Flat connections and the fundamental group

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In this post we are going to study the correspondence between flat connections and representations of the fundamental group, sometimes called the “Riemann-Hilbert correspondence”. This is a very rich result in that it draws together several fields of Mathematics and leads to new interesting problems like the study of certain moduli spaces or the relationship to Yang-Mills equation. Moreover, this correspondence is the first step to fully understand another correspondence known as the Nonabelian Hodge Theorem.

There are several ways of proving this correspondence. One is through holonomy, and it is maybe the more straightforward since it directly relates flat connections with representations of the fundamental group. In this post however, given that in my last post [CITAR] we already gave a correspondence between representations of the fundamental group and covering spaces, we are going to prove the correspondence by relating flat connections to covering spaces (also called *local systems* in this context).

With this point of view, a general “global” proof of the correspondence can be given by using the Frobenius theorem. We are going to sketch this proof, but I prefer to study in detail the proof for matrix groups, which is more simple and straightforward, and maybe fits better with the “cocycle approach” that I have been mantaining in my previous posts.

Principal bundles and local systems

During all this post we will be using the notions from my post on Torsors and Cocycles [CITAR], and also we will make use of the correspondence between covering spaces and representations that I gave in my previous post [CITAR].

Let me recall that for X a topological space and $\mathcal{G}$ a sheaf of groups over X, a $\mathcal{G}$ torsor is a sheaf of sets on X with nonempty stalks which is endowed with a free and transitive action of $\mathcal{G}$. What we saw in my post about Torsors and cocycles [CITAR] is that the functor that sends every $\mathcal{G}$-torsor to its set of transition functions gives an equivalence of categories between the category of $\mathcal{G}$ torsor and the action groupoid $[Z^1(X,\mathcal{G}), C^0(X,\mathcal{G})]$, where these two sets denote the sets of equivalence classes by refinement of pairs $(\mathfrak{U}, f)$, with $\mathfrak{U}$ an open cover of X and f a Čech 1-cocycle (a Čech 0-cochain, respectively).

For our purposes, we will fix now once and for all X a smooth manifold and G a Lie group, and consider the sheaves G, which maps any open set $U \subset X$ to the group $C^\infty(U, G)$ of G-valued smooth functions, and $\underline{G}$, which maps any open set $U \subset X$ to the group of locally constant functions $U \to G$.

We will call a G-torsor on X a *principal G-bundle* over X, and we will call a $\underline{G}$-torsor on X a *G-local system* over X.

Recall that in my previous post [CITAR] we called elements of $Z^1(\mathfrak{U}, \underline{G})$ by the name of $\mathfrak{U}$-based G-covering spaces. Therefore, a G-local system is an equivalence class by refinement of G-coverings. In that post we also proved the *monodromy theorem* which said that the groupoid $[Z^1(\mathfrak{U}, \underline{G}), C^0(\mathfrak{U}, \underline{G})]$ was equivalent to the groupoid $[\mathrm{Hom}(\pi_1(X, x_0), G), G]$ of G-representations of the fundamental groupo with the conjugation action, which we called the *Betti groupoid*, $\mathfrak{U}$ being an open cover satisfying certain “good” topological conditions. Since we are on a smooth manifold, every open cover can be refined to one that is “good” in that sense, and thus we get an equivalence between the category of G-local systems and the Betti groupoid.

The purpose of this post is to prove that the category of G-local systems is equivalent to another “differential geometric” category, the category of *flat bundles*. As a consequence, this equivalence will yield an equivalence between this category of flat bundles and the Betti groupoid.

Vector bundles

We will start by proving this equivalence for vector bundles, so we better define what these are.

Consider the *sheaf of smooth complex-valued functions* C^∞_X on X sending every open set $U \subset X$ to the set $C^\infty(X, \mathbb{C})$. A *sheaf of* C^∞_X *modules* is a sheaf of abelian groups E on X such that, on any $U \subset X$, the group $E(U)$ is a $C^\infty_X(U)$-module and such that the restriction homomorphisms $E(V) \to E(U)$ are $C^\infty_X(U)$-linear.

A *complex vector bundle of rank* n *on* X is a locally free sheaf E of C^∞_X modules. This means that there exists an open cover $\mathfrak{U}$ of X and isomorphisms $\varphi_U : E|_U \to C^\infty_X|^n_U$ for every $U \in X$. The pair $(\mathfrak{U}, \varphi)$, with $\varphi = \{\varphi_U\}_{U \in \mathfrak{U}}$ is called a *trivialization* of E.

