Continuous-time Markov chains

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Some material from:
- H. Pishro-Nik, "Introduction to probability, statistics, and random processes", available at https://www.probabilitycourse.com, Kappa Research LLC, 2014.
- https://ocw.mit.edu/courses/electrical-engineering-and-computer-science/6-262-discrete-stochastic-processes-spring-2011/course-notes/MIT6_262S11_chap02.pdf

Continuous-time Markov chain (CTMC)

- A CTMC is a continuous-time stochastic process in which the states are discrete:

$$\{X(t) \in S; \;\; t \geq 0\}; \quad S \subset \{0,1,2, \dots\}$$

- Furthermore, the process has the **Markov property**

$$\forall\, t_1 < t_2 < \cdots < t_n < t_{n+1}$$

$$\mathbb{P}(X(t_{n+1}) = j |\, X(t_n) = i, X(t_{n-1}) = i_{n-1}, \dots X(t_1) = i_1) = \mathbb{P}(X(t_{n+1}) = j |\, X(t_n) = i)$$

- We will focus on **homogeneous** CTMC's

$$\mathbb{P}(X(t+\tau) = j |\, X(t) = i) = P_{ij}(\tau)$$

Jump chain

The only memoryless rv's are:

- Geometric (discrete)
- Exponential (continuous)

- The process undergoes jumps between states at times

$$0 < t_1 < t_2 < \cdots < t_n < \cdots$$

  such that $X(t) = i_n,\ t_n \leq t < t_{n+1}$.

  - The process is **memoryless**: the probability of a jump does not depend on the time that the process has spent in state i (holding time).
  - The **holding time** for state i is an **exponentially distributed** random variable of parameter λ_i.
  - **Jump chain**: The probability of undergoing a transition between states i and j at a jump is $p_{ij} \geq 0$, with $\sum_{j \in S} p_{ij} = 1$

- Non absorbing state: $p_{ii} = 0$
- Absorbing state: $p_{ii} = 1$

Transition matrix: $\boldsymbol{P}(t)$

- The elements of the transition matrix are

$$P_{ij}(t) = \mathbb{P}(X(t) = j \mid X(0) = i), \quad i, j \in S$$

  - $0 \leq P_{ij}(t) \leq 1; i, j \in S$
  - $\sum_{j \in S} P_{ij}(t) = 1, \quad \forall t \geq 0$

    The right eigenvector corresponding to the eigenvalue 1 is $\boldsymbol{v}^T = (1, \dots, 1)$
  - The largest eigenvalue of $\boldsymbol{P}(t)$ in absolute value is 1.
  - $\boldsymbol{P}(0) = \mathbb{I}_{|S|}$

Chapman-Kolmogorov equation

$$\boldsymbol{P}(t+s) = \boldsymbol{P}(t)\boldsymbol{P}(s), \qquad \forall t, s \geq 0$$

Proof:

$$\begin{aligned}
p_{ij}(t+s) &= \mathbb{P}(X(t+s) = j \mid X(0) = i) \\
&= \textstyle\sum_{k\in S} \mathbb{P}(X(t+s) = j, X(t) = k \mid X(0) = i) \\
&= \textstyle\sum_{k\in S} \mathbb{P}(X(t+s) = j \mid X(t) = k, X(0) = i) \ \ \mathbb{P}(X(t) = k \mid X(0) = i) \\
&= \textstyle\sum_{k\in S} \mathbb{P}(X(t+s) = j \mid X(t) = k) \mathbb{P}(X(t) = k \mid X(0) = i) \\
&= \textstyle\sum_{k\in S} p_{ik}(t) p_{kj}(s)
\end{aligned}$$

Left / right eigenvectors

If $\mathbf{A}$ is real, the complex eigenvalues form pairs of complex conjugates.

