

Class-Based Expurgation Attains Csiszár's Expurgated Source-Channel Exponent

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Abstract—This paper studies expurgated error exponents for joint source-channel coding for discrete memoryless sources and channels. We consider a partition of the source messages into classes, where the codeword distributions depend on the class. We show that two carefully chosen classes suffice to achieve Csiszár's expurgated exponent.

I. INTRODUCTION

We study the transmission of non-equiprobable messages from a discrete memoryless source (DMS) with distribution $P^k(\mathbf{v}) = \prod_{i=1}^k P_V(v_i)$, where $\mathbf{v} = (v_1, \dots, v_k) \in \mathcal{V}^k$ is the source message, and \mathcal{V} is a finite discrete alphabet. The channel is a discrete memoryless channel (DMC) given by $W^n(\mathbf{y}|\mathbf{x}) = \prod_{i=1}^n W(y_i|x_i)$, $\mathbf{x} = (x_1, \dots, x_n) \in \mathcal{X}^n$ and $\mathbf{y} = (y_1, \dots, y_n) \in \mathcal{Y}^n$, where \mathcal{X} and \mathcal{Y} are discrete alphabets with cardinalities $|\mathcal{X}|$ and $|\mathcal{Y}|$, respectively. An encoder maps the length- k source message \mathbf{v} to a length- n codeword \mathbf{x}_v , which is then transmitted over the channel. We refer to $t \triangleq k/n$ as the transmission rate. Based on the channel output \mathbf{y} the decoder guesses which source message was transmitted.

We say that an error exponent $E > 0$ is achievable if there exists a sequence of codes of length n such that the error probability satisfies

$$p_e \leq e^{-nE+o(n)}, \quad (1)$$

where $\lim_{n \rightarrow \infty} o(n)/n = 0$. Most achievability results are obtained using random coding, followed by analyzing the average probability of error. It has been shown that joint source-channel coding (JSCC) can achieve lower error probabilities than separate source and channel coding [1]. In Csiszár's construction, codewords are randomly selected from a set of sequences whose composition depends on the source message composition. Later, Zhong et al. [2] studied the random coding error exponent in JSCC in more detail and proved that Csiszár's exponent corresponds to the concave hull of the E_0 function in the dual domain. A more general message set partitioning method was studied in [3], where it

was shown that two carefully chosen classes are enough to achieve Csiszár's random coding exponent.

Expurgation has received less attention than random coding bounds in JSCC. In [4], Csiszár derived two expurgated exponents, which were later recovered by Scarlett [5] using a new method based on source-type duplication and type-by-type expurgation.

Motivated by the preceding discussion, we study expurgation under a partition of the source messages. We show that, as in the random-coding case, two carefully chosen classes are sufficient to achieve Csiszár's second expurgated exponent [4].

Due to space limitations, technical details and complete proofs are omitted and are provided in the full version [6].

Notation: In this paper, scalar random variables are denoted by uppercase letters, their realizations by lowercase letters, and their alphabets by calligraphic letters. Random vectors are written in boldface. For two positive sequences f_n and g_n , we write $f_n \doteq g_n$ if $\lim_{n \rightarrow \infty} \frac{1}{n} \log \frac{f_n}{g_n} = 0$, and we write $f_n \preceq g_n$ if $\limsup_{n \rightarrow \infty} \frac{1}{n} \log \frac{f_n}{g_n} \leq 0$.

The type of a sequence $\mathbf{x} = (x_1, \dots, x_n) \in \mathcal{X}^n$ is its empirical distribution, defined by $\hat{P}_{\mathbf{x}}(x) \triangleq \frac{1}{n} \sum_{i=1}^n \mathbb{1}\{x_i = x\}$. The set of all probability distributions on an alphabet \mathcal{X} is denoted by $\mathcal{P}(\mathcal{X})$, while $\mathcal{P}_n(\mathcal{X})$ represents the set of empirical distributions for vectors in \mathcal{X}^n . For $P_X \in \mathcal{P}_n(\mathcal{X})$, the type class $\mathcal{T}^n(P_X)$ consists of all sequences in \mathcal{X}^n with type P_X .

The number of source type classes in the set \mathcal{V}^k is denoted by N_k , which grows polynomially with k [7, Lemma 2.2]. Throughout, we use the indices i and j to refer to source types, and c to indicate the class to which a source message belongs under a given partitioning. The total number of classes in the partitioning is denoted by m . Moreover, for a source type P_i , we define $R_i \triangleq tH(P_i)$.

