Good Strategies for the Iterated Prisoner's Dilemma

Ethan Akin

Mathematics Department

The City College

137 Street and Convent Avenue

New York City, NY 10031, USA

October, 2012 - July, 2013

Abstract

For the iterated Prisoner's Dilemma, there exist Markov strategies which solve the problem when we restrict attention to the long term average payoff. When used by both players these assure the cooperative payoff for each of them. Neither player can benefit by moving unilaterally any other strategy, i.e. these are Nash equilibria. In addition, if a player uses instead an alternative which decreases the opponent's payoff below the cooperative level, then his own payoff is decreased as well. Thus, if we limit attention to the long term payoff, these *good strategies* effectively stabilize cooperative behavior. We characterize these good strategies and analyze their role in evolutionary dynamics.

Keywords: Prisoner's Dilemma, Stable Cooperative Behavior, Iterated Play, Markov strategies, Zero-Determinant Strategies, Press-Dyson Equations, Evolutionary Game Dynamics.

2010 Mathematics Subject Classification: 91A05, 91A20, 91A22, 60J20.

1 The Iterated Prisoner's Dilemma

The *Prisoner's Dilemma* is a two person game that provides a simple model of a disturbing social phenomenon. It is a symmetric game in which each of the two players, X and Y, has a choice between two strategies, c and d. Thus, there are four outcomes which we list in the order: cc, cd, dc, dd, where, for example, cd is the outcome when X plays c and Y plays d. Each then receives a payoff. The following 2×2 chart describes the payoff to the X player. The transpose is the Y payoff.

$X \backslash Y$	c	d
c	R	S
d	T	P

(1.1)

Alternatively, we can define the *payoff vectors* for each player by

$$\mathbf{S}_X \quad = \quad (R, S, T, P) \qquad \text{and} \qquad \mathbf{S}_Y \quad = \quad (R, T, S, P). \tag{1.2}$$

Davis [6] and Straffin [17] provide clear introductory discussions of the elements of game theory.

Either player can use a *mixed strategy*, randomizing by choosing c with probability p_c and d with the complementary probability $1 - p_c$.

A probability *distribution* $\mathbf{v}$ on the set of outcomes is a non-negative vector with unit sum, indexed by the four states. That is, $v_i \geq 0$ for $i = 1, ..., 4$ and the dot product $< \mathbf{v} \cdot \mathbf{1} > = 1$. For example, v_2 is the probability that X played c and Y played d. In particular, $v_1 + v_2$ is the probability X played c. With respect to $\mathbf{v}$ the expected payoffs to X and Y, denoted s_X and s_Y, are the dot products with the corresponding payoff vectors:

$$s_X \quad = \quad < \mathbf{v} \cdot \mathbf{S}_X > \qquad \text{and} \qquad s_Y \quad = \quad < \mathbf{v} \cdot \mathbf{S}_Y > . \tag{1.3}$$

The payoffs are assumed to satisfy

$$T \;>\; R \;>\; P \;>\; S \qquad \text{and} \qquad 2R \;>\; T + S. \tag{1.4}$$

We will later use the following easy consequence of these inequalities.

Proposition 1.1 *If* $\mathbf{v}$ *is a distribution, then the associated expected payoffs to the two players, as defined by (1.3), satisfy the following equation.*

$$s_Y \; - \; s_X \quad = \quad (v_2 - v_3)(T - S). \tag{1.5}$$

So we have $s_Y \; = \; s_X$ *iff* $v_2 \; = \; v_3$.

In addition,

$$\frac{1}{2}(s_Y + s_X) \quad \leq \quad R, \tag{1.6}$$

with equality iff $\mathbf{v} \; = \; (1,0,0,0)$. *Hence, the following statements are equivalent.*

(i) $\frac{1}{2}(s_Y + s_X) \; = \; R.$

(ii) $v_1 \; = \; 1.$

(iii) $s_Y \; = \; s_X \; = \; R.$

Proof: Dot $\mathbf{v}$ with $\mathbf{S}_Y - \mathbf{S}_X = (0,\ T-S,\ S-T,\ 0)$ and with $\frac{1}{2}(\mathbf{S}_Y + \mathbf{S}_X) = (R,\ \frac{1}{2}(T+S),\ \frac{1}{2}(T+S),\ P)$. Observe that R is the maximum entry of the latter. $\Box$

In the Prisoner's Dilemma, the strategy c is *cooperation*. When both players cooperate they each receive the reward for cooperation ($= R$). The strategy d is *defection*. When both players defect they each receive the punishment for defection ($= P$). However, if one player cooperates and the other does not, then the defector receives the large temptation payoff ($= T$), while the hapless cooperator receives the very small sucker's payoff ($= S$). The condition $2R > T + S$ says that the reward for cooperation is larger than the players would receive by dividing equally the total payoff of a *cd* or *dc* outcome. Thus, the maximum total payoff occurs uniquely at *cc* and that location is a *strict Pareto optimum*, which means that at every other outcome at least one player does worse. The cooperative outcome *cc* is clearly where the players "should" end up. If they could negotiate a binding agreement in advance of play, they would agree to play c and each receive R. However, the structure of the game is such that, at the time of play, each chooses a strategy in ignorance of the other's choice.

This is where it gets ugly. In game theory lingo, the strategy d *strictly dominates* strategy c. This means that, whatever Y's choice is, X receives a larger payoff by playing d than by using c. In the array (1.1) each number in the d row is larger than the corresponding number in the c row above it. Hence, X chooses d, and for exactly the same reason, Y chooses d. So they are driven to the dd outcome with payoff P for each. Having firmly agreed to cooperate, X hopes that Y will stick to the agreement because X can then obtain the large payoff T by defecting. Furthermore, if he were not to play d, then he risks getting S when Y defects. All the more reason to defect, as X realizes Y is thinking the same thing.

The payoffs are often stated in money amounts or in years reduced from a prison sentence (the original "prisoner" version), but it is important to understand that the payoffs are really in units of *utility*. That is, the ordering in (1.4) is assumed to describe the order of desirability of the various outcomes to each player when all the consequences of each outcome are taken into account. Thus, if X is induced to feel guilty at the dc outcome, then the payoff to X of that outcome is reduced. Adjusting the payoffs is the classic way of stabilizing cooperative behavior. Suppose prisoner X walks out of prison, free after defecting, having consigned Y, who played c, to a 20 year sentence. Colleagues of Y might well do X some serious damage. Anticipation of such an event considerably reduces the desirability of dc for X, perhaps to well below R. If X and Y each have threatening friends, then it is reasonable for each to expect that a prior agreement to play cc will stand and so they each receive R. However, in terms of utility this is no longer a Prisoner's Dilemma. In the book which originated modern game theory, Von Neumann and Morgenstern [19], the authors developed an axiomatic theory of utility which allows us to make sense of such arithmetic relationships as the second inequality in (1.4). We won't consider this further, but the reader should remember that the payoffs are numerical measurements of desirability.

This two person collapse of cooperation can be regarded as a simple model of what Garret Hardin [7] calls *the tragedy of the commons*. This is a similar sort of collapse of mutually beneficial cooperation on a multi-person scale.

In the search for a way to avert this tragedy, attention has focused upon *repeated play*. X and Y play repeated rounds of the same game. For each round the players' choices are made independently, but each is aware of all of the previous outcomes. The hope is that the threat of future retaliation will rein in the temptation to defect in the current round.

Robert Axelrod devised a tournament in which submitted computer programs played against one another. Each program played a fixed, but unknown, number of rounds against each of the competing programs, and the resulting payoffs were summed. The results are described and analyzed in his landmark book [4]. The winning program, Tit-for-Tat, submitted by game theorist Anatol Rapaport, consists in playing in each round the strategy used by the opponent in the previous round. A second tournament yielded the same winner. Axelrod extracted some interesting rules of thumb from Tit-for-Tat and applied these to some historical examples.

At around the same time, game theory was being introduced by John Maynard Smith into biology in order to study problems in the evolution of behavior. Maynard Smith [11] and Sigmund [13] provide good surveys of the early work. Tournament play for games, which has been widely explored since, exactly simulates the dynamics examined in this growing field of evolutionary game theory. However, the tournament/evolutionary viewpoint changes the problem in a subtle way. In evolutionary game theory, what matters is how a player is doing as compared with the competing players. Consider this with just two players and suppose they are currently considering strategies with the same payoff to each. Comparing outcomes, Y would reject a move to a strategy where she does better, but which allows X to do still better than she. That this sort of *altruism* is selected against is a major problem in the theory of evolution. However, in classical game theory the payoffs are in utilities. Y simply desires to obtain the highest absolute payoff. The payoffs to her opponent are irrelevant, except as data to predict X's choice of strategy. It is the classical problem that we will mainly consider, although we will return to evolutionary dynamics in the last section.

I am not competent to summarize the immense literature devoted to these matters. I recommend the excellent book length treatments of Hofbauer and Sigmund [9], Nowak [12] and Sigmund [14]. The latter two discuss the Markov approach which we now examine.

Tit-for-Tat (hereafter, just TFT) is an example of a *memory-one strategy* which bases its response entirely on the outcome of the previous round. With the outcomes listed in order as cc, cd, dc, dd, a memory one strategy for X is a vector $\mathbf{p} = (p_1, p_2, p_3, p_4) = (p_{cc}, p_{cd}, p_{dc}, p_{dd})$ where p_z is the probability of playing c when the outcome z occurred in the previous round. If Y uses strategy vector $\mathbf{q} = (q_1, q_2, q_3, q_4)$ then the response vector is $(q_{cc}, q_{cd}, q_{dc}, q_{dd}) = (q_1, q_3, q_2, q_4)$ and the successive outcomes follow a Markov chain with transition matrix given by:

$$\mathbf{M} \quad = \quad \begin{pmatrix} p_1q_1 & p_1(1-q_1) & (1-p_1)q_1 & (1-p_1)(1-q_1) \\ p_2q_3 & p_2(1-q_3) & (1-p_2)q_3 & (1-p_2)(1-q_3) \\ p_3q_2 & p_3(1-q_2) & (1-p_3)q_2 & (1-p_3)(1-q_2) \\ p_4q_4 & p_4(1-q_4) & (1-p_4)q_4 & (1-p_4)(1-q_4) \end{pmatrix}. \tag{1.7}$$

We use the switch in numbering from the Y strategy $\mathbf{q}$ to the Y response vector because switching the perspective of the players interchanges *cd* and *dc*. This way the "same" strategy for X and for Y is given by the same probability vector. For example, *TFT* for X and for Y is given by $\mathbf{p} = \mathbf{q} = (1,0,1,0)$, but the response vector for Y is $(1,1,0,0)$. The strategy *Repeat* is given by $\mathbf{p} = \mathbf{q} = (1,1,0,0)$ with the response vector for Y equal to $(1,0,1,0)$. This strategy just repeats the previous play, regardless of what the opponent did.

We describe some elementary facts about finite Markov chains, see, e. g., Chapter 2 of Karlin and Taylor [10].

A Markov matrix like $\mathbf{M}$ is a non-negative matrix with row sums equal to 1. Thus, the vector $\mathbf{1}$ is a right eigenvector with eigenvalue 1. For such a matrix, we can represent the associated Markov chain as movement along a directed graph with vertices the states, in this case, cc, cd, dc, dd, and with a directed edge from the i^{th} state z_i to the j^{th} state z_j when $\mathbf{M}_{ij} > 0$, that is, when we can move from z_i to z_j with positive probability. In particular, there is an edge from z_i to itself iff the diagonal entry $\mathbf{M}_{ii}$ is positive.

A *path* in the graph is a state sequence $z^1, ..., z^n$ with $n > 1$ such that there is an edge from z^i to z^{i+1} for $i = 1, ..., n-1$. A set of states I is called a *closed set* when no path that begins in I can exit I. For example, the entire set of states is closed and for any z the set of states accessible via a path that begins at z is a closed set. I is closed iff $\mathbf{M}_{ij} = 0$ whenever $z_i \in I$ and $z_j \notin I$. In particular, when we restrict the chain to a closed set I, the associated submatrix of $\mathbf{M}$ still has row sums equal to 1. A minimal, nonempty, closed set of states is called a *terminal set*. A state is called *recurrent* when it lies in some terminal set and *transient* when it does not. The following facts are easy to check.

- A nonempty, closed set of states I is terminal iff for all $z_i, z_j \in I$, there exists a path from z_i to z_j.

- If I is a terminal set and $z_j \in I$, then there exists $z_i \in I$ with an edge from z_i to z_j.

- Distinct terminal sets are disjoint.

- Any nonempty, closed set contains at least one terminal set.

- From any transient state there is a path into some terminal set.

Suppose we are given an initial distribution $\mathbf{v}^1$, describing the outcome of the first round of play. The Markov process evolves in discrete time via the equation

$$\mathbf{v}^{n+1} \quad = \quad \mathbf{v}^n \cdot \mathbf{M}, \tag{1.8}$$

where we regard the distributions as row vectors.

In our game context, the initial distribution is given by the initial plays, pure or mixed, of the two players. If X uses initial probability p_c and Y uses q_c, then

$$\mathbf{v}^1 \quad = \quad (p_c q_c, p_c(1 - q_c), (1 - p_c)q_c, (1 - p_c)(1 - q_c)). \tag{1.9}$$

Thus, v_i^n is the probability that outcome z_i occurs on the n^{th} round of play. A distribution $\mathbf{v}$ is *stationary* when it satisfies $\mathbf{vM} = \mathbf{v}$. That is, it is a left eigenvector with eigenvalue 1. From Perron-Frobenius theory (see, e.g., Appendix 2 of [10]) it follows that if I is a terminal set, then there is a unique stationary distribution $\mathbf{v}$ with $v_i > 0$ iff $i \in I$. That is, the *support* of $\mathbf{v}$ is exactly I. In particular, if the eigenspace of $\mathbf{M}$ associated with the eigenvalue 1 is one dimensional, then there is a unique stationary distribution, and so a unique terminal set which is the support of the stationary distribution. The converse is also true and any stationary distribution $\mathbf{v}$ is a mixture of the $\mathbf{v}_J$'s where $\mathbf{v}_J$ is supported on the terminal set J. This follows from the fact that any stationary distribution $\mathbf{v}$ satisfies $v_i = 0$ for all transient states z_i and so is supported on the set of recurrent states. On the recurrent states the matrix $\mathbf{M}$ is block diagonal. Hence, the following are equivalent in our 4×4 case.

- There is a unique terminal set of states for the process associated with M.

- There is a unique stationary distribution vector for M.

- The matrix $M' = M - I$ has rank 3.

We will call $\mathbf{M}$ *convergent* when these conditions hold. For example, when all of the probabilities of $\mathbf{p}$ and $\mathbf{q}$ lie strictly between 0 and 1, then all the entries of $\mathbf{M}$ given by (1.7) are positive and so the entire set of states is the unique terminal state and the positive matrix $\mathbf{M}$ is convergent.

