Quantum chaos: Quantization of the standard map

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Abstract

This paper provides a basic introduction to the concept of quantum chaos and the importance of the standard map in the study of dynamical systems. Finally a general method to quantize maps is provided and applied to the standard map.

1 What is quantum chaos?

Classically, the intuitive idea of chaos is based upon the extreme sensitivity to the initial conditions, and mathematically, that is what we use to define it, quantifying it through the Lyapunov exponents. Quantum chaos began as an attempt to find chaos, in the sense of extreme sensitivity to changes in initial conditions, in quantum mechanical systems.

In quantum mechanics, trajectories cease to make sense, so we must look for another way to measure this instability. The first thing we might think of is replacing the classical trajectory with the quantum wave function and looking for a separation of two nearby wave functions caused by the system's evolution. However, the linearity of the Schrödinger equation implies that the evolution is unitary. If we have two distinct wave functions as initial conditions $|\phi(0)>$ and $|\psi(0)>$, the unitary evolution preserves the inner product in Hilbert space.

$$|<\phi(t)|\psi(t)>|^2 = |<\phi(0)|\psi(0)>|^2 \tag{1.1}$$

However, because of the correspondence principle, chaos is reflected in some ways in quantum systems. We will consider a simple system of a billiard free particle moving in a domain Ω of the plane, with border $\partial\Omega$.
Quantically the hamiltonian will be given by

$$H = -\frac{\hbar}{2m}\Delta \tag{1.2}$$

and the elastic collisions are given by a wave function that is null in the border $\partial\Omega$. If we consider Ω to be a stadium the only constant of movement is energy, the system is ergodic and presents deterministic chaos.

Quantically the energy levels and eigenstates can be calculated

$$H\psi_i = E_i\psi_i \tag{1.3}$$

Consider the energies ordered in increasing order:

$$E_1 \leq E_2 \leq \cdots \leq E_n \leq \dots \tag{1.4}$$

We can associate the density of states:

$$N(E) = \sum_i \theta(E - E_i) \tag{1.5}$$

This is a step function that jumps each time E crosses an eigenvalue.

The average behavior of this curve is obtained by taking the limit $\hbar \to 0$, where the levels get closer and closer:

$$N(E) = \sum_i \theta\left(\frac{2mL^2}{\hbar^2}E - k_i^2\right) \tag{1.6}$$

When $E \to \infty$, we have $N(E) \sim n(E)$, and then:

$$n(E) = \int_{-\infty}^{\infty} d^2p \int_\Lambda \frac{d^2q}{\hbar^2}\theta\left(E - \frac{p^2}{2m}\right) = |\Lambda|\frac{2mE}{\hbar^2} \tag{1.7}$$

This function $n(E)$ will vary from one system to another.

What happens with the fluctuations around this average? They should reveal the quantum nature of the system, that is, the fact that the $N(E)$ curve is discontinuous. If we want to analyze these fluctuations without referring to the average behavior, which is not universal, we need to make a change of variables:

$$\epsilon_i = n(E_i) \tag{1.8}$$

In these new variables, the average density of states is ϵ, meaning the average space between levels is one.

We can then study the statistical behavior of the variables ϵ_i. One of the most studied quantities is the probability distribution of the spaces $s_i = \epsilon_{i+1} - \epsilon_i$ between neighboring levels:

$$p(s)ds = \lim_{N\to\infty} \frac{1}{N} \text{ number of } s_i \in (s, s{+}ds), 1 \le i \le N \tag{1.9}$$

For chaotic systems energy levels follow Wigner-Dyson distribution

$$p(s) \sim se^{-\frac{\pi}{4}s^2} \tag{1.10}$$

This behavior contrasts with that of integrable systems, where the energy level spacings are more regular and follow a Poisson distribution. In chaotic systems, the Wigner-Dyson distribution reflects the irregular and unpredictable nature of the energy levels, analogous to the sensitivity to initial conditions seen in classical chaos.

