Torsors and cocycles

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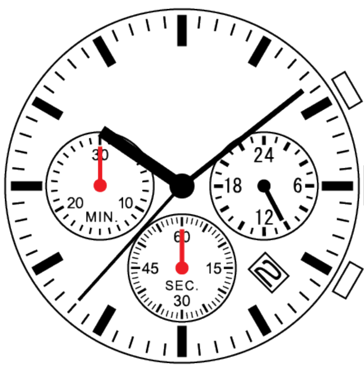
In my last post I mentioned that first Čech cohomology classes of a sheaf $\mathcal{G}$ of (maybe non-abelian) groups admit a geometric interpretation in terms of $\mathcal{G}$-torsors. In this post I am going to introduce the notion of a $\mathcal{G}$-torsor over a topological space X, and show how the set of equivalence classes of $\mathcal{G}$-torsors on X can be identified with $H^1(X, \mathcal{G})$.



The action groupoid

Before I talk about torsors, let me introduce a very simple concept, but that will be very useful in this and other posts. Take any left action $G \times S \to S$ of a group G on a set S. To this action we can associate its *action groupoid*, which is the category $[S, G]$ whose objects are precisely the elements of the set S and, for every $x, y \in S$, the set of morphisms from x to y is

$$\mathrm{Mor}_{[S,G]}(x, y) = \{g \in G : g \cdot x = y\}.$$

This is clearly a groupoid since every morphism g is an isomorphism, with inverse given by g^{-1}. The *moduli set* of this category (that is, its set of isomorphism classes) is simply the set of orbits S/G.

In the last post we saw an example of a group action. Recall that for any open covering $\mathfrak{U}$ of a topological space X and for any sheaf of groups $\mathcal{G}$ over X, we had that an action of 0-cochains on 1-cochains by "conjugation":

$$\begin{aligned} C^0(\mathfrak{U}, \mathcal{G}) \times C^1(\mathfrak{U}, \mathcal{G}) &\longrightarrow C^1(\mathfrak{U}, \mathcal{G}) \\ ((f_U)_{U \in \mathfrak{U}}, (g_{UV})_{U<V \in \mathfrak{U}}) &\longmapsto (f_U g_{UV} f_V^{-1})_{U<V \in \mathfrak{U}}. \end{aligned}$$

Moreover, we saw that this action respects cocycles. Now, as in the last post, we can use refinement maps to define the sets $C^0(X, \mathcal{G})$, $C^1(X, \mathcal{G})$ and $Z^1(X, \mathcal{G})$ as limits by refinement. Suppose that we have some element of $Z^1(X, \mathcal{G})$ represented by a pair $(\mathfrak{U}, g)$, with $g \in Z^1(\mathfrak{U}, \mathcal{G})$ and some element of $C^0(X, \mathcal{G})$ represented by another pair $(\mathfrak{V}, f)$. We can take a common refinement of both open covers by defining

$$\mathfrak{W} = \{U \cap V : U \in \mathfrak{U}, V \in \mathfrak{V}\}$$

(which is of course an open cover, since every point is in some U *and* in some V) with refinement maps given by

$$\begin{aligned} \mathfrak{W} &\longrightarrow \mathfrak{U} \\ U \cap V &\longmapsto U \end{aligned}$$

(analogously for $\mathfrak{W} \to \mathfrak{V}$). Thus, for the previously chosen elements we can take representatives $g \in Z^1(\mathfrak{W}, \mathcal{G})$ and $f \in C^0(\mathfrak{W}, \mathcal{G})$ and define

$$f \cdot g = (f_U g_{UV} f_V^{-1})_{U<V \in \mathfrak{W}}.$$

In conclusion, we have just defined an action of the group $C^0(X, \mathcal{G})$ in the set $Z^1(X, \mathcal{G})$ (of course, in the same way we can define an action on C^1, but we are particularly interested in this one). Moreover, the good properties of the direct limit guarantee that the set of orbits is precisely $H^1(X, \mathcal{G})$.

We can now consider the action groupoid $[Z^1(X, \mathcal{G}), C^0(X, \mathcal{G})]$ associated to this action, whose moduli set is the Čech cohomology set $H^1(X, \mathcal{G})$. What we are going to do now is to give an interpretation of this action groupoid in terms of $\mathcal{G}$-torsors.

Torsors

As above, let X be a topological space and $\mathcal{G}$ a sheaf of groups over X.

Definition 1. A *$\mathcal{G}$-torsor* is a sheaf of sets $\mathcal{F}$ on X endowed with an action $\mathcal{G} \times \mathcal{F} \to \mathcal{F}$ such that

1. whenever $\mathcal{F}(U)$ is nonempty, the action $\mathcal{G}(U) \times \mathcal{F}(U) \to \mathcal{F}(U)$ is free and transitive, and
2. for every $x \in X$, the stalk $\mathcal{F}_x$ is nonempty.

A *morphism of $\mathcal{G}$-torsors* $\mathcal{F} \to \mathcal{F}'$ is simply a morphism of sheaves compatible with the $\mathcal{G}$-actions (we say that it is $\mathcal{G}$-equivariant).

