i_j = (d + A)s.$$

The matrix A is called the *connection* 1*-form* of D on U.

Definition. Let D be a connection on a vector bundle E. We define the *curvature* of D as the operator

$$D^2 : E \to E \otimes \Omega^2_X.$$

The curvature is a C^∞_X-linear map since

$$D^2(fs) = D(sdf + fDs) = Ds \wedge df + df \wedge Ds + fD^2 s = fD^2 s,$$

for $s \in E(U)$ and $f \in C^\infty_X(U)$.

Locally, in some trivializing open set, we have

$$D^2(e) = D(eA) = De \wedge A + edA = e(A \wedge A + dA) = eF_A,$$

for $F_A = dA + A \wedge A$ a matrix of 2-forms which we call the *curvature* 2*-form*.

Fix E_1 and E_2 vector bundles and consider $\xi : E_1 \to E_2$ a gauge transformation. If D is a connection on E_1 we can define a connection on E_2 as

$$\xi \cdot D = \xi \circ D \circ \xi^{-1}.$$

Now, take trivializations $(\mathfrak{U}, \varphi_1)$ and $(\mathfrak{U}, \varphi_2)$ of E_1 and E_2 and $f_U = \varphi_{2,U} \circ \xi|_U \circ \varphi_{1,U}^{-1}$. Choose now local frames e_1 and e_2 of E_1 and E_2, respectively, on U, such that their images through their respective trivializations yield the canonical basis. Consider A_1 the connection 1-form of D in the frame e_1. If A_2 denotes the connection 1-form of $(\xi \cdot D)$ in the frame e_2, we have

$$e_2 A_2 = (\xi \cdot D)(e_2) = \xi D(\xi^{-1}(e_2)) = \xi D(e_1 f_U^{-1}) = \xi(De_1 f_U^{-1} + e_1 df_U^{-1}) = e_2(f_U A_1 f_U^{-1} + f_U df_U^{-1}),$$

so

$$A_2 = f_U A_1 f_U^{-1} + f_U df_U^{-1}.$$

The curvature of the gauge-transformed connection now is

$$(\xi \cdot D)^2 = \xi \circ D \circ \xi^{-1} \circ \xi \circ D \circ \xi^{-1} = \xi \circ D^2 \circ \xi^{-1}.$$

Thus, in the frames defined above,

$$\begin{aligned} e_2 F_{A_2} &= \xi(D^2(\xi^{-1}(e_2))) = \xi D^2(e_1 f_U^{-1}) = \xi D(D(e_1) f_U^{-1} + e_1 df_U^{-1}) \\ &= \xi(D^2(e_1) f_U^{-1} - D(e_1) \wedge df_U^{-1} + D(e_1) \wedge df_U^{-1}) \\ &= \xi D^2(e_1) f_U^{-1} = \xi(e_1 F_{A_2} f_U^{-1}) = e_2 f_U F_{A_1} f_U^{-1}, \end{aligned}$$

so

$$F_{A_2} = f_U F_{A_1} f_U^{-1}.$$

Flat connections

Definition. We say that a connection D on a vector bundle E is *flat* if its curvature vanishes, that is, $D^2 = 0$. A *flat vector bundle* is a pair (E, D), where E is a vector bundle over X and D is a flat connection on E.

The formulas above show that if a connection is flat then its gauge transformations are also flat, so *flatness is preserved by gauge transformations*.

We can thus define the category $\mathbf{FlatVect}_n$ whose objects are flat vector bundles of rank n on X and its morphisms are given by gauge transformations. This category is clearly a groupoid, and it is called the *de Rham groupoid*. Its set of equivalence classes $\mathcal{M}_{dR}(\mathrm{GL}(n, \mathbb{C}))$ is called the *de Rham moduli set* of $\mathrm{GL}(n, \mathbb{C})$.

We can now state and prove the main theorem of this post:

Theorem. There is an equivalence of groupoids

$$\mathbf{FlatVect}_n \longrightarrow [Z^1(X, \underline{\mathrm{GL}(n, \mathbb{C})}), C^0(X, \underline{\mathrm{GL}(n, \mathbb{C})})].$$

In particular, this gives a bijection $\mathcal{M}_{dR}(\mathrm{GL}(n, \mathbb{C})) \cong \mathcal{M}_B(\mathrm{GL}(n, \mathbb{C}))$ between the de Rham and the Betti moduli sets.