$$\mathbf{A}\,\mathbf{v} = \lambda\,\mathbf{v} \quad \Longrightarrow \quad \mathbf{A}\,\bar{\mathbf{v}} = \bar{\lambda}\,\bar{\mathbf{v}}$$

complex conjugate

$\lambda = Re(\lambda) + i\,Im(\lambda)$

$\bar{\lambda} = \lambda = Re(\lambda) - i\,Im(\lambda)$

Consider the complex valued $D \times D$ matrix $\mathbf{A}$

- Assuming it can be diagonalized:
  - Eigenvalues: $\{\lambda_i \in \mathbb{C}\}_{i=1}^{D}$
  - Right eigenvectors:

$$\mathbf{A}\,\mathbf{v}_i^{(r)} = \lambda_i\,\mathbf{v}_i^{(r)}; \quad i = 1,2,\dots,D$$

  - Left eigenvectors:

$\mathbf{A}^\dagger$ Hermitian conjugate of matrix $\mathbf{A}$ (transpose + complex conjugate)

$$\left(\mathbf{v}_i^{(l)}\right)^\dagger \mathbf{A} = \lambda_i \left(\mathbf{v}_i^{(l)}\right)^\dagger \quad \Longrightarrow \quad \mathbf{A}^\dagger\,\mathbf{v}_i^{(l)} = \bar{\lambda}_i\,\mathbf{v}_i^{(l)}; \quad i = 1,2,\dots,D$$

  - Orthogonality relations: $\left(\mathbf{v}_i^{(l)}\right)^\dagger \mathbf{v}_j^{(r)} = 0, \quad i \neq j, \quad i,j = 1,2,\dots,D$

Diagonalization of Hermitian matrices

If the matrix is Hermitian: $\mathbf{A}^{\dagger} = \mathbf{A}$

If $\mathbf{A}$ is real: $\mathbf{A}^{T} = \mathbf{A}$ (symmetric)

- The eigenvalues are real $\{\lambda_i \in \mathbb{R}\}_{i=1}^{D}$
- The left and right eigenvectors coincide:

  $\mathbf{A}\mathbf{v}_i = \lambda_i \mathbf{v}_i$

  $\mathbf{v}_i^{\dagger}\mathbf{A} = \lambda_i\, \mathbf{v}_i^{\dagger}, \quad i = 1, \dots, D$
- The eigenvectors $\{\mathbf{v}_i\}_{i=1}^{D}$ can be chosen to form an orthonormal basis of $\mathbb{C}^{D}$

$$\mathbf{v}_i^{\dagger}\mathbf{v}_j = \delta_{ij};\ i, j = 1, \dots, D.$$

- $\mathbf{A}$ can be diagonalized by the unitary transformation $\mathbf{U}$

$$\mathbf{U}^{\dagger}\mathbf{A}\,\mathbf{U} = \mathbf{\Lambda}$$

  - $\mathbf{\Lambda} = diag(\lambda_1, \dots, \lambda_D)$
  - $\mathbf{U}^{\dagger}\,\mathbf{U} = \mathbf{I}_D$

Stationary distribution

- Consider $X(t)$, a CTMC with a transition matrix $\boldsymbol{P}(t)$.
- The probability vector $\boldsymbol{\pi}$ is a stationary distribution of the chain, if $\boldsymbol{\pi}^T$ is a **left eigenvector** of $\boldsymbol{P}(t)$ with **eigenvalue 1**:

$$\boldsymbol{\pi}^T \boldsymbol{P}(t) = \boldsymbol{\pi}^T, \qquad \forall t \geq 0.$$

$0 \leq \pi_i \leq 1, i \in S; \quad \sum_{i \in S} \pi_i = 1$

Left eigenvector of $\boldsymbol{P}(t)$ = Right eigenvector of $\boldsymbol{P}(t)^T$