To any rank n complex vector bundle E, fixing a trivialization $(\mathfrak{U}, \varphi)$, we can associate its set of transition functions $g_{UV} : U \cap V \to \mathrm{GL}(n, \mathbb{C})$, defined as

$$g_{UV} = \varphi_V|_{U\cap V} \circ \varphi_U^{-1}|_{U\cap V},$$

for $U, V \in \mathfrak{U}$, $U \cap V \neq \varnothing$. Clearly the g_{UV} define a cocycle, so we get a map

$$\{\text{Rank } n \text{ vector bundles}\} \longrightarrow Z^1(\mathfrak{U}, \mathrm{GL}(n, \mathbb{C})).$$

A *gauge transformation* of a vector bundle E is a C^∞_X-linear sheaf automorphism $\xi : E \to E$. Fixing a trivialization $(\mathfrak{U}, \varphi)$, to any gauge transformation ξ we can associate the element $f \in C^0(\mathfrak{U}, \mathrm{GL}(n, \mathbb{C}))$ defined as

$$f_U = \varphi_U \circ \xi|_U \circ \varphi_U^{-1},$$

for $U \in \mathfrak{U}$.

Thus, if we denote by $\mathbf{Vect}_n$ the category whose objects are vector bundles on X of rank n and where morphisms are given by gauge transformations, by making a choice of trivialization on any vector bundle, we have defined a functor

$$\mathbf{Vect}_n \longrightarrow [Z^1(X, \mathrm{GL}(n, \mathbb{C})), C^0(X, \mathrm{GL}(n, \mathbb{C}))].$$

This functor is fully faithful. Fix E is a vector bundle and we fix $(\mathfrak{U}, \varphi)$ a trivialization of E. Clearly, the map $\xi \mapsto f$, with f defined as above gives a bijection between gauge transformations and elements of $C^0(\mathfrak{U}, \mathrm{GL}(n, \mathbb{C}))$.

This functor is essentially surjective. Let $(\mathfrak{U}, g)$, with $g \in Z^1(\mathfrak{U}, \mathrm{GL}(n, \mathbb{C}))$ be a pair representing an element of $Z^1(X, \mathrm{GL}(n, \mathbb{C}))$. The way to recover now the vector bundle is similar to how we recovered a torsor from a cocycle. Define the presheaf

$$E(U) = \coprod_{V \in \mathfrak{U}} C^\infty_X(U \cap V)^n / \sim,$$

where $f \in C^\infty_X(U \cap V)^n$ and $f' \in C^\infty_X(U \cap V')^n$, with $V \cap V'$ are related by $\sim$ if

$$f|_{U\cap V\cap V'} = g_{VV'} f'|_{U\cap V\cap V'}.$$

This presheaf verifies the sheaf condition by construction and it is clearly locally free of rank n. If $(\mathfrak{V}, \varphi)$ is any trivialization of E, then it is easy to check that, after passing to a common refinement, its transition functions are on the same orbit as the refinement of the cocycle g by a 0-cochain.

Therefore, the category $\mathbf{Vect}_n$ is equivalent to that of $\mathrm{GL}(n, \mathbb{C})$-torsors (that is, principal $\mathrm{GL}(n, \mathbb{C})$-bundles).

Connections in vector bundles

Let E be a vector bundle and consider the bundles Ω^k_X consisting on complex-valued smooth differential k-forms on X. For example, Ω^1_X is the *cotangent bundle* of X (actually, it is the sheaf of sections of the cotangent bundle, but we are already regarding bundles as locally free sheaves).

Definition. A *connection* D on E is a $\mathbb{C}$-linear operator

$$D : E \to E \otimes \Omega^1_X$$

such that

$$D(fs) = sdf + fDs,$$

for $f \in C^\infty_X(U)$ and $s \in E(U)$, for every open subset $U \subset X$.

Let D be a connection on a vector bundle E and take an open set $U \in \mathfrak{U}$ in some trivialization $(\mathfrak{U}, \varphi)$ of E. We define a *frame of* E *in* U to be a basis $\{e_1, ..., e_n\}$ of $E(U)$, given that it is a free $C^\infty_X(U)$-module. For any e_i of the frame, the connection acts as

$$De_i = \sum_j e_j A^j_i,$$

for $A^j_i \in \Omega^1_X(U)$. Using matrix notation, regarding $e = (e_i)$ as a row vector and $A = (A^j_i)$ as a square matrix, we get

$$De = eA.$$

Now, given any other section $s \in E(U)$, we can write $s = \sum_i s^i e_i$, for $s^i \in C^\infty_X(U)$ and we have

$$Ds = \sum_i (ds^i e_i + s^i De_i) = \sum_i ds^i e_i + s^i e_j A^i_j = (d + A)s.$$

The matrix A is called the *connection* 1*-form* of D on U.

