Higgs bundles twisted by a vector bundle

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For now on, assume $g \geq 2$.

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- **Moduli space of stable vector bundles**: $\mathcal{N}(n, d)$

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- $E = (\mathbb{E}, \bar{\partial}_E)$ is a holomorphic vector bundle.

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- Symplectic quotient: $\mathcal{A} /\!\!/ \mathcal{G} = \mu^{-1}(0)/\mathcal{G}$.

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Theorem (Donaldson's version of Narasimhan–Seshadri)

An irreducible h-unitary connection ∇ satisfies

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- Consequence:

$$\mathcal{N}(n,d) \longleftrightarrow \mathcal{A} /\!\!/ \mathcal{G}.$$

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- Fibres?

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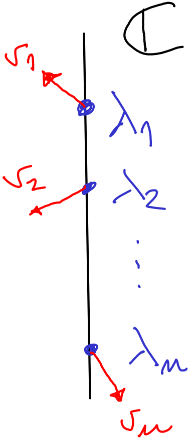

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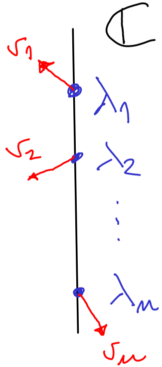

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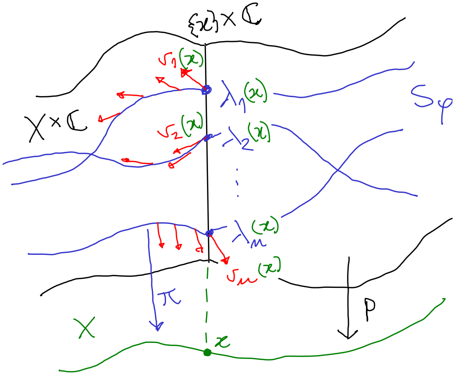

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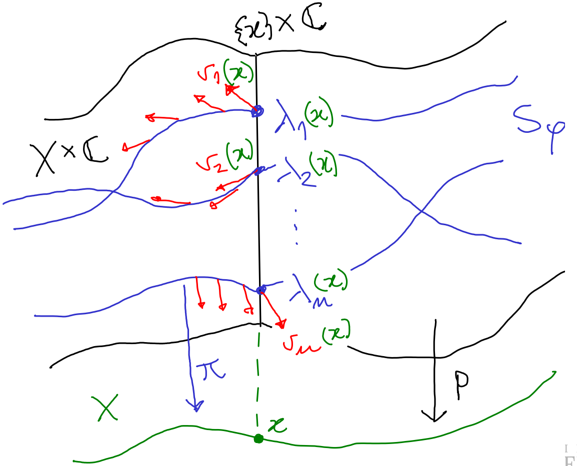

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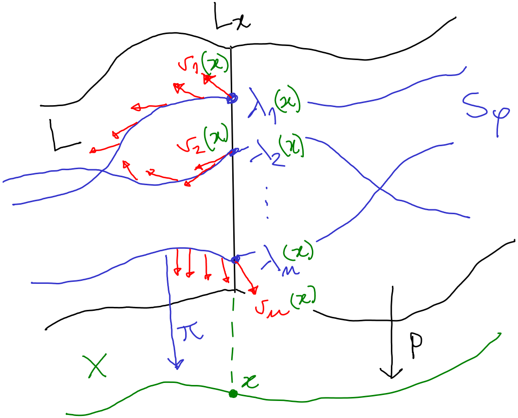

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- Hitchin (1987) for $L = K_X$.
- Beauville, Narasimhan and Ramanan (1989) for general L.

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  Wider notion of stability for pairs (E, φ).

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- Studied by D. Xie and K. Yonekura in the context of $\mathcal{N} = 1$ supersymmetric gauge theories.

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Theorem

Take $b \in \mathcal{B}_V$ such that the spectral curve S_b is integral and smooth. Then there is a bijective correspondence

$$\operatorname{Pic}(S_b) \longleftrightarrow \{[(E,\varphi)] | S_\varphi = S_b\}.$$

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- Donaldson–Thomas invariants.

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