The sequence of the Cesaro averages $\{\frac{1}{n}\Sigma_{i=1}^{n} \mathbf{v}^i\}$ of the outcome distributions always converges to some stationary distribution $\mathbf{v}$. That is,

$$Lim_{n\to\infty} \frac{1}{n}\Sigma_{k=1}^{n} \mathbf{v}^k \quad = \quad \mathbf{v}. \tag{1.10}$$

Hence, using the payoff vectors from (1.2) the long run average payoffs for X and Y converge to s_X and s_Y of (1.3) with $\mathbf{v}$ this limiting stationary distribution.

When $\mathbf{M}$ is convergent, the limit $\mathbf{v}$ is the unique stationary distribution and so the average payoffs are independent of the initial distribution. In the non-convergent case, the long term payoffs depend on the initial distribution. Suppose there are exactly two terminal sets, I and J, with stationary distribution vectors $\mathbf{v}_I$ and $\mathbf{v}_J$, supported on I and J, respectively. For any initial distribution $\mathbf{v}^1$, there are probabilities p_I and $p_J = 1 - p_I$ of entering into, and so terminating in, I or J, respectively. In that case, the limit of the Cesaro averages sequence for $\{\mathbf{v}^n\}$ is given by

$$\mathbf{v} \quad = \quad p_I\mathbf{v}_I \;+\; p_J\mathbf{v}_J, \tag{1.11}$$

and the limits of the average payoffs are given by (1.3) with this distribution $\mathbf{v}$. This extends in the obvious way when there are more terminal sets.

When Y responds to the memory-one strategy $\mathbf{p}$ with a memory-one strategy $\mathbf{q}$, we have the *Markov case* as above. We will also want to see how a strategy $\mathbf{p}$ for X fares against a not necessarily memory-one response by Y. Such a general *strategy pattern* is a potential choice of response, pure or mixed, for any sequence of previous outcomes. In that case, the sequence of Cesaro averages need not converge. We will call any limit point of the sequence <u>an</u> associated *limit distribution*. We will call s_X and s_Y, given by (1.3) with such a limit distribution $\mathbf{v}$, the *expected payoffs* associated with $\mathbf{v}$.

Call a strategy vector $\mathbf{p}$ *agreeable* when $p_1 = 1$ and *firm* when $p_4 = 0$. That is, an agreeable strategy always responds to cc with c and a firm strategy always responds to dd with d. If both $\mathbf{p}$ and $\mathbf{q}$ are agreeable, then $\{cc\}$ is a terminal set for the Markov matrix $\mathbf{M}$ given by (1.7) and so $\mathbf{v} = (1, 0, 0, 0)$ is a stationary distribution with fixation at cc. If both $\mathbf{p}$ and $\mathbf{q}$ are firm, then $\{dd\}$ is a terminal set for $\mathbf{M}$ and $\mathbf{v} = (0,0,0,1)$ is a stationary distribution with fixation at dd. Any convex combination of agreeable strategies (or firm strategies) is agreeable (respectively, firm).

The strategies $TFT = (1,0,1,0)$ and $Repeat = (1,1,0,0)$ are each agreeable and firm. The same is true for any mixture of these. If both X and Y use TFT, then the outcome is determined by the initial play. Initial outcomes cc and dd lead to immediate fixation. Either cd or dc results in period 2 alternation between these two states. Thus, $\{cd, dc\}$ is another terminal set with stationary distribution $(0, \frac{1}{2}, \frac{1}{2}, 0)$. If $a \cdot TFT + (1-a)Repeat$ is used instead by either player (with $0 < a < 1$), then eventually fixation at cc or dd results. There are then only two terminal sets instead of three. The period 2 alternation described above illustrates why we needed the Cesaro limit, i.e. the limit of averages, in (1.10) rather than the limit per se.

Because so much work had been done on this Markov model, the exciting new ideas in Press and Dyson [15] took people by surprise. They have inspired a number of responses, e. g., Stewart and Plotkin [16] and especially, Hilbe, Nowak and Sigmund [8]. I would here like to express my gratitude to Karl Sigmund whose kind, but firm, criticism of the initial draft directed me to this recent work. The result is both a substantive and expository improvement.

Our purpose here is to use these new ideas to characterize the strategies that are good in the following sense.

Definition 1.2 *A memory one strategy* $\mathbf{p}$ *for X is called* good *if it is agreeable and if for any strategy pattern chosen by Y against it and any associated limit distribution, the expected payoffs satisfy*

$$s_Y \ \geq \ R \qquad \Longrightarrow \qquad s_Y \ = \ s_X \ = \ R. \tag{1.12}$$

The strategy is called of Nash type *if it is agreeable and if the expected payoffs against any Y strategy pattern satisfy*

$$s_Y \ \geq \ R \qquad \Longrightarrow \qquad s_Y \ = \ R. \tag{1.13}$$

By Proposition 1.1, $s_Y = s_X = R$ iff the associated limit distribution is $(1,0,0,0)$. In the memory-one case, $(1,0,0,0)$ is a stationary distribution iff both strategies are agreeable. It is the unique stationary distribution iff, in addition, the matrix $\mathbf{M}$ is convergent. If $\mathbf{p}$ is not agreeable, then (5.1) can be vacuously true. For example, if X plays $AllD = (0,0,0,0)$, then $P \geq s_Y$ for any strategy pattern and the implication is true.

When both players use agreeable strategies, with initial cooperation, then the joint cooperative payoff is achieved. The pair of strategies is a Nash equilibrium exactly when the two strategies are of Nash type. That is, both players receive R and neither player can do better by playing an alternative strategy. A good strategy is of Nash type, but more is true. We will see that with a Nash equilibrium it is possible that Y can play an alternative which still yields R for herself but with the payoff to X smaller than R. That is, Y has no incentive to play so as to reach the joint cooperative payoff. On the other hand, if X uses a good strategy, then the only responses for Y that obtain R for her also yield R for X.

The strategy *Repeat* $= (1,1,0,0)$ is an agreeable strategy that is not of Nash type. If both players use *Repeat*, then the initial outcome repeats forever. If the initial outcome is cd, then $s_Y = T$ and $s_X = S$.

For a strategy $\mathbf{p}$, we define the *X Press-Dyson vector* $\tilde{\mathbf{p}} = \mathbf{p} - \mathbf{e}_{12}$, where $\mathbf{e}_{12} = (1,1,0,0)$. Considering the utility of the following result of Hilbe, Nowak and Sigmund, its proof, taken from Appendix A of [8], is remarkably simple.

Theorem 1.3 *Assume that X uses the strategy* $\mathbf{p}$ *with X Press-Dyson vector* $\tilde{\mathbf{p}}$. *If the opponent Y uses a strategy pattern that yields a sequence of distributions* $\{\mathbf{v}^n\}$*, then*

$$\begin{aligned} Lim_{n\to\infty}\frac{1}{n}\ \Sigma_{k=1}^{n}\ < \mathbf{v}^k \cdot \tilde{\mathbf{p}} > \quad &= \quad 0, \\ \textit{and so} \quad < \mathbf{v} \cdot \tilde{\mathbf{p}} > = \ v_1\tilde{p}_1 + v_2\tilde{p}_2 + v_3\tilde{p}_3 + v_4\tilde{p}_4 \ &= \ 0 \end{aligned} \tag{1.14}$$

for any associated limit distribution $\mathbf{v}$.

Proof: Let $v_{12}^n = v_1^n + v_2^n$, the probability that either cc or cd is the outcome in the n^{th} round of play. That is, $v_{12}^n =< \mathbf{v}^n \cdot \mathbf{e}_{12} >$ is the probability that X played c in the n^{th} round. On the other hand, since X is using the memory one strategy $\mathbf{p}$, p_i is the conditional probability that X plays c in the next round, given outcome z_i in the current round. Thus, $< \mathbf{v}^n \cdot \mathbf{p} >$ is the probability that X plays c in the $(n+1)^{st}$ round, i. e. it is v_{12}^{n+1}. Hence, $v_{12}^{n+1} - v_{12}^n \ = < \mathbf{v}^n \cdot \tilde{\mathbf{p}} >$. The sum telescopes to yield

$$v_{12}^{n+1} \ - \ v_{12}^1 \quad = \quad \Sigma_{k=1}^{n}\ < \mathbf{v}^k \cdot \tilde{\mathbf{p}} > . \tag{1.15}$$

As the left side has absolute value at most 1, the limit (1.14) follows. If a subsequence of the Cesaro averages converges to $\mathbf{v}$, then $< \mathbf{v} \cdot \tilde{\mathbf{p}} > = 0$ by continuity of the dot product. $\Box$

To illustrate the use of this result, we examine $TFT = (1,0,1,0)$ and another strategy which has been labeled in the literature $Grim = (1,0,0,0)$. We consider mixtures of each with $Repeat = (1,1,0,0)$.

Corollary 1.4 *Let* $1 \geq a > 0$.

(a) The strategy $\mathbf{p} = aTFT + (1-a)Repeat$ *is a good strategy with* $s_Y = s_X$ *for any limiting distribution.*

(b) The strategy $\mathbf{p} = aGrim + (1-a)Repeat$ *is good.*

Proof: (a) In this case, $\tilde{\mathbf{p}} = a(0,-1,1,0)$ and so (1.14) implies that $v_2 = v_3$. Thus, $s_Y = s_X$. From this (5.1) follows from Proposition 1.1.

(b) Now $\tilde{\mathbf{p}} = a(0,-1,0,0)$ and so (1.14) implies that $v_2 = 0$. Thus, $s_Y = v_1 R + v_3 S + v_4 P$ and this is less than R unless $v_3 = v_4 = 0$ and $v_1 = 1$. When $v_1 = 1$, $s_Y = s_X = R$, proving (5.1). $\Box$

In the next section we will prove the following characterization of the good strategies.

Theorem 1.5 *Let* $\mathbf{p} = (p_1, p_2, p_3, p_4)$ *be an agreeable strategy vector, that is not Repeat. That is,* $p_1 = 1$ *but* $\mathbf{p} \neq (1,1,0,0)$.

The strategy $\mathbf{p}$ *is of Nash type iff the following inequalities hold.*

$$\frac{T-R}{R-S} \cdot p_3 \ \leq \ (1-p_2) \qquad \textit{and} \qquad \frac{T-R}{R-P} \cdot p_4 \ \leq \ (1-p_2). \tag{1.16}$$

The strategy $\mathbf{p}$ *is good iff, in addition, both inequalities are strict.*

Corollary 1.6 *In the compact convex set of agreeable strategies, the set* $\{\mathbf{p}$ *equals Repeat or is of Nash type* $\}$ *is a closed convex set with interior the set of good strategies.*

Proof: The X Press-Dyson vectors form a cube and the agreeable strategies are the intersection with the subspace $\tilde{p}_1 = 0$. We then intersect with the half-spaces defined by

$$\frac{T-R}{R-S}\tilde{p}_3 + \tilde{p}_2 \ \leq \ 0 \qquad \text{and} \qquad \frac{T-R}{R-P}\tilde{p}_4 + \tilde{p}_2 \ \leq \ 0. \tag{1.17}$$

The result is a closed convex set with interior given by the strict inequalities. Notice that these conditions are preserved by multiplication by a positive constant $a \leq 1$ or by any larger constant so long as $a\tilde{\mathbf{p}}$ remains in the cube. Hence, *Repeat* with $\tilde{\mathbf{p}} = 0$ is on the boundary. $\square$

It is easy to compute that that

$$det \begin{pmatrix} R & R & 1 & 0 \\ S & T & 1 & 1 \\ T & S & 1 & 1 \\ P & P & 1 & 0 \end{pmatrix} \quad = \quad -2(R-P)(T-S). \tag{1.18}$$

Hence, with $\mathbf{e}_{23} = (0,1,1,0)$, we can use $\{\mathbf{S}_X, \mathbf{S}_Y, \mathbf{1}, \mathbf{e}_{23}\}$ as a basis for $\mathbb{R}^4$. For a distribution vector $\mathbf{v}$ we will write v_{23} for $v_2 + v_3 \ = < \mathbf{v} \cdot \mathbf{e}_{23} >$. From Theorem 1.3, we immediately obtain the following.

Theorem 1.7 *If* $\mathbf{p}$ *is a strategy whose X Press-Dyson vector* $\tilde{\mathbf{p}} = \alpha\mathbf{S}_X + \beta\mathbf{S}_Y + \gamma\mathbf{1} + \delta\mathbf{e}_{23}$ *and* $\mathbf{v}$ *is a limit distribution when Y plays some strategy pattern against* $\mathbf{p}$*, then the average payoffs satisfy the following* Press-Dyson Equation*.*

$$\alpha s_X \ + \ \beta s_Y \ + \ \gamma \ + \ \delta v_{23} \quad = \quad 0. \tag{1.19}$$

The most convenient cases to study occur when $\delta = 0$. Press and Dyson called such a strategy a *Zero-Determinant Strategy* (hereafter ZDS) because of an ingenious determinant argument leading to (1.19). We have used Theorem 1.3 of Hilbe-Nowak-Sigmund instead.

This representation yields a simple description of the good strategies.

Theorem 1.8 *Assume that* $\mathbf{p} = (p_1, p_2, p_3, p_4)$ *is an agreeable strategy vector with X Press-Dyson vector* $\tilde{\mathbf{p}} = \alpha\mathbf{S}_X + \beta\mathbf{S}_Y + \gamma\mathbf{1} + \delta\mathbf{e}_{23}$*. Assume that* $\mathbf{p}$ *is not Repeat, i. e.* $(\alpha, \beta, \gamma, \delta) \neq (0,0,0,0)$*. The strategy* $\mathbf{p}$ *is of Nash type iff*

$$max(\frac{\delta}{(T-S)}, \frac{\delta}{(2R-(T+S))}) \quad \leq \quad \alpha. \tag{1.20}$$

The strategy $\mathbf{p}$ *is good iff, in addition, the inequality is strict.*

Remark: Observe that $T - S \ > \ 2R - (T + S) \ > \ 0$. It follows that if $\delta \leq 0$, then $\mathbf{p}$ is good iff $\frac{\delta}{(T-S)} < \alpha$. On the other hand, if $\delta > 0$, then $\mathbf{p}$ is good iff $\frac{\delta}{(2R-(T+S))} < \alpha$.

In the next section, we will investigate the geometry of the $\{\mathbf{S}_X, \mathbf{S}_Y, \mathbf{1}, \mathbf{e}_{23}\}$ decomposition of the Press-Dyson vectors and prove the theorems.

2 Good Strategies and The Press-Dyson Decomposition

We begin by normalizing the payoffs. We can add to all a common number and multiply all by a common positive number without changing the relationship between the various strategies. We subtract S and divide by $T - S$. So from now on we will assume that $T = 1$ and $S = 0$.

The payoff vectors of (1.2) are then given by

$$\mathbf{S}_X \quad = \quad (R, 0, 1, P), \qquad \mathbf{S}_Y \quad = \quad (R, 1, 0, P), \tag{2.1}$$

and from (1.4) we have

$$1 \ > \ R \ > \ \frac{1}{2}, \quad \text{and} \quad R \ > \ P \ > \ 0. \tag{2.2}$$

After normalization Theorem 1.5 becomes the following.