Proof. More precisely, we want to prove the following.

1. Provided any pair (E, D), where E is a vector bundle of rank n and D is a flat connection on E, we have to find an open cover $\mathfrak{U}$ of X and locally constant functions $h_{UV} : U \cap V \to \mathrm{GL}(n, \mathbb{C})$, for $U, V \in \mathfrak{U}$ and $U \cap V \neq \varnothing$. This allows us to define the functor at the level of objects.

2. If we denote by $\mathrm{Hom}((E_1, D_1), (E_2, D_2))$ the set of gauge transformations between two flat bundles (E_1, D_1) and (E_2, D_2) and by $\mathrm{Hom}(h_1, h_2)$ the set of 0-cochains relating the two corresponding cocycles h_1 and h_2, we want to give a bijection $\mathrm{Hom}(D_1, D_2) \to \mathrm{Hom}(h_1, h_2)$. This would define the functor at the level of morphisms as well as prove that it is fully faithful.

3. Given any pair $(\mathfrak{U}, h)$ representing an element of $Z^1(X, \underline{\mathrm{GL}(n, \mathbb{C})})$, we have to construct a flat vector bundle (E, D) such that the corresponding cocycle h' is equivalent to h. This will show that the functor is essentially surjective.

In order to prove 1 we have to find the mentioned locally constant functions h_{UV}. Choose $(\mathfrak{U}, \varphi)$ a trivialization of E. Suppose that we could find, for every $U \in \mathfrak{U}$, a frame ϵ_U of E in U such that $D\epsilon_U = 0$. Consider as above the frame e_U of E in U defined as the inverse image through φ_U of the canonical basis. We have that e_U and ϵ_U are related by some matrix-valued function $f_U : U \to \mathrm{GL}(n, \mathbb{C})$, so that $\epsilon_U = e_U f_U$.

Therefore, if we consider in a nonempty overlap $U \cap V$ the cocycle $h_{UV} = f_V^{-1} g_{UV} f_U$, we have

$$0 = D\epsilon_U = D(e_U f_U) = D(e_V g_{UV} f_U) = D(\epsilon_V f_V^{-1} g_{UV} f_U) = D(\epsilon_V h_{UV}) = D\epsilon_V h_{UV} + \epsilon_V dh_{UV},$$

so $dh_{UV} = 0$ and h_{UV} is locally constant.

It remains to see then why we can find such a frame ϵ_U. By taking the frame e_U as above, we want to find matrix-valued functions $f_U : U \to \mathrm{GL}(n, \mathbb{C})$ satisfying

$$0 = D\epsilon_U = D(e_U f_U) = D(e_U) f_U + e_U df_U = e_U(A_U f_U + df_U),$$

where A_U is the connection 1-form in the frame e_U.

That is, we want to find f_U solutions to the differential equation

$$df_U + A_U f_U = 0.$$

[DUDOSO] If we call $\alpha = df_U + A_U f_U$, the integrability condition of this equation is $\alpha \wedge d\alpha = 0$. Now,

$$d\alpha = dA_U f_U - A_U \wedge df_U = F_{A_U} f_U - A_U \wedge A_U f_U - A_U \wedge df_U = -A_U \wedge \alpha,$$

so $\alpha \wedge d\alpha = 0$. [?????]

Let us prove 3 now. Take $(\mathfrak{U}, h)$ representing an element of $Z^1(X, \underline{\mathrm{GL}(n, \mathbb{C})})$. As above, from $(\mathfrak{U}, h)$ we can recover a vector bundle E_h whose transition functions are given precisely by h. The connection 1-form $A_U = 0$ in any $U \in \mathfrak{U}$ defines a connection on E_h which is trivially flat.

Finally, let us see the proof of 2. Choose trivializations $(\mathfrak{U}, \varphi_1)$ and $(\mathfrak{U}, \varphi_2)$ of E_1 and E_2 with the same trivializing cover. Assume that every $U \in \mathfrak{U}$ is connected and fix x_U a point on every U. Given any gauge transformation ξ, to which we associate a cochain $f \in C^0(\mathfrak{U}, \mathrm{GL}(n, \mathbb{C}))$, we can define a cochain $\tilde{f} \in C^0(\mathfrak{U}, \underline{\mathrm{GL}(n, \mathbb{C})})$ by choosing $\tilde{f}_U = f_U(x_U)$. This assignation yields a map

$$\mathrm{Hom}((E_1, D_1), (E_2, D_2)) \to \mathrm{Hom}(h_1, h_2),$$

which is trivially surjective, since $C^0(\mathfrak{U}, \underline{\mathrm{GL}(n, \mathbb{C})}) \subset C^0(\mathfrak{U}, \mathrm{GL}(n, \mathbb{C}))$.