2 Dynamical systems

A dynamical system is described by a set of states M and an evolution law telling us how to propagate these states in time, whether discrete or continuous. Mathematically speaking a dynamical system is given by a one parameter map

$$\Phi : G \times M \to M \tag{2.1}$$

We are particularlly concerned with measure preserving dynamical systems in order to introduce area preserving maps.
Our first task is to consider the basic notions of measure theory

2.1 Measure preserving dynamical systems

Definition 1 *A measurable space is a pair (M,$\mathcal{M}$) with a set M and a family $\mathcal{M}$ of subsets of M such that*

$M \in \mathcal{M}$
If $A_n \in \mathcal{M}(n \in \mathbf{N})$ then $\bigcup_{n\in\mathbf{N}} A_n \in \mathcal{M}$
If $A \in \mathcal{M}$ then $A^c := M \backslash A \in \mathcal{M}$

$\mathcal{M}$ is called a $\sigma-$algebra on M

Definition 2 *A measure on a measurable space $(M, \mathcal{M})$ is a map $\mu : \mathcal{M} \to [0, \infty)$ with $\mu(\mathcal{M}) \neq (\infty)$ which is countably additive*

$$\mu\bigg(\bigcup_{n\in\mathbf{N}} A_n\bigg) = \sum_{n\in\mathbf{N}} \mu(A_n)$$

If $A_m \cap A_n = \emptyset$ for $m \neq n$ then a measure space $(M, \mathcal{M}, \mu)$ is a measurable space $(M, \mathcal{M})$ with a measure $\mu : \mathcal{M} \to [0, \infty)$

Definition 3 *A map $T : M_1 \to M_2$ between measure spaces $(M_i, \mathcal{M}_i, \mu_i)$ is called measurable if $T^{-1}(A_2) \in \mathcal{M}_1$, $A_2 \in \mathcal{M}_2$ A measurable map $T : M_1 \to M_2$ is called measure preserving if*

$$\mu_1(T^{-1}(A_2)) = \mu_2(A_2)$$

A group action of a group G on a measurable space M is a map

$$\Phi : G \times M \to M \tag{2.2}$$

with $\phi_{Id} = Id_M$ and $\Phi_{g1} \circ \Phi_{g2} = \Phi_{g1\circ g2}$ for $\Phi_g : M \to M$, $\Phi_g(m) := \Phi(g, m)$.

A measure preserving dynamical system is a quadruple $(M, \mathcal{M}, \mu, \Phi)$ consisting of a measure space $(M, \mathcal{M}, \mu)$ and a group action Φ of an abelian group G such that the maps Φ_g are measure preserving for μ

In the case of area preserving maps in 2 dimensions the preserved measure will be that of Lebesque, we will now study the case of the standard map in closer detail.

2.2 Standard map

The standard map is a canonical example of a symplectic map, given by

$$p_{n+1} = p_n - \frac{k}{2\pi} sin(2\pi q_n) \tag{2.3}$$

$$q_{n+1} = q_n + p_{n+1} \tag{2.4}$$

The map conserves the Lebesque measure because it is a hamiltonian system. Symplectic maps that arise from hamiltonian mechanics preserve their symplectic structure, which in turn, implies conservation of the phase-space volume, which in 2D is equivalent to preserving the Lebesque measure.
The jacobian determinant of a map that conserves the phase-space volume will be equal to one.
The jacobian matrix of the standard map is given by

$$J = \begin{bmatrix} \frac{\partial p_{n+1}}{\partial p_n} & \frac{\partial p_{n+1}}{\partial q_n} \\ \frac{\partial q_{n+1}}{\partial p_n} & \frac{\partial q_{n+1}}{\partial q_n} \end{bmatrix} = \begin{bmatrix} 1 & -k\cos(2\pi q_n) \\ 1 & 1 - k\cos(2\pi q_n) \end{bmatrix}.$$