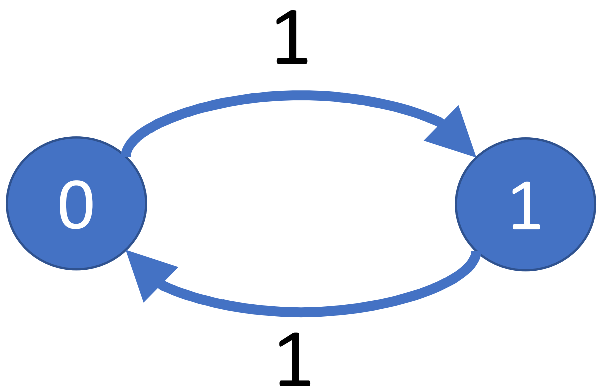
$$\boldsymbol{P}(t)^T \boldsymbol{\pi} = \boldsymbol{\pi}$$

- <u>Interpretation</u>: If the distribution of $X(0)$ is $\boldsymbol{\pi}$, then the distribution of $X(t)$ is $\boldsymbol{\pi}$ $\forall t \geq 0$

Not all Markov chains have a stationary distribution

Example

- Consider a continuous Markov chain with two states $S = \{0,1\}$, none of which is absorbing.
- Assume the holding times follow an exponential distribution with parameter $\lambda_0 = \lambda_1 = \lambda > 0$
- Find the transition matrix $\boldsymbol{P}(t)$, and the stationary distribution $\boldsymbol{\pi}$.



Answer:

- $\mathbb{P}(X(t) = 0 |\, X(0) = 0) = \mathbb{P}(X(t) = 1 |\, X(0) = 1)$

$= \sum_{k=0}^{\infty} \mathbb{P}(N(t) = 2k) = \sum_{k=0}^{\infty} \frac{1}{(2k)!} (\lambda t)^{2k} e^{-\lambda t} = \frac{e^{\lambda t} + e^{-\lambda t}}{2} e^{-\lambda t} = \frac{1 + e^{-2\lambda t}}{2}$

- $\mathbb{P}(X(t) = 1 |\, X(0) = 0) = \mathbb{P}(X(t) = 0 |\, X(0) = 1)$

$= \sum_{k=0}^{\infty} \mathbb{P}(N(t) = 2k+1) = \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} (\lambda t)^{2k+1} e^{-\lambda t} = \frac{e^{\lambda t} - e^{-\lambda t}}{2} e^{-\lambda t} = \frac{1 - e^{-2\lambda t}}{2}$

Answer: $\boldsymbol{P}(t) = \begin{pmatrix} \frac{1}{2} + \frac{1}{2} e^{-2\lambda t} & \frac{1}{2} - \frac{1}{2} e^{-2\lambda t} \\ \frac{1}{2} - \frac{1}{2} e^{-2\lambda t} & \frac{1}{2} + \frac{1}{2} e^{-2\lambda t} \end{pmatrix}, \quad \boldsymbol{\pi} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}.$

The infinitesimal generator (generator matrix)

- The infinitesimal generator is the matrix $\boldsymbol{G}(t)$ whose elements are

$$\mathbb{P}(X(t+dt)=j|\,X(t)=i)=g_{ij}(t)\,dt \qquad j\neq i$$
$$\mathbb{P}(X(t+dt)=i|\,X(t)=i)=1+g_{ii}(t)dt; \quad i,j\in S$$
$$\textstyle\sum_{j\in S} g_{ij}(t)=0$$

- For a CTMC: $$g_{ij}=\begin{cases}\lambda_i p_{ij},; & i\neq j\\ -\lambda_i; & i=j\end{cases} \qquad i,j\in S$$

Probability of a jump in $[t,t+dt]$

Proof:

$$\mathbb{P}(X(t+dt)=j|\,X(t)=i)=\left(1-e^{-\lambda_i dt}\right)p_{ij}\approx\lambda_i p_{ij}\,dt; \quad j\neq i$$
$$\mathbb{P}(X(t+dt)=i|\,X(t)=i)=e^{-\lambda_i dt}\approx 1-\ \lambda_i\,dt$$

