Supporting Text

A Direct Reciprocity

Here we derive the payoff for one strategy versus another in a mixed large population by averaging over all possible random walks of upstream reciprocity.

A.1 Altruistic Random Walks

A.1.1 Social Dynamics

We first give a precise definition of our model for the social dynamics of upstream reciprocity with direct reciprocity. Although we are interested in an infinite population, the definition of the payoff requires considering a large but finite population. Let V be the set of N players. For any $v \in V$, denote by $S_v(p_v, q_v, r_v)$ the strategy used by v.

The dynamics are defined as follows. Each player initiates an independent random walk. Consider the walk started by $v \in V$. At the first step, the walk dies with probability $(1 - q_v)$. In particular, if $q_v = 0$, player v never initiates a walk. Then a player is chosen uniformly at random in $V - \{v\}$, say w. The walk moves to w where one of three things can happen: 1) the walk dies with probability $(1 - p_w)(1 - r_w)$, 2) it is passed along to a uniformly random player in $V - \{w\}$ with probability $p_w(1 - r_w)$, or 3) it is passed back to v with probability r_w. And so on. Every time the walk reaches a player, it brings a profit b to that player. Every time the walk exits a player (without dying), it costs c to that player.

Let $N_{\text{in}}^{u\to v}$ be the number of times the walk started at u reaches v. Likewise, $N_{\text{out}}^{u\to v}$ is the number of times the walk started at u exits v (without dying). The *payoff* to player v is then

$$f_v = b \sum_{u \in V} \mathbb{E}[N_{\text{in}}^{u\to v}] - c \sum_{u \in V} \mathbb{E}[N_{\text{out}}^{u\to v}],$$

where $\mathbb{E}$ is the expectation under the process described above.

A.1.2 Payoff Difference

We now consider an infinite population made of two types. Type 1 players, of relative abundance x, use strategy $S_1(p_1, q_1, r_1)$. Type 2 players use strategy $S_2(p_2, q_2, r_2)$. By symmetry, the payoff to each player of a given type is the same. Denote $f_1(x)$ the payoff to a player of type 1 and $f_2(x)$ the payoff to a player of type 2. The fitness difference is

$$f(x) := f_1(x) - f_2(x).$$

We show next that f is of the form

$$f = \frac{\alpha\beta}{\gamma},$$

where

$$\begin{aligned}
\alpha &= q_1(1 - p_2)(1 - r_2) - q_2(1 - p_1)(1 - r_1) \\
\beta &= (br_2 - c)(1 - r_1) - x(b - c)(r_2 - r_1) \\
\gamma &= [(1 - p_1)(1 - r_1(p_2(1 - r_2) + r_2))x + (1 - p_2)(1 - r_2(p_1(1 - r_1) + r_1))(1 - x)](1 - r_1)(1 - r_2).
\end{aligned}$$

A.2 Technical Proofs

In this Section, we derive expressions for the payoffs f_1, f_2. The calculations rely on Markov Chain theory (see e.g. [1]).

A.2.1 Aggregate Payoff and Extended State Space

We go back to a finite population. Assume type 1 players, denoted $V_1 \subseteq V$, use strategy $S_1(p_1, q_1, r_1)$ and comprise $N_1 = xN$ players. Likewise, type 2 players, denoted $V_2 \subseteq V$, use strategy $S_2(p_2, q_2, r_2)$ and comprise $N_2 = (1-x)N$ players. By symmetry, the payoff to each player in V_1 is the same, and we have in fact,

$$f_v = f_1(x) = \overline{f}_{12}(x) := \frac{\sum_{w \in V_1} f_w}{N_1},$$

for all $v \in V_1$. Likewise,

$$f_v = f_2(x) = \overline{f}_{21}(x) := \frac{\sum_{w \in V_2} f_w}{N_2},$$

for all $v \in V_2$.

Therefore, rather than computing the payoff to each v in V_i individually, it suffices to compute the total payoff to subpopulation V_i and divide by N_i. We call this the *aggregate payoff*. Note that this argument uses the linearity of expectations. Note also that we can get $\overline{f}_{12}(x)$ from $\overline{f}_{21}(x)$ by permuting the parameters and taking $x \leftrightarrow (1-x)$. So our main task is to compute $\overline{f}_{21}(x)$.