and the jacobian determinant is given by

$$det[J] = 1 - kcos(2\pi q_n) + kcos(2\pi q_n) = 1$$

The jacobian determinant of the standard map has a constant value and it equals 1, this shows that the map preserves the area in the phase-space.
Let's try to construct the standard map from the hamiltonian.
The standard map describes the motion of a kicked

rotator, therefore we should start by considering the general equation for kicked systems

$$H(q,p) = f(p) + \sum_{n=-\infty}^{\infty} \delta(t/\tau - n)V(q) \quad (2.5)$$

where for the kicked rotor, the free kinetic term $f(p) = \frac{p^2}{2}$ and the potential produced by the kick $V(q) = -\frac{g}{4\pi^2} cos(2\pi q)$.
τ is the period of each kick and g is the kick-intensity parameter.
In order to obtain the map from the hamiltonian we have to consider hamilton equations

$$\dot{p} = -\frac{\partial H}{\partial q} \quad (2.6)$$

$$\dot{q} = \frac{\partial H}{\partial p} \quad (2.7)$$

now integrating over a period

$$\int_{n^+}^{(n+1)^+} \frac{dp}{dt} dt = \int_{n^+}^{(n+1)^+} -\frac{\partial H}{\partial q} dt$$

$$\int_{n^+}^{(n+1)^+} \frac{dq}{dt} dt = \int_{n^+}^{(n+1)^+} \frac{\partial H}{\partial p} dt$$

We can split the time into two; the free dynamics in $n^+ < t/\tau < (n+1)^-$ and the part in $(n+1)^- < t/\tau < (n+1)^+$.
Where + means the time is right after the kick and - that the time is right before the kick.
This way we have four equations

$$\int_{n^+}^{(n+1)^-} \frac{dp}{dt} dt = 0$$

$$\int_{(n+1)^-}^{(n+1)^+} \frac{dp}{dt} dt = -\int_{(n+1)^-}^{(n+1)^+} \delta(t/\tau - n)V^{`}(q) dt = -\tau V^{`}(q_{n+1}^+)$$

$$\int_{n^+}^{(n+1)^-} \frac{dq}{dt} dt = \int_{n^+}^{(n+1)^-} \frac{df(p)}{dp} dt = \tau f^{`}(p_n)$$

$$\int_{(n+1)^-}^{(n+1)^+} \frac{dq}{dt} dt = 0$$

Combining the solutions we find the general iterating map for kicked systems

$$q_{n+1} = q_n + \tau \frac{d}{dp} f(p_n) \quad (2.8)$$

$$p_{n+1} = p_n - \tau \frac{d}{dq} V(q_{n+1}) \quad (2.9)$$

and considering that for the kicked rotor $f(p) = \frac{p^2}{2}$ and $V(q) = -\frac{g}{4\pi^2} cos(2\pi q)$ we end up finding the standard map

$$p_{n+1} = p_n - \frac{k}{2\pi} sin(2\pi q_{n+1}) \quad (2.10)$$

$$q_{n+1} = q_n + p_n \quad (2.11)$$

with $k = \tau^2 g$ and $\tau p \rightarrow p$
Finally we may consider a dynamical system such as the kicked pendulum and study how it can be described by the standard map.
Let's consider a kicked pendulum under a gravitational field acting at times nT. The equation of movement will be

$$\ddot{\theta} = -\frac{g}{l} sin(\theta) \sum_{n=-\infty}^{+\infty} \delta(\frac{t}{T} - n) \quad (2.12)$$

if

$$p = ml^2\dot{\theta}$$

is the momentum , then the equations of motion become

$$ml^2\dot{\theta} = p$$

$$\dot{p} = -mglsin(\theta) \sum_{n=-\infty}^{+\infty} \delta(\frac{t}{T} - n)$$

defining then θ_n for the angle at time nT i $p_n^{\pm}$ for the momentum at time $nT \pm 0$ we have:

$$p_n^+ = p_{n+1}^-$$

$$p_n^+ - p_n^- = \int_{nT-0}^{nT+0} \dot{p} dt = -mglTsin(\theta_n)$$

$$ml^2(\theta_{n+1} - \theta_n) = \int_{nT}^{(n+1)T} p dt = Tp_n^+$$

the dynamic can then be reduced to the study of the standard map

$$p_{n+1} = p_n - \frac{k}{2\pi} sin(2\pi q_n) \quad (2.13)$$

$$q_{n+1} = q_n + p_{n+1} \quad (2.14)$$

with

$$p_n = p_n^- \frac{T}{2\pi ml^2}$$

$$q_n = \frac{\theta_n}{2\pi}$$

$$k = \frac{T^2 g}{l}$$

Another dynamical system that evolves according to the standard map is the Frenkel-Kontorova model in solid state physics.
Imagine a chain of atoms, coupled by springs, placed on the surface of a crystal represented by a potential:

$$V(x) = \frac{k}{4\pi^2} \cos(2\pi x) \quad (2.15)$$

The energy of the configuration is then given by:

$$W = \sum_n \frac{1}{2}(q_n - q_{n-1})^2 + \frac{k}{4\pi^2} \cos(2\pi q_n) \quad (2.16)$$

The equilibrium configurations are given by $\frac{\partial W}{\partial q_n}$, satisfying the equation:

$$(q_n - q_{n+1}) + (q_n - q_{n-1}) - \frac{k}{2\pi}\sin(2\pi q_n) = 0 \quad (2.17)$$

These equations are equivalent to those of the standard application if we write $p_n = q_n - q_{n-1}$.

3 Heisenberg group

We should consider the basic notions about quantization on $\mathbf{R^{2n}}$ via representations of the Heisenberg group.
Let's consider $\mathbf{R^{2n+1}}$ with coordinates

$$(p_1, ..., p_n, q_1, ..., q_n, t) := (p, q, t) := (x, t)$$

then we can define the Heisenberg Lie algebra h_n as the vector space $\mathbf{R^{2n+1}}$ with Lie bracket

$$[(x, t), (y, s)] = (0, 0, w(x, y)) \qquad (3.1)$$

where x = (p,q), y = ($p^{'}$, $q^{'}$) and w is the symplectic 2-form

$$w(x, y) = \sum_{l=0}^{n}(p_l q_l^{'} - q_l p_l^{'})$$

Then, the Heisenberg group, $\mathbf{H_n}(\mathbf{R})$, is the simply connected Lie group with lie algebra h_n.
It is important to note that if the standard basis for $\mathbf{R^{2n+1}}$ is $P_1, ...P_n, Q_1, ..., Q_n, T$ then, the Lie algebra structure is given by

$$[P_i, P_k] = [Q_i, Q_k] = [P_j, T] = [Q_J, T] = 0$$

and

$$[P_j, Q_k] = \delta_j^k \cdot T$$

The momentum, position and constant observable in both quantum mechanics and classical mechanics span a Lie algebra isomorphic to $h_n \cdot h_n$, which is a nilpotent Lie algebra and can be identified with a subgroup of $M_{n+2}(\mathbf{R})$ throught the map

$$\Psi(p, q, t) := \begin{bmatrix} 0 & p_1 & p_2 & t \\ 0 & 0 & 0 & q_1 \\ 0 & 0 & 0 & q_2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad (3.2)$$

Since h_n is two-step nilpotent we get the group law

$$e^{\Psi(x,t)} \cdot e^{\Psi(y,s)} = e^{\Psi(x+y, t+s+\frac{1}{2}w(x,y))} \qquad (3.3)$$