It remains to prove that this map is injective. Choose ξ_1 and ξ_2 such that $(\xi_1 \cdot D_1) = (\xi_2 \cdot D_1) = D_2$. Therefore, for $\xi = \xi_1^{-1} \circ \xi_2$, we have $(\xi \cdot D_1) = D_1$. Therefore, if f is the 0-cochain associated to ξ, we have, in $U \in \mathfrak{U}$,

$$A_U = f_U A_U f_U^{-1} + f_U df_U^{-1}.$$

Now, if the images of ξ_1 and ξ_2 coincide we have that $f_{1,U}(x_U) = f_{2,U}(x_U)$ for every $U \in \mathfrak{U}$, so $f_U(x_U) = I$, the identity matrix. Therefore,

$$A_U(x_U) = A_U(x_U) + df_U^{-1}(x_U),$$

so $df_U^{-1} = 0$ on x_U. Now, maybe by refining the covering $\mathfrak{U}$, we get $f_U = I$ and $f_{1,U} = f_{2,U}$.

This finishes the proof of the theorem. ■

The correspondence for matrix groups

We have established the correspondence between G-flat bundles and G-representations of the fundamental group for $G = \mathrm{GL}(n, \mathbb{C})$. We are going to sketch now some of the ideas underlying this same correspondence for a general Lie group G. Let us start by considering *groups of complex matrices*, that is, Lie groups G acting on $\mathbb{C}^n$ for some n by multiplication, so that $G \subset \mathrm{GL}(n, \mathbb{C})$.

If G is a group of complex matrices, then the equivalence between rank n vector bundles and principal $\mathrm{GL}(n, \mathbb{C})$-bundles descends to an equivalence between principal G-bundles and some rank n vector bundles *with extra structure* with gauge transformations respecting that structure.

For example, suppose that $G = \mathrm{U}(n)$, the group of $n \times n$ complex matrices A such that $A^\dagger A = I$, where I denotes the identity matrix and $\dagger$ stands for taking transpose and complex-conjugating the entries. These matrices are called *unitary matrices* and $\mathrm{U}(n)$ is the *unitary group*. The unitary matrices are precisely those which preserve the canonical Hermitian product of $\mathbb{C}^n$:

$$(x_1, ..., x_n) \cdot (y_1, ..., y_n) = \bar{x}_1 y_1 + ... + \bar{x}_n y_n.$$

Recall that a *Hermitian product* in a complex vector space V is a mapping $h : V \times V \to \mathbb{C}$ which is linear on the second component and anti-linear on the first (that is, $h(\lambda u, v) = \bar{\lambda} h(u, v)$), definite positive and such that $h(u, v) = \overline{h(v, u)}$. A *Hermitian structure* on a complex vector bundle E is a smoothly varying Hermitian product h_x on each E_x. A pair (E, h) where E is a complex vector bundle and h is a Hermitian structure on E is called a *Hermitian vector bundle*.

If E is the vector bundle associated to a principal $\mathrm{U}(n)$-bundle, since the transition functions are unitary, E will have a canonical Hermitian structure h associated to it. On the other hand, if we consider only gauge transformations that respect Hermitian metrics, we get a category $\mathbf{HermVect}_n$ of Hermitian vector bundles of rank n and an equivalence of categories

$$\mathbf{HermVect}_n \longrightarrow \text{Principal } \mathrm{U}(n)\text{-bundles}.$$

A *Hermitian connection* is a connection respecting the Hermitian metric and in the same way as above we can give an equivalence of categories

$$\mathbf{FlatHermVect}_n \longrightarrow [Z^1(X, \underline{\mathrm{U}(n)}), C^0(X, \underline{\mathrm{U}(n)})].$$

The same arguments work for other matrix groups like $\mathrm{O}(n)$ (with Riemannian metrics), for example.