Theorem 2.1 *Let $\mathbf{p} = (p_1, p_2, p_3, p_4)$ be an agreeable strategy vector, that is not Repeat. That is, $p_1 = 1$ but $\mathbf{p} \neq (1, 1, 0, 0)$.*

The strategy $\mathbf{p}$ is of Nash type iff the following inequalities hold.

$$\frac{1-R}{R} \cdot p_3 \ \leq \ (1 - p_2) \qquad \textit{and} \qquad \frac{1-R}{R-P} \cdot p_4 \ \leq \ (1 - p_2). \tag{2.3}$$

The strategy $\mathbf{p}$ is good iff, in addition, both inequalities are strict.

Proof: We first eliminate the possibility $p_2 = 1$. If $1 - p_2 = 0$, then the inequalities would yield $p_3 = p_4 = 0$ and so $\mathbf{p} = \textit{Repeat}$, which we have excluded. On the other hand, if $p_2 = 1$, then $\mathbf{p} = (1, 1, p_3, p_4)$. If against this Y plays $\textit{AllD} = (0, 0, 0, 0)$, then $\{cd\}$ is a terminal set with stationary distribution $(0,1,0,0)$ and so with $s_Y = 1$ and $s_X = 0$. Hence, $\mathbf{p}$ is not of Nash type. Thus, if $p_2 = 1$, then neither is $\mathbf{p}$ of Nash type, nor do the inequalities hold for it. We now assume $1 - p_2 > 0$.

Observe that

$$\begin{aligned} s_Y - R \quad &= \quad (v_1 R + v_2 + v_4 P) \ - \ (v_1 R + v_2 R + v_3 R + v_4 R) \\ &= \quad v_2(1-R) \ - \ v_3 R \ - \ v_4(R-P). \end{aligned} \tag{2.4}$$

Hence, multiplying by the positive quantity $(1 - p_2)$, we have

$$s_Y \ >= \ R \quad \Longleftrightarrow \quad (1-p_2)v_2(1-R) \ >= \ v_3(1-p_2)R \ + \ v_4(1-p_2)(R-P), \tag{2.5}$$

where this notation means that the inequalities are equivalent and the equations are equivalent.

Since $\tilde{p}_1 = 0$, equation (1.14) implies $v_2\tilde{p}_2 + v_3\tilde{p}_3 + v_4\tilde{p}_4 = 0$ and so $(1 - p_2)v_2 = v_3 p_3 + v_4 p_4$. Substituting in the above inequality and collecting terms we get

$$\begin{gathered} s_Y \ >= \ R \qquad \Longleftrightarrow \qquad Av_3 \ >= \ Bv_4 \qquad \text{with} \\ A \ = \ [p_3(1-R) - (1-p_2)R] \qquad \text{and} \qquad B \ = \ [(1-p_2)(R-P) - p_4(1-R)]. \end{gathered} \tag{2.6}$$

Observe that the inequalities of (2.3) are equivalent to $A \leq 0$ and $B \geq 0$. The proof is completed by using a sequence of little cases.

Case(i) $A = 0, B = 0$: In this case, $Av_3 = Bv_4$ holds for any strategy for Y. So for any Y strategy, $s_Y = R$ and $\mathbf{p}$ is of Nash type. If Y chooses any strategy that is not agreeable, then $\{cc\}$ is not a closed set of states and so $v_1 \neq 1$. From Proposition 1.1, $s_X < R$ and so $\mathbf{p}$ is not good.

Case(ii) $A < 0, B = 0$: The inequality $Av_3 \geq Bv_4$ holds iff $v_3 = 0$. If $v_3 = 0$, then $Av_3 = Bv_4$ and so $s_Y = R$. Thus, $\mathbf{p}$ is Nash.

Case(iia) $B \leq 0$, any A: Assume Y chooses a strategy that is not agreeable and is such that $v_3 = 0$. For example, if Y plays $AllD = (0,0,0,0)$, then no state moves to dc. With such a Y choice, $Av_3 \geq Bv_4$ and so $s_Y \geq R$. As above, $v_1 \neq 1$ because the Y choice is not agreeable. Again $s_X < R$ and $\mathbf{p}$ is not good. Furthermore, $v_3 = 0, v_1 < 1, p_2 < 1$, and $(1 - p_2)v_2 = v_4 p_4$ imply that $v_4 > 0$. So if $B < 0$, then $Av_3 > Bv_4$ and so $s_Y > R$. Hence, $\mathbf{p}$ is not Nash when $B < 0$.

Case(iii) $A = 0, B > 0$: The inequality $Av_3 \geq Bv_4$ holds iff $v_4 = 0$. If $v_4 = 0$, then $Av_3 = Bv_4$ and $s_Y = R$. Thus, $\mathbf{p}$ is Nash.

Case(iiia)$A \geq 0$, any B: Assume Y chooses a strategy that is not agreeable and is such that $v_4 = 0$. For example, if Y plays $(0,1,1,1)$, then no state moves to dd. With such a Y choice, $Av_3 \geq Bv_4$ and so $s_Y \geq R$. As before, $v_1 \neq 1$ implies $s_X < R$ and the strategy is not good. Furthermore, $v_4 = 0, v_1 < 1, p_2 < 1$, and $(1-p_2)v_2 = v_3 p_3$ imply that $v_3 > 0$. So if $A > 0$, then $Av_3 > Bv_4$ and so $s_Y > R$. Hence, $\mathbf{p}$ is not Nash when $A > 0$.

Case(iv) $A < 0, B > 0$: The inequality $Av_3 \geq Bv_4$ implies $v_3, v_4 = 0$. So $(1-p_2)v_2 = v_3 p_3 + v_4 p_4 = 0$. Since $p_2 < 1$, $v_2 = 0$. Hence, $v_1 = 1$. That is, $s_Y \geq R$ implies $s_Y = s_X = R$ and so $\mathbf{p}$ is good. $\square$

Remarks: (a) Since $1 > R > \frac{1}{2}$, it is always true that $\frac{1-R}{R} < 1$. On the other hand, $\frac{1-R}{R-P}$ can be greater than 1 and the second inequality requires $p_4 \leq \frac{R-P}{1-R}$. In particular, if $p_2 = 0$, then the strategy is good iff $p_4 < \frac{R-P}{1-R}$. For example, the strategy $(1,0,0,1)$ is, in the literature, labeled $Pavlov$, or $WinStay, LoseShift$. This strategy always satisfies the first inequality strictly, but it satisfies the second strictly, and so is good, iff $1 - R < R - P$.

(b) In Case(i) of the proof, the payoff $s_Y = R$ is determined by $\mathbf{p}$ independent of the choice of strategy for Y. In general, strategies that fix the opponent's payoff in this way were described by Press and Dyson [15] and, earlier, by Boerlijst, Nowak and Sigmund [5], where they are called *equalizer strategies*. The agreeable equalizer strategies have $\tilde{\mathbf{p}} = a(0, -\frac{1-R}{R}, 1, \frac{R-P}{R})$ with $1 \geq a > 0$.

Christian Hilbe suggests a nice interpretation of the above results:

Corollary 2.2 *Let* $\mathbf{p}$ *be an agreeable strategy vector with* $p_2 < 1$.

(a) If $\mathbf{p}$ *is good, then using any strategy* $\mathbf{q}$ *that is not agreeable forces* Y *to get a payoff* $s_Y < R$.

(b) If $\mathbf{p}$ *is not good, then by using at least one of the two strategies* $\mathbf{q} = (0,0,0,0)$ *or* $\mathbf{q} = (0,1,1,1)$*,* Y *can certainly obtain a payoff* $s_Y \geq R$*, and force* X *to get a payoff* $s_X < R$.

(c) If $\mathbf{p}$ *is not Nash, then by using at least one of the two strategies* $\mathbf{q} = (0,0,0,0)$ *or* $\mathbf{q} = (0,1,1,1)$ Y *can certainly obtain a payoff* $s_Y > R$*, and force* X *to get a payoff* $s_X < R$.

Proof: (a): If $\mathbf{p}$ is good, then $s_Y \geq R$ implies $s_Y = s_X = R$ which requires $v = (1,0,0,0)$. This is only stationary when $\mathbf{q}$ as well as $\mathbf{p}$ is agreeable.

(b) and (c) follow from the analysis of cases in the above proof. $\square$

Remark: If $p_2 = p_1 = 1$, then the strategy $\mathbf{p}$ is not Nash. As observed in the proof above, if Y plays $\mathbf{q} = (0,0,0,0)$, then cd is a terminal set with stationary distribution $\mathbf{v} = (0,1,0,0)$ and so with $s_Y = 1, s_X = 0$. However, if, in addition, $p_4 = 0$, e. g., if X uses *Repeat*, then dd is also a terminal set. Thus, if X plays $\mathbf{p}$ with $1 - p_4 = p_2 = p_1 = 1$ and Y always defects, then fixation occurs immediately at either cd with $s_Y = 1$ and $s_X = 0$, or else at dd with $s_Y = s_X = P$. The result is determined by the initial play of X.

We now consider the Press-Dyson representation, using the normalized payoff vectors of (2.1). If $\tilde{\mathbf{p}} = \alpha \mathbf{S}_X + \beta \mathbf{S}_Y + \gamma \mathbf{1} + \delta \mathbf{e}_{23}$ is the X Press-Dyson vector of a strategy $\mathbf{p}$, then it must satisfy two sorts of constraints.

The *sign constraints* require that the first two entries be nonpositive and the last two be nonnegative. That is,

$$\begin{aligned} (\alpha+\beta)R + \gamma &\leq 0, \\ \beta + \gamma + \delta &\leq 0, \\ \alpha + \gamma + \delta &\geq 0, \\ (\alpha+\beta)P + \gamma &\geq 0. \end{aligned} \tag{2.7}$$

Lemma 2.3 *If $\tilde{\mathbf{p}} = \alpha \mathbf{S}_X + \beta \mathbf{S}_Y + \gamma \mathbf{1} + \delta \mathbf{e}_{23}$ satisfies the sign constraints, then*

$$\begin{array}{lcl} \alpha+\beta \leq 0 & \textit{and} & \gamma \geq 0, \\ \alpha+\beta = 0 & \Leftrightarrow & \gamma = 0. \end{array} \tag{2.8}$$

Proof: Subtracting the fourth inequality from the first we see that $(\alpha+\beta)(R-P) \leq 0$ and so $R - P > 0$ implies $\alpha + \beta \leq 0$. Then the fourth inequality and $P > 0$ imply $\gamma \geq 0$. The first and fourth imply $\alpha + \beta = 0$ iff $\gamma = 0$. $\square$

Remark: Notice that both $\tilde{p}_1$ and $\tilde{p}_4$ vanish iff $\alpha + \beta = \gamma = 0$. These are the cases when the strategy $\mathbf{p}$ is both agreeable and firm.

In addition, the entries of an X Press-Dyson vector have absolute value at most 1. These are the *size constraints*. If a vector satisfies the sign constraints then, multiplying by a sufficiently small positive number, we obtain the size constraints as well. Any vector in $\mathbb{R}^4$ that satisfies both the sign and the size constraints is an X strategy Press-Dyson vector. Call $\mathbf{p}$ a *top strategy* if $|\tilde{p}_i| = 1$ for some i. For any strategy $\mathbf{p}$, other than *Repeat*, which has Press-Dyson vector 0, $\mathbf{p} = a(\mathbf{p}^t) + (1-a)Repeat$ for a unique top strategy vector $\mathbf{p}^t$ and a unique positive $a \leq 1$. Equivalently, $\tilde{\mathbf{p}} = a\tilde{\mathbf{p}}^t$.

Observe that $\mathbf{p}$ is agreeable iff $\tilde{p}_1 = 0$ and so iff $(\alpha+\beta)R + \gamma = 0$. In that case, $\beta = -\alpha - \gamma R^{-1}$. Substituting into (1.19), we obtain the following corollary of Theorem 1.7.

Corollary 2.4 *If* $\mathbf{p}$ *is an agreeable strategy with X Press-Dyson vector* $\tilde{\mathbf{p}} = \alpha\mathbf{S}_X + \beta\mathbf{S}_Y + \gamma\mathbf{1} + \delta\mathbf{e}_{23}$*, then the payoffs with any limit distribution satisfy the following version of the* Press-Dyson Equation*.*

$$\gamma R^{-1}s_Y \ + \ \alpha(s_Y - s_X) \ - \ \delta v_{23} \quad = \quad \gamma. \tag{2.9}$$

Now we justify the description in Theorem 1.8. Notice that if we label by ${\mathbf{S}_X}^0$ and ${\mathbf{S}_Y}^0$ our original payoff vectors before normalization then ${\mathbf{S}_X}^0 = (T-S)\mathbf{S}_X + S\mathbf{1}$, ${\mathbf{S}_Y}^0 = (T-S)\mathbf{S}_Y + S\mathbf{1}$ and so if $(\alpha,\beta,\gamma,\delta)$ are the coordinates of $\tilde{\mathbf{p}}$ with respect to the basis $\{\mathbf{S}_X, \mathbf{S}_Y, \mathbf{1}, \mathbf{e}_{23}\}$ then $(\alpha^0,\beta^0,\gamma^0,\delta^0) = (\alpha/(T-S), \beta/(T-S), \gamma - (\alpha+\beta)S/(T-S), \delta)$ are the coordinates with respect to $\{{\mathbf{S}_X}^0, {\mathbf{S}_Y}^0, \mathbf{1}, \mathbf{e}_{23}\}$. In particular, $\alpha >= k\delta$ iff $\alpha^0 >= k\delta^0/(T-S)$ for any k. Furthermore, the constant $k = \frac{T-S}{2R-(T+S)}$ is independent of normalization. So it suffices to prove the normalized version of the theorem which is the following.

Theorem 2.5 *Assume that* $\mathbf{p} = (p_1,p_2,p_3,p_4)$ *is an agreeable strategy vector with X Press-Dyson vector* $\tilde{\mathbf{p}} = \alpha\mathbf{S}_X + \beta\mathbf{S}_Y + \gamma\mathbf{1} + \delta\mathbf{e}_{23}$*. Assume that* $\mathbf{p}$ *is not Repeat, i. e.* $(\alpha,\beta,\gamma,\delta) \neq (0,0,0,0)$*. The strategy* $\mathbf{p}$ *is of Nash type iff*

$$max(\delta, (2R-1)^{-1}\delta) \quad \leq \quad \alpha. \tag{2.10}$$

The strategy $\mathbf{p}$ *is good iff, in addition, the inequality is strict.*

Proof: Since $\beta = -\alpha - \gamma R^{-1}$, we have

$$\begin{aligned} (1-p_2) \ &= \ -\tilde{p}_2 \ = \ -\beta - \gamma - \delta \ = \ \alpha + \frac{1-R}{R}\gamma - \delta, \\ p_3 \ = \ \tilde{p}_3 \ &= \ \alpha + \gamma + \delta, \qquad p_4 \ = \ \tilde{p}_4 \ = \ \frac{R-P}{R}\gamma. \end{aligned} \tag{2.11}$$

The inequality $(1-R)p_3 \leq R(1-p_2)$ becomes $(1-R)(\alpha+\gamma+\delta) \leq R\alpha+(1-R)\gamma-R\delta$. This reduces to $\delta \leq (2R-1)\alpha$. Similarly, the inequality $(1-R)p_4 \leq (R-P)(1-p_2)$ reduces to $\delta \leq \alpha$. $\square$

Remarks: (a) Thus, when $\delta \leq 0$, $\mathbf{p}$ is good iff $\delta < \alpha$. When $\delta > 0$, $\mathbf{p}$ is good iff $\frac{\delta}{2R-1} < \alpha$.