If (x,t) $\in \mathbf{R}^{2n+1}$ is identified with the matrix $e^{\Psi(x,t)}$, then the Heisenberg group is realized as $\mathbf{R^{2n+1}}$ with the group law

$$(x, t) \cdot (y, s) = (x + y, t + s + \frac{1}{2}w(x, y)) \qquad (3.4)$$

With this, the exponential map is banally the identity and the inverse element of (x,t). The center of the group is therefore given by

$$\mathbf{Z}_n = \{(0, 0, t) : t \in \mathbf{R}\}$$

Given the Lebesque measure μ the unitary representations of $\mathbf{H}_n(\mathbf{R})$ on $\mathbf{L}^2(\mathbf{R}^n, \mu)$ are defined as follows

$$(T_\hbar(p, q, t)f)(x) := e^{2\pi iht+2\pi iqx+\pi ihpq} \cdot f(x + hp) \qquad (3.5)$$

$\forall h \neq 0$ and $\forall f \in \mathbf{L^2}(\mathbf{R}^n, \mu)$, $\forall(p, q, t) \in h_n$. For any $h \in \mathbf{R}$, $T_\hbar$ is a unitary representation, the Schrödinger representation, the only unitary irreducible representations of $\mathbf{H}_n(\mathbf{R})$ non trivial on the center for $h \neq 0$.
If h = 0, the representation factors through the quotient group

$$\mathbf{H}_n(\mathbf{R})/\mathbf{Z}_n \cong \mathbf{R}^{2n}$$

in other words, the representation is one dimensional, and therefore a homomorphismf of $\mathbf{R}^{2n}$ into the circle. Therefore we have the following theorem

Theorem 1 *If π is an irreducible unitary representation of $\mathbf{H}_n(\mathbf{R})$, then it has to be equivalent to one of the following representations.*

-$T_\hbar$ acting on $\mathbf{L}^2(\mathbf{R}^n, \mu)$, $h \neq 0$
-$\sigma_{ab}(p, q, t) = e^{2\pi i(ap+qb)}$, a,b$\in \mathbf{R}^n$ acting on $\mathbf{C}$

The canonical quantization of any classical observable, in other words, any smooth phase space function $f : \mathbf{R}^{2n} \to \mathbf{C}$ is then obtained in the following way.
Considering an smooth function $f : \mathbf{R}^{2n} \to \mathbf{C}$ that can be written through a Fourier representation in the following way

$$f(p, q) = \int_{\mathbf{R}^{2n}} \tilde{f}(\eta, \xi) \cdot e^{2\pi i(\eta q+\xi p)} d\xi d\eta \qquad (3.6)$$

then the quantized observable is an element $\mathbf{O}p_\hbar(f)$ in the algebra defined as

$$\mathbf{O}p_\hbar(f) = \int \tilde{f}(\eta, \xi) \cdot T_\hbar(\xi, \eta) d\eta d\xi \qquad (3.7)$$

k functions in the Hilbert space.Then, the matrix elements of the quantum observables are related to the functions in phase space, in other words, the classical observables, through the matrix elements of the representation $T_\hbar$

4 Quantization of the standard map

We will consider the quantization of measure preserving maps of the two dimensional torus $\mathbf{T}^2 = \mathbf{R^2} \backslash \mathbf{Z^2}$ We will concentrate on the semiclassical properties of the torus and the mathematical aspects of the quantization process.

4.1 Quantization of the torus

The way we are quantizing the torus is employing the weyl quantization procedure on $\mathbf{R^2}$ through the study of the Heisenberg group.
We consider, as Hilbert space of states the distributions $\psi(q)$ on the real line $\mathbf{R}$, those are periodical in the position and momentum representation, the momentum representation being defined as follows:

$$\mathcal{F}_N\psi(p) = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty}\psi(q)e^{-i\hbar^{-1}qp}dq \qquad (4.1)$$

by imposing

$$\psi(q+1) = \psi(q)$$

$$\mathcal{F}_N\psi(p+1) = \mathcal{F}_N\psi(p)$$

the periodicity in the position representation yields

$$\psi(q) = \sum_{n\in\mathbf{Z}} c_n e^{2\pi inq}$$

and applying $\mathcal{F}_N$ we finally get

$$\mathcal{F}_N\psi(q) = \sqrt{2\pi\hbar}\sum_{n\in\mathbf{Z}} c_n\delta(p-2\pi n\hbar) \qquad (4.2)$$