Probability of zero jumps in $[t,t+dt]$

Probability of transition $i\to j$

Forward equation

- From the Chapman-Kolmogorov equation

$$\boldsymbol{P}(t+dt) = \boldsymbol{P}(t)\boldsymbol{P}(dt), \quad \forall t, s \geq 0$$

$$\begin{aligned} p_{ij}(t+dt) &= \sum_{k \in S} p_{ik}(t) p_{kj}(dt) = p_{ij}(t) p_{jj}(dt) + \sum_{k \neq j} p_{ik}(t) p_{kj}(dt) \\ &= p_{ij}(t) + p_{ij}(t) g_{jj} dt + \sum_{k \neq j} p_{ik}(t) g_{kj}\, dt \\ &= p_{ij}(t) + \sum_{k \in S} p_{ik}(t) g_{kj}\, dt \end{aligned}$$

$$\frac{d}{dt} p_{ij}(t) = \lim_{dt \to 0} \frac{p_{ij}(t+dt) - p_{ij}(t)}{dt} = \sum_{k \in S} p_{ik}(t) g_{kj} \Longrightarrow \frac{d}{dt}\boldsymbol{P}(t) = \boldsymbol{P}(t)\boldsymbol{G}$$

Backward equation

- From the Kolmogorov-Chapman equation

$$\boldsymbol{P}(t+dt) = \boldsymbol{P}(dt)\boldsymbol{P}(t), \quad \forall t, s \geq 0$$

$$\begin{aligned} p_{ij}(t+dt) &= \textstyle\sum_{k\in S} p_{ik}(dt)p_{kj}(t) = p_{ii}(dt)\, p_{ij}(t) + \sum_{k\neq i} p_{ik}(dt)p_{kj}(t) \\ &= p_{ij}(t) + g_{ii}\, p_{ij}(t)dt + \textstyle\sum_{k\neq i} g_{ik}\, p_{kj}(t)\, dt \\ &= p_{ij}(t) + \textstyle\sum_{k\in S} g_{ik}\, p_{kj}(t)\, dt \end{aligned}$$

$$\frac{d}{dt} p_{ij}(t) = \lim_{dt\to 0} \frac{p_{ij}(t+dt) - p_{ij}(t)}{dt} = \sum_{k\in S} g_{ik}\, p_{kj}(t) \Longrightarrow \frac{d}{dt}\boldsymbol{P}(t) = \boldsymbol{G}\, \boldsymbol{P}(t)$$

Solution

The solution of the forward and backward equations

$$\frac{d}{dt}\boldsymbol{P}(t) = \boldsymbol{P}(t)\boldsymbol{G}$$

$$\frac{d}{dt}\boldsymbol{P}(t) = \boldsymbol{G}\,\boldsymbol{P}(t)$$

with the initial condition $\boldsymbol{P}(0) = \mathbb{I}_{|S|}$ is

$$\boldsymbol{P}(t) = e^{\boldsymbol{G}t} = \sum_{n=0}^{\infty} \frac{1}{n!}\boldsymbol{G}^n t^n$$

Stationary distribution in terms of $\boldsymbol{G}$

The stationary distribution of a CTMC whose infinitesimal generator is $\boldsymbol{G}$ is $\boldsymbol{\pi}$ if and only if

$$\boldsymbol{\pi}^T \boldsymbol{G} = \boldsymbol{0}$$

Proof [if $\boldsymbol{\pi}$ is stationary then $\boldsymbol{\pi}^T \boldsymbol{G} = \boldsymbol{0}$]

- From the definition of stationary distribution: $\boldsymbol{\pi}^T \boldsymbol{P}(t) = \boldsymbol{\pi}^T$
- Taking the derivative w.r.t. t: $\boldsymbol{\pi}^T \frac{d}{dt} \boldsymbol{P}(t) = \boldsymbol{0}$
- Using the backwards equation: $\boldsymbol{\pi}^T \boldsymbol{G}\boldsymbol{P}(t) = \boldsymbol{0}$
- Since the equation is valid for all t, let's evaluate at $t = 0$: $\boldsymbol{\pi}^T \boldsymbol{G}\boldsymbol{P}(0) = \boldsymbol{0}$
- Since $\boldsymbol{P}(0) = \mathbb{I}_{|S|}$: $\boldsymbol{\pi}^T \boldsymbol{G} = \boldsymbol{0}$