(b) From the proof we see that the equalizer case, when both inequalities of (2.3) are equations, occurs when $\delta = \alpha = (2R-1)^{-1}\delta$. Since $2R-1 < 1$, this reduces to $0 = \delta = \alpha$.

In the ZDS case, when $\delta = 0$, we can rewrite (2.9) as

$$\kappa \cdot (s_X - R) \quad = \quad s_Y - R \tag{2.12}$$

with $\kappa = \frac{\alpha R}{\gamma+\alpha R}$. Thus, the condition $\alpha > 0$ is equivalent to $0 < \kappa \leq 1$. In [8] these strategies are introduced and called *complier strategies*. The equation and the condition $\kappa > 0$ make it clear that such strategies are good. In addition, if $s_Y < R$, then it follows that $s_X \leq s_Y$ with strict inequality when $\gamma > 0$ and so $\kappa < 1$. The strategy ZGTFT-2 analyzed in Stewart and Plotkin [16] is an example of a complier strategy. When X plays a complier strategy, then either both s_X and s_Y are equal to R, or else both are below R. This is not true for good strategies in general. If X plays the good strategy $Grim = (1,0,0,0)$ and Y plays $(0,1,1,1)$, then fixation at dc occurs with $v = (0,0,1,0)$ and so with $s_Y = 0$ ($< R$ as required by Corollary 2.2), but with $s_X = 1 > R$.

Let us look at the geometry of the Press-Dyson representation.

We begin with the *exceptional strategies* which are defined by $\gamma = \alpha+\beta = 0$. The sign constraints yield $\alpha = -\beta \geq |\delta|$ and $\tilde{\mathbf{p}} = (0, \delta-\alpha, \delta+\alpha, 0)$. As remarked after Lemma 2.3, the exceptional strategies are exactly those strategies that are both agreeable and firm. In the xy plane with $x = \tilde{p}_2$ and $y = \tilde{p}_3$ they form a square with vertices *Repeat* ($\tilde{\mathbf{p}} = (0,0,0,0)$), *Grim* ($\tilde{\mathbf{p}} = (0,-1,0,0)$), TFT ($\tilde{\mathbf{p}} = (0,-1,1,0)$) and what we will call *Lame* ($\tilde{\mathbf{p}} = (0,0,1,0)$). Thus, $Lame = (1,1,1,0)$. The top strategies consist of the segment that connects TFT with $Grim$ together with the segment that connects TFT with $Lame$.

On the $Grim - TFT$ segment $\delta - \alpha = -1$ and $0 \leq \delta + \alpha \leq 1$. That is, $\delta = \alpha - 1$ and $-\frac{1}{2} \leq \delta \leq 0$. By Theorem 2.5, the strategies in the triangle with vertices Grim, TFT and Repeat are all good except for Repeat itself.

On the $Lame - TFT$ segment $-1 \leq \delta - \alpha \leq 0$ and $\delta + \alpha = 1$. That is, $\delta = 1-\alpha$ and $\frac{1}{2} \geq \delta \geq 0$. Such a strategy is the mixture $tTFT + (1-t)Lame = (1, 1-t, 1, 0)$ with $t = 2\alpha - 1$. By Theorem 2.1, this strategy is good iff $t > \frac{1-R}{R}$. The strategy on the $TFT - Lame$ segment with $t = \frac{1-R}{R}$, and so with $2\alpha = R^{-1}$, we will call $Edge = (1, \frac{2R-1}{R}, 1, 0)$. The strategies in the $TFT - Edge - Repeat$ triangle that are not on the $Edge - Repeat$ side are good strategies. The strategies in the complementary $Edge - Lame - Repeat$ triangle are not good.

Now assume $\gamma > 0$, and define

$$\bar{\alpha} \quad = \quad \alpha/\gamma, \qquad \bar{\beta} \quad = \quad \beta/\gamma, \qquad \bar{\delta} \quad = \quad \delta/\gamma. \tag{2.13}$$

with the sign constraints

$$\begin{aligned} -P^{-1} \quad &\leq \quad \bar{\alpha} + \bar{\beta} \quad \leq \quad -R^{-1}, \\ \bar{\beta} \quad &\leq \quad -1 - \bar{\delta} \quad \leq \quad \bar{\alpha}. \end{aligned} \tag{2.14}$$

For any triple $(\bar{\alpha}, \bar{\beta}, \bar{\delta})$ that satisfies these inequalities, we obtain an X Press-Dyson vector vector $\tilde{\mathbf{p}} = \alpha \mathbf{S}_X + \beta \mathbf{S}_Y + \gamma \mathbf{1} + \delta \mathbf{e}_{23}$ that satisfies the size constraints as well by using $(\alpha, \beta, \gamma, \delta) = \gamma \cdot (\bar{\alpha}, \bar{\beta}, 1, \bar{\delta})$ with $\gamma > 0$ small enough. When we use the largest value of γ such that the size constraints hold, we obtain the top strategy associated with $(\bar{\alpha}, \bar{\beta}, \bar{\delta})$. The others are mixtures of the top strategy with Repeat. For a strategy with this triple the Press-Dyson Equation (1.19) becomes

$$\bar{\alpha} s_X + \bar{\beta} s_Y + \bar{\delta} v_{23} + 1 \quad = \quad 0. \tag{2.15}$$

The points $(x, y) = (\bar{\alpha}, \bar{\beta})$ lie in the *Strategy Strip*. This consists of the points of the xy plane with $y \leq x$ and that lie on or below the line $x + y = -R^{-1}$ and on or above the line $x + y = -P^{-1}$. Then $\bar{\delta}$ must satisfy $-1 - x \leq \bar{\delta} \leq -1 - y$. Alternatively, we can fix $\bar{\delta}$ to be arbitrary and intersect the Strategy Strip with the fourth quadrant when the origin is at $(-1 - \bar{\delta}, -1 - \bar{\delta})$, i. e. the points with $y \leq -1 - \bar{\delta} \leq x$.

Together with the exceptional strategies those with $(\bar{\alpha}, \bar{\beta})$ on the line $x + y = -R^{-1}$ are exactly the agreeable strategies. Together with the exceptional strategies those on the line $x + y = -P^{-1}$ are exactly the firm strategies.

Let us look at the good ZDS's, i.e. those with $\delta = 0$. In the exceptional case with $\gamma = 0$, the top good strategy is TFT. When $\delta = 0$ and $\gamma > 0$, the good strategies are those that satisfy $\bar{\alpha} + \bar{\beta} = -R^{-1}$ and $\bar{\alpha} > 0$. As mentioned above, these are the complier strategies.

Proposition 2.6 *Given $\bar{\alpha} > 0$, the associated agreeable ZDS top strategy is given by*

$$\mathbf{p} \quad = \quad (1, \frac{2R-1}{R(\bar{\alpha}+1)}, 1, \frac{R-P}{R(\bar{\alpha}+1)}). \tag{2.16}$$

Proof: The agreeable strategy $\mathbf{p}$ with $\gamma, \bar{\alpha} > 0$ and $\bar{\delta} = 0$ has X Press-Dyson vector

$$\tilde{\mathbf{p}} \quad = \quad (0, -\gamma(\bar{\alpha} + R^{-1} - 1), \gamma(\bar{\alpha}+1), \gamma(1 - P \cdot R^{-1})). \tag{2.17}$$

With $\bar{\alpha}$ fixed, the largest value for γ so that the size constraints hold is $(\bar{\alpha}+1)^{-1}$. This easily yields (2.16) for the top strategy. $\Box$

When $\bar{\delta} = 0$, the vertical line $\bar{\alpha} = 0$ intersects the strip in points whose strategies are all the *equalizers*, as discussed by Press and Dyson [15] and by Boerlijst, Nowak and Sigmund [5]. Observe that with $\bar{\delta} = 0$ and $\bar{\alpha} = 0$ the Press-Dyson Equation (2.15) becomes $\bar{\beta} s_Y + 1 = 0$, and so $s_Y = -\bar{\beta}^{-1}$ regardless of the choice of strategy for Y. The agreeable case has $\bar{\beta} = -R^{-1}$. The vertical line of equalizers cuts the line of agreeable strategies, separating it into the unbounded ray with good strategies and the segment with strategies that are not even of Nash type.

Finally, we call a strategy $\mathbf{p}$ *generous* when $p_2 > 0$ and $p_4 > 0$. That is, whenever Y defects there is a positive probability that X will cooperate. The complier strategies given by (2.16) are generous.

Proposition 2.7 *Assume that X plays $\mathbf{p}$, a generous strategy of Nash type. If Y plays strategy $\mathbf{q}$ of Nash type and either (i) $\mathbf{q}$ is generous, or (ii) $q_3 + q_4 > 0$, then $\{cc\}$ is the unique terminal set for the associated Markov matrix $\mathbf{M}$. Thus, $\mathbf{M}$ is convergent.*

Proof: Since $\mathbf{p}$ and $\mathbf{q}$ are both agreeable, $\{cc\}$ is a terminal set for $\mathbf{M}$.

Since $\mathbf{p}$ is of Nash type, it is not *Repeat* and so (2.3) implies that $p_2 < 1$.

For the first case, we prove that if $p_1 = 1, p_2 < 1, p_4 > 0$ and $\mathbf{q}$ satisfies analogous conditions and not both $\mathbf{p}$ and $\mathbf{q}$ are of the form $(1, 0, 1, a)$, then $\mathbf{M}$ is convergent.

Recall that Y responds to cd using q_3 and to dc using q_2.

The assumptions $p_4, q_4 > 0$ imply that there is an edge from dd to cc, and so that dd is transient. There is an edge from dc to dd if $p_3 < 1$ since $q_2 < 1$. If $p_3 = 1$ and $q_2 > 0$, then there is an edge to cc. There remains the case that $p_3 = 1, q_2 = 0$ with the only edge from dc going to cd. Similarly, there is an edge from cd to either dd or cc except when $p_2 = 0, q_3 = 1$. Thus, the only case when $\mathbf{M}$ is not convergent is when both $\mathbf{p}$ and $\mathbf{q}$ are of the form $(1, 0, 1, a)$. In that case, $\{cd, dc\}$ is an additional terminal set. In particular, if either p_2 or q_2 is positive, then $\{cc\}$ is the only terminal set. This completes case (i). It also shows that if $\mathbf{p}$ is generous and $q_4 > 0$, then $\mathbf{M}$ is convergent.

To complete case(ii), we assume that $\mathbf{p}$ is generous and $q_3 > 0$. Since $p_2 > 0$ and $q_3 > 0$, there is an edge from cd to cc and so cd is transient. Since $p_4 > 0$, there is an edge from dd either to cc or to cd and so dd is transient. Finally, $q_2 < 1$ implies there is an edge from dc to cd or to dd. Thus, dc is transient as well. $\square$

This result indicates the advantage which the good strategies that are generous have over the good exceptional strategies like $Grim$ and TFT. The latter are firm as well as agreeable. Playing them against each other yields a nonconvergent matrix with both $\{cc\}$ and $\{dd\}$ as terminal sets. Initial cooperation does lead to immediate fixation at cc, but an error might move the sequence of outcomes on a path leading to another terminal set. When generous good strategies are used against each other, $\{cc\}$ is the unique terminal set. Eventual fixation at cc occurs whatever the initial distribution is, and if an error occurs, then the strategies move the successive outcomes along a path that returns to cc. It is easy to compute the expected number of steps T_z from transient state z to cc.

$$T_z \quad = \quad 1 \ + \ \Sigma_{z'} p_{zz'} T_{z'}, \tag{2.18}$$

where we sum over the three transient states and $p_{zz'}$ is the probability of moving along an edge from z to z'. Thus, with $\mathbf{M'} = \mathbf{M} - I$, we obtain the formula for the vector $\mathbf{T} = (T_2, T_3, T_4)$:

$$\mathbf{M}'_t \cdot \mathbf{T} \quad = \quad -\mathbf{1}. \tag{2.19}$$

where $\mathbf{M}'_t$ is the invertible 3×3 matrix obtained from $\mathbf{M}'$ by omitting the first row and column.

Consider the case when X and Y both use the strategy given by (2.16), so that $\mathbf{p} = \mathbf{q} = (1, p_2, 1, p_4)$. The only edges coming from cd connect with

cc or with dc and similarly for dc. Symmetry will imply that $T_{cd} = T_{dc}$. So with T this common value we obtain from (2.18) $T = 1 + (1 - p_2)T$. Hence, from (2.16) we get

$$T \;=\; T_{cd} \;=\; T_{dc} \quad = \quad \frac{1}{p_2} \;=\; \frac{\bar{\alpha} + 1}{2 - R^{-1}}. \tag{2.20}$$

Thus, the closer the strategy is to the equalizer strategy with $\bar{\alpha} = 0$ the shorter the expected recovery time from an error leading to a dc or cd outcome. From (2.18) one can see that

$$T_{dd} \quad = \quad 1 \;+\; 2p_4(1 - p_4) \cdot T \;+\; (1 - p_4)^2 \cdot T_{dd}. \tag{2.21}$$

We won't examine this further as arriving at dd from cc implies errors on the part of both players.

Of course, one might regard such departures from cooperation not as noise or error but as ploys. Y might try a rare move to cd in order to pick up the temptation payoff for defection as an occasional bonus. But if this is strategy rather than error, it means that Y is departing from the good strategy to one with q_1 a bit less than 1. Corollary 2.2(a) implies that Y loses by executing such a ploy.

3 Competing Zero Determinant Strategies

We now examine the ZDS's in more detail. Recall that a strategy $\mathbf{p}$ is a ZDS when $\delta = 0$ in the Press-Dyson decomposition of the X Press-Dyson vector $\tilde{\mathbf{p}} = \mathbf{p} - \mathbf{e}_{12}$. With the normalization (2.1) the inverse matrix of $(\ \mathbf{S}_X\ \mathbf{S}_Y\ \mathbf{1}\ \mathbf{e}_{23}\ )$ is

$$\frac{-1}{2(R-P)} \begin{pmatrix} -1 & R-P & P-R & 1 \\ -1 & P-R & R-P & 1 \\ 2P & 0 & 0 & -2R \\ 1-2P & P-R & P-R & 2R-1 \end{pmatrix} \tag{3.1}$$

and so if $\tilde{\mathbf{p}} = \alpha\mathbf{S}_X \;+\; \beta\mathbf{S}_Y \;+\; \gamma\mathbf{1} \;+\; \delta\mathbf{e}_{23}$,

$$2(R-P)\delta \;=\; (2P-1)\tilde{p}_1 \;+\; (R-P)(\tilde{p}_2 + \tilde{p}_3) \;-\; (2R-1)\tilde{p}_4. \tag{3.2}$$

Thus, for example, if $R + P = 1$, both *AllD* with $\tilde{\mathbf{p}} = (-1, -1, 0, 0)$ and *AllC* with $\tilde{\mathbf{p}} = (0, 0, 1, 1)$ are ZDS.