and the periodicity in both coordinates implies that $2\pi\hbar N = 1$ for all $N \in \mathbf{N}$ and $c_{n+N} = c_n$ Let's consider now a more general approach.
The 2-dimensional torus can be understood as a submanifold of $\mathbf{C^2}$ under the map

$$\pi : \mathbf{R^2} \to \mathbf{T^2} : \pi(q,p) = (\eta,\xi) = (e^{2\pi iq}, e^{2\pi ip}) \qquad (4.3)$$

the action of π on the form $w = dp \times dq$ gives us

$$\Omega = \frac{d\eta \times d\xi}{(2\pi i)^2\xi\eta}$$

The symplectic form Ω induces a Lie structure on the space of the smooth functions defined on the torus

$$\{f(\eta,\xi), g(\eta,\xi)\} \equiv (2\pi i)^2\xi\eta[\partial_\eta f\partial_\xi g - \partial_\eta g\partial_\xi f] = \Omega^{-1}(df,dg) \qquad (4.4)$$

defining

$$\xi^{n_2}\eta^{n_1} \equiv \chi$$

and $n = (n_1,n_2)$, we have

$$\{\chi(m),\chi(n)\} = (2\pi i)^2 w(n,m)\chi(n,m)$$

where $w(n,m) = n_1m_2 - n_2m_1$ If f and g have the following form:

$$f = \sum_{n\in\mathbf{Z^2}} f_n\chi(n)$$

$$g = \sum_{n\in\mathbf{Z^2}} g_n\chi(n)$$

then their poisson bracket is

$$\{f,g\} = \sum_{n,m} f_n g_m (2\pi i)^2 w(n,m)\chi(n,m) \qquad (4.5)$$

In order to canonically quantize the observables defined on the torus, according to the procedure above, the objects we have to look for are the ones obtained through the quotient by $\mathbf{Z^2}$ of the unique representation of the Heisenberg group $\tilde{\mathbf{A}}_\mathbf{h}$ over the plane.

Definition 4 $\forall h \in \mathbf{R}\backslash 0$ *the discrete Heisenberg group* $\mathbf{H_1} := \mathbf{H_1}(\mathbf{Z})$ *is the subgroup topologically equivalent to* $\mathbf{Z^2} \times \mathbf{R}$ *with group law*

$$(n,t)(m,s) = (n+m, t+s+\frac{1}{2}w(n,m))$$

The discrete Heisenberg albegra $\mathbf{A_h}$ *is the subalgebra of* $\tilde{\mathbf{A}}_\mathbf{h}$ *defined as the unitary algebra over* $\mathbf{C}$ *generated by the group.*

$$T_h = \{T(n)\}_{n\in\mathbf{Z^2}}$$

where

$$T(n)^* = T(-n) \qquad (4.6)$$

$$T(n)T(m) = e^{i\pi hw(n,m)}T(n,m) \qquad (4.7)$$

(we are using the abbreviation $T_h = T$*)*

The canonical quantization is obtained upon classification of all irreducible representations of $\mathbf{A_h}$ defined by the relations (4.6),(4.7) into the unitary operators acting in the hilbert spaces $\mathbf{L^2}(\mathbf{S^1};\lambda)$ for some measure λ in S^1 that has to be determined. From (4.7) we obtain the expression for the commutation of the algebra

$$[T(n(,T(m)] = 2isin(\pi hw(n,m))T(n+m) \qquad (4.8)$$

then, to each $f = \sum_{n\in\mathbf{Z^2}} f_n\chi(n)$ we associate the element of the algebra $\mathbf{O}p\hbar(f)$ that we obtain by replacing $\chi(n)$ by $T(n)$ in the fourier expansion.
Now we return to the description of the irreductible representations of the discrete Heisenberg group, to this end, we consider the generators of the algebra:

$$t_1 := T(1,0)$$

$$t_2 := T(0,1)$$

and then we have

$$T(n) = e^{\pi i\frac{n_1n_2}{N}}t_2^{n_2}t_1^{n_1} \qquad (4.9)$$

with $t_2 t_1 = e^{-2\pi i h} t_1 t_2$.
If $h = 1/N$ then t_1^N and t_2^N are the generators of the center and each one has to be mapped into a unitary scalar multiple of the identity by any irreducible representation.
The unique infinite dimensional unitary representation of the standard Heisenberg group $\tilde{A}_h$ splits into a direct integral of finite dimensional, non equivalent, unitary representations.
In order to represent them let's consider yet again the N-dimensional Hilbert space $\mathbf{L^2}(S^1, \mu_n)$ where for all $h = N^{-1}$, $\mu_N(x)$ is the atomic measure on the circle and is defined by

$$\mu_N(x). = \frac{1}{N} \sum_{k_0}^{N-1} \delta(x - \frac{k}{N})$$

The vectors $|k >= \psi_k(x) = \delta^x_{k/N}$ for $k = 0, 1, ...N$ are a basis of the hilbert space with the inner product between ψ, ϕ given by

$$< \psi, \phi >= \int_{S^1} \psi^*(x)\phi(x) d\mu_N(x)$$

and the action of the Fourier transform on the hilbert space is

$$(\mathcal{F}_N \psi)_m := \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} e^{\frac{2i\pi mn}{N}} \psi_n \tag{4.10}$$

now writing

$$\mathbf{Q}_N := \{0, 1/N, 2/N, ..., (N-1)/N\}$$

and

$$\mathbf{Z}_n := \mathbf{Z}/N\mathbf{Z} = \{0, 1, 2, ..., N-1\}$$

we can associate $\mathbf{L^2}(S^1, \mu_n)$ with $\mathbf{L^2}(\mathbf{Z}_N, \mu_n)$ where $\psi \in \mathbf{L^2}(\mathbf{Z}_N, \mu_n)$ is a vector in $\mathbf{C}^N$: $\psi = (\psi_0, ..., \psi_{N-1})$.
Now for any $\theta = (\theta_1, \theta_2) \in \mathbf{T^2}$ we finally define the representations of our algebra on $\mathbf{L^2}(S^1, \mu_n)$:

$$t_1 |l >= e^{\frac{2i\pi(\theta_1 + l)}{N}} |l >, t_1^N = e^{2\pi i \theta_1} \cdot Id \tag{4.11}$$

$$t_2 |l >= e^{\frac{2i\pi\theta_2}{N}} |l + 1 >, t_2^N = e^{2\pi i \theta_2} \cdot Id \tag{4.12}$$

these representations are irreducible, non equivalent for different values of θ and they are the only possible ones.
For any fixed $2\pi\hbar = 1/N$ and $\theta \in \mathbf{T^2}$ we denote by $T_{N,\theta}$ the representation of $T(n)$

$$T_{N,\theta}(n) = e^{\frac{\pi i n_1 n_2}{N}} t_2^{n_2} t_1^{n_1} \tag{4.13}$$

Our main interests now are about the asymptotic properties presented by both the eigenvalues and the eigenfunctions for quantized maps in the semiclassical limit $\hbar \to 0$ (which given the condition we found $2\pi\hbar = 1/N$ we can rewrite as $N \to \infty$), we will therefore often assume periodic boundary conditions for the quantum states.
The generators t_1 and t_2 correspond to the exponential of the usual position and momentum operators

$$\hat{q}\psi(q) := q\psi(q)$$

$$\hat{p}\psi(q) := -i\hbar \frac{d\psi}{dq}(q)$$

The action of $T_N(n)$ on a wave function $\psi \in \mathbf{L^2}(\mathbf{Z}_N)$ is