By symmetry, it is enough to consider a walk on two states $\{V_1, V_2\}$. Note that, because of direct reciprocity, each move of any given walk depends on the present state *and* the previous state. To apply the Markov property, we need to expand the state space and consider all pairs of states instead. Denote by

$$U \quad = \quad \{u_1 = (V_1, V_1),\ u_2 = (V_1, V_2),\ u_3 = (V_2, V_1),\ u_4 = (V_2, V_2)\},$$

the extended state space. Here the state (V_i, V_j) indicates a pair of consecutive states where V_i is the previous state and V_j is the present state. Then, in the large N limit, the substochastic transition matrix (excluding the very first step) is given by

$$P = (P_{ij})_{i,j=1}^4 = \begin{pmatrix} r_1 + p_1(1-r_1)x & p_1(1-r_1)(1-x) & 0 & 0 \\ 0 & 0 & r_2 + p_2(1-r_2)x & p_2(1-r_2)(1-x) \\ p_1(1-r_1)x & r_1 + p_1(1-r_1)(1-x) & 0 & 0 \\ 0 & 0 & p_2(1-r_2)x & r_2 + p_2(1-r_2)(1-x) \end{pmatrix},$$

where P_{ij} is the probability to move from u_i to u_j.

Let M_{ij} be N times the aggregate payoff to a type 2 player when the walk starts on V_i and is conditioned to move to V_j at the first step. (In particular, this ignores the q-value.) In other words, M_{ij} is N times the aggregate payoff to a type 2 player that results from a single initiated random walk that starts at V_i and moves to V_j at the first step. Define

$$m_{11} = 0, \quad m_{22} = \frac{b-c}{1-x}, \quad m_{12} = \frac{b}{1-x}, \quad m_{21} = -\frac{c}{1-x}.$$

In words, m_{ij} is N times the change in aggregate payoff to a type 2 player after the walk moves from V_i to V_j. Let

$$\mu_1 = xM_{11} + (1-x)M_{12}, \quad \mu_2 = xM_{21} + (1-x)M_{22}.$$

Then, in the large N limit, by the Markov property, we have

$$\begin{pmatrix} M_{11} \\ M_{12} \\ M_{21} \\ M_{22} \end{pmatrix} = \begin{pmatrix} m_{11} \\ m_{12} \\ m_{21} \\ m_{22} \end{pmatrix} + P \begin{pmatrix} M_{11} \\ M_{12} \\ M_{21} \\ M_{22} \end{pmatrix},$$

or equivalently,

$$\begin{aligned} M_{11} &= m_{11} + r_1 M_{11} + p_1(1-r_1)\mu_1, && (1) \\ M_{12} &= m_{12} + r_2 M_{21} + p_2(1-r_2)\mu_2, && (2) \\ M_{21} &= m_{21} + r_1 M_{12} + p_1(1-r_1)\mu_1, && (3) \\ M_{22} &= m_{22} + r_2 M_{22} + p_2(1-r_2)\mu_2. && (4) \end{aligned}$$

Moreover,

$$\overline{f}_{21}(x) = xq_1\mu_1 + (1-x)q_2\mu_2. \quad (5)$$

A.2.2 Solving the System

To compute $\overline{f}_{21}(x)$ we reduce the system (1) - (4) to the variables $\mu_1' \equiv x\mu_1$ and $\mu_2' \equiv (1-x)\mu_2$ only. We start with the following identity.

Proposition 1 (Identity 1) *The following identity holds*

$$\mathcal{F}_1\mu_1' + \mathcal{F}_2\mu_2' = \mathcal{E}, \quad (6)$$

where $\mathcal{F}_1 = (1-p_1)(1-r_1)$, $\mathcal{F}_2 = (1-p_2)(1-r_2)$, *and* $\mathcal{E} = b - c$.

