Higgs bundles twisted by a vector bundle

Guillermo Gallego Sánchez

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Chapter 1

Vector bundles on Riemann surfaces

§ 1.1. Topological classification of vector bundles

The first step towards the classification of vector bundles on Riemann surfaces is their "topological" classification. That is, we want to classify smooth complex vector bundles on Riemann surface up to C^∞ isomorphism. This is indeed pretty easy to do, since the problem can be reduced to the classification of line bundles.

Theorem 1.1.1. *If E is a rank n smooth complex vector bundle over a compact Riemann surface X then it is isomorphic to* $\det E \oplus (X \times \mathbb{C}^{n-1})$.

Proof. We will proceed by induction on n. Of course, if E is a line bundle, $E \cong \det E$. Now, let $n > 1$, suppose that it is true for any vector bundle of rank $n-1$ and let E be a rank n vector bundle.

Lemma 1. *E has a nowhere vanishing section.*

Proof. Let s_0 be the zero section of E and $S_0 = s_0(X) \subset E$. By the transversality theorem [REF: Hirsch], we can densely choose a section $s \in \Gamma(E)$ transversal to S_0. Now, if s vanishes at some point $x \in X$, then $s(x) \in S_0$ and, since s is transversal to S_0,

$$d_x s(T_x X) + T_{s(x)} S_0 = T_{s(x)} E.$$

But $\dim_{\mathbb{R}} E = 2n + 2$, $\dim_{\mathbb{R}} S_0 = 2$ and $\dim_{\mathbb{R}} d_x s(T_x X) \leq \dim_{\mathbb{R}} X = 2$, so, if $n > 1$, dimensions do not add up to verify the above equality and therefore we get a contradiction. $\square$

Let us continue with the proof of the theorem. Since E has a nowhere vanishing section s, we can define the line bundle

$$L = \bigsqcup_{x \in X} \operatorname{span}(s(x)),$$

with

$$\begin{aligned} \pi : L &\longrightarrow X \\ \lambda s(x) &\longmapsto x. \end{aligned}$$

Therefore, we can decompose $E = E' \oplus L$, with E' a rank $n-1$ vector bundle. For example, fixing a metric on E, we can define E' to be the orthogonal complement of L. But observe now that L is isomorphic to the trivial line bundle: the bundle morphism

$$\begin{aligned} X \times \mathbb{C} &\longrightarrow L \\ (x, \lambda) &\longmapsto \lambda s(x), \end{aligned}$$

is in fact an isomorphism. This can be proven by defining a metric on L and normalizing $s \mapsto s/\|s\|$. Then we can define the inverse $y \in L \mapsto (\pi(y), \langle s(\pi(y)), y\rangle) \in X \times \mathbb{C}$.

Thus, we have shown that $E \cong E' \oplus (X \times \mathbb{C})$. Now, applying the induction hypothesis, $E' \cong \det E' \oplus (X \times \mathbb{C}^{n-2})$, so $E \cong \det E' \oplus (X \times \mathbb{C}^{n-1})$. Finally, via transition functions it can be easily shown that $\det E \cong \det E'$. □

This last theorem says that vector bundles can be topologically classified by their rank and their determinant, which is a line bundle. Let us proceed then with the classification of line bundles. Recall that all the data of a vector bundle can be recovered by the transition functions $\{g_{\alpha\beta} \in C^\infty(U_\alpha \cap U_\beta, \mathbb{C})\}$ defining it, where $\{U_\alpha\}$ is an open cover of X. These functions verifiy the cocycle condition

$$g_{\alpha\beta} = g_{\gamma\beta} \cdot g_{\alpha\gamma}.$$

Also recall that an isomorphism of vector bundles induces a coboundary on the transition functions

$$\tilde{g}_{\alpha\beta} = f_{\delta\beta}^{-1} \cdot g_{\gamma\delta} \cdot f_{\gamma\alpha}.$$

Therefore, topological (C^∞) isomorphism classes of vector bundles are parametrized by the Čech cohomology group

$$H^1(X, C_X^{\infty,*}).$$

To obtain more information about this cohomology group we are going to introduce a very powerful tool: the first Chern class of a vector bundle.