II. CSISZÁR'S EXPURGATED ERROR EXPONENTS

In this section, we introduce definitions and summarize existing expurgated results in JSCC that are relevant to this work. The source reliability function [7, Ch. 9] is given by

$$e(R, P_V) \triangleq \min_{Q: H(Q) \geq R} D(Q \| P_V), \quad (2)$$

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or, in the dual domain [8], as

$$e(R, P_V) = \sup_{\rho \geq 0} \left\{ \rho R - E_s(\rho, P_V) \right\}, \quad (3)$$

where

$$E_s(\rho, P_V) \triangleq \log \left(\sum_{v \in \mathcal{V}} P_V(v)^{\frac{1}{1+\rho}} \right)^{1+\rho}. \quad (4)$$

The weak channel expurgated exponent is given by

$$E'_{\text{ex}}(Q, R) \triangleq \min_{\substack{P_{X\bar{X}}: P_X=Q \\ I_P(X; \bar{X}) \leq R}} \mathbb{E}_P[d_B(X, \bar{X})] + I_P(X; \bar{X}) - R, \quad (5)$$

where $d_B(x, \bar{x}) \triangleq -\log \sum_y \sqrt{W(y|x)W(y|\bar{x})}$ is the Bhattacharyya distance. This expression is referred to as “weak” since it is looser than the Csiszár-Körner-Martón (CKM) expurgated exponent [9], given by

$$E_{\text{ex}}(Q, R) \triangleq \min_{\substack{P_{X\bar{X}} \\ P_X = P_{\bar{X}} = Q \\ I_P(X; \bar{X}) \leq R}} \mathbb{E}_P[d_B(X, \bar{X})] + I_P(X; \bar{X}) - R. \quad (6)$$

The dual expression of (5), as given in [5, Eq. 9], is

$$E'_{\text{ex}}(Q, R) = \sup_{\rho \geq 1} E'_x(Q, \rho) - \rho R, \quad (7)$$

where

$$E'_x(Q, \rho) \triangleq \min_{Q'} -\rho \sum_x Q(x) \log \sum_{\bar{x}} Q'(\bar{x}) e^{-\frac{d_B(x, \bar{x})}{\rho}}. \quad (8)$$

Using the packing lemma [1, Lemma 6], Csiszár [4] derived two achievable expurgated exponents for joint source-channel coding. The first exponent was derived by assigning the same composition to all source messages, in a similar way to Gallager’s [10, Problem 5.16] for the i.i.d. ensemble. The expression is given by

$$E_{J,1}^{\text{ex}}(t, P_V) \triangleq \max_Q \min_R \left[t e \left(\frac{R}{t}, P_V \right) + E_{\text{ex}}(Q, R) \right]. \quad (9)$$

The second exponent requires partitioning of the source sequences and is given by

$$E_{J,2}^{\text{ex}}(t, P_V) \triangleq \min_R \left[t e \left(\frac{R}{t}, P_V \right) + \max_Q E'_{\text{ex}}(Q, R) \right]. \quad (10)$$

It is not known which of the two exponents is higher. Csiszár partitions the source sequences according to their type (i.e., $m = N_k$), assigning a distinct codeword distribution to each one. A more general partitioning was later proposed in [3], where they derived the maximum a posteriori (MAP) random-coding error exponent for arbitrary partitioning of the source sequence set. It was further shown that a specific two-class partitioning suffices to recover Csiszár’s random-coding exponent. The work in [11] considers a less general partitioning, where each class contains one or more full source type classes, and shows that this recovers the MAP

random-coding exponent of [3] using the universal maximum mutual information (MMI) decoder. In this work, we adopt the partitioning of [11], and show that two carefully chosen classes are also sufficient to achieve $E_{J,2}^{\text{ex}}(t, P_V)$.

The work in [5] revisits expurgation in joint source-channel coding and rederives Csiszár’s exponents from [4] using a new expurgation method. The key result from [5] is summarized in the following lemma.