Proof: By (1) and (4), we have

$$(1-r_1)M_{11} = m_{11} + p_1(1-r_1)\mu_1, \quad (1-r_2)M_{22} = m_{22} + p_2(1-r_2)\mu_2.$$

By adding up (2) and (3), we also have

$$(1-r_1)M_{12} + (1-r_2)M_{21} = m_{12} + m_{21} + p_1(1-r_1)\mu_1 + p_2(1-r_2)\mu_2.$$

Plugging the previous equations into the definitions of μ_1 and μ_2 leads to

$$\begin{aligned} &x(1-r_1)\mu_1 + (1-x)(1-r_2)\mu_2 \\ &\quad = x^2(1-r_1)M_{11} + x(1-x)(1-r_1)M_{12} + x(1-x)(1-r_2)M_{21} + (1-x)^2(1-r_2)M_{22} \\ &\quad = [xb - xc + (1-x)(b-c)] + xp_1(1-r_1)\mu_1 + (1-x)p_2(1-r_2)\mu_2. \end{aligned}$$

It is easy to check that the term in square brackets is $b-c$. Therefore,

$$\begin{aligned} b - c &= [x(1-r_1)\mu_1 + (1-x)(1-r_2)\mu_2] - [xp_1(1-r_1)\mu_1 + (1-x)p_2(1-r_2)\mu_2] \\ &= x(1-p_1)(1-r_1)\mu_1 + (1-x)(1-p_2)(1-r_2)\mu_2. \end{aligned}$$

This concludes the proof.

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A second equation is given in the next proposition.

Proposition 2 (Identity 2) *The following identity holds*

$$\mathcal{D}_1\mu_1' - \mathcal{D}_2\mu_2' = \mathcal{C}, \tag{7}$$

where

$$\mathcal{D}_1 = (1-r_2)\left\{(1-r_1)(1+r_1) + p_1(1-r_1)[(1-x)(1-r_1) - x(1+r_1)]\right\},$$

$$\mathcal{D}_2 = (1-r_1)\left\{(1-r_2)(1+r_2) + p_2(1-r_2)[x(1-r_2) - (1-x)(1+r_2)]\right\},$$

and

$$\mathcal{C} = x(1-r_1)(1-r_2)(b+c) - (1-x)(1-r_1)(1+r_2)(b-c).$$

Proof: Let

$$\mathcal{A} = \left(\frac{x}{1-x}\right)\frac{1+r_1}{1-r_1}, \qquad \mathcal{B} = \left(\frac{1-x}{x}\right)\frac{1+r_2}{1-r_2}.$$

Adding up equations (2) and $\mathcal{A}$ times (1) and then subtracting equations (3) and $\mathcal{B}$ times (4), one gets

$$\begin{aligned}\left[\frac{1+r_1}{1-x} + p_1(1-r_1)(1-\mathcal{A})\right]\mu_1 - \left[\frac{1+r_2}{x} + p_2(1-r_2)(1-\mathcal{B})\right]\mu_2 \\ = m_{12} - m_{21} + \mathcal{A}m_{11} - \mathcal{B}m_{22}.\end{aligned}$$

Note that

$$\begin{aligned}x(1-x)(1-r_1)(1-r_2)&\left[m_{12} - m_{21} + \mathcal{A}m_{11} - \mathcal{B}m_{22}\right] \\ &= (1-r_1)(1-r_2)x(1-x)[m_{12} - m_{21}] + (1-r_2)(1+r_1)x^2 m_{11} \\ &\quad -(1-r_1)(1+r_2)(1-x)^2 m_{22} \\ &= \mathcal{C}.\end{aligned}$$

Also, it follows immediately that

$$x(1-x)(1-r_1)(1-r_2)\left[\frac{1+r_1}{1-x} + p_1(1-r_1)(1-\mathcal{A})\right] = x\mathcal{D}_1,$$

and

$$x(1-x)(1-r_1)(1-r_2)\left[\frac{1+r_2}{x} + p_2(1-r_2)(1-\mathcal{B})\right] = (1-x)\mathcal{D}_2.$$

Replacing above yields the result.

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Putting Propositions 1 and 2 together, we get the following.