The exceptional ZDS's, which have $\gamma = 0$ as well as $\delta = 0$, are mixtures of TFT and *Repeat*. Otherwise, $\gamma > 0$ and we can write $\tilde{\mathbf{p}} = \gamma(\bar{\alpha}\mathbf{S}_X \ + \ \bar{\beta}\mathbf{S}_Y \ + \ \mathbf{1})$. When $(\bar{\alpha}, \bar{\beta})$ lies in the *ZDSstrip* defined by

$$ZDSstrip \quad = \quad \{(x, y) : x \geq -1 \geq y \text{ and } -R^{-1} \geq x + y \geq -P^{-1}\}, \quad (3.3)$$

then the sign constraints are satisfied. The size constraints hold as well when $\gamma > 0$ is small enough. For Z with $P \leq Z \leq R$ the intersection of the ZDSstrip with the line $x + y = -Z^{-1}$ is a *value line* in the strip.

Lemma 3.1 *Assume that* $(\bar{\alpha}, \bar{\beta})$ *in the ZDS strip, with* $\bar{\alpha} + \bar{\beta} = -Z^{-1}$. *We then have* $-\bar{\beta} \geq max(1, |\bar{\alpha}|)$ *and* $-\bar{\beta} = |\bar{\alpha}|$ *iff* $\bar{\alpha} = \bar{\beta} = -1$. *If* $(\bar{a}, \bar{b})$ *is also in the strip then* $D = \bar{\beta}\bar{b} \ - \ \bar{\alpha}\bar{a} \geq 0$ *with equality iff* $\bar{\alpha} = \bar{\beta} = \bar{a} = \bar{b} = -1$.

Proof: By definition of Z, $-\bar{\beta} = \bar{\alpha} + Z^{-1} > \bar{\alpha}$. Also, the sign constraints imply $-\bar{\beta} \geq 1 \geq -\bar{\alpha}$, and so $-\bar{\beta} \geq -\bar{\alpha}$ with equality iff $\bar{\alpha} = \bar{\beta} = -1$. $D \geq (-\bar{\beta})(-\bar{b}) \ - \ |\bar{\alpha}||\bar{a}| \geq 0$ and the inequality is strict unless $\bar{\alpha} = \bar{\beta} = \bar{a} = \bar{b} = -1$. $\square$

Remark: Because $R > \frac{1}{2}$ it is always true that $-R^{-1} > -2$, but $-2 \geq -P^{-1}$ iff $\frac{1}{2} \geq P$. Hence, $(-1, -1)$ is in the ZDSstrip iff $\frac{1}{2} \geq P$.

For a ZDS there are useful ways to rewrite the Press-Dyson equation (2.15).

Proposition 3.2 *Assume that X uses a strategy vector* $\mathbf{p}$ *with X Press-Dyson vector* $\tilde{\mathbf{p}} = \gamma(\bar{\alpha}\mathbf{S}_X \ + \ \bar{\beta}\mathbf{S}_Y \ + \ \mathbf{1})$, $\gamma > 0$. *Let* $-Z^{-1} = \bar{\alpha} + \bar{\beta}$, *so that* $P \leq Z \leq R$.

For any strategy pattern played by Y,

$$\bar{\alpha}Z(s_X \ - \ s_Y) \quad = \quad (s_Y \ - \ Z). \quad (3.4)$$

If $\kappa = \bar{\alpha}Z/(1 + \bar{\alpha}Z)$, *then* $1 > \kappa$ *and* κ *has the same sign as* $\bar{\alpha}$. *For any strategy pattern played by Y,*

$$\kappa(s_X \ - \ Z) \quad = \quad (s_Y \ - \ Z). \quad (3.5)$$

Proof: Notice that $1+\bar{\alpha}Z = -\bar{\beta}Z \geq Z \geq P > 0$. Multiplying (2.15) by Z and substituting for $\bar{\beta}Z$ easily yields (3.4) and then (3.5). $\Box$

If $\bar{\alpha} = 0$, which is the equalizer case, $s_Y = Z$ and s_X is undetermined. When $\bar{\alpha} > 0$, the payoffs s_X and s_Y are on the same side of Z, while they are on opposite sides when $\bar{\alpha} < 0$. To be precise, we have the following.

Corollary 3.3 *Assume that X uses a strategy vector* $\mathbf{p}$ *with X Press-Dyson vector* $\tilde{\mathbf{p}} = \gamma(\bar{\alpha}\mathbf{S}_X + \bar{\beta}\mathbf{S}_Y + \mathbf{1})$, $\gamma > 0$. *Let* $-Z^{-1} = \bar{\alpha} + \bar{\beta}$. *Assume that Y uses an arbitrary strategy pattern.*

(a) If $\bar{\alpha} = 0$ *then* $s_Y = Z$. *If* $\bar{\alpha} \neq 0$ *then the following are equivalent*

(i) $s_Y = s_X$.

(ii) $s_Y = Z$.

(iii) $s_X = Z$.

(b) If $s_Y > s_X$ *then*

$$\begin{cases} \bar{\alpha} > 0, & \Rightarrow \quad Z > s_Y > s_X. \\ \bar{\alpha} = 0, & \Rightarrow \quad Z = s_Y > s_X. \\ \bar{\alpha} < 0, & \Rightarrow \quad s_Y > Z > s_X. \end{cases} \tag{3.6}$$

(c) If $s_X > s_Y$ *then*

$$\begin{cases} \bar{\alpha} > 0, & \Rightarrow \quad s_X > s_Y > Z. \\ \bar{\alpha} = 0, & \Rightarrow \quad s_X > s_Y = Z. \\ \bar{\alpha} < 0, & \Rightarrow \quad s_X > Z > s_Y. \end{cases} \tag{3.7}$$

Proof: (a) If $\bar{\alpha} = 0$ then $s_Y = Z$ by (3.4). If $\bar{\alpha} \neq 0$ then (i) $\Leftrightarrow$ (ii) by (3.4) and (ii) $\Leftrightarrow$ (iii) (3.5).

(b), (c) If $\bar{\alpha} \neq 0$ then by (3.4) $s_Y - Z$ has the same sign as that of $\bar{\alpha}(s_X - s_Y)$. $\Box$

For $Z = R$ (3.5) is (2.12). When $\bar{\alpha} > 0$ these are the complier strategies, the generous, good strategies described in Proposition 2.6.

For $Z = P, \bar{\alpha} > 0$ the strategies are firm. These were considered by Press and Dyson who called them *extortion strategies*. The name comes from the observation that whenever Y chooses a strategy so that her payoff is above P, the bonus beyond P is divided between X and Y in a ratio of $1 : \kappa$. They point out that the best reply against such an extortion play by X is for Y is to play $AllC = (1,1,1,1)$ which gives X a payoff above R. At first glance, it seems hard to escape from this coercive effect. I believe that the answer is for Y to play a generous good strategy like the compliers above. With repeated play, each player receives enough data to estimate statistically the strategy used by the opponent. Y's good strategy represents a credible invitation for X to switch to an agreeable strategy and receive R, or else be locked below R. Hence, it undercuts the threat from X to remain extortionate.

In order to compute what happens when both players use a ZDS, we need to examine the symmetry between the two players. Let $Switch : \mathbb{R}^4 \to \mathbb{R}^4$ be defined by $Switch(x_1, x_2, x_3, x_4) = (x_1, x_3, x_2, x_4)$. Notice that $Switch$ interchanges the vectors $\mathbf{S}_X$ and $\mathbf{S}_Y$. If X uses $\mathbf{p}$ and Y uses $\mathbf{q}$ then recall that the response vectors used to build the Markov matrix $\mathbf{M}$ are $\mathbf{p}$ and $Switch(\mathbf{q})$. Now suppose that the two players exchange strategies so that X uses $\mathbf{q}$ and Y uses $\mathbf{p}$. Then the X response is $\mathbf{q} = Switch(Switch(\mathbf{q}))$ and the Y response is $Switch(\mathbf{p})$. Hence, the new Markov matrix is obtained by transposing both the second and third rows and the second and third columns. It follows that if $\mathbf{v}$ was a stationary vector for $\mathbf{M}$, then $Switch(\mathbf{v})$ is a stationary vector for the new matrix. Hence, Theorem 1.3 applied to the X Press-Dyson vector $\tilde{\mathbf{q}}$ implies that $0 =< Switch(\mathbf{v}) \cdot \tilde{\mathbf{q}} >=< \mathbf{v} \cdot Switch(\tilde{\mathbf{q}}) >$. Furthermore, if $\tilde{\mathbf{q}} = a\mathbf{S}_X + b\mathbf{S}_Y + g\mathbf{1} + \delta\mathbf{e}_{23}$, then $Switch(\tilde{\mathbf{q}}) = b\mathbf{S}_X + a\mathbf{S}_Y + g\mathbf{1} + \delta\mathbf{e}_{23}$.

For a strategy $\mathbf{q}$, we define *Y Press-Dyson vector* $\tilde{\tilde{\mathbf{q}}} = Switch(\tilde{\mathbf{q}}) = Switch(\mathbf{q}) - \mathbf{e}_{13}$, where $\mathbf{e}_{13} = (1,0,1,0)$. For any strategy pattern for X and any limiting distribution $\mathbf{v}$ when Y uses $\mathbf{q}$ we have $< \mathbf{v} \cdot \tilde{\tilde{\mathbf{q}}} >= 0$. The strategy $\mathbf{q}$ is a ZDS associated with $(\bar{a}, \bar{b})$ in the ZDSstrip when $\tilde{\tilde{\mathbf{q}}} = g(\bar{b}\mathbf{S}_X + \bar{a}\mathbf{S}_Y + \mathbf{1})$ with some $g > 0$.

Now we compute what happens when X and Y use ZDS strategies associated, respectively, with points $(\bar{\alpha}, \bar{\beta})$ and $(\bar{a}, \bar{b})$ in the ZDS strip. This means that for some $\gamma > 0, g > 0$, $\tilde{\mathbf{p}} = \gamma(\bar{\alpha}S_X + \bar{\beta}S_Y + \mathbf{1})$ and $\tilde{\tilde{\mathbf{q}}} = g(\bar{b}S_X + \bar{a}S_Y + \mathbf{1})$. We obtain two Press-Dyson equations which hold simultaneously

$$\begin{array}{rcl} \bar{\alpha}s_X \ + \ \bar{\beta}s_Y & = & -1, \\ \bar{b}s_X \ + \ \bar{a}s_Y & = & -1. \end{array} \tag{3.8}$$

If $\bar{\alpha} = \bar{\beta} = \bar{a} = \bar{b} = -1$, which we will call the *Vertex strategy*, then the two equations are the same. In that case, since the two players use the same strategy, $s_X = s_Y$. Then the single equation of (3.8) yields $s_X = s_Y = \frac{1}{2}$. Recall that the Vertex strategy is only defined when $P \leq \frac{1}{2}$.

Otherwise, Lemma 3.1 implies that the determinant $D = \bar{\beta}\bar{b} - \bar{\alpha}\bar{a}$ is positive and by Cramer's Rule we get

$$\begin{aligned} s_X = D^{-1}(\bar{a} - \bar{\beta}), \quad s_Y = D^{-1}(\bar{\alpha} - \bar{b}), \\ \text{and so} \qquad s_Y - s_X = D^{-1}[(\bar{\alpha} + \bar{\beta}) - (\bar{a} + \bar{b})]. \end{aligned} \tag{3.9}$$

Notice that s_X and s_Y are independent of γ and g.

Proposition 3.4 *Assume that* $\tilde{\mathbf{p}} = \gamma(\bar{\alpha}S_X + \bar{\beta}S_Y + \mathbf{1})$ *and* $\tilde{\tilde{\mathbf{q}}} = g(\bar{b}S_X + \bar{a}S_Y + \mathbf{1})$. *Let* $\bar{\alpha} + \bar{\beta} = -Z_X^{-1}$ *and* $\bar{a} + \bar{b} = -Z_Y^{-1}$.

(a) The points $(\bar{\alpha}, \bar{\beta}), (\bar{a}, \bar{b})$ *lie on the same value line* $x + y = -Z^{-1}$, *i.e.* $Z_X = Z_Y$, *iff* $s_X = s_Y$. *In that case,* $Z_X = s_X = s_Y = Z_Y$.

(b) $s_Y > s_X$ *iff* $Z_X > Z_Y$.

(c) Assume $Z_X > Z_Y$. *The following implications hold.*

$$\begin{cases} \bar{\alpha} > 0, & \Rightarrow \quad Z_X > s_Y > s_X. \\ \bar{\alpha} = 0, & \Rightarrow \quad Z_X = s_Y > s_X. \\ \bar{\alpha} < 0, & \Rightarrow \quad s_Y > Z_X > s_X. \end{cases}$$
$$\begin{cases} \bar{a} > 0, & \Rightarrow \quad s_Y > s_X > Z_Y. \\ \bar{a} = 0, & \Rightarrow \quad s_Y > s_X = Z_Y. \\ \bar{a} < 0, & \Rightarrow \quad s_Y > Z_Y > s_X. \end{cases} \tag{3.10}$$

Proof: If both players use Vertex, then $Z_X = s_X = s_Y = Z_Y = \frac{1}{2}$. Now we can exclude this case and so assume $D > 0$.

(a) Assume $Z_X = Z_Y$. From (3.9) we see that $s_Y - s_X = 0$.

When $s_X = s_Y$ Corollary 3.3(a) implies that $s_X = s_Y = Z_X$. By using the XY symmetry we see that the common value is Z_Y as well. Hence, $Z_X = Z_Y$ and the points lie on the same line.

(b) Since $D > 0$, (b) follows from (3.9).

(c) From (b), $s_Y - s_X > 0$. The first part follows from (3.6) with $Z = Z_X$. The second follows from (3.7) by using the XY symmetry with $\bar{\alpha}, \bar{\beta}, Z$ replaced by $\bar{a}, \bar{b}, Z_Y$. $\square$

4 Dynamics Among Zero Determinant Strategies

In this section we move beyond the classical question which motivated our original interest in good strategies. We consider now the evolutionary dynamics among memory one strategies. We follow Hofbauer and Sigmund [9] Chapter 9 and Akin [2].