Stationary distribution in terms of $\boldsymbol{G}$

Proof [if $\boldsymbol{\pi}^T \boldsymbol{G} = \boldsymbol{0}$ then $\boldsymbol{\pi}$ is stationary]

- Let $\boldsymbol{\pi}^T \boldsymbol{G} = \boldsymbol{0}$
- Right-multiply both sides by $\boldsymbol{P}(t)$**:** $\boldsymbol{\pi}^T \boldsymbol{G}\boldsymbol{P}(t) = \boldsymbol{0}$
- Using the backward equation, we get $\boldsymbol{\pi}^T \frac{d}{dt}\boldsymbol{P}(t) = \boldsymbol{0}$
- If $|S|$ is finite: $\frac{d}{dt}\left(\boldsymbol{\pi}^T \boldsymbol{P}(t)\right) = \boldsymbol{0}$
- Therefore, $\boldsymbol{\pi}^T \boldsymbol{P}(t)$ is independent of t. In particular, $\boldsymbol{\pi}^T \boldsymbol{P}(t) = \boldsymbol{\pi}^T \boldsymbol{P}(\boldsymbol{0})$
- Since $\boldsymbol{P}(0) = \mathbb{I}_{|S|}$: $\boldsymbol{\pi}^T \boldsymbol{P}(t) = \boldsymbol{\pi}^T$. Therefore, $\boldsymbol{\pi}$ is stationary

Example

- Consider a continuous Markov chain with two states $S = \{0,1\}$, none of which is absorbing.
- Assume the holding times follow an exponential distribution with parameter $\lambda_0 = \lambda_1 = \lambda > 0$
- Find the infinitesimal generator $\boldsymbol{G}$.
- Compute the stationary distribution $\boldsymbol{\pi}^T$ from $\boldsymbol{G}$.
- Compute $\boldsymbol{P}(t)$ from $\boldsymbol{G}$.

Answer

- $\boldsymbol{G} = \begin{pmatrix} -\lambda & \lambda \\ \lambda & -\lambda \end{pmatrix}.$

- $\boldsymbol{\pi}^T \boldsymbol{G} = \boldsymbol{0} \implies (\pi_0 \quad \pi_1) \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} = (0 \quad 0) \Longrightarrow \boldsymbol{\pi}^T = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \end{pmatrix}$
- $\boldsymbol{G}^n = -\frac{1}{2}(-2\lambda)^n \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}; n \geq 1$
- $\boldsymbol{P}(t) = e^{\boldsymbol{G}t} = \sum_{n=0}^{\infty} \frac{1}{n!} \boldsymbol{G}^n t^n = \mathbb{I} + \sum_{n=1}^{\infty} \frac{1}{n!} \boldsymbol{G}^n t^n = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n!} (-2\lambda t)^n \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{1}{2}\left(e^{-2\lambda} - 1\right) \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} + \frac{1}{2} e^{-2\lambda t} & \frac{1}{2} - \frac{1}{2} e^{-2\lambda t} \\ \frac{1}{2} - \frac{1}{2} e^{-2\lambda t} & \frac{1}{2} + \frac{1}{2} e^{-2\lambda t} \end{pmatrix}$$

Detailed balance

The probability vector $\boldsymbol{\pi}$ satisfies the detailed balance condition if

$$\pi_i g_{ij} = \pi_j g_{ji}, \qquad i, j \in S$$

The flux of probability per unit time from i to j is the same as the one from j to i

<u>Theorem</u>: If $\boldsymbol{\pi}$ satisfies the detailed balance condition, then it is the stationary distribution.

