# Arrival processes

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# Arrival processes

- An arrival process is a sequence of increasing non-negative random variables

$$0 < S_1 < S_2 < \cdots < S_n < S_{n+1} < \cdots$$

Arrival times or arrival epochs

- One can also be interested in
  - The **sequence of interarrival times:**

$$S_n = \sum_{i=1}^{n} T_i$$

$$\{T_i = S_i - S_{i-1}; \;\; i = 1,2, \dots\}$$

Since $S_i > S_{i-1}, \quad T_i > 0$

  - The **counting process** $N(t) \in \{0,1,2, \dots\}$, which is defined by the relation

$$\mathbb{I}[N(t) < n] = \mathbb{I}[S_n > t]$$

Or, equivalently,
$\mathbb{I}[N(t) \geq n] = \mathbb{I}[S_n \leq t]$

# Poisson process: A type of renewal process

- Renewal process: An arrival process in which $\{T_i\}_{i\geq 1}$ are iidrv's
- Homogenous Poisson process:

  A renewal process in which the interarrival times follow an exponential distribution

$$T_i \sim f_T(t) = \lambda e^{-\lambda t}$$

$\lambda$: rate or inverse scale

$\mathbb{E}[T] = 1/\lambda$
$Var[T] = 1/\lambda^2$

# Homogeneous Poisson process

A homogenous Poisson process with rate $\lambda$ is a counting process $\{N(t); \quad t \geq 0\}$ with the following properties:

- $N(0) = 0$.
- $N(t)$ has independent increments.

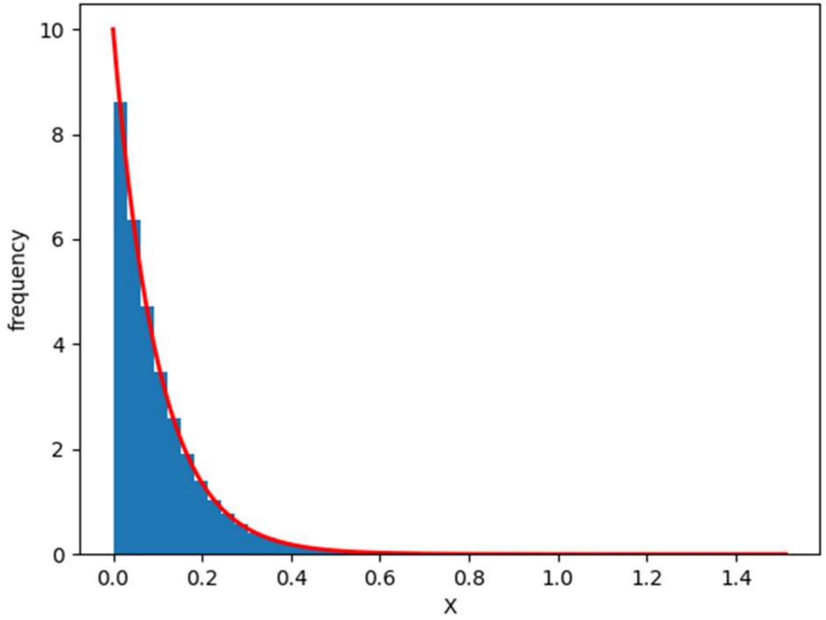
$$\big(N(t_2) - N(t_1)\big) \perp \big(N(t_4) - N(t_3)\big), \text{ for all } t_1 < t_2 < t_3 < t_4$$

- $N(t+\tau) - N(t) \sim Poisson(\lambda\tau)$ for all $t > 0, \quad t \geq 0$.

$\mathbb{E}[N(t+\tau) - N(t)] = \lambda\tau; \quad Var[N(t+\tau) - N(t)] = \lambda\tau;$

# Exponential distribution

- pdf: $f_T(\tau) = \lambda e^{-\lambda\tau}$
- cdf: $F_T(\tau) = 1 - e^{-\lambda\tau}$
- inv: $F_T^{-1}(p) = -\frac{1}{\lambda}\log(1-p)$
- $\mathbb{E}[T] = 1/\lambda$
- $Var[T] = 1/\lambda^2$




- Simulation of $\{T_i\}_{i=1}^{N}$
  - $U_i \sim U[0,1]$
  - $T_i = -\frac{1}{\lambda}\log U_i\,; \quad i = 1,2,\dots,N$

# Density of arrival times

- Density of interarrival times

$$T_i \sim Exponential(\lambda): \quad f_T(\tau) = \lambda e^{-\lambda\tau}$$

- Density of arrival times

$$S_n \sim f_{S_n}(t) = \frac{1}{(n-1)!}\lambda^n t^{n-1} e^{-\lambda t}$$

Erlang pdf
(Gamma pdf whose shape parameter is a positive integer):
pdf of the sum of exponentials with the same rate parameter.