The dynamics that we consider takes place in the context of a symmetric two-person game, but generalizing our initial description, we merely assume that there is a set of strategies indexed by a finite set $\mathcal{I}$. When players X and Y use strategies with index $i, j \in \mathcal{I}$, respectively, then the payoff to player X is given by A_{ij} and the payoff to Y is A_{ji}. Thus, the game is described by the payoff matrix $\{A_{ij}\}$. We imagine a population of players each using a particular strategy for each encounter and let π_i denote the ratio of the number of i players to the total population. The frequency vector $\{\pi_i\}$ lives in the unit simplex $\Delta \subset \mathbb{R}^{\mathcal{I}}$, i.e. the entries are nonnegative and sum to 1. The vertex $v(i)$ associated with $i \in \mathcal{I}$ corresponds to a population consisting entirely of i players. We assume the population is large so that we can regard π as changing continuously in time.

Now we regard the payoff in units of *fitness*. That is, when an i player meets a j player in an interval of time dt, the payoff A_{ij} is an addition to the background reproductive rate ρ of the members of the population. So the i player is replaced by $1 + (\rho + A_{ij})dt$ i players. Averaging over the current population distribution, the expected relative reproductive rate for the subpopulation of i players is $\rho + A_{i\pi}$, where

$$\begin{aligned} A_{i\pi} &= \Sigma_{j \in \mathcal{I}} \, \pi_j A_{ij} \qquad \text{and} \\ A_{\pi\pi} = \Sigma_{i \in \mathcal{I}} \, \pi_i A_{i\pi} &= \Sigma_{i,j \,\in\, \mathcal{I}} \, \pi_i \pi_j A_{ij}. \end{aligned} \tag{4.1}$$

The resulting dynamical system on Δ is given by the *Taylor-Jonker Game*

Dynamics Equations introduced in Taylor and Jonker [18].

$$\frac{d\pi_i}{dt} \quad = \quad \pi_i(A_{i\pi} \ - \ A_{\pi\pi}). \tag{4.2}$$

This system is an example of the *replicator equations* studied in great detail in Hofbauer and Sigmund [9].

We will need some general game dynamic results for later application. Fix the game matrix $\{A_{ij}\}$.

A subset A of Δ is called *invariant* if $\pi(0) \in A$ implies that the entire solution path lies in A. That is, $\pi(t) \in A$ for all $t \in \mathbb{R}$. An invariant point is is an *equilibrium*.

Each nonempty subset $\mathcal{J}$ of $\mathcal{I}$ determines the *face* $\Delta_{\mathcal{J}}$ of the simplex consisting of those $\pi \in \Delta$ such that $\pi_i = 0$ for all $i \not\in \mathcal{J}$. Each face of the simplex is invariant because $\pi_i = 0$ implies that $\frac{d\pi_i}{dt} = 0$. In particular, for each $i \in \mathcal{I}$ the vertex $v(i)$, which represents fixation at the i strategy, is an equilibrium. In general, π is an equilibrium when, for all $i, j \in \mathcal{I}$, $\pi_i, \pi_j > 0$ imply $A_{i\pi} = A_{j\pi}$. This implies that $A_{i\pi} = A_{\pi\pi}$ for all i such that $\pi_i > 0$. That is, for all i in the *support* of π.

An important example of an invariant set is the *omega limit point set of an orbit.* Given an initial point $\pi \in \Delta$ with associated solution path $\pi(t)$, it is defined by intersecting the closures of the tail values.

$$\omega(\pi) \quad = \quad \bigcap_{t>0} \overline{\{\pi(s) : s \geq t\}}. \tag{4.3}$$

By compactness this set is nonempty. A point is in $\omega(\pi)$ iff it is the limit of some sequence $\{\pi(t_n)\}$ with $\{t_n\}$ tending to infinity. The set $\omega(\pi)$ consists of a single point π^* iff $Lim_{t \to \infty} \pi(t) = \pi^*$. In that case, $\{\pi^*\}$ is an invariant point, i. e. an equilibrium.

Definition 4.1 *We call a strategy i^* an* evolutionarily stable strategy *(hereafter, an ESS) when*

$$A_{ji^*} \ < \ A_{i^*i^*} \qquad \textit{for all} \ \ j \neq i^* \quad \textit{in} \ \ \mathcal{I}. \tag{4.4}$$

We call a strategy i^* an evolutionarily unstable strategy *(hereafter, an EUS) when*

$$A_{ji^*} \ > \ A_{i^*i^*} \qquad \textit{for all} \ \ j \neq i^* \quad \textit{in} \ \ \mathcal{I}. \tag{4.5}$$

The ESS condition above is really a special case of a more general notion, see page 63 of [9], and is referred to there as a *strict Nash equilibrium*. We will not need the generalization and we use the term to avoid confusion with the strategies of Nash type considered in the previous sections.

Proposition 4.2 *If i^* is an ESS then the vertex $v(i^*)$ is an attractor, i.e. a locally stable equilibrium, for the system (4.2). In fact, there exists $\epsilon > 0$ such that*

$$1 \ > \ \pi_{i^*} \ \geq \ 1 - \epsilon \qquad \Longrightarrow \qquad \frac{d\pi_{i^*}}{dt} \ > \ 0. \tag{4.6}$$

Thus, near the equilibrium $v(i^*)$, which is characterized by $\pi_{i^*} = 1$, $\pi_{i^*}(t)$ increases monotonically, converging to 1 and the alternative strategies are eliminated from the population in the limit.

If i^* is an EUS then the vertex $v(i^*)$ is a repellor, i.e. a locally unstable equilibrium, for the system (4.2). In fact, there exists $\epsilon > 0$ such that

$$1 \ > \ \pi_{i^*} \ \geq \ 1 - \epsilon \qquad \Longrightarrow \qquad \frac{d\pi_{i^*}}{dt} \ < \ 0. \tag{4.7}$$

Thus, near the equilibrium $v(i^*)$ $\pi_{i^*}(t)$ decreases monotonically, until the system enters, and then remains in, the region where $\pi_{i^*} < 1 - \epsilon$.

Proof: When i^* is an ESS, $A_{i^*i^*} > A_{ji^*}$ for all $j \neq i^*$. It then follows for $\epsilon > 0$ sufficiently small that $p_{i^*} \geq 1 - \epsilon$ implies $A_{i^*\pi} > A_{j\pi}$ for all $j \neq i^*$. If also $1 > p_{i^*}$, then $A_{i^*\pi} > A_{\pi\pi}$. So (4.2) implies (4.6).

The EUS case is similar. Notice that no solution path can cross $\Delta \cap \{\pi_{i^*} = 1 - \epsilon\}$ from $\{\pi_{i^*} < 1 - \epsilon\}$. $\square$

Definition 4.3 *For $\mathfrak{J}$ a nonempty subset of $\mathfrak{I}$ we say a strategy i* weakly dominates *a strategy j in $\mathfrak{J}$ when $i, j \in \mathfrak{J}$ and*

$$A_{jk} \ \leq \ A_{ik} \qquad \text{for all} \ \ k \in \ \mathfrak{J}, \tag{4.8}$$

and the inequality is strict either for $k = i$ or $k = j$. If the inequalities are strict for all k then we say that i dominates *j in $\mathfrak{J}$.*

We say that $i \in \mathfrak{J}$ dominates a sequence $\{j_1, ..., j_n\}$ in $\mathfrak{J}$ when i dominates j_1 in $\mathfrak{J}$ and for $p = 2, ..., n$, i dominates j_p in $\mathfrak{J} \setminus \{j_1, ..., j_{p-1}\}$.

When $\mathfrak{J}$ equals all of $\mathfrak{I}$ we will omit the phrase "in $\mathfrak{J}$".

For $i, j \in \mathcal{I}$, define the set Q_{ij} and on it the real valued function L_{ij} by

$$\begin{aligned} Q_{ij} &= \{\pi \in \Delta : \pi_i, \pi_j > 0\} \\ L_{ij}(\pi) &= \ln(\pi_i) - \ln(\pi_j). \end{aligned} \tag{4.9}$$

Lemma 4.4 *(a) If i weakly dominates j then $dL_{ij}/dt \ > \ 0$ on the set Q_{ij}.*

(b) If i dominates j in $\mathcal{J}$ then there exists $\epsilon > 0$ such that $dL_{ij}/dt \ > \ 0$ on the set $Q_{ij} \cap \{\pi \in \Delta : \Sigma_{k \notin \mathcal{J}} \ \pi_k \ \leq \ \epsilon\}$.

Proof: Observe that

$$dL_{ij}/dt \ = \ A_{i\pi} - A_{j\pi} \ = \ \Sigma_{k \in \ \mathcal{I}} \pi_k (A_{ik} - A_{jk}) \tag{4.10}$$

(a) Since $\pi_i, \pi_j > 0$ in Q_{ij} and $A_{ik} - A_{jk} \geq 0$ for all k with strict inequality for $k = i$ or $k = j$, it follows that the derivative is positive.

(b) Define

$$\begin{aligned} m &= min\{A_{ik} - A_{jk} : k \in \mathcal{J}\} \ > \ 0, \\ M &= max\{|A_{ik} - A_{jk}| : k \notin \mathcal{J}\}, \\ \pi_{\mathcal{J}} &= \Sigma_{k \in \ \mathcal{J}} \pi_k, \\ \pi_{k|\mathcal{J}} &= \pi_k / \pi_{\mathcal{J}} \quad \text{for } k \in \mathcal{J}. \end{aligned} \tag{4.11}$$

Observe that $\Sigma_{k \notin \ \mathcal{J}} \pi_k = 1 - \pi_{\mathcal{J}}$.

For any $\pi \in Q_{ij}$

$$\begin{aligned} A_{i\pi} - A_{j\pi} \ &= \ \pi_{\mathcal{J}} \Sigma_{k \in \ \mathcal{J}} \ \pi_{k|\mathcal{J}} (A_{ik} - A_{jk}) \ + \ \Sigma_{k \notin \ \mathcal{J}} \pi_k (A_{ik} - A_{jk}) \\ &\geq \ \ \pi_{\mathcal{J}} m \ - \ (1 - \pi_{\mathcal{J}}) M. \end{aligned} \tag{4.12}$$

So if ϵ is chosen with $0 < \epsilon < m/(m + M)$ then $A_{i\pi} - A_{j\pi} > 0$ when $\pi \in Q_{ij} \cap \{\pi \in \Delta : (1 - \pi_{\mathcal{J}}) \ \leq \ \epsilon\}$. $\square$

Lemma 4.5 *If $\pi(t)$ is a solution path with $\pi(0) \in Q_{ij}$ and there exists $T \in \mathbb{R}$ such that $dL_{ij}/dt \ > \ 0$ on the set $Q_{ij} \cap \overline{\{\pi(t) : t \geq T\}}$, then*

$$Lim_{t \to \infty} \ \pi_j(t) \quad = \quad 0. \tag{4.13}$$

Proof: By assumption, $L_{ij}(\pi(t))$ is a strictly increasing function of t for $t \geq T$. Thus, as a t tends to infinity $L_{ij}(\pi(t))$ approaches $\ell = sup\{L_{ij}(\pi(t)) : t \geq T\}$ with $L_{ij}(\pi(T)) < \ell \leq +\infty$.

We must prove that $\pi_j = 0$ on the omega limit set. Assume instead that $\pi^* \in \omega(\pi(0))$ with $\pi_j^* > 0$. If π_i^* were 0 then $L_{ij}(\pi(t))$ would not be bounded below on $\{\pi(t) : t \geq T\}$. Hence, π^* lies in Q_{ij} with $\ell = L_{ij}(\pi^*) < \infty$. So on the invariant set $\omega(\pi(0)) \cap Q_{ij}$, which contains π^* and so is nonempty, L_{ij} would be constantly $\ell < \infty$. Since this set is invariant, dL_{ij}/dt would equal zero. This contradicts our assumption that the derivative is positive on $\omega(\pi(0)) \cap Q_{ij}$. $\square$

Proposition 4.6 *For $i \in \mathcal{I}$, let $\pi(t)$ be a solution path with $\pi_i(0) > 0$*

(a) If i weakly dominates j then $Lim_{t\to\infty}\ \pi_j(t) \ = \ 0$.

(b) If i dominates the sequence $\{j_1, ..., j_n\}$ then for $j = j_1, ..., j_n$, $Lim_{t\to\infty}\ \pi_j(t) \ = \ 0$.

Proof: (a) If $\pi_j(0) = 0$, then $\pi_j(t) = 0$ for all t and so the limit is 0. Hence, we may assume $\pi_j(0) > 0$ and so that $\pi(0) \in Q_{ij}$. By Lemma 4.4 (a), $dL_{ij}/dt \ > \ 0$ on Q_{ij} and so Lemma 4.5 implies $Lim_{t\to\infty}\ \pi_j(t) \ = \ 0$.

(b) We prove the result by induction on n.

By part (a) $Lim_{t\to\infty}\ \pi_j(t) \ = \ 0$. for $j = j_1$.

Now assume the limit result is true for $j = j_1, ..., j_{p-1}$ with $1 < p \leq n$. We prove the result for $j = j_p$.

Let $\mathcal{J} = \mathcal{I} \setminus \{j_1, ..., j_{p-1}\}$. By assumption, i dominates j_p in $\mathcal{J}$. Hence, with $j = j_p$ Lemma 4.4 (b) implies there exists $\epsilon > 0$ such that $dL_{ij}/dt \ > \ 0$ on the set $Q_{ij} \cap \{\pi \in \Delta : \Sigma_{k \not\in \mathcal{J}} \pi_k \leq \epsilon\}$.

By induction hypothesis, there exists T such that $\Sigma_{k \not\in\, \mathcal{J}} \pi_k(t) \leq \epsilon$ for all $t \geq T$. Hence, $\overline{\{\pi(t) : t \geq T\}} \subset \{\pi : \Sigma_{k \not\in \mathcal{J}} \pi_k(t) \leq \epsilon\}$.

As in part (a), we can assume $\pi \in Q_{ij}$ and then apply Lemma 4.5 to conclude $Lim_{t\to\infty}\ \pi_j(t) \ = \ 0$. This completes the inductive step. $\square$

Now we specialize the iterated Prisoner's Dilemma. By a *strategy* we will mean a memory-one strategy vector $\mathbf{p}$ together with an initial play, pure or mixed. To apply the Taylor-Jonker dynamics to our case, we suppose that $\mathcal{I}$ indexes a finite collection of strategies. We then use

$$A_{ij} \quad = \quad s_X \qquad \text{so that} \qquad A_{ji} \quad = \quad s_Y. \tag{4.14}$$

That is, when the X player uses the i strategy and the Y player uses the j strategy then the players receive the payoffs s_X and s_Y, respectively, as additions to their reproductive rate. When the associated Markov matrix is convergent, there is a unique terminal set, and the long term payoffs, s_X, s_Y depend only on the strategy vectors and not on the initial plays.