<u>Proof:</u>

$$\left(\boldsymbol{\pi}^T \boldsymbol{G}\right)_j = \sum_{i \in S} \pi_i g_{ij} = \sum_{i \in S} \pi_j g_{ji} = \pi_j \sum_{i \in S} g_{ji} = 0, \qquad j = 1, \dots, |S|$$

$$\boldsymbol{\pi}^T \boldsymbol{G} = \boldsymbol{0}$$

Limiting distribution

- Consider $X(t)$, a CTMC with a transition matrix $\boldsymbol{P}(t)$.
- The probability vector $\boldsymbol{\pi}$ is the limiting distribution of the chain, if

$$\pi_j = \lim_{t\to\infty} \mathbb{P}(X(t) = j \mid X(0) = i), \qquad i, j \in S$$

$0 \leq \pi_i \leq 1, i \in S; \quad \sum_{i\in S} \pi_i = 1$

- Interpretation: The distribution of $X(t)$ tends to $\boldsymbol{\pi}$ as $t \to \infty$, independently of the distribution of $X(0)$.

- Not all Markov chains have a limiting distribution.
- If a MC have it, the **limiting distribution is stationary**.

The limiting distribution is stationary

Proof [not entirely rigorous]:

- From the definition of limiting distribution

$$\lim_{s\to\infty} \boldsymbol{P}^T(s) = \boldsymbol{\pi}^T = \left(\pi_1\ \pi_2\ \ldots \pi_{|S|}\right)$$

- From the Chapman-Kolmogorov equation

$$\boldsymbol{P}^T(t+s) = \boldsymbol{P}^T(t)\boldsymbol{P}^T(s) \quad \forall t, s \geq 0$$

- Taking the limit $s \to \infty$ for any $t \geq 0$.

Needs to be justified!

$$\lim_{s\to\infty} \boldsymbol{P}^T(t+s) = \lim_{s\to\infty}\left(\boldsymbol{P}^T(t)\boldsymbol{P}^T(s)\right) = \boldsymbol{P}^T(t) \lim_{s\to\infty}\left(\boldsymbol{P}^T(s)\right) \Longrightarrow$$

$$\left(\pi_1\ \pi_2\ \ldots \pi_{|S|}\right) = \boldsymbol{P}^T(t)\left(\pi_1\ \pi_2\ \ldots \pi_{|S|}\right) \Longrightarrow \boldsymbol{\pi}^T = \boldsymbol{\pi}^T \boldsymbol{P}(t)$$

Irreducible continuous-time Markov chain

The CTMC $X(t)$ with an infinitesimal generator $\boldsymbol{G}$ is irreducible if for all $i, j \in S$, there is a finite sequence of states

$$i_0 = i, i_1, i_2, \dots, i_N = j \text{ such that } g(i_n, i_{n+1}) > 0, \quad n = 0,1, \dots, N-1.$$

<u>Theorem:</u> Let $X(t)$ be an irreducible CTMC whose stationary distribution is $\boldsymbol{\pi}$, then

$$\lim_{t\to\infty} \mathbb{P}(X(t) = j |\, X(0) = i) = \pi_j$$

Furthermore, let $f: S \longrightarrow \mathbb{R}$ be a function such that $\sum_{i\in S} \pi_i |f(i)| < \infty$

$$\lim_{t\to\infty} \frac{1}{t} \sum_{\tau=0}^{t} f\big(X(t)\big) = \mathbb{E}_{\boldsymbol{\pi}}[f] = \sum_{i\in S} \pi_i f(i)\,.$$

Derivation of the limiting distribution

All states communicate

Positive recurrent: The expected time of return to state j is finite

- Let the irreducible, positive recurrent **jump chain** characterized by the transition matrix $\left\{p_{ij} \geq 0;\ \ i,j \in S,\ \ \sum_{j \in S} p_{ij} = 1\right\}$ have the stationary distribution $\left\{\tilde{\pi}_i \geq 0;\ \ i \in S,\ \sum_{i \in S} \tilde{\pi}_i = 1\right\}$.