Proposition 3 (Solution) *We have*

$$\mu_1' = \frac{\mathcal{E}\mathcal{D}_2 + \mathcal{C}\mathcal{F}_2}{\mathcal{D}_1\mathcal{F}_2 + \mathcal{F}_1\mathcal{D}_2}, \qquad \mu_2' = \frac{\mathcal{E}\mathcal{D}_1 - \mathcal{C}\mathcal{F}_1}{\mathcal{D}_1\mathcal{F}_2 + \mathcal{F}_1\mathcal{D}_2}.$$

Moreover, let

$$\widetilde{\mathcal{C}} = (1-x)(1-r_1)(1-r_2)(b+c) - x(1-r_2)(1+r_1)(b-c).$$

Then

$$\overline{f}_{21}(x) = \frac{\mathcal{E}(q_1\mathcal{D}_2 + q_2\mathcal{D}_1) + \mathcal{C}(q_1\mathcal{F}_2 - q_2\mathcal{F}_1)}{\mathcal{D}_1\mathcal{F}_2 + \mathcal{F}_1\mathcal{D}_2}, \qquad \overline{f}_{12}(x) = \frac{\mathcal{E}(q_1\mathcal{D}_2 + q_2\mathcal{D}_1) - \widetilde{\mathcal{C}}(q_1\mathcal{F}_2 - q_2\mathcal{F}_1)}{\mathcal{D}_1\mathcal{F}_2 + \mathcal{F}_1\mathcal{D}_2}.$$

Proof: The expressions for μ_1' and μ_2' follow immediately from Propositions 1 and 2. Note that the operation

$$(p_1, q_1, r_1) \leftrightarrow (p_2, q_2, r_2) \qquad x \leftrightarrow (1-x),$$

acts as follows

$$\mathcal{D}_1 \leftrightarrow \mathcal{D}_2 \qquad \mathcal{F}_1 \leftrightarrow \mathcal{F}_2 \qquad \mathcal{E} \leftrightarrow \mathcal{E} \qquad \mathcal{C} \leftrightarrow \widetilde{\mathcal{C}}.$$

This gives $\overline{f}_{12}(x)$ from $\overline{f}_{21}(x)$.

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A.2.3 Fitness Difference

Using Proposition 3, we derive an analytic expression for the payoff difference f.

Proposition 4 (Payoff Difference) *In the large N limit, the payoff difference is given by*

$$f(x) = \frac{\alpha\beta}{\gamma},$$

where α, β, γ are defined in Section A.1.2.

Proof: From Proposition 3, we get

$$f(x) = f_1(x) - f_2(x) = \overline{f}_{12}(x) - \overline{f}_{21}(x) = \frac{(-\widetilde{\mathcal{C}} - \mathcal{C})(q_1\mathcal{F}_2 - q_2\mathcal{F}_1)}{\mathcal{D}_1\mathcal{F}_2 + \mathcal{F}_1\mathcal{D}_2}.$$

The denominator is a linear function of x. By the type 1 vs. type 2 symmetry, it suffices to compute the value at $x = 0$ where it is

$$\begin{aligned}
&(1-p_2)(1-r_2)^2(1-r_1)[1+r_1+p_1(1-r_1)] + (1-p_1)(1-r_1)^2(1+r_2)[1-r_2-p_2(1-r_2)] \\
&\qquad = (1-p_2)(1-r_2)(1-r_1)[(1+[p_1(1-r_1)+r_1])(1-r_2) + (1-p_1)(1-r_1)(1+r_2)] \\
&\qquad = 2(1-p_2)(1-r_2)(1-r_1)(1-r_2[p_1(1-r_1)+r_1]).
\end{aligned}$$

Therefore, for a general x, we have

$$\begin{aligned}
\mathcal{D}_1\mathcal{F}_2 + \mathcal{D}_2\mathcal{F}_1 \quad = \quad & 2(1-r_2)(1-p_1)(1-r_1)(1-r_1[p_2(1-r_2)+r_2])x \\
& +2(1-r_1)(1-p_2)(1-r_2)(1-r_2[p_1(1-r_1)+r_1])(1-x).
\end{aligned}$$

Finally note that,

$$\begin{aligned}
-\widetilde{\mathcal{C}} - \mathcal{C} \quad &= \quad -(b+c)(1-r_1)(1-r_2) + (b-c)[x(1+r_1)(1-r_2) + (1-x)(1+r_2)(1-r_1)] \\
&= \quad 2(br_2 - c)(1-r_1) - 2x(b-c)(r_2-r_1)
\end{aligned}$$

This concludes the proof.