Theorem 4.7 *Let $\mathfrak{I}$ index a finite set of strategies for the iterated Prisoner's Dilemma. Suppose that associated with $i^* \in \mathfrak{I}$ is a good strategy $\mathbf{p}^{i*}$ together with initial cooperation. If for no other $j \in \mathfrak{I}$ is the strategy $\mathbf{p}^j$ agreeable, then i^* is an ESS for the associated game $\{A_{ij} : i, j \in \mathfrak{I}\}$ and so the vertex $v(i^*)$ is an attractor for the dynamic.*

Proof: Since i^* is associated with an agreeable strategy and initial cooperation, $A_{i^*i^*} = R$. Since $\mathbf{p}^{i^*}$ is good and $\mathbf{p}^j$ is not agreeable for $j \neq i^*$, it follows from Corollary 2.2(a) that $A_{ji^*} < R$ for $j \neq i^*$. Thus, i^* is an ESS. $\square$

There are other cases of ESS which are far from good.

Lemma 4.8 *(a) Assume that X uses a strategy $\mathbf{p} = (p_1, p_2, 0, 0)$ with $p_1, p_2 < 1$. If Y uses any strategy $\mathbf{q}$ which is not firm, then*

$$s_Y \ < \ P \ < s_X. \tag{4.15}$$

(b) Assume that X uses a strategy $\mathbf{p} = (1, p_2, 0, 0)$ with $p_2 < 1$. If Y uses any strategy $\mathbf{q}$ which is neither firm nor agreeable, then (4.15) holds.

(c) Assume $P < \frac{1}{2}$ and that X uses a firm, non-exceptional ZDS with $\bar{\alpha} < 0$. If Y uses any strategy $\mathbf{q}$ which is not firm, then (4.15) holds.

Proof:(a) and (b): Since $p_3 = p_4 = 0$ the set $\{dc, dd\}$ is closed. If $q_4 > 0$ then $\{dd\}$ is not closed and so is not a terminal set.

(a): Since $p_2 < 1$ there is an edge from cd to either dc or dd. Hence, cd is transient. Similarly, $p_1 < 1$ implies cc is transient. Hence, for any stationary distribution $\mathbf{v}$, $v_1 = v_2 = 0$. Since $\mathbf{q}$ is not firm, $q_4 > 0$ and so $v_4 < 1$. Hence, $s_Y = v_4 P < P$ and $s_X = v_3 + v_4 P = (1 - v_4) + v_4 P > P$.

(b): As before $p_2 < 1$ implies that cd is transient. Now $\mathbf{q}$ is not agreeable and so $q_1 < 1$. This implies there is an edge from cc to the transient state cd and so cc is transient. The proof is completed as in (a).

(c): Because $P < \frac{1}{2}$, the smallest entry in $\frac{1}{2}(\mathbf{S}_X + \mathbf{S}_Y)$ is P and so $\frac{1}{2}(s_X + s_Y) \leq P$ can only happen when $v_4 = 1$ which implies $s_X = s_Y = P$. This requires that Y play a firm strategy so that $\{dd\}$ is a terminal set. Compare Proposition 1.1.

From (3.5) we see that with $\bar{\alpha} + \bar{\beta} = -Z^{-1}$ and $\kappa = \bar{\alpha}Z/(1 + \bar{\alpha}Z)$

$$\frac{1}{2}(1+\kappa)(s_X \ - \ Z) \quad = \quad (\frac{1}{2}(s_X + s_Y) \ - \ Z). \tag{4.16}$$

When $Z = P$, $P < \frac{1}{2}$ and $-1 \geq \bar{\alpha}$ imply that $(1+\kappa) = (1+2\bar{\alpha}P)/(1+\bar{\alpha}P) > 0$. Hence, $s_X \leq P$ implies $\frac{1}{2}(s_X + s_Y) \leq P$. Since the Y strategy is not firm, this does not happen. Hence, $s_X > P$. Since $\kappa < 0$, (3.5) implies that $s_Y < P$. $\square$

Theorem 4.9 *Let $\mathcal{I}$ index a finite set of strategies for the IPD.*

(a) Suppose that associated with $i^* \in \mathcal{I}$ is a strategy $\mathbf{p}^{i*} = (p_1, p_2, 0, 0)$ with $p_1, p_2 < 1$ together with any initial play. If for no other $j \in \mathcal{I}$ is the strategy $\mathbf{p}^j$ firm, then i^* is an ESS for the associated game $\{A_{ij} : i, j \in \mathcal{I}\}$.

(b) Suppose that associated with $i^* \in \mathcal{I}$ is a strategy $\mathbf{p}^{i*} = (1, p_2, 0, 0)$ with $p_2 < 1$ together with any initial play. If for no other $j \in \mathcal{I}$ is the strategy $\mathbf{p}^j$ either agreeable or firm, then i^* is an ESS for the associated game $\{A_{ij} : i, j \in \mathcal{I}\}$.

(c) Assume that $P < \frac{1}{2}$. Suppose that associated with $i^* \in \mathcal{I}$ is a firm, non-exceptional ZDS with $\bar{\alpha} < 0$ together with any initial play. If for no other $j \in \mathcal{I}$ is the strategy $\mathbf{p}^j$ firm, then i^* is an ESS for the associated game $\{A_{ij} : i, j \in \mathcal{I}\}$.

Proof: (a) If both players use p^{i^*} then there is an edge from dc to dd and so dc, cd and cc are all transient. Thus, $\{dd\}$ is the unique terminal set and so $A_{i^*i^*} = P$ regardless of the initial plays. By Lemma 4.8(a) $A_{ji^*} < P$ for all $j \neq i^*$.

(b) If both players use p^{i^*} then there are edges from cd to dd and from dc to dd. The two terminal sets are $\{cc\}$ and $\{dd\}$. Hence, $R \geq A_{i^*i^*} \geq P$. This time Lemma 4.8(b) implies $A_{ji^*} < P$ for any $j \neq i^*$.

(c) $A_{i^*i^*} = Z_{i^*} = P$ since the i^* strategy is firm. Lemma 4.8(c) implies $A_{ji^*} < P$ for any $j \neq i^*$. $\square$

Thus, $p^{i^*} = AllD = (0, 0, 0, 0)$ with any initial play is an ESS when played against strategies which are not firm. If $p^{i^*} = Grim = (1, 0, 0, 0)$ then with any initial play i^* is an ESS when played against strategies which are neither agreeable nor firm.

At the other extreme we have the following.

Theorem 4.10 *Let $\mathcal{I}$ index a finite set of strategies for the iterated Prisoners' Dilemma. Assume that $P < \frac{1}{2}$. Suppose that associated with $i^* \in \mathcal{I}$ is an extortionate strategy $\mathbf{p}^{i*}$ together with initial defection. That is, $\mathbf{p}^{i^*}$ is a firm ZDS with $\bar{\alpha} > 0$. If for no other $j \in \mathcal{I}$ is the strategy $\mathbf{p}^j$ firm, then i^* is an EUS for the associated game $\{A_{ij} : i, j \in \mathcal{I}\}$ and so the vertex $v(i^*)$ is a repellor for the dynamic.*

Proof: Because $P < \frac{1}{2}$, the smallest entry in $\frac{1}{2}(\mathbf{S}_X + \mathbf{S}_Y)$ is P and so $s_X, s_Y \leq P$ implies $s_X = s_Y = P$ and this can only happen when $v_4 = 1$ which requires that Y play a firm strategy so that $\{dd\}$ is a terminal set. Compare Proposition 1.1.

Since the i^* strategy is firm with initial defection, $A_{i^*i^*} = P$.

If Y plays any strategy which is not firm then (3.5) with $z = P$ and $\bar{\alpha} > 0$ shows that if $s_Y \leq P$ then $s_X \leq P$ as well. Because $P < \frac{1}{2}$ this can only happen when $s_X = s_Y = P$ and $v_4 = 1$. But the Y strategy is not firm. It follows that $s_Y > P$. Thus, for any $j \neq i^*$, $A_{ji^*} > A_{i^*i^*}$. This says that strategy i^* is an EUS. $\square$

We specialize to the case when all the strategies indexed by $\mathcal{I}$ are ZDS's with the exceptional strategies excluded. We can thus regard $\mathcal{I}$ as listing a finite set of points $(\bar{\alpha}_i, \bar{\beta}_i)$ in the ZDSstrip. We define $Z_i = -(\bar{\alpha}_i + \bar{\beta}_i)^{-1}$. That is, the point $(\bar{\alpha}_i, \bar{\beta}_i)$ lies on the value line $x + y = -(Z_i)^{-1}$.

X uses $\mathbf{p}$ associated with $(\bar{\alpha}_i, \bar{\beta}_i)$ when $\tilde{\mathbf{p}} = \gamma_i(\bar{\alpha}_i \mathbf{S}_X + \bar{\beta}_i \mathbf{S}_Y + \mathbf{1})$ and Y uses $\mathbf{q}$ associated with $(\bar{\alpha}_j, \bar{\beta}_j)$ when $\tilde{\mathbf{q}} = \gamma_j(\bar{\beta}_j \mathbf{S}_X + \bar{\alpha}_j \mathbf{S}_Y + \mathbf{1})$ for some $\gamma_i, \gamma_j > 0$. Notice the XY switch.

If both players use a Vertex strategy with $(\bar{\alpha}_i, \bar{\beta}_i) = (-1, -1)$ then $A_{ii} = \frac{1}{2} = Z_i$. Recall that $(-1, -1)$ lies in the ZDSstrip iff $P \leq \frac{1}{2}$. Otherwise, we apply (3.8) with $(\bar{\alpha}, \bar{\beta}) = (\bar{\alpha}_i, \bar{\beta}_i)$ and $(\bar{a}, \bar{b}) = (\bar{\alpha}_j, \bar{\beta}_j)$. Then from (3.9) we get

$$\begin{array}{rcl} A_{ij} = s_X & = & K_{ij}(\bar{\alpha}_j - \bar{\beta}_i) \\ \text{with} \quad K_{ij} = K_{ji} & = & (\bar{\beta}_i \bar{\beta}_j - \bar{\alpha}_i \bar{\alpha}_j)^{-1} > 0. \end{array} \tag{4.17}$$

Note that the payoffs are independent of the choice of γ_i, γ_j.

By Proposition 3.4(a)

$$A_{ii} \quad = \quad Z_i \qquad \text{for all } i \in \mathcal{I}. \tag{4.18}$$

We begin with some degenerate cases.

First, if all of the points $(\bar{\alpha}_i, \bar{\beta}_i)$ lie on the same value line $x + y = -Z^{-1}$, i.e. all the Z_i's are equal, then by Proposition 3.4 (a) $A_{ij} = Z$ for all i, j and so $\frac{d\pi}{dt} = 0$ and every population distribution is an equilibrium. In general, if for two strategies i, j $A_{ij} = A_{ji} = Z$ then by Proposition 3.4(a) both points lie on $x + y = -Z^{-1}$ and it follows that $A_{ii} = A_{jj} = Z$ as well. In general, if $\mathcal{I}_Z = \{i : Z_i = Z\}$ contains more than one $i \in I$ then the dynamics is degenerate on the face $\Delta_{\mathcal{I}_Z}$ of the simplex.

Second, if all of the points satisfy $\bar{\alpha}_i = 0$ then all the strategies are equalizer strategies. In this case the payoff matrix need not be constant but A_{ij} depends only on j. This implies that for all i $A_{i\pi} = A_{\pi\pi}$ and so again $\frac{d\pi}{dt} = 0$ and every population distribution is an equilibrium.

We will now see that the line $\bar{\alpha} = 0$ separates different interesting dynamic behaviors.

Theorem 4.11 *Let $\mathcal{I}$ index a set of non-exceptional ZDS strategies. Thus, each $i \in \mathcal{I}$ is associated with a point $(\bar{\alpha}_i, \bar{\beta}_i)$ in the ZDS strip and $\bar{\alpha}_i + \bar{\beta}_i = -(Z_i)^{-1}$.*

Assume either

Case (+): $\bar{\alpha}_i > 0$ for all $i \in \mathcal{I}$ and for some $i^* \in \mathcal{I}$, $Z_{i^*} > Z_j$ for all $j \neq i^*$;

or

Case (-): $\bar{\alpha}_i < 0$ for all $i \in \mathcal{I}$ and for some $i^* \in \mathcal{I}$, $Z_{i^*} < Z_j$ for all $j \neq i^*$.

The strategy i^* is an ESS and if $\pi_{i^*}(0) > 0$ then the solution path converges to the vertex $v(i^*)$.

Proof: List the strategies $j_1, ..., j_n$ of $\mathcal{I} \setminus \{i^*\}$ so that in Case(+) $Z_{j_1} \leq Z_{j_2} \leq ... \leq Z_{j_n} < Z_{i^*}$ and in Case(-) $Z_{j_1} \geq Z_{j_2} \geq ... \geq Z_{j_n} > Z_{i^*}$. For both cases we apply Proposition 3.4. It first implies that if $Z_i = Z_j$ then

$$A_{ii} \ = \ Z_i \ = \ A_{ji} \ = \ A_{ij} \ = \ Z_j \ = \ A_{jj}. \tag{4.19}$$

Case(+) If $Z_i > Z_j$, then, because $\bar{\alpha}_i, \bar{\alpha}_j > 0$, Proposition 3.4 implies that

$$A_{ii} \ = \ Z_i \ > \ A_{ji} \ > \ A_{ij} \ > \ Z_j \ = \ A_{jj}. \tag{4.20}$$

Hence, if $Z_i > Z_k \geq Z_j$ then $A_{ii} > A_{ji}$ and $A_{ik} > A_{kk} \geq A_{jk}$.

It follows that i^* dominates the sequence $\{j_1, ..., j_n\}$. Hence, Proposition 4.6 (b) implies that $Lim_{t \to \infty} \pi_j(t) = 0$ for $j = j_1, ..., j_n$ when $\pi_{i^*}(0) > 0$. Consequently, $\pi_{i^*}(t) = 1 - \Sigma_{p=1}^n \pi_{j_p}(t)$ tends to 1. That is, $\pi(t)$ converges to $v(i^*)$.

Case(-) If $Z_i < Z_j$, then, because $\bar{\alpha}_i, \bar{\alpha}_j < 0$, Proposition 3.4 implies that

$$A_{ij} \ > \ Z_j \ = \ A_{jj} > \ A_{ii} \ = \ Z_i \ > \ A_{ji}. \tag{4.21}$$

It again follows that i^* dominates the sequence $\{j_1, ..., j_n\}$ and convergence to $v(i^*)$ again follows from Proposition 3.4.

In both cases, it is clear that i^* is an ESS. $\Box$

Thus, when only $\bar{\alpha} > 0$ ZDS strategies are competing with one another, the ones on the highest value line win. Among $\bar{\alpha} < 0$ ZDS strategies the ones on the lowest value line win.

The local stability of an ESS good strategy will not be global when both signs occur. To illustrate this, consider the case of two strategies indexed by $\mathcal{I} = \{1, 2\}$. Letting $w = \pi_1$, it is an easy exercise to show that (4.2) reduces to

$$\frac{dw}{dt} \quad = \quad w(1-w)[(A_{11} - A_{21})w + (A_{12} - A_{22})(1-w)]. \tag{4.22}$$

Proposition 4.12 *Assume that $Z_1 > Z_2$ and that $\bar{\alpha}_1 \cdot \bar{\alpha}_2 < 0$. There is an equilibrium population $\pi^* = (w^*, (1-w^*))$ which contains both strategies. with*

$$w^*/(1-w^*) \quad = \quad (A_{22} - A_{12})/(A_{11} - A_{21}). \tag{4.23}$$

This equilibrium is stable if $\bar{\alpha}_1 < 0$ and is unstable if $\bar{\alpha}_1 > 0$.