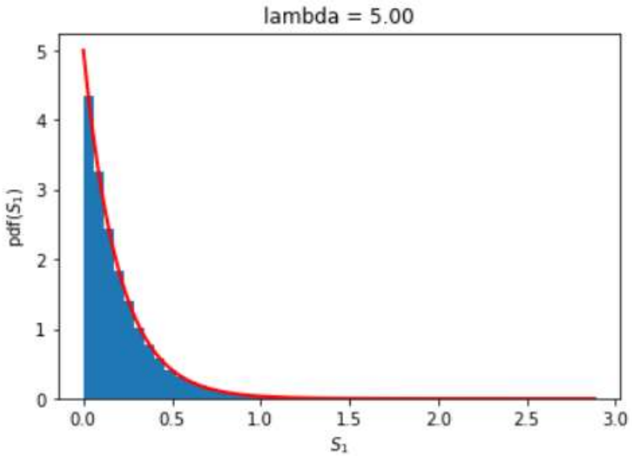
Proof: $f_{S_1,\dots,S_n}(s_1,\dots,s_n) = f_{T_1,\dots,T_n}(T_1 = s_1, T_2 = s_2 - s_1, \dots, T_n = s_n - s_{n-1})$

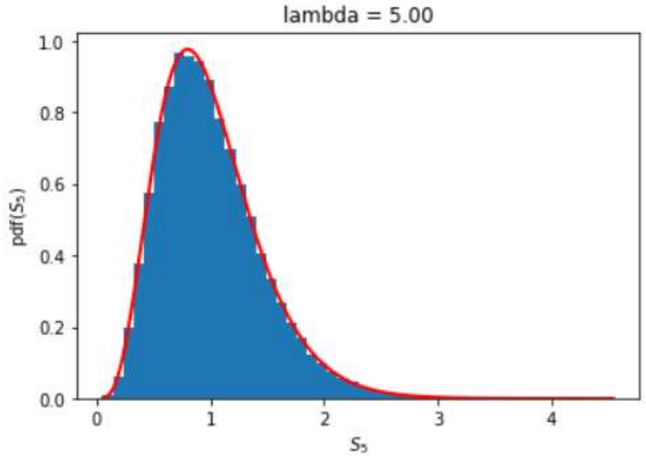
$$= f_{T_1}(T_1 = s_1)\, f_{T_2}(T_2 = s_2 - s_1) \dots f_{X_n}(T_n = s_n - s_{n-1}) = \lambda^n e^{-\lambda s_n}$$

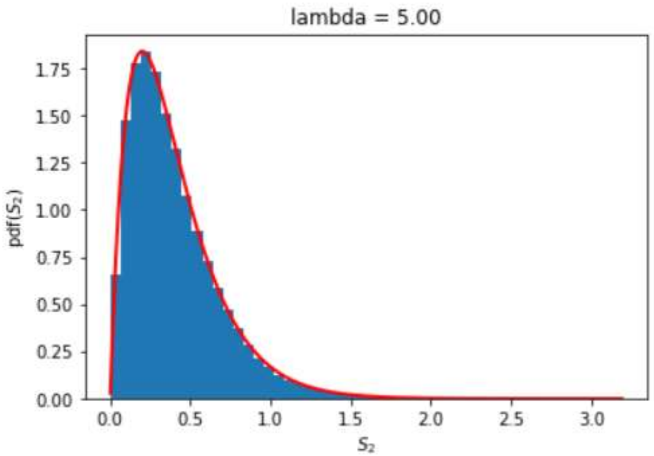
$$f_{S_n}(t) = \int_0^t ds_{n-1} \int_0^{s_{n-1}} ds_{n-2} \dots \int_0^{s_2} ds_1\, f_{S_1,\dots,S_n}(s_1,\dots,s_{n-1},t)$$

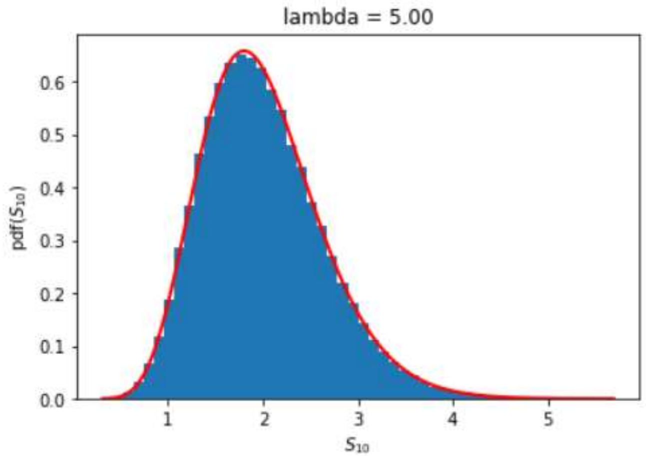
$$= \int_0^t ds_{n-1} \int_0^{s_{n-1}} ds_{n-2} \dots \int_0^{s_2} ds_1\, \lambda^n e^{-\lambda} = \frac{t^{n-1}}{(n-1)!}\lambda^n e^{-\lambda t}$$