Recall from Chern-Weil theory [REF:Griffiths-Harris,Wells] that for any complex vector bundle E over X and for any connection ∇ on E with associated curvature form F, the 2-form $\mathrm{tr}F$ is closed and its de Rham cohomology class $[\mathrm{tr}F] \in H^2_{\mathrm{dR}}(X)$ does not depend on the choice of the connection, so it is an invariant of the vector bundle E. If we normalize this form to get an integer cohomology class, we define the ***first Chern class*** of E as the cohomology class:

$$c_1(E) = \left[\frac{i}{2\pi}\mathrm{tr}F\right] \in H^2_{\mathrm{dR}}(X).$$

We define the ***degree*** of a vector bundle E as the pairing of $c_1(E)$ with the fundamental class of X, that is

$$\deg E = \int_X \frac{i}{2\pi}\mathrm{tr}F.$$

The next proposition [REF:Wells] summarizes the most important properties about the degree that we are going to use:

Proposition 1.1.2 (Propierties of the degree)**.** *Let E and F be complex vector bundles over a compact Riemann surface X.*

1. $\deg(E)$ *depends only on the isomorphism class of E.*
2. $\deg(E) \in \mathbb{Z}$.
3. $\deg(E \oplus F) = \deg(E) + \deg(F)$.
4. $\deg(E \otimes F) = \mathrm{rk}F \deg E + \mathrm{rk}E \deg F$.
5. $\deg E = \deg(\det E)$.
6. *Let δ be the connecting homomorphism of the long exact sequence in cohomology induced by the exponential sheaf exact sequence*

$$\mathbb{Z} \longrightarrow C_X^\infty \xrightarrow{\exp} C_X^{\infty,*},$$

where $\exp(f) = e^{2\pi i f}$. *The diagram*

$$\begin{tikzcd} H^1(X, C_X^{\infty,*}) \arrow[r, "\delta"] \arrow[dr, "c_1"'] & H^2(X,\mathbb{Z}) \arrow[d, hook] \\ & H^2_{\mathrm{dR}}(X), \end{tikzcd}$$

is commutative.

This last property will be crucial in the classification of line bundles. Let us consider again the exponential sheaf exact sequence

$$\mathbb{Z} \longrightarrow C_X^\infty \xrightarrow{\exp} C_X^{\infty,*}.$$

The existence of smooth partitions of unity implies that the sheaf C_X^∞ is fine, so $H^1(X, C_X^\infty) = H^2(X, C_X^\infty) = 0$. Therefore, the connecting map $\delta : H^1(X, C_X^{\infty,*}) \to H^2(X,\mathbb{Z})$ is an isomorphism. If we now consider the set of second de Rham cohomology classes with integer coefficients $H^2_{\mathrm{dR}}(X,\mathbb{Z})$, which is just the image of $H^2(X,\mathbb{Z})$ by the inclusion $H^2(X,\mathbb{Z}) \hookrightarrow H^2_{\mathrm{dR}}(X)$, we get an isomorphism $c_1 : H^1(X, C_X^{\infty,*}) \to H^2_{\mathrm{dR}}(X,\mathbb{Z})$. Now, the isomorphism $H^2_{\mathrm{dR}}(X) \cong \mathbb{C}$ given by integration on X, descends to an isomorphism $H^2_{\mathrm{dR}}(X,\mathbb{Z}) \cong \mathbb{Z}$. Summarizing, we have the diagram

$$\begin{tikzcd} H^1(X, C_X^{\infty,*}) \arrow[r, "c_1"] \arrow[rr, bend right, "\deg"'] & H^2_{\mathrm{dR}}(X,\mathbb{Z}) \arrow[r, "\int_X"] & \mathbb{Z}. \end{tikzcd}$$

That is, the degree gives an isomorphism between the set isomorphism classes of smooth line bundles and $\mathbb{Z}$. This concludes the topological classification of vector bundles, which we can gather in the next theorem

Theorem 1.1.3. *Smooth complex vector bundles over a compact Riemann surfaces are classified, up to* C^∞ *isomorphism by their rank and their degree.*

§ 1.2. The problem of classification of holomorphic vector bundles