Connections in principal bundles

Finally, we are going to generalize the correspondence to the setting of a principal G-bundle with a general Lie group G. In order to do this we are going to leave the "cocycle approach" and work in a more "global" approach, which means that I am going to work with the usual "topological" definitions of principal G-bundle and vector bundle. Since I do not want to repeat things that very well known and appear on every textbook, I refer the reader to Wikipedia for these definitions.

Principal G-bundles and vector bundles as we know them (that is, as G-torsors or as locally free sheaves, respectively) are recovered from the topological definitions by considering its sheaves of sections. The equivalence is completed if we use the usual procedure, that we have already described several times, of "gluing" with transition functions.

Just to make everything clear, in this new approach we change our notation: if we refer to a vector bundle E, we mean the topological space, while its sheaf of sections will be denoted by Γ_E.

In order to define a connection in a principal bundle, let us take another look at how we defined connections in vector bundles. Consider the natural projection map $p : E \to X$ and its differential $p_* : TE \to TX$. This gives a natural morphism $TE \to p^*TX$ of vector bundles over E. The kernel of this map is the bundle p^*E, so we have an exact sequence of vector bundles over E

$$0 \to p^*E \to TE \to p^*TX \to 0.$$

Now, if $D : \Gamma_E \to \Gamma_E \otimes \Omega^1_X$ is a connection on E, we can define a map of vector bundles over E,

$$\omega : TE \longrightarrow p^*E,$$

sending an element of TE of the form $(v, (\xi, w))$, with $v \in E$, $\xi \in T_{p(v)}X$ and $w \in E_{p(v)}$ to the element $(v, wA(\xi)) \in p^*E$, for A the connection 1-form. Therefore, a connection D gives an splitting,

$$TE = \ker \omega \oplus p^*E.$$

The idea is to regard ω as a p^*E-valued 1-form on E, that is $\omega \in \Omega^1(E, p^*E)$ and get the curvature as the 2-form

$$\Omega = d\omega + \omega \wedge \omega.$$

Acting in vector fields $\xi, \eta \in \Gamma(E, TE)$, we have

$$\Omega(\xi, \eta) = d\omega(\xi, \eta) + [\omega(\xi), \omega(\eta)].$$

Suppose now that $\xi, \eta \in \ker \omega$, then

$$\Omega(\xi, \eta) = d\omega(\xi, \eta) = \xi\omega(\eta) - \eta\omega(\xi) - \omega([\xi, \eta]) = -\omega([\xi, \eta]).$$

Therefore, $\Omega(\xi, \eta) = 0$ if and only if $[\xi, \eta] \in \ker \omega$. This means that the curvature 2-form Ω vanishes if and only if the distribution $\ker \omega$ is involutive.

We are now going to generalize the same ideas to principal bundles. Consider $p : P \to X$ a principal G-bundle.

The differential of the map p induces a morphism in the tangent bundles $p_* : TP \to TX$, let us call $T_F P = \ker p_*$ the *tangent bundle along the fibres.* A *connection* A on P is a G-equivariant splitting of the following exact sequence of vector bundles on P

$$0 \to T_F P \to TP \to p^* TX \to 0.$$

That is, a connection A defines a distribution $T_A P$ such that $TP \cong T_F P \oplus T_A P$. Being G-equivariant means means that the splitting is preserved by the natural action of G in the exact sequence. A connection A is associated to a map $\omega_A : TP \to T_F P$, so that $T_A P = \ker \omega_A$.

The map ω_A can be seen also as a $T_F P$-valued 1-form on P. We define the curvature of A to be the 2-form

$$F_A = d\omega + \omega \wedge \omega.$$

As before, we see that the curvature F_A vanishes if and only if the distribution $T_A P$ is involutive.

The correspondence with local systems is now easy. Clearly, if we regard any G-covering space $Y \to X$ as a principal G-bundle, we can define in it the trivial connection since $TY \cong TX$. On the other hand, the Frobenius theorem says that any involutive distribution is integrable. Therefore, if A is a flat connection (that is, if $F_A = 0$) on some principal G-bundle P, then the distribution $T_A P$ is integrable. What this integrability property means is that there exists some smooth submanifold $Y \hookrightarrow P$ with $T_A P_y = T_y Y$ for every $y \in P$.

Now, on every point we have $T_y P = T_F P_y \oplus T_{p(y)} X$, and thus $T_y Y \cong T_{p(y)} X$. By the inverse function theorem, we have that $p|_Y$ gives a local diffeomorphism between Y and X. This implies that $Y \to X$ is a G-covering space of X.

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