Lemma 1. *There exists a code \mathcal{C} such that for any source type $P_i \in \mathcal{P}_k(\mathcal{V})$ and any sequence $\mathbf{v} \in \mathcal{T}^k(P_i)$, the error probability satisfies*

$$p_e(\mathbf{v}, \mathcal{C}) \leq \sum_{j=1}^{N_k} \mathbb{E} \left[p_e(\mathbf{v}, j, \mathcal{C})^{\frac{1}{\rho_{ij}}} \right]^{\rho_{ij}}, \quad (11)$$

where $\rho_{ij} > 0$, and $p_e(\mathbf{v}, j, \mathcal{C})$ refers to the probability that, given $\mathbf{V} = \mathbf{v}$, there exists some $\bar{\mathbf{v}} \in \mathcal{T}^k(P_j)$ that yields a decoding metric at least as high as that of \mathbf{v} . Moreover, the expectation is taken with respect to the given (not necessarily distinct) codeword distributions $\{P_{\mathbf{X}}^{(j)}\}_{j=1}^{N_k}$.

III. MAIN RESULTS

In this section, we present the main results. We begin with the results in the primal domain, followed by direct derivations of their counterparts in the dual domain; their equivalence follows from Lagrange duality [12]. Direct dual-domain derivations have the advantages of remaining valid for arbitrary alphabets, and error exponents remain achievable for any choice of the optimization parameters.

In this paper, we consider the partitioning of the source sequences into m disjoint sets $\mathcal{A}_1, \dots, \mathcal{A}_m$, each containing one or more full type classes, associated with a codeword distribution $Q_c \in \mathcal{P}_n(\mathcal{X})$, where the size of the type class must satisfy $|\mathcal{T}^n(Q_c)| \geq |\mathcal{A}_c|$ for all $c = 1, \dots, m$. We denote by $\Lambda_c \subset \mathcal{P}_k(\mathcal{V})$ the set of source types included in \mathcal{A}_c . For each source sequence $\mathbf{v} \in \mathcal{A}_c$, the codeword $\mathbf{x}_{\mathbf{v}} \in \mathcal{X}^n$ is drawn independently and uniformly from the type class $\mathcal{T}^n(Q_c)$.

A. Primal-Domain

As a first step, we consider an arbitrary partition of the source sequences and derive an error bound for the code in Lemma 1.

Lemma 2. *Let $\mathcal{A}_1, \dots, \mathcal{A}_m$ be a given arbitrary partition of the source sequences, each associated with a codeword distribution $Q_c \in \mathcal{P}_n(\mathcal{X})$. Then there exists a codebook such that*

$$p_e \leq \sum_{c=1}^m \sum_{i \in \Lambda_c} \exp \left(-n \left[t e \left(\frac{R_i}{t}, P_V \right) + E'_{\text{ex}}(Q_c, R_i) \right] \right). \quad (12)$$

The next theorem considers the partitioning that maximizes the exponent in Lemma 2, for the given codeword distributions.

Theorem 1. For a given set of codeword distributions $\mathcal{Q}_m = \{Q_1, Q_2, \dots, Q_m\}$, there exists a partition of the source sequences into m disjoint sets such that the following exponent is achievable

$$E_{J,\text{ex}}^{\text{primal}}(t, \mathcal{Q}_m) \triangleq \min_R \left[te\left(\frac{R}{t}, P_V\right) + \max_{Q \in \mathcal{Q}_m} E'_{\text{ex}}(Q, R) \right]. \quad (13)$$

Remark 1. If the distribution in \mathcal{Q}_m that maximizes $E'_{\text{ex}}(Q, R)$ is independent of R , then

$$E_{J,\text{ex}}^{\text{primal}}(t, \mathcal{Q}_m) = \max_{Q \in \mathcal{Q}_m} \min_R \left[te\left(\frac{R}{t}, P_V\right) + E'_{\text{ex}}(Q, R) \right]. \quad (14)$$

Note that the exponent in (14) can be achieved without partitioning, by generating all codewords from the type class of the distribution in \mathcal{Q}_m that maximizes $E'_{\text{ex}}(Q, R)$. However, if it were known in advance that all codewords share the same composition, then $E'_{\text{ex}}(Q, R)$ in (14) would be replaced by $E_{\text{ex}}(Q, R)$, resulting in a higher exponent.

B. Dual-Domain

The following lemma presents the analogue of Lemma 2 in the dual domain.