Proof: If $\bar{\alpha}_1 < 0$ and $\bar{\alpha}_2 > 0$ then Proposition 3.4 implies that $A_{11} - A_{21} = Z_1 - A_{21} < 0$ and $A_{12} - A_{22} = A_{12} - Z_2 > 0$. Reversing the signs reverses the inequalities. The result then easily follows from equation (4.22). Just graph the linear function of w in the brackets and observe where the result is positive or negative. $\Box$

Question 4.13 *Suppose we restrict to the case where $\mathcal{I}$ indexes ZDS's lying on different value lines to avoid degeneracies. We ask:*

- How large a population can coexist? If N is the size of $\mathcal{I}$, the number of competing strategies, then for what N do there exist examples with an interior equilibrium, that is, an equilibrium π such that $\pi_i > 0$ for all $i \in \mathcal{I}$? When is there a locally stable interior equilibrium? For how large an N can *permanence* occur (see [9] Section 3), that is, the boundary of Δ be a repellor? The Brouwer Fixed Point Theorem implies that such a permanent system always admits an interior equilibrium. When an interior equilibrium does not exist there is always some sort of dominance among the mixed strategies of the game $\{A_{ij}\}$. See [1] and [3].

- Can there exist a stable, closed invariant set containing no equilibria, e.g. a stable limit cycle?

There is alternative version of the dynamics which explicitly considers for X not the payoff s_X but the advantage that X has over Y. That is, the addition to the growth rate is given not by s_X but by the difference $s_X - s_Y$. This amounts to replacing A_{ij} by the anti-symmetric matrix $S_{ij} = A_{ij} - A_{ji}$ so that the game becomes zero-sum. In this case, we define $\xi_i = -Z_i^{-1} = \bar{\alpha}_i + \bar{\beta}_i$. Thus, ξ_i varies in the interval $[-P^{-1}, -R^{-1}]$. Define $\xi_\pi = \Sigma_{i \in \mathcal{I}} \, \pi_i \xi_i$. From (4.17) we get

$$S_{ij} \quad = \quad K_{ij}(\xi_j \; - \; \xi_i). \tag{4.24}$$

Since $\{S_{ij}\}$ is antisymmetric, $S_{\pi\pi} \; = \; 0$.

For this system the behavior is always like the $\bar{\alpha} < 0$ case for the previous system.

Theorem 4.14 *Let $\mathcal{I}$ index a finite list of non-exceptional ZDS strategies. For the system with*

$$\frac{d\pi_i}{dt} \quad = \quad \pi_i(S_{i\pi} - S_{\pi\pi}) \quad = \quad \pi_i S_{i\pi}, \tag{4.25}$$

we have

$$\begin{aligned} \frac{d\xi_\pi}{dt} \quad &\leq \quad 0, \\ \text{with equality iff} \qquad & \pi_i, \pi_j > 0 \quad \Longrightarrow \quad \xi_i \; = \; \xi_j. \end{aligned} \tag{4.26}$$

Assume now that $Z_{i^*} < Z_j$, or equivalently $\xi_{i^*} < \xi_j$ for all $j \neq i^*$. The strategy i^* is an ESS and if $\pi_{i^*}(0) > 0$ then the solution path converges to the vertex $v(i^*)$.

Proof: Because K_{ij} is symmetric and positive, $\frac{d\xi_\pi}{dt}$ equals

$$\begin{aligned} -\Sigma_{i,j\,\in\,\mathcal{I}}\ \pi_i\pi_j K_{ij}\xi_i(\xi_i-\xi_j) \quad &= \\ -\frac{1}{2}[\Sigma_{i,j\,\in\,\mathcal{I}}\ \pi_i\pi_j K_{ij}\xi_i(\xi_i-\xi_j) - \Sigma_{i,j\,\in\,\mathcal{I}}\ \pi_j\pi_j K_{ij}\xi_j(\xi_i-\xi_j)] \\ = \quad -\frac{1}{2}\Sigma_{i,j\,\in\,\mathcal{I}}\ \pi_i\pi_j K_{ij}(\xi_i-\xi_j)^2 \quad &\leq \quad 0. \end{aligned} \tag{4.27}$$

Equality holds iff $\pi_i\pi_j(\xi_i-\xi_j)^2 = 0$ for all $i,j \in \mathcal{I}$. That is, when $\xi_i = \xi_j$ for all i,j with $\pi_i, \pi_j > 0$.

If $\xi_i < \xi_j$ then

$$S_{ij} \ > \ 0 \ = \ S_{jj} = \ S_{ii} \ > \ S_{ji}. \tag{4.28}$$

If $\xi_i < \xi_k \leq \xi_j$, then $S_{ik} > S_{kk} \geq S_{jk}$. Let $\{j_1, ..., j_n\}$ list $\mathcal{I} \setminus \{i^*\}$ with $\xi_{j_1} \geq .... \geq \xi_{j_n}$. As in Case(-) of Theorem 4.11 it follows that i^* dominates the sequence $\{j_1, ..., j_n\}$. If $\pi_{i^*}(0) > 0$ then $\pi(t)$ converges to $v(i^*)$ by Proposition 3.4. $\square$

5 Appendix: Strictly Firm Strategies

We describe a class of memory-one strategy vectors which are dual to the good strategies and use them to extend the results of Theorem 4.9.

Definition 5.1 *A memory one strategy* $\mathbf{p}$ *for* X *is called* strictly firm *if it is firm and if for any strategy pattern chosen by* Y *against it and any associated limit distribution* $\mathbf{v}$*, we have*

$$s_Y \ \geq \ P \qquad \Longrightarrow \qquad v_4 = 1 \qquad \textit{and so} \quad s_Y \ = \ s_X \ = \ P. \tag{5.1}$$

Theorem 5.2 *Let* $\mathbf{p} = (p_1, p_2, p_3, p_4)$ *be a firm strategy vector.*

The strategy $\mathbf{p}$ *is of strictly firm iff the following inequalities hold.*

$$p_3 < \frac{P-S}{R-P} \cdot (1-p_1) \qquad \textit{and} \qquad p_3 < \frac{P-S}{T-P} \cdot (1-p_2). \tag{5.2}$$

If $\mathbf{p}$ *is a firm strategy vector with with X Press-Dyson vector* $\tilde{\mathbf{p}} = \alpha \mathbf{S}_X + \beta \mathbf{S}_Y + \gamma \mathbf{1} + \delta \mathbf{e}_{23}$*, the strategy* $\mathbf{p}$ *is strictly firm iff*

$$-\delta \quad > \quad \frac{1}{T-S} max(\alpha, (T+S-2P)\alpha). \tag{5.3}$$

As before, we normalize using (2.1) and so prove the normalized version.

Theorem 5.3 *Let* $\mathbf{p} = (p_1, p_2, p_3, p_4)$ *be a firm strategy vector.*
The strategy $\mathbf{p}$ *is of strictly firm iff the following inequalities hold.*

$$p_3 < \frac{P}{R-P} \cdot (1-p_1) \qquad and \qquad p_3 < \frac{P}{1-P} \cdot (1-p_2). \tag{5.4}$$

If $\mathbf{p}$ *is a firm strategy vector with with X Press-Dyson vector* $\tilde{\mathbf{p}} = \alpha \mathbf{S}_X + \beta \mathbf{S}_Y + \gamma \mathbf{1} + \delta \mathbf{e}_{23}$*, the strategy* $\mathbf{p}$ *is strictly firm iff*

$$-\delta \quad > \quad max(\alpha, (1-2P)\alpha). \tag{5.5}$$

Proof: We proceed as in Theorem 2.1.

$$\begin{aligned} s_Y - P \quad &= \quad (v_1 R + v_2 + v_4 P) \ - \ (v_1 P + v_2 P + v_3 P + v_4 P) \\ &= \quad v_1(R-P) \ + \ v_2(1-P) \ - \ v_3 P. \end{aligned} \tag{5.6}$$

So that

$$s_Y \ >= \ P \qquad \Longleftrightarrow \qquad v_1(R-P) \ + \ v_2(1-P) \ \geq \ v_3 P. \tag{5.7}$$

This time $p_4 = 0$ and so that Theorem 1.3 implies

$$v_1(1-p_1) + v_2(1-p_2) \ = \ v_3 p_3. \tag{5.8}$$

Multiplying by $p_3 \geq 0$ and substituting we obtain

$$\begin{gathered} s_Y \ >= \ P \qquad \Longrightarrow \qquad A v_1 \ >= \ B v_2 \qquad \text{with} \\ A \ = \ [p_3(R-P) - (1-p_1)P] \qquad \text{and} \qquad B \ = \ [(1-p_2)P - p_3(1-P)]. \end{gathered} \tag{5.9}$$

The reverse implication holds if $p_3 > 0$.

The inequalities (5.4) say that $A < 0$ and $B > 0$. When they hold, $s_Y \geq P$ implies, by (5.9) that $v_1 = v_2 = 0$ and so from (5.7) that $v_3 = 0$ and so $v_4 = 1$ as required.

If $A \geq 0$ and Y plays $\mathbf{q} = (1,1,1,1)$ then $v_4 \neq 1$ since $\mathbf{q}$ is not firm. Furthermore, no edge ends at cd and so $v_2 = 0$. If $p_3 > 0$ then the reverse implication in (5.9) holds and so $s_Y \geq P$ without $v_4 = 1$. If $p_3 = 0$ then $A \geq 0$ implies $p_1 = 1$ and so $\mathbf{p} = (1, p_2, 0, 0)$. Against the agreeable strategy $(1,1,1,1)$ $\{cc\}$ is a terminal set with $s_Y = R > P$.

If $B \leq 0$ and Y plays $\mathbf{q} = (0,0,0,1)$ then again $v_4 \neq 1$. This time no edge ends at cc and so $v_1 = 0$. If $p_3 > 0$ then the reverse implication in (5.9) holds and so $s_Y \geq P$ without $v_4 = 1$. If $p_3 = 0$ then $B \leq 0$ implies $p_2 = 1$ and so $\mathbf{p} = (p_1, 1, 0, 0)$. Against $(0,0,0,1)$ $\{cd\}$ is a terminal set with $s_Y = 1 > P$.

For (5.5) we proceed as in Theorem 2.5. Because $\mathbf{p}$ is firm, $\beta = -\alpha - \gamma P^{-1}$ and we have

$$\begin{aligned} (1-p_1) \;&=\; -\tilde{p}_1 \;=\; \frac{R-P}{P}\gamma, \\ (1-p_2) \;=\; -\tilde{p}_2 \;&=\; -\beta-\gamma-\delta \;=\; \alpha + \frac{1-P}{P}\gamma - \delta, \\ p_3 \;&=\; \tilde{p}_3 \;=\; \alpha+\gamma+\delta. \end{aligned} \tag{5.10}$$

The inequality $(1-P)p_3 < P(1-p_2)$ becomes $(1-2P)\alpha + \delta < 0$. The inequality $(R-P)p_3 < P(1-p_1)$ reduces to $\alpha + \delta < 0$. $\square$

Remark: If $\mathbf{p}$ is a ZDS, i. e. $\delta = 0$, then $\mathbf{p}$ can be strictly firm only when $1 - 2P > 0$. That is, $P < \frac{1}{2}$. In that case, a ZDS is strictly firm iff $\alpha < 0$.

Clearly, *AllD* $= (0,0,0,0)$ is strictly firm, but *Grim* $= (1,0,0,0)$ is not.

Now we extend Theorem 4.9

Theorem 5.4 *Let $\mathcal{I}$ index a finite set of strategies for the IPD.*

Suppose that associated with $i^* \in \mathcal{I}$ is a strictly firm strategy $\mathbf{p}^{i*}$ with initial play d. If for no other $j \in \mathcal{I}$ is the strategy $\mathbf{p}^j$ firm, then i^* is an ESS for the associated game $\{A_{ij} : i, j \in \mathcal{I}\}$.

Proof: Since i^* is associated with a firm strategy and initial defection we have $A_{i^*i^*} = P$. Since $\mathbf{p}^{i^*}$ is firm and no opponent plays a firm strategy it follows that for any $j \neq i^*$ that $A_{ji^*} < P$. Hence, i^* is an ESS. $\square$

References

1. E. Akin, (1980) Domination or equilibrium, Math. Biosciences **50:** 239-250.

2. —, (1990) The differential geometry of population genetics and evolutionary games, In S. Lessard (ed.) **Mathematical and statistical developments of evolutionary theory**, Kluwer, Dordrecht : 1-93.

3. E. Akin, J. Hofbauer, (1982) *Recurrence of the unfit*, Math. Biosciences **61:** 51-63.

4. R. Axelrod, *The Evolution of Cooperation*, Basic Books, New York, NY, 1984.

5. M. Boerlijst, M. Nowak, K. Sigmund, (1997) Equal pay for all prisoners, Amer. Math. Monthly, **104** 303-305.

6. M. D. Davis, *Game Theory: A Nontechnical Introduction*, Dover Publications, Mineola, NY, 1983.

7. G. Hardin, (1968) The tragedy of the commons, Science, **162** 1243-1248.

8. C. Hilbe, M. Nowak, K. Sigmund, (2013) The evolution of extortion in iterated Prisoner's Dilemma games, PNAS, **110** no. 17, 6913-6918.

9. J. Hofbauer, K. Sigmund, *Evolutionary Games and Population Dynamics*, Cambridge Univ. Press, Cambridge, UK, 1998.

10. S. Karlin, H. Taylor, *A First Course in Stochastic Processes*, second edition, Academic Press, New York, NY, 1975.

11. J. Maynard Smith, *Evolution and the Theory of Games*, Cambridge Univ. Press, Cambridge, UK, 1982.

12. M. Nowak, *Evolutionary Dynamics*, Harvard Univ. Press, Cambridge, MA, 2006.

13. K. Sigmund, *Games of life*, Oxford Univ. Press, Oxford, UK, 1993.

14. —, *The Calculus of Selfishness*, Princeton Univ. Press, Princeton, NJ, 2010.

15. W. Press, F. Dyson, (2012) Iterated Prisoner's Dilemma contains strategies that dominate any evolutionary opponent, PNAS, **109** no. 26, 10409-10413.

16. A. Stewart, J. Plotkin, (2012) Extortion and cooperation in the Prisoner's Dilemma, PNAS, **109** no. 26, 10134-10135.

17. P. D. Straffin, *Game Theory and Strategy*, Mathematical Association of America, Washington, DC, 1993.

18. P. Taylor, L. Jonker, (1978) Evolutionarily stable strategies and game dynamics, Math. Biosciences **40:** 145-156.

19. J. Von Neumann, O. Morgenstern, *Theory of Games and Economic Behavior*, Princeton Univ. Press, Princeton, NJ, 1944.