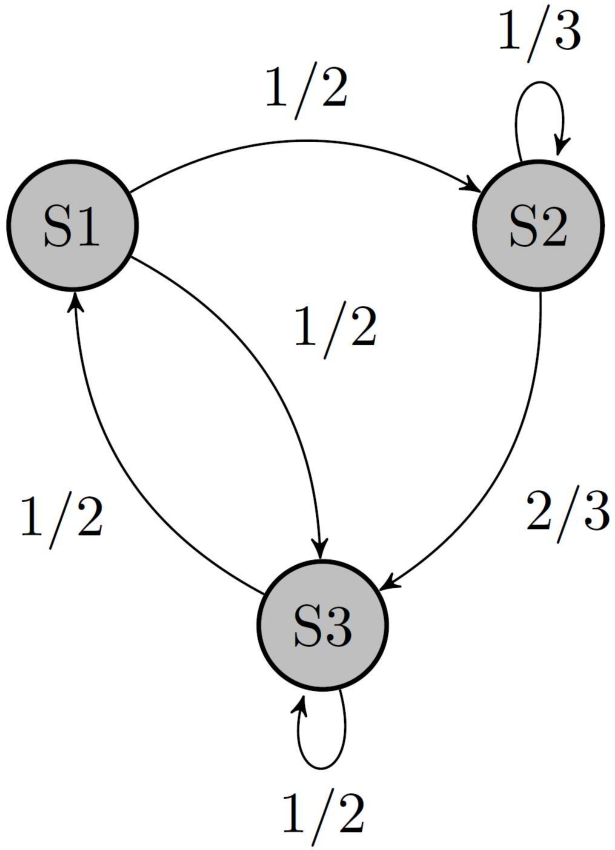
$$\sum_{i \in S} \tilde{\pi}_i\, p_{ij} = \tilde{\pi}_j; \qquad i \in S$$

- Assuming $0 < \sum_{k \in S} \frac{\tilde{\pi}_k}{\lambda_k} < \infty$

$\frac{1}{\lambda_j}$ is the average time the chain spends in state j

$$\pi_j = \lim_{t \to \infty} \mathbb{P}(X(t) = j \,|\, X(0) = i) = \frac{\frac{\tilde{\pi}_j}{\lambda_j}}{\sum_{k \in S} \frac{\tilde{\pi}_k}{\lambda_k}}; \qquad j \in S$$

Example



$$\tilde{P} = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 0 & 1/3 & 2/3 \\ 1/2 & 0 & 1/2 \end{pmatrix}$$

$$\lambda_1 = 2; \ \lambda_2 = 3; \ \lambda_3 = 4.$$

Stationary distribution of the jump chain

Left eigenvector with eigenvalue 1:

$$(\tilde{\pi}_1\ \tilde{\pi}_2\ \tilde{\pi}_3)\begin{pmatrix} 0 & 1/2 & 1/2 \\ 0 & 1/3 & 2/3 \\ 1/2 & 0 & 1/2 \end{pmatrix} = (\tilde{\pi}_1\ \tilde{\pi}_2\ \tilde{\pi}_3)$$

[1] $\tilde{\pi}_1 = \frac{1}{2}\tilde{\pi}_3$;

[2] $\tilde{\pi}_2 = \frac{1}{2}\tilde{\pi}_1 + \frac{1}{3}\tilde{\pi}_2 \Longrightarrow \frac{2}{3}\tilde{\pi}_2 = \frac{1}{4}\tilde{\pi}_3 \Longrightarrow \tilde{\pi}_2 = \frac{3}{8}\tilde{\pi}_3$

[3] $\tilde{\pi}_3 = \frac{1}{2}\tilde{\pi}_1 + \frac{2}{3}\tilde{\pi}_2 + \frac{1}{2}\tilde{\pi}_3 \Longrightarrow \tilde{\pi}_3 = \tilde{\pi}_1 + \frac{1}{4}\tilde{\pi}_2 + \frac{1}{2}\tilde{\pi}_3$ (not independent of [1] + [2]])

- Normalization: $\tilde{\pi}_1 + \tilde{\pi}_2 + \tilde{\pi}_3 = 1 \Longrightarrow \left(\frac{1}{2} + \frac{3}{8} + 1\right)\tilde{\pi}_3 = 1 \Longrightarrow \tilde{\pi}_3 = \frac{8}{15}$
- Stationary distribution of the jump chain:

$$\tilde{\pi}_1 = \frac{4}{15};\quad \tilde{\pi}_2 = \frac{1}{5};\ \tilde{\pi}_3 = \frac{8}{15}$$