$$T_N(n)\psi(q) := e^{\frac{i\pi n_1 n_2}{N}} e^{\frac{2\pi i n_2 q}{N}} \psi(q + n_1) \tag{4.14}$$

with $q \in \mathbf{Z}_N$ We can now construct quantum observables for any smooth classical observable $f \in \mathbf{C}^\infty(\mathbf{T^2})$ with Fourier expansion $\chi(n) = e^{2\pi I(n_1 q + n_2 p)}$.
Given

$$f(q, p) = \sum_{n \in \mathbf{Z^2}} f_n \chi(n)$$

we define its quantization

$$\mathbf{O}p_N(f) := \sum_{n \in \mathbf{Z^2}} f_n T_N(n) \tag{4.15}$$

wich ultimately means that there is a well defined map

$$f(q, p) \in \mathbf{C}^\infty(\mathbf{T^2}) \to \mathbf{O}p_N(f) \tag{4.16}$$

which allows to associate a suitable operator on $\mathcal{H}_N$ to any smooth observable on phase space

4.2 Quantization of the standard map

We now focus on the problem of defining a suitable quantum operator corresponding to a given area preserving dynamics, in this particular case, given by the standard map.
We will specify the standard map using a generating function $S(q', q)$ so that, for an area preserving map $P : \mathbf{T^2} \to \mathbf{T^2}$ we have

$$p = -\frac{\partial S(q', q)}{\partial q}$$

$$p^{'} = \frac{\partial S(q', q)}{\partial q'}$$

and for the standard map (2.3),(2.4) the generating function is

$$S(q^{'}, q) = \frac{1}{2}(q - q^{'})^2 + \frac{k}{4\pi^2} cos(2\pi q) \tag{4.17}$$

A quantum map is a sequence of unitary operators U_N, $N \in \mathbf{N}$ on a hilbert space $\mathcal{H}_N$
The quantum map is a quantization of a classical map P on $\mathbf{T^2}$, if the egorov property is fullfilled,

$$lim_{N\to\infty} ||U_N^{-1} \mathbf{O}p(f) U_N - \mathbf{O}p(f \circ P)|| = 0 \tag{4.18}$$

for all $f \in \mathbf{C}^{\infty}(\mathbf{T^2})$, then it means that the semi-classically quantum evolution and classical time evolution commute.
So the goal is to find, for a given map, a corresponding sequence of unitary operators.
We will determine U_N corresponding to a given area preserving map using its generating functions to define

$$(U_N)_{j',j} := < q_{j'}|U_N|q_j >= \frac{1}{\sqrt{N}} \left| \frac{\partial^2 S(\tilde{q}^{'}, \tilde{q})}{\partial \tilde{q}^{'} \partial \tilde{q}} \right|^{1/2}_{\tilde{q}^{'}=q_j^{'}, \tilde{q}=q_j} e^{2\pi i N S(q_j^{'}, q_j)} \tag{4.19}$$

with $q_j = j/N$, $q_j^{'} = j^{'}/N$ and $j, j^{'} = 0, 1, ..., N-1$
For the standard map we finally obtain

$$(U_N)_{j',j} = \frac{1}{\sqrt{N}} e^{\frac{2\pi i}{N}(j^{'2} - j^{'} j + j^2) + iN\frac{k}{2\pi} sin(2\pi j/N)} \tag{4.20}$$

Which is our final result.

5 References

[**1**] Wimberger, S. *Nonlinear Dynamics and Quantum Chaos: An Introduction*. Graduate Texts in Physics, Springer (2014).

[**2**] Degli Esposti, M. and Graffi, S. *The Mathematical Aspects of Quantum Maps*. Lecture Notes in Physics, vol. 618, Springer (2003).

[**3**] Signor, E.M. "Chaology of Quantum Maps Applied to the Standard Map." *Physica A: Statistical Mechanics and its Applications*, vol. 516, pp. 192–202 (2019).