# Density of arrival times



# Memoryless property

- The positive rv $T$ has the memoryless property if

$$\mathbb{P}(T > t + s) = \mathbb{P}(T > t)\mathbb{P}(T > s)$$

- Conditional distribution for a memoryless rv

$$\mathbb{P}(T > t + s |\, T > t) = \mathbb{P}(T > s)$$

- The exponential distribution $f_T(\tau) = \lambda e^{-\lambda}$ has the memoryless property

$$\mathbb{P}(T > t) = \int_t^{\infty} f_T(\tau) d\tau = e^{-\lambda t}$$

$$\Rightarrow \mathbb{P}(T > t + s) = e^{-\lambda(t+s)} = e^{-\lambda t} e^{-\lambda} = \mathbb{P}(T > t)\mathbb{P}(T > s)$$

The only memoryless rv's are:

- Geometric (discrete)
- Exponential (continuous)

# Probability of counts

- Probability mass function: $N(t) \sim Poisson(\lambda)$:

$$\mathbb{P}(N(t) = n) = \frac{\lambda^n}{n!}\ t^n e^{-\lambda t}$$

$$f_{S_n}(t) = \frac{\lambda^n}{(n-1)!} t^{n-1} e^{-\lambda t}$$

$$\mathbb{P}[S_n > t] = \frac{\lambda^n}{(n-1)!} \int_t^\infty s^{n-1} e^{-\lambda s} ds$$

Proof: From the definition of the count process

$$\mathbb{I}[N(t) < n] = \mathbb{I}[S_n > t] \Rightarrow \mathbb{P}[N(t) < n] = \mathbb{P}[S_n > t]$$

$$\mathbb{P}[N(t) < n] = \sum_{m=0}^{n-1} \mathbb{P}(N(t) = m) = \frac{\lambda^n}{(n-1)!} \int_t^\infty s^{n-1} e^{-\lambda s} ds$$

$$\mathbb{P}(N(t) = n) = \sum_{m=0}^{n} \mathbb{P}(N(t) = m) - \sum_{m=0}^{n-1} \mathbb{P}(N(t) = m)$$

$$= \frac{\lambda^{n+1}}{n!} \int_t^\infty s^n e^{-\lambda s} ds - \frac{\lambda^n}{(n-1)!} \int_t^\infty s^{n-1} e^{-\lambda s} ds = \frac{\lambda^n}{n!} \int_t^\infty \left(\lambda s^n - n s^{n-1}\right) e^{-\lambda s} ds$$

$$= \frac{\lambda^n}{n!} \left(-s^n e^{-\lambda}\ \right)\Big|_t^\infty = \frac{\lambda^n}{n!} t^n e^{-\lambda t}$$

# Simulation of homogenous Poisson process: Method 1

Generate samples from $Poisson(\lambda)$ in $[t_0, t_1]$:

1. $i \coloneqq 1$
2. Generate $T_i \sim Exponential(\lambda)$
3. $aux \coloneqq t_0 + T_i$
4. While $aux < t_1$

    $S_i \coloneqq aux$

    $i \coloneqq i + 1$

    Generate $T_i \sim Exponential(\lambda)$

    $aux \coloneqq S_{i-1} + T_i$

# Simulation of homogenous Poisson process: Method 2

Generate samples from $Poisson(\lambda)$ in $[t_0, t_1]$:

1. Sample $n$ from the Poisson distribution

$$\mathbb{P}(N(t_1) - N(t_0) = n) = \frac{\lambda^n}{n!}(t_1 - t_0)^n e^{-\lambda(t_1 - t_0)}$$

1. Generate $n$ independent samples from a distribution whose density is uniform in $[t_0, t_1]$ : $\{U_i \sim U[t_0, t_1]; \;\; i = 1, \dots, n\}$
3. Order the samples so that they take increasing values and correspond to a sequence of consecutive arrival times: $0 < S_1 < S_2 < \cdots < S_n$.