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A.2.4 Adaptive Dynamics

We derive the dynamical system for the adaptive dynamics which we obtain by differentiating the following function

$$W(x) = \overline{f}_{21}(x) - (b-c)s_1,$$

where the last term is the payoff in a population made only of type 1 players. Note that at $x = 1$, we have

$$\begin{aligned} \mathcal{E} &= b - c, \\ \mathcal{C} &= (b+c)(1-r_1)(1-r_2), \end{aligned}$$

and

$$\begin{aligned} &q_1\mathcal{D}_2 + q_2\mathcal{D}_1 = q_2(1-r_2)(1+r_1)(1-p_1)(1-r_1) + q_1(1-r_1)(1-r_2)(1+[p_2(1-r_2)+r_2]), \\ &q_1\mathcal{F}_2 - q_2\mathcal{F}_1 = q_1(1-p_2)(1-r_2) - q_2(1-p_1)(1-r_1), \\ &\mathcal{D}_1\mathcal{F}_2 + \mathcal{D}_2\mathcal{F}_1 = 2(1-r_2)(1-p_1)(1-r_1)(1-r_1[p_2(1-r_2)+r_2]), \end{aligned}$$

where the expression on the last line follows from the proof of Proposition 4. A straightforward calculation gives

$$\overline{f}_{21}(1) \quad = \quad (b - [p_2(1-r_2)+r_2]c)\frac{s_1(1-r_1)}{1-r_1[p_2(1-r_2)+r_2]} + (br_1 - c)\frac{q_2}{1-r_1[p_2(1-r_2)+r_2]}.$$

Therefore, setting $(p_1, q_1, r_1) = (p, q, r)$, it follows easily that

$$\left.\frac{\partial W(1)}{\partial p_2}\right|_{(p_2,q_2,r_2)=(p,q,r)} = \frac{q(br-c)}{(1-p)(1-r[p(1-r)+r])},$$

$$\left.\frac{\partial W(1)}{\partial q_2}\right|_{(p_2,q_2,r_2)=(p,q,r)} = \frac{br-c}{1-r[p(1-r)+r]},$$

and

$$\left.\frac{\partial W(1)}{\partial r_2}\right|_{(p_2,q_2,r_2)=(p,q,r)} = \frac{q(br-c)}{(1-r)(1-r[p(1-r)+r])}.$$

Up to a speed factor, this system is identical to the one derived in the main text.

B Spatial Reciprocity

Here we derive the payoff functions for two adjacent infinite clusters on the line by summing up all possible random walks of upstream reciprocity.

B.1 Altruistic Random Walks

B.1.1 Social dynamics

We first recall our model for the social dynamics of upstream reciprocity. Imagine an infinite set of players arranged on a line. Each player is assigned a pair of probabilities, (p_v, q_v), $\forall v \in V$, where $0 \leq p_v, q_v \leq 1$ have the following interpretation:

- p_v : probability to pass along a wave of generosity to a uniformly chosen neighbor, i.e. with probability $p_v/2$ the walk moves to the left and with probability $p_v/2$ the walk moves to the right (see below);

- q_v : probability to initiate a wave of generosity.

More precisely, the dynamics are defined as follows. Each player initiates a random walk. Walks initiated at different vertices are independent from each other. Consider the walk started at v. At the first step, the walk dies with probability $(1 - q_v)$. In particular, if $q_v = 0$, player v never initiates a walk. Then a uniform player is chosen among all neighbours of v, say w, i.e. with probability $1/2$, w is the left neighbour of v and with probability $1/2$, w is the right neighbour of v. The walk moves to w where it then dies with probability $1 - p_w$ or is passed along to a uniform neighbour of w with probability p_w, and so on. Every time the walk enters a player, it brings a profit of b to this player. Every time the walk exits a player (without dying), it costs c to the player. Let $N_{\text{in}}^{u\to v}$ be the number of times the walk started at u enters v. Likewise, $N_{\text{out}}^{u\to v}$ is the number of times the walk started at u exits v. The payoff to player v is then

$$f_v = b\sum_{u\in V}\mathbb{E}[N_{\text{in}}^{u\to v}] - c\sum_{u\in V}\mathbb{E}[N_{\text{out}}^{u\to v}].$$