Definition. Let D be a connection on a vector bundle E. We define the *curvature* of D as the operator

$$D^2 : E \to E \otimes \Omega^2_X.$$

The curvature is a C^∞_X-linear map since

$$D^2(fs) = D(sdf + fDs) = Ds \wedge df + df \wedge Ds + fD^2 s = fD^2 s,$$

for $s \in E(U)$ and $f \in C^\infty_X(U)$.

Locally, in some trivializing open set, we have

$$D^2(e) = D(eA) = De \wedge A + edA = e(A \wedge A + dA) = eF_A,$$

for $F_A = dA + A \wedge A$ a matrix of 2-forms which we call the *curvature* 2*-form*.

Fix E a vector bundle and consider the group $\mathcal{G}_E$ of gauge transformations of E and the set $\mathcal{A}_E$ of all connections on E. We have a natural action of $\mathcal{G}_E$ on $\mathcal{A}_E$ by conjugation: if $s \in E(U)$,

$$(\xi \cdot D)(s) = \xi \circ D \circ \xi^{-1}.$$

Now, take $U \in \mathfrak{U}$ for $(\mathfrak{U}, \varphi)$ some trivialization and $f_U = \varphi_U \circ \xi|_U \circ \varphi_U^{-1}$. If we choose the local frame e as the inverse image through φ_U of the canonical basis and we consider A the connection 1-form of D in this frame we have

$$e(\xi \cdot A) = (\xi \cdot D)(e) = \xi D(\xi^{-1}(e)) = \xi D(ef_U^{-1}) = \xi(Def_U^{-1} + edf_U^{-1}) = e(f_U A f_U^{-1} + f_U df_U^{-1}),$$

so

$$\xi \cdot A = f_U A f_U^{-1} + f_U df_U^{-1}.$$

The curvature of the gauge-transformed connection now is

$$(\xi \cdot D)^2 = \xi \circ D \circ \xi^{-1} \circ \xi \circ D \circ \xi^{-1} = \xi \circ D^2 \circ \xi^{-1}.$$

Thus, in the frame defined above,

$$\begin{aligned} eF_{\xi \cdot A} &= \xi(D^2(\xi^{-1}(e))) = D^2(ef_U^{-1})f_U = D(D(e)f_U^{-1} + edf_U^{-1})f_U \\ &= (D^2(e)f_U^{-1} - D(e) \wedge df_U^{-1} + D(e) \wedge df_U^{-1})f_U = D^2(e) = eF_A. \end{aligned}$$

Flat connections

Definition. We say that a connection D on a vector bundle E is flat if its curvature vanishes, that is, $D^2 = 0$.

If we denote by $\mathcal{F}_E$ the set of flat connections on a vector bundle E, the formulas above show that the action of the group of gauge transformations $\mathcal{G}_E$ on $\mathcal{A}_E$ descends to an action on $\mathcal{F}_E$.

We can now state and prove the main theorem of this post:

Theorem. Fix E a vector bundle over X. There is an equivalence of groupoids

$$[\mathcal{F}_E, \mathcal{G}_E] \longrightarrow [Z^1(X, \underline{\mathrm{GL}(n, \mathbb{C})}), C^0(X, \underline{\mathrm{GL}(n, \mathbb{C})})].$$

Proof. More precisely, we want to prove the following.

1. Provided any flat connection D on E, we have to find an open cover $\mathfrak{U}$ of X and locally constant functions $h_{UV} : U \cap V \to \mathrm{GL}(n, \mathbb{C})$, for $U, V \in \mathfrak{U}$ and $U \cap V \neq \varnothing$. This allows us to define the functor at the level of objects.

2. If we denote by $\mathrm{Hom}(D_1, D_2)$ the set of gauge transformations between two flat connections D_1 and D_2 and by $\mathrm{Hom}(h_1, h_2)$ the set of 0-cochains relating the two corresponding cocycles h_1 and h_2. We want to give a bijection $\mathrm{Hom}(D_1, D_2) \to \mathrm{Hom}(h_1, h_2)$.

3. Given any pair $(\mathfrak{U}, h)$ representing an element of $Z^1(X, \underline{\mathrm{GL}(n, \mathbb{C})})$, we have to construct a flat connection D on E such that the corresponding cocycle h' is equivalent to h.