Stationary distribution of the continuous time Markov chain

- Stationary distribution of the jump chain:

$$\tilde{\pi}_1 = \frac{4}{15}; \quad \tilde{\pi}_2 = \frac{1}{5}; \; \tilde{\pi}_3 = \frac{8}{15}$$

- Rates: $\lambda_1 = 2; \;\; \lambda_2 = 3; \;\; \lambda_3 = 4.$
- Stationary distribution for the CTMC

$$\pi_1 \propto \frac{\tilde{\pi}_1}{\lambda_1} = \frac{2}{15};$$

$$\pi_2 \propto \frac{\tilde{\pi}_2}{\lambda_2} = \frac{1}{15};$$

$$\pi_3 \propto \frac{\tilde{\pi}_3}{\lambda_3} = \frac{2}{15}$$

- Normalization: $\pi_1 + \pi_2 + \pi_3 = 1$

$$\pi_1 = \frac{2}{5}; \quad \pi_2 = \frac{1}{5}; \quad \pi_3 = \frac{2}{5}$$

Birth-death Markov chain

A birth-death Markov chain is a CTMC with $S = \{0,1,2,\dots,N\}$ and infinitesimal generator

$$\boldsymbol{G} = \begin{pmatrix} -\lambda_0 & \lambda_0 & 0 & & & \\ \mu_1 & -(\mu_1+\lambda_1) & \lambda_1 & \cdots & 0 & \\ 0 & \mu_2 & -(\mu_2+\lambda_2) & & & \\ & \vdots & & \ddots & \vdots & \\ & 0 & & \cdots & -(\mu_{N-1}+\lambda_{N-1}) & \lambda_{N-1} \\ & & & & \mu_N & -\mu_N \end{pmatrix}$$

Birth rates: λ_0,λ_1,…, λ_{N-1}

Death rates: $\mu_1, \mu_2, \dots, \mu_N$

Stationary distribution of a birth-death MC

- From the detailed balance equation

$$\pi_{n-1}\lambda_{n-1} = \pi_n\mu_n, \qquad n \geq 1$$

one gets

$$\pi_n = \frac{\lambda_{n-1}}{\mu_n}\pi_{n-1},\ n \geq 1$$

which iterated yields

$$\pi_n = \prod_{i=1}^{n} \frac{\lambda_{i-1}}{\mu_i}\pi_0$$

- Using the fact that the stationary distribution is normalized:

$$\pi_0 = \left(1 + \sum_{n=1}^{\infty}\prod_{i=1}^{n}\frac{\lambda_{i-1}}{\mu_i}\right)^{-1}$$

Examples of birth-death Markov chains

- Poisson process (does not have a stationary distribution):
  - $\lambda_n = \lambda > 0, \quad n \geq 0$
  - $\mu_n = 0, \quad n \geq 1$
- *M/M/1* queue:
  - $\lambda_n = \lambda > 0, \quad n \geq 0$
  - $\mu_n = \mu > 0, \quad n \geq 1$

  $\mu > \lambda; \qquad \pi_n = \left(\frac{\lambda}{\mu}\right)^n \left(1 - \frac{\lambda}{\mu}\right), \qquad n \geq 0.$
- *M/M/∞* queue
  - $\lambda_n = \lambda > 0, \quad n \geq 0$
  - $\mu_n = n\mu > 0, \quad n \geq 1$

  $\pi_n = \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n e^{-\frac{\lambda}{\mu}} \qquad n \geq 0.$

Number of customers in a system with a single server, in which job arrivals follow a Poisson process (rate λ) rate and service times have an exponential distribution (rate μ).

Positive recurrent

Number of customers in a system with unlimited servers, in which arrivals follow a Poisson process (rate λ) and service times have an exponential distribution (rate μ).

Poisson distribution