B.1.2 Payoff

Let $R^{u\to v}$ be the probability that the walk started at u reaches v after time 0. In particular, $R^{v\to v}$ is the probability of return to v. The following proposition gives a characterization of the payoff in terms of return probabilities. For this, it is useful to consider the model where $\pi_u = (p_u, q_u)$ is replaced with $\widetilde{\pi}_u = (p_u, p_u)$ for all $u \in V$. We denote with a $\widetilde{\ }$ the corresponding quantities.

Proposition 5 *Let*

$$K_v = \frac{S_v}{1 - \widetilde{R}^{v\to v}},$$

where

$$S_v = \sum_{u\in V} R^{u\to v}.$$

Then, the payoff to v is

$$f_v = b(K_v) - c(q_v + p_v K_v). \tag{8}$$

Proof: By the strong Markov property,

$$\mathbb{E}[N_{\text{in}}^{u\to v}] = R^{u\to v}[1 + \mathbb{E}[\widetilde{N}_{\text{in}}^{v\to v}]],$$

where

$$\mathbb{E}[\widetilde{N}_{\text{in}}^{v\to v}] = \widetilde{R}^{v\to v}[1 + \mathbb{E}[\widetilde{N}_{\text{in}}^{v\to v}]],$$

which solves for

$$\mathbb{E}[\widetilde{N}_{\text{in}}^{v\to v}] = \frac{\widetilde{R}^{v\to v}}{1 - \widetilde{R}^{v\to v}}.$$

So,

$$\sum_{u\in V}\mathbb{E}[N_{\text{in}}^{u\to v}] = \left(\frac{1}{1 - \widetilde{R}^{v\to v}}\right)\sum_{u\in V} R^{u\to v}.$$

Likewise, for $u \neq v$,

$$\mathbb{E}[N_{\text{out}}^{u\to v}] = R^{u\to v}\mathbb{E}[\widetilde{N}_{\text{out}}^{v\to v}],$$

and, for $u = v$,

$$\mathbb{E}[N_{\text{out}}^{v\to v}] = q_v + R^{v\to v}\mathbb{E}[\widetilde{N}_{\text{out}}^{v\to v}],$$

where

$$\mathbb{E}[\widetilde{N}_{\text{out}}^{v\to v}] = p_v + \widetilde{R}^{v\to v}\mathbb{E}[\widetilde{N}_{\text{out}}^{v\to v}],$$

which solves for

$$\mathbb{E}[\widetilde{N}_{\text{in}}^{v\to v}] = \frac{p_v}{1-\widetilde{R}^{v\to v}}.$$

So,

$$\sum_{u\in V}\mathbb{E}[N_{\text{out}}^{u\to v}] = q_v + p_v\left(\frac{1}{1-\widetilde{R}^{v\to v}}\right)\sum_{u\in V}R^{u\to v}.$$

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B.1.3 Evolutionary Dynamics.

Assume that there are two classes of players C_0, C_1. Let $0 \le p_0, p_1, q_0, q_1 \le 1$. Players $v \in \{0, -1, -2, \ldots\}$ have $\pi_v = \pi_0 \equiv (p_0, q_0)$ and players $v \in \{1, 2, \ldots\}$ have $\pi_v = \pi_1 \equiv (p_1, q_1)$. We consider the following evolutionary dynamics. At each time step, one of the two players at the $C_0 - C_1$ boundary is chosen at random and replaced with one of its neighbours according to their fitness in the limit of weak selection. It is clear that, as time goes by, the configuration remains the same up to a translation and we only need to determine which side is more likely to invade the other side in the initial state. In the weak selection case, that is when the fitness is of the form $1 + \rho f_i$ for some small $\rho > 0$, it is easy to show that, at the first order in ρ, C_1 is more likely to invade C_0 in the imitation process if and only if

$$f_{-1} + 3f_0 < 3f_1 + f_2,$$

where the condition refers to the initial setup.