$\{S_i = U_{(i)}; \quad i = 1, \dots n\}$ are the order statistics of $\{U_i; \quad i = 1, \dots n\}$

# Merging Poisson processes

Let $\{N_1(t), N_2(t), \dots, N_M(t)\}$ be $M$ independent Poisson processes

$$N_m(t) \sim Poisson(\lambda_m), \qquad\qquad m = 1, \dots, M$$

The process

$$N(t) = \sum_{m=1}^{M} N_m(t) \sim Poisson\left(\sum_{m=1}^{M} \lambda_m\right)$$

# Splitting Poisson processes

- Consider the Poisson process $N(t)$ with rate $\lambda$
- To each arrival:
  - Assign label 1 with probability $p$
  - Assign label 2 with probability 1-$p$
- Sequence of arrivals labeled 1: Poisson process $N_1(t)$ with rate $\lambda_1 = p\lambda$
- Sequence of arrivals labeled 2: Poisson process $N_2(t)$ with rate $\lambda_2 = (1-p)\lambda$
- The processes $N_1(t), N_2(t)$ are independent:

$$\mathbb{P}(N_1(t) = n, N_2(t) = m) = \mathbb{P}(N_1(t) = n, N_2(t) = m \mid N(t) = n + m)\mathbb{P}(N(t) = m + n)$$

$$= \binom{n+m}{n} p^n (1-p)^m \frac{(\lambda t)^{n+m}}{(n+m)!} e^{-\lambda t} = \frac{(\lambda p t)^n}{n!} e^{-\lambda p t} \frac{(\lambda(1-p)t)^m}{m!} e^{-\lambda(1-p)t}$$

$$= \mathbb{P}(N_1(t) = n)\, \mathbb{P}(N_2(t) = m)$$

# Inhomogenous Poisson process

Consider the Poisson process $\{N(t);\ t \geq 0\}$ with continuous arrival rate $\lambda(t)$

$$\mathbb{P}(N(t) - N(s) = n) = \frac{\left(\int_s^t \lambda(\tau)d\tau\right)^n}{n!}\ e^{-\int_s^t \lambda(\tau)d\tau}; \qquad t > s > 0.$$

- Consider the random variable $S_i \mid S_{i-1}$

$$\mathbb{P}\big((S_i \mid S_{i-1}) \leq t\big) = 1 - \mathbb{P}(N(t) - N(S_{i-1}) = 0) = 1 - e^{-\int_{S_{i-1}}^t \lambda(\tau)d\tau}, \qquad t > S_{i-1}.$$

- Consider the random variable $Y_i = \int_{S_{i-1}}^{S_i} \lambda(\tau)\, d\tau$ with $S_{i-1}$ fixed

$$\mathbb{P}(Y_i \leq y) = \mathbb{P}\big((S_i \mid S_{i-1}) \leq t\big) = 1 - e^{-y}, \text{ with } y = \int_{S_{i-1}}^t \lambda(\tau)\, d\tau$$

Therefore, $Y_i \sim Exponential(1)$

$$\mathbb{P}(Y_i \leq y) = 1 - e^{-y}, \quad y > 0.$$

# Simulation of inhomogenous Poisson process: Method 1

**Simulation in interval $[0, T]$**

1. Set $S_0 = 0,\ i = 0.$
2. Repeat until $S_i$ larger than $T$

   $i = i + 1$

   Generate $Y_i \sim Exponential(1)$

   Determine $S_i$, such that $Y_i = \int_{S_{i-1}}^{S_i} \lambda(\tau)\, d\tau$

For instance, using Newton-Raphson to find the zero of

$$f(t) = \int_{S_{i-1}}^{t} \lambda(\tau)\, d\tau - Y_i$$

(i.e. $f(S_i) = 0$)

# Simulation of inhomogenous Poisson process: Method 2

**Simulation in interval** $[t_0, t_1]$

1. Sample $n$ from $\mathbb{P}(N(t_1) - N(t_0) = n) = \frac{\left(\int_{t_0}^{t_1} \lambda(\tau)d\tau\right)^n}{n!} e^{-\int_{t_0}^{t_1} \lambda(\tau)d\tau}$
2. Generate $n$ independent samples in $[t_0, t_1]$ from a distribution whose density is
$$Y_i \sim \frac{\lambda(t)}{\int_{t_0}^{t_1} \lambda(\tau)d\tau} \mathbb{I}[t_0 \leq t \leq t_1]; \quad i = 1, \dots, n$$
For a homogeneous Poisson process, $\lambda(\tau) = \lambda$: $Y_i \sim \frac{1}{t_1 - t_0} \mathbb{I}[t_0 \leq t \leq t_1] \sim \mathbb{I}[t_0, t_1]$
3. Order the samples so that they take increasing values and correspond to a sequence of consecutive arrival times: $0 < S_1 < S_2 < \cdots < S_n$

$\{S_i = Y_{(i)}; \quad i = 1, \dots n\}$ are the order statistics of $\{Y_i; \quad i = 1, \dots n\}$