Lemma 3. Consider a given partitioning of the source sequences as described in Lemma 2. Then, there exists a codebook such that for any choice of parameters $\{\lambda_{ic}, \rho_{ic}\}$ satisfying $\lambda_{ic} > 0$ and $\rho_{ic} \geq 1$,

$$p_e \leq \sum_{c=1}^m \sum_{i \in \Lambda_c} \exp \left(-n \left[E'_x(Q_c, \rho_{ic}) + (\lambda_{ic} - \rho_{ic}) R_i - t E_s(\lambda_{ic}, P_V) \right] \right). \quad (15)$$

The next corollary expresses the result of Lemma 3 in the same form as Theorem 1 in [3].

Corollary 1. The expression in (15) can also be expressed as

$$p_e \leq \sum_{c=1}^m \exp \left(-\sup_{\rho_c \geq 1} \left\{ n E'_x(Q_c, \rho_c) - t E_s^{(c)}(\rho_c, P^k) \right\} \right), \quad (16)$$

where

$$E_s^{(c)}(\rho, P^k) \triangleq \log \left(\sum_{\mathbf{v} \in \mathcal{A}_c} P^k(\mathbf{v})^{\frac{1}{1+\rho}} \right)^{1+\rho}. \quad (17)$$

Next theorem considers the partitioning that maximizes the exponent in Lemma 3.

Theorem 2. Consider a given set of codeword distributions $\mathcal{Q}_m = \{Q_1, Q_2, \dots, Q_m\}$, as described in Theorem 1. Then, there exists a partition of the source sequences into m disjoint sets such that the following exponent is achievable

$$E_{J,\text{ex}}^{\text{dual}}(t, \mathcal{Q}_m) \triangleq \sup_{\lambda \geq 1} \left\{ \bar{E}'_x(\mathcal{Q}_m, \lambda) - t E_s(\lambda, P_V) \right\}, \quad (18)$$

where

$$\bar{E}'_x(\mathcal{Q}_m, \lambda) \triangleq \min_R \sup_{\rho \geq 1} E'_x(\mathcal{Q}_m, \rho) + (\lambda - \rho) R, \quad (19)$$

and $E'_x(\mathcal{Q}_m, \rho) \triangleq \max_{Q \in \mathcal{Q}_m} E'_x(Q, \rho)$.

Theorem 2 is the dual counterpart of Theorem 1, and their equivalence follows from Lagrange duality. The following result extends [2, Lemma 3] to the expurgated exponent for a given set of codeword distributions.

Theorem 3. The function $\bar{E}'_x(\mathcal{Q}_m, \rho)$ is the concave hull on the interval $[1, \infty)$ of the function $E'_x(\mathcal{Q}_m, \rho)$.

Remark 2. If $E'_x(\rho, \mathcal{Q}_m)$ is concave in ρ , or equivalently, if the distribution in \mathcal{Q}_m that maximizes $E'_x(\rho, Q)$ is independent of ρ , then $\bar{E}'_x(\rho, \mathcal{Q}_m) = E'_x(\rho, \mathcal{Q}_m)$, and hence

$$E_{J,\text{ex}}^{\text{dual}}(t, \mathcal{Q}_m) = \sup_{\rho \geq 1} \left\{ E'_x(\rho, \mathcal{Q}_m) - t E_s(\rho, P_V) \right\}. \quad (20)$$

The next corollary considers Csiszár's setting, where the partitioning is based on source types, and presents the corresponding dual expression of $E_{J,2}^{\text{ex}}(t, P_V)$ given in (10).

Corollary 2. If the partitioning is based on source type and the associated distributions $\{Q_i\}_{i=1}^{N_k}$ are chosen according to

$$Q_i = \arg \max_{\tilde{Q}} \sup_{\rho \geq 1, \lambda > 0} E'_x(\tilde{Q}, \rho) + (\lambda - \rho) R_i - t E_s(\lambda, P_V), \quad (21)$$

then

$$E_{J,\text{ex}}^{\text{dual}}(t, \{Q_i\}) = \sup_{\lambda \geq 1} \left\{ \bar{E}'_x(\lambda) - t E_s(\lambda, P_V) \right\}, \quad (22)$$

where $\bar{E}'_x(\lambda)$ is the concave hull of $E'_x(\lambda) \triangleq \max_Q E'_x(Q, \lambda)$.

C. Two-Class Partition of the Source Sequences

Now, we assume that we only have two classes, i.e. $m = 2$.