More precisely, given a morphism of $\mathcal{G}$-torsors $\varphi : \mathcal{F} \to \mathcal{F}'$, being $\mathcal{G}$-equivariant means that, if $\mathcal{F}(U)$ is nonempty, for every $p \in \mathcal{F}(U)$ we have

$$\varphi_U(g \cdot p) = g \cdot \varphi_U(p).$$

The best way to unravel this definition is by looking at **examples**. The simplest example of a $\mathcal{G}$-torsor is the *trivial $\mathcal{G}$-torsor*, which is $\mathcal{F} = \mathcal{G}$ with the natural action given by the group operation. A key fact is now the following:

Proposition 1. *Let $\mathcal{F}$ be a $\mathcal{G}$-torsor. If $\mathcal{F}$ admits a global section, that is, if $\mathcal{F}(X)$ is nonempty, then it is isomorphic to the trivial $\mathcal{G}$-torsor.*

Proof. Choose $f \in \mathcal{F}(X)$. Since $f|_U \in \mathcal{F}(U)$ for every open subset $U \subset X$, the action $\mathcal{G}(U) \times \mathcal{F}(U) \to \mathcal{F}(U)$ is free and transitive. Therefore, every $h_U \in \mathcal{F}(U)$ can be written in a unique way as $h_U = g_U \cdot f|_U$, for $g_U \in \mathcal{G}(U)$. Thus, the map

$$\begin{aligned} \mathcal{F}(U) &\longrightarrow \mathcal{G}(U) \\ h_U &\longmapsto g_U, \end{aligned}$$

which is clearly equivariant, defines a sheaf isomorphism. ■

Note now that since for every $x \in X$, we have that $\mathcal{F}_x \neq \varnothing$, there is an open cover $\mathfrak{U}$ of X such that, for every $U \in \mathfrak{U}$, the set $\mathcal{F}(U)$ is nonempty. Therefore, on every $U \in \mathfrak{U}$, the sheaf $\mathcal{F}|_U$ is isomorphic to the trivial $\mathcal{G}|_U$-torsor. In conclusion, what property 2 in the definition of $\mathcal{G}$-torsor actually means is that every $\mathcal{G}$-torsor is, in some way, "locally trivial". The open cover $\mathfrak{U}$ is called a *trivializing cover*.

There are other examples of of $\mathcal{G}$-torsors that the reader could be familiar with. To introduce these examples, first consider a topological group G. Associated to this group we can define two different sheaves. One is the *sheaf of G-valued functions*, which we denote simply by G, and is defined by

$$G(U) = \{\text{Continuous maps } U \to G\}.$$

The other one is the *sheaf of **locally constant** G-valued functions*, denoted by $\underline{G}$, and defined by

$$\underline{G}(U) = \{\text{Continuous maps } U \to G \text{ that are locally constant}\}.$$

Note that these two sheaves are essentially different, although they coincide if the group G is endowed with the discrete topology. Now, G-torsors are best known as *principal G-bundles* (or rather, as their sheaves of sections). On the other hand $\underline{G}$-torsors can be identified with *G-covering spaces*. I will say a lot about these two examples in future posts. If the reader is familiar with principal bundles, maybe it is useful for them to think of a $\mathcal{G}$-torsor as a generalization of a principal bundle in the sense that the structure group depends continuously on the base point.

Now, let us see how there is a groupoid naturally associated with $\mathcal{G}$-torsors:

Proposition 2. Every morphism of $\mathcal{G}$-torsors is an isomorphism.

Proof. Consider $\varphi : \mathcal{F} \to \mathcal{F}'$ a morhpism of $\mathcal{G}$-torsors. First, we will see that the map is injective. Suppose that there are $p_1, p_2 \in \mathcal{F}(U)$ such that $\varphi_U(p_1) = \varphi_U(p_2)$. Since the action on $\mathcal{F}(U)$ is transitive and free, there exists a unique $g \in \mathcal{G}(U)$ such that $p_1 = g \cdot p_2$ and, since φ is equivariant, $\varphi_U(p_1) = g \cdot \varphi_U(p_2)$. But the group $\mathcal{G}(U)$ also acts freely and transitively on $\mathcal{F}'(U)$, so $g = 1$ and $p_1 = p_2$. On the other hand, to see that it is surjective take any element $p \in \mathcal{F}(U)$. Since the action is transitive we can write any other element $p' \in \mathcal{F}(U)$ as $p' = g \cdot \varphi_U(p)$, for some $g \in \mathcal{G}(U)$. Therefore, since φ is equivariant, $p' = \varphi_U(g \cdot p)$. ■

What we have just shown is that if we consider the category whose objects are $\mathcal{G}$-torsors and whose morphisms are morphisms of $\mathcal{G}$-torsors, this category is in fact a groupoid. The main purpose of this post is to show that this groupoid is equivalent to the action groupoid $[Z^1(X, \mathcal{G}), C^0(X, \mathcal{G})]$. In particular, this equivalence will yield a bijective correspondence between isomorphism classes of $\mathcal{G}$-torsors and cohomology classes in $H^1(X, \mathcal{G})$.

Transition functions

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