Stochastic Systems --- Discrete Time Systems

Ejercicios internedios

Solutions and comments

(Perdón por el inglés... escribo más rápido y con menos errores así)

i) Most of you got the right matrix. I had a couple of cases in which you forgot that the sum of the rows must be 1 and that there is (implicit) an arc from a state to itself that balances the probability so that the sum is 1.

ii) The code was quite easy and most of you nailed (even programming things more complicated than would be becessary). The interpretation of the results had a few glitch. The hs were easy, but the $k = \infty$ gave a few problems. The numerical result not always made sense, and had to be interpreted. In this case, I suspect that most people did a *a posteriori* interpretation: knowing that certain ks had to be ∞ interpreted the results accordingly. In general the thing is complicated. One option is to run a simulation with several values of the final simulation time T, and see that these k grow linearly with T while the ks that are not ∞ stay more or less constant. A simpler way is to notice that, even with a very long time ($T = 1000$ for example), not all trials make it to the state. This observation, alone, is enough to decree that $k = \infty$. Simply putting the average value obtained from the simulation without comment is, in this case, an error.

iii) The linear systems are quite straightforward but in most cases, we can find the theoretical solution with a bit of reasoning, without too many calculations.

$q = 0$ Let us begin with h_0^2. If we sart from state 0, the only way we can move out of it is going to state 1. The probability of staying in 0 is 0.7 so the probability of staying t steps in 0 is 0.7^t. This goes to zero so the probability of leaving 0 sometimes in the future is 1. By the same reasoning, once we have arrived to state 1, at some point in the future we shall leave it (with probability 1... this is not the same thing as saying that we shall **always** leave it, but for our purposes is is enough). When we leave, we have the same probability of moving to 2 or to 4 and, in this case, we shall never reach 2, since 4 is part of an absorbing set os which 2 is not a part. We reach 2 from 0 only if, when we are in state 1, we transition to 2. Therefore

$$\boxed{q = 0 \;\longrightarrow\; h_0^2 = \frac{1}{2}}$$

(Have you seen how I make the results very noticeable? Many of you hid them so well in the text that they gave me the impression of not wanting that I should find them. You should learn to write good report, with the important stuff well highlighted: it is important.)

*   *   *

Let us now compute h_0^5. By the same reasoning as before, if we leave the cycle $\begin{tikzcd} 0 \arrow[r] & 1 \arrow[r] & 2 \arrow[ll, bend right] \end{tikzcd}$ we shall end up in state 5. the cycle as a probability $0.3 \times 0.15 \times 0.25 = 0.011$ so, woth probability 1, we shall eventually leave it and arrive to 5. Therefore

$$\boxed{q = 0 \ \longrightarrow\ h_0^5 = 1}$$

*   *   *

On to the average times. For k_0^2, to go go from 0 to 2, we have to go through 1, which has an equation

$$k_1^2 = 1 + 0.15k_2^2 + 0.15k_4^2 + 0.7k_1^2 \tag{1}$$

But state 2 is unreachable fom 4, so $k_4^2 = \infty$. Consequently

$$\boxed{q = 0 \ \longrightarrow\ k_0^2 = \infty}$$

This will also give us the solution for the other case:

$$\boxed{q = 0 \ \longrightarrow\ k_4^2 = \infty}$$

$q = 0.1$ In this case, there are not abssorbing states or sets of states, therefore we shall visit them all with probability 1

$$\boxed{q = 0.1 \ \longrightarrow\ h_0^2 = 1}$$

$$\boxed{q = 0.1 \ \longrightarrow\ h_0^5 = 1}$$

*   *   *

For the k_i^j we have no other choice but to write down the equations:

$$\begin{aligned}
k_0^2 &= 1 + 0.3k_1^2 + 0.7k_0^2 \\
k_1^2 &= 1 + 0.15k_2^2 + 0.15k_4^2 + 0.7k_1^2 \\
k_2^2 &= 0 \\
k_3^2 &= 1 + 0.1k_1^2 + 0.8k_3^3 + 0.1k_4^2 \\
k_4^2 &= 1 + 0.5k_4^2 + 0.5k_5^2 \\
k_5^2 &= 1 + 0.25k_3^2 + 0.5k_4^2 + 0.25k_5^2
\end{aligned} \tag{2}$$

The result is

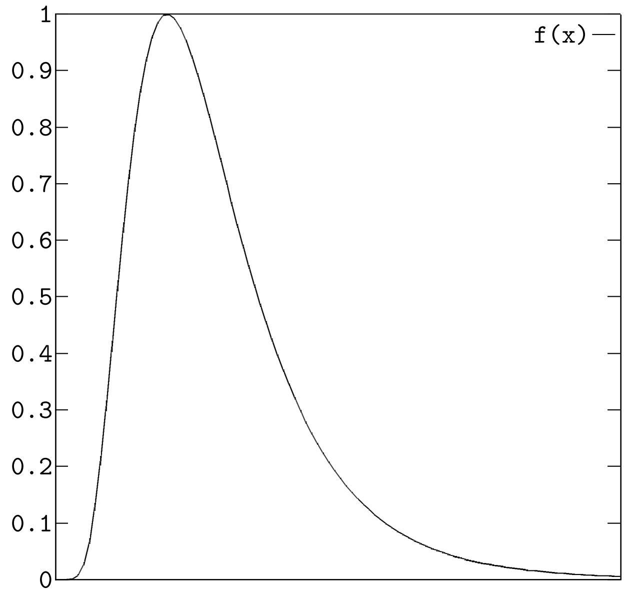
$$\boxed{q = 0.1 \;\longrightarrow\; k_0^2 = \frac{130}{3} \approx 43.33}$$

and

$$\boxed{q = 0.1 \;\longrightarrow\; k_4^2 = \frac{220}{3} \approx 73.33}$$

The simulation gave more or less the same results, with the aforementioned caveat for the ∞ values.

iv) Here there were a few bizarre results... most of you got it right and got something like this:



Others have plotted something that looks more like a cumulative distribution, which is not quite what I was asking

v) Several people have used numerical results from the simulations to answer this one. This is wrong. The word *sempre* in the statement was quite clear: always means always, and it is something you can't get from a few simulation. In this case, you were supposed to write no code and reason: The only way to get to state 5 is from state 4 so, unless your initial state is 5, you have to go through 4 at least one step before you get to 5. This is the case, since we are starting from 0. You could even make the stronger statement

$$H_o^{\{5\}} \geq H_o^{\{4\}} + 1$$