Lemma 4. For a pair of distributions $\mathcal{Q} = \{Q, Q'\}$, the partition that achieves $E_{J,\text{ex}}^{\text{dual}}(t, \mathcal{Q})$ (or $E_{J,\text{ex}}^{\text{primal}}(t, \mathcal{Q})$), is

$$\mathcal{A}_1 = \left\{ \mathbf{v} : |\mathcal{T}^k(\hat{P}_{\mathbf{v}})| \leq e^{nR_0} \right\} \quad (23)$$

$$\mathcal{A}_2 = \left\{ \mathbf{v} : |\mathcal{T}^k(\hat{P}_{\mathbf{v}})| > e^{nR_0} \right\}, \quad (24)$$

for some $R_0 > 0$ that depends on t, P_V, W , and \mathcal{Q} .

The following corollary is the expurgated analogue of [3, Corollary 1]. As in the random-coding case, a direct application of Carathéodory's theorem [13, Cor. 17.1.5] implies that for any λ_0 , the value $\bar{E}'_x(\lambda_0)$ can be expressed as a convex combination of two points on the graph of $E'_x(\lambda)$. Let these points correspond to λ_1 and λ_2 , and let Q_1 and Q_2 be the respective maximisers of $E'_x(Q, \lambda)$ at λ_1 and λ_2 . It follows that both points also lie on the graph of $E'_x(\{Q_1, Q_2\}, \lambda)$, and thus $\bar{E}'_x(\lambda_0)$ also lies on its concave hull. This shows that a

two-class partitioning suffices to achieve $\overline{E}'_x(\lambda_0)$. Optimizing over all such pairs gives the following result.

Corollary 3. *There exists a partition of the source sequences into two classes, such that $E_{J,2}^{\text{ex}}(t, P_V)$ in (10) is achievable.*

An immediate implication of Corollary 3 is that $E_{J,2}^{\text{ex}}(t, P_V)$ can be achieved with only two classes, in contrast to Csiszár's construction [4], which requires $m = N_k$ classes, with N_k growing polynomially in k .

IV. SINGLE CLASS CODING

This section presents the dual-domain counterparts of our results in the absence of partitioning, where all codewords share the same composition. The following lemmas present the counterparts of Theorem 2 and Corollary 2 in this setting

Lemma 5. *Assume that all codewords are drawn independently and uniformly from the type class $\mathcal{T}^n(Q)$, for some given $Q \in \mathcal{P}_n(\mathcal{X})$. Then, there exists a codebook such that*

$$p_e \leq \exp \left(-n \left[\sup_{\rho \geq 1} \left\{ E_x(Q, \rho) - tE_s(\rho, P_V) \right\} \right] \right), \quad (25)$$

where

$$E_x(Q, \rho) \triangleq \sup_{a(\cdot)} -\rho \sum_x Q(x) \log \sum_{\bar{x}} Q(\bar{x}) \left(e^{-d_B(x, \bar{x})} \frac{e^{a(\bar{x})}}{e^{a(x)}} \right)^{\frac{1}{\rho}}. \quad (26)$$

Corollary 4. *If the codeword distribution in Lemma 5 is chosen as*

$$Q = \operatorname{argmax}_{\tilde{Q}} \left\{ \sup_{\rho \geq 1} E_x(\tilde{Q}, \rho) - tE_s(\rho, P_V) \right\}, \quad (27)$$

then there exists a codebook achieving the following exponent

$$E_{G,\text{ex}}^{\text{dual}}(t) \triangleq \sup_{\rho \geq 1} \left\{ E_x(\rho) - tE_s(\rho, P_V) \right\}, \quad (28)$$

where $E_x(\rho) \triangleq \max_Q E_x(Q, \rho)$.

The expression in (28) is the dual counterpart of $E_{J,1}^{\text{ex}}(t, P_V)$ given in (9). This result was previously derived by Gallager in [10, Problem 5.16], where the i.i.d. ensemble was used instead of constant-composition coding.

The result of Corollary 4 cannot be directly compared with Corollary 2, since the concave hull of $E'_x(\lambda)$ can, in general, be larger than $E_x(\lambda)$, even though $E_x(\lambda) \geq E'_x(\lambda)$ always holds. Thus, whether partitioning can generally improve the achievable expurgated exponent remains an open question, as pointed out in [4], [5]. This contrasts with the case of the random-coding exponent, where it is known that partitioning can strictly improve the exponent [2], [3].

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