In order to prove 1 we have to find the mentioned locally constant functions h_{UV}. Choose $(\mathfrak{U}, \varphi)$ a trivialization of E. Suppose that we could find, for every $U \in \mathfrak{U}$, a frame ϵ_U of E in U such that $D\epsilon_U = 0$. Consider as above the frame e_U of E in U defined as the inverse image through φ_U of the canonical basis. We have that e_U and ϵ_U are related by some matrix-valued function $f_U : U \to \mathrm{GL}(n, \mathbb{C})$, so that $\epsilon_U = e_U f_U$.

Therefore, if we consider in a nonempty overlap $U \cap V$ the cocycle $h_{UV} = f_V^{-1} g_{UV} f_U$, we have

$$0 = D\epsilon_U = D(e_U f_U) = D(e_V g_{UV} f_U) = D(\epsilon_V f_V^{-1} g_{UV} f_U) = D(\epsilon_V h_{UV}) = D\epsilon_V h_{UV} + \epsilon_V dh_{UV},$$

so $dh_{UV} = 0$ and h_{UV} is locally constant.

It remains to see then why we can find such a frame ϵ_U. By taking the frame e_U as above, we want to find matrix-valued functions $f_U : U \to \mathrm{GL}(n, \mathbb{C})$ satisfying

$$0 = D\epsilon_U = D(e_U f_U) = D(e_U) f_U + e_U df_U = e_U (A_U f_U + df_U),$$

where A_U is the connection 1-form in the frame e_U.

That is, we want to find f_U solutions to the differential equation

$$df_U + A_U f_U = 0.$$

[DUDOSO] If we call $\alpha = df_U + A_U f_U$, the integrability condition of this equation is $\alpha \wedge d\alpha = 0$. Now,

$$d\alpha = dA_U f_U - A_U \wedge df_U = F_{A_U} f_U - A_U \wedge A_U f_U - A_U \wedge df_U = -A_U \wedge \alpha,$$

so $\alpha \wedge d\alpha = 0$. [?????]

Let us prove 3 now. Take $(\mathfrak{U}, h)$ representing an element of $Z^1(X, \underline{\mathrm{GL}(n, \mathbb{C})})$. It is clear that we can construct (for example, by an inductive process) a frame ϵ_U on every $U \in \mathfrak{U}$ so that $\epsilon_U = \epsilon_V h_{UV}$ on the nonempty intersections $U \cap V$. Define now the connection D so that $D\epsilon_U = 0$. Now, the locally constant cocycle corresponding to this connection is precisely h.

Finally, let us see the proof of 2. Choose a trivialization $(\mathfrak{U}, \varphi)$ of E and fix x_U a point on every connected component of every $U \in \mathfrak{U}$. Given any gauge transformation ξ, to which we associate a cochain $f \in C^0(\mathfrak{U}, \mathrm{GL}(n, \mathbb{C}))$, we can define a cochain $\tilde{f} \in C^0(\mathfrak{U}, \underline{\mathrm{GL}(n, \mathbb{C})})$ by choosing $\tilde{f}_U = f_U(x_U)$. This assignation restricts to a map

$$\mathrm{Hom}(D_1, D_2) \to \mathrm{Hom}(h_1, h_2),$$

which is trivially surjective, since $C^0(\mathfrak{U}, \underline{\mathrm{GL}(n, \mathbb{C})}) \subset C^0(\mathfrak{U}, \mathrm{GL}(n, \mathbb{C}))$.

It remains to prove that this map is injective. Choose ξ_1 and ξ_2 such that $(\xi_1 \cdot D_1) = (\xi_2 \cdot D_1) = D_2$. Therefore, for $\xi = \xi_1^{-1} \circ \xi_2$, we have $(\xi \cdot D_1) = D_1$. Therefore, if f is the 0-cochain associated to ξ, we have, in $U \in \mathfrak{U}$,

$$A_U = f_U A_U f_U^{-1} + f_U df_U^{-1}.$$

Now, if the images of ξ_1 and ξ_2 coincide we have that $f_{1,U}(x_U) = f_{2,U}(x_U)$ for every $U \in \mathfrak{U}$, so $f_U(x_U) = I$, the identity matrix. Therefore,

$$A_U(x_U) = A_U(x_U) + df_U^{-1}(x_U),$$

so $df_U^{-1} = 0$ on x_U. Now, maybe by refining the covering $\mathfrak{U}$, we get $f_U = I$ and $f_{1,U} = f_{2,U}$.

This finishes the proof of the theorem. ∎

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