Let

$$f(p) = \frac{8 + 2p + 8\sqrt{1-p^2}}{3 + 4p + \sqrt{1-p^2}},$$

and

$$g(p) = \frac{p}{1+2p}\frac{3 + 3p + \sqrt{1-p^2}}{1 + p - \sqrt{1-p^2}}.$$

We show the following:

- **Generous Cooperators vs. Classical Defectors.** This is the case $(p_0, q_0) = (0, 0)$ and $(p_1, q_1) = (p, 1)$ for $p > 0$. Generous cooperators are favored if

$$\frac{b}{c} > f(p).$$

- **Generous Cooperators vs. Classical Cooperators.** This is the case $(p_0, q_0) = (0, 1)$ and $(p_1, q_1) = (p, 1)$ for $p > 0$. Generous cooperators are favored if

$$\frac{b}{c} > f(p).$$

  Note that this turns out to be the same criterion as in the first case.

- **Generous Cooperators vs. Passers-On.** This is the case $(p_0, q_0) = (p, 0)$ and $(p_1, q_1) = (p, 1)$ for $p > 0$. Generous cooperators are favored if

$$\frac{b}{c} > g(p).$$

B.2 Technical Proofs

By symmetry, $3f_1 + f_2$ is the same as $f_{-1} + 3f_0$ with (p_0, q_0) and (p_1, q_1) interchanged. Therefore, we only need to compute f_1 and f_2. By Proposition 5, it suffices to compute $R^{u\to 1}$, $R^{u\to 2}$, $\widetilde{R}^{1\to 1}$ and $\widetilde{R}^{2\to 2}$ for all $u \in V$. The following quantities will be useful to compute return probabilities. Let A_1 (resp. A_0) be the probability that the walk started at 1 (resp. 0) reaches 0 (resp. 1) before dying, with the corresponding $\sim$ quantity $\widetilde{A}_1$ (resp. $\widetilde{A}_0$). Also, let B_1 (resp. B_0) be the probability that the walk started at 1 (resp. 0) reaches 2 (resp. -1) before dying, with the corresponding $\sim$ quantity $\widetilde{B}_1$ (resp. $\widetilde{B}_0$). Let

$$H_1 = 1 + \sqrt{1 - p_1^2},$$

with a similar expression for H_0.

B.2.1 General Results

We have the following.

Proposition 6 *Under the above setup, we have*

$$\widetilde{A}_1 = \frac{p_1}{H_1} \qquad A_1 = \frac{q_1}{H_1} \qquad \widetilde{B}_1 = \frac{H_0 p_1}{2H_0 + p_0 p_1} \qquad B_1 = \frac{H_0 q_1}{2H_0 + p_0 p_1}.$$

Similar expressions hold for $\widetilde{A}_0$, A_0, $\widetilde{B}_0$, and B_0.

Proof: By the strong Markov property, it follows that

$$\widetilde{A}_1 = \frac{p_1}{2} + \frac{p_1}{2}\widetilde{A}_1^2,$$

which solves for

$$\widetilde{A}_1 = \frac{1 - \sqrt{1 - p_1^2}}{p_1} = \frac{p_1}{1 + \sqrt{1 - p_1^2}},$$

(the right expression applies to $p_1 = 0$ as well). Therefore,

$$A_1 = \frac{q_1}{p_1}\widetilde{A}_1 = \frac{q_1}{1 + \sqrt{1 - p_1^2}},$$

(the right expression applies to $p_1 = 0$ as well). Similar expressions hold for A_0 and $\widetilde{A}_0$. By the strong Markov property again, we have

$$\widetilde{B}_1 = \frac{p_1}{2} + \frac{p_1}{2}\widetilde{A}_0\widetilde{B}_1,$$

or

$$\widetilde{B}_1 = \frac{p_1}{2 - p_1\widetilde{A}_0} = \frac{(1 + \sqrt{1 - p_0^2})p_1}{2(1 + \sqrt{1 - p_0^2}) + p_0 p_1}.$$

Also,

$$B_1 = \frac{q_1}{p_1}\widetilde{B}_1 = \frac{q_1}{2 - p_1\widetilde{A}_0} = \frac{(1+\sqrt{1-p_0^2})q_1}{2(1+\sqrt{1-p_0^2}) + p_0p_1},$$

(the right expression applies to $p_1 = 0$ as well). Similar expressions hold for B_0 and $\widetilde{B}_0$.

■

We now tackle the return probabilities. Let

$$F_0 = \frac{A_0}{1-\widetilde{A}_0} = \frac{q_0}{H_0 - p_0},$$

if $q_0 > 0$ (including the case $p_0 = 1$), and $F_0 = 0$ otherwise. Define F_1 similarly. Also, let

$$G_0 = \frac{A_0\widetilde{B}_1}{1-\widetilde{A}_0} = \frac{q_0p_1H_0}{(H_0-p_0)(2H_0+p_0p_1)},$$

if $q_0 > 0$ and $p_1 > 0$ (including the case $p_0 = 1$), and $G_0 = 0$ otherwise. Define G_1 similarly.

Proposition 7 *Under the setup above, we have*

$$\widetilde{R}^{1\to1} = \frac{p_1}{2}(\widetilde{A}_0 + \widetilde{A}_1) \qquad \widetilde{R}^{2\to2} = \frac{p_1}{2}(\widetilde{A}_1 + \widetilde{B}_1)$$

$$S_1 = \frac{q_1}{2}(\widetilde{A}_0 + \widetilde{A}_1) + F_0 + F_1 \qquad S_2 = B_1 + \frac{q_1}{2}(\widetilde{B}_1 + \widetilde{A}_1) + G_0 + F_1.$$

Similar expressions hold for $\widetilde{R}^{0\to0}$, $\widetilde{R}^{-1\to-1}$, $\sum_{u\in V} R^{u\to0}$, *and* $\sum_{u\in V} R^{u\to-1}$.

Proof: Note first that by the Markov property,

$$\widetilde{R}^{1\to1} = \frac{p_1}{2}\widetilde{A}_0 + \frac{p_1}{2}\widetilde{A}_1 \qquad \widetilde{R}^{2\to2} = \frac{p_1}{2}\widetilde{A}_1 + \frac{p_1}{2}\widetilde{B}_1.$$

We have

$$R^{u\to1} = \begin{cases} A_1(\widetilde{A}_1)^{u-2} & \text{if } u > 1 \\ \frac{q_1}{2}(\widetilde{A}_0 + \widetilde{A}_1) & \text{if } u = 1 \\ A_0(\widetilde{A}_0)^{-u} & \text{if } u < 1 \end{cases}$$

Note that

$$\sum_{u>1}(\widetilde{A}_1)^{u-2} = \frac{1}{1-\widetilde{A}_1} \qquad \sum_{u<1}(\widetilde{A}_0)^{-u} = \frac{1}{1-\widetilde{A}_0}.$$

Therefore,

$$\sum_{u\in\mathbb{Z}} R^{u\to1} = \frac{q_1}{2}(\widetilde{A}_0 + \widetilde{A}_1) + F_0 + F_1.$$

Similarly, we have

$$R^{u\to2} = \begin{cases} A_1(\widetilde{A}_1)^{u-3} & \text{if } u > 2 \\ \frac{q_1}{2}(\widetilde{B}_1 + \widetilde{A}_1) & \text{if } u = 2 \\ B_1 & \text{if } u = 1 \\ A_0(\widetilde{A}_0)^{-u}\widetilde{B}_1 & \text{if } u < 1 \end{cases}$$

Therefore,

$$\sum_{u\in\mathbb{Z}} R^{u\to2} = B_1 + \frac{q_1}{2}(\widetilde{B}_1 + \widetilde{A}_1) + G_0 + F_1.$$

■

B.2.2 Special Cases

The evolutionary dynamics results mentioned above can be obtained by plugging the values of p_0, q_0, p_1, q_1 in the formulas above. The calculations are straightforward, and we omit the details.

References

[1] Feller, William, An introduction to probability theory and its applications, John Wiley, 1961.